C. V. STANOJEVIĆ

Mielnik's Probability Spaces and Characterization of Inner Product Spaces

Les rencontres physiciens-mathématiciens de Strasbourg - RCP25, 1973, tome 15
« Conférences de : R. Balian, H.J. Borchers, J.J. Duistermaat J.P. Eckmann, H. Goldschmidt et C.V. Stanojević », , exp. nº 6, p. 1-8

<http://www.numdam.org/item?id=RCP25_1973__15__A6_0>

MIELNIK'S PROBABILITY SPACES AND CHARACTERIZATION OF INNER PRODUCT SPACES

C. V. Stanojević

B. Mielnik pointed out the insufficiency of the classical approach, [1], [2], and proposed a geometric approach to the foundation of general quantum mechanics [3]. For that purpose he introduced an abstract space of states as a setting for his geometric theory of quantum states.

Let S be a non-empty set, and let p be a real-valued function defined on S x S such that

(A) $0 \leq p(a,b) \leq 1$ and $a=b \Longleftrightarrow p(a,b) = 1$

(B) $p(a,b) = p(b,a)$ for all $a, b, \in S$.

Two elements a and b in S are orthogonal if $p(a,b) = 0$. A subset R of S is an orthogonal system if any two distinct elements of R are orthogonal. From Zorn's lemma it follows that there exists a maximal orthogonal system B which is called a basis in S.

Let B be a basis and F_B be the class of all finite subsets F of B. Define $p(a,F) = \sum_{b \in F} p(a,b)$ for all $a \in S$. Then the following property of (S,p) is also postulated:

(C) For each basis B and for each $a \in S$,

$$\sup_{F \in F_B} p(a,F) = 1$$

Any pair (S,p) satisfying axiom (A), (B) and (C) is called a probability space.

Let B_1 and B_2 be two bases in S, then, as shown in [3], B_1 and B_2 have the same cardinal number. This cardinal number is called <u>the dimension</u> of (S,p).

The existence of a representation of states by vectors in a Hilbert space imposes strong limitations on the geometric structure of the space of states. This space must be a Hilbert space over one of the three fields: real numbers, complex numbers or quaternions. An analogous situation arises in connection with Mielnik's theory. A particular probability space structure, imposed on a subset S of a normed linear space N, can turn N into an inner product space. For instance, in [4] we have the result:

Let S be the unit sphere of a normed real linear space N and let $p(a,b) = \frac{\|a+b\|^2}{4}$ for a, b in S. Then N is an inner product space if and only if (S,p) is a probability space of dimension 2.

The theorem that follows generalizes the above result.

<u>Theorem</u> Let S be the unit sphere of a normed real linear space N. If (S,p) is a probability space of dimension 2, and

(1.1) $p(x,y) \leq 1/4 \|x+y\|^2$, for all x, y S, then N is an inner product space.

The above result motivates an effort to characterize inner product spaces in terms of probability spaces in the sense of B. Mielnik.

We shall give a necessary and sufficient condition for N to be an inner product space in terms of probability space structure imposed over the unit sphere S of N.

2. Lemmas. Let $[0,2]$ be the domain and $[0,1]$ the range of f, and let f be continuous and increasing, with $f(0) = 0$ and $f(2) = 1$. The class of all such functions we denote by F.

Let $[0,2]$ be both the domain and the range of continuous and decreasing function g, and let $g(0) = 2$, $g(2) = 0$. The class of these functions we denote by G.

Lemma 2.1 Let $f \in F$. Then there exists a $g \in G$, such that

(2.1) $\quad f + f \circ g = 1$

where $\quad (f \circ g)(t) = f(g(t))$.

The converse problem of finding $f \in F$ for given $g \in G$, from (2.1), is much more difficult. In the following lemma we shall solve that problem for a class of functions wide enough for our purpose.

Lemma 2.2 Let h be continuous and decreasing over $[0,2]$ and let $h(0) = 2$, $h(2) = 0$. If

$$g(t) = h^{-1}(2-\lg\left[e^2-e^{2-h(t)}+1\right]) \tag{2.2}$$

then there exists $f \in F$ such that

$$f(t) + f(g(t)) = 1 \tag{2.1}$$

In the next lemma we give a characterization of inner product spaces.

Lemma 2.3 Let N be a normed real linear space, and let $S = \left\{ x \mid \|x\| = 1 \right\}$. Then N is an inner product space if and only if

$$(2.4) \qquad f(\|x+y\|) + f(\|x-y\|) = 1$$

for some $f \in F$ and all $x, y \in S$.

Corollary 2.1 Let h satisfy the condition of lemma 2.2, let N be a normed real linear space, and let $S = \left\{ x \mid \|x\| = 1 \right\}$.

A necessary and sufficient condition for N to be an inner product space is that

$$(2.5) \qquad e^{-h(\|x+y\|)} + e^{-h(\|x-y\|)} = 1 + e^{-2}$$

for all $x, y \in S$.

