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On the Functional Equation $f(\lambda) + f(\omega\lambda)f(\omega^{-1}\lambda) = 1$**,** $(\omega^5 = 1)$

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On the functional equation $f(\lambda)+f(\omega\lambda)f(\omega^{-1}\lambda)=1$, $(\omega^5=1)$

By

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by Yasutaka Sibuya

1. Stokes multipliers of subdominant solutions: In a study of asymptotic solutions of the differential equation

(1.1) $$y'' - P(x)y = 0, \quad P(x) = x^m+a_1x^{m-1}+\dots+a_{m-1}x+a_m \qquad (a_j \in \mathbb{C}),$$

P.F.Hsieh and Y.Sibuya [2] (cf. also, Y.Sibuya [6]) constructed a solution $\mathcal{Y}_m(x,a)$ of (1.1) such that

(i) $\mathcal{Y}_m$ is entire in (x,a) ;

(ii) $\mathcal{Y}_m$ and its derivative $\mathcal{Y}_m'$ with respect to x admit asymptotic representations:

(1.2) $$\mathcal{Y}_m = x^{r_m}\left[1 + O(x^{-\frac{1}{2}})\right]\exp\left[-E_m(x,a)\right],$$

and

(1.3) $$\mathcal{Y}_m' = x^{\frac{1}{2}m+r_m}\left[-1 + O(x^{-\frac{1}{2}})\right]\exp\left[-E_m(x,a)\right],$$

respectively, uniformly on each compact set in the a-space as $x \to \infty$ in the sector

(1.4) $$|\arg x| \leq \frac{3}{m+2}\pi - \delta \qquad (\delta > 0),$$

where

(1.5) $$E_m(x,a) = \frac{2}{m+2}x^{\frac{1}{2}(m+2)} + \sum_{1\leq h<\frac{1}{2}(m+2)} \frac{2}{m+2-2h} b_h(a)x^{\frac{1}{2}(m+2-2h)}$$

(1.6) $$r_m = \begin{cases} -\frac{1}{4}m & \text{if } m \text{ is odd,} \\ -\frac{1}{4}m - b_{\frac{1}{2}(m+2)}(a) & \text{if } m \text{ is even}, \end{cases}$$

(1.7) $$\left(1+\sum_{j=1}^{m} a_j x^{-j}\right)^{\frac{1}{2}} = 1 + \sum_{h=1}^{\infty} b_h(a)x^{-h} .$$

The solution $\mathcal{Y}_m$ is subdominant in the sector

(1.8) $$|\arg x| < \frac{\pi}{m+2},$$

since $\mathcal{Y}_m \longrightarrow 0$ as $x \longrightarrow \infty$ in (1.8).

Set

$$(1.9)\quad \begin{cases} \omega = \exp(i2\pi/(m+2)) , \\ G^k(a) = (\omega^{-k}a_1, \omega^{-2k}a_2, \ldots, \omega^{-mk}a_m) , \quad k \in \mathbb{Z} , \\ \mathcal{Y}_{m,k}(x,a) = \mathcal{Y}_m(\omega^{k}x, G^k(a)) . \end{cases}$$

Then, $\mathcal{Y}_{m,k}$ are solutions of (1.1), and admit asymptotic representations as $x \longrightarrow \infty$ in $\left|\arg x - \frac{2k\pi}{m+2}\right| \leqq \frac{3\pi}{m+2} - \delta$ ($\delta > 0$), respectively. In particular, $\mathcal{Y}_{m,k}$ are subdominant in $\left|\arg x - \frac{2k\pi}{m+2}\right| < \frac{\pi}{m+2}$, resp..

