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The Generalized Three Circle - And Other Convexity Theorems with Application to the Construction of Envelopes of Holomorphy

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The generalized three circle- and other convexity theorems with application to the construction of envelopes of holomorphy

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Summary: If $G_1 \subset \mathbb{C}^n$ and $H_1 \subset \mathbb{C}^m$ are natural domains and if $G_o \subset G_1$ and $H_o \subset H_1$ are domains then we will construct the envelope of holomorphy of $G_o \times H_1 \cup G_1 \times H_o$. On the way we will prove convexity theorems for the logarithms of the moduli of holomorphic functions. The connection between the convexity theorems and the construction of envelopes of holomorphy will be established by technics of Hilbert-spaces of holomorphic functions.

Résumé: Si $G_1 \subset \mathbb{C}^n$ et $H_1 \subset \mathbb{C}^m$ sont des domaines naturels d'holomorphie et si G_o et H_o sont des domaines respectivement contenus dans G_1 et H_1, on construit l'enveloppe d'holomorphie de $G_o \times H_1 \cup G_1 \times H_o$. On démontre simultanément des théorèmes de convexité pour les logarithmes des modules de fonctions holomorphes. La relation entre les théorèmes de convexité et la construction des enveloppes d'holomorphie est établie au moyen de techniques d'espaces de Hilbert de fonctions holomorphes.

I. Introduction

In some examples of constructive field theory the euclidean version of this theory has been used, and in particular the measure theoretic version of it. These examples have revived the interest in this field, in particular in the question whether every Wightman field theory in the euclidean region can be represented by a measure or whether this is a particularity of special models. Lately J. Yngvason and the author [1] gave necessary and sufficient condition that a Wightman field theory has such a representation. These conditions are given in terms of growth estimates of the Wightman functions at Schwinger points, these are points where the time co-ordinates are purely imaginary and the space components are real. One gets the Wightman functions at these points by analytic continuation starting from the real (Minkowski) region.

The real region is also the physical space where the axioms of field theory are valid. Therefore the proof of estimates in the complex has to start from the reals where one can get estimates from the assumptions of the theory. Afterwards methods of analytic completion have to be used in order to carry these estimates into the complex.

The basic estimates follow usually from positivity conditions of the theory which are consequences of the probability interpretation of quantum mechanics. These positivity conditions do allow the use Cauchy-Schwarz inequality and in many cases one obtains estimates on domains of the form $G_o \times H_1 \cup G_1 \times H_o$ where $G_o \subset G_1 \subset \mathbb{C}^n$ and $H_o \subset H_1 \subset \mathbb{C}^m$. Since the same estimate holds in the envelope of holomorphy one would like to know the answer for this problem.

In all examples which have been solved so far the answer has the form $\bigcup_{\lambda=0}^{1} G_\lambda \times H_{1-\lambda}$ where G_λ resp. H_λ are interpolating domains of the pair G_o, G_1 resp. H_o, H_1. It is the aim of this paper to prove that the answer to the above problem is always of this form provided the pairs G_o, G_1 and H_o, H_1 have some properties which will be defined in the next section.

In the next section we give a characterization of these pairs and define an interpolating family of domains for such pairs. Furthermore we show that these definitions have some universal properties. From these properties we derive in section 3 a generalization of the Hadamrd three circle theorem and other convexity results for holomorphic functions. In section 4 we will treat Hilbert-spaces of analytic functions, which we need in section 5 as a tool for converting the convexity theorems into theorems of envelopes of holomorphy.

II. Interpolating families of domains of holomorphy

We start our investigations with some notations and remarks

II. 1. Notations:

Let G be a domain in $\mathbb{C}^n$ then we denote by

a) $A(G)$ the set of functions which are holomorphic in G. $A(G)$ is furnished with the topology of uniform convergence on compact subsets of G. With this topology $A(G)$ is a nuclear locally convex topological vector space.

b) $P(G)$ the set of functions which are pluri-subharmonic on G.

c) Let $F \subset P(G)$ be a family of pluri-subharmonic functions, such that the elements of F are uniformly bounded on every compact set of G, then there exists a pluri-subharmonic majorant $p(z, F) \in P(G)$.

The function $p(z) = \sup \left\{ f(z) \; ; \; f \in F \right\}$ will not be upper semi-continuous is general, therefore we put

$$p(z, F) = \lim_{z' \to z} \sup \; p(z)$$

(see e.g. [3]).

d). Let $M \subset \mathbb{C}^n$ be any set then we denote by $\overline{M}$ the closure of M and by M^o the interior points of M.

With these notations we introduce the following concepts:

II. 2. Definitions:

1) Assume $G_o \subset G_1 \subset \mathbb{C}^n$ such that G_1 is a domain of holomorphy. We call G_o, G_1 an Hadamard pair and write $G_o \overset{H}{\subset} G_1$ if the following conditions are fulfilled:

a) $G_o = \left\{ \overline{G}_o \cap G_1 \right\}^o$.

b) For every connected component Γ of G_1 we have $G_o \cap \Gamma \neq \emptyset$

c) To every point $z_o \in G_1 \setminus G_o$ and every neighbourhood U of z_o exists a plurisubharmonic function $p \in P(G_1)$ with the properties
   (i) $p(z) \leq 1$ on G_1
   (ii) $P(z) \leq 0$ for $z \in G_o$
   (iii) there exists a point $z_1 \in U$ (the neighbourhood of z_o) with $p(z_1) > 0$

2) Let G_1 be a domain of holomorphy and $G_o \overset{H}{\subset} G_1$, denote by $F \subset P(G_1)$ the set of pluri-subharmonic functions fulfilling the condition c(i) and c(ii) of definition 1) then this family contains a pluri-subharmonic majorant which we denote by $p_m(z, G_o, G_1)$.

3) Let $G_1 \subset \mathbb{C}^n$ be a domain of holomorphy and let $G_o \overset{H}{\subset} G_1$. Furthermore let $p_m(z)$ be the pluri-subharmonic majorant $p_m(z, G_o, G_1)$ then follows (since $f(z) = 0$ is pluri-subharmonic) from a) and c) that $G_o = \left\{ z \in G_1 ;\ p_m(z) = 0 \right\}^o$. We define for $0 < \lambda \leq 1$

$$G_\lambda = \left\{ z ;\ p_m(z) < \lambda \right\}$$

All the G_λ are domains of holomorphy [2] and they form an interpolating family of domains because of the maximum principle.

It is our aim to study this interpolating family in some detail. We want to show that this definition has some universal properties, and that for this family an ananalogon of the Hadamard three circle theorem is fulfilled. We start with some preparations.

II.3. Lemma:

Let $G_1^i \subset G_1^{i+1} \subset G_1$, $i = 1,2,\ldots$ be domains of holomorphy. In addition let $G_o^i \subset G_o^{i+1} \subset G_o$ be such that $G_o^i \overset{H}{\subset} G_1^i$, $i = 1,2,\ldots$ and $G_o \overset{H}{\subset} G_1$. If G_λ^i are the interpolating domains of G_o^i and G_1^i then follows

$$G_\lambda^i \subset G_\lambda^{i+1} \subset G_\lambda .$$

If furthermore $\bigcup_i G_o^i = G_o$ and $\bigcup_i G_1^i = G_1$ holds, then follows for every $\lambda \in [0, 1]$

$$\bigcup_i G^i_\lambda = G_\lambda .$$

Proof:

Let $p^i_m(z)$ be the pluri-subharmonic majorant belonging to the pair G^i_o, G^i_1 (Def. II. 2. 2) then we know that $p^i_m(z)$ is defined on G^i_1 . From G^{i+1}_1 G^i_1 and the maximality of $p^i_m(z)$ follows

$$p_m(z) \leqslant p^{i+1}_m(z) \leqslant p^i_m(z) \quad \text{on } G^i_1$$

This implies by definition of G_λ the relation

$$G^i_\lambda \subset G^{i+1} \subset G_\lambda .$$

For the second statement we remark that $p^i_m(Z)$ is a decreasing sequence. Thus

$$f(z) = \lim_{i \to \infty} p^i_m(z) \geqslant p_m(z)$$

is a pluri-subharmonic function in the region where it is defined. From $\bigcup_i G^i_1 = G_1$ follows that $f(z)$ is defined on G_1 and that $f(z) \leqslant 1$ holds because it is true for all $p^i_m(z)$. From $\bigcup_i G^i_o = G_o$ follows furthermore the equation $f(z) = o$ for $z \in G_o$. Hence we get by maximality of $p_m(z)$ the inequality

$$f(z) \leqslant p_m(z)$$

which implies together with the above inequality the relation $f(z) = p_m(z)$. In terms of domains this means

$$\bigcup_i G^i_\lambda = G_\lambda$$

In order to derive further consequences of the definition of the family of interpolating domains we need some preparations. The last lemma suggest that it is sufficient to look at bounded domains. So the first step would be to show that we can approximate G_o and G_1 by bounded domains. But before doing this we want to show that G_o is a Runge domain in G_1 . (We say G_o is a Runge domain in G_1 if $A(G_1)$ is dense in $A(G_o)$).

II. 4. Lemma:

Let $G_o \overset{H}{\subset} G_1$ then follows that G_o is a Runge domain in G_1 . But the converse is not true in general.

Proof:

Let us first show the second statement. Assume $G_1 = \mathbb{C}^1$ and G_o is the unit-circle then it is clear that G_o is a Runge domain in $\mathbb{C}^1$. Let now D_R be the circle of radius $R > 1$ then $D_1 \overset{H}{\subset} D_R$, since the conditions of definition II. 2 are obviously fulfilled by the function $\log R^{-1} \log|z|$. Using the Hadamard three circle theorem, which also holds for subharmonic functions one concludes

$$p_m(z, D_1, D_R) = \begin{cases} (\log R)^{-1} \log|z| & , 1 \leq |z| < R \\ 0 & , |z| \leq 1 . \end{cases}$$

From this follows that

$$\lim_{R \to \infty} p_m(z, D_1, D_R) = 0$$

which implies by Lemma II. 3 that $D_1, \mathbb{C}^1$ is not an Hadamard pair.

In order to prove the first part, we have to show that the $A(G_1)$ -hull of every compact set in G_o lies in G_o. Let $d(z)$ be a distance in $\mathbb{C}^n$ depending only on $|z_i|$ and $K \subset G_o$ be a compact set of G_o then follows:

$$\delta = \inf \{ d(z-w) ; z \in K, w \in \mathbb{C}^n \setminus G_o \} > 0$$

Let now $\varphi(z) \in C^\infty(\mathbb{C}^n)$ be such that

a) $\varphi > 0$ for $d(z) < \frac{\delta}{2}$

b) $\varphi = 0$ for $d(z) \geq \frac{\delta}{2}$

c) $\int \varphi(z) \, d\lambda = 1$ where $d\lambda$ denotes the Lebesgue measure on $\mathbb{C}^n$ and

d) $\varphi = \varphi(|z_1|, |z_2|, \cdots, |z_n|)$.

