David Ruelle

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Les rencontres physiciens-mathématiciens de Strasbourg - RCP25, 1969, tome 9
« Conférences de J. Carmona, M. Froissart, D.W. Robinson et D. Ruelle », , exp. n° 4,
p. 1-54
<http://www.numdam.org/item?id=RCP25_1969__9__A4_0>

INTEGRAL REPRESENTATION OF STATES ON A C^*-ALGEBRA

David RUELLE
Institut des Hautes Etudes Scientifiques
91. BURES-sur-YVETTE - France

Abstract. Let E be the compact set of states on a C^*-algebra $\mathfrak{A}$ with identity. We discuss the representations of a state ρ as barycenter of a probability measure μ on E. Examples of such representations are the central decomposition and the ergodic decomposition. They are associated with an abelian von Neumann algebra $\mathfrak{B}$ in the commutant $\pi(\mathfrak{A})'$ of the image of $\mathfrak{A}$ in the representation canonically associated with ρ. This situation is studied in general and a number of applications are discussed.

0. Introduction

Let $\mathfrak{A}$ be a C^*-algebra with identity, E the set of states on $\mathfrak{A}$. In a number of situations of mathematical physics, a state ρ is "decomposed" into other states σ, i.e. ρ is exhibited as the resultant of a probability measure μ on E, or ρ has an integral representation of the form

$$\rho = \int_E \mu(d\sigma)\, \sigma$$

The measure μ is usually defined through a von Neumann algebra $\mathfrak{B}$ in the Hilbert space of the cyclic representation π canonically associated with ρ; $\mathfrak{B}$ is abelian and contained in the commutant $\pi(\mathfrak{A})'$ of the image of $\mathfrak{A}$. In Section 1 we describe the relation between $\mathfrak{B}$ and μ. In Section 2 we show, under certain separability conditions, how $\mathfrak{B}$ is diagonalized by a direct Hilbert space integral. In the following sections we consider some examples: decomposition of states invariant under a group into ergodic states, central decomposition, etc.

One can often (under suitable separability assumptions) show that μ is carried by a special class of states: ergodic states, factor states, etc. Otherwise, the various decompositions have their particular problems and properties. For instance in the case of the ergodic decomposition of a G-invariant state ρ on a G-abelian algebra, the mapping $\rho \to \mu_\rho$ is affine, but for other decompositions (e.g. central) such a property does not hold in general.

There is quite a bit of recent literature on the subject matter of this article, besides the classical literature on ergodic theory and dynamical systems (which deals essentially with the case of abelian $\mathfrak{A}$). In order to be reasonably readable, informative and self-contained, we have included here a relatively large amount of material which is not original (in particular much of Section 3). The main results of this work are the general theory of Sections 1 and 2 and the study of "multiperiodic" decomposition in Section 4 and decomposition "at infinity" in Section 5. Section 4 presents an extension of the theory of dynamical systems with discrete spectrum; in particular Theorem 4.1 shows that the "equicontinuous part" of the action of a locally compact abelian group can be so to say isolated and exhibited as translations on a torus. In Section 5 we consider C^*-algebras with "quasi-local" structure. In such an algebra it makes some sense to say that two elements A, A' are "far away" ; a state σ may be called clustering if $\sigma(A A')$ is close to $\sigma(A).\sigma(A')$ when A and A' are far away. Theorems 5.3 and 5.4 say essentially that every state ρ has a natural decomposition into clustering states.

For the organization of the article, we mention that Section 5 and 6.1, 6.2, 6.3 may be read independently after Section 2. A number of results used in the present work have been collected in Appendix A for easy reference. On the other hand the reader is assumed to be familiar with the basic results on von Neumann algebras and C^*-algebras. Appendix B contains technical developments needed in Section 2.

1. General theorems.

Throughout this note we use the following notation and assumptions. $\mathfrak{A}$ is a C^*-algebra with identity, $\mathfrak{A}'$ is the dual of $\mathfrak{A}$ with the w^*-topology, $E \subset \mathfrak{A}'$ is the (compact) set of states on $\mathfrak{A}$. If $A \in \mathfrak{A}$, the function $\hat{A}$ on E is defined by

$$\hat{A}(\sigma) = \sigma(A)$$

A fixed state $\rho \in E$ is chosen; the canonical cyclic representation associated with ρ is $(\mathfrak{H}, \pi, \Omega)$[*)].

1.1. Theorem. (a) *Let the von Neumann algebra* $\mathfrak{B}$ *satisfy*

$$\mathfrak{B} \subset \pi(\mathfrak{A})' \qquad\qquad \mathfrak{B} \subset \mathfrak{B}' \tag{1.1}$$

Then the orthogonal projection P *on the closure of* $\mathfrak{B}\,\Omega$ *in* $\mathfrak{H}$ *is such that*

$$P\,\Omega = \Omega \qquad\qquad P\,\pi(\mathfrak{A})P \subset [P\,\pi(\mathfrak{A})P]' \tag{1.2}$$

(b) *Let* P *be an orthogonal projection in* $\mathfrak{H}$ *satisfying* (1.2), *then the von Neumann algebra* $\mathfrak{B} = [\pi(\mathfrak{A}) \cup \{P\}]'$ *satisfies* (1.1).

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*) In this triple $\mathfrak{H}$ is a complex Hilbert space, π a representation of $\mathfrak{A}$ in $\mathfrak{H}$, $\Omega \in \mathfrak{H}$, and the following conditions are satisfied

(i) $\|\Omega\| = 1$

(ii) $\pi(\mathfrak{A})\Omega$ is dense in $\mathfrak{H}$ (Ω is a cyclic vector for $\pi(\mathfrak{A})$)

(iii) $(\forall A \in \mathfrak{A}) \qquad \rho(A) = (\Omega, \pi(A)\Omega)$.

(c) The relations between $\mathfrak{B}$ and P established by (a) and (b) are the inverse of each other.

Let the von Neumann algebra $\mathfrak{B}$ satisfy (1.1) and let P be the orthogonal projection on the closure of $\mathfrak{B}\Omega$ in $\mathfrak{H}$. We note the following facts

(i) $P \in \mathfrak{B}'$

[Let $B, B_1 \in \mathfrak{B}$, we have $B PB_1 \Omega = B B_1\Omega = PBB_1\Omega = PBPB_1\Omega$ and, since $\mathfrak{B}\Omega$ is dense in $P\mathfrak{H}$, $BP = PBP$. Therefore $BP = PB$].

(ii) Multiplication by P yields an isomorphism

$$[\pi(\mathfrak{A}) \cup \{P\}]' \longrightarrow P[\pi(\mathfrak{A}) \cup \{P\}]'$$

[Let $B \in [\pi(\mathfrak{A}) \cup \{P\}]'$, then $BP = 0 \Rightarrow B\,\Omega = 0 \Rightarrow B\pi(\mathfrak{A})\Omega = 0 \Rightarrow B = 0$].

(iii) $\quad P[\pi(\mathfrak{A}) \cup \{P\}]' = P[P\,\pi(\mathfrak{A})P]'$

[This follows from the formula $(\mathcal{A}')_P = (\mathcal{A}_P)'$ (see A.1) with $\mathcal{A} = [\pi(\mathfrak{A}) \cup \{P\}]''$].

(iv) $P\mathfrak{B} = P(P\mathfrak{B})' = P[P\pi(\mathfrak{A})P]'' = P[P\pi(\mathfrak{A})P]'$

[The restriction of $P\mathfrak{B}$ to $P\mathfrak{H}$ is abelian and has the cyclic vector Ω; by A.2 it is thus equal to its commutant. Thus $P\mathfrak{B} = P(P\mathfrak{B})'$. The set $P\pi(\mathfrak{A})P$ restricted to $P\mathfrak{H}$ commutes with $P\mathfrak{B}$, and has the cyclic vector Ω, therefore

$$P(P\mathfrak{B})' \supset P[P\pi(\mathfrak{A})P]'' \quad \text{or} \quad P[P\pi(\mathfrak{A})P]' \supset P\mathfrak{B}$$

and, by A.2, $P[P\pi(\mathfrak{A})P]'' = P[P\pi(\mathfrak{A})P]'$].

(v) $\mathfrak{B} = [\pi(\mathfrak{A}) \cup \{P\}]'$

[(i) yields $\mathfrak{B} \subset [\pi(\mathfrak{A}) \cup \{P\}]'$, (iii) and (iv) yield $P\mathfrak{B} = P[\pi(\mathfrak{A}) \cup \{P\}]'$, it suffices then to apply (ii)].

Part (a) of the theorem and one half of part (c) follow from (iv) and (v) respectively.

Let now P be an orthogonal projection in $\mathfrak{h}$ satisfying (1.2). We note the following facts.

(iv) $P[P\pi(\mathcal{A})P]'' = P[P\pi(\mathcal{A})P]'$

[By A.2 because the restriction of $[P\pi(\mathcal{A})P]''$ to $P\mathfrak{h}$ is abelian and has the cyclic vector Ω].

(vii) $P[P\pi(\mathcal{A})P]' = P[\pi(\mathcal{A}) \cup \{P\}]'$

[The proof is the same as for (iii)]

(viii) Multiplication by P yields an isomorphism

$$[\pi(\mathcal{A}) \cup \{P\}]' \longrightarrow P[\pi(\mathcal{A}) \cup \{P\}]'$$

[The proof is the same as for (ii)]

(ix) The closure of $[\pi(\mathcal{A}) \cup \{P\}]'\Omega$ is the range of P .

[Because $[\pi(\mathcal{A}) \cup \{P\}]'\Omega = P[\pi(\mathcal{A}) \cup \{P\}]'\Omega = P[P\pi(\mathcal{A})P]''\Omega \supset P\pi(\mathcal{A})\Omega$ by (vi), (vii)].

It follows from (vi), (vii), (viii) that $[\pi(\mathcal{A}) \cup \{P\}]'$ is abelian, proving part (b) of the theorem. The second half of part (c) follows from (ix).

1.2. Theorem. Let $\mathfrak{B}$ and P be as in Theorem 1.1.

(a) Multiplication by P yields an isomorphism $\mathfrak{B} \longrightarrow P\mathfrak{B}$

(b) $P\mathfrak{B} = P(P\mathfrak{B})' = P[P\pi(\mathcal{A})P]'' = P[P\pi(\mathcal{A})P]'$

(c) There is a morphism $\alpha : \mathcal{C}(E) \longrightarrow \mathfrak{B}$ of C^*-algebras such that $P\,\alpha(\hat{A}) = P\pi(A)P$ for all $A \in \mathcal{A}$. This morphism is unique, its image is strongly dense in $\mathfrak{B}$.

$$
\begin{tikzcd}
\mathscr{C}(E) \arrow[r, "\alpha"] & \mathfrak{B} \arrow[r, "\simeq"] & P\,\mathfrak{B} \\
\mathfrak{A} \arrow[u, "\wedge"] \arrow[rr] & & P\pi(\mathfrak{A})P \arrow[u, "\subset"']
\end{tikzcd}
\tag{1.3}
$$

Part (a) and (b) of the theorem follow respectively from (ii) and (iv) in the proof of Theorem 1.1.

To prove (c) let first $A_1,\dots,A_n$ be self-adjoint elements of $\mathfrak{A}$ and $\mathcal{P}$ be a complex-polynomial in n-variables. Consider a simultaneous spectral decomposition of $P\pi(A_1)P,\dots,P\pi(A_n)P$:

$$P = \int F(dx_1\dots dx_n)$$

$$P\pi(A_k)\,P = \int x_k\, F(dx_1\dots dx_n)$$

We have then

$$
\begin{aligned}
&\|\mathcal{P}\,(P\pi(A_1)P,\dots,P\pi(A_n)P)\| \\
&= \Big\|\int \mathcal{P}(x_1,\dots,x_n)\; F(dx_1\dots dx_n)\Big\| \\
&\le \sup_{\Psi\,\in\, P\mathfrak{h},\ \|\Psi\|=1} \big|\mathcal{P}\,(\Psi,\ \pi(A_1)\Psi),\dots,(\Psi,\ \pi(A_n)\Psi))\big| \\
&\le \sup_{\sigma\,\in\, E}\ \big|\,\mathcal{P}\,(\sigma(A_1),\dots,\sigma(A_n))\big| = \|\mathcal{P}(\hat{A}_1,\dots,\hat{A}_n)\|
\end{aligned}
\tag{1.4}
$$

The polynomials $\mathcal{P}\,(\hat{A}_1,\dots,\hat{A}_n)$ are dense in $\mathscr{C}(E)$ and therefore (1.4) implies the existence of a unique morphism β: $\mathscr{C}(E) \to P\,\mathfrak{B}$ such that

$$\beta(\hat{A}) \;=\; P\;\pi(A)P$$

The image of β is strongly dense in $P\mathfrak{B}$. In view of (a) there is a unique morphism $\alpha : \mathscr{C}(E) \longrightarrow \mathfrak{B}$ such that for all $\varphi \in \mathscr{C}(E)$,

$$\beta(\varphi) = P\;\alpha(\varphi)$$

If the $\beta(\varphi)$ are uniformly bounded and converge strongly to PB, the $\alpha(\varphi)$ are uniformly bounded and for each $A \in \mathfrak{A}$ the $\alpha(\varphi)\pi(A)\Omega = \pi(A)\beta(\varphi)\Omega$ converge, hence the $\alpha(\varphi)$ converge strongly to P, proving part (c) of the theorem.

