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LOCALITY AND COVARIANCE OF THE SPECTRUM

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Abstract

It will be shown that locality and spectrum condition together imply that the support of the spectrum of the translations is a set invariant under Lorentz transformation. On the way we develop a local method for finding real points in the envelope of holomorphy for the opposite edge of the wedge problem.

Introduction

V. Glaser and R. Streater [1,2] have observed that the domain of holomorphy of the n-point Wightman function is invariant under complex Lorentz transformation. This result follows alone from spectrum condition and locality. It is not necessary for this result to assume that Lorentz transformations are a symmetry of the theory under consideration.

Some years later the author has observed [3] that in the vacuum sector the spectrum of the translations has a support which is invariant under Lorentz transformations also if the theory is not Lorentz invariant or if one has an invariant theory, but one is dealing with a representation with spontaneously broken Lorentz symmetry. In a recent paper by D. Buchholz and myself [4] it has been shown that in every particle sector (in which the spectrum condition is fulfilled) the lower boundary of the spectrum is invariant under Lorentz transformations. We have conjectured that also gaps in the spectrum, if they exist must necessarily invariant. This paper is devoted to the investigation of this question and it will be shown that this conjecture is correct.

The next section deals with the study of the edge of the wedge problem. We study a situation which is not covered by the Jost-Lehmann-Dyson representation [5, 6]. In particular, we look at the situation where the coincidence domain is the union of two domains. One of them is such that it can be treated by the Jost-Lehman-Dyson technique, and we investigate its influence on the second domain. These investigations generalize in part unpublished results obtained in collaboration with R. Stora (see e.g. the remark in [7]).

In section three we apply our result to a special situation which appears regulary in physics. This is the situation where Jost-Lehman-Dyson domain is given by the points which are spacelike with respect to an order interval (double cone). Finally in the last section we show with the help of the results obtained in section three that also the holes in the spectrum of the translations are sets invariant under Lorentz-transformations.

Part of this note has been prepared during my visit at Z.i.F. I like to thank Professor Streit for his kind hospitality.

II. On the edge of the wedge problem

Let V denote the forward light cone in the Euclidean space R^n:

$$V = \{x \in \mathbb{R}^n;\ x_o^2 - \sum_1^{n-1} x_i^2 \geqq 0 \quad , \quad x_o \geqq 0\}$$

and let V^o be the interiour of V. The forward tube $T^+ \subset \mathbb{C}^n$ denotes the tube with basis V^o:

$$T^+ = \{z \in \mathbb{C}^n \ ; \quad \operatorname{Im} z \in V^o\} \quad ,$$

and the backward tube T^- is the negative of T^+ i.e. $T^- = -T^+ = \overline{T^+}$, where the bar denotes the conjugate complex.

Assume G is an open domain in $\mathbb{R}^n$ and one has two functions f^+ and f^- where f^+ is holomorphic in T^+, and f^- in T^- with the properties:

(i) f^+ and f^- have in G boundary values in the sense of distributions.

(ii) These boundary values coincide on G (in the distributional sense).

In this case it follows from the edge of the wedge theorem [8] that both functions are analytic in a complex neighbourhood of G and that they coincide there as analytic functions. Therefore

f^+ and f^- are different representations of one function f holomorphic in $T^+ \cup T^- \cup N(G)$ where N(G) is the complex neighbourhood of G obtained by the edge of the wedge theorem.

The set $T^+ \cup T^- \cup N(G)$ will almost never be a natural domain. The envelope of holomorphy of this set will be denoted by H(G), and by $H_r(G)$ we denote $H(G) \cap \mathbb{R}^n$ which is mostly the set of interest in physical applications.

The light-cone V introduces in $\mathbb{R}^n$ a semi-ordering. A domain $G \subset \mathbb{R}^n$ is called order convex if it fulfills the relation:

$$G = (G + V) \cap (G - V) \quad .$$

We will call a domain locally order convex if $b - a \in V$ and $a + \lambda(b - a) \in G$ for $0 \leq \lambda \leq 1$ implies $(a + V) \cap (b - V) \subset G$. From the so-called double-cone-theorem [9, 10] one knows that for every coincidence domain G the set $H_r(G)$ is locally order convex.

A hyperboloid h(u, m) is the set

$$h(u, m) = \{z \in \mathbb{C}^n \quad , (z - u)^2 - m^2 = 0\}$$

where z^2 denotes the Minkowski square $z^2 = z_o^2 - \sum_1^{n-1} z_i^2$. Here we are dealing only with the case $u \in \mathbb{R}^n$ and $m^2 \geq 0$ so that h(u, m) can be characterized by a point in $\mathbb{R}^n \times \mathbb{R}^+$. For any domain $G \subset \mathbb{R}^n$ we put

$$\Gamma(G) = \{ (u,m) \in \mathbb{R}^n \times \mathbb{R}^+ ; (x-u)^2 - m^2 < 0 \quad \text{for all } x \in G\} .$$

In the following a domain G will be called a Jost-Lehman-Dyson domain or short J.L.D. domain if it is order convex and if there exist a hyperboloid h(u, m) such that G lies between the two branches of h(u, m). (Not necessarily all points between the two branches belong to G). For J.L.D. domains the envelope of holomorphy of the edge of the wedge problem is known:

Let G be a J.L.D. domain then H(G) are the interiour points of the set

$$\mathbb{C}^n \setminus \bigcup \{h(u, m) \quad ; \quad (u, m) \in \Gamma(G)\}$$

This result has first been proved by F. Dyson [6] for functions having boundary values as tempered distributions which was a generalization of a result of Jost and Lehmann [5]. That this describes the envelope of holomorphy is a result of Bros, Messiah, and Stora [11]. Nowadays it can be shown using general arguments which are due to Pflug [12].

