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THE POSITIVITY CONDITION IN MOMENTUM SPACE

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A B S T R A C T

A formulation of the positivity condition within the framework of the general field theory is given in momentum space. It is shown how the usual requirements of locality and spectrum can be partially incorporated in order to represent the "absorptive parts" of the Green's functions by positive operators of the Hilbert-Schmidt type operating on a suitably defined Hilbert space of analytic functions introduced into mathematics by Bergman and Bochner. The application of this method to the x space positivity condition formulated by Wightman is not discussed in this paper. As an illustration, two simple examples are discussed in the last Section.

1 . INTRODUCTION .

As it is well known, the study of a field theory either in the formulation of Wightman [1] or that of Haag-Araki [2] can be reduced to the study of the set of functions

$$(\Omega\ ,\ A(x_{\pi 1})\ A(x_{\pi 2})\ \ldots\ A(x_{\pi n})\Omega)\ ,\ n = 1,\ 2,\ 3,\ \ldots \qquad (1.1)$$

where $A(x_i)$ is the field operator attached to the space-time point x_i , π is any permutation of the indices, and Ω is the vaccum state [*] . All the physical properties imposed on the system can be translated into functional properties of the set of Wightman functions (1.1) . Thus, invariance under space-time translations, spectral condition and local commutativity entail analyticity of the functions in a certain "primitive" domain of the complexified variables $x_i - x_j$. The search for a representation that would embody automatically these three properties constitutes the so-called "linear program". The "positivity condition" which has to be added to these linear properties, expresses the fact that the set (1.1) is a positive functional on the algebra of field operators, i.e., that these functions are matrix elements of operators in a Hilbert space with positive metric. This condition, which interconnects different Wightman functions, was first explicitly stated and investigated by Wightman himself.

In their approach to field theory, Bogoliubov and his co-workers [3] lay, however, stress on the S matrix as the fundamental physical quantity - the fields appear rather as a reponse of particles to external perturbations :

$$(\Box - m^2)\ A(x) \equiv j(x) = S^* \frac{1}{i}\ \frac{\delta}{\delta A_{in}(x)}\ S \qquad (1.2)$$

and the locality condition as a causal propagation of these perturbations :

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[*] Only the case of a single neutral scalar field will be discussed in this paper.

$$\frac{1}{i}\ \frac{8}{8A_{in}(x_1)} \cdots \frac{1}{i}\ \frac{8}{8A_{in}(x_{n-1})}\ A(x_n) \equiv R(x_1, \ldots, x_{n-1} ; x_n) = 0 \qquad (1.3)$$

unless $x_n - x_i \in \overline{V}_+$, $i = 1, 2, \ldots, n-1$

where V_+ denotes the future light-cone. The study of the vacuum expectation values of "Wightman products" $A(x_1) \ldots A(x_n)$ can be replaced by that of retarded functions :

$$(\Omega\ ,\ R_n^i(x)\Omega) = r_n^i(x),\ R_n^i(x) = R(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n ; x_i) \ (1.4)$$

which have now analiticity properties in momentum space in view of the support properties (1.3) in x space. The positivity condition is replaced by a stronger one which is a generalization of the unitary condition $S^*S = 1$ and which includes also the L.S.Z. reduction formulae [4] and the assumption of the completeness of the asymptotic states. From the set of functions (1.4) satisfying these conditions one also can reconstruct the field (1.2) having all the resuired properties [5] . This approach, although less general, has the advantage of being more closely connected to the physically observalbe scattering amplitudes.

The purpose of this article is to formulate the positivity condition in the second framework and to study its connection with analyticity properties of the off and on mass-shell scattering amplitudes in momentum space. The incentive for this investigation were the papers by Martin [6] in which he showed the interplay of analyticity and positivity leads to an enlargement of the domain of analyticity of the four-point scattering amplitude in the framework of general field theory. As the reader will notice, the positivity condition in momentum space takes a slightly different form than the corresponding one in x space and resembles very much to the theory of Bergman [7] kernels for Hilbert spaces of functions analytic in a given domain in $\mathbb{C}_N$ [8] . The method of Hilbert spaces of analytic functions developed here can be also applied to the set of Wightman functions in x space, but that case will not be considered in this paper.

Next Section will very briefly remind the reader of some properties of the generalized retarded functions and state the positivity condition. Section 3 will discuss the "continuation" of the positivity condition into the complex domain in several different forms. It contains the main substance of this paper. In the final Section we will try to illustrate the usefulness of the general concepts by rederiving a very well-known positivity property of the absorptive part of the scattering amplitude used repeatedly by Martin in [6].

2. THE POSITIVITY CONDITION.

The retarded functions (1.4) have the following formal expression [5),9)] in terms of the fields A

$$r(x_2, \ldots, x_n ; x_1) = (\Box_1 - m^2) \ldots (\Box_n - m^2) \sum_{\pi} \theta(x_1^o - x_{\pi 2}^o) \ldots \tag{2.1}$$

$$\theta(x^o_{\pi(n-1)} - x^o_{\pi n})(\Omega ,[\ldots[A(x_1), A(x_2)] \ldots A(x_{\pi n})] \Omega)$$

where $\Box_i - m_i^2$ is the Klein-Gordon operator referring to the variables $x_i = (x_i^o , \vec{x}_2)$, $\theta(t)$ is the usual step function and the square brackets indicate commutators. It is still unknown whether starting from the Wightman axioms, these formal expressions for $n > 3$ have a se se as tempered distributions satisfying the same algebraic relations, support properties, etc., as they would do if the Wightman functions were genuine functions [*)].

In the L.S.Z. formalism they are assumed to do so. However, by replacing in (1.1) the "sharp" fields $A(x)$ by smeared out fields $A_f(x) = (A * f)(x)$, $f \in \mathcal{D}(\mathbb{R}_4)$ (infinitely differentiable functions having compact support) or even by local observables attached to finite space-time regions as proposed by Haag and Araki [12)], the just mentioned difficulty can be avoided. Moreover, the reduction formulae can be then deduced in a rigorous way from the Wightman

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*) This problem is a generalization of the problem of renormalization in perturbation theory, and was studied especially by Steinmann [10)]. The problem for $n = 2$ is trivial, the case $n = 3$ was solved by Stora [11)].

or Haag-Araki axioms, as shown by Hepp [12] *). Also the analiticity properties remain unaltered by this "smearing out" since the support of (1.3) gets only shifted by a finite amount. We will, therefore, place ourselves in what follows indifferently in any of the shemes just described.

Although, as already mentioned, the set of retarded functions (1.4) [or the set of advanced functions obtained by substituting the step function $\theta_-(t) = \theta(-t)$ for $\theta(t)$ in the expression (2.1)] is rich enough to reconstruct the field operators themselves by making use also of the uniterity relations, it was recognized by Steinmann [14] that in order to exploit completely the linear properties of the n point function a more general set of "retarded" products had to be introduced. In the Bogoliubov formalism they can be described as generated from the field $A(x)$ by taking functional derivatives with respect to the A_{in} and A_{out} fields in all possible combinations and permutations [compare (1.3)]. They can also be defined as a linear combination of multiplied by suitable step functions [compare (1.1)] **). For our purposes the following qualitative remarks will do.

Let us denote by $R_n^i(x)$ the generalized retarded product of n fields, and by $r_n^i(x)$ the corresponding vacuum expectation values. The functions (distributions) r_n^i depend only on the differences $x_i - x_j$ of their arguments and have supports in certain cones C_n^i. Consequently, their Fourier transforms $\tilde{r}_n^i(p)$ defined by

$$r_n^i(x) = (2\pi)^{-n} \int \delta_4(p_1 + \dots + p_n)\, \tilde{\pi}_n^i(p)\, e^{-ipx}\, dp$$

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*) As in Ruelle's [13] proof of the existence of the S matrix, one has to suppose here that the energy has a finite gap above the vacuum state and that the asymptotic states are complete in the underlying Hilbert space.

**) A detailed study of the different and somewhat involved properties of the generalized retarded functions, which in wome way reflect the rather complicated kinematics of the n body problem, was done also by Ruelle [15], Araki [16], Araki and Burgoyne [17] and Bros [18]. For a recommended review article, see [19].

where

$$px = p_1 x_1 + \dots + p_n x_n \quad , \quad dp = d^4p_1 \dots d^4p_n$$

are boundary values (in the sense of distributions) of functions $\tilde{r}^i_n(k)$, $k = p + iq$ analytic in the tubes $\mathcal{T}^i_n = \{k : \mathrm{Im}\, k \equiv q \in \tilde{C}^i_n\}$ ($\tilde{C}^i_n$ is the dual cone of the cone C^i_n). This analyticity is an expression of the locality of the theory. The spectral condition finds its expression in the coincidence of all the boundary values $\tilde{r}^i_n(p) = \tilde{r}^j_n(p)$ (n fixed !) in a certain region $p \in R^s_n$ of the real momentum space $\mathbb{R}_{4(n-i)}$ depending on the masses of the particles described by the theory. From the edge of the wedge theorem [3),20)] , it follows then that all the functions $\tilde{r}^i_n(p)$ are different boundary values of one and the same function $\tilde{r}_n(k)$ analytic in the envelope of holomorphy of the domain $\bigcup_i \mathcal{T}_n \cup [R^s_n]$, where $[R^s_n]$ is a complex neighbourhood of the region of coincidence R^s_n . The S matrix elements involving n particles are restrictions of a well chosen function $\tilde{r}^i_n(p)$ to the mass shell manifold $p^2_j = m^2$ $(j = 1, 2, \dots, n)$, m being the mass of the particles involved *) . The analyticity domain of the scattering amplitude is the intersection of the complex mass shell manifold $k^2_j = m^2$ with the domain of holomorphy of the function $\tilde{r}_n(k)$ (one has proved so far only the case $n = 4$ that this intersection is non-empty). We mention also that for each (real) momentum p there is a function $\tilde{r}^i_n(p)$ which coincides in a neighbourhood of that point with the Fourier transform of the truncated vacuum expectation value of the time-ordered product T_n . As a final remark, the generalized retarded functions satisfy some linear identities called the Steinmann relations. We shall mention them later when needed.