1.3. Theorem. (a) A probability measure μ on E is defined by

$$\mu(\varphi) = (\Omega, \alpha(\varphi)\,\Omega) \tag{1.5}$$

The resultant of μ is ρ.

(b) There is a unique mapping $\tilde{\alpha} : L^{\infty}(E, \mu) \to \mathfrak{B}$ such that

1. if $\varphi \in \mathcal{C}(E)$, then $\tilde{\alpha}(\varphi) = \alpha(\varphi)$ *)
2. $\tilde{\alpha}$ is continuous from the topology of weak dual of $L^1(E, \mu)$ on $L^{\infty}(E, \mu)$ to the weak operator topology on $\mathfrak{B}$.

The mapping $\tilde{\alpha}$ is onto, is an isomorphism of C^*-algebras and, for every $A \in \mathfrak{A}$, $\psi \in L^{\infty}(E, \mu)$,

$$\mu(\hat{A}\,\psi) = (\Omega, \pi(A)\,\tilde{\alpha}(\psi)\,\Omega) \tag{1.6}$$

Part (a) is checked immediately. We prove (b).
Let $\mathfrak{X} = \alpha(\mathcal{C}(E))$, X the spectrum of $\mathfrak{X}$, $B : \mathcal{C}(X) \to \mathfrak{X}$ the inverse of the Gel'fand isomorphism. We may identify X to a subset of E such that $\alpha(\varphi) = B(\varphi|_X)$. Then $\operatorname{supp}\mu = X$ and (b) follows from A.3.

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*) That is, if p is the canonical mapping $\mathcal{C}(E) \to L^{\infty}(E, \mu)$, then $\alpha = \tilde{\alpha} \circ p$.

1.4. Corollary. Let $\{B_j\}$ be a finite set of positive elements of $\mathfrak{B}$ such that $\Sigma B_j = 1$. We define $\alpha_j \geq 0$ and $\sigma_j \in E$ by

$$\alpha_j = (\Omega, B_j \Omega) \qquad \alpha_j \sigma_j(A) = (\Omega, \pi(A)B_j \Omega)$$

and introduce a probability measure $\mu_{\{B_j\}} = \sum_j \alpha_j \delta_{\sigma_j}$ on E (δ_σ is the unit mass at σ). If $\{B'_k\}$ is the set of partial sums corresponding to some partition of $\{B_j\}$ we write $\{B_j\} \geq \{B'_k\}$. Given two sets $\{B'_k\}$, $\{B''_\ell\}$, there exists $\{B_j\} \geq \{B'_k\}$, $\{B''_\ell\}$ (take $\{B_j\} = \{B'_k B''_\ell\}$). The directed system $(\mu_{\{B_j\}})$ converges to μ in the vague topology of measures on E *). This follows from Theorem 1.3 (b) and A.4. If $\{B'_k\} \leq \{B_j\}$ then, using the order $\prec$ of Bishop-de Leeuw (see A.5) we have $\mu_{\{B'_k\}} \prec \mu_{\{B_j\}} \prec \mu$.

1.5. Corollary. Let $\tilde{\mathfrak{B}}$ be an abelian von Neumann algebra $\subset \pi(\mathfrak{A})'$. If we associate with it a measure $\tilde{\mu}$ on E by the above theory, we have $(\tilde{\mathfrak{B}} \subset \mathfrak{B}) \Leftrightarrow (\tilde{\mu} \prec \mu)$.

[Corollary 1.4 shows that $(\tilde{\mathfrak{B}} \subset \mathfrak{B}) \Rightarrow (\tilde{\mu} \prec \mu)$. Conversely, if $\tilde{\mu} \prec \mu$, theorem 1 of [6]**) shows that if $\tilde{\psi} \in \mathcal{C}(E)$ there exists $\psi \in L^\infty(\mu)$ such that for all $A \in \mathfrak{A}$

$$\mu(\hat{A}\,\psi) = \tilde{\mu}(\hat{A}\,\tilde{\psi})$$

By (1.6), this gives $\tilde{\alpha}(\tilde{\psi}) = \tilde{\alpha}(\psi)$, hence $\tilde{\mathfrak{B}} \subset \mathfrak{B}$]

*) i.e. in the w^*-topology of the space of measures considered as dual of $\mathcal{C}(E)$.

**) Reference [6] was pointed out to the author by J. Dixmier.

If E *is metrizable and* $\tilde{\mu} \prec \mu$ *there is a family* $(T_\sigma)_{\sigma \in E}$ *of probability measures on* E *such hat*

(a) *the resultant of* T_σ *is* σ

(b) *if* $\varphi \in \mathcal{C}(E)$ *then* $\sigma \to T_\sigma(\varphi)$ *is a Borel function, and*

$$\mu(\varphi) = \int_E T_\sigma(\varphi) \quad \tilde{\mu}(d\sigma) \tag{1.7}$$

[This results from Theorem 2 of [6]]

Formula (1.7) may be written $\mu = \int T_\sigma \tilde{\mu}(d\sigma)$ and shows that if $\tilde{\mathfrak{B}} \subset \mathfrak{B}$, the decomposition of ρ associated with $\mathfrak{B}$ may be accomplished in two steps, via the decomposition associated with $\tilde{\mathfrak{B}}$.

1.6. *Sources.* The use of (1.5) as definition of a measure μ giving an integral representation of ρ appears in Ruelle [34] for the case of ergodic decomposition; a form of the same idea is already present in Sakai [37] for central decomposition. Further references are given for each specific application. A version of Theorems 1.2 and 1.3 for the case $\mathfrak{B} \subset \pi(\mathfrak{A})' \cap \pi(\mathfrak{A})''$ has been obtained independently of the present work by Doplicher, Guichardet and Kastler [13].

2. Reduction theory *)

In this Section we let $\mathfrak{A}$, P be as in Section 1 and we make the following separability assumption.

Condition S. For $k = 1,\dots,n$ there are countable families $(\mathcal{A}_{\alpha_1\dots\alpha_k})$ and $(\mathcal{J}_{\alpha_1\dots\alpha_k})$ of sub-C^*-algebras of $\mathcal{A}$ such that

(i) $\mathcal{A}_{\alpha_1\dots\alpha_k\alpha_{k+1}} \subset \mathcal{J}_{\alpha_1\dots\alpha_k}$,

(ii) $\cup_{\alpha_{k+1}} \mathcal{A}_{\alpha_1\dots\alpha_k\alpha_{k+1}}$ is dense in $\mathcal{J}_{\alpha_1\dots\alpha_k}$ and $\cup_{\alpha_1} \mathcal{A}_{\alpha_1}$ is dense in $\mathcal{A}$,

(iii) $\mathcal{J}_{\alpha_1\dots\alpha_k}$ is a closed two-sided ideal of $\mathcal{A}_{\alpha_1\dots\alpha_k}$,

(iv) $\mathcal{J}_{\alpha_1\dots\alpha_n}$ is separable,

(v) the restriction of ρ to each $\mathcal{J}_{\alpha_1\dots\alpha_n}$ has norm 1.

Define

$$\mathcal{F}_{\alpha_1\dots\alpha_n} = \{\sigma \in E : \text{the restriction of } \sigma \text{ to } \mathcal{J}_{\alpha_1\dots\alpha_n} \text{ has norm } 1\}$$

$$\mathcal{F} = \cap_{\alpha_1\dots\alpha_n} \mathcal{F}_{\alpha_1\dots\alpha_n}$$

Let also (A_i) be a sequence in $\mathcal{A}$ such that each $\mathcal{J}_{\alpha_1\dots\alpha_n}$ contains a dense subsequence. We shall denote by $(\mathfrak{H}_\sigma, \pi_\sigma, \Omega_\sigma)$ the cyclic representation of $\mathcal{A}$ associated with $\sigma \in E$. It is convenient to think of a special case of condition S, namely that of separable $\mathcal{A}$. We may then take $\mathcal{F} = E$ and for (A_i) any dense sequence in $\mathcal{A}$. The further complications which arise in the general case are dealt with in Appendix B (Proposition B.3).

*) See Dixmier [8] Ch.2, and [9] Appendix A.

Let $\sigma \in \mathcal{F}$, for any A_i, A_j in the sequence (A_i) define $\Psi_{ij} \in \mathfrak{h}$, $\Psi_{ij}(\sigma) \in \mathfrak{h}_\sigma$ by

$$\Psi_{ij} = \pi(A_i) P \pi(A_j)\, \Omega \tag{2.1}$$

$$\Psi_{ij}(\sigma) = \sigma(A_j)\, \pi_\sigma(A_i)\, \Omega_\sigma \tag{2.2}$$

The vectors Ψ_{ij} (resp. $\Psi_{ij}(\sigma)$) are dense in $\mathfrak{h}$, resp. $\mathfrak{h}(\sigma)$. With the help of the family $(\Psi_{ij}(\cdot))$ a <u>direct Hilbert space integral</u>

$$\int^{\oplus} \mu\,(d\sigma)\, \mathfrak{h}_\sigma \tag{2.3}$$

may be constructed[*)]. It is the Hilbert space consisting of functions $\Phi : \sigma \in E \longrightarrow \mathfrak{h}_\sigma$ such that, for every i, j, the complex function $\sigma \rightarrow (\Psi_{ij}(\sigma),\ \Phi(\sigma))$ is μ-measurable and $\sigma \longrightarrow \|\Phi(\sigma)\|$ is square-integrable; the norm is $[\int \mu(d\sigma) \|\Phi(\sigma)\|^2\ ]^{1/2}$. The Hilbert space (2.3) does not depend on the choice of (A_i). It follows from (2.1), (2.2) and (1.5) that

$$(\Psi_{i'j'},\ \Psi_{ij}) = \int \mu(d\sigma)\ (\Psi_{i'j'}(\sigma), \Psi_{ij}(\sigma))$$

There is thus a linear isometry of $\mathfrak{h}$ into $\int^{\oplus} \mu(d\sigma) \mathfrak{h}_\sigma$ extending $\Psi_{ij} \longrightarrow \Psi_{ij}(\cdot)$. This isometry is onto : suppose that we have

$$0 = \int \mu(d\sigma)(\Psi_{ij}(\sigma), \Phi(\sigma)) = \int \mu(d\sigma)\ \hat{A}_j(\sigma)^{*}\ (\pi_\sigma(A_i)\ \Omega_\sigma, \Phi(\sigma))$$

Since the $\hat{A}_j$ are dense in $L^2(E,\mu)$ [by (1.5) the mapping $\varphi \rightarrow \alpha(\varphi)\Omega$ is isometric from $L^2(E,\mu)$ to $P\mathfrak{h}$, the continuous φ are dense in $L^2(E,\mu)$ and the $P\pi(A)\Omega$ are dense in $P\mathfrak{h}$, therefore the $\hat{A}$ are dense in $L^2(E,\mu)$] we obtain μ-almost every where

*) See Dixmier [8] Ch. 2, § 1, Proposition 4.

$$(\pi_\sigma(A_i)\Omega_\sigma\ ,\ \Phi(\sigma)) = 0$$

and therefore

$$\Phi(\sigma) = 0$$

For each σ, let $T(\sigma)$ be a bounded operator on $\mathfrak{H}_\sigma$; for every i, j, i', j' let $\sigma \longrightarrow (\Psi_{i'j'}(\sigma), T(\sigma)\Psi_{ij}(\sigma))$ be measurable and let $\sigma \longrightarrow \|T(\sigma)\|$ be essentially bounded.