Every pair $\{\mathcal{Y}_{m,k}\ \mathcal{Y}_{m,k+1}\}$ is a set of two linearly independent solution of (1.1), where $\mathcal{Y}_{m,m+2} = \mathcal{Y}_{m,0}$. If we set

$$(1.10)\qquad \mathcal{Y}_{m,k}(x,a) = C_k(a)\,\mathcal{Y}_{m,k+1}(x,a) + \widetilde{C}_k(a)\,\mathcal{Y}_{m,k+2}(x,a) ,$$

then

(i) the quantities $C_k(a)$ and $\widetilde{C}_k(a)$ are entire in a ;

(ii) $C_k(a) = C_0(G^k(a))$, $\widetilde{C}_k(a) = \widetilde{C}_0(G^k(a))$;

(iii)

$$(1.11)\qquad \widetilde{C}_0(a) = \begin{cases} -\omega & \text{if } m \text{ is odd} , \\ -\omega^{1-2b_{\frac{1}{2}(m+2)}(a)} & \text{if } m \text{ is even} . \end{cases}$$

Set

$$(1.12)\qquad S_k(a) = \begin{bmatrix} C_k(a) & 1 \\ \widetilde{C}_k(a) & 0 \end{bmatrix} .$$

Then

$$(1.13)\qquad (\mathcal{Y}_{m,k} , \mathcal{Y}_{m,k+1}) = (\mathcal{Y}_{m,k+1}, \mathcal{Y}_{m,k+2})\, S_k(a) ,$$

and

$$(1.14)\qquad S_{m+1}(a)S_m(a) \ldots S_1(a)S_0(a) = I_2 ,$$

where I_2 is the 2x2 identity matrix. The identity (1.14) is due to the fact that the monodromy group of (1.1) is trivial.

Remark 1.1: If we define $u_1, \ldots, u_m$ by

$$(1.15)\quad (x+s)^m + a_1(x+s)^{m-1} + \cdots + a_{m-1}(x+s) + a_m = s^m + u_1 s^{m-1} + \cdots + u_{m-1}s + u_m ,$$

then

$$(1.16)\qquad C_0(u) = \begin{cases} C_0(a) & \text{if } m \text{ is odd} , \\ \exp 2E_m(x,a)\, C_0(a) & \text{if } m \text{ is even} \end{cases}$$

(cf. Y.Sibuya [6 ; Theorem 21.2, p.84]).

2. **Examples**: The main interest is in the study of (1.14). In this section we shall consider the cases m = 0,1,2,3 and 4.

(1) m = 0: In this case differential equation (1.1) is

$$(2.1)\qquad y'' - y = 0$$

and

$$(2.2)\qquad \begin{cases} \mathcal{Y}_0 = e^{-x} , & \omega = \exp[i\pi] = -1 , \\ \mathcal{Y}_{00} = e^{-x} , & \mathcal{Y}_{01} = e^{x} . \end{cases}$$

Matrices (1.12) are

$$(2.3)\qquad S_0(a) = \begin{bmatrix} C_0 & 1 \\ 1 & 0 \end{bmatrix} , \qquad S_1(a) = \begin{bmatrix} C_1 & 1 \\ 1 & 0 \end{bmatrix} ,$$

and identity (1.14) is

$$(2.4)\qquad S_1 S_0 = I_2 .$$

This implies that

$$S_1 = S_0^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & -C_0 \end{bmatrix} ,$$

and hence $C_0 = 0$ and $C_1 = 0$. This result simply means that

$$(e^{-x}, e^{x}) = (e^{x}, e^{-x}) \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} .$$

(2) $m = 1$: In this case differential equation (1.1) is

(2.5) $$y'' - (x+\lambda)y = 0$$

and matrices (1.12) are

(2.6) $$S_k(\lambda) = \begin{bmatrix} C_k(\lambda) & 1 \\ -\omega & 0 \end{bmatrix} , \quad \omega = \exp[i2\pi/3] .$$

Since

$$S_2(\lambda)S_1(\lambda)S_0(\lambda) = \begin{bmatrix} [C_2(\lambda)C_1(\lambda)-\omega]C_0(\lambda) - C_2(\lambda)\omega & C_2(\lambda)C_1(\lambda) - \omega \\ -\omega[C_1(\lambda)C_0(\lambda)-\omega] & -\omega C_1(\lambda) \end{bmatrix}$$

identity (1.14) implies that

(2.7) $$C_0(\lambda) = C_1(\lambda) = C_2(\lambda) = -\omega^2 .$$

Thus we could find the Stokes multipliers of Airy functions from (1.14) (cf. W.Wasow [9]).