After these preliminaries, we are ready to formulate the positivity condition. By denoting

$$R^i_n(f) = \int R^i_n(x_1, \dots, x_n)\, f^i_n(x_1, \dots, x_n)\, d^4x_1 \dots d^4x_n \quad , \qquad (2.2)$$

where f^i_n is a test function $\in \mathcal{S}(\mathbb{R}_{4n})$, (i.e., infinitely differentiable and of fast descrease at infinity), the positivity condition reads :

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*) The restriction of a distribution to a manifold makes in general no sense. It was proved by Hepp in the quoted article [12)] that it does in this case.

$$\| \sum_{i,n} R_n^i(f)\, \Omega \|^2 = \sum_{i,n_{ij},m} (\Omega\, ,\, R_n^{i*}(f)\, R_m^d(f)\, \Omega) \geq 0 \qquad (2.3)$$

for every choice of the f_n^i . We suppose here that the sum extends only over a finite number of terms. This condition was first written and studied by Wightman [1] in the case of products of fields. To avoid notational complications we shall study it in the special case of one term instead of the sum. What we will have to say will extend in a straightforward way to the general case (see end of Section 3) .

In this simpler case, (2.3) becomes (we drop the indices) :

$$\int \widetilde{A}(x_1, \ldots, x_{2n})\, \bar{f}(x_1, \ldots, x_n)\, f(x_{n+1}, \ldots, x_{2n})\, dx_1 \ldots dx_{2n} \geq 0$$

for all $f \in \mathcal{S}(\mathbb{R}_{4n})$ (2.4)

with $\widetilde{A}(x_1, \ldots, x_{2n}) = (\Omega\, ,\, R^*(x_1, \ldots, x_n)\, R(x_{n+1}, \ldots, x_{2n})\, \Omega)$.

By going to momentum space, we get

$$\int \delta_4(k_1 + \ldots + k_{2n}) A(k_1, \ldots, k_{2n}) \overline{\widetilde{f}}(-k_1, \ldots, -k_n) \widetilde{f}(k_{n+1}, \ldots, k_{2n}) dk_1 \ldots dk_{2n} \qquad (2.5)$$

where we have introduced the Fourier transforms as follows

$$f(x) = \int e^{-ik_1x_1 - \ldots - ik_nx_n}\, \widetilde{f}(k)\, dk_1 \ldots dk_n$$

$$\widetilde{A}(x) = (2\pi)^{-2n} \int \delta_k(k_1 + \ldots + k_{2n})\, A(k_1, \ldots, k_{2n})\, e^{-ik_1x_1 - \ldots - ik_{2n}x_{2n}}\, dk_1 \ldots dk_{2n} \quad .$$

Let us now introduce the total energy and momentum co-ordinates σ , σ' and the relative momenta p_ν , q_ν of the two "clusters" R^* and R by the formula

$$k_\nu = -\frac{\sigma}{n} - p_\nu\ , \ (\nu = 1, \ldots, n)\ , \quad \sigma = -\sum_1^n k_\nu\ , \quad \text{with } \sum_1^\nu p_\nu = 0$$

$$k_{n+\nu} = \frac{\sigma'}{n} + q_\nu\ , \ (\nu = 1, \ldots, n)\ , \quad \sigma' = \sum_1^n k_{n+\nu}\ , \quad \text{with } \sum_1^\nu q_\nu = 0 \qquad (2.6)$$

Then (2.5) takes the following form :

$$\int A_\sigma(p,q)\, \bar{\varphi}(\sigma\, ,\, p)\, \varphi(\sigma\, ,\, q)\, dp\, dq \geq 0 \tag{2.7}$$

where $A_\sigma(p\, ,\, q) = A(-\frac{\sigma}{n} - p_1, \dots, -\frac{\sigma}{n} - p_n\, ,\, \frac{\sigma}{n} + q_1, \dots, \frac{\sigma}{n} + q_n)$,

$$\varphi(\sigma\, ,\, p) = \tilde{f}(\frac{\sigma}{n} + p_1, \dots, \frac{\sigma}{n} + p_n)$$

and $dp = dp_1 \dots dp_{n-1}$, $dq = dq_1 \dots dq_{n-1}$.

Since the n vectors p_ν and the n vectors q_ν satisfy the two relations (2.6) , we have (arbitrarily) chosen the first $n-1$ as linearly independent.

The spectral condition tells us that the function A_σ (tempered distribution more precisely) has its support in $\sigma \in \bar{V}_+(M)$ where $\bar{V}_+(M) = \{\sigma : \sigma_0 \geq 0\, ,\, \sigma^2 \geq M^2\}$, and M is the lowest mass of the intermediary states that can be inserted between R^* and R in (2.4) . Since the Fourier transform of $(\Omega\, ,\, R\, R^*\, \Omega)$ expressed in the variables (2.6) has, as immediately seen, its support in $\{\sigma \in \bar{V}_-(M) = -\bar{V}_+(M)\}$, $A_\sigma(p\, ,\, q)$ can also be regarded as the Fourier transform of the commutator

$$A_\sigma(p\, ,\, q) = \mathcal{F}(\Omega\, ,\, [R^*\, ,\, R]\, \Omega) \qquad \text{for} \quad \sigma \in V_+ \tag{2.8}$$.

The function $(\Omega,\, R^*(x)\, R(x')\Omega)$ is only partially a retarded function : it has support properties only separately in the variables x and x' but none in $x-x'$. Therefore $A_\sigma(p\, ,\, q)$ is the boundary value of a function analytic in p and q but not in σ . More precisely if we consider $A_\sigma(p\, ,\, q)$ as a member of the family of functions

$$A_\sigma^{ij} = \mathcal{F}(\Omega\, ,\, R_n^i\; R_n^j\, \Omega)$$

where i and j run independently over all the generalized retarded n point products, it turns out, on the basis of the spectral condition and the edge of the wedge theorem, that the functions $A_\sigma^{ij}(p\ ,\ q)$ are for fixed σ (this term will have to be specified later) different boundary values of one and the same function [which, for simplicity we will again denote by $A_\sigma(p\ ,\ q)$] analytic in a domain $\{(p\ ,\ q) \in \Omega_\sigma \times \Omega_\sigma\}$. Here Ω_σ is the envelope of holomorphy of the n point function in which the region of coincidence $R_n^C = \{p : \sum_{i\in I} p_i)^2 < M_I^2$ for all $I \in (1, 2, \ldots, n)\}$ has been replaced by the "shifted" region $R_n^C(\sigma) = \{p : \sum_{i\in I} (p_i + \frac{\sigma}{n})^2 < M_I^2$, for all $I \in (1, 2, \ldots, n)\}$. Note that $\Omega_\sigma \times \Omega_\sigma$ is a <u>topological product</u> of twice the same region. The reader will immediatly notice that for $n = 2$, $(A_\sigma(p\ ,\ q)$ reduces to the well-known absorptive part of the four-point function studied first by Bogoliubov and co-workers in connection with the proof of dispersion relations [3] and then by Lehmann [21] , who found also its envelope of holomorphy Ω_σ . The above statements are simple generalizations *) .

On the other hand, the algebra of the generalized retarded functions tells us that the commutator $(\Omega\ ,\ [R_n^i\ ,\ R_n^j\ ]\Omega)$ can be expressed as the difference of (in general different) pairs of fully retarded functions :

$$(\Omega\ ,\ [R_n^i\ ,\ R_n^j\ ]\Omega) = r_{2n}^\alpha - r_{2n}^\beta \qquad . \tag{2.9}$$

Translated into momentum space this tells us, in view of (2.8) , that the "absorptive part" A_σ is the difference of two different boundary values of the full 2n point function $\tilde{r}_{2n}(k)$. This fact adds, in principle, new information for the function A ; it can in particular result in an enlargement of the domain $\Omega_\sigma \times \Omega_\sigma$ described above. Indeed, that is what happens in the case of the four-point function as shown by Martin [6] .

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*) As a matter of fact, a rigorous proof of these statements requires some gymnastics with distribution theory and analytic completion.