There is an operator*)

$$T = \int^{\oplus} \mu(d\sigma)\ T(\sigma)$$

such that, if $\Phi = \int^{\oplus} \mu(d\sigma)\Phi(\sigma)$, then $T\Phi = \int^{\oplus} \mu(d\sigma)\ T(\sigma)\Phi(\sigma)$.

If $T(\sigma)$ is a multiple $\lambda(\sigma)$ of the identity for all σ , then T is called <u>diagonalizable</u> ; if λ is continuous, T is called <u>continuously diagonalizable</u> .

2.1. <u>Theorem.</u> <u>There is a unique identification</u>

$$\mathfrak{H} = \int^{\oplus} \mu(d\sigma)\,\mathfrak{H}_\sigma \tag{2.4}$$

<u>such that</u>

$$\Omega = \int^{\oplus} \mu(d\sigma)\ \Omega_\sigma \tag{2.5}$$

<u>and for all</u> $A \in \mathfrak{A}$

$$\pi(A) = \int^{\oplus} \mu(d\sigma)\ \pi_\sigma\ (A) \tag{2.6}$$

<u>With this identification,</u> $\mathfrak{B}$ <u>becomes the von Neumann algebra of diagonalizable operators, in particular</u>

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*) See Dixmier [8] Ch. 2, § 2.

$$\alpha(\hat{A}) = \int^{\oplus} \mu(d\sigma)\ \hat{A}(\sigma) \tag{2.7}$$

If we identify $\mathfrak{H}$ and $\int^{\oplus} \mu(d\sigma)\, \mathfrak{H}_\sigma$ by the isometry extending $\Psi_{ij} \longrightarrow \Psi_{ij}(\cdot)$ which we discussed above, we have

$$\alpha(\hat{A}')\pi(A)\ \Omega = \pi(A)\ P\pi(A')\ \Omega = \int \mu(d\sigma)\ \sigma(A')\ \pi_\sigma(A)\ \Omega_\sigma \tag{2.8}$$

for any A, A' in the sequence (A_j) and therefore for any $A, A' \in \mathfrak{A}$ (the sequence may be enlarged to include them), (2.5), (2.6) and (2.7) follow from (2.8). The identification (2.4) is uniquely determined by (2.5), (2.6) because $\pi(\mathfrak{A})\Omega$ is dense in $\mathfrak{H}$. The von Neumann algebra $\mathfrak{B}$ is the strong closure of $\alpha(\mathcal{C}(E))$ by Theorem 1.2(c), by (2.7) it is thus the weak closure of the algebra of continuously diagonalizable operators, which is precisely the von Neumann algebra of diagonalizable operators[*)].

Let (T_i) be a sequence of bounded operators in $\mathfrak{H}$ such that

$$T_i = \int^{\oplus} \mu(d\sigma)\ T_i(\sigma)$$

If $\mathcal{N}_\sigma$ is the von Neumann algebra generated by the $T_i(\sigma)$, the operators of the form

$$T = \int^{\oplus} \mu(d\sigma)\ T(\sigma)$$

with $T(\sigma) \in \mathcal{N}_\sigma$ form a von Neumann algebra $\mathcal{N}$ which is said to be decomposable and is denoted by

$$\mathcal{N} = \int^{\oplus} \mu(d\sigma) \mathcal{N}_\sigma$$

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*) See Dixmier [8] Ch. 2, § 2, Proposition 8.

$\mathcal{N}$ is generated by the T_i and the diagonalizable operators*).

2.2. Theorem. (a) Let $\mathcal{N}$ be a decomposable von Neumann algebra :

$$\mathcal{N} = \int^{\oplus} \mu(d\sigma)\mathcal{N}_{\sigma}$$

Then $\mathcal{N}'$ is decomposable and

$$\mathcal{N}' = \int^{\oplus} \mu(d\sigma)\mathcal{N}'_{\sigma} \tag{2.9}$$

(b) Let $(\mathcal{N}_i)$ be a sequence of decomposable von Neumann algebras:

$$\mathcal{N}_i = \int^{\oplus} \mu(d\sigma)\, \mathcal{N}_{i\sigma}$$

Then

$$\cap_i \mathcal{N}_i = \int^{\oplus} \mu(d\sigma)\ (\cap_i \mathcal{N}_{i\sigma}) \tag{2.10}$$

This theorem is proved in Dixmier [7] (Ch. 2, § 3, Théorème 4) in the case of a (Radon) measure μ on a locally compact space with countable basis. The result hold however without countability hypothesis on E as follows from a paper by Effrös [18]**)

2.3. Sources. The direct integral $\int^{\oplus} \mu(d\sigma)\mathfrak{h}_{\sigma}$ was considered by Sakai [37] for the central decomposition of a state on a separable C^*-algebra. (The absence of separability condition in the note by Wils [42] on the same subject is puzzling.) The case of separable $\mathfrak{A}$, and $\mathfrak{B} \subset \pi(\mathfrak{A})' \cap \pi(\mathfrak{A})''$, is considered in [13].

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*) See Dixmier [8] Ch. 2, § 3.

**) This reference was pointed out to the author by J. Dixmier.

3. Ergodic decomposition.

Let G be a group and τ a representation of G in aut $\mathfrak{A}$. We define an action τ of G on E by

$$\tau_g \, \sigma(A) = \sigma(\tau_g^{-1} A) \tag{3.1}$$

and let $I \subset E$ be the set of G-invariant*) states, i.e., of states such that $\tau_g \, \sigma = \sigma$ for all $g \in G$.

We assume that $\rho \in I$; there is then a unique unitary representation U of G in $\mathfrak{H}$ such that

$$U(g) \, \Omega = \Omega \tag{3.2}$$

$$U(g) \, \pi(A) \, U(g^{-1}) = \pi(\tau_g A) \tag{3.3}$$

We let P be the orthogonal projection on the subspace of $\mathfrak{H}$ constituted of the vectors invariant under U; (3.2) yields

$$P \, \Omega = \Omega \tag{3.4}$$

3.1. Theorem**). The following conditions are equivalent

(a) $$P \, \pi(\mathfrak{A}) P \subset [P \, \pi(\mathfrak{A}) P]' \tag{3.5}$$

(b)***) Let $A_1, A_2 \in \mathfrak{A}$ and let $\Phi \in P \, \mathfrak{H}$. Then, given

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*) "τ-invariant" would be more correct but "G-invariant" will cause no confusion.

**) See Lanford and Ruelle [27].

***) One might in (b) suppose A_1, A_2 self-adjoint and/or replace the expectation value for Φ by a matrix element between $\Phi_1, \Phi_2 \in P \, \mathfrak{H}$.

$\varepsilon > 0$, <u>there exist</u> $\lambda_i \geq 0$, $g_i \in G$ <u>such that</u> $\Sigma \lambda_i = 1$ <u>and</u>

$$|(\Phi, [\sum_i \lambda_i \pi(\tau_{g_i} A_1), \pi(A_2)] \Phi)| < \varepsilon \tag{3.6}$$

The proof will result from the following facts

(i) If $\Psi_1, \Psi_2 \in \mathfrak{H}$ and $\varepsilon > 0$ there exist $\lambda_i \geq 0$, $g_i \in G$ such that $\Sigma \lambda_i = 1$ and, for $\alpha = 1, 2$,

$$\| \sum_{j,i} \lambda'_j \lambda_i U(g'_j g_i) \Psi_\alpha - P \Psi_\alpha \| < \varepsilon$$

where the $\lambda'_j \geq 0$, $g'_j \in G$ are arbitrary subject to $\Sigma \lambda'_j = 1$

[Using A.6, we may suppose $\| \sum_i \lambda_i U(g_i) \Psi_\alpha - P\Psi_\alpha \| < \varepsilon$ hence $\|(\sum_j \lambda'_j U(g'_j)) (\sum_i \lambda_i U(g_i)\Psi_\alpha - P\Psi_\alpha)\| < \varepsilon$].

(ii) Let $A_1, A_2 \in \mathfrak{A}$ be such that $\|A_1\| \leq 1$, $\|A_2\| \leq 1$.
Let $\Phi_1, \Phi_2 \in P \mathfrak{H}$, be such that $\|\Phi_1\| \leq 1$ $\|\Phi_2\| \leq 1$. Given $\varepsilon > 0$ one can find $\lambda_i \geq 0$, $g_i \in G$ such that $\Sigma \lambda_i = 1$ and

$$|(\Phi_1, [\pi(A_1)P\pi(A_2) - \pi(A_2) P\pi(A_1)]\Phi_2)$$

$$- (\Phi_1, [\sum_j \lambda'_j \pi(\tau_{g'_j} A'_1), \pi(A_2)] \Phi_2)| < \varepsilon$$

where $A'_1 = \sum_i \lambda_i \tau_{g_i} A_1$ and the λ'_j ; $g'_j \in G$ are arbitrary subject to $\Sigma \lambda'_j = 1$.

[This follows from (i) with $\Psi_1 = \pi(A_1^*)\Phi_1$, $\Psi_2 = \pi(A_1)\Phi_2$].

(iii) (a) $\Leftrightarrow$ (b)

[Notice that, by polarization, (a) is equivalent to "$(\Phi,[\pi(A_1) P\pi(A_2) - \pi(A_2)P\pi(A_1)]\Phi) = 0$ for all $\Phi \in P \mathfrak{H}$" . Putting $\lambda'_1 = 1$, $g'_1 = 1$ and $\Phi_1 = \Phi_2 = \Phi$ in (ii) yields the implication (a) $\Rightarrow$ (b) . To prove (b)$\Rightarrow$ (a) we use again (ii) : if (b) holds we may choose λ'_j, g'_j so that

$$|(\Phi, [\sum_j \lambda_j' \pi(\tau_{g_j'} A_1'), \pi(A_2)] \Phi)| < \varepsilon$$

and (a) follows].

3.2. Corollary. If the conditions of Theorem 3.1 are satisfied with respect to a closed subgroup H of G, they are satisfied with respect to G.

[This is immediately verified for (b)].

3.3. Corollary. The conditions of Theorem 3.1 are implied by the following

(c) Let $A_1, A_2 \in \mathfrak{A}$ and $\Phi \in P\mathfrak{H}$ then

$$\inf_{g\in G} |(\Phi, \pi([\tau_g A_1, A_2])\Phi)| = 0 \tag{3.7}$$

[This is immediately verified for (b)].

3.4. Theorem*). Consider the following conditions on the G-invariant state ρ.

(a) ρ is ergodic, i.e., ρ is an extremal point of I.

(b) The set $\pi(\mathfrak{A}) \cup U(G)$ is irreducible in $\mathfrak{H}$.

(c) P is one dimensional.

We have (a) $\Leftrightarrow$ (b) $\Leftarrow$ (c). If ρ satisfies the conditions of Theorem 3.1, then (a), (b), and (c) are equivalent.

The existence of a self-adjoint operator $C\in[\pi(\mathfrak{A})\cup U(G)]'$,

*) See [16], [34], [27].

such that $0 \leq C \leq 1$ and C is not a multiple of 1, is equivalent by A.7 to non (a) and non (b); thus (a) $\Leftrightarrow$ (b). If (c) holds, (1.2) is verified and (c) $\Rightarrow$ (b) by Theorem 1.2 (a). If the conditions of Theorem 3.1 are satisfied, (1.2) is verified and Theorem 1.1 gives (b) $\Rightarrow$ (c).

3.5. Proposition. *If* $A \in \mathfrak{A}$, *define*

$$\operatorname{conv}(\tau_G A) = \{ \Sigma \lambda_i \tau_{g_i} A : \lambda_i \geq 0, \Sigma \lambda_i = 1, g_i \in G \} \tag{3.8}$$

Then

$$\inf_{C \in \operatorname{conv}(\tau_G A)} \rho(C^* C) = (\Omega, \pi(A^*) P \pi(A) \Omega) \tag{3.9}$$

The proof results from A.6 and the inequality*)

$$\rho(C^* C) = \|\pi(C)\Omega\|^2 \geq \|P\,\pi(C)\Omega\|^2$$

$$= \|P\,\pi(A)\Omega\|^2 = (\Omega, \pi(A^*) P\,\pi(A)\,\Omega)$$

3.6. Theorem. *Let the conditions of Theorem 3.1 be satisfied, so that the theory of Section 1 applies***).

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*) This simple proof was communicated to the author by H. Araki.