(3) $m = 2$: In this case we consider the differential equation

(2.8) $$y'' - (x^2 + \lambda) y = 0 .$$

Then, matrices (1.12) are

(2.9) $$S_k(0,\lambda) = S_0(0, (-1)^k\lambda) = \begin{bmatrix} \varphi((-1)^k\lambda) & 1 \\ \psi((-1)^k\lambda) & 0 \end{bmatrix} ,$$

where $\psi(\lambda) = (-i)\exp(-\frac{1}{2}\pi\lambda i)$. (Note also that $S_k(a) = S_0(G^k(a))$ (cf. (1.12) and property (ii) of $C_k(a)$ and $\tilde{C}_k(a)$), and that $\omega = \exp(i\frac{1}{2}\pi) = i$). Hence identity (1.14) becomes

(2.10) $$S_0(0,-\lambda)S_0(0,\lambda)S_0(0,-\lambda)S_0(0,\lambda) = I_2 .$$

This means that, if we set $A = S_0(0,-\lambda)S_0(0,\lambda)$, then $A = A^{-1}$. Since

$$A = \begin{bmatrix} \varphi(-\lambda)\varphi(\lambda) + (-i)\exp(-\tfrac{1}{2}\pi\lambda i) & \varphi(-\lambda) \\ (-i)\varphi(\lambda)\exp(\tfrac{1}{2}\pi\lambda i) & (-i)\exp(\tfrac{1}{2}\pi\lambda i) \end{bmatrix},$$

we must have $\varphi(-\lambda)\varphi(\lambda) + (-i)\exp(-\tfrac{1}{2}\pi\lambda i) = -(-i)\exp(\tfrac{1}{2}\pi\lambda i)$, i.e.

$$\varphi(-\lambda)\varphi(\lambda) = 2i\cos(\tfrac{1}{2}\pi\lambda) = \frac{2\pi i}{\Gamma(\tfrac{1}{2}(1-\lambda))\,\Gamma(\tfrac{1}{2}(1+\lambda))} \quad . \tag{2.11}$$

In fact,

$$\varphi(\lambda) = 2^{\frac{1}{2}\lambda}(2\pi)^{\frac{1}{2}}\frac{1}{\Gamma(\tfrac{1}{2}(1+\lambda))}\exp(-\tfrac{1}{4}\pi(\lambda-1)i) \quad . \tag{2.12}$$

To derive (2.12) from (2.11), we need some additional informations on φ (cf. Y.Sibuya [6]).

(4) $m = 3$: In this case we consider the differential equation

$$y'' - (x^3 + ax + b)y = 0 \quad . \tag{2.13}$$

Then, $\omega = \exp(i2\pi/5)$, and matrices (1.12) are

$$S_k(0,a,b) = \begin{bmatrix} C(\omega^{-2k}a,\ \omega^{-3k}b) & 1 \\ -\omega & 0 \end{bmatrix}, \tag{2.14}$$

where $C(a,b) = C_0(0,a,b)$. Identity (1.14) implies that

$$S_4S_3S_2S_1 = S_0^{-1} = \begin{bmatrix} 0 & -\omega^{-1} \\ 1 & \omega^{-1}C(a,b) \end{bmatrix} .$$

Since

$$S_{k+1}S_k = \begin{bmatrix} C(\omega^{-2(k+1)}a,\ \omega^{-3(k+1)}b)C(\omega^{-2k}a,\omega^{-3k}b) - \omega & C(\omega^{-2(k+1)}a,\omega^{-3(k+1)} \\ -\omega C(\omega^{-2k}a,\ \omega^{-3k}b) & -\omega \end{bmatrix}$$

we must have

$$\begin{cases} \left[C(\omega^2 a, \omega^{-2} b)C(\omega^{-1} a, \omega b) - \omega\right]\left[C(\omega a, \omega^{-1} b)C(\omega^{-2} a, \omega^2 b) - \omega\right] = \omega C(\omega^{-2} a, \omega^2 b)C(\omega^2 a, \omega^{-2} b \\ \left[C(\omega^2 a, \omega^{-2} b)C(\omega^{-1} a, \omega b) - \omega\right] C(\omega a, \omega^{-1} b) - \omega C(\omega^2 a, \omega^{-2} b) = -\omega^{-1} , \\ \left[C(\omega a, \omega^{-1} b)C(\omega^{-2} a, \omega^2 b) - \omega\right] C(\omega^{-1} a, \omega b) - \omega C(\omega^{-2} a, \omega^2 b) = -\omega^{-1} , \\ -\omega C(\omega^{-1} a, \omega b)C(\omega a, \omega^{-1} b) + \omega^2 = \omega^{-1} C(a,b) . \end{cases}$$