The above remarks about the analyticity properties of A being needed in the next Section, let us turn back to the inequality (2.7). If we choose the test function there to be of the form $\varphi(\sigma, p) = \chi(\sigma) f(p)$, with $\chi \in S(\mathbb{R}_4)$ and $f \in S(\mathbb{R}_4)$, that condition takes the form :

$$\int d\sigma \, |\chi(\sigma)|^2 \int A_\sigma(p, q) \, \overline{f}(p) \, f(q) \, dp \, dq \geq 0 \qquad (2.10)$$

By a well-known theorem of Schwartz [22], the positivity of (2.10) implies that A_σ is a positive measure with respect to σ. The term "σ fixed" will therefore have to be understood in the sense of a convolution with a positive test function $|\chi(\sigma)|^2$ having its support centered sufficiently closely to the desired value. It can actually be shown that the regularization with respect to only a timelike direction will suffice. Also the symbol Ω_σ will have to be understood as

$$" \Omega_\sigma " = \bigcap_{\sigma' \in \mathcal{U}} \Omega_{\sigma'} \quad , \quad \mathcal{U} = \operatorname{supp} |\chi|^2 \quad .$$

In what follows we will simply suppress the integration over σ in (2.10).

3. EXTENSION OF THE POSITIVITY CONDITION INTO THE COMPLEX DOMAIN.

The aim of this Section is to "extend" the condition

$$\int A(p, q) \, \overline{f}(p) \, f(q) \, dp \, dq \geq 0 \qquad (3.1)$$

for all $\qquad f \in \mathcal{S}(\mathbb{R}_N) \quad , \quad N = 4(n-1)$

into the whole domain of analyticity of the function $A(p, q)$ (σ being fixed once and for all we shall drop it in the notation). As it was discussed in the preceding paragraph, A is certainly analytic in a domain of the type $\Omega \times \Omega$. We shall need also the following two simple properties of the domain

Ω : (a)Ω is invariant under complex conjugation $\Omega = \Omega^*$, and (b)Ω contains real points of analyticity. Here Ω^* denote the set $\Omega^* = \{p : \bar{p} \in \Omega\}$ where the bar indicates complex conjugation : if $p = \mathrm{Re}\, p + i\, \mathrm{Im}\, p$, then $\bar{p} = \mathrm{Re}\, p - i\, \mathrm{Im}\, p$ *) .

The property (a) is certainly true for the primitive domain of the n point function and therefore also for its envelope of holomorphy Ω_σ . The property (b) follows from the fact that the region of coincidence $R_n^s(\sigma)$ is non-empty : by the edge of the wedge theorem all its points are (real) points of analyticity.

We are now ready to formulate the

<u>THEOREM 1</u> .

If the function $A(p , q)$ is analytic in a schlicht domain $\Omega \times \Omega^* \subset \mathbb{C}_N \times \mathbb{C}_N$ sucht that Ω contains a <u>real</u> open set $R \in \mathbb{R}_N$ and if on $R \times R \subset \mathbb{R}_N \times \mathbb{R}_N$ $A(p , q)$ satisfies the positivity condition (3.1) for every test function $f \in \mathcal{D}(R)$, then it satisfies also the three **following** conditions :

A) - $\displaystyle\int_{\omega\times\omega} A(p , \bar{q})\, \bar{g}(p)\, g(q)\, d\lambda_p\, d\lambda_q \geq 0$

for all $g \in L_2(\omega)$ and all $\omega \subset\subset \Omega$, ω being any open set in $\mathbb{C}_N$ with compact closure such that closure $\omega \subset \Omega$, and $d\lambda_p$,

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*) Our notation is somewhat inconsistent since the same letters p and q stand for real variables as in (3.1) , where $A(p , q)$ represents the boundary value of an analytic function, and for complex variables varying in $\Omega \times \Omega$, where they serve as arguments of the (unique !) analytic extension of the boundary value in question. This analytic extension is again denoted by $A(p , q)$. It is hoped that this notational simplification will not lead to confusion ; from the context it should be clear whether the variables in question are to be considered as real or complex.

resp. $d\lambda_q$, denoting the Lebesgue measure in $\mathbb{C}_N$ [i.e., $d\lambda_p = (2i)^{-N}\, d\bar{p} \wedge dp$, resp. $d\lambda_q = (2i)^{-N}\, d\bar{q} \wedge dq$] *) .

B) - There exists a sequence of functions $f_\nu(p) \in \mathcal{A}(\Omega)$, $\nu = 1, 2, 3, \ldots$, $\mathcal{A}(\Omega)$ denoting the set of functions analytic in Ω, such that

$$A(p\,,\,\bar{q}) = \sum_{\nu=1}^{\infty} f_\nu(p)\, \bar{f}_\nu(q)$$

the series being uniformly convergent in $\Omega \times \Omega$ **) .

C) - The quadratic form

$$\mathcal{A}_p(a\,,\,a) = \sum_{\alpha,\beta} \frac{a_\alpha}{\alpha!} \frac{\bar{a}_\beta}{\beta!}\, \partial^\alpha_\beta\, \partial^\beta_{\bar{p}}\, A(p\,,\,\bar{p})$$

defined on all finite sequences of complex numbers $\{a_\alpha\}$ is positive definite for all $p \in \Omega$.

All these three conditions are equivalent.

The formula C) needs some explanations. In it the notation of Schwartz [22] for multi-indices was used : $\alpha = \{\alpha_1, \alpha_2, \ldots, \alpha_n\} \subset Z^+_N$ denotes a sequence of N non-negative integers ;

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*) Note that $A(p\,,\,q)$ being analytic in $\Omega \times \Omega^*$, $A(p\,,\,q)$ is analytic in $p \in \Omega$ and anti-analytic in $q \in \Omega$.

**) A series of functions is said to be uniformly convergent in an open set U if it converges uniformly in every compact subset of U.

$$\alpha\,! = \alpha_1! \dots \alpha_N! \quad , \quad \beta\,! = \beta_1! \dots \beta_N!$$

$$\partial^\alpha_\beta = \left(\frac{\partial}{\partial p_1}\right)^{\alpha_1} \dots \left(\frac{\partial}{\partial p_N}\right)^{\alpha_N} \quad , \quad \partial^\beta_{\bar{p}} = \left(\frac{\partial}{\partial \bar{p}_1}\right)^{\beta_1} \dots \left(\frac{\partial}{\partial \bar{p}_N}\right)^{\beta_N} \quad .$$

For later use, we note also

$$|\alpha| = \alpha_1 + \dots + \alpha_N \quad , \quad p^\alpha = p_1^{\alpha_1} \dots p_N^{\alpha_N} \quad .$$

In C) , the summation extends formally over all $Z_N^+ \times Z_N^+$ but only a finite number of the a_α is supposed to be $\neq 0$: $a_\alpha = 0$ for all $|\alpha| > n$ for some n . The factorials $\alpha\,!\ \beta\,!$ were introduced for later convenience. We want to show that the condition C) involves only values of the function $\mathbf{A}(p\ ,\ q)$ on the "diagonal plane" $q = \bar{p}$, which is a linear subspace of $\mathbb{C}_{2N} = \mathbb{C}_N \times \mathbb{C}_N$ of real dimension $2N$. Let us denote it by $\mathbb{D}_{2N}$. $\mathbb{D}_{2N}$ can be parametrized either by the real and imaginary on the vector $p = x + iy$, $x = (p + \bar{p})/2$, $y = (p - \bar{p})/2i$, or formally by p and $\bar{p}$. Given an arbitrary C^∞ function f on an open set $\omega \subset \mathbb{D}_{2N}$, its value in a point p can be denoted by $f(x\ ,\ y)$ or formally by $f(p\ ,\ \bar{p})$; its derivatives $\partial^\alpha_p\, \partial^\beta_{\bar{p}}\, f$ are by definition to be computed by using the formulae :

$$\frac{\partial}{\partial p_i} = \frac{1}{2}\left(\frac{\partial}{\partial x_i} - i\,\frac{\partial}{\partial y_i}\right) , \quad \frac{\partial}{\partial p_i} = \frac{1}{2}\left(\frac{\partial}{\partial x_i} + i\,\frac{\partial}{\partial y_i}\right) \quad .$$

The function f will be said to be <u>real</u> analytic in ω if for every $p_o \in \omega$ it can be represented by its Taylor series

$$f(p\ ,\ \bar{p}) = \sum_{\alpha,\beta} \frac{(p - p_o)^\alpha}{\alpha\,!}\, \frac{(\bar{p} - \bar{p}_o)^\beta}{\beta\,!}\, \partial^\alpha_p\, \partial^\beta_{\bar{p}}\, f(p\ ,\ \bar{p}_o) \qquad (3.2)$$

absolutely converging in a sufficiently small polydisc

$$P = \{p : |(p - p_o)_i| < R_i \quad , \quad i = 1, \dots, N\} \quad .$$

If we replace $\bar{p}$ by q in the series (3.2), the series will continue to converge absolutely for $(p, q) \in P \times P^*$ defining there an analytic function $f(p, q)$ - the unique analytic continuation of $f(p, \bar{p})$. In this case the formal notation acquires a real meaning and proves our assertion.

