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# Causality and Non-Localisable Fields

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# Causality and Non-localisable Fields

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E R R A T A   T O

"Causality and

Non-localisable Fields"

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(1) Page 2 lines 1 and 2 to read:

decrease of the commutator outside the light cone.

instead of:

decrease of the commutator outside the light cone as well as the range of the existence.

(2) Page 5 lines 1 and 2 to read:

If the commutator applies to test functions which actually increase for large space-like ...

instead of:

If it is to test functions which actually increase for large space-like ...

Abstract

We give a model-independent and Lorentz invariant prescription for the manner in which a non-localisable quantum field extends outside the light cone, in particular specifying the order of this extension. We show how our definition applies to several examples, including the non-localisable free field and certain functions of the massive free field; non-localisable functions of the massless free field presents a difficulty which has not yet been resolved.

## §1. Introduction

There has been much recent discussion of non-polynomial functions of the free field, both with respect to mathematical problems arising in setting up such a functional calculus[1] and also with respect to the application of such a calculus to non-polynomial chiral and gravitational theories.[2] One of the most important questions involved in such problems is that of the properties that such non-polynomial functions possess, especially causality and positive-definiteness of the metric in the state space.[3] In this paper we attempt an analysis of causality, in particular investigating the manner in which it is broken by non-localisable functions of the free field.

These non-localisable functions have physical interest since they can arise when particular field co-ordinates are chosen in either the chiral or gravitational interactions. A class of these functions has been investigated elsewhere and shown[4] to possess various useful properties, such as the existence of a PCT operator and of a scattering theory, as the vestiges of local commutativity. However, such non-localisable fields cannot satisfy strict causality, and a description of the extension of the commutator bracket outside the light cone has been given for the zero mass case.[5] We wish here to give a general discussion of this extension of the commutator bracket of a non-localisable field outside the light cone, both from a mathematical and a physical point of view. This necessitates the introduction of a new class of test functions which can suitably probe the behaviour of the commutator bracket both outside and inside the light cone. This allows us to indicate, in a Lorentz invariant fashion, both the rate of decrease of the commutator outside the light cone as well as the range of the extension. In other words, we specify the degree of non-causality in an invariant fashion, and expect the range defined in this way to have physical interest and so determine the energy at which violations of causality can appear.

The detailed plan of the paper is as follows. In the next section we give a general discussion of the way the extension of the commutator bracket can be described in a Lorentz invariant and mathematically precise fashion. We give a definition which classifies the degree of non-localisability of non-localisable fields. In section 3 we show how functions of the zero mass free field nearly satisfy conditions for this definition to apply, and consider a special class of non-zero mass fields in the subsequent section for which our definition can be used. A more complete discussion of the non-zero mass case is given in section 5. The physical significance of the discussion is considered in the final section.

## §2. Non-localisable Fields

In this section we wish to formulate the properties of non-localisable fields in a manner independent of the fact that the most interesting applications are to functions of a free field. We will consider only one neutral scalar field $\phi(x)$, the extension to complex spin fields presenting no conceptual difficulty. We suppose that the field $\phi(x)$ is an operator-valued generalised function over a suitable test function space $C$ of functions $f(x)$, so that $\phi(f) = \int \phi(x)\, f(x)\, dx$ is an operator for which the usual Wightman axioms,[6] except that of local commutativity, apply. Thus there are a sequence of Wightman functions $W_n\ (x_1, \ldots x_n)$, defined by

$$W_n(x_1, \ldots , x_n) = < 0 \mid \phi(x_1) \ldots (x_n) \mid \ 0 >$$

from which the fields may be reconstructed.

The usual tempered field theory results if C is the space $\mathcal{S}$ of indefinitely differentiable functions decreasing at infinity faster than any polynomial; the localisable case occurs if C contains a dense subset of functions of compact support. The non-localisable situation in which we are interested in here corresponds to C being comprised of analytic functions due to their rapid decrease in momentum space. This fall-off is required in order that the rapid high energy increase of the Fourier transforms $\bar{W}_n$ of the Wightman functions can be satisfactorily taken account of by the Fourier transforms of the functions of C, which we denote by ʻʻ . If $\bar{W}_n$ increases like $\exp\left[|p|^{\alpha}\right]$, for $\alpha < 1$, a satisfactory choice for ʻʻ is any space $S^{\beta}$, for $\beta < \alpha$ of Gelfand and Shilov(7), being the set of indefinitely differentiable functions, which together with all derivations, are bounded at infinity by $\exp\left[-b\ |p|^{1/\beta}\right]$ for some positive b (which depends on the function considered). A more precise specification(8) is given by the indicatrix function $g(p^2)$ which is an entire function satisfying

$$\int_1^{\infty} \frac{\log g(t^2)}{t^2}\ dt < \infty \quad . \tag{1}$$

The related function space, which we denote $C^g$ in momentum space or $C_g$ in co-ordinate space, is composed of momentum space functions $\phi(p)$

which, together with all derivatives, are bounded by $g^{-1}(Ap^2)$ for some constant A. If (1) is not satisfied the space $C_g$ is composed of analytic functions, so again corresponds to the non-localisable case.

We suppose, then, that each Wightman function $W_n$ belongs to the dual of one of the above spaces $C(\underline{R}^{4(n-1)})$ in 4(n-1) variables. As usual we denote this by

$$W_n \in . C'(\underline{R}^{4(n-1)}). \tag{2}$$

It is not possible, in the non-localisable case, even to formulate local commutativity. Analysis of special cases(5) has shown that there may still be a trace of local commutativity in that commutator brackets of the field operators at two points, whilst not zero for space-like separations, can decrease fast as the space-like separation increases. In fact we will be led to consider an exponential fall-off. If we denote by $\phi(x)$ the non-localisable field of interest, then we expect(5) for large space-like values of $(x - y)$, that

$$| < 0 \mid [\phi(x), \phi(y)]_- \mid 0 > | \leq e^{-a|(x-y)^2|^{1/2\gamma}} \tag{3}$$

for some positive constant a and $\gamma$ , with $\gamma < 1$. We note that (3) does not contradict the result of Pohlmeyer and Borchers(9) that such a fast fall off implies strict local commutativity, since this result was obtained on the basis of analyticity of Wightman functions in space-time; such properties no longer persist in the non-localisable case.