Denote furthermore as usual

$$G^\varepsilon = \{ z \in G ; d(z-w) > \varepsilon \quad \text{for all} \quad w \in \mathbb{C}^n \setminus G \}$$

Now, the function $p_m(z, G_o, G_1) * \varphi = p(z)$ is pluri-subharmonic on $G_1^{\delta/2}$. From construction follows $p(z) = o$ for $z \in G_o^{\delta/2}$ and $p(z) > o$ for $z \in G_1^{\delta/2} \setminus \overline{G_o^{\delta/2}}$. Since K is a compact set in $G_o^{\delta/2}$ it follows that the $P(G_1^{\delta/2})$ hull of K stays in G_o. But the $P(G_1^{\delta/2})$ and the $A(G_1^{\delta/2})$ hull coincide (see e.g. [6] Theorem 4.3.4) which implies that the $A(G_1^{\delta/2})$ hull of K is compact in G_o. On the other hand it is well known that $G_1^{\delta/2}$ is a Runge domain in G_1, which implies that $A(G_1)$ is dense in $A(G^{\delta/2})$ and hence the $A(G_1)$ hull of K is compact in G_o, which proves the lemma.

After this preparation we show:

II. 5. Lemma:

Let $G_o \overset{H}{\subset} G_1$, then we can find increasing sequences of domains G_o^i, G_1^i $i = 1,2,\ldots$ with the properties:

a) $G_o^i \subset G_1^i$ and G_o^i is relatively compact in G_1^i

b) $G_o^i \subset G_o^{i+1} \subset G_o$ such that $\bigcup_i G_o^i = G_o$ and G_o^i is relatively compact in G_o

c) $G_1^i \subset G_1^{i+1} \subset G_1$ such that $\bigcup_i G_1^i = G_1$ and G_1^i is relatively compact in G_1

d) G_o^i and G_1^i are the interior points of their closure and these closures are all $A(G_1)$ convex.

Proof:

According to well known theorems we can find an increasing sequence of domains G_1^i fulfilling the condition c) and d) of the lemma (take for instance analytic poly-hedrons, see e.g. [5] th. II. 6. 6.). Without loss of generality we might assume $G_1^i \cap G_o = \Gamma^i \neq \emptyset$. Let now K be a compact set in and $\hat{K}$ its $A(G_1)$ hull, then follows $\hat{K} \subset G_o$ since G_o is a Runge domain in G_1 (Lemma II. 4) and also $\hat{K} \subset G_1^i$ since G_1^i is a Runge domain in G_1 by construction. Hence $\hat{K} \subset \Gamma^i$. Now $(\Gamma^i)^\varepsilon$ is relatively compact in Γ^i and also $A(G_1)$ convex. Hence we can find a domain G_o^i such that

$$(\Gamma^i)^{\frac{1}{i}} \subset G_o^i \subset (\Gamma^i)^{1/2i}$$

such that its closure is $A(G_1)$-convex and it is the interior of its closure. Since $\bigcup_i \Gamma^i = G_o \cap G_1 = G_o$ follows that all conditions of the lemma are fulfilled.

II. 6. Remark:

Since the closure of G_o^i is $A(G_1)$ convex it follows immediately that $G_o^i \overset{H}{\subset} G_1^i$. This lemma together with lemma II. 3 does allow to reduce all further investigations to bounded domains which are relatively compact in G_1 and also $A(G_1)$ convex, this means to such domains G for which the bounded analytic functions are dense in $A(G)$.

Our next aim will be the investigation and characterization of the interpolating family of such domains.

II. 7. Lemma:

Let $G_0 \overset{H}{\subset} G_1 \subset \mathbb{C}^n$, $H_0 \overset{H}{\subset} H_1 \subset \mathbb{C}^m$ and let G_λ resp. H_λ be their interpolating families. Assume

$$f(z) = \{ f_1(z), \cdots f_n(z) \} \in A^m(G_1)$$

is such that

$$f(G_0) \subset H_0 \qquad \text{and} \qquad f(G_1) \subset H_1$$

then follows $f(G_\lambda) \subset H_\lambda$.

Proof:

Let $p_m(w, H_0, H_1)$ be the maximal pluri-subharmonic function belonging to H_0 and H_1 then follows that $p_m(f(z), H_0, H_1)$ is pluri-subharmonic on G_1 and bounded by 1. Since $f(G_0) \subset H_0$ it follows that $p_m(f(z); H_0, H_1)$ vanishes on G_0. This implies

$$p_m(f(z); H_0, H_1) \leq p_m(z, G_0, G_1)$$

and hence we get for $z \in G_\lambda$, the inequality $p_m(f(z); H_0, H_1) \leq p_m(z, G_0, G_1) < \lambda$ which implies $f(z) \in H_\lambda$.

First we will investigate absolutely convex domains. The reason for this is that we need the following result in the next section. Recall a set G is called absolutely convex if it is convex in the usual sense and if it contains with z also λz with $|\lambda| \leq 1$.

II. 8. Lemma:

Let $G_0 \subset G_1 \subset \mathbb{C}^n$ be bounded absolutely convex domains then we have $G_0 \overset{H}{\subset} G_1$.

For $a \in \mathbb{C}^n$ denote by $(a, z) = \sum_{i=1}^{n} a_i z_i$; and by $m_i(a) = \sup \{ |(a, z)| ;\ z \in G_i \}$, $i = 0, 1$ then we have

$$G_\lambda = \{ z \in G_1 ;\ |(a, z)| < m_0^{1-\lambda}(a)\ m_1^{\lambda}(a) \ \text{for all} \ a \neq 0 \}$$

In addition the function $p_m(z, G_0, G_1)$ is continuous on G_1.

If we define for $z \in \partial G_0$ (the boundary of G_0) the function

$$r(z) = \begin{cases} \sup\{\mu ; \ \mu > 0 \ , \ \mu \cdot z \in G_1\} & z \in \partial G_0 \cap G_1 \\ 1 & z \in \partial G_0 \cap \partial G_1 \end{cases}$$

we have also

$$G_\lambda = \{\mu z \ ; \ z \in \partial G_0 \text{ and } 0 \leq \mu < r^\lambda(z)\}$$

Proof:

Since G_o is absolutely convex it follows that every point in the complement of G_o is separated from G_o by a linear functional. Since G_1 is bounded it follows that this functional is bounded on G_1 which implies $G_o \overset{H}{\subset} G_1$.

Let now $f(z)$ be a bounded non-negative pluri-subharmonic function on G_1 and $z_0 \neq 0$ with $z_0 \in G_1$ then $g(w) = f(w \cdot z)$ is sub-harmonic in $w \in \mathbb{C}^1$. Define $n_i(z_0) = \sup\{|w| \ ; \ wz_0 \in G_i\}$, $i = 0, 1$ and $m_i(z_0, f) = \sup\{g(w) \ ; \ |w| < n_i(z_o)\}$ then we get by the Hadamard three circle theorem:

$$\sup\{g(w) ; |w| \leq n_0^{1-\lambda}(z_0) n_1^\lambda(z_0)\} \leq \lambda m_1(z_0, f) + (1-\lambda) m_0(z_0, f)$$

If we take in particular $f(z) = p_m(z, G_o, G_1)$ then follows $m_o(z_o, f) = o$, $m_1(z_o, f) = 1$ and hence

$$\sup\{p_m(w \cdot z_0, G_0, G_1) ; |w| \leq n_0^{1-\lambda}(z_0) \, n_1^\lambda(z_0)\} \leq \lambda$$

From this we get by maximality of $p_m(z, G_o, G_1)$ $w z_o \in G_\lambda$ exactly if $|w| < n_0^{1-\lambda}(z_0) \, n_1^\lambda(z_0)$. Using the fact that G_o and G_1 are absolutely convex then we get from this the first characterization of G_λ.

If we choose $z_o \in \partial G_o$ then we have $n_o(z_o) = 1$ and $n_1(z_o) = r(z_o)$ and we get the second characterization.

Let now $\|z\|$ be a norm on $\mathbb{C}^n$. It follows from the convexity that $\|z\|$ is a continuous function on ∂G_1 and ∂G_o. Hence $r(z)$ which is the quotient of these function is continuous. From the second definition of G_λ and from $p_m(z, G_o, G_1) = \sup\{\lambda \ ; \ z \in G_\lambda\}$ follows the continuity of p_m.

As a next step let us drop the assumption that G_o and G_1 are bounded, but, assume further on that they are absolutely convex.

II. 9. Lemma:

Let $G_o \subset G_1 \subset \mathbb{C}^n$ be absolutely convex domains. Let L_1 be the maximal linear subspace contained in G_1, then $G_o \overset{H}{\subset} G_1$ if and only if $L_1 \subset G_o$.

Since L_1 is also absolutely convex it is isomorphic to some $\mathbb{C}^m$. Hence we can write $\mathbb{C}^n = \mathbb{C}^m \times \mathbb{C}^{m'}$, $m + m' = n$, $G_0 = \mathbb{C}^m \times G_0'$ and $G_1 = \mathbb{C}^m \times G_1'$ with G_0', G_1' bounded and absolutely convex. If G_λ' are their interpolating domains then we abtain $G_\lambda = \mathbb{C}^m \times G_\lambda'$.

Proof:

Since G_1 is absolutely convex follows from the bi-polar-theorem that G_1 is a cylinder this means $G_1 + L_1 \subset G_1$. Since $\mathbb{C}^n$ is finite dimensional we can write $G_1 = \mathbb{C}^m \times G_1'$ with $\mathbb{C}^m$ isomorphic to L_1. Therefore if $L_1 \subset G_0$ then follows $G_0 \overset{H}{\subset} G_1$ and the structure of G_λ from the previous lemma. If we assume on the other hand $G_0 \overset{H}{\subset} G_1$ then follows from the argument given in the proof of Lemma II.4 that $L_1 \subset G_0$.

In the next step we are turning to more general domains.

II.10. Lemma:

Let $G \subset \mathbb{C}^n$ be a domain of holomorphy and let $G_0 \subset G_1 \subset G$ be such that

a) G_0 is relatively compact in G_1 and G_1 is relatively compact in G.

b) Both domains coincide with the interior of their closures.

c) $\overline{G}_0$ and $\overline{G}_1$ are $A(G)$ convex.

d) Each component of G_1 contains a component of G_0.

Then we have $G_0 \overset{H}{\subset} G_1$.

If we define for every $f \in A(G)$

$$M(f) = \sup \{|f(z)| ; z \in G_1\} \quad \text{and} \quad m(f) = \sup\{|f(z)| ; z \in G_0\}$$

then we obtain

$$G_\lambda = \{z \in G ; |f(z)| \leq m(f)^{1-\lambda} M(f)^\lambda \text{ for all } f \in A(G)\}^0$$

Proof:

Since $\overline{G}_0$ and $\overline{G}_1$ are compact sets in G it follows that M(f) and m(f) are finite numbers. Since $\overline{G}_0$ is A(G) convex there exists for every $z_0 \in G \setminus \overline{G}_0$ a function $f \in A(G)$ with $|f(z_0)| > m(f)$.

Hence we have $G_0 \overset{H}{\subset} G_1$.