Before proceeding to the proof we wish to make still a few remarks. It is evident that the representation B) displays positivity in the most explicit way. From it, A) and B) follow immediatly. For example, in order to get C), we have only to apply the differential operator $P(\partial_p)\,\bar{P}(\partial_q)$ to the series C) and put $q = p$, where P is given by

$$P(\partial_p) = \sum_\alpha \frac{a_\alpha}{\alpha!}\,\partial_p^\alpha \qquad .$$

The function $A(p, \bar{q})$ resembles very much to a Bergman kernel associated with a suitably defined Hilbert space of functions analytic in a domain Ω [7),8)*]. The Bergman theory proceeds, roughly speaking, in the direction A) ⇒ B) ⇒ C), while our proof of Theorem 1 will follow the direction (3.1) ⇒ C) ⇒ A) ⇒ B).

<u>Proof of C)</u>.

In order to show C), let us insert for f in formula (3.1) the expression

$$f(p) = \sum_{|\alpha|\le n} \frac{\bar{a}_\alpha}{\alpha!}\,(-)^{|\alpha|}\,\partial_p^\alpha\,\delta_N(p - p') \qquad , \qquad p' \in R \tag{3.3}$$

or, to be more precise, let us take a sequence of functions $f_\nu \in \mathfrak{D}(R)$ converging to the distribution (3.3) in the topology of $\mathfrak{D}'(R)$. Since $A(p, q)$ is C^∞ in $R \times R$, the result will be the same. Here δ_N is the N dimensional Dirac function. As p' is any fixed point in R we obtain

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*) Most of the author's knowledge about the Bergman-Bochner theory derives from a book by Meschkowski [23)], especially from Chapters IV and XII.

$$\sum_{\alpha,\beta} \frac{a_\alpha}{\alpha !} \frac{\bar{a}_\beta}{\beta !} \partial_p^\alpha \partial_q^\beta \; A(p , q) /_{p = q} \equiv Q_p(a , a) \geq 0 \tag{3.4}$$

for all $p \in R$.

In this inequality all the $a_\alpha = 0$ for $|\alpha| > n$, some n , but we are allowed to drop this condition provided the resulting series converges absolutely. That is what we will do presently. Let $p \in R$, and let D be a polydisc $D = \{z \in \mathbb{C}_N : |z_i| < R_i \;, \; i = 1, \ldots, N\}$ such that $\{p\} + D \subset \Omega$ and let $z \in D$. Then by introducing

$$a_\alpha = \sum_{|\gamma| \leq n} \frac{b_\gamma}{\gamma !} (\alpha - \gamma) ! \; z^{\alpha-\gamma} \qquad \text{for all } \alpha \in z_N^+ \;, \; b_\gamma \in \mathbb{C}_1$$

into the series (3.4) we get the inequality $Q_{p+z}(b , b) \geq 0$ if we note that the Taylor series of the function $A(p + z , p + \bar{z})$ as well as all its derivatives converge. Since any point $p \in \Omega$ can be connected to a given point $p_o \in R$ by a finite chain of polydiscs, a finite number of repetitions of the above substitution will yield us the inequality $Q_p(a , a) \geq 0$, which proves C) .

<u>Partial proof of B)</u> .

We will prove C) $\Rightarrow$ B) for Ω = a polydisc P . Let $P = \{p_o\} + D$, $p_o \in R$, D a polydisc with its centre at the origin such that $p \subset\subset \Omega$. Let us introduce, following Bergman and Bochner [7),8),23)] , the Hilbert space $\mathcal{H}_P = \mathcal{A}(P) \cap L_2(P)$ of functions <u>analytic</u> and square integrable in P with the scalar product

$$(f , g) = \int_P \bar{f}(p) \; g(p) \; d \; \lambda_p \tag{3.5}$$

and norm $\|f\|^2 = (f , f)$.

The fundamental property of this space of functions is the fact that the value of the function f at a point $p \in P$ is a continuous functional of f considered as an element of $\mathcal{H}_P$:

$$|f(p)| \leq M_p \|f\| \qquad \text{with} \qquad M_p = C \; d(p)^{-N/2} \tag{3.6}$$

where $d(p)$ is the distance of the point p to the boundary of P , C a numerical constant. From this, it follows **immediately** that the strong convergence $f_n \to f$ in $\mathcal{H}_P$ implies the pointwise convergence $f_n(p) \to f(p)$ uniformly in every compact subset of P . Equation (3.6) is an easy consequence of the Cauchy integral representation for analytic functions (Ref. [23], Chapter IV) . It is also immediate to verify that the powers

$$\varphi_\alpha = (p - p_o)^\alpha N_\alpha^{-1} = Z^\alpha N_\alpha^{-1} \quad , \quad \alpha \in Z_N^+$$

where

$$N_\alpha^2 = \int_D |Z^\alpha|^2 \, d \lambda(Z)$$

are normalization constants, form a complete orthonormal set in $\mathcal{H}_P$.

Let us associate to the function $A(p , \bar{q})$ the bounded linear operator $A \in \mathfrak{L}(\mathcal{H}_P)$ by the formula

$$(A f)(p) = \int_P A(p , \bar{q}) \, f(q) \, d \lambda_2 \quad . \tag{3.7}$$

A is evidently bounded since $A(p , q)$ is an analytic and hence bounded function in the closure of $P \times P$. It is also Hermitian and positive. This can be seen by introducing into the formula

$$(f , A f) = \int_{P \times P} \bar{f}(p) \, A(p , \bar{q}) \, f(q) \, d \lambda_p \, d \lambda_q$$

for f the polynomial $f = \sum_{|\alpha| \leq n} a_\alpha (p - p_o)^\alpha N_\alpha^{-1}$ and for $A(p , \bar{q})$ its Taylor series centered at p_o . One gets $(f , Af) = Q_{p_o}(a , a)$, where Q is the quadratic form (3.4) , which was shown to be positive. Since polynomials are dense in $\mathcal{H}_P$, it follows that $(f , Af) \geq 0$ for all $f \in \mathcal{H}_P$. This is the inequality A) for $\omega = P$ and $f \in \mathcal{A}(P) \cap L_2(P)$. Finally, A has a finite trace. Indeed, by computing the trace with the help of the orthonormal system $\{\varphi_\alpha\}$ using again the Taylor expansion for $A(p , \bar{q})$, one obtains

$$\mathrm{tr}\, A = \int_P A(p , \bar{p}) \, d \lambda_p < \infty$$

All these properties imply that A is of the Hilbert-Schmidt type and has therefore a purely point spectrum. The set of all the eigenfunctions of A

$$\int_P A(p\ ,\ \bar{q})\ g_\nu(q)\ d\lambda_q = \lambda_\nu\ g_\nu(p) \quad , \qquad \nu = 1,\ 2,\ 3,\ \ldots \tag{3.8}$$

is a complete set of orthonormal functions in $\mathcal{H}_P$. $(f\ ,\ Af) \geq 0$ entails $\lambda_\nu \geq 0$. Since for fixed $\bar{q}$ $A(p\ ,\ \bar{q}) \equiv f_{\bar{q}}(p)$ is an element of $\mathcal{H}_P$, we may expand $f_{\bar{q}}$ into a Fourier series with respect to the orthonormal set $\{g_\nu\}$: $f_{\bar{q}} = \sum_1^\infty a_\nu\ g_\nu$, with $a_\nu = (g_\nu\ ,\ f_{\bar{q}}) = \lambda_\nu\ \bar{g}_\nu(q)$ because of (3.8) . Now, according to (3.6), strong convergence entails uniform pointwise convergence, and therefore the series

$$A(p\ ,\ \bar{q}) = \sum_1^\infty \lambda_\nu\ g_\nu(p)\ \bar{g}_\nu(q) = \sum_1^\infty f_\nu(p)\ \bar{f}_\nu(q)$$

$$\text{with}\quad f_\nu = \lambda_\nu^{1/2}\ g_\nu \tag{3.9}$$

converges uniformly in $p \in P$ for each fixed $q \in P$. It remains only to be shown that the series (3.9) converges uniformly in $P \times P$. It is useful for later purposes to proceed as follows. By putting $q = p$ in (3.9) we get

$$A(p\ ,\ \bar{p}) = \sum_1^\infty |f_\nu(p)|^2 \tag{3.10}$$

and, since this is a series of positive continuous functions in P converging to a continuous function in P , Dini's theorem tells us that the series converges uniformly in P . The Cauchy inequality yields :

$$|A(p\ ,\ \bar{q})|^2 \leq \{\sum_\nu |f_\nu(p)\ \bar{f}_\nu(q)|\}^2 = \sum_\nu |f_\nu(p)|^2 \sum_\mu |\ f_\mu(q)|^2 = A(p\ ,\ \bar{p})\ A(q\ ,\ \bar{q}) \tag{3.11}$$

This shows that the series (3.9) automatically converges absolutely and, as a little reflection shows, also uniformly in $P \times P$ since the two series on the right-hand side of the inequality do so .

The uniform convergence of (3.9) evidently implies the inequality A) for any $\omega \subset\subset P$ and any $f \in L_2(\omega)$. Thus, Theorem 1 is proved in the special case $\Omega = P$.

The proof of Theorem 1 will be complete if we prove the following theorem, which has an independent interest (compare Ref. [23], Chapter XII) .

THEOREM 2.