We will now discuss how we may formulate (3) in a general fashion. The idea behind our approach is to determine how far outside C we may extend the commutator bracket (3). If it is to test functions which actually increase for large space-like values like the inverse of the r.h.s. of (3) then we can conclude that the decrease of the commutator is roughly given by the r.h.s. of (3). This extension has to be investigated in detail for each particular field being considered, but we will attempt in this section to give a general formulation of it which is model independent. To do that let us take the case when C is required to be an $S^{\alpha}$ space, with $\alpha < 1$. We consider the commutator brackets

$$C_{n,j}(x_1 \ldots x_n) = W(x_1 \ldots x_j, x_{j+1} \ldots x_n) - W(x_1 \ldots x_{j+1} x_j \ldots x_n) \quad (4)$$

If $\phi$ were local then causality would indicate that each $C_j$ vanishes for $\xi_j = x_j - x_{j+1}$ space-like, but would be non-zero for time-like spearations. Thus we have to choose test functions which are in $S^{\alpha}$ for time-like values; we denote this space of test functions by $S^{\alpha}_X$. Since the set $X = \{\xi \ ; \ \xi^2 \geq 0\}$ is a closed set we have to define $S^{\alpha}_X$ by a suitable limiting procedure. It may be possible to use the closed set X directly, without using the following construction, but there are various point which need to be resolved before that can be done. We will not consider that further here but use the better known inductive limit approach.

Let $S^{\alpha B}(\Omega_\mu)$ be the set of functions $\phi$ of the four-vector x which are defined on the set $\Omega_\mu = \{x \ ; \ x^2 > -\mu^2\}$, and satisfy there

$$|\underline{x}^{\underline{k}} \ \phi^{(q)}(x)| \leq C_{\underline{k}} \ B^{|q|} \ q^{q\alpha}$$

where $\underline{x}^{\underline{k}} = \prod_{i=1}^{4} x_i^{k_i}$, $|q| = q_1 + q_2 + q_3 + q_4$, $q^{q\alpha} = \prod_{i=1}^{4} q_i^{q_i\alpha}$, and B, $C_{\underline{k}}$ are given constants. In order to take account of the constant a on the r.h.s. of (3) we should take B to be independent of $\phi$ in (5). However, that does not give a Lorentz covariant space, but initially we allow only the constants $C_{\underline{k}}$ to vary with $\phi$. We now define

$$S_x^{\alpha} = \bigcup_{\substack{\mu>0 \\ B=1,2,\ldots}} S^{\alpha B}(\Omega_\mu) \tag{6}$$

Each space $S^{\alpha B}(\Omega_\mu)$ is a complete, countably normed perfect space with the norms

$$\| \phi \|_{k,\rho,\mu} = \sup_{\substack{x\varepsilon\Omega_\mu \\ q}} \left| \underline{x}^{\underline{k}} \phi^{|q|}(x) \right| / \left[ (B + \rho)^{|q|} q^{q\alpha} \right]$$

for $k = 0, 1, 2, \ldots$, $\rho = 1, \frac{1}{2}, \ldots$, being a standard $S^{\alpha B}$ space but on the open set $\Omega_\mu$ (10). Since $S^{\alpha B_1}(\Omega_\mu) \subseteq S^{\alpha B_2}(\Omega_\nu)$ for $\mu \geq \nu$ and $B_2 \geq B_1$ with topological inclusion, then we may take the topology on $S_x^{\alpha}$ as the inductive limit as $\mu$ tends to zero and B to infinity of these topologies on the $S^{\alpha B}(\Omega_\mu)$, so that a sequence $\{\phi_n\}$ converges to zero in $S_x^{\alpha}$ if all the functions $\phi_n$ belong to some space $S^{\alpha B}(\Omega_\mu)$ for some $\mu$ and B, and converge to zero in its topology. We note that if $\alpha > 1$ the space $S^{\alpha B}(\Omega_\mu)$ is not complete, and the quantity $\| \phi \|_{k,\rho,\mu}$ is not a true norm but only a pre-norm. However we do not wish to apply our construction of $S^{\alpha B}(\Omega_\mu)$ and $S_x^{\alpha}$ to the localisable case.

We construct the space of test functions appropriate to the commutator $C_{n,j}$ as

$$T_{n,j}(\alpha) = S^{\alpha}(\underline{R}^{4(n-2)}) \otimes S_x^{\alpha} \tag{7}$$

where the functions $\phi(\xi_1,\ldots,\xi_{n-1})$ in $T_{n,j}(\alpha)$ are in $S^\alpha(\underline{R}^{4(n-2)})$ with respect to the variables $\xi_i$ $(= x_i - x_{i+1})$ for $i = 1,2, \ldots,$ $j - 1$, $j + 1$, $\ldots$, $n - 1$ and are in $S_x^\alpha$ in $\xi_j$. At this stage it is not evident how this space $T_{n,j}(\alpha)$ is relevant to the exponential decrease given by the r.h.s. of (3). This will become apparent from the following lemma.

Lemma 1.

Any function $\phi \in S_x^\alpha$ has an analytic continuation to the whole of $\underline{R}^4 + i\underline{R}^4 = \underline{C}^4$, and increases at infinity at most like $\exp 4(B \;\| x \|)^{\frac{1}{1+\alpha}}$ where $\| x \| = \sup_{1 \leqslant i \leqslant 4} |x_i|$, for some constant B.