It is not difficult to verify that these four realtions are equivalent to

(2.15) $$C(\omega^{-1} a, \omega b)C(\omega a, \omega^{-1} b) - \omega = -\omega^{-2} C(a,b) .$$

If we set $F(a,b) = \omega^2 C(a,b)$, (2.15) becomes

(2.16) $$F(a,b) + F(\omega^{-1} a, \omega b)F(\omega a, \omega^{-1} b) = 1 .$$

In particular, if we set $f(\lambda) = F(\lambda, 0)$ or $F(0,\lambda)$, (2.16) becomes

(2.17) $$f(\lambda) + f(\omega\lambda)f(\omega^{-1}\lambda) = 1 .$$

(5) m = 4: We consider only the differential equation

(2.18) $$y'' - (x^4 + \lambda)y = 0 .$$

In this case, $\omega = \exp(i2\pi/6)$, and identity (1.14) is

(2.19) $$S(\omega^{-20}\lambda)S(\omega^{-16}\lambda)S(\omega^{-12}\lambda)S(\omega^{-8}\lambda)S(\omega^{-4}\lambda)S(\lambda) = I_2 ,$$

where

(2.20) $$S(\lambda) = \begin{bmatrix} \varphi(\lambda) & 1 \\ -\omega & 0 \end{bmatrix} , \quad \varphi(\lambda) = C_0(0,0,0,\lambda) .$$

Setting $H(\lambda) = S(\omega^{-4}\lambda)S(\lambda)$, we write (2.19) as

(2.21) $$H(\omega^{-4}\lambda)H(\omega^{-2}\lambda)H(\lambda) = I_2 .$$

Now, observe that

$$H(\lambda) = \begin{bmatrix} \varphi(\omega^2\lambda)\varphi(\lambda) - \omega & \varphi(\omega^2\lambda) \\ -\omega\varphi(\lambda) & -\omega \end{bmatrix} ,$$

$$H(\omega^{-2}\lambda)H(\lambda)$$
$$= \begin{bmatrix} [\varphi(\lambda)\varphi(\omega^{-2}\lambda) - \omega][\varphi(\omega^2\lambda)\varphi(\lambda) - \omega] - \omega\varphi(\lambda)^2 & [\varphi(\lambda)\varphi(\omega^{-2}\lambda) - \omega]\varphi(\omega^2\lambda) - \omega\varphi(\lambda) \\ -\omega[\varphi(\omega^2\lambda)\varphi(\lambda) - \omega]\varphi(\omega^{-2}\lambda) + \omega^2\varphi(\lambda) & -\omega\varphi(\omega^{-2}\lambda)\varphi(\omega^2\lambda) + \omega^2 \end{bmatrix}$$

and

$$H(\omega^{-4}\lambda)^{-1} = \omega^{-2} \begin{bmatrix} -\omega & -\varphi(\omega^{-2}\lambda) \\ \omega\varphi(\omega^{-4}\lambda) & \varphi(\omega^{-2}\lambda)\varphi(\omega^{-4}\lambda) - \omega \end{bmatrix} .$$

Thus, we conclude that (2.21) is equivalent to

$$(2.22) \quad \varphi(\lambda)\varphi(\omega^{-2}\lambda)\varphi(\omega^2\lambda) - \omega[\varphi(\lambda) + \varphi(\omega^{-2}\lambda) + \varphi(\omega^2\lambda)] = 0 .$$

If we set

$$(2.23) \qquad \Delta(\lambda) = \varphi(\omega^2\lambda)\varphi(\lambda) - \omega ,$$

then

$$(2.24) \quad \Delta(\lambda)\Delta(\omega^{-2}\lambda)\Delta(\omega^2\lambda) = \omega^2[\Delta(\lambda) + \Delta(\omega^{-2}\lambda) + \Delta(\omega^2\lambda)] + 2\omega^3$$

(cf. A.Voros [7,8]). Note that

$$(\mathcal{Y}_{4,0}, \mathcal{Y}_{4,1}) = (\mathcal{Y}_{4,2}, \mathcal{Y}_{4,3})\, H(\lambda) .$$

3. $f(\lambda) + f(\omega\lambda)f(\omega^{-1}\lambda) = 1$, $\omega = \exp(i2\pi/5)$: In this section, we shall state some known facts concerning equation (2.17).