If $A(p , \bar{p})$ is real analytic in Ω and if some polydisc P_o (centered at $p_o \subset \Omega$) the representation (3.10) is valid with the f_ν analytic in P , then all the functions f_ν can be continued analytically into all of Ω and the representation (3.10) remains valid in Ω in the sense of uniform convergence. The function $A(p , q)$ defined by the series

$$A(p , q) = \sum_\nu f_\nu(p) \bar{f}_\nu(\bar{q}) \tag{3.12}$$

converging uniformly and absolutely in $\Omega \times \Omega^*$ continues analytically the function $A(p , \bar{p}) = A(p , q)|_{q=\bar{p}}$ into $\Omega \times \Omega^*$.

Proof.

As already explained, $A(p , \bar{p})$ real analytic in Ω means that for any $p_i \in \Omega$ there is a (non-empty) polydisc P_i centered at p_i in which the Taylor series of $A(p , \bar{p})$

$$A(p , \bar{p}) = \sum_{\alpha,\beta} A^i_{\alpha\beta} (p - p_i)^\alpha (\bar{p} - \bar{p}_i)^\beta \tag{3.13}$$

converges absolutely. Since by the Heine-Borel lemma any point $p \in \Omega$ can be connected to $p_o \in P_o$ by a finite chain of polydiscs P_r such that $p_r \in P_{r-1}$ $(r = 1, \ldots, m)$ and $p \in P_m$, it is sufficient to show that (3.10) can be "continued" from P_o to P_1 . Let $R = (R_1, \ldots, R_N)$ be the radius of $P_1 = \{p : |p_i - p_{1i}| < R_i \;,\; i = 1, \ldots, N\}$. Let

$$f_\nu(p) = \sum_\alpha f_{\nu\alpha}(p - p_1)^\alpha \tag{3.14}$$

the Taylor series of f_ν ; they converge all a polydisc $P_1' \subset P_o \cap P_1$ centered at p_1 . We have on the one hand, the Cauchy inequalities

$$|A^1_{\alpha\beta}| < M/R^{\alpha+\beta} \qquad \text{for some} \quad M \tag{3.15}$$

since (3.13) for $i = 1$ converges in P_1 , and on the other the representation

$$A^1_{\alpha\beta} = \sum_{\nu=1}^{\infty} f_{\nu\alpha}\, \overline{f}_{\nu\beta} \quad , \tag{3.16}$$

which we get by inserting the series (3.14) into (3.10) by inverting summation signs. This we are allowed to do since Theorem 2 is valid in P_o by the remarks following formula (3.9) . Putting $\alpha = \beta$, (3.15) and (3.16) yield the inequality

$$A^1_{\alpha\alpha} = \sum_\nu |f_{\nu\alpha}|^2 < M/R^{2\alpha} \tag{3.17}$$

and a fortiori $|f_{\nu\alpha}| < M^{\frac{1}{2}}/R^\alpha$. This shows that the Taylor series (3.14) keeps on converging absolutely in all of P_1 . We have to show that the series (3.10) behaves likewise. For that purpose let us apply to the series

$$\sum_\alpha f_{\nu\alpha} Z^\alpha = \sum_\alpha \{f_{\nu\alpha} R^\alpha (Z/R)^{\alpha/2}\}(Z/R)^{\alpha/2} \quad , \quad Z = p - p_1$$

the Schwartz inequality. We get

$$|f_\nu(p)|^2 \le \sum_\alpha |f_{\nu\alpha}|^2 R^{2\alpha} |\frac{Z}{R}|^\alpha \cdot \sum_\beta |\frac{Z}{R}|^\beta \quad .$$

Combining this with the inequality (3.17) , we obtain

$$\sum_\nu |f_\nu(p)|^2 \le M(\sum_\alpha |\frac{Z}{R}|^\alpha)^2 = M \prod_{i=1}^{N} (1 - \frac{|p_i - p_{1i}|}{R_i})^{-2} < \infty$$

for all $p \in P_1$.

This proves the (3.10) part of the theorem. But the remaining part follows **immediatly** from what has been said in the proof of formula (3.9) .

The Theorem 2 shows that the "sensitive" points of the analytic function $A_\sigma(p\ ,\ q)$ lie in the immediate neighbourhood of the "diagonal plane" $\mathbb{D}_{2N} = \{(p\ ,\ q) : p = \bar{q}\}$. If by using any other information we succed to enlarge the domain of analyticity of A in a neighbourhood, however "thin" of the diagonal plane, this enlargement becomes automatically a topological product. The following Theorem 3 will strengthen this conclusion since in it not even continuity of the function $A(p\ ,\ \bar{p})$ will be required. This theorem is inspired by the Bernstein theorem of classical analyses on the convergence of the Taylor series of a function having all its derivatives positive *) . But in spite of its apparent generality, the author believes that Theorem 2 could be more useful for practical applications.

THEOREM 3.

Let $A(p\ ,\ \bar{p}) \in \mathfrak{D}'(\Omega)$ satisfy - in the sense of distributions - the positivity condition C) :

$$\sum_{\alpha,\beta} \frac{a_\alpha \bar{a}_\alpha}{\alpha!\ \beta!} \partial_p^\alpha \partial_{\bar{p}}^\beta A(p\ ,\ \bar{p}) \equiv Q_p(a\ ,\ a) \geq 0 \qquad (3.18)$$

everywhere in Ω for all finite sequences a_α . Then $A(p\ ,\ \bar{p})$ is the restriction to the "diagonal plane" $p = \bar{q}$ of a function $A(p\ ,\ q)$ analytic in $\Omega \times \Omega^*$.

Proof.

It is enough to show that $A(p\ ,\ \bar{p})$ is real analytic in Ω , for then the methods of proof of Theorems 1 and 2 will evidently lead to the statement of Theorem 3.

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*) The author would like to thank D . Bessis for drawing his attention to the Bernstein theorem during the elaboration of this work. His thanks are also due to Dr. H. Epstein, who suggested the distribution part of Theorem 3.

By noting that

$$\Delta_i = 4 \frac{\partial^2}{\partial p_i \, \partial \bar{p}_i}$$

are two-dimensional Laplace operators, we get, as a consequence of (3.18) , the inequalities :

$$\Delta_1^{n_1} \; \Delta_2^{n_2} \cdots \Delta_N^{n_N} \geq 0 \qquad \text{for all} \qquad (n_1, \ldots, n_N) \in Z_N^+ \qquad ,$$
$$(\Delta_i^o = 1) \tag{3.19}$$

and also

$$\mu_n \equiv \Delta^n A \geq 0 \qquad \text{for} \qquad n = 0, 1, 2, \ldots \tag{3.20}$$

where $\Delta = \Delta_1 + \Delta_2 + \ldots + \Delta_N$ is the Laplace operator with respect to all the variables.

Equation (3.20) implies $A \in C^\infty(\Omega)$. To see this, we shall use a classical argument [22),24)] . We first notice that the μ_n being positive distributions are measures. The second remark is that the elementary solution $G_n(r)$ of the equation $\Delta^n F = \delta_{2N}$ is in $C^{2n-2N-1}(\mathbb{R}_{2N})$ provided $n > N$ [22),24)]. Here $r^2 = |p_1|^2 + \ldots + |p_N|^2$. Therefore $G_n*(\vartheta\mu_n)$ is also in $C^{2n-2N-1}(\mathbb{R}_{2N})$ for any $\vartheta \in \mathfrak{D}(\Omega)$. By choosing $\vartheta(p) = 1$ in a sphere $S \subset\subset \Omega$, we find that the measure $F_n = A - G*(\vartheta\mu_n)$ satisfies the equation $\Delta^n F_n = 0$ in S . But any solution of the last equation is $C^\infty(S)$ and therefore $A \in C^{2n-2N-1}(S)$. n being arbitrary and $S \subset\subset \Omega$ also arbitrary we conclude that $A \in C^\infty(\Omega)$.

Knowing this, we want to show that the Taylor series of A at any given point $p_o \in \Omega$ converges absolutely in every polydisc contained in Ω . Without loss of generality we shall take p_o to be the origin and it will be sufficient to show that the coefficients of the formal power series

$$S(p) = \sum_{\alpha,\beta} \frac{p^\alpha \bar{p}^\beta}{\alpha! \; \beta!} \; \partial_p^\alpha \; \partial_{\bar{p}}^\beta \; A(0 \; , \; 0) \equiv \sum_{\alpha,\beta} A_{\alpha\beta} \, p^\alpha \, \bar{p}^\beta$$

satisfy the Cauchy inequalities

$$|A_{\alpha\beta}| < M/R^{\alpha+\beta} \tag{3.21}$$

where $R = (R_1, \ldots, R_N)$ is the radius of any polydiscs contained in Ω.