It is now our aim to say something more about the structure of H(G) in the neighbourhood of real point when G is a J.L.D. domain. To this end we introduce the following notation:

II.1 Definition

(1) Let G be a J.L.D. domain, and let $x \in \mathbb{R}^n$ denote

$\Gamma_x(G) = \{(u, m) \in \Gamma(G);\ x \in h(u, m)\}$.

(Remark: $(u, m) \in \Gamma_x(G)$ then $(x - u)^2 \geq 0)$.

(2) Denote by:

$$\Delta_x = \{x - u;\ (u, m) \in \Gamma_x(G) \text{ for a suitable m and } x > u\}$$
$$\bigcup \{u - x;\ (u, m) \in \Gamma_x(G) \text{ for a suitable m and } x < u\}$$

where the order denotes the order given by the light-cone V.

(3) Let $\tilde{K}_x = \mathrm{Co} \bigcup_{\lambda \geq 0} \lambda\ \Delta_x$

where Co indicates the convex hull of this cone.

(4) For any open set U define

$$K_U = \text{closure Co} \bigcup_{x' \in U} \tilde{K}_{x'} \quad .$$

(5) Define

$$K_x = \bigcap_{x \in U} K_U \quad .$$

(6) Denote by

$$C_u = \hat{K}_U \quad \text{and}$$

$$C_x = \hat{K}_x \quad ,$$

where $\hat{K}$ is the open dual cone i.e.:

$$\hat{K} = \{y;\ (y, x) > 0 \quad \text{for all } x \in K, \quad x \neq 0\}$$

if K is a proper cone, and $\hat{K} = \mathbb{R}^n$ if $K = \{0\}$.

Remark:

(1) By construction we see, that $\tilde{K}_x$ is a subcone of V hence the cone C_x contains the interiour of V.

(2) The reason for passing from $\tilde{K}_x$ to K_x is to obtain the cones C_x semi-continuous from the interiour which corresponds to the semi-continuity from below for functions. This semi-continuity is given in the following

II.2 Lemma:

Let G be a J.L.D. domain and let C_x respectively C_U be the cones described in Definition II.1. Assume $y \in C_x$ then we can find an open set U containing x such that $y \in C_U$.

Proof: If $x \in H_r(G)$ then $C_x = \mathbb{R}_n$. Since $H_r(G)$ is open then this is true for a neighbourhood of x. If $x \notin H_r(G)$ then $\tilde{K}_x$ contains non-trivial elements and hence also $K_x \supset \tilde{K}_x$. Let now $u_\varepsilon(x) = \{x'; |x - x'| < \varepsilon\}$ then $K_x = \bigcap_{\varepsilon>0} K_{u_\varepsilon(x)}$ and since $K_{u_\varepsilon(x)}$ is monotonic decreasing follows $\hat{K}_x = \bigcup_{\varepsilon<0} \hat{K}_{u_\varepsilon(x)}$ which implies that every $y \in \hat{K}_x$ belongs to some $\hat{K}_{u_\varepsilon(x)}$ which is the statement of the Lemma by means of the definition of C_x and C_u.

Next we show the following crucial result on the cone C_x:

II.3 Proposition

Let G be a J.L.D. domain then:

1) $y \in C_x$ implies there exist $\lambda > 0$ such that

$$x + i\rho y \in H(G) \quad \text{for} \quad 0 < \rho < \lambda \quad .$$

2) For $(u, m) \in \Gamma_x(G)$ denote by $T_x(u, m)$ the tangent hyperplane at x and by $\bigcup T_x$ the union of these tangent hyperplanes.

Assume:

(α) $\quad x + i\rho y \in H(G)$ for $0 < \rho < \lambda$ for some $\lambda > 0$.

(β) There exist a neighbourhood U of x such that $\mathbb{R}^n - \bigcup_{x'} \bigcup T_{x'}$, $x' \in U$ contains not more than two components, then either

$$y \in C_x \quad \text{or} \quad -y \in C_x \quad .$$

Remark: The domain H(G) is always invariant under complex conjugation. Therefore with $x + iy \in H(G)$ also $x - iy$ belongs to H(G).

Condition 2β shall avoid complications at points where two coincidence domains touch each other.

$\lambda(x,y)$ is semi-continuous from below.

<u>Proof:</u> For this proof introduce the notation

$$\tilde{h}_z = \{(u, m) \in \mathbb{R}^n \times \mathbb{R}^+ \quad ; \quad z \quad h(u, m)\} \quad .$$

If $z = x + iy$ then $(u, m) \in \tilde{h}_z$ implies the two equations $(x - u, y) = 0$ and $(x - u)^2 - m^2 = y^2$. This we can view as an hyperboloid in $(x + y^{\perp}) \times \mathbb{R}$ which has the mass y^2, and its center at x. Since T^+ always belongs to H(G) and since V^o is contained in C_x for every x we are only interested in the cases $y^2 \leq 0$. If $y^2 = 0$ then the set $\tilde{h}_z$ reduces to the line

$$\tilde{h}_z = \{(x + \lambda y, 0) \quad ; \quad \lambda \in \mathbb{R}\} \quad ,$$

otherwise it is the hyperboloid with negative mass square described above.