Proof

To prove this we note that $\phi \in S^{\alpha B}(\Omega_\mu)$, for some B and $\mu$. Thus by (5)

$$| \phi^{(q)}(o)| \leqslant C_{\underline{o}} \; B^{|q|} \; q^{q\alpha} \tag{8}$$

Thus the series

$$\psi(x) = \sum_q \underline{x}^q \quad \phi^{(q)}(o)/_{q!}$$

where

$$\underline{q}! = \prod_{i=1}^{4} q_i \;!, \; \underline{x}^q = \prod_{i=1}^{4} x_i^{q_i}, \; \phi^{(q)}(o) = \left( \prod_{i=1}^{4} \partial^{q_i}/\partial x_i^{q_i} \right)\phi \;\Big|_{x=o}$$

has infinite radius of convergence and is bounded at infinity, by (8), by

$$C_o \sum_q B^{|q|} q^{q\alpha} |\underline{x}|^q / q! \leq C_o \exp \left[4B \|x\|^{1/1-\alpha}\right] \tag{9}$$

where

$$|\underline{x}|^q = \prod_{i=1}^{4} |x_i|^{q_i}$$

Since $\psi$ coincides with the analytic function $\phi$ in $\Omega_\mu$ then the lemma is proved. We see that if we can extend $C_{n,j}$ from $S^\alpha(\underline{R}^{4(n-1)})$ to $T_{n,j}(\beta)$ then we obtain the decrease like the r.h.s. of (3) with $\gamma = 1 - \beta$ and $a = 4(B)^{1/(1-\beta)}$. Thus we define the degree of extension of the commutator bracket outside the light cone as follows.

<u>Definition 1</u>

If for each n and each j we may extend $C_{n,j}$ from $S^\alpha(\underline{R}^{4(n-1)})$ to $T_{n,j}(\beta)$, where $\beta$ may be chosen independently of n and j, then the largest such values of $\gamma = 1 - \beta$ will be called the order of extension of the commutator bracket outside the light cone.

We note that this definition is obviously Lorentz invariant, the space $S^\alpha_x$ evidently being so.

We extend this definition to the finer specification given by and indicatrix function $g(p^2)$. Let

$$g(p^2) = \sum_{r=0}^{\infty} c_{2r}\, p^{2r}$$

Then the space $C^{Bg}(\underline{R}^4)$ is defined as

$$C^{Bg}(\underline{R}^4) = \left[\phi : |\underline{x}^{\underline{k}}\, \phi^{(q)}(x)| \leq C_{\underline{k}}\, B^{|q|}\, c_q^{-q}\right] \tag{10}$$

We modify this space to $C^g(\Omega_\mu)$ as before, and define

$$C_x^g = \bigcup_{\substack{\mu>0 \\ B<\infty}} C^{Bg}(\Omega_\mu)$$

Similarly we define the space $T_{n,j}(g) = C^g(\underline{R}^{4(n-2)}) \otimes C_x^g$, and extend the definition 1 to

## Definition 2

If for each n and j we may extend $C_{n,j}$ from $C^g(\underline{R}^{4(n-1)})$ to $T_{n,j}(g)$, where g may be chosen independently of n and j, then the smallest g which may be chosen defines the nature of decrease of the extension of the commutator outside the light cone.

To see this nature of decrease in detail, we extend Lemma 1 to

## Lemma 2

Any function $\phi \in C_x^g$ has an analytic continuation to the whole of $\underline{C}^4$ and increases at infinity at most like $\prod_{i=1}^{4} G(B|x_i|)$, when $G(x) = \sum_{n\geq 0} c_n^{-n} \frac{(Bx)^n}{n!}$ has an infinite radius of convergence, and B depends on $\phi$.

The proof of this Lemma follows that of Lemma 1 almost identically, where we assume that g does not satisfy (1), so we are dealing with the non-localisable situation. It need not be the case that any $\phi(x) \varepsilon C^g_x$ has an <u>analytic</u> continuation to all $\underline{C}^4$ in general, since $\phi$ need at most be quasi-analytic in $\Omega_\mu$ for some $\mu$. That is why we need to impose the condition on the infinite radius of convergence of the series $G(x)$. However, definition 2 may be used even if it is only quasi-analytic but the extension process can only be performed infinitesimally outside the light cone. It does not seem possible to specify the rate of decrease of the commutator in this more general situation. As examples we note that if $c_n = n^{-\alpha}$ we return to the $S^\alpha$ spaces, whilst if $c_n = n^{-1} \log n$ it has an infinite radius of convergence. However if $c_n = n^{-1} (\log n)^{-1}$ the function $G(x)$ has zero radius of convergence, so that analytic continuation of any $\phi \varepsilon C^g_x$ need not be possible to arbitrary space-like points.

We can enlarge the above approach to include the <u>range</u> of decrease of the commutator bracket outside the light cone if we consider the various Wightman functions in terms of their invariant variables; the discussion is so similar to the above, except for replacement of four-vector variables by invariants, that we need not give that discussion here.

## §3. Functions of the Massless Free Field

Let us turn now to specific examples to indicate how the above formulation of non-causality actually applies. We consider in this section an infinite series in normal-ordered powers of the free massless scalar field

$$\phi(x) = \sum_{n=o} \frac{d_n}{n!} : \phi_o^n (x): \tag{11}$$

where : : denotes the normal ordering and the $d_n$ are real coefficients. Then in the notation of section 4 of reference (5),

$$C_{n,k} (x_1, \ldots, x_n) = \sum_{r_{ij}=o} \Big( \prod_{\substack{i=1 \\ i \neq k,k+1}}^{r} d_{R_i} \Big) T^{R/r_{k,k+1}} \left[R!/_{r_{k,k+1}!}\right]^{-1} \times$$
$$\times \; 2\pi i \, \varepsilon(x_o) \sum_{r_{k,k+1}=o}^{\infty} \frac{d_{R_k} d_{R_{k+1}} (-1)^{r_{k,k+1}} \delta^{(r_{k,k+1}-1)} (\xi_k^2)}{r_{k,k+1}! \, (r_{k,k+1} - 1)! \, (4\pi^2)^{r_{k,k+1}}} \tag{12}$$

where

$$T^{R/r_{k,k+1}} = \prod_{\substack{1 \leq i \leq j \leq n \\ (i,j) \neq (k,k+1)}} \left[\frac{1}{i} \Delta_+ (x_i - x_j)\right]^{r_{ij}} \tag{13}$$