**) It is interesting to notice that here $\mathfrak{B} = [\pi(\mathfrak{A}) \cup U(\cdot)]'$, we shall not make explicit use of this fact.

(a) The measure μ defined by (1.5) is the unique maximal measure on I (with respect to the order of Bishop- de Leeuw, see A.5) with resultant ρ .

(b) If the condition S of Section 2 is satisfied (e.g. if $\mathfrak{A}$ is separable), the measure μ is carried by ergodic states.

The proof results from the following facts

(i) supp $\mu \subset I$

[By Corollary 1.4, μ is limit of measures $\mu_{\{B_j\}}$ carried by finitely many points $\sigma_j \in E$ where

$$\sigma_j(A) = (\Omega, B_j \Omega)^{-1} (\Omega, \pi(A) B_j \Omega)$$

and $B_j \in \mathfrak{B}$; using $(\Omega, \pi(A)B_j \Omega) = (\Omega,\pi(A)P B_j \Omega)$ we find $\sigma_i \in I$]

(ii) (a) holds

[We have to show that if $\tilde{\mu}$ is any probability measure on I with resultant ρ and φ a convex continuous function on I , then $\tilde{\mu}(\varphi) \leq \mu(\varphi)$. In view of A.4 we may suppose that $\tilde{\mu}$ has finite support: $\tilde{\mu}(\varphi) = \sum \alpha_i \varphi(\rho_i)$ where $\alpha_i \geq 0$, $\rho_i \in I$, $\sum \alpha_i = 1$, $\sum \alpha_i \rho_i = \rho$, but then (see A.7) $\tilde{\mu}$ is of the form $\mu_{\{B_j\}}$ of Corollary 1.4 with $B_j \in \pi(\mathfrak{A})' \cap U(G)'$ and, since $U(G)' \subset \{P\}'$ by A.6, $B_j \in \mathfrak{B}$. Corollary 1.4 gives then $\tilde{\mu}(\varphi) \leq \mu(\varphi)$].

(iii) If $\sigma \in I$, let P_σ be the projection on the subspace of G-invariant vectors in $\mathfrak{H}_\sigma$. For any $A \in \mathfrak{A}$, the following quantity vanishes μ-almost every where in σ

$$(\pi_\sigma(A)\ \Omega_\sigma,\ P_\sigma\ \pi_\sigma(A)\ \Omega_\sigma) - |(\Omega_\sigma,\pi(A)\ \Omega_\sigma)|^2$$

[Since this quantity is a priori ≥ 0 , it suffices to remark that

$$\int \mu(d\sigma)[(\pi_\sigma(A)\Omega, P_\sigma\, \pi_\sigma(A)\Omega_\sigma) - \hat{A}^*(\sigma)\hat{A}(\sigma)]$$
$$= \int \mu(d\sigma)[\inf_{C \in \mathrm{conv}(\tau_G A)} \sigma(C^* C)] - (\Omega, \alpha(\hat{A}^*)\alpha(\hat{A})\Omega)$$
$$\leq \inf_{C \in \mathrm{conv}(\tau_G A)} \rho(C^* C) \quad (\Omega, \pi(A^*)P\pi(A)\Omega) = 0$$

where we have used twice Proposition 3.5].

(iv) (b) holds.

[In view of Proposition B.3 (a) the sequence $\pi_\sigma(A_i)\Omega_\sigma$ is dense in $\mathfrak{H}_\sigma$ μ-almost every where, and (iii) shows that P_σ is almost every where the projection on Ω_σ]

3.7. G-abelian algebras.

If the conditions of Theorem 3.1 are satisfied, the integral representation of ρ given by μ will be called ergodic decomposition (this terminology is justified by Theorem 3.6 (b)). We shall say that $\mathfrak{A}$ is G-abelian if the conditions of Theorem 3.1 are satisfied for every G-invariant state ρ. The following characterization is readily deduced from Theorem 3.1 : $\mathfrak{A}$ is G-abelian if and only if for all $\sigma \in I$ and $\epsilon > 0$ there exist $\lambda_i \geq 0$, $g_i \in G$ such that $\Sigma \lambda_i = 1$ and

$$|\sigma([\sum_i \lambda_i \tau_{g_i} A_1, A_2])| < \epsilon$$

3.8. Theorem*). If $\mathfrak{A}$ is G-abelian, then I is a simplex in the sense of Choquet (see A.5).

This follows immediately from Theorem 3.6(a) and the definitions.

3.9. Theorem**). Let $\mathfrak{A}$ be G-abelian and let $\mathfrak{B}$ be contained

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*) See Lanford and Ruelle [27].

**) This theorem was proved originally by Størmer [39] under the assumption that $\alpha(\hat{A})$ is contained in the strong operator closure of conv $\pi(\tau_G A)$ for each $A \in \mathfrak{A}$ and each invariant state ρ . Here we follow [36], Exercise 6.D.

<u>in</u> $\pi(\mathfrak{A})''$ <u>for each invariant state</u> ρ . <u>Then two ergodic states</u> ρ_1 <u>and</u> ρ_2 <u>cannot be quasi-equivalent if they are distinct.</u>

Let $(\mathfrak{H}_1, \pi_1, \Omega_1)$, $(\mathfrak{H}_2, \pi_2, \Omega_2)$ be the canonical cyclic representations associated with ρ_1 and ρ_2 . The states ρ_1, ρ_2 are called quasi-equivalent if there is an isomorphism δ of $\pi_1(\mathfrak{A})''$ onto $\pi_2(\mathfrak{A})''$ such that $\delta\, \pi_1(A) = \pi_2(A)$ when $A \in \mathfrak{A}$. Let now ρ_1, ρ_2 be ergodic, distinct, and take $\rho = \frac{1}{2}\rho_1 + \frac{1}{2}\rho_2$; by A.7 and A.6 there exist B_1 , $B_2 \in \mathfrak{B}$ with $0 \leq B_1$, $0 \leq B_2$, $B_1 + B_2 = 1$, and $\frac{1}{2}\rho_i(\cdot) = (\Omega, \pi(A)\, B_i\, \Omega)$. Since the ρ_i are ergodic we have $B_1 B_2 = 0$ so that B_1 and B_2 are mutually orthogonal projections, we may identify $\mathfrak{H}_i$ with the range of B_i in $\mathfrak{H}$ and write $\pi_i(\cdot)\Omega_i = \sqrt{2}.\pi(\cdot)B_i\,\Omega$. We have $B_i \in \mathfrak{R} \subset \pi(\mathfrak{A})''$, let thus $\pi(A) \longrightarrow B_1$, then

$$\pi(A).|_{\mathfrak{H}_2} \longrightarrow B_1|_{\mathfrak{H}_2} = 0|_{\mathfrak{H}_2}$$

But if ρ_1 and ρ_2 were quasi-equivalent we would have the contradiction

$$\pi(A)|_{\mathfrak{H}_2} = \delta\, \pi(A)|_{\mathfrak{H}_1} \longrightarrow \delta\ 1_{\mathfrak{H}_1} = 1_{\mathfrak{H}_2}$$

3.10. <u>Sources.</u> For the case of abelian $\mathfrak{A}$ (decomposition of an invariant measure into ergodic measures) see for instance Phelps [30] Section 10. For the extension to non-abelian $\mathfrak{A}$ see Ruelle [34] , and in a different spirit Kastler and Robinson [23] where an "abstract" decomposition is discussed. The present treatment largely follows Lanford and Ruelle [27] with some improvements in Theorem 3.6 and the addition of Theorem 3.9 ("Størmer's theorem" [39]). For further results see [25], [14], [40], [41], [33], [17] . A review and applications to statistical mechanics are given in [36] Ch 6 and 7. In the examples of ergodic decomposition which occur in statistical mechanics, G is typically the Euclidean group or the translation group in 3 dimensions; a G-ergodic state is interpreted as "pure thermodynamic phase" , and ergodic decomposition is the decomposition of a "mixture" into pure thermodynamic phases. In physical applications the algebra $\mathfrak{A}$ is not always separable, but the states of physical interest satisfy a form of condition S . For instance it may be that $\mathfrak{A}_\alpha$, $\mathfrak{J}_\alpha$ are sub-C^*-algebras of $\mathfrak{A}$ such that $\mathfrak{A}_\alpha$ is isomorphic to the bounded operators and $\mathfrak{J}_\alpha$ to the compact operators of some Hilbert space $\mathcal{H}_\alpha$; a state ρ which has a restriction of norm 1 to each $\mathfrak{J}_\alpha$ is then called <u>locally normal</u> (see [35], [20], [36] Ch 7).

4. Multiperiodic decomposition.

Let G be a locally compact abelian group noted multiplicatively. As in Section 3 we let τ be a representation of G in aut $\mathfrak{A}$, we assume that the state ρ is G-invariant and we let U be the unitary representation of G in $\mathfrak{h}$ satisfying (3.2) and (3.3). We assume that U is strongly continuous*) and we let $E(\cdot)$ be the spectral measure on the charactergroup $\hat{G}$ such that**)

$$U(g) = \int_{\hat{G}} X(g)\, E(dX) \tag{4.1}$$

Let $\mathfrak{X}$ be the subset of $\hat{G}$ consisting of the points X such that the corresponding projection does not vanish: $E(\{X\}) \neq 0$. For simplicity we write $E(\{X\}) = E[X]$. Then

$$\mathfrak{X} = \{X \in \hat{G} : E[X] \neq 0\} \tag{4.2}$$

We define the projection

$$P = \sum_{X\in\hat{G}} E[X] = \sum_{X\in\mathfrak{X}} E[X] \tag{4.3}$$

From (3.2) we obtain then

$$P\,\Omega = \Omega \tag{4.4}$$

It is known***) that the range $P\mathfrak{h}$ of P consists of the almost periodic vectors of $\mathfrak{h}$, i.e. of the vectors Ψ with a relatively compact orbit $U(G)\Psi$.

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*) If for each $A \in \mathfrak{A}$ and $\sigma \in E$ the function $g \longrightarrow \sigma(\tau_g A)$ is continuous on G , then it can be shown that U is strongly continuous.

**) The existence of $E(\cdot)$ is asserted by the S.N.A.G. theorem, see for instance Maurin [29] p. 218.

***) See for instance [21] Ch 1 §7.

4.1. Theorem. The following conditions are equivalent.

(a) $$P\pi(\mathfrak{A})P \subset [P\ \pi(\mathfrak{A})P]' \tag{4.5}$$

(b) Let $A_1, A_2 \in \mathfrak{A}$, and let $X_1, X_2, X_3 \in \hat{G}$, then

$$E[X_1]\pi(A_1)\ E[X_1\ X_3]\ \pi(A_2)\ E[X_2]$$
$$= E[X_1]\pi(A_2)\ E[X_3^{-1}\ X_2]\ \pi(A_1)\ E[X_2] \tag{4.6}$$

(c) Let $A_1, A_2 \in \mathfrak{A}$, let $\Phi_1, \Phi_2 \in P\mathfrak{H}$, and let $X \in \hat{G}$. Then, given $\varepsilon > 0$, there exist $\lambda_i \geq 0$, $g_i \in G$ such that $\Sigma\,\lambda_i = 1$ and

$$\left|(\Phi_1, [\sum_i \lambda_i\ X(g_i)^{-1}\pi(\tau_{g_i} A_1), \pi(A_2)]\ \Phi_2)\right| < \varepsilon \tag{4.7}$$

The proof will result from the following facts;

(i) If S is a finite subset of $\mathfrak{H}$, $X \in \hat{G}$ and $\varepsilon > 0$ there exist $\lambda_i \geq 0$, $g_i \in G$ such that $\Sigma\,\lambda_i = 1$ and

$$\left\| \sum_i \lambda_i\ X(g_i)^{-1}\ U(g_i)\ \Psi - E[X]\Psi \right\| < \varepsilon \tag{4.8}$$

for all $\Psi \in S$.

[Notice that $E[X]$ is the projection on the space of invariant vectors for the representation $g \rightarrow X(g)^{-1}\ U(g)$ of G in $\mathfrak{H}$. It suffices then to use A.6].