Let us put $p_j = r_j\, e^{i\varphi_j}$ ($r_j \geq 0$, φ_j real) and let us study the mean value of the function A over the angles :

$$\mathcal{M}(r_1, \ldots, r_N) = (2\pi)^{-N} \int_0^{2\pi} d\varphi_1 \ldots \int_0^{2\pi} d\varphi_N\, A(r_1\, e^{i\varphi_1}, \ldots, r_N\, e^{i\varphi_N}) \; .$$

We shall consider only the case $N = 1$ in detail. With the help of the Green's formula, we get :

$$d\mathcal{M}(r)/dr = (2\pi r)^{-1} \int_0^r \frac{\partial}{\partial r} A(r\, e^{i\varphi}) r\, dr = r^{-1} \int_0^r dr'\, r'(2\pi)^{-1} \; .$$
$$\cdot \int_0^{r'} \Delta\, A(r'\, e^{i\varphi}) d\varphi \equiv r^{-1} \int_0^r dr'\, r'\, \mathcal{M}_1(r')$$

or by integrating :

$$\mathcal{M}(r) = \mathcal{M}(0) + \int_0^{r'} dr' \int_0^{r'} dr'' \frac{r''}{r'} \mathcal{M}_1(r'') \quad . \tag{3.22}$$

Here we have introduced the notation

$$\mathcal{M}_\alpha(r) = (2\pi)^{-1} \int_0^{2\pi} \Delta^\alpha A(r\, e^{i\varphi})\, d\varphi \quad , \quad \alpha = 0, 1, 2, \ldots, \quad \mathcal{M}_0 = \mathcal{M}$$

Equation (3.22) can be also written in the form

$$\mathcal{M}_\alpha(r) = \mathcal{M}_\alpha(0) + \int_0^r G(r\, ,\, r') \mathcal{M}_{\alpha+1}(r')\, dr' \quad ,$$

with

$$G(r\, ,\, r') = r'\, \ln r/r' \geq 0 \quad .$$

By n fold iteration we obtain :

$$\mathcal{M}_0(r) = \sum_{\alpha=0}^{n} \mathcal{M}_\alpha(0) \int_0^r G_\alpha(r\, ,\, r') dr' + \int_0^r G_n(r\, ,\, r')\, \mathcal{M}_n(r') dr' \tag{3.23}$$

where G_α is the α times iterated kernel. One finds easily :

$$\int_0^r G_\alpha(r\ ,\ r')dr' = (r/4)^{2\alpha}/(\alpha!)^2$$

and

$$\mathcal{M}_\alpha(0) = \Delta^\alpha\ A(0) = 4^\alpha\ \partial_p^\alpha\ \partial_{\overline{p}}^\alpha\ A(0\ ,\ 0) \qquad .$$

Now, since G_n is positive and by (3.19) all the $\mathcal{M}_\alpha(r)$ are also positive, we obtain by dropping the last term in (3.23), the inequality

$$\mathcal{M}_o(r) \geq \sum_{|\alpha|\leq n} A_{\alpha\alpha}\ r^{2\alpha}$$

for all $0 \leq r \leq R$ and all n , which, in turn, implies the Cauchy inequality

$$A_{\alpha\alpha} \leq M/R^{2\alpha} \qquad , \qquad M = \mathcal{M}_o(R) \qquad .$$

It is almost evident that this last formula is also valid in the case $N > 1$ if we replace in it the index α by the corresponding multi-index $\alpha = (\alpha_1,\ \ldots,\ \alpha_N)$ and R by $R = (R_1,\ \ldots,\ R_N)$; to see this, one has only to apply formula (3.23) to each of the variables $r_1,\ \ldots,\ r_N$ separately.

In order to get the full Cauchy inequality (3.21), we notice that the positivity condition (3.18) implies $A_{\alpha\beta} = \overline{A}_{\beta\alpha}$ and $|A_{\alpha\beta}|^2 \leq A_{\alpha\alpha}\ A_{\beta\beta}$.

Thus the convergence of the formal power series at each $p \in \Omega$ is established. It remains only to be shown that these series converge to the function $R(p\ ,\ \overline{p})$. But this is an immediate consequence of the classical Taylor formula with the rest term. Therefore, Theorem 3 is proved *) .

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*) After the completion of this work, Professor P. Lelong has Kindly informed me at the last Strasbourg meeting that in 1948, by using somewhat different methods, he had proved the following theorem : if a C^∞ function satisfies the conditions (3.20) in a domain, then the function is real analytic there [25] .

The rest of this Section will be taken up with the generalization to the case when functions with a different number of fields are involved in the positivity condition. Our remarks will be only sketchy, since, except for one point, essentially only notational questions will be at stake.

Let us denote the Fourier transform of $(\Omega, R_n^* R_m \Omega)$ by

$$A_{nm}(\sigma ; p_n , q_m) = \mathcal{F}(\Omega , R_n^* R_m \Omega) \tag{3.24}$$

$$(n = 1, 2, \dots ; m = 1, 2, \dots)$$

Here again $p_n \in \mathbb{R}_{4(n-1)}$ are the internal momentum variables of the "cluster" R_n^* and $q_m \in \mathbb{R}_{4(m-1)}$ those of the cluster R_m . σ is the total energy and momentum created by any of the clusters from the vacuum state. We will keep it again fixed and hence drop it from our notation.

The discussion at the end of Section 2 shows that A_{nm} is a boundary value of a function analytic at least in the domain $\Omega_n(\sigma) \times \Omega_m(\sigma)$, where $\Omega_i(\sigma)$ is the "σ-shifted" domain of analyticity of the i point function . Each of these domains contains real points and is invariant under complex conjugation. The original positivity condition reads

$$\sum_{\substack{1\le n\le N \\ 1\le m\le N}} \int A_{nm}(p_n , q_m) \, \bar{f}_n(p_n) \, f_m(q_m) \, dp_n \, dq_m \geq 0 \tag{3.25}$$

for all $f_n \in S(\mathbb{R}_{4(n-1)})$, $n = 1, \dots, N$ [note that $S(\mathbb{R}_o) = \mathbb{C}_1$ = the set of complex numbers].

In order to formulate the generalization of the condition A) , of Theorem 1, let us introduce the (pre-) Hilbert space $\mathcal{H}_\omega$, whose elements are N-tuples of analytic functions :

$$f = \{f_1(p_1) , f_2(p_2), \dots, f_N(p_N)\} \quad \text{with} \quad f_n \in \mathcal{A}(\omega_n) \cap L_2(\omega_n) ,$$

$$\omega_n \subset\subset \Omega_n$$

and with the scalar product defined by :

$$(f\ ,\ g) = \sum_{n=1}^{N} \int_{\omega_n} \bar{f}_n\,(p_n)\; g_n(p_n)\; d\,\lambda_{pn} \tag{3.26}$$

where $d\,\lambda_{pn}$ is the Lebesgue measure in $\mathbb{C}_{4(n-1)}$. $\mathcal{H}_\omega$ is simply a finite direct sum of Bergman-Hilbert spaces and therefore itself a Hilbert space. $\mathcal{H}_\omega$ inherits the, for us, essential property of a Bergman-Hilbert space : the value $f_n(p_n)$ at the point $(n\ ,\ p_n)$ of an $f \in \mathcal{H}_\omega$ is a continuous linear fonctional of f :

$$|f_n(p_n)| \le M_n(p_n)\{\int_{\omega_n} |f_n(p_n)|^2\; d\,\lambda_n\}^{1/2} \le M_n(p_n)\;\|f\| \tag{3.27a}$$

with $M_n(p_n) = C_n\; d(p_n)^{-4(n-1)/2}$, C_n a numerical constant [compare formula (3.6)] .

To the $N \times N$ matrix of functions (A_{nm}) , we associate now the linear operator Λ on $\mathcal{H}_\omega$ defined by

$$(\Lambda f)_n(p_n) = \sum_{m=1}^{N} \int_{\omega_m} A_{nm}(p_n\ ,\ \bar{q}_m\ ,\ f_m(q_m)\; d\,\lambda_{qm}$$

which is obviously bounded since $\omega_n \subset\subset \Omega_n$.

What one wants to prove is that (3.25) implies the generalized condition

A') $\quad (f\ ,\ \Lambda f) \ge 0 \quad$ for all $\quad f \in \mathcal{H}_\omega$.

As it was seen in the proof of Theorems 1 and 2 , it is actually sufficient to prove this inequality for $\omega_n = \mathbf{P}_n$ = any polydisc $\subset\subset \Omega_n$ centered at some real point of analyticity $p'_n \in R_n$ $(n = 1, \ldots, N)$. But that can be easily done by inserting for the test functions in (3.25) linear combinations of δ functions as in formula (3.3) and then choosing, as in proof of Theorem 1, powers as a complete orthonormal system of functions in $\mathcal{H}_P$.

Since

$$\mathrm{tr}\, A = \sum_{n=1}^{N} \int_{\omega_n} A_{nn}(p_n\, ,\, \bar{p}_n)\, d\,\lambda_n \tag{3.27b}$$

is also obviously finite, the diagonalization of A yields a complete orthonormal system of functions $\{\varphi_\nu\}$: $\mathbf{A}\,\varphi_\nu = \lambda_\nu\,\varphi_\nu$, $\lambda_\nu \geq 0$, $\nu = 1,2,3,\ldots$. By introducing the functions $f_\nu = \lambda_\nu^{\frac{1}{2}}\,\varphi_\nu = (f_{1\nu}(p_1),\ldots, f_{N\nu}(p_N))$ the fundamental inequality (3.27a) again leads to the uniformly converging representation :

$$A_{nm}(p_n\, ,\, \bar{q}_m) = \sum_{\nu=1}^{\infty} f_{n\nu}(p_n)\,\bar{f}_{m\nu}\,(q_m) \qquad ,$$

$$(n\, ,\, m = 1,\, \ldots,\, N) \tag{3.28}$$

Here the functions $f_{n\nu}(p_n)$ are analytic in $P_n(n = 1,\, \ldots,\, N)$.