We see from (12) in this case that if

$$d_n^2 = \lambda^n \, \Gamma(1 + n\alpha) \tag{14}$$

then $C_{n,k}$ would appear to be in $T_{n,k}(\beta)$ provided $\beta < 2 - \alpha$; the non-localisable situation corresponds to $\alpha > 1$ so the $\beta$ can be chosen less than one to satisfy this. However, there is a difficulty in this approach which we will clarify for the particular case of the two-point, n = 2. For then, using (14),

$$C_2(x_1,x_2) = \sum_{n\geq 0} \frac{d_n^2}{n!} \left[\Delta_+(x_1 - x_2)^n - \Delta_-(x_1 - x_2)^n\right]$$

$$= \sum_{n\geq 0} \int_0^\infty \frac{t^{n\alpha} e^{-t}}{n!}\, dt \left[\Delta_+(x_1 - x_2)^n - \Delta_-(x_1 - x_2)^n\right]$$

$$= \int_0^\infty dt\; e^{-t} \left[e^{t^\alpha \Delta_+(x_1-x_2)} - e^{t^\alpha \Delta_-(x_1-x_2)}\right] \tag{15}$$

We may form $\langle C_2,\phi\rangle$ , for $\phi$ analytic, by using that

$$\Delta_\pm(x) = \lim_{\varepsilon\to 0} \left[(x_0 \mp i\varepsilon)^2 - \underline{r}^2\right]^{-1}$$

with $x = (x_0, \underline{r})$ so that

$$\langle C_2,\phi\rangle = \lim_{\varepsilon\to 0} \int_0^\infty dt\; e^{-t} \int_{C_\varepsilon} dx_0 \int d^3r\; e^{t^\alpha/x^2}\; \phi(x) \tag{16}$$

where $C_\varepsilon$ is a contour in the complex $x_0$ - plane composed of two parts, as shown in fig. 1.

$x_0$ - plane

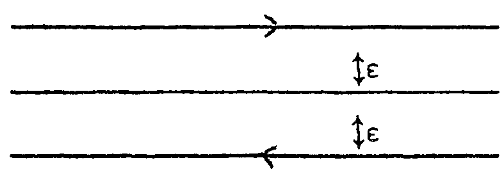


fig. 1.

Thus $\langle C_2,\phi\rangle = \lim_{\varepsilon\to 0} \int_0^\infty dt\; e^{-t} \int_{C_\varepsilon} dx_0 \int d^3\underset{\sim}{r} \sum_{n\geq 0} \frac{t^{n\alpha}}{n!} \frac{1}{(x^2)^n} \cdot \phi(x).$

For $|\underset{\sim}{r}| \neq 0$ the poles of $(x^2)^{-n}$ are at $x_0 = \pm\, |\underline{r}|$, and the contour

$C_\varepsilon$ may be shrunk to that of fig. 2.

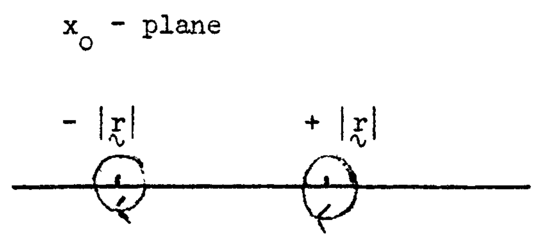


fig. 2

Thus we have

$$<C_2,\phi> = \int_o^\infty dt\ e^{-t} \sum_n 2\pi i \frac{t^{n\alpha}}{(n!)^2} \int \frac{d^3\underset{\sim}{r}}{|\underset{\sim}{r}|^n} \left[ \frac{\partial^n \phi}{\partial x_o^n}\bigg|_{x_o=|\underline{r}|} + (-1)^n \frac{\partial^n \phi}{\partial x_o^n}\bigg|_{x_o=|\underline{r}|} \right] \tag{17}$$

Provided that for all $\underset{\sim}{r} \varepsilon \underline{R}^3$

$$|\underset{\sim}{r}|^k \left| \frac{\partial^n \phi}{\partial x_o^n} \right|_{x_o=\pm|\underline{r}|} \leq C_k\ n^{n\beta}(B)^n \tag{18}$$

then

$$|<C_2,\phi>| \leq \text{constant x} \int_o^\infty dt\ e^{-t} \sum_n B^n \frac{t^{n\alpha}}{n^{n(2-\beta)}} < \infty \qquad \text{if } \alpha + \beta < 2.$$

This is the same condition which arose above, but now we see the defect arising from both of these approaches. This is that the space integral in (17) is divergent for $n > 2$, owing to the factor $|\underset{\sim}{r}|^{-n}$. Thus the condition (18) or the stronger condition $\phi \;\varepsilon\; S_x^\beta$ does not lead to a definite value for $<C_2, \phi>$ . Thus our discussion of functions of the free massless field is deficient at this point. Whilst this is unsatisfactory we will see that a similar situation does not arise in the massive case, so we turn to that now.

## §4. The Massive Case

In this section we will consider two very simple examples of non-localisable fields which have been discussed already in the localisable case. The first of these is the generalised free field $\phi(x)$ for which the commutator bracket is

$$[A(x), A(y)]_- = \int_{m_o^2}^{\infty} d\kappa^2 \; \rho(\kappa^2) \; \Delta(x - y, \kappa^2) \tag{19}$$

where $\Delta(x, \kappa^2)$ is the invariant propagator for mass $\kappa$. The function $\rho(\kappa^2)$ is allowed to increase for large $\kappa$ with order of growth at most one for localisability. We can see this by considering the r.h.s. of (19) when applied to a test function $\phi$, by means of Fourier transforms, for it takes the value

$$\begin{aligned} <[A(x), A(o)]_-, \phi> &= \int_{m_o^2}^{\infty} d\kappa^2 \; \rho(\kappa^2) \int d^4x \;\; \Delta(x,\kappa^2) \; \phi(x) \\ &= \int_{m_o^2}^{\infty} d\kappa^2 \;\; \rho(\kappa^2) \;\; \Delta_\phi(\kappa^2) \end{aligned} \tag{20}$$

where the functions $\Delta_\phi$ of $\kappa^2$ is defined by

$$\Delta_\phi(\kappa^2) = \int d^4x\ \phi(x)\ \Delta(x,\kappa^2)$$

$$= \int d^4p\ \tilde{\phi}(p)\ \delta(p^2 - \kappa^2)\quad \varepsilon(p_0) \tag{21}$$

where $\phi$ is the Fourier transform of $\phi$ . We now prove

<u>Lemma 3</u>

If $\phi \in S^{\alpha B}$ then $|\Delta_\phi(\kappa^2)| \leq e^{-(B\kappa)^{1/\alpha}}$ . (constant)