Every $f(z) \in A(G)$ maps G_0 into the circle $|w| < m(f)$ and G_1 into the circle $|w| < M(f)$. Hence we get from Lemma II.7. the inequality

$$|f(z)| \leq m(f)^{1-\lambda} M(f)^{\lambda} \qquad \text{for} \quad z \in G_\lambda$$

If we define for every f with $M(f) \neq m(f)$ the pluri-subharmonic function

$$p_f(z) = \left(\log \frac{M(f)}{m(f)} \right)^{-1} \cdot \log \frac{|f(z)|}{m(f)}$$

and by $q(z)$ the pluri-subharmonic majorant of all $p_f(z)$ then we get from the above argument

$$q(z) \leq p_m(z, G_0, G_1)$$

In order to show that the two functions are equal we make use of an argument due to H. Bremermann [4] showing that the functions $\lambda \log f(z), \lambda > 0$ are total in $P(G)$ if G is a domain of holomorphy. If we denote by D_r the circle of radius r in $\mathbb{C}^1$ then the envelope of holomorphy of $G_0 \times D_1 \cup G_1 \times D_{1/e}$ is given by

$$H = \left\{ (z, w) ; \ z \in G_1 \quad \text{and} \quad |w| < e^{-p_m(z, G_0, G_1)} \right\}$$

If $F(z,w) \in A(H)$ then it can be written as $F(z,w) = \sum f_n(z) w^n$
The radius of convergence $r(z)$ is given by

$$\log \frac{1}{r(z)} = \limsup_{n \to \infty} \frac{1}{n} \log | f_n(z)|$$

If $\log \frac{1}{\rho(z)}$ denotes the upper semi-continuous majorant then we have

$$p_m(z, G_0, G_1) \geq \log \frac{1}{\rho(z)}$$

and $p_m(z, G_0, G)$ is the pluri-subharmonic majorant of all the $\log \frac{1}{\rho(z)}$

Since G_1 is $A(G)$ convex we obtain a dense set of function $F(z,w) = \sum f_n(z) w^n \in A(H)$ by choosing $f_n(z) \in A(G)$.

Since $G_0 \times D_1 \subset H$ and $G_1 \times D_{1/e} \subset H$ follows

$$\limsup_{n \to \infty} \log m(f_n) \leq 0 \quad \text{and}$$

$$\limsup_{n \to \infty} \log M(f_n) \leq 1$$

and consequently we get from previous inequality

$$\frac{1}{n} \log |f_n(z)| \leq \frac{1}{n}\{(1-\lambda)\log m(f_n) + \lambda \log M(f_n)\}\,; \quad z \in G_\lambda$$

which means

$$\log \frac{1}{r(z)} < \lambda \qquad \text{for} \quad z \in G_\lambda$$

Since this holds for all F we get

$$p_m(z, G_0, G_1) = q(z)$$

Since the majorant of the $\log \frac{1}{\varsigma(z)}$ coincides with p_m.
This shows the lemma.

The last lemma gives us for the special situation some more information. We obtain

II. 11. <u>Corollary:</u>

Under the assumptions of Lemma II. 10. we get for $0 \neq \lambda \neq 1$:

a) $G_\lambda = (\overline{G_\lambda})^o$ and $\overline{G_\lambda}$ is A(G) convex

b) G_λ is relatively compact in G_1 and

c) G_o is relatively compact in G_λ .

d) If we extend $p_m(z, G_o, G_1)$ to $\overline{G}_1$ by putting it equal to one on ∂G_1 , then $p_m(z, G_o, G_1)$ is continuous on $\overline{G}_1$.

<u>Proof:</u>

Let us first show statement b).
Since G_o is relatively compact in G_1 follows that for every $f \in A(G)$ we have $m(f) \neq M(f)$ except for the constant function. Therefore for f not constant the function

$$p(z, f) = \max\left[0 \, , \left(\log \frac{M(f)}{m(f)}\right)^{-1} \cdot \log \frac{|f(z)|}{m(f)} \right]$$

is well defined, pluri-subharmonic and continuous. $p_m(z, G_o, G_1)$ is the pluri-subharmonic majorant of the $p(z, f)$ on G_1. Since $\overline{G}_1$ is A(G)-convex there exists for every $z_o \in \partial G_1$ a function f with $p(z_o, f) > 1 - \frac{\varepsilon}{2}$. Since f is continuous there exists a neighbourhood $\mathcal{U}_{z_o}$ of z_o such that $p(z, f) > 1 - \varepsilon$ for $z \in \mathcal{U}_{z_o}$. Since $\partial \overline{G}$ is compact there exists a finite covering $\mathcal{U}_{z_i}$, $i = 1 \ldots n$ of ∂G_1 such that $\max\{p(z, f_i)\} > 1 - \varepsilon$ in $\bigcup_i \mathcal{U}_{z_i}$. Choosing $\varepsilon < 1 - \lambda$ we see that G_λ is relatively compact in G_1. We also see that $p_m(z, G_o, G_1)$ is continuous at the boundary of G_1.

Since $p(z,f)$ is continuous follows that the set $\{z ; p(z, f) \leq \lambda\}$ is closed. Hence follows that

$$\Gamma_\lambda = \{z ; p(z, f) \leq \lambda \text{ for all } f \in A(G)\}$$

is a closed compact $A(G)$ convex set. Let $\lambda > 0$ be fixed and $\varepsilon > 0$ then we can find to every point $z_0 \in \partial \Gamma_\lambda$ again a function $f(z)$ with $p(z_0, f) > \lambda - \varepsilon$. Therefore we find by compactness of Γ_λ and the same arguments as above

$$\Gamma_{\lambda'} \subset \Gamma_\lambda^0 \qquad \text{for} \qquad \lambda' < \lambda$$

Since $G_\lambda = \Gamma_\lambda^0$ follows from this

$G_{\lambda'}$ is relatively compact in G_λ for $\lambda' < \lambda$

but from this follows that $p_m(z, G_o, G_1)$ is a continuous function on G_1 and by the above argument also in $\overline{G}_1$. This proves d). The other statements of Corollary are easy consequences of this.

II. 12. Corollary:

Under the assumption of Lemma II. 10 we get for $0 \leq \lambda_1 < \lambda_2 \leq 1$

a) $$G_{\lambda_1} \overset{H}{\subset} G_{\lambda_2}$$

b) If we denote $H_o = G_{\lambda_1}$ and $H_1 = G_{\lambda_2}$ then we have

$$H_\mu = G_{(1-\mu)\lambda_1 + \mu\lambda_2} \quad , \qquad 0 \leq \mu \leq 1$$

Proof:

Statement a) is obtained by applying Lemma II. 10. to the results of Corollary II. 11. The proof of b) will be obtained in three steps.

First step:

Let $\lambda_1 = 0$, $\lambda_2 \neq 1$, then we find:

$$p_m(z, G_o, H_1) = \frac{1}{\lambda_2} p_m(z, G_o, G_1) \quad \text{for} \quad z \in H_1 .$$

Proof:

We have $\frac{1}{\lambda_2} p_m(z, G_0, G_1) \leq p_m(z; G_0, H_1)$ in H_1

Since the right hand-side is the pluri-subharmonic majorant.

Define the function $f(z)$ on G_1 by

$$f(z) = \begin{cases} \lambda_2 \, p_m(z, G_0, H_1) & z \in H_1 = G_{\lambda_2} \\ p_m(z, G_0, G_1) & z \in G_1 \setminus \overline{H}_1 . \end{cases}$$

Since the functions on the right hand-side are taking both the value λ_2 on the boundary of H_2 follows that $f(z)$ is continuous. Furthermore we know that $f(z)$ is pluri-subharmonic with the possible exception of the points in ∂H_1. But we want to show that it is also pluri-subharmonic in these points. Let $z_0 \in \partial H_1$ and $w \in \mathbb{C}^n$ such that $z_0 + \tau w \subset G_1$ for $|\tau| \leq 1$. (Such w exist since $H_1 = G_{\lambda_2}$ is relatively compact in G_1). By the first inequality and the definition of $f(z)$ we have $f(z) \geq p_m(z, G_0, G_1)$. Hence we get

$$f(z_0) = p_m(z_0, G_0, G_1) \leq \frac{1}{2\pi} \int p_m(z_0 + e^{i\varphi} w, G_0, G_1) \, d\varphi$$

$$\leq \frac{1}{2\pi} \int f(z_0 + e^{i\varphi} w) \, d\varphi$$

This shows $f(z)$ is pluri-subharmonic in G_1 and consequently $f(z) \leq p_m(z, G_0, G_1)$ which implies $\lambda_2 p_m(z, G_0, H_1) \leq p_m(z, G_0, G_1)$ on H_1 and hence

$$p_m(z, G_0, H_1) = \frac{1}{\lambda_2} p_m(z, G_0, G_1).$$

Second step:

Let $\lambda_1 \neq 1$ and $\lambda_2 = 1$ and define

$$q_m(z, \lambda_1) = \begin{cases} \lambda_1 & \text{for} \quad z \in G_{\lambda_1} \\ p_m(z, G_0, G_1) & \text{for} \quad z \in G_1 \setminus G_{\lambda_1} \end{cases}$$

then we obtain

$$p_m(z, H_0, G_1) = \frac{1}{1 - \lambda_1} \left(q_m(z, \lambda_1) - \lambda_1 \right).$$

Proof:

By maximality of $p_m(z, H_o, G_1)$ we obtain

$$p_m(z, H_0, G_1) \geqslant \frac{1}{1-\lambda_1}\left(q_m(z,\lambda_1) - \lambda_1\right)$$

Define again a function f(z) by:

$$f(z) = \begin{cases} p_m(z, G_0, G_1) & \text{for} \quad z \in H_0 = G_{\lambda_1} \\ \lambda_1 + (1-\lambda_1)\, p_m(z, H_0, G_1) & \text{for} \quad z \in G_1 \setminus \overline{H_0} \end{cases} .$$

We obtain again by the continuity of the two functions p_m that also f(z) is a continuous function and takes the values λ_1 on ∂H_0. In order to show that f(z) is pluri-subharmonic we only have to consider points of ∂H_0. We remark again that $f(z) \geqslant p_m(z, G_o, G_1)$ and therefore we obtain as before f(z) is pluri-subharmonic. Therefore we find $f(z) = p_m(z, G_o, G_1)$ which is equivalent to the statement we are looking for.

Last step:

By the second step we have for $\lambda_1 \neq 1$

$$p_m(z, G_{\lambda_1}, G_1) = \frac{1}{1-\lambda_1}\left(q_m(z,\lambda_1) - \lambda_1\right) ,$$

From this follows that G_{λ_2} is a member of the interpolating family of the pair G_{λ_1}, G_1. So we can use step one for the tripel G_{λ_1}, G_{λ_2}, G_1 and obtain

$$p_m(z, G_{\lambda_1}, G_{\lambda_2}) = \frac{1-\lambda_1}{\lambda_2-\lambda_1}\; p_m(z, G_{\lambda_1}, G_1)$$

$$= \frac{1}{\lambda_2-\lambda_1}\left(q_m(z,\lambda_1) - \lambda_1\right) .$$

Using the definition of H_μ and of $q_m(z, \lambda_1)$ we obtain the desired result.