(1) From Lemma II.2 we know that there exist a neighbourhood $U_x \ni x$ such that $y \in C_{U_x}$. Since C_{U_x} is open there exists a neighbourhood $U_y \ni y$ such that $U_y \subset C_{U_x}$. Assume first $y^2 = 0$ then

$((x'-u'), y) \neq 0$ for $(u', m') \in \Gamma_{x'}$ with $x' \in U_x$ and $x' - u' \neq 0$ and hence $\tilde{h}_{x'+iy} \cap \Gamma(G) = \emptyset$ for $x' \in U_x$. Next assume $y^2 < 0$ and assume $y^2 = -1$. Remark first that the set $\tilde{h}^o_{x,y} = \{(u, m) \in \mathbb{R}^n \times \mathbb{R}^+ \; ; \; x \in h(u, m) \text{ and } (x - u, y) = 0\}$ has an empty intersection with $\Gamma(G)$ if y belongs to C_x since y is not tangent at x to any hyperboloid through x which does not enter G (by construction of $\tilde{K}_x \subset K_x$). The set $\tilde{h}^o_{x,y}$ can be viewed in $(x - y^{\perp}) \times \mathbb{R}$ as a hyperpoloid with mass zero and center x. We look now at the set $\bigcup_{x' \subset U_x} \tilde{h}^o_{x',y}$. Let $r(x')$ be the distance of x' from the complement of U_x then we see that

$\left[\bigcup_{x' \in U_x} \tilde{h}^o_{x',y}\right] \cap (x' + y^{\perp}) \times \mathbb{R}$ contains the hyperboloids

$$(x' - u)^2 - m^2 = - \rho r^2(x') \quad , \quad 0 < \rho < 1 \quad .$$

Since $\bigcup_{x' \in U_x} \tilde{h}^o_{x',y} \cap \Gamma(G) = \emptyset$ we obtain that $\tilde{h}_{x'+i\rho r(x')y} \cap \Gamma(G) = \emptyset$ for $0 < \rho < 1$.

But since this is true by the above remark for a neighbourhood of y one finds $x' + i\,\rho r(x')\,y \in H(G)$ for $0 < \rho < 1$ (If we do not suppose $y^2 = -1$ then the dependence would be

$$\tilde{h}_{x'+i\rho r(x')\frac{y'}{\sqrt{-y'^2}}} \cap \Gamma(G) = \emptyset$$

for $x' \in U_x$ and $y' \in U_y$ with $y'^2 \neq 0$).

(2) If $x + i\varrho y \in H(G)$ for $0 < \varrho < \lambda$ then it follows that $\tilde{h}_{x+i\rho y} \cap \Gamma(G) = \emptyset$. Since now $\tilde{h}^o_{x,y} = \lim_{\rho\to 0} h_{x+i\varrho y}$ we get that y is not tangent at x to any hyperboloid through x which does not enter G. From this it follows that either y or $-y$ belongs to $\hat{\tilde{K}}_x$. At this point it is essential that $\mathbb{R}^n \setminus \bigcup T_x$ has only two components. Since $H(G)$ is open it is easy to see that y or $-y$ belongs also to $C_x = \hat{K}_x$.

From this proposition we obtain immediately

<u>II.4 Theorem.</u>

Let G_1 be a J.L.D. domain and let G_2 be an arbitrary domain. Let $x \in \partial G_2$ and U be an open neighbourhood of x and let

$$H_U(G \cap U) = \text{locally } C_U \text{ order convex hull of } G \cap U$$

then $H_u(G \cap U) \cap U \subset H_r(G_1 \cup G_2)$

In not quite precise terms this means at every $x \in \partial H_r(G_1 \cup G_2)$ the tangent hyperlane at x does not enter $x + C_x$.

Proof

We apply the local version of the double cone theorem* to the coincidence domain $G \cap U$ and the cone C_U. Choosing a sequence U_n with $\bigcap U_n = \{x\}$ we see that the second statement is correct if $H(G_1 \cup G_2)$ has a tangent-plane at x.

III. Applications to commutator functions

In this section we want to explore Theorem II.4 for a situation which appears regularly in quantum field theory.

Let $a \in V^o$ then we denote by D_a the double cone

$$D_a = (a - V) \cap (-a - V)$$

and by

$$D'_a = \{x \in \mathbb{R}^n ;\ (x - y)^2 < 0 \quad \text{for all } y \in D_a\} .$$

D'_a is a J.L.D. domain which appears in configuration space because of the locality of the theory, and in momentum space because of the spectrum condition.

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* Although the double cone theorem is normally proved with a whole cone it is easily to be seen that only some complex neighbourhood of the real points is needed in the proof.

In this case one finds:

$$\Gamma(D_a') = D_a \times \mathbb{R}^+ \quad .$$

We identify D_a' with the domain G_1 of theorem II.4 and we will take for G_2 a domain which is contained in $(-a + V) \cup (a - V)$. In order to simplify life we will assume that G_2 is order convex and order restricted this means again a J.L.D. domain.