<u>Proof</u>

If $\phi \in S^{\alpha B} \Rightarrow \tilde{\phi} \in S_{\alpha B} \Rightarrow |\ \tilde{\phi}(p)| \leq e^{-(B\ \|\ p\ \|\ )^{1/\alpha}}$ ,

where $\|\ p\ \|$ is any norm on $\underline{R}^4$ consistent with the usual Euclidean topology. Let us take $\|\ p\ \|^{1/\alpha} = |p|^{1/\alpha} + |p_0|^{1/\alpha}$ . Then

$$\Delta_\phi(\kappa^2) = \int \frac{d^3p}{2|p|}\quad [\ \tilde{\phi}(\sqrt{p^2 + \kappa^2},\ p) - \tilde{\phi}(-\ \sqrt{p^2 + \kappa^2}),\ p)\ ] \tag{22}$$

so

$$|\Delta_\phi(\kappa^2)| \leq \int \frac{d^3p}{2|p|}\quad e^{-(B\ \|p\ \|)^{1/\alpha}} - (B\sqrt{p^2 + \kappa^2})^{1/\alpha}$$

$$\leq \text{(constant)} \times e^{-(B\kappa)^{1/\alpha}}$$

as required.

Thus the integral on the r.h.s. of (20) is finite at infinity for all $\phi \in S^{\alpha,B+\epsilon}$ and $\epsilon > 0$, provided that

$$|\rho| \sim e^{(\kappa^2)^{1/2\alpha}} \qquad \text{as } \kappa^2 \to \infty \tag{23}$$

To take account of possible singularities in $\rho$ we have to discuss the differentiability properties of $\Delta_\phi(\kappa^2)$.

Lemma 4

If $\phi \in S^{\alpha,B}(\underline{R}^4)$ then $\Delta_\phi(\kappa^2) \in S_{\alpha,B}((0,\infty))$, Gelfand-Shilov $S_{\alpha,B}$-space, but defined on the open interval $(0,\infty)$.

Proof

This follows directly from (22), using the differentiability properties of $\phi$ :

$$|\Delta_\phi^{(q)}(\kappa^2)| \leq \int \frac{d^3p}{2|p|} \left| \left[ \frac{1}{2\sqrt{p^2+\kappa^2}} \cdot \frac{\partial}{\partial\sqrt{\kappa^2+p^2}} \right]^{(q)} \left[\tilde{\phi}(\sqrt{p^2+\kappa^2},p) - \tilde{\phi}(-\sqrt{p^2+\kappa^2},p)\right] \right| < \infty$$

so that $\Delta_\phi(\kappa^2) \in C^\infty((0,\infty))$, the set of indefinitely differentiable functions on the open interval $(0,\infty)$. Including the results of Lemma 3 proves the Lemma.

We have proved that if $\rho \in S_{\alpha B}(0,\infty)$ then the commutator brackets (as well as all the Wightman functions, as can easily be seen) are all in the appropriate $S^{\alpha B'}$ spaces of the relevant four-vector variables.

When we turn to the extension problem, with $\alpha < 1$, we see that the previous discussion using Fourier transforms can no longer be given, since the test functions of $S_x^{\alpha}$ or $S_x^{g}$ may increase too rapidly outside the light cone for their Fourier transforms even to be defined. Thus we need to rephrase the preceeding discussion purely in terms of co-ordinate space. We do that in the following lemma.

Lemma 5

If $\phi \in S^{Bg}(\Omega_\mu)$ for some B and $\mu$ then $\Delta_\phi(\kappa^2) \in S_{B^{-1}g}(0,\infty) =$

$$\left[\phi \;:\; |x^k \phi^{(q)}(x) \leq c_q B^k (c_k)^{-k}, \quad x \in (0,\infty)\right]$$

Proof

Let us consider $\Delta_\phi(\kappa^2)$ of (21), and form

$$\kappa^{2n} \Delta_\phi(\kappa^2) = \int d^4x\, \kappa^{2n} \Delta(x,\kappa^2)\, \phi(x) = \int d^4x\, (\Box^2)^n \Delta(x,\kappa^2)\, \phi(x)$$

$$= \int d^4x\, \Delta(x,\kappa^2)\, (\Box^2)^n\, \phi(x) \tag{24}$$

Using the standard representation for $\Delta(x,\kappa^2)$:

$$\Delta(x,\kappa^2) = \frac{1}{4\pi r} \frac{\partial}{\partial r} \{[\epsilon(x_0)\, J_0(\kappa\sqrt{x^2})] \quad \theta(x^2)\}$$

we obtain

$$\kappa^{2n}\ \Delta_\phi(\kappa^2) = \int_{-\infty}^{+\infty} dx_o\ \varepsilon(x_o) \int_o^{|x_o|} r^2 dr\, \frac{\partial}{\partial r}\ \left[\frac{1}{r}\ \hat{\phi}_n(x_o,r)\right] . J_o(\kappa\sqrt{x_o^2-r^2})$$

where

$$\hat{\phi}_n(x_o,r) = \int_{|\underline{r}| = r} d\Omega(\underset{\sim}{r})\ \ (\Box^2)^n\ \phi.$$