(ii) If (4.8) holds and if $\lambda'_j \geq 0$, $g'_j \in G$ are such that $\Sigma\,\lambda'_j = 1$, then $\left\| \sum_{i,j} \lambda_i\ \lambda'_j\ X(g_i\cdot g'_j)^{-1}\ U(g_i\cdot g'_j)\Psi - E[X]\Psi \right\| < \varepsilon$

[Because if (4.8) holds and $g \in G$, then

$$\left\| \sum_i \lambda_i\ X(g_i\cdot g)^{-1}\ U(g_i\cdot g)\Psi - E[X]\Psi \right\|$$
$$= \left\| X(g)^{-1}\ U(g)\ [\ \sum_i \lambda_i\ X(g_i)^{-1}\ U(g_i)\ \Psi - E[X]\ \Psi\right\| < \varepsilon\ ]$$

(iii) Let $A_1, A_2 \in \mathfrak{A}$ be such that $\|A_1\| \leq 1$, $\|A_2\| \leq 1$. Let $X_1, X_2, X_3 \in \hat{G}$ and $\Phi_1, \Phi_2 \in \mathfrak{H}$ be such that $\|\Phi_1\| \leq 1$, $\|\Phi_2\| \leq 1$ and $E[X_1]\Phi_1 = \Phi_1$, $E[X_2]\Phi_2 = \Phi_2$. Given $\epsilon > 0$, there exist $\lambda_i \geq 0$, $g_i \in G$ such that $\Sigma \lambda_i = 1$ and

$$|(\Phi_1, [\pi(A_1)\, E[X_1\, X_3]\, \pi(A_2) - \pi(A_2)\, E[X_3^{-1}\, X_2]\, \pi(A_1)]\Phi_2)$$

$$- (\Phi_1, [\sum_j \lambda_j'\, X_3(g_j')\, \pi(\tau_{g_j'}\, A_1'), \pi(A_2)]\Phi_2)| < \epsilon$$

where $A_1' = \sum_i \lambda_i\, X_3(g_i)\tau_{g_i}\, A_1$ and the $\lambda_j' \geq 0$, $g_j' \in G$ are arbitrary subject to $\Sigma \lambda_j' = 1$.

[In view of (i) and (ii) one can choose the λ_i, g_i such that

$$\| \sum_j \lambda_j' \sum_i \lambda_i\, X_1(g_i\, g_j')^{-1}\, X_3(g_i\, g_j')^{-1}\, U(g_i\, g_j')\, \pi(A_1^*)\Phi_1$$

$$- E[X_1\, X_3]\, \pi(A_1^*)\, \Phi_1\| < \epsilon/2$$

$$\| \sum_j \lambda_j' \sum_i \lambda_i\, X_3(g_i\, g_j')\, X_2(g_i\, g_j')^{-1}\, U(g_i\, g_j')\, \pi(A_1)\, \Phi_2$$

$$- E[X_3^{-1}\, X_2]\, \pi(A_1)\Phi_2\| < \epsilon/2$$

This yields immediately the result].

(iv) (c) $\Rightarrow$ (b) $\Rightarrow$ (a)

[(iii) yields the first implication, the second results from summation over $X_3, X_1, X_2 \in \hat{G}$ in (4.6)].

(v) (a) $\Rightarrow$ (b)

[Let $\Phi_1 \in E[X_1]\mathfrak{H}$, $\Phi_2 \in E[X_2]\mathfrak{H}$, (a) gives

$$(\Phi_1, \pi(\tau_g\, A_1)\, P\, \pi(A_2)\Phi_2) = (\Phi_1, \pi(A_2)\, P\pi(\tau_g\, A_1)\Phi_2) .$$

Writing $P = \sum_X E[X_1 X] = \sum_X E[X^{-1}\, X_2]$ yields then

$$\sum_{X \in \hat{G}} X(g)^{-1} (\Phi_1, \pi(A_1) E[X_1 X] \pi(A_2)\Phi_2)$$

$$= \sum_{X \in \hat{G}} X(g)^{-1} (\Phi_1, \pi(A_2) E[X^{-1} X_2] \pi(A_1)\Phi_2)$$

and (b) follows].

(vi) Let $\Phi_1 \in E[X_1]\mathfrak{h}$, $\Phi_2 \in E[X_2]\mathfrak{h}$ and $\varepsilon > 0$, (b) implies the existence of $\lambda_i \geq 0$, $g_i \in G$ such that $\Sigma \lambda_i = 1$ and

$$|(\Phi_1, [\sum_{i,j} \lambda_i \lambda_j' X(g_i g_j')^{-1} \pi(\tau_{g_i g_j'} A_1), \pi(A_2)] \Phi_2)| < \varepsilon$$

where the $\lambda_j' \geq 0$, $g_j' \in G$ are arbitrary subject to $\Sigma \lambda_j' = 1$.

[This follows directly from (iii)].

(vii) (b) $\Rightarrow$ (c)

[It suffices to prove (c) for the case of finite sums $\Phi_1 = \sum_X \Phi_1^X$, $\Phi_2 = \sum_X \Phi_2^X$ where Φ_1^X, $\Phi_2^X \in E[X]\mathfrak{h}$, and this follows from (vi)].

4.2. Corollary. <u>If the conditions of Theorem</u> 4.1 <u>are satisfied with respect to a closed subgroup</u> H <u>of</u> G, <u>they are satisfied with respect to</u> G.

[This is immediately verified for (c)].

4.3. Corollary. <u>The conditions of Theorem</u> 4.1 <u>are implied by the following</u>

(d) <u>Let</u> A_1, A_2 <u>be self-adjoint elements of</u> $\mathfrak{A}$ <u>and</u> $\Phi_1, \Phi_2 \in P\mathfrak{h}$, <u>then</u>

$$\inf_{g \in G} |(\Phi_1, \pi([\tau_g A_1, A_2])\Phi_2)| = 0 \tag{4.9}$$

[This is immediately verified for (c)].

4.4. Theorem*) *Let the conditions of Theorem* 4.1 *be satisfied, then*

(a) *The conditions of Theorem* 3.1 *are satisfied*

(b) $\mathcal{X} = \mathcal{X}^{-1}$

(c) *If* ρ *is ergodic, then* $E[X]$ *is one dimensional for every* $X \in \mathcal{X}$ *and* $\mathcal{X}$ *is a subgroup of* $\hat{G}$.

From Theorem 4.1 (b) we obtain

$$[E[X]\ \pi(A_1)\ E[X],\ E[X]\ \pi(A_2)\ E[X]] = 0 \tag{4.10}$$

$$E[1]\ \pi(A_1)\ E[X]\ \pi(A_2)\ E[1] = E[1]\ \pi(A_2)\ E[X^{-1}]\ \pi(A_2)\ E[1] \tag{4.11}$$

$$E[X_1]\ \pi(A_1)\ E[X_1\ X_2]\ \pi(A_2)\ E[X_2] = E[X_1]\ \pi(A_2)\ E[1]\ \pi(A_1)\ E[X_2] \tag{4.12}$$

Inserting $X = 1$ into (4.10) we obtain (3.5), proving (a). Part (b) of the theorem results from (4.11). By Theorem 3.4, the ergodicity of ρ implies the irreducibility of $\pi(\mathcal{O}\!\mathcal{l}) \cup U(G)$, therefore the algebra

$$E[X]\ [E[X]\ \pi(\mathcal{O}\!\mathcal{l})\ E[X]]''$$

restricted to the range of $E[X]$ is irreducible and since it is abelian by (4.10), $E[X]$ is one dimensional. In particular $E[1]$ is one dimensional and (4.12) gives $\mathcal{X} \cdot \mathcal{X} \subset \mathcal{X}$, which together with (b) proves that $\mathcal{X}$ is a group.

4.5. Equicontinuous actions.

If the conditions of Theorem 4.1 are satisfied, equations (4.4), (4.5) hold and therefore the theory of Section 1 applies. In particular

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*) See [23]

there is a natural integral representation of ρ given by a probability measure μ on E (see Theorem 1.3). We call this integral representation the <u>multiperiodic decomposition</u> of ρ. We shall show (Theorem 4.7) that if ρ is ergodic and $P\mathcal{H}$ separable, the action of G on the measure μ is equivalent to a certain equicontinuous action of G on the Haar measure m of a compact abelian group M. This will justify the phrase "multiperiodic decomposition".

Let K be a compact space and τ a <u>continuous action</u> of G on K, i.e., $\tau : G \times K \longrightarrow K$ is continuous and is a representation of G by homeomorphisms of K. We say that the action τ is <u>equicontinuous</u> if, for each $\varphi \in \mathcal{C}(K)$, the set $\{\varphi \circ \tau_g = g \in G\}$ is relatively compact in $\mathcal{C}(K)$.

Let $\hat{G}^*$ be obtained by replacing the original topology by the discrete topology on $\hat{G}$. The character group $\overline{G}$ of $\hat{G}^*$ is the <u>compact group associated with</u> G. Define a group isomorphism $\gamma : G \longrightarrow \overline{G}$ such that $(\gamma g)(X) = X(g)$ for all $X \in \hat{G}$, then γ is continuous and has dense image. For every continuous group homomorphism $\eta : G \longrightarrow H$ where H is compact, there is a continuous homomorphism $\bar{\eta} : \overline{G} \longrightarrow H$ such that $\eta = \bar{\eta}\gamma$ *).

4.6. <u>Theorem.</u> <u>Let</u> τ <u>be an equicontinuous action of</u> G <u>on the compact space</u> K.

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*) For a proof see [9] 16.1.

(a) There exists a continuous action $\bar{\tau}$ of $\bar{G}$ on K such that $\bar{\tau}_{\gamma g} = \tau_g$ if $g \in G$.

(b) If m is a probability measure on K, invariant and ergodic with respect to the action τ of G, then the support M of m in K is a homogeneous space of $\bar{G}$ (for the action $\bar{\tau}$) and m restricted to M is the Haar measure of this homogeneous space.

(c) Conversely, let $\bar{\tau}$ be a continuous action of $\bar{G}$ on a homogeneous space M and let m be the Haar measure on M. Then the action $g \longrightarrow \bar{\tau}_{\gamma g}$ of G on M is equicontinuous and m is ergodic with respect to it.

We prove successively the three parts of the theorem

(a) The equicontinuity of the action of G implies that the closure of the set of operators $T_g : \varphi \longrightarrow \varphi \circ \tau_g$ in $\mathscr{C}(K)$, with respect to the strong operator topology, is a compact group H *). Therefore there exists a continuous homomorphism $\bar{T}.: \bar{G} \longrightarrow H$ such that for all $g \in G$ we have $T_g = \bar{T}_{\gamma g}$. By continuity H consists of automorphisms of $\mathscr{C}(K)$; there is thus a homeomorphism $\bar{\tau}_{\bar{g}}$ of K such that

$$\bar{T}_{\bar{g}}\, \varphi = \varphi \circ \bar{\tau}_{\bar{g}}$$

The mapping $\bar{\tau} : \bar{G} \times K \longrightarrow K$ is continuous and $\bar{\tau}_{\gamma g} = \tau_g$ if $g \in G$

(b) If M were not a homogeneous space of $\bar{G}$, we could find $x, y \in M$ such that $x \notin \bar{\tau}_{\bar{G}}\, y$ (where $\bar{\tau}_{\bar{G}}\, y$ is compact). There would then

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*) See for instance Jacobs [22] p 112.

exist a compact neighbourhood L of x such that $L \cap \bar{\tau}_{\bar{G}}\, y = \emptyset$ or equivalently $y \notin \bar{\tau}_{\bar{G}} L$. Then $\bar{\tau}_{\bar{G}} L$ would b a compact set with x in its interior and $y \notin \bar{\tau}_{\bar{G}}\, L$. Because of the ergodicity of m with respect to the action $\bar{\tau}$ of $\bar{G}$, m would be carried by $\bar{\tau}_{\bar{G}}\, L$ or the complement of this set in M , in contradiction with the fact that M is the support of m .