3. Probability spaces and inner product spaces. It remains to determine the extent of limitation that a very general probability space structure imposes upon a normed real linear space N. We shall show that a certain probability space structure imposed upon $S = \left\{ x \mid \|x\| = 1 \right\} \subset N$, makes N into an inner product space.

Theorem 3.1 Let N be a normed real linear space, $S = \left\{ x \mid \|x\| = 1 \right\}$ and let $p(x,y) = f(\|x+y\|)$, where $f \in F$, and $f + f \circ g = 1$ for some $g \in G$.

Then N is an inner product space if and only if (S,p) is a probability space of dimension 2.

Corollary 3.1 Let h satisfy the conditions of lemma 2.2 and let N and S be as in theorem 3.1.

Then N is an inner product space if and only if

$$(S, \frac{e^{2-h(\| x+y \|)}-1}{e^2-1})$$

is a probability space of dimension 2.

Corollary 3.2 Let the conditions of corollary 3.1 be satisfied. Then for

$$h(t) = -\lg\left[t^2/4(1-e^{-2}) + e^{-2}\right]$$

we have the result proved in [4].

4. Remarks

4.1 In view of theorem 3.1 the condition (1.1) become

$$(4.1) \qquad p(x,y) \leq f(\| x+y \|)$$

4.2 Lemma 2.3 is essentially a Lorch type of condition [5], and it is related to M. M. Day's condition [6] in the same way as the original Lorch condition is to that of P. Jordan and J von Neumann [7].

The condition (2.4) of lemma 2.3 is a direct generalization of the well known parallelogram law [7], and it is more natural then that of Senechalle [8]. One can get the contion (2.4) from Senechalle's condition but one has first to derive and solve the functional equation

$$f + f \circ g = 1$$

for given g G. As one may infer from lemma 2.2 that solving for f is more difficult than the more direct technique of lemma 2.3.

For $h(t) = -\lg\left[t^2/4(1-e^{-2}) + e^{-2}\right]$, the condition (2.5) reduces to that of M. M. Day.

4.3 If in theorem 3.1 we omit the condition that $f \in F$, we still can get some restrictions about N.

Let f be continuous at 0 and 2, f(2) = 1, and let 0 be the only point such that f(0) = 0. We have the following result.

<u>Theorem 4.1</u> Let N be a normed real linear space, $S = \left\{ x \,\middle|\, \|x\| = 1 \right\}$ and let f be as above. If $(S, f(\|x+y\|))$ is a probability space of dimension 2, then N is locally uniformly convex.

4.4 It appears that if we impose limitations of the Piron kind or the Mielnik Kind, the space in which representation is taking place undergoes some restrictions, which range from locally uniformly convex (Theorem 4.1) to the existence of an inner product. (e.g., Theorem 3.1)

It should be pointed out that this remark concerns only Mielnik's probability spaces of dimension 2. One can show that in the case of Mielnik's probability space of dimension ≥ 3, and $p(x,y) = f(\|x+x\|)$, where f is as in theorem 4.1, it follows that N cannot be locally uniformly convex.

4.5 Consider a linear topological space T and a suitably chosen subset S of T. One could try to find the conditions for p(x,y) such that if (S,p) is a probability space then T is a normed linear space. For some indications is this direction see the second Mielnik Paper [9].

4.6 The inequality (4.1) indicates that p(x,y) might be related to the semi-inner product in the sense of G. Lumer [10] and J. R. Giles [11]. It would be of interest to know what the implications might be of the relations between the geometry of quantum states and characterizations of semi-inner product spaces.

REFERENCES

1. G. Birkhoff and J. von Neumann; Ann. Math. 37, 823-843 (1936).
2. C. Piron; Helv. Phys. Acta. 37, 439-468 (1964).
3. B. Mielnik, Geometry of Quantum States, Commun. Math. Physics 9, 55-80 (1968).
4. C. V. Stanojević, Mielnik's Probability Manifolds and Inner Product Spaces; Bulletin De L'Académie Polonaise Des Sciences, Série Des Sciences Math. Astron. Et Phys. vol XVIII, No. 9, 1970.
5. E. R. Lorch, On certain implications which characterize Hilbert space; Ann. of Math. (2) (1948) 523-532.
6. M. M. Day, Some Characterizations of Inner Product Spaces, Trans. Amer. Math. Soc. 62 (1947) 320-337.
7. P. Jordan and J. von Neumann, On inner products in linear spaces; Ann. of Math. (2) 36 (1935), 719-732.
8. D. A. Senechalle, A Characterization of Inner Product Spaces; Proc. Amer. Math. Soc. vol. 19, No. 6 (1968), 1306-1312.
9. B. Mielnik, Theory of filters; Commun. Math. Phys. 15, 1-46, (1969).
10. G. Lumer, Semi-inner point product spaces, Trans. Amer. Math. Soc. 100 (1961) 29-43.
11. J. R. Giles, Classes of semi-inner product spaces; Trans. Amer. Math. Soc. 129 (1967) 436-446.

University of Missouri - Rolla, Missouri 65401