$d\Omega(\underline{r})$ being the measure of the sperical surface $|\underline{r}| = r$. In terms of the variables $\lambda = \sqrt{x_o^2 - \underset{\sim}{r}^2}$ and r we have

$$\kappa^{2n}\ \Delta_\phi(\kappa^2) = \int_o^\infty \lambda d\lambda \int_o^\infty r^2 dr (r^2+\lambda^2)^{-\frac{1}{2}}\ (\partial/\partial r)\ \left[r^{-1}\ \ \bar{\phi}_n^{(+)}(\lambda,r)\ -\right.$$

$$\left. r^{-1}\ \ \bar{\phi}_n^{(-)}\ (\lambda,r)\right]\ \ J_o(\kappa\lambda) \tag{25}$$

where $\bar{\phi}_n^{\pm}\ \ (\lambda,r) = \hat{\phi}_n(\pm\sqrt{\lambda^2+r^2},r)$. From (10) we have

$$r^{-1}|\bar{\phi}_n^{+}(\lambda,r) - \bar{\phi}_n^{-}(\lambda,r)| \leqslant (\lambda^2 + r^2 + 1)^{-N}\ B^{2n}(c_{2n})^{-2n} \tag{26}$$

for a suitable positive integer N,

so

$$|\kappa^{2n}\ \Delta_\phi(\kappa^2)| \leqslant\ \ (\text{constant})\ B^{2n}(c_{2n})^{-2n} \tag{27}$$

The derivative $\kappa^{2n}\ \Delta_\phi^{(q)}\ (\kappa^2)$ can be handled exactly as above, with

$$\kappa^{2n}\,\Delta_\phi^{(q)}(\kappa^2) = \int_0^\infty d\lambda\,\lambda^{q+1}\int_0^\infty r^2 dr(\lambda^2 + r^2)^{-\frac{1}{2}}\,(\partial/\partial r)\,r^{-1}\,[\bar{\phi}_n^{-+}(\lambda,r) -$$

$$\bar{\phi}_n^{--}(\lambda,r)]\,J_0^{(q)}\,(\lambda\kappa)$$

The bound (26) with $N \geq q + 2$ will thus prove

$$|\kappa^{2n}\,\Delta_\phi^{(q)}\,(\kappa^2)| \leq C_q\,B^{2n}(c_{2n})^{-2n}$$

Thus $\Delta_\phi(\kappa^2) \in S_{B^{-1}g}\,((0,\infty))$

We may thus extend $[A(x), A(0)]_-$ as a generalised function-valued operator to the space $S_x^g$ if $\rho \in S_g'$ (since $\bigcap_B S_{B^{-1}g} = S_g'$). We may specify the increase of $\rho(\kappa^2)$ in this case straightforwardly from (27), since then

$$|\Delta_\phi(\kappa^2) \leq \sup_{n\geq 0} C_0\,(B^{2n}(c_{2n})^{-2n}/\kappa^{2n}) = C_0\,\bar{G}(\kappa^2/B) \tag{28}$$

so the increase of $\rho$ for larger $\kappa^2$ must be slower than $\bar{G}(\kappa^2)^{-1}$. In the case of $c_n = n^{-\alpha}$ we obtain the familiar value $\bar{G}(\kappa^2)^{-1} = e^{\kappa^{1/2\alpha}}$.
We have thus proved

Theorem 1

The generalised free field (19) has an extension whose order of decrease outside the light cone is specified by the smallest possible $g$ for which $\rho \in S_g'$. The bound on $\rho$ for large $\kappa^2$ is essentially

$\bar{G}(\kappa^2)^{-1}$, where $\bar{G}(\kappa^2)$ is given by (28).

Let us now turn to another simple example of a non-localisable field, for the case

$$B(x) = g(\Box^2) A(x) \tag{29}$$

where $A(x)$ is a tempered field (though A could be localisable) and g is a suitable function. To consider this case we note that the Wightman functions for B are trivially related to those for A

$$W_n^{(B)}(x_1,\ldots,x_n) = \prod_{i=1}^{n} g(\Box_i^2)\, W_n^{(A)}(x_1,\ldots,x_n)$$

let us consider specifically the two point function

$$W_2^{(B)}(x_1,x_2) = g^2(\Box^2_{x_1-x_2})\; W_2^A(x_1,x_2)$$

We may write $W_2^A$ by means of a Lehmann representation

$$W_2^{(A)}(x) = \int_{m_o^2}^{\infty} d\kappa^2\, \rho^{(A)}(\kappa^2)\, \Delta^{(+)}(x,\kappa^2) \tag{30}$$

so that $W_2^{(B)}$ also has such a representation, though now with weight function $\rho^{(B)}(\kappa^2) = g^2(\kappa^2)\,\rho^{(A)}(\kappa^2)$. Thus if g is an entire function with exponential growth of order $({}^1/_{2\alpha})$ and $\rho^{(A)}$ is a measure then $\rho^{(B)}(\kappa^2) \in S_\alpha{}'$. By the discussion for the generalised free field the commutator

$$C_2^{(A)}(x) = \int_{m_o^2}^{\infty} d\kappa^2\, \rho^{(A)}(\kappa^2)\, \Delta(x,\kappa^2)$$

can be extended to some space $S^{\alpha}_{x}$. For the general commutator

$$C^{(B)}_{n,j} \quad (x_1,\ldots,x_n) = \prod_{i=1}^{n} \quad g(\Box^2{}_{x_i}) \; C^{(A)}_{n,j} \quad (x_1,\ldots,x_n) \tag{31}$$

the tempered distribution $C^{(A)}_{nj}$ vanishes outside the light cone in the variable $\xi_j = x_j,\ldots,x_{j+1}$. The generalised function $g(\Box^2_{\xi_j})\, C^{(A)}_{nj}\ (\xi_j)$ will have an extension outside the light cone which can be determined immediately, since

$$< \prod_{i=1}^{n} g(\Box^2_i)\; C_{nj},\phi> = <C_{nj}, \prod_{i=1}^{n} \quad g(\Box^2_i)\phi> \tag{32}$$

The details of the extension are given by

Lemma 6

If $g$ is an entire function order of growth $^1/_{2\alpha}$ then $\prod_{i=1}^{n} g(\Box^2_i)\phi \ \varepsilon\ S(\underline{R}^{4(n-1)})$ for any $\phi\ \varepsilon\ S^{\beta}(\underline{R}^{4(n-1)})$, any $\beta < \alpha$, whilst if $\phi\ \varepsilon\ T_{nj}(\beta)$ then $\prod_{i=1}^{n} \ g(\Box^2_i)\phi\ \varepsilon\ S^{x}_{nj}(\underline{R}^{4(n-1)})$ for any $\beta < \alpha$. Here $S^{x}_{nj}\ (R^{4(n-1)}) =$