If $x \in M$, the Haar measure m_x on M is defined by

$$m_x(\varphi) = \int_{\bar{G}} d\bar{g}\ \varphi(\bar{\tau}_{\bar{g}}\, x)$$

The measure m_x is independent of the choice of x because of the transitivity of $\bar{\tau}_{\bar{G}}$ on M and the invariance of the Haar measure on $\bar{G}$. Notice now that the invariance of m with respect to the action τ of G implies its invariance with respect to the action $\bar{\tau}$ of $\bar{G}$: $m(\varphi \circ \tau_g) = m(\varphi)$ gives by continuity $m(\varphi \circ \bar{\tau}_{\bar{g}}) = m(\varphi)$.
We have thus

$$m(\varphi) = \int_K m(dx)\ \varphi(x) = \int_{\bar{G}} d\bar{g}\Big[\int_K m(dx)\ \varphi(\bar{\tau}_{\bar{g}}\, x)\Big]$$

$$= \int_K m(dx)\ \Big[\int_{\bar{G}} d\bar{g}\ \ \varphi(\bar{\tau}_{\bar{g}}\, x)\Big] = m_x(\varphi)$$

and therefore $m = m_x$.

(c) If $\varphi \in \mathscr{C}(K)$, $\varphi \circ \bar{\tau}_{\bar{G}}$ is compact, hence $\varphi \circ \bar{\tau}_{\gamma G}$ is relatively compact, and $g \longrightarrow \tau_g = \bar{\tau}_{\gamma g}$ is equicontinuous. Since there is a measure on M invariant under G (namely m), there exists also an ergodic measure on M , but such an ergodic measure is by (b) necessarily the Haar measure m , therefore m is ergodic.]

4.7. Theorem. Let the conditions of Theorem 4.1 be satisfied, let ρ be ergodic and let $\mathcal{P}\mathfrak{H}$ be separable.

Replacing the original topology of $\mathcal{X}$ (defined by (4.2)) by the discrete topology we obtain a group $\mathcal{X}^*$; we let M be the compact character group of $\mathcal{X}^*$ and m the normalized Haar measure on M . We define a continuous homomorphism $\delta : G \longrightarrow M$ with dense image by $(\delta g)(X) = X(g)$ for all $X \in \mathcal{X}$. The action $(g, x) \rightarrow x \, . \, \delta g$ of G on M is equicontinuous and m is ergodic with respect to it.

There exists a mapping $f : M \longrightarrow E$ with the following properties.

(a) f transforms m into μ in the sense that the mapping $\varphi(\cdot) \longrightarrow \varphi(f)$ is isometric from $L^2(E, \mu)$ onto $L^2(M, m)$.

(b) For all $A \in \mathfrak{A}$, $g \in G$

$$f_{x.\delta g}(A) = f_x(\tau_g^{-1} A) \tag{4.13}$$

m-almost everywhere with respect to x .

If $A \in \mathfrak{A}$, $X \in \mathcal{X}$ we define

$$A^X = \sum_{X' \in \mathcal{X}} E[XX'] \, \pi(A) \, E[X'] \tag{4.14}$$

We let $\mathfrak{Q}$ be the C^*-algebra generated by the A^X and define a representation τ of G into aut $\mathfrak{Q}$ by

$$\tau_g Q = U(g) \, Q \, U(g)^{-1} \tag{4.15}$$

We have in particular

$$\tau_g A^X = X(g) \, . \, A^X \tag{4.16}$$

The proof of the theorem will result from the following facts.

(i) $\mathcal{Q}$ is abelian

[Using (4.6) we have $[A_1^{X_1}, A_2^{X_2}]$

$$= \sum_{X\in\mathcal{X}} [E[XX_1X_2]\pi(A_1)E[XX_2]\pi(A_2)E[X] - E[XX_1X_2]\pi(A_2)E[XX_1]\pi(A_1)E[X]] = 0] .$$

(ii) Let M be the spectrum of $\mathcal{Q}$, we denote by $Q \longrightarrow [Q]$ the Gel'fand isomorphism $\mathcal{Q} \longrightarrow \mathcal{C}(M)$. The action τ of G on M defined by $[Q](\tau_g x) = [\tau_g^{-1} Q](x)$ is equicontinuous.

[It follows from (4.16) that the mapping $g \longrightarrow \tau_g Q$ is continuous and the orbit $\tau_g Q$ relatively compact for the norm topology of $\mathcal{Q}$].

(iii) The algebra $P\mathfrak{B}$ (see Theorem 1.1 and Theorem 1.2 (b)) is equal to the weak closure $\mathcal{Q}^{-1}$ of $\mathcal{Q}$.

[We have $P\,\pi(A)P = \sum_{X\in\mathcal{X}} A^X$ in the sense of strong convergence, hence $P\mathfrak{B} \subset \mathcal{Q}^-$. The restriction of $P\mathfrak{B}$ to $P\mathfrak{h}$ is abelian and has the cyclic vector Ω , hence it is maximal abelian and contains the restriction of $\mathcal{Q}$ to $P\mathfrak{h}$ (which commutes with it); therefore $P\mathfrak{B} \supset \mathcal{Q}$].

(iv) A measure m on M is defined by

$$m([Q]) = (\Omega, Q\Omega) \tag{4.17}$$

m is ergodic and its support is M .

[If m were not ergodic there would exist a G-invariant vector Ψ in the closure of $\mathcal{Q}\,\Omega$ such that Ψ is not a multiple of Ω , in contradiction with the ergodicity of ρ (see Theorem 4.4(a) and Theorem 3.4). Let $0 \le Q \in \mathcal{Q}$, then $m([Q]) = 0 \Rightarrow Q^{1/2}\,P\,\pi(A)P\,\Omega = 0$ (because of (i) and (iii)) $\Rightarrow Q^{1/2}\,\pi(A)\,\Omega = 0 \Rightarrow Q^{1/2} = 0$; therefore supp $m = M$].

(v) M can be identified to the character group of $\mathcal{X}^*$ so that m is the Haar measure and $\tau_g x = x.\,\delta g$ (here $\mathcal{X}^*$ and δ are defined as in the statement of the theorem).

[By (ii), (iv) and Theorem 4.6 (a), (b), τ extends to a continuous action $\bar{\tau}$ of $\bar{G}$ on M and one may identify M with $\bar{G}/H$ where $H = \{\bar{g} \in \bar{G} : \bar{\tau}_{\bar{g}} = 1\}$; in this identification m is the Haar measure of G/H and $\tau_g \langle\bar{g}\rangle = \langle\bar{g}.\gamma g\rangle$ where $\langle\cdot\rangle : \bar{G} \longrightarrow \bar{G}/H$ is the quotient mapping. From (4.16) it follows that $H = \{\bar{g} \in \bar{G} : X \in \mathcal{X} \Rightarrow \bar{g}(X) = 1\}$ and we may therefore identify $\bar{G}/H$ to the character group of $\mathcal{X}^*$. The image of $\langle\gamma g\rangle$ in $\mathcal{X}^*$ is δg so that $\tau_g x = x.\delta g$].

(vi) The space $\mathcal{C}(M)$ is separable

[Because the separability of $P\mathfrak{h}$ implies that the character group $\mathcal{X}^*$ of M is countable].

(vii) The Gel'fand isomorphism $\mathcal{Q} \to \mathcal{C}(K)$ extends uniquely to a morphism of C^*-algebras $\mathcal{Q}^- \to L^\infty(M, m)$, again denoted by $[\cdot]$, such that

$$m([R]) = (\Omega, R\,\Omega) \tag{4.18}$$

This morphism is an isomorphism onto.

[This results from A.3 applied to the restriction of $\mathcal{Q}$ to $P\mathfrak{h}$].

(viii) There is a mapping $f_\cdot : M \to E$ such that for all $A \in \mathcal{O}\!\!\mathcal{l}$, $x \to f_x(A)$ is measurable and for all $\psi \in L^1(M, m)$

$$\int m(dx)\psi(x)[P\pi(A)P](x) = \int m(dx)\psi(x)\; f_x(A) \tag{4.19}$$

We have m-almost everywhere

$$[P\alpha(\hat{A})](\cdot) = [P\pi(A)P](\cdot) = f_\cdot(A) = \hat{A}(f_\cdot) \tag{4.20}$$

[The function $f_\cdot$ defined by A.8 satisfies (4.19); since $\sup_x \|f_x\| = 1$ and $f_\cdot(1) = 1$ m-almost everywhere we may assume that $f_\cdot$ maps M into

E ; (4.20) follows from (4.19) and Theorem 1.2].

(ix) Property (a) of the theorem holds.

[Since polynomials in the $\hat{A}$ are dense in $\mathcal{C}(M)$ and since $\varphi \longrightarrow [P\alpha(\varphi)]$ is a morphism $\mathcal{C}(E) \longrightarrow L^{\infty}(M, m)$ (by Theorem 1.2, (iii) and (vii)), (4.20) gives $[P\,\alpha(\varphi)]\,(\cdot) = \varphi(f_{\cdot})$ m-almost everywhere if $\varphi \in \mathcal{C}(E)$. Therefore

$$\mu(\varphi) = (\Omega,\alpha(\varphi)\Omega) = \int m(dx)[P\alpha(\varphi)](x) = \int m(dx)\varphi(f_x)$$

Therefore the mapping $\varphi \longrightarrow \varphi(f_{\cdot})$ is isometric $L^2(E, \mu) \longrightarrow L^2(M, m)$. The image of $\mathcal{C}(M)$ in $\mathcal{Z}^{-}$ by $\varphi \longrightarrow P\,\alpha(\varphi)$ is strongly dense (Theorem 1.2 and (iii)). Since the morphism $[\cdot] : \mathcal{Z}^{-} \longrightarrow L^{\infty}(M, m)$ is onto by (vii) and since the norm of $[R]$ in $L^2(M, m)$ is $\|[R]\|_2 = \|R\,\Omega\|$ by (4.18), we find that the image of $\mathcal{C}(M)$ by $\varphi \longrightarrow [P\,\alpha(\varphi)]$ is dense in $L^2(M, m)$. Therefore the isometry $L^2(E, \mu) \longrightarrow L^2(M, m)$ is onto].

(x) Property (b) of the theorem holds

[In view of (4.20) we have m-almost everywhere in x

$$f_{x.\delta g}(A) = [P\pi(A)P]\,(x.\delta g) = [P\pi(A)P]\,(\tau_g\, x)$$
$$= [U(g^{-1})P\pi(A)P\,U(g)](x) = [P\,\pi(\tau_g^{-1}\,A)P](x) = f_x(\tau_g^{-1}\,A) \qquad]$$

4.8. Remarks on Theorem 4.7.

(a) Define unitary representations V and W of G in $L^2(E, \mu)$ and $L^2(E, m)$ respectively by

$$V(g)\,\varphi(\sigma) = \varphi(\tau_g^{-1}\,\sigma)$$
$$W(g)\,\psi(x) = \psi(x.(\delta g)^{-1})$$

Define further the mapping $T : L^2(E, \mu) \longrightarrow L^2(M, m)$ by

$$T\,\varphi(x) = \varphi(f_x)$$

Part (a) of Theorem 4.6 expresses that T is an isometry of $L^2(E, \mu)$ onto $L^2(M, m)$ and part (b) that

$$TV(g) = W(g)\,T$$

(b) Let the conditions of Theorem 4.1 be satisfied and ρ be ergodic. Let $\tilde{\mathfrak{X}}$ be any subgroup of $\mathfrak{X}$ and define

$$\tilde{P} = \sum_{X\in\tilde{\mathfrak{X}}} E(\{X\})$$

Then (4.6) gives

$$\tilde{P}\,\pi(\mathfrak{A})\,\tilde{P} \subset [\tilde{P}\,\pi(\mathfrak{A})\,\tilde{P}]'$$

Furthermore Theorem 4.7 remains true if $\mathfrak{X}$ and P are replaced everywhere by $\tilde{\mathfrak{X}}$ and $\tilde{P}$.

(c) Suppose that $\mathfrak{X}$ is a discrete subgroup of $\hat{G}$ and define

$$H = \{g \in G : X \in \mathfrak{X} \Rightarrow X(g) = 1\}$$

then H is a closed subgroup of G, G/H is compact, and $P\mathfrak{H}$ consists exactly of the vectors invariant under H. The multiperiodic decomposition is in that case an ergodic decomposition with respect to H and it will follow from 6.4 that μ is carried by H-ergodic states.

4.9. Sources. Much interest has been paid to dynamical systems with discrete spectrum and to the discrete part of the spectrum of dynamical systems (see for instance Arnold and Avez [3] 9.13, Appendix 7, and references quoted there). A version of Theorem 4.4 with non commutative $\mathfrak{A}$ was proved by Kastler and Robinson [23], see also [15]. A first attempt at understanding the decomposition studied here was made by

Doplicher, Gallavotti and Ruelle [12] .