(I) There exists a non-trivial entire solution $f(\lambda)$ of (2.17) such that

$$(3.1) \quad f(\lambda) = -[1 + o(1)] \exp[K(1+\omega^{-\frac{5}{6}})\lambda^{\frac{5}{6}}] \text{ as } \lambda \to \infty \text{ in the sector}$$

$$(3.2) \qquad -\frac{4}{5}\pi + \delta \leqq \arg\lambda \leqq 2\pi - \frac{4}{5}\pi - \delta \qquad (\delta > 0) ,$$

where

$$K = \int_0^{+\infty} \left[(t^3+1)^{\frac{1}{2}} - t^{\frac{3}{2}}\right] dt > 0 \quad ;$$

(3.3) $$f(\lambda) = -\left\{ (1+o(1))\exp\left[K(1+\omega^{-\frac{5}{6}})\lambda^{\frac{5}{6}}\right] + (1+o(1))\exp\left[K(1+\omega^{\frac{5}{6}})\lambda^{\frac{5}{6}}\right] \right\}$$

as $\lambda \to \infty$ in the sector

(3.4) $$\left|\arg\lambda + \frac{4}{5}\pi - 2\pi\right| \leqq \delta \qquad (\delta > 0) ;$$

and

(3.5) $$f(0) = \omega^2 + \omega^{-2} .$$

Remark 3.1: $\alpha = \omega + \omega^{-1}$ and $\beta = \omega^2 + \omega^{-2}$ are two zeros of $X^2 + X - 1$.

(II) Y.Sibuya and R.H.Cameron [5] constructed a solution of the form:

(3.6) $$f(\lambda) = \alpha\left\{1 + (1+\alpha^2)\frac{\xi(\lambda)\eta(\lambda) - \beta^2}{[\omega^2\xi(\lambda) + \omega^{-2}\eta(\lambda) - 1][\omega^{-2}\xi(\lambda) + \omega^2\eta(\lambda) - 1]}\right\} ,$$

where ξ and η are arbitrary functions such that $\xi(\omega\lambda) = \omega\xi(\lambda)$ and $\eta(\omega\lambda) = \omega^{-1}\eta(\lambda)$. Similarly, we can construct another solution of the form:

(3.7) $$f(\lambda) = \beta\left\{1 + (1+\beta^2)\frac{\xi(\lambda)\eta(\lambda) - \alpha^2}{[\omega\xi(\lambda) + \omega^{-1}\eta(\lambda) - 1]\,[\omega^{-1}\xi(\lambda) + \omega\eta(\lambda) - 1]}\right\} ,$$

where ξ and η are arbitrary functions such that $\xi(\omega\lambda) = \omega^2\xi(\lambda)$ and $\eta(\omega\lambda) = \omega^{-2}\eta(\lambda)$. Note that if we choose ξ and η so that $\xi(0)=0$ and $\eta(0)=0$, then (3.6) and (3.7) respectively yield

(3.8) $$f(0) = \alpha\left[1 - (1+\alpha^2)\beta^2\right] = \beta$$

and

(3.9) $$f(0) = \beta\left[1 - (1+\beta^2)\alpha^2\right] = \alpha .$$

(III) Utilizing the identity

$$\begin{vmatrix} A & a \\ B & b \end{vmatrix}\begin{vmatrix} C & c \\ D & d \end{vmatrix} - \begin{vmatrix} C & a \\ D & b \end{vmatrix}\begin{vmatrix} A & c \\ B & d \end{vmatrix} = \begin{vmatrix} A & C \\ B & D \end{vmatrix}\begin{vmatrix} a & c \\ b & d \end{vmatrix}$$

we can derive

$$(3.10)\quad \begin{vmatrix} \omega\, a(\omega^2\lambda) & a(\omega\lambda) \\ \omega^{-1} b(\omega^2\lambda) & b(\omega\lambda) \end{vmatrix} \begin{vmatrix} \omega\, a(\lambda) & a(\omega^{-1}\lambda) \\ \omega^{-1} b(\lambda) & b(\omega^{-1}\lambda) \end{vmatrix}$$