$\bigcup_{\mu>0} \ S(\underline{R}^{4(n-2)}) \otimes S(\Omega_{\mu})$ where $S(\Omega_{\mu})$ is in the variable $\xi_j$. The first part of this lemma follows immediately by Fourier transformation, whilst both it and the second part can be derived by direct computation by using the defining properties of the spaces $S^{\alpha}$ and $S$ . Since the support of $C_{nj}$ is the interior of the light cone in $\xi_j$ then the right hand side of (32) is defined for $\phi\ \varepsilon\ T_{nj}(\beta)$ for any $\beta < \alpha$. We have thus proved

Theorem 2

The massive field (29) has an extension of order of decrease $\alpha$.

It is evidently possible to obtain further results on this extension, such as its range or its order, in the case of $S^g$ spaces, but we will not do that here, because the results are not of essential interest. This is especially so for both of these examples in that the first, the generalised free field, has the trivial unit S-matrix whilst the example (29) has the same S-matrix elements as the field $A(x)$, as can easily be seen in momentum space. We leave these cases, then, and turn to functions of the free massive field. These also have trivial S-matrix, but have a great deal more complexity.

## §5. Functions of the Free Massive Field

As before we consider functions of the form

$$B(x) = \sum_{n\geq o} \frac{d_n}{n!} \; : A(x)^n : \tag{33}$$

where $A(x)$ is a free scalar field of mass $m$. We will only investigate the two point function here for simplicity, especially because this situation is already quite complicated. We have, as for the massless field, and taking (14), that

$$\langle C_2, \phi \rangle = \int_o^\infty dt \; e^{-t} \lim_{\varepsilon\to o} \int_{C_\varepsilon} dx_o \int d^3x \; e^{t^\alpha \Delta(x,m^2)} \; \phi(x) \tag{34}$$

where $A(x)$ is a free scalar field of mass m. We will only investigate the two point function here for simplicity, especially because this situation is already quite complicated. We have, as for the massless field, and taking (14), that

$$\langle C_2, \phi \rangle = \int_0^\infty dt\, e^{-t} \lim_{\varepsilon \to 0} \int_{C_\varepsilon} dx_0 \int d^3x\, e^{t^\alpha \Delta(x,m^2)} \phi(x) \tag{34}$$

We have that $\Delta(x,m^2) = m(-x^2)^{-\frac{1}{2}} K_1(m(-x^2)^{\frac{1}{2}})$, so that for $|\underline{r}| \neq 0$ the contour $C_\varepsilon$ is to be taken as in fig. 3. We can express the most


fig. 3.

singular part of $\Delta$ as

$$\Delta(x,m^2) = -\frac{1}{x^2} - \frac{m}{2} \log\left(\frac{1}{2} m \sqrt{-x^2}\right) + \log(-x^2)\, O(x^2)$$

Thus

$$e^{t^\alpha \Delta(x,m^2)} = e^{-\frac{t^\alpha}{x^2}} \left(t^2 \sqrt{-x^2}\right)^{-\frac{m}{2}} e^{t^\alpha \log(-x^2)\, O(x^2)} \tag{35}$$

The integration contour $C_\varepsilon$ may be modified so as to include two small circles, one round $|\underset{\sim}{r}|$, the other round $-|\underset{\sim}{r}|$, together with the remainder, so giving the contours of fig. 4. The contribution from the circles may be evaluated as for the massless case in section 3 by expansion of the first factor in (35). Except for rather special values of m this will give

fig. 4

no contribution at all, neglecting the third factor on the r.h.s. of (25) very near x = 0. So the main contribution to (34) is completely different in the massive case from the massless situation; only the contribution from the contours outside $\pm |\underline{r}|$ in fig. 4 are to be considered.

To obtain the extension of (34) we expand the exponential in (34) and use an integral representation(11) for powers of $\Delta(x,m^2)$:

$$\langle C_2,\phi\rangle = 2 \int_0^\infty dt\ e^{-t} \int d^4x \quad \phi(x) \sum_{n\geq 0} \frac{t^{n\alpha}}{n!} \int_{(nm)^2}^\infty d\kappa^2 \Delta(x,\kappa^2)\ \Omega_n^{(m)}\ (\kappa^2) \tag{36}$$

where $\Omega_n^{(m)}\ (\kappa^2)$ is the phase space for n particles of mass m and total squared centre of mass energy $\kappa^2$. We use the result of lemma 5, so that if $\phi \in S_x^\beta$, with $\beta < 1$, then $\Delta_\phi(\kappa^2) \in S_{B\beta}(R_+)$, for some B. Using the bound(11)

$$|\Omega_n^{(m)}(\kappa^2)| \leq (\text{constant}) \cdot \frac{\kappa^{\frac{n-3}{2}}(\kappa-nm)^{\frac{3n-s}{2}}}{\Gamma(2n)} \tag{37}$$

to within a function of slow increase in n and $\kappa$, which we can neglect without error,

$$|\langle C_2,\phi\rangle| \leq (\text{constant}) \times \int_0^\infty dt\, e^{-t} \sum_{n\geq o} \frac{t^{n\alpha}}{n!} \int_{(nm)^2}^\infty d\kappa^2 \times \tag{38}$$

$$\times \quad \kappa^{\frac{n-3}{2}} (\kappa-nm)^{\frac{3n-s}{2}} e^{-(B\kappa)^{1/\beta}} \,/\, [\Gamma(2n)]$$

We may evaluate the $\kappa^2$-integral on the r.h.s. of (38) by the change of variable $\kappa$ = nm x to give

$$(nm)^{2n-2} \int_1^\infty dx\, x^{\frac{n-1}{2}} (x-1)^{\frac{3n-s}{2}} e^{-(bn\, mx)^{1/\beta}} \tag{39}$$