If the ideas expressed by Landau and Lifshitz about the nature of turbulence in hydrodynamics ([26] § 27) are correct, the multiperiodic decomposition may be useful in the description of a turbulent state. Other applications exist in statistical mechanics (see [36]). The interesting situations are those for which $\mathcal{X}$ is not a discrete subgroup of $\hat{G}$, this corresponds for physical systems to the existence of periods with irrational ratios.

5. Quasi-local structure and decomposition at infinity.

When a family $(\mathfrak{A}_\Lambda)$ of sub-C^*-algebras of $\mathfrak{A}$ is given, we may say that a quasi-local structure is defined on $\mathfrak{A}$. The following theorem is then often useful.

5.1. Theorem. Let $\mathcal{L}$ be a directed*) ordered set and let $(\mathfrak{B}_\Lambda)_{\Lambda \in \mathcal{L}}$ be a decreasing family of von Neumann algebras in $\mathfrak{H}$. Define $\mathfrak{B} = \bigcap_{\Lambda \in \mathcal{L}} \mathfrak{B}_\Lambda$ and assume $\mathfrak{B} \subset \pi(\mathfrak{A})'$. The following conditions are equivalent.

(a) $\mathfrak{B}$ consists of the multiples of 1.

(b) Given $A \in \mathfrak{A}$ there exists $\Lambda \in \mathcal{L}$ such that

$$B \in \mathfrak{B}_\Lambda \Rightarrow |(\Omega, \pi(A)B\,\Omega) - \rho(A)(\Omega, B\,\Omega)| < \| B \|$$

(c) Given $\varepsilon > 0$ and $A \in \mathfrak{A}$ there exists $\Lambda \in \mathcal{L}$ such that

$$B \in \mathfrak{B}_\Lambda \Rightarrow |(\Omega, \pi(A)B\,\Omega) - \rho(A)(\Omega, B\,\Omega)| \le \varepsilon \| B \|$$

Using the replacement $A \longrightarrow A/\varepsilon$ one verifies (b) $\Leftrightarrow$ (c). The proof of (a) $\Leftrightarrow$ (b) is obtained by observing the equivalence of the following conditions [To obtain (iv) $\Rightarrow$ (iii) use the compacity of the set of operators of norm ≤ 1 in the weak operator topology].

(i): non (a)

(ii): there exist $A_1, A_2 \in \mathfrak{A}$ and $B \in \mathfrak{B}$ such that

$$(\Omega, \pi(A_1)B\,\pi(A_2)\Omega) \neq (\Omega, \pi(A_1 A_2)\Omega)\,(\Omega, B\,\Omega)$$

(iii): there exist $A \in \mathfrak{A}$ and $B \in \mathfrak{B}$ such that

$$\| B \| \le 1 \quad \text{and} \quad |(\Omega, \pi(A)B\,\Omega) - (\Omega, \pi(A)\Omega)\,(\Omega, B\,\Omega)| \ge 1$$

---

*) $\mathcal{L}$ is directed if, given $\Lambda_1, \Lambda_2 \in \mathcal{L}$ there exists $\Lambda \in \mathcal{L}$ such that $\Lambda_1, \Lambda_2 \le \Lambda$.

(iv): there exists $A \in \mathfrak{A}$ and for every Λ there exists $B_\Lambda \in \mathfrak{R}_\Lambda$ such that

$$\|B_\Lambda\| \leq 1 \quad \text{and} \quad |(\Omega, \pi(A)B_\Lambda \Omega) - (\Omega, \pi(A)\Omega)(\Omega, B_\Lambda \Omega)| \geq 1$$

(v) : non (b) .

5.2. Quasi-local structure.

We shall now study an example where algebras $\mathfrak{R}_\Lambda$ are constructed from a quasi-local structure.

Let $\mathfrak{L}$ be an ordered set where a relation $\Lambda \perp M$ may hold between pairs of elements, and let $(\mathfrak{A}_\Lambda)_{\Lambda \in \mathfrak{L}}$ be a family of sub-C^*-algebras of $\mathfrak{A}$. We assume that the following conditions are satisfied.

QL 1. If $\Lambda_1 \leq \Lambda_2$ and $\Lambda_2 \perp M$, then $\Lambda_1 \perp M$

QL 2. The set $\mathfrak{L}$ is directed and if $\Lambda \perp M_1$, $\Lambda \perp M_2$, there exists $M \in \mathfrak{L}$ such that $M_1, M_2 \leq M$ and $\Lambda \perp M$.

QL 3. If $\Lambda \perp M$, then $[\mathfrak{A}_\Lambda, \mathfrak{A}_M] = 0$

QL 4. $\cup_{\Lambda \in \mathfrak{L}} \mathfrak{A}_\Lambda$ is dense in $\mathfrak{A}$.

We define

$$\mathfrak{A}_\Lambda^\perp = \cup_{M : \Lambda \perp M} \mathfrak{A}_M \tag{5.1}$$

By QL 2, $\mathfrak{A}_\Lambda^\perp$ is a self-adjoint algebra and QL 1 gives

$$(\Lambda_1 \leq \Lambda_2) \Rightarrow (\mathfrak{A}_{\Lambda_1}^\perp \supset \mathfrak{A}_{\Lambda_2}^\perp) \tag{5.2}$$

Define also

$$\mathfrak{B}_\Lambda = \pi(\mathfrak{A}_\Lambda^\perp)'' \qquad \mathfrak{B} = \cap_{\Lambda \in \mathfrak{L}} \mathfrak{R}_\Lambda \tag{5.3}$$

Clearly $\mathfrak{R} \subset \pi(\mathfrak{A})''$. On the other hand QL 3 and (5.1) give $[\mathfrak{A}_\Lambda^\perp, \mathfrak{A}_\Lambda] = 0$, hence $[\mathfrak{B}, \pi(\mathfrak{A}_\Lambda)] = 0$ and, by QL 4, $\mathfrak{B} \subset \pi(\mathfrak{A})'$. We shall call $\mathfrak{B}$ the algebra at infinity ; we have just shown that the algebra at infinity is contained in the center of $\pi(\mathfrak{A})''$. In particular the theory of Section 1 applies. The corresponding decomposition of ρ given by μ (see Theorem 1.3) will be called decomposition at infinity ; under suitable separability assumptions μ is carried by states with a trivial algebra at infinity (see Theorem 5.4 below). From (5.2) we get

$$(\Lambda_1 \leq \Lambda_2) \Rightarrow (\mathfrak{B}_{\Lambda_1} \supset \mathfrak{B}_{\Lambda_2})$$

Therefore Theorem 5.1 holds, it characterizes the cases where the algebra at infinity is trivial, we reformulate this theorem as follows.

5.3. Theorem*). We let $(\mathfrak{A}_\Lambda)_{\Lambda \in \mathfrak{L}}$ satisfy QL 1 - QL 4, and use the notation (5.1), (5.3). The following conditions are equivalent.

(a) The algebra at infinity $\mathfrak{B}$ consists of the multiples of 1.

(b) Given $\epsilon > 0$ and $A \in \mathfrak{A}$ there exists $\Lambda \in \mathfrak{L}$ such that if $A' \in \mathfrak{A}_\Lambda^\perp$, then

$$|\rho(A\,A') - \rho(A)\,\rho(A')| \leq \epsilon \,\|A'\|$$

Let $\mathfrak{D}_\Lambda$ be the weak closure of $\pi(\mathfrak{A}_\Lambda^\perp)$ and P_Λ be the largest projector in $\mathfrak{D}_\Lambda$. Every $B \in \mathfrak{B}_\Lambda$ is of the form $B=B_1+\lambda(1-P_\Lambda)$ with $B_1 \in \mathfrak{D}_\Lambda$, $\|B_1\| \leq \|B\|$, $|\lambda| \leq \|B\|$; therefore $B = \lambda\, 1+B'$ where $B' = B_1 - \lambda P_\Lambda \in \mathfrak{D}_\Lambda$, $\|B'\| \leq \|B_1\| + |\lambda| \leq 2\,\|B\|$. From Theorem 5.1, we see thus that (a) is equivalent to

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*) This theorem is of the Sinai-Powers type (see Sinai [38], Powers [31], Lanford and Ruelle [28]).

(b') Given $A \in \mathfrak{A}$ there exists $\Lambda \in \mathcal{L}$ such that

$$B' \in \mathfrak{D}_\Lambda \Rightarrow |(\Omega, \pi(A)B'\Omega) - \rho(A)(\Omega, B'\Omega)| \leq \varepsilon \|B'\|$$

Using Kaplansky's density theorem *) we may write equivalently

$$A' \in \mathfrak{A}_\Lambda^\perp \Rightarrow |\rho(A\,A') - \rho(A)\,\rho(A')| \leq \varepsilon \|\pi(A)\|$$

This in turn is equivalent to (b) because if $A' \in \mathfrak{A}_\Lambda^\perp$ there exists $A'' \in \mathfrak{A}_\Lambda^\perp$ such that $\pi(A') = \pi(A'')$ and $\|A''\|$ is arbitrarily close to $\|\pi(A'')\|$ **) .

5.4. Theorem. Let $(\mathfrak{A}_\Lambda)_{\Lambda \in \mathcal{L}}$ be a countable family of sub-C^*-algebras of $\mathfrak{A}$ satisfying the conditions QL 1 - QL 4 . If either of the conditions (a), (b) below is satisfied, the measure μ is carried by states σ with trivial algebra at infinity.

(a) $\mathfrak{A}$ is separable.

(b) For each $\Lambda \in \mathcal{L}$ there is a separable closed two-sided ideal $\mathfrak{J}_\Lambda$ of $\mathfrak{A}_\Lambda$ such that the restriction of ρ to $\mathfrak{J}_\Lambda$ has norm 1 .

In both cases, the condition S is satisfied and we may use the results of Section 2. For each Λ let (A_{Λ_j}) be a dense sequence in $\mathfrak{A}_\Lambda$ (case (a)) or in $\mathfrak{J}_\Lambda$ (case (b)). The von Neumann algebra $\mathfrak{B}_\Lambda$ is generated by the $\pi(A_{Mj})$ with $\Lambda \perp M$ and contains the diagonalizable operators (Theorem 2.1); furthermore the von Neumann algebra generated by the $\pi_\sigma(A_{Mj})$ is $\mathfrak{B}_{\Lambda\sigma} = \pi_\sigma(\mathfrak{A}_\Lambda^\perp)''$. We may therefore write

*) See Dixmier [8] Ch 1, § 3, Théorème 3.

**) See Dixmier [9] Proposition 1.8.2.

$$\mathfrak{A}_\Lambda = \pi(\mathfrak{A}^\perp_\Lambda)'' = \int^\oplus \mu(d\sigma)\ \pi_\sigma(\mathfrak{A}^\perp_\Lambda)''$$

Using Theorem 2.2 (b) this gives

$$\mathfrak{A} = \cap\ \pi(\mathfrak{A}^\perp_\Lambda)'' = \int^\oplus \mu(d\sigma)\ \cap\ \pi_\sigma(\mathfrak{A}^\perp_\Lambda)''$$

Since $\mathfrak{A}$ is the algebra of diagonalizable operators we find that $\cap\ \pi_\sigma(\mathfrak{A}^\perp_\Lambda)''$ consists of the multiples of the identity operator in $\mathfrak{H}_\sigma$ μ-almost everywhere in σ.

5.5. Sources. The concept of quasi-local structure originates in local quantum field theory (see for instance Araki [1]) where $\mathfrak{L}$ consists of the bounded open regions in Minkowski space ordered by inclusion and $\Lambda \perp M$ if Λ and M are space-like regions. Similar situations arise in statistical mechanics (see for instance [36]), the definition of K-systems (see Sinai [38]), or the study of canonical (anti-) commutation relations (see Powers [31]). In statistical mechanics, Theorem 5.4 may be used to describe the decomposition of equilibrium states invariant under space translations into clustering equilibrium states (see Dobrushin [10], [11], Lanford and Ruelle [28]). When such a decomposition is non trivial, symmetry breakdown is said to occur, concrete and non trivial examples of symmetry breakdown have been worked out by Dobrushin [11]. The case (b) in Theorem 5.4 is useful in dealing with states of physical interest, for instance locally normal states (see 3.9).