For the formulation of the results we need some notations.

III.1 Definition

(1) Let G_1, G_2 be two J.L.D. domains.

Define

$$\tilde{\tilde{G}} = \mathbb{R}^n \setminus \bigcup \{h(u, m) \quad ; \; (u, m) \in \Gamma(G_1) \cap \Gamma(G_2)\}$$

and

$$\tilde{G}_2 = \bigcup \text{ components of } \tilde{\tilde{G}} \text{ containing a component of } G_2,$$

(2) For $b - a \quad V^o$ we denote by $D_{a,b}$ the set $(a + V) \cap (b - V)$.

This implies the set D_a defined before is the same as $D_{-a,a}$ in this notation. After these preparations we are able to formulate the following result:

III.2 Theorem: Let $D_{b,c}$ be in the interiour of $-a + V$ and assume $c - b \in V^o$, then $\tilde{D}^o_{b,c}$ is contained in $H_r(D_a' \cup D^o_{b,c})$.

The proof of this theorem will need some preparations. Let us describe first the technique we want to employ for the proof of the Theorem.

<u>III.3 Lemma:</u>

Assume G_2 is a connected domain and $x \in \mathbb{R}^n$ such we can find a family of curves $\mathcal{C}_\mu$, $0 \leq \mu \leq 1$ with the properties:

(i) For every fixed μ the curve $\mathcal{C}_\mu : \{x_\mu(\lambda) \ , \ 0 \leq \lambda \leq 1\}$ is a continuous piecewise C^1-curve.

(ii) The right and left tangents at $x(\lambda)$ belong to the cone $C_{x(\lambda)}$.

(iii) $x_\mu(0)$ and $x_\mu(1) \in G_2$.

(iv) The function $x_\mu(\lambda)$ is continuous in both variables on $[0, 1]^2$.

(v) $\mathcal{C}_o \subset G_2$.

(vi) $x \in \mathcal{C}_1$.

Then $x \in H_r(G_1 \cup G_2)$.

<u>Proof:</u> Assume we know that $\bigcup_{\mu<\mu_o} \mathcal{C}_\mu$ and a real neighbourhood of this belongs to $H_r(G_1 \cup G_2)$. But since the tangents at $\mathcal{C}_{\mu_o}$ belong to $C_{x_{\mu_o}}(\lambda)$ it follows from Theorem II.4 and Lemma II.2 that also some neighbourhood of $\mathcal{C}_{\mu_o}$ belongs to H_r. Since this neighbourhood contains $\mathcal{C}_\mu$ for $\mu_o \leq \mu < \mu_1$ for some $\mu_1 > \mu_o$ we see by induction that also $\mathcal{C}_1$ belongs to $H_r(G_1 \cup G_2)$ and consequently the point x, which belongs to $\mathcal{C}_1$.

Next we have to describe the set $\tilde{D}_{b,c}$ in some detail. As we have remarked earlier, the set $\Gamma(D'_a)$ is nothing else than $D_a \times \mathbb{R}^+$.

From this knowledge one easily can characterize the cone C_x. For $x \in D'_a$ one has $C_x = \mathbb{R}^n$. For $x \in D_a$ one gets $C_x = V^o$. If $x \in (-a+V) \setminus D_a$ then $\tilde{K}_x$ is the cone generated by the elements $x - u$ with $u \in D_a \cap x - V$. It is easy to see that $\tilde{K}_x$, which is convex, changes continuously with x. Hence $\tilde{K}_x$ and K_x coincide. From this we get that C_x is the open dual cone of $\tilde{K}_x$.

For what follows it is convenient to introduce a special description. Denote by $a^{\perp}$ the hyperplane perpendicular to a (with respect to the Minkowski scalar product). Then every point in $\mathbb{R}^n$ can uniquely be presented in the form $x = y + \alpha a$, $y \in a^{\perp}$, $\alpha \in \mathbb{R}$. Let K_a denote the set $K_a = D_a \cap a^{\perp}$ and $\Delta_{b,c}$ the projection of $\tilde{D}_{b,c}$ into $a^{\perp}$ so that we can write $\tilde{D}_{b,c} = \{y + \alpha a;\ y \in \Delta_{b,c}$, and $\alpha \in I(y)\}$ where $I(y)$ is a closed interval of $\mathbb{R}$. By $\partial^{\pm} I(y)$ we denote the largest respectively the smallest value of $I(y)$ and by $\partial^{\pm}\tilde{D}_{b,c} = \{y + \alpha^{\pm}(y)a;\ y \in \Delta_{b,c},$ $\alpha^{\pm}(y) \in \partial^{\pm} I(y)\}$.