Next we want to generalize the result of the last corollary to arbitrary Hadamard pairs of domains. As a preparation we prove first the following

II. 13. Lemma:

Let $G_1 \subset \mathbb{C}^n$ be a domain of holomorphy and $G_o \overset{H}{\subset} G_1$. Let $0 < \lambda_1 < 1$ then we obtain $G_o \overset{H}{\subset} G_{\lambda_1}$ and $G_{\lambda_1} \overset{H}{\subset} G_1$.

Proof:

The first statement is trivial since $G_o \overset{H}{\subset} G_1$. Since we know the existence of the function $p_m(z, G_o, G_1)$ follows that the conditions b) and c) of Definition II. 2. are fulfilled. It remains to show condition a) i.e. we have to show that $G_{\lambda_1} = \left\{ \overline{G}_{\lambda_1} \cap G_1 \right\}^o$ holds. Assume the contrary, then exists a point $z_o \in \left\{ \overline{G}_{\lambda_1} \cap G_1 \right\}^o$ which does not belong to G_{λ_1}. Since z_o is an interior point of an open set exists a neighbourhood U of this point which belongs to the same open set. The points of U which do not belong to G_{λ_1} form a relatively closed set without interior points. Therefore we can find $w \in \mathbb{C}^n$ such that $z_o + e^{i\varphi} w \subset \mathcal{U}$ and such that the set

$$\left\{ \varphi \; ; \; z_0 + e^{i\varphi} w \in \mathcal{U} \setminus G_{\lambda_1} \right\}$$

has Lebesgue measure zero. Since $p_m(z, G_o, G_1) < \lambda_1$ for $z \in G_{\lambda_1}$ follows

$$p_m(z_0, G_0, G_1) \leq \frac{1}{2\pi} \int p_m(z_0 + e^{i\varphi} w, G_0, G_1) d\varphi < \lambda_1 .$$

This proves the lemma.

Now we are prepared for the main result of this section

II. 14. Theorem:

Let $G_1 \subset \mathbb{C}^n$ be a domain of holomorphy and assume $G_o \overset{H}{\subset} G_1$. If we choose

$$0 \leq \lambda_1 < \lambda_2 \leq 1$$

then we have $G_{\lambda_1} \overset{H}{\subset} G_{\lambda_2}$. If we denote $H_o = G_{\lambda_1}$ and $H_1 = G_{\lambda_2}$, then we find the relation

$$H_\mu = G_{(1-\mu)\lambda_1 + \mu \lambda_2} \qquad \text{for} \quad 0 \leq \mu \leq 1 .$$

Proof:

The first statemant follows directly from Lemma II.13. The second statement follows from Corollary II.12 and the approximation results Lemma II.3 and II.5.

III. The generalized three circle- and other convexity theorems

In this section we want to show that the definition of the interpolating domains lead to a series of estimates for holomorphic functions. They are of the type of the Hadamard three circle theorem and its generalization to Reinhardt domains. All these results are consequences of the maximality of the function $p_m(z, G_o, G_1)$ which has as geometric version the Theorem II.14.

We start with the correspondence of the three-circle theorem

III.1. Theorem:

Let $G_1 \subset \mathbb{C}^n$ be a domain of holomorphy and let $G_o \overset{H}{\subset} G_1$ and let G_λ be their interpolating family of domains.

For $p(z) \in \mathcal{P}(G_1)$ denote by

$$m(\lambda, p) = \sup \{ p(z) \; ; \; z \in G_\lambda \}$$

then follows that $m(\lambda, p)$ is a convex function of λ.

The usual estimate for holomorphic functions are obtained by taking $p(z) = \log |f(z)|$.

Proof:

If $m(\lambda) = \infty$ then this is true also for all $\lambda' \geq \lambda$. Hence there exists λ_o with $m(\lambda) = \infty$ for $\lambda > \lambda_o$ and $m(\lambda) < \infty$ for $\lambda < \lambda_o$. Let now $\lambda_1 < \lambda_2 < \lambda_o$ and assume $m(\lambda_1) < m(\lambda_2)$. Under these conditions is

$$f(z) = (m(\lambda_2) - m(\lambda_1))^{-1} (p(z) - m(\lambda_1))$$

a pluri-subharmonic function with $f(z) \leq 1$ for $z \in G_{\lambda_2}$ and $f(z) \leq o$ for $z \in G_{\lambda_1}$ and we get

$$f(z) \leq p_m(z, G_{\lambda_1}, G_{\lambda_2}).$$

For $\lambda_1 \leq \lambda \leq \lambda_2$ we obtain by Theorem II. 14

$$\sup_{z \in G_\lambda} p_m(z, G_{\lambda_1}, G_{\lambda_2}) = \frac{\lambda - \lambda_1}{\lambda_2 - \lambda_1}$$

and hence by difinition of f(z)

$$\sup_{z \in G_\lambda} f(z) = \frac{m(\lambda) - m(\lambda_1)}{m(\lambda_2) - m(\lambda_1)} \leq \frac{\lambda - \lambda_1}{\lambda_2 - \lambda_1}$$

which proves that $m(\lambda)$ is a convex function of λ. Since $m(\lambda)$ increases with λ follows that $m(\lambda)$ is convex in λ in all situations.

This theorem allows some converse

III. 2. Lemma

Let $G_1 \subset \mathbb{C}^n$ be a domain of holomorphy and assume $G_o \overset{H}{\subset} G_1$ with $G_o \neq G_1$. Let $p(z) \in P(G_1)$ be such that $p(z) \leq 1$ for $z \in G_1$ and $p(z) \leq o$ for $z \in G_o$. Define for $o < \lambda < 1$

$$H_\lambda = \{ z \in G_1 \; ; \; p(z) < \lambda \}$$

and for $f \in P(G_1)$

$$m(\lambda, f) = \sup \{ f(z) \; ; \; z \in H_\lambda \} .$$

Assume for every $f \in P(G_1)$ the expression $m(\lambda, f)$ is a convex function of λ, then follows $H_\lambda = G_\lambda$.

Proof:

Since $p_m(z, G_o, G_1) \leq 1$ for $z \in G_1$ and $= o$ for $z \in G_o$ follows by assumption

$$\sup_{z \in H_\lambda} p_m(z, G_o, G_1) \leq \lambda$$

and consequently $H_\lambda \subset G_\lambda$. But using Theorem III. 1 we get

$$\sup_{z \in G_\lambda} p(z) \leq \lambda$$

and hence $G_\lambda \subset H_\lambda$, which proves the lemma.

Our next aim is to discuss convexity theorems on direct products of domains. We start with some preparation concerning absolutely convex domains.

III. 3. Lemma:

Let $G_0 \subset G_1 \subset \mathbb{C}^n$ and $H_0 \subset H_1 \subset \mathbb{C}^m$ be bounded absolutely convex domains. Assume L_n and L_m are injective complex linear mappings of $\mathbb{C}^n$ resp. $\mathbb{C}^m$ into $\mathbb{C}^N$ and denote for $x, y \in \mathbb{C}^N$ the sum $\sum x_i y_i = (x, y)$ then we have with the abbreviation

$$m(\lambda, \mu) = \sup \{ |(L_n z, L_m w)| ; \ z \in G_\lambda \text{ and } w \in H_\mu \}$$

the function $\log m(\lambda, \mu)$ is convex on $[0, 1]^2$.

Proof:

Assume (λ, μ) and (λ', μ') are two points in $[0, 1]^2$ then it is sufficient to prove the inequality

$$\log m\left(\frac{\lambda+\lambda'}{2}, \frac{\mu+\mu'}{2}\right) \leq \frac{1}{2}\left\{\log m(\lambda, \mu) + \log m(\lambda', \mu')\right. .$$

If we put $\lambda_0 = \min(\lambda, \lambda')$, $\lambda_1 = \max(\lambda, \lambda')$ and similar expressions for μ then we can restrict ourselves to the rectangle $\lambda_0 \leq \lambda \leq \lambda_1$ and $\mu_0 \leq \mu \leq \mu_1$. Using Theorem II. 14 we may identify (λ_0, μ_0) with $(0, 0)$ and (λ_1, μ_1) with $(1, 1)$. This reduces the proof of the lemma to the two cases

$$m(\tfrac{1}{2}, \tfrac{1}{2}) \leq m(0,0)^{1/2} m(1,1)^{1/2} \quad \text{and}$$

$$m(\tfrac{1}{2}, \tfrac{1}{2}) \leq m(1,0)^{1/2} m(0,1)^{1/2}.$$

Since the domains in question are absolutely convex we have a characterization of $G_{1/2}$ and $H_{1/2}$ given in Lemma II. 8. With the notation of that lemma we have for $z \in \partial G_0$ and $w \in \partial H_0$

$$\varrho z \in G_{1/2} \text{ for } \varrho < r^{1/2}(z) \qquad \text{and } \varrho w \in H_{1/2} \quad \text{for } \varrho < r^{1/2}(w).$$

From this we get:

$$m(\tfrac{1}{2}, \tfrac{1}{2}) = \sup \{ |(L_n z, L_m w)| \, r^{1/2}(z) \, r^{1/2}(w) ; \ z \in \partial G_0, w \in \partial H_0 \}.$$

Writing now

$$|(L_n z, L_m w)| \, r^{1/2}(z) r^{1/2}(w) = |(L_n z, L_m w)|^{1/2} \left[|(L_n z, L_m w)| \, r(z) r(w) \right]^{1/2} \text{ or}$$

$$= \left[|(L_n z, L_m w)| \, r(z) \right]^{1/2} \left[|(L_n z, L_m w)| \, r(w) \right]^{1/2},$$

we obtain, by taking the supremum of each factor, the two inequalities

$$m(\tfrac{1}{2},\tfrac{1}{2}) \le m(0,0)^{1/2}\, m(1,1)^{1/2} \qquad \text{or}$$
$$\le m(1,0)^{1/2}\, m(0,1)^{1/2} \;.$$

If we combine this lemma with the result of Lemma II. 7., then we obtain the basis for the general convexity theorem

III. 4. Corollary:

Assume $G_o \overset{H}{\subset} G_1 \subset \mathbb{C}^n$ and $H_o \overset{H}{\subset} H_1 \subset \mathbb{C}^m$ where G_1 and H_1 are domains of holomorphy. Let $F = (f_1, \dots f_N) \in A(G_1)^N$ and $G = (g_1 \dots g_N) \in A(H_1)^N$ be such that the functions f_i and g_j are bounded. If we define

$$m(\lambda,\mu) = \sup \left\{ |(F(z), G(w))| ; \quad z \in G_\lambda \text{ and } \quad w \in H_\mu \right\}$$

then we have: $\log m(\lambda, \mu)$ is a convex function on $[0, 1]^2$.