Denoting by $g(x)\, x^{-3/2}$ the integrand of (39) we may put a bound on (39) by finding the positive of the maximum of g, which is at the solution of

$$g'(x) = \left[\frac{n+2}{2x} + \frac{(3n-s)}{2x-1} - \frac{(bnm)^{1/\beta}}{\beta} x^{\frac{1}{\beta}-1}\right] g(x) = 0$$

The solution of this for large n is very close to x = 1. (in the range $1 \leq x \leq \infty$), and has value $x = 1 + \varepsilon$, $\varepsilon \sim (3n-s)\beta/2(bnm)^{1/\beta}$, which is as small as we please for n large enough. Then (39) is bounded for all n by

$$(nm)^{2n-2} (1+\varepsilon)^{\frac{n+2}{2}} \varepsilon^{\frac{3n-s}{2}} e^{-(bm)^{1/\beta}[n(1+\varepsilon)]^{1/\beta}} \int_1^\infty dx/x^{3/2}$$

with crucial contribution proportional to

$$n^{2n} \left[(3n-s)\beta/2\right]^{\frac{3n-s}{2}} n^{-(\frac{3n-s}{2})/\beta} e^{-(bmn)^{1/\beta}}$$

or $$n^{\frac{3n}{2}(1-\frac{1}{\beta})+2n} e^{-(bm)^{1/\beta} n^{1/\beta}}$$

Thus we have the bound on (38) given by

$$|\langle C_2,\phi\rangle| \leq \int_0^\infty dt\, e^{-t} \sum_{n\geq 0} \frac{t^{n\alpha}}{n!} n^{\frac{3n}{2}(1-\frac{1}{\beta})} e^{-(bmn)^{1/\beta}} \tag{40}$$

and this is finite if $\beta < 1$, since then the summation in (40) gives a function increasing at infinity slower than $\exp(t)$. Thus we need to choose any $\beta < 1$ for the extension of $\langle C_2,\phi\rangle$ from $S^\alpha$ to $S^\beta_x$ to be possible. We have thus proved

Theorem 3

The two point commutator bracket of the function (33) of the massive free field has an extension of order $\beta$ for any $\beta < 1$, where $\alpha$ is defined by (14) and $\alpha > 1$.

We note that the massless case does not give the same limitation if the above method is used here, but only the condition $\alpha < 1$; this approach only works in that case for a localisable theory. We see that the above method could be extended to the indicatrix spaces, and also to higher point functions, though we will not do the latter of those here since no further insight into the situation is expected to be gained. <u>However, we can sharpen the results of theorem 3</u> so as to relate to the discussion of Rieckers[12]. If we assume, with Rieckers, that for large n,

$$\frac{d_n^2}{n!} \sim e^{n^{\frac{1}{\alpha}}} \tag{41}$$

then we have to replace the expression (36) by

$$\langle C_2, \phi \rangle = \sum_{n \geq o} \frac{d_n^2}{n!} \int_{(nm)^2}^{\infty} \Delta_\phi\ (\kappa^2)\ \Omega_n^{(m)}\ (\kappa^2)\ d\kappa^2$$

so

$$|\langle C_2, \phi \rangle| \leq \sum_n e^{an^{1/\alpha}} \int_{(nm)^2}^{\infty} d\kappa^2\ \frac{\kappa^{\frac{n-3}{2}} (\kappa - nm)^{\frac{3n-s}{2}}}{\Gamma(2n)}\ e^{-(b\kappa)^{1/\beta}}$$

Using the previous method we obtain

$$|\langle C_2, \phi \rangle| \leq \sum_n e^{an^{1/\alpha} - (bmn)^{1/\beta}}\ n^{\frac{n}{2}(1-\frac{3}{\beta})} \tag{42}$$

This is convergent for any $\beta < \alpha < 1$ (agreeing with Rieckers[12] results for the Wightman Functions) so proving

<u>Theorem 4</u>

The two point commutator bracket of the function (33) with coefficients

$d_n$ satisfying (41) has a decrease outside the light cone of order $\alpha$, and is damped like $\exp(-|x^2|^{1/2\alpha})$ as $x^2 \to -\infty$.

## §6. Discussion

We have obtained a prescription for describing how the commutator bracket of a non-localisable field extends outside the light cone, and shown that it is applicable to various models. There are two difficulties associated with this. The first is that we have not been able to show that our prescription does actually work for the case of functions of a massless free field. This is rather surprising since we expect that case to be simpler than the massive one. This problem is associated with that of defining $S^{(n)}(x^2)$ for $n > 2$, and of course related to the fact that in the massless case all the higher particle thresholds coalesce onto the single particle one. We do not at present see any way of satisfactorily treating this question, though feel it rather pressing especially because of all the work involved in applications of non-polynomial lagrangians which use the massless case[2].

The second difficulty is that we have not been able to present a realistic model of a truly non-localisable field, that is, one for which the S-matrix is not the same as that arising from some localisable one. Only if that can be done can we expect that there is any possible physical trace of non-causality. Indeed we have discussed[13] recently the manner in which the notion of Borcher's equivalence classes of fields[14] can be extended to include non-localisable ones. What is needed is a proof that any non-localisable field is equivalent in this extended sense to some localisable field. This can, indeed, be done if indefinite metric localisable fields are allowed, but it is not known if such a theorem is true for positive metric fields.

Even if it is not possible to say whether or not non-localisable fields actually appear as such in nature can we say anything about the expected sizes of the range and order of the extension, if it exists? A natural range would be that determined by the radius of an interaction. There is far greater difficulty about a natural value for the order of the extension, it being a dimensionless quantity. However, the dimensionless quantities of interest are the coupling constants of the various interactions. But present evidence indicates that the order of the non-localisability depends heavily on the nature of the interaction; it may well be that among all equivalent interactions the least order of decrease is determined by the dimensionless strength.

We must realise, of course, that non-causality need not destroy many of the usual results which follow from causality, such as analyticity[14] and even polynomial boundedness of S-matrix elements may still be valid. There may be observable effects of non-polynomiality in the behaviour of form factors, as discussed in the localisable case by Jaffe[15]. We hope to discuss this and related questions in more detail elsewhere.

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