If b and c are both belonging to D_a then $\tilde{D}_{b,c}$ and $D_{b,c}$ coincide, in this case nothing has to be proved. If b belongs to D_a then $\partial^{-}\tilde{D}_{b,c}$ is a subset of $b + \partial V$. Assume $c \in \{-a + V\} \setminus D_a$ then $\partial^{+} D_{b,c}$ is given by the envelope of hyperboloids containing c and which have their center in $D_a \cap c - V$. This gives us for $\alpha(y)$ the equation

$$\alpha^{+}(y) = \inf \{\alpha\ ; \exists\, u \in D_a \cap c - V \quad \text{and}$$

$$(c - u)^2 = (y + \alpha a - u)^2\} \quad .$$

If $b \in \{-a + V\} \setminus D_a$ then we get for $\alpha^-(y)$ a similar formula but c replaced by b and the infimum replaced by the supremum. The set $D_a \cap c - V$ is compact and convex so that the above infimum is taken at a boundary point of $D_a \cap (c - V)$. From the equation $(c - u)^2 - (y + \alpha a - u)^2 = 0$ we obtain

$$\frac{\partial \alpha}{\partial (d,u)} = -\frac{(d, c - (y + \alpha a)}{(a, \alpha a - u)} \quad .$$

Since $(a, \alpha a - u) > 0$ we see that $\alpha^+(y)$ is obtained at the point where the hyperplane perpendicular to $c - (y + \alpha^+ a)$ is tangent to $D_a \cap (c - V)$ and moreover if u_o is the point of contact then

$$(u - u_o \ , c - (y + \alpha^+ a)) \geq 0 \quad . \qquad (*)$$

From this we find the following location for u_o:

(i) For $(c - u_o)^2 \neq 0$ we have $u \in \partial K_a \cap c - V$

(ii) For $(c - u_o)^2 = 0$ but $(c - y - \alpha^+ a)^2 \neq 0$ we have

$$u \in \partial K_a \cap c - \partial V$$

(iii) For $(c - u_o)^2 = 0$ and $(c - y - \alpha^+ a)^2 = 0$ the three vectors $c - u_o$, $c - y - \alpha^+ a$, and $y + \alpha^+ a - u_o$ are all multiples of each other and light-like. In this case u_o is not uniquely defined and we can choose $u_o \in K_a \cap c - \partial V$.

From this consideration we learn that the point u_o depends only on the direction of $y - c_o$, where c_o denotes the projection of c onto $a^{\perp}$. Hence the curve

$$C_1: x(\lambda) = y + \lambda(c_o - y) + \alpha^+(y + \lambda(c_o - y))a$$

is lying on a fixed hyperboloid and is therefore representing a differentiable curve. Moreover we obtain:

III.4 Lemma: Let C_1 be the above curve on $\partial^+\tilde{D}_{b,c}$ and let λ_o be such that $x(\lambda_o) - b \in V^o$ then $x(\lambda)$ for $\lambda_o \geqq \lambda \geqq 0$ belongs also to $\partial^+\tilde{D}_{b,x(\lambda_o)}$.

Proof: Let us introduce $\beta^+(y)$ by the relation

$$y + \beta^+(y) \cdot a \in \partial^+\tilde{D}_{b,x(\lambda_o)} \quad .$$

Then β^+ is defined in the same manner like α^+ but only replacing c by $x(\lambda_o)$. Since the direction of $c_o - y$ and $\{(y + \lambda_o(c_o - y)\} - y)$ coincide it follows that the infimum is taken at the same point u_o. But since $(c - u_o)^2 = (x(\lambda_o) - u_o)^2$ we see that $\alpha^+(y(\lambda))$ and $\beta^+(y(\lambda))$ coincide for $\lambda_o \geqq \lambda \geqq 0$ and $y(\lambda) = y + \lambda(c_o - y)$.

Remark: This Lemma remains still true if we replace the point b by a point $b' \in \partial^-\tilde{D}_{b,c}$ provided $x(\lambda_o) - b'$ is time-like.

Using this last Lemma we see that we can replace the point c by $x(\lambda)$. Therefore equation (*) becomes for $x(\lambda) \in C_1$

$$(u - u_o, x(\lambda_1) - x(\lambda_o)) \geqslant 0 \quad \text{for } \lambda_1 > \lambda_o$$

and hence if we denote by $t(\lambda)$ the tangent at the curve C_1, we find

$$(u - u_o, t(\lambda)) \geq 0 .$$

Since we have $(x(\lambda) - u_o , t(\lambda)) = 0$ by construction we obtain $(x(\lambda) - u_o , t(\lambda)) \geq 0$ so that $t(\lambda) \in \partial C_{x(\lambda)}$ for every $x(\lambda) \in C_1$.

We obtain $\alpha^+(y(\lambda))$, with $y(\lambda) = y - \lambda(c_o - y)$, by the equation

$$(c - u_o)^2 = (y(\lambda) - u_o + \alpha^+(y(\lambda))a)^2 ,$$

which gives us:

$$\alpha^{+2}(y(\lambda))\, a^2 = (c - u_o)^2 - (y(\lambda) - u_o)^2 ,$$

since u_o belongs to $a^{\perp}$. From this we obtain

$$2\alpha^+ \frac{d\alpha^+}{d\lambda} \cdot a^2 = -2(y'(\lambda), y(\lambda) - u_o) = -2(c_o - y, y(\lambda) - u_o)$$

and hence

$$t(\lambda) = c^o - y - \frac{(c_o - y, y(\lambda) - u_o)}{\alpha^+ \cdot a^2} a .$$