Proof:

Using the same argument as in the proof of the last lemma, which was based on Theorem II. 14, we need only to prove the two inequalities

$$m(\tfrac{1}{2},\tfrac{1}{2}) \le m(0,0)^{1/2}\, m(1,1)^{1/2} \qquad \text{and}$$
$$\le m(1,0)^{1/2}\, m(0,1)^{1/2} \;.$$

In order to prove these inequalities we remark first: Let M_1, M_2 be bounded sets in $\mathbb{C}^N$ and $\Gamma(M_i)$ their absolutely convex hulls then one gets

$$\sup \left\{ |(x,y)| ; x \in M_1, y \in M_2 \right\} = \sup \left\{ |(x,y)| ; x \in \Gamma(M_1), y \in \Gamma(M_2) \right\}.$$

The second remark we have to make is the following: If $\Gamma(F(G_0))$ lies in some complex linear subspace $\mathcal{L}$ of $\mathbb{C}^N$, then $\Gamma(F(G_1))$ lies in the same linear subspace, because for any element $a \in \mathcal{L}^\perp$ the equation $(a, F(z)) = 0$ on G_o has an analytic extension to G_1.

If we put $\widetilde{G}_o = \Gamma(F(G_0))$ and $\widetilde{G}_1 = \Gamma(F(G_1))$ and denote by $\widetilde{G}_\lambda$ the interpolating family of $\widetilde{G}_o$ and $\widetilde{G}_1$ then we find by Lemma II. 7. $F(G_{1/2}) \subset \widetilde{G}_{1/2}$. Since the same arguments hold for the domains H we can use Lemma III. 3. and obtain:

$$m(\tfrac{1}{2},\tfrac{1}{2})^2 = \left\{\sup\left[|(F(z), G(w))| \;;\; z \in G_{1/2} \;,\; w \in H_{1/2}\right]\right\}^2$$

$$\leq \left\{\sup\left[|(x,y)| \;;\; x \in \tilde{G}_{1/2} \;,\; y \in \tilde{H}_{1/2}\right]\right\}^2$$

$$\leq \begin{cases} \sup\left[|(x,y)| \;;\; x \in \tilde{G}_0 , y \in \tilde{H}_0\right] \cdot \sup\left[|(x,y)| ; x \in \tilde{G}_1 , y \in \tilde{H}_1\right] \\ \sup\left[|(x,y)| \;;\; x \in \tilde{G}_1 , y \in \tilde{H}_0\right] \cdot \sup\left[|(x,y)| , x \in \tilde{G}_0 , y \in \tilde{H}_1\right]. \end{cases}$$

From this we get by the first remark

$$m(\tfrac{1}{2},\tfrac{1}{2})^2 \leq m(0,0)\cdot m(1,1)$$
$$\leq m(1,0)\cdot m(0,1)$$

We are now prepared for proving the main results of this section. The first one is a characterization of interpolating domains of direct products and the second result is a general convexity theorem for the logarithms of the moduli of holomorphic functions.

III. 5. Theorem:

Let $G_0^i \subset G_1^i \subset \mathbb{C}^{n_i}$, $i = 1, 2, \dots, N$ be such that G_1^i are domains of holomorphy, then we get

$$G_0^1 \times G_0^2 \times \dots \times G_0^N \overset{H}{\subset} G_1^1 \times G_2^1 \times \dots \times G_1^N$$

and the interpolating family is given by

$$(G^1 \times G^2 \times \dots \times G^N)_\lambda = G_\lambda^1 \times G_\lambda^2 \times \dots \times G_\lambda^N .$$

Proof:

It is sufficient to prove this statement for N = 2. The general result follows by iteration of the special one.

For simplifying the notation we will work with the domains $G_0 \overset{H}{\subset} G_1$ and $H_0 \overset{H}{\subset} H_1$. Let $p_m(z, G_0, G_1)$ and $p_m(w, H_0, H_1)$ be the pluri-subharmonic majorants belonging to the two pairs. Each one defines also a pluri-subharmonic function on $G_1 \times H_1$ which does not depend on the other variable. Therefore

$$p(z,w) = \max\left\{ p_m(z, G_0, G_1) , p_m(w, H_0, H_1)\right\}$$

is a pluri-subharmonic function on $G_1 \times H_1$. From construction of this function follows $p(z, w) \leq 1$ on $G_1 \times H_1$ and $p(z, w) = o$ on $G_o \times H_o$. If $(z_0, w_0) \in G_1 \times H_1 \setminus \overline{G_o \times H_o}$ we have $p(z_0, w_0) > o$. These properties imply $G_o \times H_o \overset{H}{\subset} G_1 \times H_1$.

For proving the second statement assume first that $G_o \subset G_1 \subset G$ are relatively compact in G and G_o and G_1 are both $A(G)$ convex and the same for $H_o \subset H_1 \subset H$. Then follows that $G_o \times H_o \subset G_1 \times H_1 \subset G \times H$ are relatively compact with $A(G \times H)$ convex closures. For this case we can use Lemma II. 10 for the determination of the interpolating domains $(G \times H)_\lambda$. Since the space $A(G \times H)$ is a complete nuclear vector space follows $A(G \times H) = A(G) \hat{\otimes}_\pi A(H)$ (the complete π -tensor-product of the two spaces $A(G)$, $A(H)$). This means every function $f(z, w)$ can be approximated by sums $\sum_{i=1}^{N} f_i(z) g_i(w)$ converging uniformly on every compact set, in particular on $\overline{G_1 \times H_1}$. Denoting $m(\lambda, \Sigma) = \sup \{ | \sum f_i(z) g_i(w) | ; z \in G_\lambda, w \in H_\lambda \}$ we obtain from Corollary III. 4

$$m(\lambda, \Sigma) \leq m(0, \Sigma)^{1-\lambda} m(1, \Sigma)^{\lambda} ,$$

Since the sums are dense in $A(G \times H)$ we obtain

$$|f(z,w)| \leq m(0,f)^{1-\lambda} m(1,f)^{\lambda} \quad \text{for} \quad z,w \in G_\lambda \times H_\lambda \quad \text{and} \quad f \in A(G \times H)$$

This implies by Lemma II. 10. the relation

$$G_\lambda \times H_\lambda \subset (G \times H)_\lambda$$

Using on the other hand the special functions $f(z) \cdot g(w)$ we get by the characterization of G_λ and H_λ the relation $G_\lambda \times H_\lambda \supset (G \times H)_\lambda$. So we have

$$G_\lambda \times H_\lambda = (G \times H)_\lambda$$

first for this special situation, but using the approximations of domains given in Lemma II. 3. and II. 5. we see that the result is true also for the general case.

Now we can prove the general convexity property for holomorphic functions.

III. 6. Theorem:

Let $G_0^i \overset{H}{\subset} G_1^i \subset \mathbb{C}^{n_i}$, $i = 1, \dots N$ be domains of holomorphy and let G_λ^i be the corresponding interpolating families.

Denote for $F(z_1,\dots, z_N) \in A(G_1^1 \times G_1^2 \times \dots \times G_1^N)$ and $\underline{\lambda} \in [0,1]^N$

$$m(\underline{\lambda},F) = \sup \{|F(z_1 ,\dots, z_N)| \; ; \; z_i \in G^i_{\lambda_i}\}$$

then follows $\log m(\underline{\lambda},F)$ is a convex function on $[0,1]^N$.

<u>Proof</u> : $\underline{\lambda}^1$ and $\underline{\lambda}^2$ are two points in $[0,1]^N$ it is sufficient to show the inequality

$$m(\frac{\underline{\lambda}^1 + \underline{\lambda}^2}{2}, F) \leq m(\underline{\lambda}^1,F)^{\frac{1}{2}} m(\underline{\lambda}^2,F)^{\frac{1}{2}} .$$

If the i-th component of $\underline{\lambda}^1$ and $\underline{\lambda}^2$ coincide then the domain $G^i_{\lambda_i}$ is a common factor in all considerations, so that we have to deal in reality only with a problem in N-1 variables. Therefore we may assume without loss of generality that all components of $\underline{\lambda}^1$ and $\underline{\lambda}^2$ are different .

If we put $\underline{\lambda}_o = (\min(\lambda_i^1 , \lambda_i^2))$ and $\underline{\lambda}_1 = (\max(\lambda_i^1 , \lambda_i^2))$ then by Theorem II.14. the situation can be reduced to $\underline{\lambda}_o = (0,0,\dots 0)$; $\underline{\lambda}_1 = (1,1,\dots 1)$. Renaming the indices we get

$$\underline{\lambda}^1 = (0,0,\dots 0,1,1,\dots 1) \quad ; \quad \underline{\lambda}^2 = (1,1,\dots 1,0,0,\dots 0)$$

and
$$\tfrac{1}{2}(\underline{\lambda}^1 + \underline{\lambda}^2) = (\tfrac{1}{2},\tfrac{1}{2} ,\dots, \tfrac{1}{2})$$

where we have in $\underline{\lambda}^1$ K zeros and N-K ones and zeros and ones interchanged for $\underline{\lambda}^2$.

Introducing now

$$G_o = G_o^1 \times\dots\times G_o^K \quad , \quad G_1 = G_1^1 \times\dots\times G_1^K$$
$$H_o = G_o^{K+1} \times\dots\times G_o^N , \quad H_1 = G_1^{K+1} \times\dots\times G_1^N$$

then by Theorem III.6. we get

$$G_\lambda = G_\lambda^1 \times\dots\times G_\lambda^K \quad \text{etc.}$$

so that we only have to prove the inequality

$$m(\tfrac{1}{2},\tfrac{1}{2},F) \leq m(1,0,F)^{\frac{1}{2}} m(0,1,F)^{\frac{1}{2}}$$

for two pairs of domains.

Now we approximate these domains from inside by an increasing family. If we denote by $m^i(\lambda, \mu, f)$ the maximum of $|f|$ on $G^i_\lambda \times H^i_\mu$ we get by Corollary III. 4. and the same density argument, as in the proof of the previous theorem, the relation

$$m^i(\tfrac{1}{2}, \tfrac{1}{2}, f) \leq \{ m^i(1, 0, f)\, m^i(0, 1, f) \}^{1/2}$$

for all $f \in A(G_1 \times H_1)$. Taking the limit $i \to \infty$ we obtain the desired result.

IV. Interpolating domains and Hilbert spaces of holomorphic functions

It is our aim to convert the general convexity theorem of the last section into statements of finding envelopes of holomorphy. In order to clarify the situation let us assume $G_o \overset{H}{\subset} G_1$ and $H_o \overset{H}{\subset} H_1$ and we have to compute the envelope of holomorphy of $G_o \times H_1 \cup G_1 \times H_o$. We know that both domains $G_o \times H_1$ and $G_1 \times H_o$ are Runge domains in $G_1 \times H_1$. Therefore we can approximate every function given on the union of the two small domains by function in $A(G_1 \times H_1)$ as well on $G_o \times H_1$ as on $G_1 \times H_o$. If we succeed to find an approximation on the union of both small domains simultaneously then the convexity theorem gives us an extension of the given function into a bigger domain. That such approximations exist, at least for sufficiently many domains, we will show by means of Hilbert spaces of analytic functions. (For an introduction to the theory of Hilbert spaces of analytic functions see e.g. [7]).