By considering the diagonal term $n = m$ it follows from <u>Theorem 2</u> that the functions $f_{n\nu}$ can be analytically continued to the whole of Ω_n $(n = 1,\, \ldots,\, N)$, and the Schwartz inequality

$$|A_{nm}(p_n\, ,\, \bar{q}_m)|^2 \leq \{\sum_\nu |\, f_{n\nu}(p_n)\, \bar{f}_{m\nu}(q_m)|\}^2 \leq \sum_\nu |f_{n\nu}(p_n)|^2 \cdot \sum_\nu |f_{m\nu}(q)|^2 =$$

$$= A_{nm}(p_n\, ,\, \bar{p}_n)\; A_{mm}(q_m\, ,\, \bar{q}_m) \tag{3.29}$$

tells us, again through Theorem 2, that the series automatically converges uniformly in the whole of $\Omega_n \times \Omega_m$.

Therefore we have proved the following theorem :

<u>THEOREM 4</u>.

Let a set of $N \times N$ continuous functions $A_{nm}(p_n\, ,\, q_m)$ be defined in $R_n \times R_m$, where R_n is a (real) neighbourhood of a point $p_n = p_n'$ in $\mathbb{R}_{4(n-1)}$; let this set of functions satisfy condition (3.25) for all $f_n \in \mathfrak{D}\,(R_n)$ $(n = 1,\, \ldots,\, N)$; let all the functions A_{nm} be analytic in a complex neighbourhood of the points $p_n = p_n'$, $q_m = p_m'$; let further the "diagonal" functions $A_{nn}(p\ ,\ \bar{p}_n)$ be real-analytic in domains $\Omega_n \in \mathbb{C}_{4(n-1)}$ containing the real point $p_n = p'_n$ $(n = 1, \ldots, N)$. Then the functions $A_{nm}(p_n\ ,\ q_m)$ can be analytically continued into $\Omega_n \times \Omega_m^*$ $(n\ ,\ m = 1, \ldots, N)$. Furthermore the matrix of functions $A = (A_{nm})$ can be "diagonalized" there, that is, there exists a sequence of functions $f_{n\nu} \in \mathcal{H}(\Omega_n)$ $(\nu = 1, 2, 3, \ldots)$, $(n = 1, \ldots, N)$ such that the representation (3.28) is valid in $\Omega_n \times \Omega_m$ in the sense of the uniform convergence of the series.

Generalizations and outlook.

The main tool for getting Theorems 1 to 4 was the introduction of a suitably defined Hilbert space of analytic functions on which the "absorptive amplitude" turned out to be a positive operator of the Hilbert-Schmidt type. The use of these Hilbert space techniques was essentially local : the domains ω in the definition of $\mathcal{H}_\omega$ [see (3.26)] were relatively compact subdomains of the domains of holomorphy Ω. The measures λ were also rather arbitrarily taken to be Lebesgue measures.

We can now ask the question whether the initial positivity condition (3.25) can be formulated in a form that would - at least partially - respect analyticity (that is locality and the spectral condition) in a more explicit way. The answer is obviously to be sought in a suitable choice of the Hilbert space $\mathcal{H}_\omega(\mu)$ of analytic "test functions". Here μ denotes the set of measures which are to replace the Lebesgue measure in formula (3.26). The most natural choice seems to be $\omega = \Omega$ = the set of envelopes of holomorphy Ω_n , and

$$d\mu_n = e^{-\varphi_n} d\lambda_n \quad , \qquad (n = 1, 2, 3, \ldots) \qquad (3.30)$$

(compare Hörmander [26], Ch. IV), with φ_n some real, let us say continuous, function in Ω_n such that

$$\int_{\Omega_n} A_{nn}(p_n\ ,\ \bar{p}_n)\, d\mu_n < \infty \qquad (3.31)$$

Thanks to the (appropriately generalized) formula (3.27b), the last condition namely automatically ensures that the positive operator A_N associated to the $N \times N$ matrix (Λ_{mn}) has a finite trace for any finite N, and hence that A_N is the diagonalizable Hilbert-Schmidt type. For a given field such a choice of measures is always possible : one only has to choose the functions φ_n of sufficiently rapid increase near the boundary of Ω_n (including points at infinity) (compare Hörmander, loc. cit.). One can then obviously choose the φ_n also in such a way that the operator A_∞ is of finite trace ; hence the representation (3.28) is valid for <u>all</u> n, m, since the fundamental property (3.27a) of a Bergman-Hilbert space is - with a slight modification - preserved also in this case. Thus we get the

<u>THEOREM 4'.</u>

Let the set of "absorptive amplitudes" satisfy the positivity condition (3.25). Then there exists a double sequence of functions $f_{n\nu} \in \mathcal{A}(\Omega_n)$ $(\nu = 1, 2, 3, \ldots ; n = 1, 2, 3, \ldots)$ such that the representation (3.28) is valid in $\Omega_n \times \Omega_m$ for all n and m in the sense of uniform convergence.

What we would still like to achieve is to find systems of measures μ_n such that <u>every</u> positive operator A with finite trace acting on $\mathcal{H}_\Omega(\mu)$ gives rise to a system of Λ_{nm}'s with boundary values satisfying the initial <u>physical</u> conditions(3.25). For that purpose it is necessary and sufficient that the Λ_{nm}'s so defined do not increase faster than an inverse power of the distance to the boundary when approaching their physical boundary values (compare, e.g., Streater and Wightman [1], Theorem 2, or Epstein [19]) - appropriate modifications of the asymptotic behaviour at infinity are to be made in the case of the Haag-Araki theory). Now the worse possible behaviour of an analytic function $f \in \mathcal{A}(\Omega) \cap L_2(\Omega, \mu)$ near a boundary point of Ω is determined - as seen by repeating the derivation of the formulas (3.6) and (3.27) in this slightly more general case - (compare [23], Ch. IV) - by the local behaviour of the inverse "weight function" e^{φ}. Thus e^{φ_n} should essentially behave in Ω_n as the upper bound of moduli $|F_n|$ of the set $\mathcal{F}$

of all the n point functions satisfying the conditions of the linear program with some <u>fixed</u> growth properties in their primitive domain of analyticity. In other words φ_n should be chosen as the (maybe somewhat smeared out) plurisubharmonic function

$$\varphi_n(p_n) = \sup_{F_n \in \mathcal{F}} \ln |F_n(p_n)| \equiv \ln M_n(p_n) \qquad .$$

Unfortunately, neither the envelopes of holomorphy Ω_n nor the functions M_n are explicitly known in all of Ω_n, . The situation can be remedied - at least partially - by replacing in the above considerations the holomorphy envelopes Ω_n by the corresponding <u>primitive</u> domains Ω_n^o (containing a <u>finite</u> complex neighbourhood of the real points of coincidence). As to the M_n's , although known only in the initial tubes, they can also be extended to Ω_n^o with the help of the edge of the wedge theorem using an appropriate trick *) .

These last sketchy observations require a more detailed investigation and will have to be treated elsewhere. Let us only stress here, as a final remark, that the above methods can be applied - hopefully with more profit - also to the set of Wightman functions in x space.

4 . <u>TWO EXAMPLES</u>.

As an illustration of the expounded theory let us consider first the case when A is of the form

$$A(p\ ,\ q) = A(p - q) \qquad . \qquad (4.1)$$

The Wightamn two-point function in x space $A(x - y) = (\Omega\ ,\ A(x)\ A(y))$ is such an example [1] (we have to set $x = p$, $y = q$) .

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*) To be contained in a forthcoming paper by Epstein and Glaser.

The function $A(p - \bar{p}) = A(2i \operatorname{Im} p)$ does not depend on the real part of p, hence the diagonal plane $\mathbb{D}_{2N} = \mathbb{R}_N \oplus i\,\mathbb{R}_N$ actually reduces to its imaginary subspace $i\,\mathbb{R}_N$. If we put $p - q = z = x + iy$, the differential form of the positivity condition becomes :

$$\sum_{\alpha,\beta} a_\alpha \bar{a}_\beta \frac{\partial^{\alpha+\beta}}{\partial y^{\alpha+\beta}} A(iy) \geq 0 \quad . \qquad (4.2)$$

From Theorem 3 it then follows :

COROLLARY 1.

If a distribution $f(y) \equiv A(iy) \in \mathcal{D}'(B)$, where B is an open set in $\mathbb{R}_N$, satisfies condition (4.2) in B, then f is the restriction to the imaginary plane $\{x = 0\}$ of a function $A(x + iy)$ analytic in the tube $\mathcal{T}_B = \{z = x + iy \in \mathbb{C}_N : y \in B\}$.

Let us call the set $\hat{B} = \{z : \operatorname{Re} z = 0 \ , \ \operatorname{Im} z \in B\}$ the "generating set" of A. From Theorem 2, we then conclude :

COROLLARY 2.