Next we define the curve $\hat{C}_1$ by:

$$\hat{C}_1 : \hat{x}(\lambda) = x(\lambda) - (2 - \lambda)\varepsilon \cdot a$$

and obtain for its tangent

$$\hat{t}(\lambda) = t(\lambda) + \varepsilon \cdot a .$$

From this we obtain:

$$(\hat{x}(\lambda) - u, \hat{t}(\lambda)) = (x\,(\lambda) - u - (2 - \lambda)\varepsilon \cdot a,\ \ t(\lambda) + \varepsilon a)$$

$$= (x\,(\lambda) - u,\ t(\lambda)) + \varepsilon(x(\lambda) - u - (2 - \lambda)\ t(\lambda),\ a)$$

$$- (2 - \lambda)\ \varepsilon^2\ a^2 \quad .$$

The first term is always non-negative and takes the value zero for $u = u_o$. Hence the curve $\hat{C}_1$ has its tangents $\hat{t}(\lambda)$ in $C_{\hat{x}(\lambda)}$ if the two remaining terms are positive. The term $(x(\lambda) - u, a)$ is strict positive, since $(x(\lambda) - u)$ is time- or light-like and $a \in V^o$, and moreover since c does not belong to D_a. Hence there exist a constant $m > 0$ with

$$(x(\lambda) - u,\ a) \geqq m > 0 \quad .$$

The next term $-(2 - \lambda)(t(\lambda), a)$ becomes $(2 - \lambda)\ \dfrac{(c_o - y, y(\lambda) - u_o)}{\alpha}$ which can have either sign.

α^+ is bounded below and $y(\lambda)$ is between c_o and y this means the above expression tends linearly to zero with $\| c_o - y \|$.
Hence there exist $\delta > 0$ such that

$$|-(2 - \lambda)(t(\lambda),\ a)| < \frac{m}{2} \qquad \text{for } \| c_o - y \| \leqq \delta \quad .$$

From this we learn for $\varepsilon \leqq \dfrac{m}{8a^2}$ and $\| c_o - y \| \leqq \delta$ we have $\hat{t}(\lambda) \in C_{\hat{x}(\lambda)}$. Putting everything together we get:

<u>III.5 Lemma:</u> With the notation of the Theorem we obtain:
Let $x \in b + V^{o} \cap \tilde{D}^{o}_{b,c}$ then $x \in H_r(D'_a \cup D^{o}_{b,c})$ or let $x \in c - V^{o} \cap \tilde{D}^{o}_{b,c}$ then we get the same statement.

<u>Proof of the Lemma:</u> Assume $x = y + \alpha a$ is such that $\|c - y\| < \delta$ with the above δ. Let ε be such that it fulfills the following three conditions

(i) $\varepsilon \leq \frac{m}{8a^2}$, (ii) $x \in b + 3\varepsilon \cdot a + V$, $b + \varepsilon a \in D_{b,c}$

(iii) $\alpha \leq \alpha^{+}(y) - 3\varepsilon$, $c - \varepsilon a \in D_{b,c}$. Define a family of curves C_μ as follows.

$$C_\mu : \begin{cases} b + \varepsilon a + 3\lambda\mu(x - \varepsilon a - [b + \varepsilon a]) & , \ 0 \leq \lambda \leq \frac{1}{3} ; \\ \mu x + (1 - \mu)b + (1 - 2\mu)\varepsilon \cdot a + (3\lambda - 1)\{[\mu y + (1 - \mu)c_o \\ \quad + \alpha^{+}(\mu y + (1 - \mu)c_o) - (1 + \mu)\ \varepsilon \cdot a] \\ \quad - [\mu x + (1 - \mu)b + (1 - 2\mu)\ \varepsilon \cdot a]\} & , \ \frac{1}{3} \leq \lambda \leq \frac{2}{3} , \\ \mu y + (1 - \mu)c_o + \mu(3\lambda - 2)(c_o - y) \\ \quad + \alpha^{+}\{\mu y + (1 - \mu)c_o + \mu(3\lambda - 2)(c_o - y)\} \\ \quad - [1 + \mu - \mu(3\lambda - 2)]\ \varepsilon \cdot a & , \ \frac{2}{3} \leq \lambda \leq 1 . \end{cases}$$

From the choice of δ and ε it follows that the tangent at the curves C_μ belong to the cone $C_{x_{\mu,\lambda}}$. So by Lemma III.3 the point x belongs to the real points of the envelope of holomorphy.
Let now $c' \in \partial \tilde{D}_{b,c}$ with $c - c' < \delta$ and $(c' - b)^2 > 0$

then $\mathring{D}_{b,c'}$ belongs to H_r. Repeating the argument with this double cone we obtain by Lemma III.4 an inductive procedure which exhaust all of $\tilde{D}^{o}_{b,c} \cap b + V^{o}$. This procedure stops because condition (ii) can no longer be fulfilled. The second statement can be proved in the same way if b does not belong to D_a. If it is in D_a then the second statement is empty.

Proof of the Theorem

In $\{\partial\tilde{D}_{b,c} \cap b + V\} \cup \{\partial\tilde{D}_{b,c} \cap c - V\}$ there are new pairs b',c' which are timelike to each other. Applying Lemma III.5 to $D_{b',c'}$ we obtain an inductive procedure which exhausts all of $\tilde{D}^{o}_{b,c}$ (by Lemma III.4).