IV. 1. Notations

In the following we denote by G always a domain of holomorphy.

a) Let μ be a measure on G, then we say μ is a regular measure if the set

$$\{ f \in A(G) \; ; \; \int_G |f(z)|^2 \, d\mu < \infty \}$$

is a closed subspace of $\mathcal{L}^2(G, \mu)$. We denote this subspace by $\mathcal{H}(G, \mu)$.

b) If μ is a regular measure on G and if $\mathcal{H}(G,\mu)$ contains not only the function o, then the kernelfunction is defined by means of an orthonormal basis $\{f_i\}$ through the formula

$$K(w,z) = \sum_i \overline{f_i(w)}\, f_i(z).$$

This function is independent of the basis, defined on G x G, and analytic in z and anti-analytic in w.

c) If μ is a regular measure in G then we call μ completely regular if $\mathcal{H}(G,\mu)$ is a dense subspace of A(G).

IV. 2. Lemma

Let μ be a regular measure on G.

a) Let $t \in A'(G)$, then $f \to (t,f)$ defines a continuous linear functional i(t) on $\mathcal{H}(G,\mu)$. The vector i(t) is defined by the formula

$$i(t) = \overline{(t_z, K(w,z))}.$$

b) The map i defines a continuous antilinear mapping from $A'(G)$ into $\mathcal{H}(G,\mu)$ such that the image of a compact convex set in $A'(G)$ is a compact set in $\mathcal{H}(G,\mu)$.

c) The image of i is always dense in $\mathcal{H}(G,\mu)$ and i is injective if and only if μ is completely regular.

d) For every continuous Hilbert semi-norm p on A(G) exist a compact operator $\rho_p \geq 0$ acting on $\mathcal{H}(G,\mu)$ such that for every $f \in \mathcal{H}(G,\mu)$ we get the identity

$$p(f)^2 = (f, \rho_p f).$$

e) Denote by $\overline{\mathcal{H}}$ the closure of $\mathcal{H}(G,\mu)$ in A(G), and let $p(\cdot)$ be a Hilbert seminorm on A(G). The corresponding operator $\rho_{\overline{p}}$ has an (unbounded) inverse if p restricted to $\overline{\mathcal{H}}$ is a norm on $\overline{\mathcal{H}}$.

Proof:

a) Let $f \in \mathcal{H}(G,\mu)$ be such that $\|f\| = 1$, then it is member of some orthonormal basis. Consequently we get for any compact subset of G

$$\sup\{|f(z)|;\ z \in K\} \leq \sup\{K(z,z)^{1/2};\ z \in K\} = C(K) < \infty,$$

So we get in general

$$\sup\left\{ |f(z)| \, ; \, z \in K \right\} \leq C(K) \, \|f\| .$$

If t is a continuous linear functional on A(G) then exists a compact set K in G with

$|(t,f)| \leq m \sup\left\{ |f(z)|; \; z \in K \right\}$ $m > 0$ and hence

we get for $f \in \mathcal{H}(G,\mu)$:

$|(t,f)| \leq m \; C(K) \; \|f\|$. Therefore exists by the Riesz representation theorem a vector $i(t) \in \mathcal{H}(G,\mu)$ with $(t,f)_A = (i(t), f)_{\mathcal{H}}$.
If $\{f_i\}$ is a basis of $\mathcal{H}(G,\mu)$ then we find

$\|i(t)\|^2 = \sum |(t,f_i)|^2$ which implies

$$\overline{i(t,w)} = \sum \overline{f_i(w)} (t f_i) = (t, K(w,z)) .$$

b) The antilinearity of i is clear. Let j be the natural injection of $\mathcal{H}(G,\mu)$ into $A(G)$, then j is continuous since we have

$$\sup\left\{ |f(z)| \, ; \; z \in K \right\} \leq C(K) \, \|f\| ,$$

Since i is the transposed of j follows the continuity of i.
Since i is continuous follows that it maps compact sets onto compact sets.

c) The density of $i(A(G))$ is trivial. The map is injective if $i(t) = 0$ holds only for $t = 0$. But $i(t) = 0$ if and only if $(i(t), f)_{\mathcal{H}} = 0 = (t,f)_A$ for all $f \in \mathcal{H}(G,\mu)$. Therefore $i(t) = 0$ if and only if $(t,g) = 0$ for all $g \in \overline{\mathcal{H}}$ (the closure of $\mathcal{H}$ in A(G)). Therefore i is injective if and only if $\overline{\mathcal{H}} = A(G)$.

d) Let $h(\cdot)$ be a continuous Hilbert semi-norm then exist $m > 0$ and a compactum $K \subset G$ with

$$h(f) \leq \frac{1}{m} \sup\left\{ |f(z)| \, ; \, z \in K \right\} \leq \frac{C(K)}{m} \|f\|$$

where the last inequality holds only for elements in $\mathcal{H}(G,\mu)$. Since h is a Hilbert semi-norm exists a linear operator ρ_h on $\mathcal{H}(G,\mu)$ with $\rho_h \geq 0$ and $h(f)^2 = (f, \rho_h f) \leq \frac{C^2(K)}{m^2} \|f\|^2$.
The set $\left\{ f \in A(G); \; h(f) < 1 \right\}$ is open and has therefore a compact polar denoted by $\tilde{K}$. Here we have used that A(G) is a Montel space. By the bi-

polar theorem we get for $f \in \mathcal{H}(G,\mu)$:

$$(f, \varrho_h f)^{1/2} = h(f) \qquad = \sup \{ (i(t), f) \; ; \; i(t) \in i(\tilde{K}) \}.$$

Let $\varrho^{1/2} = \int_0^{\|\varrho^{1/2}\|} \lambda \, dE_\lambda$, then follows for $f \in (1-E_\varepsilon)\mathcal{H}(G,\mu)$

$$\|f\| \geq \frac{1}{\varepsilon} \|\varrho^{1/2} f\| = \frac{1}{\varepsilon} \sup \{ (i(t), f) \; ; \; i(t) \in i(\tilde{K}) \}.$$

Since $i(K)$ is compact in $\mathcal{H}(G,\mu)$ follows $(1-E_\varepsilon)\mathcal{H}(G,\mu)$ is finite dimensional and this implies $\varrho_h^{1/2}$ is a compact operator.

e) If $p(\cdot)$ is a norm on $\bar{\mathcal{H}}$ then we have for $f \in \mathcal{H}(G,\mu)$

$$p(f)^2 = (f, \varrho_p f) \neq 0 \qquad \text{for } f \neq 0$$

and hence ϱ_p is invertible.

Now we want to apply the results of the last lemma to pairs of domains. We want to make for the rest of this section the following

IV. 3. Assumptions and notations

We choose $G_o \subset G_1 \subset G \subset \mathbb{C}^n$ such that

a) G is a domain of holomorphy

b) G_o is relatively compact in G_1 and
G_1 is relatively compact in G

c) $\bar{G}_o = \{\bar{G}_o\}^o$ and $G_1 = \{\bar{G}_1\}^o$

d) G_o and G_1 are $A(G)$ convex

e) dv denotes the Lebesgue measure on $\mathbb{C}^n$

f) We write for short $\mathcal{H}_1 = \mathcal{H}(G_1, dv)$ and $\mathcal{H}_0 = \mathcal{H}(G_o, dv)$

IV. 4. Lemma:

Assume IV. 3, then we can find numbers $\sigma_i \geq 1$ and an orthonormal basis $\{f_i\}$ of $\mathcal{H}_1$, such that $\{\sigma_i f_i\}$ is an orthonormal basis of $\mathcal{H}_0$.

Proof:

Since G_o is compact in G_1 follows that every $f \in A(G_1)$ is bounded on G_o. Hence $p(f) = \left\{ \int f(z)^2 \, \cancel{dv} \, dv \right\}^{1/2}$ is a Hilbert semi-norm on $A(G_1)$. Hence by Lemma IV. 2. d exist a compact operator ρ_p on $\mathcal{H}_1$

with $$(f, \rho_p f)_1 = \int_{G_0} |f(z)|^2 \, dv = (f, f)_0 .$$

Since $(f, f)_o = p^2(f) = o$ holds only for $f = o$ follows that ρ_p is invertible, this means all eigenvalues of ρ_p are positive. This implies we can find an orthonormal basis $\{f_i\}$ of $\mathcal{H}_1$

with $$\rho_p f_i = \sigma_i^{-2} f_i \qquad \sigma_i^{-2} > 0$$

Now we get:

$$(\sigma_i f_i, \sigma_j f_j)_0 = \sigma_i \sigma_j (f_i, f_j)_0 = \sigma_i \sigma_j (f_i, \rho_p f_j)_1$$
$$= \sigma_i \sigma_j \sigma_j^{-2} (f_i, f_j)_1 = \delta_{ij}$$

This shows $\{\sigma_i f_i\}$ is an orthonormal system in $\mathcal{H}_0$. Since G_o is $A(G)$ convex follows that the set of functions which are bounded on G_o are dense in $\mathcal{H}_0$ but these functions can be approximated by the $\{\sigma_i f_i\}$ and therefore they form a basis in $\mathcal{H}_0$. From the definition of $p(g)$ follows immediately $\|\rho_p\| \leq 1$ which implies $\sigma_i \geq 1$.

As we will see in the next section, this lemma leads together with the convexity theorem of the last section to the following result: Let $G_o \overset{H}{\subset} G_1$ and $H_o \overset{H}{\subset} H_1$ then the envelope of holomorphy of $G_o \times H_1 \cup G_1 \times H_o$ is exactly $\bigcup_\lambda G_\lambda \times H_{1-\lambda}$. We will need this result in the next lemma. But we need it only in a special form which is covered by the known semi-tube theorem.

IV. 5. Lemma:

Let σ_i be the numbers and $\{f_i\}$ the orthonormal basis described in the last lemma. Define

$$K_\lambda(w, z) = \sum_i \sigma_i^{2(1-\lambda)} \overline{f_i(w)} f_i(z)$$

then the sum converges on $G_\lambda \times G_\lambda$ and defines a kernel function on G_λ.

Proof:

The function $K_\varphi(w,z) = \sum_i \sigma_i^{\varphi} \overline{f_i(\bar{w})} f_i(z)$ is for Re $\varphi \leq 0$ defined on $\overline{G}_1 \times G_1$, since $\sigma_i \geq 1$. For Re $\varphi \leq 1$ it is defined on $\overline{G}_0 \times G_0$. The interpolating family of $\overline{G}_0 \times G_0$ and $\overline{G}_1 \times G_1$ is $\overline{G}_\lambda \times G_\lambda$ by Theorem III. 5 ($\overline{G}_\lambda$ denotes here the complex conjugate domain of G_λ). Since this function is analytic in (φ, w, z) follows that it is also analytic in the envelope of holomorphy of these two domains. This can be computed by the theorem to be proven in the next section or the semi-tube theorem. Using the semi-tube result, we have to compute the maximal pluri-subharmonic function which is zero on $\overline{G}_0 \times G_0$ and bounded by 1 on $\overline{G}_1 \times G_1$. But this is exactly the function which characterizes the interpolating domains. Hence

$K_\varphi(\bar{w}, z)$ is also holomorphic in

Re $\varphi \leq 1-\lambda$ and $(\bar{w}, z) \in \overline{G}_\lambda \times G_\lambda$.