Let $B_1 \subset B_2$ be two connected open sets $\subset \mathbb{R}_N$, let $\hat{B}_i = \{z \in \mathbb{C}_N : : \operatorname{Re} z = 0 \ , \ \operatorname{Im} z \in B_i\}$ and $\mathcal{T}_{B_i} = \{z \in \mathbb{C}_N : \operatorname{Im} z \in B_i\}$. If $A(p - q) = A(z)$ satisfying the positivity condition in $\mathcal{T}_{B_1}$ [e.g., in the form (4.2)] is analytic in a neighbourhood of the set $\hat{B}_2$, then it is also analytic in the tube $\mathcal{T}_{B_2}$.

The integral form of the positivity condition can be cast into the following form ;

$$\int A(x - x' + iy)\, \bar{f}(x)\, f(x')\, dx\, dx' \geq 0$$

for all $y \in B$ and all $f \in \mathcal{D}(\mathbb{R}_N)$. (4.3)

This has landed us in the very well-known theory of functions of positive type studied by Bochner [27] and Schwartz [22] and applied to field theory by Wightman.

Our second example will be an application to the two-body scattering amplitude. But before discussing it we shall have to state the following rather trivial

LEMMA 1 .

The positivity conditions A) , B) and C) are invariant under an analytic substitution of variables $p = Tu$, $q = Tv$. Here T is an analytic mapping from ω into Ω , where ω is a domain in $\mathbb{C}_M$, and $\Omega \in \mathbb{C}_N$ the domain of definition of A (in general $M \neq N$) .

The proof is immediatly obtained by looking at the "diagonal" representation of A . If we denote $A(Tu , Tv) = \hat{A}(u , \bar{v})$ and $f_\nu(Tu) = \hat{f}_\nu(u)$ one gets :

$$\hat{A}(u , \bar{v}) = \sum_\nu \hat{f}_\nu(u) \bar{\hat{f}}_\nu(v) \qquad .$$

From here, the conditions in the forms A) and C) with p , q replaced by u , v immediately follow.

Note.

Being an inequality, the positivity condition is determined only up to a factor. Indeed, if we replace A by $F(p) A(p,\bar{q}) \bar{F}(q) = A_1(p , \bar{q})$, where F is any function analytic in Ω such that $F \not\equiv 0$, nothing will change. We also want to warn the reader that the set of functions f_ν which diagonalizes A is by no means unique.

We are now ready to study the absorptive part of the scattering amplitude of a process

$$A(k_1 , m_1) + B(k_2 , m_2) \rightarrow A(k_3 , m_1) + B(k_4 , m_2)$$

where the four-momenta and masses of the two scalar particles involved are indicated in brackets. In agreement with the notational conventions used in (2.5) and (2.6) we have

$$A(k_3 , k_4 , k_2) = A_\sigma(p , q)$$

with $\sigma = k_1 + k_2 = - k_3 - k_4$ and $k_{1,2} = \sigma/2 \pm q$, $k_{3,4} = -(\sigma/2 \pm p)$. The (complex) mass shell manifold is given by the equations

$$k_1^2 = k_2^2 = m_1^2 \qquad , \qquad k_2^2 = k_4^2 = m_2^2 \qquad .$$

If we fix σ in the form $\sigma = (\sqrt{s} , \vec{0})$ with $s \geq (m_1 + m_2)^2$, then the resulting manifold can be parametrized as follows

$$p = (A , R\vec{x}) \quad , \quad q = (A , R\vec{y}) \quad , \quad \vec{x}^2 = \vec{y}^2 = 1 \qquad (4.4)$$

where A and R are two real constants depending only on s and the masses, and $\vec{x}$ and $\vec{y}$ very independently on the unit sphere. Since the particles are supposed to be scalar , A is - at least on the mass shell - a Lorentz invariant function and hence depends only on s and the scalar product $\vec{x}\,\vec{y}$. Therefore, without loss of information, we may specialize (4.4) to

$$\begin{aligned} p &= (A , R \cos u , R \sin u , 0) \equiv Tu \\ q &= (A , R \cos v , R \sin v , 0) \equiv Tv \end{aligned} \qquad (4.5)$$

and we get

$$A_\sigma(Tu , Tv) = F(\cos(u - v)) \qquad . \qquad (4.6)$$

T is evidently an analytic mapping, so Lemma 1 may be applied and therefore also Corollaries 1 and 2 . Denoting by $\vartheta = u - v = \vartheta_1 + i\,\vartheta_2$ the scattering angle we can conclude that the "generating set" of A is a purely imaginary interval $\hat{I} = \{ \vartheta : \vartheta_1 = 0 , \vartheta_2 \in I\}$. The interval I has to contain the origin since the origin corresponds to physical points. I is also symmetrical about the origin because Λ is an even function of ϑ. Therefore the tube $\mathcal{T}_I$ of corollaries 1 and 2 is the strip $\mathcal{T}_I = \{ \vartheta : -\alpha < \mathrm{Im}\,\vartheta < +\alpha \}$. The image of this strip in the variable $z = \cos\vartheta$ is the ellipse with foci at $z = \pm 1$ and the major semi-axis $a = \mathrm{ch}\,\alpha$ introduced into physics by Lehmann [21], while the image of the set $\hat{I}$ is the real interval $1 \leq z < a$, with $a = \mathrm{ch}\,\alpha$. Thus we have proved the theorem discovered by Jin and Martin [28].

COROLLARY 2' .

If the absorptive part of the scattering amplitude of two scalar particles is analytic for fixed s in a neighbouhood of the real interval $1 \leq z < a$, where $z = \cos\vartheta$, then it is analytic also in the whole Lehmann ellipse with the major semi-axis a .

This fact can be stated also in the differential form of Corollary 1 .

COROLLARY 1' .

If a distribution $F(x) = F(\mathrm{ch}\, y) \in \mathcal{D}'(-\alpha < y < +\alpha)$ satisfies in $-\alpha < y < +\alpha$ the set of inequalities

$$\sum_{n=0}^{\infty} \frac{\partial^n}{\partial y^n} F(\mathrm{ch}\, y) \sum_{\alpha+p=n} a_\alpha \bar{a}_\beta \geq 0 \tag{4.7}$$

for all $a_\alpha \in \mathbb{C}_1$ such that $a_\alpha = 0$ for all $\alpha >$ some N , then F is the restriction of a function $F(z)$ analytic in the Lehmann ellipse $|z - 1| + |z + 1| < 2a = 2\,\mathrm{ch}\,\alpha$.

The inequalities (4.7) can be put, in principle, into a form involving only derivatives $d^nF(x)/dx^n$. What we want to show is that (4.7) implies the inequalities :

$$\frac{d^n F(Z)}{d\,Z^n} \geq 0 \quad \text{for all} \quad 1 \leq Z < a \quad \text{and} \quad n = 0, 1, 2, \ldots \tag{4.8}$$

used extensively by Martin [6).

Instead of manipulation the expression (4.7) directly we shall start from the integral condition (4.3) applied to the variable ϑ [which is a consequence of (4.7) !]*) .

The image of the strip $-\alpha < \operatorname{Im} \vartheta < \alpha$ in the complex plane of the variable $\xi = e^{i\vartheta}$, $z = \frac{1}{2}(\xi + \xi^{-1})$, is the corona $e^{-\alpha} < |\xi| < e^{\alpha}$. Since F is analytic there, it can be expanded into a convergent Laurent series

$$F = \sum_{\nu=-\infty}^{+\infty} c_\nu \, \xi^\nu = \sum_{\nu=0}^{\infty} a_\nu \, \tfrac{1}{2}(\xi^\nu + \xi^{-\nu}) \quad . \tag{4.9}$$

The last form follows from the symmetry of F under the substitution $\xi \to \xi^{-1}$. Let us apply to (4.9) the integral inequality (4.3) by substituting in it $x = \vartheta_1$, $x' = \vartheta_1'$, $y = \vartheta_2 = 0$, $f = e^{i\nu\vartheta_1}$ and integrate from 0 to 2π . We get

$$a_\nu \geq 0 \quad \text{for all} \quad \nu = 0, 1, 2, \ldots \quad .$$

The last inequality implies (4.8) for $n = 0$. If we show that dF/dz can also be represented by a series of the form (4.9) with the coefficients a_ν' all positive, the inequality (4.8) will follow by induction. But this is a consequence of the identity

$$\frac{d}{dZ} \frac{1}{2} (\xi^\nu + \xi^{-\nu}) = \nu \, \frac{\xi^\nu - \xi^{-\nu}}{\xi - \xi^{-1}} = \nu \sum_{\mu=0}^{\nu-1} \xi^{\nu-2\mu} \tag{4.10}$$

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*) The following proof is due to Epstein. The author would like to thank Dr. Epstein for the permission to include it in this paper.

The termwise differentiation of the series (4.9) and a rearrangement of terms shows namely, by virtue of (4.10) , that the coefficients a'_ν are positive linear combinations of the coefficients a_ν .

As the final comment, let us remark that the inequalities (4.8) are <u>weaker</u> than the set of inequalities (4.7) . While (4.7) implies, according to the general theory, analiticity in the whole Lehmann ellipse, the inequalities (4.8) imply only analyticity in the disc $|z - 1| < a$ with positive coefficients of the corresponding power series expansion. The last assertion follows from the Bernstein theorem (compare, e.g., [29]) .

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