Remarks:

(1) Since our method is purely local we have not treated the case where the second domain is an arbitrary J.L.D. domain, since I expect that there might occur a similar phenomenon then the "reentrence nose" effect in the J.L.D. problem.

(2) If the domain $D_{b,c}$ is sufficient large compared to D_a, then $\tilde{D}_{b,c}$ has two components. Nothing is said about this second component. If it appears I suspect that the envelope of homomorphy is no longer schlicht.

IV On the spectrum of the translations

In this section we want to apply Theorem III.2 to the theory of local observables in the sense of Araki, Haag, and Kastler. Let $\{A, A(0)\ ,\ \mathbb{R}^n, \alpha\}$ denote this theory and let $\{H\ , \pi, U\}$ be covariant representation fulfilling spectrum condition. The representation of the translation group can be choosen to fulfill the following conditions see [4] .

(i) $U(a)$ is a continuous group representation (by assumption).

(ii) The spectrum of U is contained in the forward lightcone (by assumption).

(iii) $U(a) \in \pi(A)''$

(iv) $U(a)$ implements the automorphism a i.e.

$$U(a)\ \pi(x) U\ (a) = \pi(\alpha_a\ x)$$

(v) $U(a)$ can be chosen minimal this means if V(a) implements the translations and fulfills the spectrum condition then V(a) U (a) has its spectrum in the forward light cone V.

(vi) In [4] we have shown: If $U(a)$ is the minimal translation and if the projection $F \in Z(\pi'')$ (the center of π'') then the lower boundary of the spectrum of $U(a)\cdot F$ is invariant under Lorentz transformations.

In this section we intend to prove the following

IV.1 Theorem

Let $\{A, A(0), \mathbb{R}^n, \alpha\}$ be a theory of local observables and let $\{H, \pi, U\}$ be a respresentation fulfilling spectrum condition. Assume in addition U is the unique minimal representation of the translation then for every projection $F \in Z(\pi'')$ the support of the spectrum of $U(a) \cdot F$ is invariant under Lorentz transformations. Before entering into the details of the proof we prepare some notations and describe the methods used for proving the theorem.

Let $U(a)$ be the representation of the translation group described before. Let

$$U(a) = \int \exp i(ap)\, dE(p)$$

be its integral representation. If Δ is a Borel set then its spectral projection is denoted by $E(\Delta)$. For a vector $\psi \in H$ we say ψ has support in Δ if $E(\Delta)\psi = \psi$ holds.

IV.2 Corollary to the invariance of the lower boundary:

Let ℓ be a lightlike vector and let $\varepsilon > 0$ and k_ε the ball of radius ε . Denote by

$$\Lambda_{\ell,\varepsilon} = \{ \bigcup_{\lambda \geq 0} (\lambda\ell + k_\varepsilon)\} \cap V$$

then one obtains central support of $E(\Lambda_{\ell,\varepsilon}) = 1$.

Proof: Let F be the central support of $E(\Delta_{\ell,\varepsilon})$, assume $F \neq 1$ then follows $(1 - F)\, E\,(\Delta_{\ell,\varepsilon}) = 0$. But the lower boundary of the spectrum of $(1 - F)\, U(a)$ is invariant and hence it is a hyperboloid with mass M. Choose now a coordinate system then $\ell = \{1, \vec{\ell}\}$ with $\vec{\ell}^2 = -1$. Let ℓ' be the vector $\ell' = \{1, -\vec{\ell}\}$ then $\lambda\ell + \alpha\ell' \in \Delta_{\ell,\varepsilon}$ for $\lambda > 0$ and $0 \leq \alpha \leq \frac{\varepsilon}{\sqrt{2}}$. But we have $(\lambda\ell + \alpha\ell')^2 = 4\cdot\lambda\alpha$. Choosing $\alpha = \frac{\varepsilon}{2}$ and $\lambda = \frac{M^2}{2\varepsilon}$ we see

$$(\lambda\ell + \alpha\ell')^2 = M^2$$

so that $(1 - F)\, E\,(\Delta_{\ell,\varepsilon}) \neq 0$ contradicting the assumptions.

Let $x \in \mathcal{A}(D_b^o)$ for some timelike b and $\psi \in H$. We denote

$$F^+_{x,\psi}(a) = (\psi, \pi(x^*)\; U(a)\; \pi(x)\psi) \quad ,$$

and

$$F^-_{x,\psi}(a) = (\psi,\, \pi(\alpha_a x)\; \pi(x^*)\; U(a)\psi)$$

$$= (\psi,\, U(a)\; \pi(x)\; U(-a)\; \pi(x^*)\; U(a)\psi) \quad .$$

From the locality condition we obtain

$$F_{x,\psi}(a) = F^+_{x,\psi}(a) - F^-_{x,\psi}(a) = 0 \quad \text{for} \quad a \in D'_{2b} \quad .$$

Therefore we can "cut" $F^+_{x,\psi}(a) - F^-_{x,\psi}(a)$ into two pieces:

$$F_{x,\psi}(a) = F^+_{x,\psi}(a) - F^-_{x,\psi}(a) = G^+_{x,\psi}(a) - G^-_{x,\psi}(a)$$

with supp $G^{+}_{x,\psi}$ (a) $\subset -\ell b + V$ and supp $G^{-}_{x,\psi}\mathcal{U}(a) \subset \ell b - V$.