This shows $K_\lambda(w, z)$ is defined on $G_\lambda \times G_\lambda$.

In order to show that K_λ is a kernel function we must proof the positivity condition $\sum_{\alpha,\beta} \bar{a}_\alpha K_\lambda(z_\alpha, z_\beta) a_\beta \geq 0$ (see [7] Satz V. 1.).

We get

$$\sum_{\alpha,\beta} \bar{a}_\alpha K_\lambda(z_\alpha, z_\beta) a_\beta = \sum_{\alpha,\beta} \bar{a}_\alpha \sum_i \sigma_i^{2(1-\lambda)} \overline{f_i(z_\alpha)} f_i(z_\beta)$$
$$= \sum_i \overline{\left(\sum_\alpha a_\alpha \sigma_i^{(1-\lambda)} f_i(z_\alpha)\right)} \left(\sum_\beta a_\beta \sigma_i^{(1-\lambda)} f_i(z_\beta)\right) \geq 0$$

This proves the lemma.

Since we have a kernel function on G_λ we also have a Hilbert space of holomorphic functions. But, we can not expect this to coincide with $\mathcal{H}(G_\lambda, dv)$. The reason for this is the fact that the pluri-subharmonic function $K_1(z, z)$ does not define the domains G_λ, this means, in the general situation there will be no functional relation between $K_1(z, z)$ and $p_m(z, G_0, G_1)$. But nevertheless we can use these kernel functions to prove the following

IV. 6. Lemma:

Let $\{\sigma_i\}$ and $\{f_i\}$ as in Lemma IV. 3. then for every $\mu > 0$ we have

a) $\sum_i \sigma_i^{-\mu} < \infty$

b) for every $z \in G_\lambda$ with $\lambda < 1$ we find for $\varepsilon > 0$ $\{\sigma_i^{1-\lambda-\varepsilon} f_i(z)\} \in l_1$ and there exists a constant $M(\lambda, \varepsilon)$ with $\sum \sigma_i^{1-\lambda-2\varepsilon} |f_i(z)| \leq M(\lambda, \varepsilon) < \infty$ for all $z \in G_\lambda$.

Proof:

a) Since all $\sigma_i \geq 1$ follows that the sum is decreasing with increasing μ. Hence we can restrict ourselves to the case $0 < \mu < 2$. Putting $\mu = 2\lambda$ we have $0 < \lambda < 1$ and we write

$$\sum \sigma_i^{-\mu} = \sum \sigma_i^{-2\lambda} = \sum \sigma_i^{2(1-\lambda)} \cdot \sigma_i^{-2}(f_i, f_i)_1$$
$$= \sum \sigma_i^{2(1-\lambda)}(f_i, f_i)_0 = \sum \sigma_i^{2(1-\lambda)} \int_{G_0} |f_i(z)|^2 \, dv$$
$$= \int_{G_0} \sum_i \sigma^{2(1-\lambda)} |f_i(z)|^2 dv = \int_{G_0} K_\lambda(z, z) \, dv$$

Since according to Corollary II.11. G_0 is relatively compact in G_λ follows that $K_\lambda(z, z)$ is bounded on G_0 and thus the integral is finite.

b) From the existence of the kernel function follows

$$\sigma_i^{1-\lambda} |f_i(z)| \in L_2 \qquad \text{for} \quad z \in G_\lambda .$$

By a) we have $\{\sigma_i^{-\varepsilon}\} \in l_1 \subset l_2$, hence we get

$$\left\{ \sigma_i^{1-\lambda-\varepsilon} f_i(z) \right\} \in l_1 \quad \text{for} \quad z \in G_\lambda \quad \text{with} \quad \lambda < 1 \quad .$$

But for $1 > \lambda' > \lambda$ the set of vectors $\{\sigma_i^{1-\lambda'} \; |f_i(z)|\}$ is a bounded set in l_2. Since $K_{\lambda'}(z, z)$ is bounded in G_λ. Hence $\left\{ \sigma_i^{1-\lambda-2\varepsilon} f_i(z) \right\}$ is a bounded set in l_1 for $z \in G_\lambda$.

With this lemma we can prove the main convergence theorem of this section.

IV.7. Theorem:

Assume IV.3. and let $\{\sigma_i\}$ be the set of numbers and $\{f_i(z)\}$ be the orthonormal basis described in Lemma IV.4.

a) Let $S(z) = \sum a_i f_i(z)$ be a sequence such that

$$\limsup_{i \to \infty} \frac{\log|a_i|}{\log \sigma_i} = \mu < 1$$

and let $\mu' = \max(0, \mu)$,

then $S(z)$ converges in $G_{1-\mu'}$ and it converges uniformly in every $G_{\lambda'}$ with $\lambda' < 1-\mu'$.

b) Assume on the other hand $\lambda > o$ and $F(z) \in A(G_\lambda)$ then $F(z)$ has a representation

$$F(z) = \sum a_i f_i(z)$$

with

$$\limsup_i \frac{\log |a_i|}{\log \sigma_i} \leq 1 - \lambda$$

By a) follows that this sequence converges uniformly on every $G_{\lambda'}$ with $\lambda' < \lambda$.

Remark:

Since we do not know enough about the functions $f_i(z)$, we cannot claim ($\mu \geq o$) that the series in a) diverges for $z \notin \overline{G_{1-\mu}}$. But b) tells us that there exists at least some sequences fulfilling a) which diverge outside of $G_{1-\mu}$ (Because there exists functions in $A(G_{1-\mu})$ which have $G_{1-\mu}$ as their exact domain of definition.)

Proof:

a) For every $\varepsilon > o$ we have by assumption

$\frac{\log |a_i|}{\log \sigma_i} < \mu + \varepsilon$ for almost all i .

This implies

$|a_i| < \sigma_i^{\mu+\varepsilon}$ except for a finite number of terms.

Hence we get:

$$\left| \sum a_i f_i(z) \right| \leq \sum |a_i| \, |f_i(z)| \leq \sum \sigma_i^{\mu+\varepsilon} |f_i(z)| .$$

By the previous lemma this series converges in $G_{1-\mu-\varepsilon}$ and uniformly in $G_{1-\mu-2\varepsilon}$. Since ε was arbitrary follows the result.

b) Let $F(z) \in G_\lambda$ then by compactness of $G_{\lambda'}$ in G_λ for $\lambda' < \lambda$ follows $F(z)$ is bounded in $G_{\lambda'}$. Hence it is an element of the Hilbert space defined by the kernel function $K_{\lambda'}$. So $F(z)$ has a development

$$F(z) = \sum a_n f_n(z) = \sum b_n \sigma_n^{1-\lambda'} f_n(z)$$

which converges on $G_{\lambda'}$ in the sense of that Hilbert space. Hence we have $|b_n| \in l_2$. This implies

$$|a_n| < \sigma_n^{1-\lambda'} \quad \text{for almost all } n$$

or

$$\limsup_{n \to \infty} \frac{\log |a_n|}{\log \sigma_n} < 1 - \lambda' .$$

Since this holds for all $\lambda' < \lambda$ we obtain

$$\limsup_{n \to \infty} \frac{\log |a_n|}{\log \sigma_n} \leq 1 - \lambda .$$

V. Construction of envelopes of holomorphy

Combining now the technics of the last section with the convexity theorems of section III we obtain a series of results, which contain the tube theorem, the theorem on Reinhardt domains and the semi-tube theorem as special cases. The two first results are based on Lemma IV.4. only and they contain the information needed for the proof of Lemma IV.5.

V.1. Theorem:

Let $G_1 \subset \mathbb{C}^n$ and $H_1 \subset \mathbb{C}^m$ be domains of holomorphy and assume $G_o \overset{H}{\subset} G_1$ and $H_o \overset{H}{\subset} H_1$, then the envelope of holomorphy of $G_o \times H_1 \cup G_1 \times H_c$ has the following representation

$$\text{hull} \quad (G_o \times H_1 \cup G_1 \times H_o) = \bigcup_{\lambda=0}^{1} G_\lambda \times H_{1-\lambda} .$$

Proof:

First let us show that the right hand side represents a domain of holomorphy. The function

$$p(z, w) = p_m(z, G_o, G_1) + p_m(w, H_o, H_1)$$

is defined on $G_1 \times H_1$ and is pluri-subharmonic. Hence the set

$$\{(z, w) \in G_1 \times H_1 \; ; \; p(z, w) < 1\}$$

defines a domain of holomorphy. But, by definition of the interpolating families this domain coincides with $\bigcup_\lambda G_\lambda \times H_{1-\lambda}$.

For the other part we have to show that every function $F(z, w)$ defined and holomorphic on $G_o \times H_1 \cup G_1 \times H_o$ can be extended analytically into $\bigcup_\lambda G_\lambda \times H_{1-\lambda}$. To this end we make use of Lemma II.5. which states that we can approximate the G's and the H's from inside by relatively compact domains which fulfill the conditions of Lemma IV.4. Let G_o^α, G_1^α, H_o^α, H_1^α = 1,2,... be these domains then $F(z, w)$ is bounded on $G_o^\alpha \times H_1^\alpha$ and $G_1^\alpha \times H_o^\alpha$. Let $f_i^\alpha(z)$ be the basis and σ_i^α be the sequence described in Lemma IV.4. then we can find for $F(z, w)$ the developments

$$F(z,w) = \sum f_i^\alpha(z)\, g_i'^\alpha(w) \quad \text{in } G_1^\alpha \times H_0^\alpha$$

$$= \sum \sigma_i^\alpha f_i^\alpha(z)\, g_i^\alpha(w) \quad \text{in } G_0^\alpha \times H_1^\alpha$$

where the $g_i^\alpha(W)$ are holomorphic in H_1. From the identity on $G_o \times H_o$ follows $g_i'^\alpha(w) = \sigma_i^\alpha g_i^\alpha(w)$. This implies the second sum converges in $G_o^\alpha \times H_1^\alpha \cup G_1^\alpha \times H_o^\alpha$. By choice of the domains follows that the sum converges absolutely in $G_o^{\alpha-1} \times H_1^{\alpha-1} \cup G_1^{\alpha-1} \times H_o^{\alpha-1}$ and hence by the convexity Theorem III.6. in $G^{-1} \times H_{1-}^{-1}$. Since $G = G$ by Lemma II.3. follows that $F(\ , \ )$ has an extension into $G \times H_{1-}$

A simple generalization of this result is the

V.2. <u>Theorem</u> on generalized Reinhard domains

Let $G_1^i \subset \mathbb{C}^{n_i}$, i = 1, ..., N be domains of holomorphy and assume $G_o^i \overset{H}{\subset} G_1^i$. Denote for $\underline{\lambda} \in [0, 1]^N$ the domain

$$G_{\underline{\lambda}} = G^1_{\lambda_1} \times G^2_{\lambda_2} \times \ldots \times G^N_{\lambda_N} .$$

Let $S \subset [0, 1]^N$ be a closed set and $Co\, S$ its convex hull then we get

$$\text{hull} \bigcup_{\underline{\lambda} \in S} G_{\underline{\lambda}} = \bigcup_{\underline{\lambda} \in Co\, S} G_{\underline{\lambda}}$$

<u>Proof:</u>

From the last theorem we find together with Theorem III.5. the result

$$\text{hull } G_{\underline{\lambda}_1} \cup G_{\underline{\lambda}_2} = \bigcup_{\mu=0}^{1} G_{\mu\underline{\lambda}_1 + (1-\mu)\underline{\lambda}_2}$$

This shows that the envelope of holomorphy we are looking for contains the union of the right hand side. So it remains to show that the right hand side is a domain of holomorphy.