$$+ \begin{vmatrix} \omega\, a(\omega\lambda) & a(\lambda) \\ \omega^{-1} b(\omega\lambda) & b(\lambda) \end{vmatrix} \begin{vmatrix} \omega^3 a(\omega^2\lambda) & a(\omega^{-1}\lambda) \\ \omega^{-3} b(\omega^2\lambda) & b(\omega^{-1}\lambda) \end{vmatrix}$$

$$= \begin{vmatrix} \omega^3 a(\lambda) & a(\omega^2\lambda) \\ \omega^{-3} b(\lambda) & b(\omega^2\lambda) \end{vmatrix} \begin{vmatrix} \omega^3 a(\omega^{-1}\lambda) & a(\omega\lambda) \\ \omega^{-3} b(\omega^{-1}\lambda) & b(\omega\lambda) \end{vmatrix} .$$

Therefore, if

$$\begin{vmatrix} \omega^3 a(\omega^3\lambda) & a(\lambda) \\ \omega^{-3} b(\omega^3\lambda) & b(\lambda) \end{vmatrix} = 1 ,$$

then

$$f(\lambda) = \begin{vmatrix} \omega\, a(\omega\lambda) & a(\lambda) \\ \omega^{-1} b(\omega\lambda) & b(\lambda) \end{vmatrix}$$

is a solution of (2.17) and $f(0) = \beta$. More generally, if $\rho^5 = 1$, $\rho \neq 1$, and

$$\begin{vmatrix} \rho\, a(\omega^3\lambda) & a(\lambda) \\ \rho^{-1} b(\omega^3\lambda) & b(\lambda) \end{vmatrix} = 1 ,$$

then

$$f(\lambda) = \begin{vmatrix} \rho^2 a(\omega\lambda) & a(\lambda) \\ \rho^{-2} b(\omega\lambda) & b(\lambda) \end{vmatrix}$$

is a solution of (2.17) and $f(0) = \rho + \rho^{-1}$.

As far as entire solutions of (2.17) are concerned, the converse of the results given above is also true, owing to the following result obtained by W.Messing and Y.Sibuya [4] :

Let $H(\lambda)$ be an n-by-n matrix whose entries are entire in λ, and let ω be a complex number such that $\omega^r = 1$ for a positive integer, but $\omega^p \neq 1$ for any positive integer p less than r. If H satisfies the condition

$$(3.11) \qquad H(\omega^{r-1}\lambda)H(\omega^{r-2}\lambda)\dots H(\omega\lambda)H(\lambda) = I_n ,$$

where I_n is the n-by-n identity matrix, then there exist two n-by-n matrices $E(\lambda)$ and C such that

(i) the entries of $E(\lambda)$ and $E(\lambda)^{-1}$ are entire in λ ;

(ii) C is a constant matrix satisfying the condition $C^r = I_n$;

(iii) $H(\lambda) = E(\omega\lambda)^{-1} C\, E(\lambda)$.

Remark 3.2:

(a) This result is a generalization of Theorem 90 of Hilbert (cf.E.R.Kolchin[3; Chap.V, §12]);

(b) we believe that we can prove a similar result in several variables;

(c) if we set $E^{-1} dE/d\lambda = -A(\lambda)$, then

$$(3.12) \qquad dH/d\lambda = \omega A(\omega\lambda)H(\lambda) - H(\lambda)A(\lambda) , \qquad H(0) = C .$$

4. Remarks on entire solutions: Let us consider a relation

$$(4.1) \qquad f(\lambda) + f(c\lambda)f(c^{-1}\lambda) = h ,$$

where $f(\lambda)$ is entire in λ ; c and h are complex numbers; and $c \neq 0$.

(i) If f does not have any zero, then f does not take h ; and hence f must be a constant identically due to a theorem of Picard, if $h \neq 0$.

(ii) If $f(\lambda_0) = 0$, then $f(c\lambda_0) = h$ and $f(c^{-1}\lambda_0) = h$; hence $h^2 = h$. This means that $h = 1$ or 0 .