This cutting we do for investigating the properties of the Fouriertransforms of F^{+} and F^{-}. To this end assume Δ is a compact subset at spectrum $\mathcal{U}(a)$ and ψ has support in Δ then

$$\text{supp } \mathcal{F}^{-1} F^{+}_{x,\psi}(a) \subset \text{spectrum } \mathcal{U}(a)$$

and

$$\text{supp } \mathcal{F}^{-1} F^{-}_{x,\psi}(a) \subset 2\Delta \text{ -spectrum } \mathcal{U}(a) \quad ,$$

and hence

$$\text{supp } \mathcal{F}^{-1} F_{x,\psi}(a) \subset \{\text{spectr. } \mathcal{U}\} \cup \{2\Delta \text{ -spectr. } \mathcal{U}\}.$$

Next $(\mathcal{F}^{-1}G^{+}_{x,\psi})(P)$ is the boundary value of an analytic function holomorphic in T^{+} and $(\mathcal{F}^{-1}G_{x,\psi})(P)$ has an analytic continuation into T^{-}. From

$$F_{x,\psi}(a) = G^{+}_{x,\psi}(a) - G^{-}_{x,\psi}(a)$$

we obtain

$$(\mathcal{F}^{-1}G_{x,\psi})(P) = (\mathcal{F}^{-1}G_{x,\psi})(P)$$

for p in the complement of

$$\{\text{spectr.} \mathcal{U}\} \cup \{2\Delta\text{-spectr. } \mathcal{U}\}$$

and hence by analytic continuation in $H_r(C\ [\text{spectr. } \mathcal{U} \cup 2\Delta\text{-spectr.} \mathcal{U}]$

Remark next that $\mathcal{F}^{-1}F^{+}$ and $\mathcal{F}^{-1}F$ coincide in that part of V which is outside of 2Λ-V. From this we find:

If $P \in V \setminus 2\Delta - V$ then

$$(F^{-1}F^{+}_{x,\psi})(P) = 0 \quad \text{for} \quad P \in H_r(C[\text{spectr.}\,\mathcal{U} \cup 2\Delta\text{-spectr.}\ \mathcal{U}]).$$

Proof of the Theorem:

Remark that the only restriction in the last equation is that $x \in A(D^O_b)$ for some b. But $\bigcup A(D^O_b)$ is dense in A in the norm topology, so that $\pi(A)\ E(\Delta)H$ is dense in FH where F is the central support of $E(\Delta)$ so that the above equation reads:

Let $\Delta' \subset H_r\ (C\ [\text{spectr.}\ \mathcal{U} \cup 2\Delta\text{-spectr.}\ \mathcal{U}])$ and assume $\Delta' \cap 2\Delta - V = \emptyset$ then $E(\Delta')\ F = 0$ where F = central support of $E(\Delta)$. With the help of the Jost-Lehmann-Dyson method this leads to the result proved in [4] that the lower boundary of the spectrum is invariant.

Now we want to assume that we have a hole in the spectrum and assume for simplicity that this is an orderinterval $D_{b,c}$. Choose a lightlike direction ℓ and $\varepsilon > 0$ such that $2\Delta_{\ell,\varepsilon} \cap D^O_{b,c} = \emptyset$. Take any vector $a \in \Delta_{\ell,\varepsilon}$ then one has that $2\ D_{0,a} \cap D^O_{b,c} = \emptyset$. From Corollary IV.2 follows that $E(D_{0,a+\lambda\ell}) \neq 0$ for sufficiently large λ and that

$$\lim_{\lambda\to\infty} F(D_{0,a+\lambda\ell}) = 1$$

where $F(D_{0,a+\lambda\ell})$ is the central support of $E(D_{0,a+\lambda\ell})$.

From this we obtain by Theorem III.4

$$\{E(\tilde{D}^O_{b,c}) \cdot F(D_{0,a+\lambda\ell}) \quad ; \quad a \in \Delta_{\ell,\varepsilon} \quad , \quad \lambda > 0\} = 0 \quad .$$

Taking the limit $\varepsilon \to 0$ then $\tilde{D}_{b,c}$ becomes independent of λ provided λ is sufficient large. Therefore we obtain

$$E(\hat{D}^{o}_{b,c}) = 0$$

where $\hat{D}^{o}_{b,c}$ is the limit of $\tilde{D}_{b,c}$ for $\varepsilon \to 0$ and $\lambda \to \infty$.

The boundary of $\hat{D}_{b,c}$ is given by

(i) The hyperboloid $h(0,\sqrt{b^2})$

(ii) The hyperboloid $h(0,\sqrt{c^2})$

(iii) The hyperboloid $h(\lambda_b \ell,0)$

where $\lambda_b \ell = \{\lambda \ell\} \cap b - \partial V$

(iv) The hyperboloid $h(\lambda_c \ell,0)$ with the corresponding definition of λ_c.

From this we see that in $\hat{D}^{o}_{b,c}$ are new double cones $D^{o}_{b',c'}$ with $b'^2 = b^2$ and $c'^2 = c^2$.

From this we obtain that the only set stable under this construction is

$$S\ (\sqrt{b^2},\sqrt{c^2}\ ) = \{\ P;\ \ b^2 < P^2 < c^2\ \ ,\ \ P_o > 0\}\ .$$

This proves the Theorem.

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