To this end remark that $[0, 1]^N$ becomes a semi-ordered space by introducing the relation

$$\underline{\lambda}_1 \leq \underline{\lambda}_2 \qquad \text{iff} \quad (\underline{\lambda}_1)_i \leq (\underline{\lambda}_2)_i \qquad \text{for } i = 1, 2, \dots N$$

From definition of the $G_{\underline{\lambda}}$ follows with this semi-ordering $G_{\underline{\lambda}_1} \subset G_{\underline{\lambda}_2}$ iff $\underline{\lambda}_1 \leq \underline{\lambda}_2$. For $S \subset [0, 1]^N$ define $\hat{S}$ as follows

$$\hat{S} = \{ \underline{\lambda} \ ; \ \exists \ \underline{\lambda}' \in S \quad \text{with} \quad \underline{\lambda} \leq \underline{\lambda}' \}$$

then we always get

$$\bigcup_{\underline{\lambda} \in S} G_{\underline{\lambda}} = \bigcup_{\underline{\lambda} \in \hat{S}} G_{\underline{\lambda}}$$

If S is convex then this is obviously also true for $\hat{S}$. If $\hat{S}$ is convex then it can be written as intersection of sets in $[0, 1]^N$ which are bounded by boundery points of $[0, 1]^N$ and a hyperplane. But there appear only such hyperplanes which have a normal vector $\underline{n}$ lying in $[0, 1]^N$.

Since the intersection of domains of holomorphy defines again a domain of holomorphy, we have reduced the problem to the situation where S is given by

$$S = \{ \underline{\lambda} \in [0, 1]^N \ ; \ (\underline{n}, \underline{\lambda}) \leq c \}$$

and $c \leq \sum n_i$. If we put for short writing $p^i(z_i) = p_m(z_i, G_o^i, G_1^i)$ and define

$$p(z_1, z_2, \dots z_N) = \sum n_i \, p^i(z_i)$$

then this represents a pluri-subharmonic function on $G_1^1 \times \dots \times G_1^N$. Therefore

$$\{ (z_1, \dots, z_N) \ ; \ p(z_1, \dots, z_N) < c \}$$

defines a domain of holomorphy. But looking at the definition of G_{λ}^i we find that this domain coincides with $\bigcup_{\underline{\lambda} \in S} G_{\underline{\lambda}}$.

This proves the theorem.

Next we want to give two generalizations of this theorem. The first one is a generalized semi-tube theorem.

V.3. Theorem:

Let $H \subset \mathbb{C}^n$ and $G_1 \subset \mathbb{C}^m$ be domains of holomorphy and assume $G_0 \overset{H}{\subset} G_1$. Let $\Gamma \subset \mathbb{C}^{n+m}$ be defined as follows:

$$\Gamma = \{ (z, w) ; z \in H \text{ and } w \in G_{\lambda(z)}$$

Then Γ is a domain of holomorphy exactly if $\lambda(z)$ is a pluri-superharmonic function on H.

Proof:

Assume first that $\lambda(z)$ is pluri-superharmonic function on H. Then follows that

$$p(z, w) = 1 - \lambda(z) + p_m(w, G_o, G_1)$$

is a pluri-subharmonic function on $H \times G_1$. But from the definition of G_λ follows

$$\Gamma = \{ (z, w) \in H \times G_1 ; p(z, w) < 1 \} .$$

Since $p(z, w)$ is pluri-subharmonic follows that Γ is a domain of holomorphy.

For proving the converse statement we remark first, that the function $\lambda(z)$ in the definition of Γ has to be lower semi-continuous in order that Γ becomes a domain. If G_o^i, G_1^i is an increasing approximation of G_o, G_1 such that $\bigcup_i G_\lambda^i = G_\lambda$, and we have shown that the theorem holds for

$$\Gamma^i = \{ (z, w) ; z \in H , w \in G^i_{\lambda(z)} \}$$

then it is true also for Γ, since $\bigcup_i \Gamma^i = \Gamma$.

If $G_o^{'i}$, $G_1^{'i}$ is an increasing approximation as described in Lemma II.5. then we put $G_o^i = G^i_{1/i}$ and $G_1^i = G_1^i$ in order that we can use the convergence Theorem IV.7. Γ^i is supposed to be a domain of holomorphy then (with the notation of Theorem IV.7.) $F(z, w) \in A(\Gamma^i)$ posessed a development

$$F(z, w) = \sum_i a_i(z) \, f_i(w)$$

with $a_i(z) \in A(H)$ and

$$\limsup_{i \to \infty} \frac{\log |a_i(z)|}{\log \sigma_i} \leq 1 - \lambda(z)$$

Denoting by $p(z, F)$ the pluri-subharmonic limit of the left hand side and by $p(z)$ the pluri-subharmonic majorant of all the $p(z, F)$ then we have $p(z) \leq 1 - \lambda(z)$. But since Γ^i is a domain of holomorphy follows that there exists functions with Γ^i as their natural domains. Hence we get $p(z) = 1 - \lambda(z)$. This proves the theorem.

We want to end this paper with a generalization of the first theorem of this section. There we have constructed the envelope of holomorphy of $G_o \times H_1 \cup G_1 \times H_o$ where $G_o \overset{H}{\subset} G_1$, $H_o \overset{H}{\subset} H_1$ are all domains of holomorphy. In many applications we find a more general situation namely one has to construct the domain of holomorphy of $G_o \times H_1 \cup G_1 \times H_o$ where all four domains are natural domains but where the G's and the H's do not form Hadamard pairs. For the treatment of this problem the last theorem plays an essential role. Before we can state the result, we need some notations.

Let G_1 be a domain of holomorphy and $G_o \subset G_1$ a domain, then the set $F \subset P(G_1)$

$$F = \left\{ (p(z) \in P(G_1) ; \quad p(z) \leq 1 \quad \text{and} \quad p(z) \leq 0 \text{ for } z \in G_o \right\}$$

is well defined. This contains a pluri-subharmonic majorant $p_m(z)$.
If we define $\widetilde{G}_o = \left\{ z \in G_1 ; \ p_m(z) \leq o \right\}^o$ then we have $\widetilde{G}_o \overset{H}{\subset} G_1$ and $p_m(z) = p_m(z, \widetilde{G}_o, G_1)$. With $\widetilde{G}_\lambda$ we denote the interpolating family of the pair $\widetilde{G}_o \overset{H}{\subset} G_1$.

V.4. <u>Theorem:</u>

Let $G_1 \subset \mathbb{C}^n$ and $H_1 \subset \mathbb{C}^m$ be domains of holomorphy and assume $G_o \subset G_1$ and $H_o \subset H_1$ are domains (not necessarily domains of holomorphy) then we obtain with the above notation

$$\text{hull } G_o \times H_1 \cup G_1 \times H_o = \bigcup_\lambda \widetilde{G}_\lambda \times \widetilde{H}_{1-\lambda} \ ,$$

Proof:

Let us denote the envelope of holomorphy we are surching for by Γ. Then we define

$$\widehat{G}_\lambda = \{ z \in G_1 \; ; \; z \times \widetilde{H}_{1-\lambda} \subset \Gamma \}^0$$

From Theorem V.4. follows that $\widehat{G}_\lambda$ is characterized by a pluri-subharmonic function which implies that the $\widehat{G}_\lambda$ are itselves domains of holomorphy. Furthermore we have by assumption $\widehat{G}_0 \supset G_0 \neq \emptyset$, so that we are not talking about empty sets.

Let us denote by $D_r \subset \mathbb{C}^n$ the poly-circle of radius r and let $z_0 \in \widehat{G}_\lambda$ then exists r_1 such that $z_0 + D_{r_1} \subset G_1$ and r_0 with $z_0 + D_{r_0} \subset \widehat{G}_\lambda$. Since $\widehat{G}_\lambda \subset G_1$ follows $r_1 \geqslant r_0$. Therefore we have

$$z_0 + D_{r_0} \times \widetilde{H}_{1-\lambda} \cup z_0 + D_{r_1} \times H_0 \subset \Gamma \quad ,$$

and therefore also

$$\text{hull} \quad z_0 + D_{r_0} \times \widetilde{H}_{1-\lambda} \cup z_0 \, D_{r_1} \times H_0 \subset \Gamma$$

Since $D_{r_0} \overset{H}{\subset} D_{r_1}$ follows by theorem V.3. that this hull is given by the maximal pluri-subharmonic function $\lambda(w)$ which is bounded by 1 on $\widetilde{H}_{1-\lambda}$ and zero on H_0 with $D_r = D_{r_0}{}^{\lambda(w)} \; r_1{}^{(1-\lambda(w))}$. This implies together with Theorem II.14 and the definition of $\widetilde{H}_\lambda$

$$z_0 + D_{r_0} \times \widetilde{H}_{1-\lambda} \times \quad z_0 + D_{r_1} \times \widetilde{H}_0 \subset \Gamma .$$

Taking the union over all $\widehat{D}_\lambda$ we see that

$$G_1 \times \widetilde{H}_0 \subset \Gamma .$$

But by symmetry we get $G_1 \times \widetilde{H}_0 \cup \widetilde{G}_0 \times H_1 \subset \Gamma$ and the result follows from Theorem V.1.

References

[1] Borchers, H.J. and J. Yngvason:
Necessary and Sufficient Conditions for Integral Representations of Wightman Functionals at Schwinger Points. Commun. Math. Phys. 47, 197 (1976).

[2] Bremermann, H.:
Über die Äquivalenz der pseudokonvexen Gebiete und der Holomorphiegebiete im Raum n komplexer Veränderlicher. Math. Ann. 128, 63 (1954)

[3] - " - :
Complex Convexity.
Trans. Amer. Math. Soc. 82, 17 (1956).

[4] - " - :
On the Conjecture of the Equivalence of Pluri-Subharmonic Functions and the Hartogs Functions. Math. Ann. 131, 76 (1956).

[5] Grauert, H. und F. Fritsche:
Einführung in die Funktionentheorie mehrerer Veränderlicher.
Hochschultext Springer; Berlin, Heidelberg, New York (1974).

[6] Hörmander, L.:
An Introduction to Complex Analysis in Several Variables.
D. van Nostrand; Princeton N. J. (1966).

[7] Meschkowski, H.:
Hilbertsche Räume mit Kernfunktionen.
Springer; Berlin, Göttingen, Heidelberg (1962)

[8] Pietsch, A.:
Nukleare lokalkonvexe Räume.
Akademie-Verlag; Berlin (1969).