(iii) $f(\lambda)+f(c\lambda)f(c^{-1}\lambda) = 0$ implies $f(c\lambda)+f(c^2\lambda)f(\lambda)=0$, and hence $f(c\lambda) - f(c^2\lambda)f(c\lambda)f(c^{-1}\lambda) = 0$. Therefore, if $f(\lambda)$ is not identically zero, we must have $f(c^2\lambda)f(c^{-1}\lambda) = 1$ identically. This, in turn, implies that $c = \exp(i2\pi/6)$ and $f(\lambda) = -\exp\left[\lambda\Phi(\lambda^6) + \lambda^5\Psi(\lambda^6)\right]$, where $\Phi(u)$ and $\Psi(u)$ are entire in u. Note that two zeros of $X + X^2$ are 0 and -1. The equation $f(\lambda) + f(c\lambda)f(c^{-1}\lambda) = 0$ does not have non-trivial entire solution satisfying $f(0) = 0$.

(iv) The following question is still unanswered: "Does exist a non-trivial entire function $f(\lambda)$ such that $f(\lambda) + f(\omega\lambda)f(\omega^{-1}\lambda) = 1$, $f(0) = \omega+\omega^{-1}$, where $\omega = \exp(i2\pi/5)$? "

5. Riemann-Birkhoff problem: There are many non-trivial entire solutions of (2.17). In fact if $a(\lambda)$ and $b(\lambda)$ are entire functions such that $a(\omega\lambda)=\omega a(\lambda)$ and $b(\omega\lambda) = \omega^{-1}b(\lambda)$, then $f(\lambda) = F(a(\lambda),b(\lambda))$ satisfies (2.17), where F is the function in (2.16). In order to understand the general nature of this solution, it would be helpful to quote the following result concerning the Riemann-Birkhoff problem:

Let $\omega = \exp(i2\pi/(m+2))$ for an odd integer m, and let

$$P_k = \begin{bmatrix} \gamma_k & 1 \\ -\omega & 0 \end{bmatrix}, \qquad \gamma_k \in \mathbb{C} .$$

If $P_{m+1}P_mP_{m-1}\cdots P_1P_0 = I_2$, then there exists $a = (a_1, \ldots, a_m) \in \mathbb{C}^m$ such that $C_k(a) = P_k$ ($k = 0, 1, \ldots, m+1$)(cf. Y.Sibuya[6]).

Furthermore, we can choose $a_1 = 0$. There is a similar result for an even integer m (cf. Y.Sibuya[6]). After some normalization being made, I.Bakken[1] showed that the correspondence between a and γ is locally bi-holomorphic. Probably, most of the mysteries surrounding the equation $f(\lambda) + f(\omega\lambda)f(\omega^{-1}\lambda) = 1$ are hidden in the function $F(a.b)$.

<u>References:</u>

1) I.Bakken, A multiparameter eigenvalue problem in the complex plane, Amer. J. of Math., 99(1977) 1015-1044;

2) P.F.Hsieh and Y.Sibuya, On the asymptotic integration of second order linear ordinary differential equations with polynomial coefficients, J. Math. Ana. Appl., 16(1966) 84-103;

3) E.R.Kolchin, Differential Algebra and Algebraic Groups, Academic Press,1973

4) W.Messing and Y.Sibuya, A generalization of Theorem 90 of Hilbert, under preparation;

5) Y.Sibuya and R.H.Cameron, An entire solution of the functional equation $f(\lambda) + f(\omega\lambda)f(\omega^{-1}\lambda) = 1$, ($\omega^5 = 1$), Proc. of Symposium on Ordinary Differential Equations at Univ. of Minnesota, May 29-30, 1972, Lecture Notes in Math., No.312, 194-202, Springer-Verlag, 1073;

6) Y.Sibuya, Global Theory of a Second Order Linear Ordinary Differential Equation with a Polynomial Coefficient, Math. Studies 18, North-Holland, 1975;

7) A.Voros, The return of the quartic oscillator. The complex WKB method, Ann. Inst. Henri Poincaré, Section A: Physique théorique, 39(1983) 211-338;

8) A.Voros, The zeta function of the quartic oscillator, Nuclear Physics B165(1980) 209-236;

9) W.Wasow, Asymptotic Expansions for Ordinary Differential Equations, John Wiley, 1965.