6. Further decompositions.

In Sections 3-5 we have discussed some typical integral representations of states on a C^*-algebra. We consider here briefly some further examples. Many more applications of the general theory of Sections 1 and 2 are of course possible, the choice of $\mathfrak{B}$ depending on the extra structure present on $\mathfrak{A}$ *).

6.1. Canonical representation of states on an abelian C^*-algebra.

If $\mathfrak{A}$ is abelian, we can apply the theory of Section 1 with $\mathfrak{B} = \pi(\mathfrak{A})''$. In that case μ is carried by the set of extremal points of E , i.e. the spectrum of $\mathfrak{A}$, and $\rho \longrightarrow \mu$ is the adjoint of the Gel'fand isomorphism.

6.2. Central decomposition.

If $\mathfrak{B} = \pi(\mathfrak{A})' \cap \pi(\mathfrak{A})''$ (i.e. $\mathfrak{B}$ is the center of $\pi(\mathfrak{A})''$) the theory of Section 1 applies. The integral representation of ρ given by μ is called <u>central decomposition</u> . If $\mathfrak{B}$ consists of the multiples of 1 (i.e. if $\pi(\mathfrak{A})''$ is a factor), ρ is called a factor state . Suppose that condition S of Section 2 is satisfied, then μ is carried by the factor states. It follows indeed from Theorem 2.2 that

$$\mathfrak{B} = \pi(\mathfrak{A})' \cap \pi(\mathfrak{A})'' = \int^{\oplus} \mu(d\sigma) \; [\pi_\sigma(\mathfrak{A})' \cap \pi_\sigma(\mathfrak{A})''] \tag{6.1}$$

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*) If a quasi-local structure is given, various decompositions, analogous to that of Section 5, arise naturally. If a group of automorphisms is given, a decomposition of quasi-invariant states, similar to the ergodic decomposition of invariant states, has been discussed [13] .

and since $\mathfrak{B}$ consists of the diagonalizable operators (Theorem 2.1), $\pi_\sigma(\mathcal{O})' \cap \pi_\sigma(\mathcal{O})''$ consists of the multiples of 1 μ-almost everywhere in σ .

6.3. Relation with the disintegration of measures.

Let K be a metrizable compact space, $\mathcal{O}_1 = \mathcal{C}(K)$ the separable C^*-algebra of complex continuous functions on K and δ: $\mathcal{O}_1 \to \pi(\mathcal{O})' \cap \pi(\mathcal{O})''$ a morphism of $\mathcal{O}_1$ into the center of $\pi(\mathcal{O})''$ such that $\delta 1 = 1$. A probability measure μ_1 on K is defined by

$$\mu_1(\psi) = (\Omega, \delta(\psi)\Omega) \tag{6.2}$$

If $\mathfrak{B} = \delta(\mathcal{O}_1)''$, the theory of Section 1 applies and we shall show that there is a mapping $f_\cdot : K \to E$ such that $f_\cdot(A)$ is μ_1-measurable for $A \in \mathcal{O}$, and

$$(\Omega,\alpha(\varphi)\ \delta(\psi)\Omega) = \int_K \mu_1(dx)\ \psi(x)\ \varphi(f_x) \tag{6.3}$$

for $\varphi \in \mathcal{C}(E)$, $\psi \in \mathcal{C}(K)$. In particular

$$\mu(\varphi) = \int_K \mu_1(dx)\ \varphi(f_x)$$

Let $B \in \mathfrak{B}$ and $\psi \in \mathcal{C}(K)$, then

$$|(\Omega,\ B\ \delta(\psi)\Omega)| \le \|B\| (\Omega, |\delta(\psi)|\Omega) \le \|B\|\ \mu_1(|\psi|) \tag{6.4}$$

Therefore there is a unique $[B] \in L^\infty(K, \mu_1)$ such that

$$(\Omega, B\delta(\psi)\Omega) = \int \mu_1(dx)\ \psi(x)\ [B]\ (x) \tag{6.5}$$

and one can see that $[\cdot]$ is a morphism (using A.3).

If $\psi \in \mathcal{C}(K)$ let $F_\psi \in \mathcal{O}'$ be defined by

$$F_\psi(A) = (\Omega, \pi(A)\ \delta(\psi)\Omega) = (\Omega, \alpha(\hat{A})\ \delta(\psi)\Omega) \tag{6.6}$$

Then (6.4) gives $\| F_\psi \| \le \mu_1(|\psi|)$ and $F_\bullet$ has a unique extension to a continuous mapping from $L^1(K, \mu_1)$ to the strong dual of $\mathfrak{A}$; A.8 gives the existence of $f_\bullet : K \to \mathfrak{A}'$ such that $f_\bullet(A)$ is μ_1-measurable, $\| f_\bullet \| \le 1$ and

$$F_\psi(A) = \int \mu_1(dx)\ \psi(x)\ f_x(A) \tag{6.7}$$

Since $\| f_\bullet \| \le 1$ and $\int \mu_1(dx)\ f_x(1) = 1$ we have μ_1-almost everywhere $f_\bullet \in E$; by a change of definition on a set of measure zero we assume now $f_x \in E$ for all $x \in K$. Using (6.5) and (6.6) we may rewrite (6.7) as

$$\int \mu_1(dx)\ \psi(x)\ [\alpha(\hat{A})](x) = \int \mu_1(dx)\ \psi(x)\ \hat{A}(f_x)$$

so that we have μ_1-almost everywhere in x

$$[\alpha(\hat{A})](x) = \hat{A}(f_x)$$

Since α and $[\cdot]$ are morphisms and the polynomials are dense in $\mathcal{C}(E)$ we have, for all $\varphi \in \mathcal{C}(E)$,

$$[\alpha(\varphi)]\ (x) = \varphi(f_x)$$

μ_1-almost everywhere in x , yielding (6.3).

The problem of disintegrating a measure with respect to a mapping (see for instance Bourbaki [5] §3, n° 1) corresponds to the special case $\mathfrak{A} = \mathcal{C}(L)$ where L is compact and metrizable.

6.4. Decomposition with respect to a normal subgroup.

Let G be a topological group and $\tau_\bullet$ a representation of G in aut $\mathfrak{A}$ such that the functions $g \to \sigma(\tau_g A)$ are continuous (with $\sigma \in E$, $A \in \mathfrak{A}$). Let also H be a closed normal subgroup of G such that G/H is compact. We assume that the state ρ is G-ergodic (see Theorem 3.4) and that $\mathfrak{A}$ is H-abelian (see 3.7). If μ is the measure giving the ergodic decomposition of ρ (with respect to H), then <u>the support of</u> μ <u>is a homogeneous space of</u> G/H <u>and</u> μ <u>is the Haar measure of this homogeneous space</u>. [The support of μ consists of H-invariant states on which G/H acts continuously, μ is ergodic for this action and the proof proceeds as for part (b) of Theorem 4.6].

Let $\sigma \in \operatorname{supp} \mu$, then

$$\rho(A) = \int_{G/H} d\bar{g} \quad \tau_g \sigma(A) \tag{6.8}$$

where $\bar{g}$ is the class of g in G/H . <u>The support of</u> μ <u>consists of H-ergodic states</u>. [By A.9, we may assume that σ is an extremal point of the closed convex hull of $\operatorname{supp} \mu$. Let $\sigma = \frac{1}{2}\sigma_1 + \frac{1}{2}\sigma_2$ where σ_1 , σ_2 are H-invariant states. Define probability measures μ_1, μ_2 by

$$\mu_i(\varphi) = \int_{G/H} d\bar{g} \quad \tau_g \sigma(\varphi)$$

The ergodicity of ρ implies that it is the resultant of μ_1 and μ_2 ; Theorem 3.6 (a) yields then μ_1 , $\mu_2 \prec \mu$ and, since $\mu = \frac{1}{2}\mu_1 + \frac{1}{2}\mu_2$, $\mu_1 = \mu_2 = \mu$. This shows that σ_1 , $\sigma_2 \in \operatorname{supp} \mu$. But since σ is an extremal point of the closed convex hull of $\operatorname{supp} \mu$ we have $\sigma_1 = \sigma_2$].

6.5. <u>Sources.</u> Central decomposition has been studied by Sakai [37] in the case of separable $\mathfrak{A}$, see also [42] ; for physical applications see Araki and Miyata [2], Haag, Kastler and Michel [19]. The decomposition in 6.4 of a G-ergodic state into H-ergodic states improves a theorem of Ginibre (for which see [33]) by weakening the continuity conditions.

Appendix A.

A.1. Let $\mathcal{A}$ be a von Neumann algebra in $\mathfrak{h}$, $P \in \mathcal{A}$ a projection. Let $\mathcal{A}_P$ be the restriction of $P\mathcal{A}P$ to $P\mathfrak{h}$ and $(\mathcal{A}')_P$ the restriction of $P\mathcal{A}'$ to $P\mathfrak{h}$. Then $\mathcal{A}_P$ and $(\mathcal{A}')_P$ are von Neumann algebras in $P\mathfrak{h}$ and $(\mathcal{A}')_P = (\mathcal{A}_P)'$. [See Dixmier [8] Ch 1, §2, n° 1].

A.2. A von Neumann algebra $\mathfrak{M}$ is called maximal abelian if $\mathfrak{M} = \mathfrak{M}'$. If an abelian von Neumann algebra has a cyclic vector, then it is maximal abelian. [See Dixmier [8] Ch1, § 6, n°3, Corollaire 2].

A.3. Extension of the Gel'fand isomorphism.

Let $\mathcal{X}$ be an abelian C^*-algebra of operators on the Hilbert space $\mathfrak{h}$, $\Omega \in \mathfrak{h}$ a cyclic vector for the commutant $\mathcal{X}'$ of $\mathcal{X}$. We denote by X the spectrum of $\mathcal{X}$, by $\mathcal{C}(X)$ the space of complex continuous functions vanishing at infinity, by $B : \mathcal{C}(X) \longrightarrow \mathcal{X}$ the inverse of the Gel'fand isomorphism, and by m the measure on X such that

$$m(f) = (\Omega, B(f)\Omega)$$

The mapping B extends by continuity to a unique mapping $B: L^{\infty}(X,m) \longrightarrow \mathfrak{B}$ where $L^{\infty}(X,m)$ has the topology of weak dual of $L'(X,m)$ and $\mathfrak{B}$ is the weak closure of $\mathcal{X}$ with the weak operator topology ; B thus extended is onto and is an isomorphism of C^*-algebras. [See Dixmier [8] Ch 1, § 7].

A.4. Let E be a convex compact set in a locally convex space and let (ψ_j) be a continuous partition of unity on E (i.e. a finite family of continuous functions $\psi_j \geq 0$ such that $\sum_j \psi_j = 1$). If μ is a probability measure on E , let $\alpha_j = \mu(\psi_j)$ and σ_j be the resultant of $\alpha_j^{-1} \psi_j \mu$. Define

$$\tilde{\mu} = \sum_j \alpha_j \delta_{\sigma_j}$$

where δ_σ is the unit mass at σ . The measure μ can be approximated in the vague topology by measures of the form $\tilde{\mu}$. [Take (ψ_j) subordinate to a sufficiently fine open covering of E , see Bourbaki [4] p. 217 Prop. 3].

A.5. Integral representations on convex compact sets.

Let E be a convex compact set in a locally convex space. An order relation $\prec$ is defined (Bishop and de Leeuw) on the probability measures on E by $\mu_1 \prec \mu_2$ if $\mu_1(\varphi) \leq \mu_2(\varphi)$ for all convex continuous function φ on E . If $\mu_1 \prec \mu_2$ then μ_1 and μ_2 have the same resultant.

E is said to be a simplex (Choquet) if for every $\rho \in E$ there is a unique probability measure μ_ρ on E which has resultant ρ and is maximal for the order $\prec$. [See Choquet et Meyer [7]].

A.6. Theorem of Alaoglu-Birkhoff.

Let $\mathcal{U}$ be a semi-group of contractions of a Hilbert space, and let P be the orthogonal projection on the space of vectors invariant under every $U \in \mathcal{U}$; then P is contained in the strong operator closure of the convex hull of $\mathcal{U}$ [See Riesz and Nagy [32] n° 146].

A.7. Let ρ be a state on $\mathfrak{A}$ and f a positive linear form on $\mathfrak{A}$ such that $f \leq \rho$; then there exists $T \in \pi(\mathfrak{A})'$ such that

$$f(\cdot) = (\Omega, \pi(\cdot) T \Omega)$$

T is unique and $0 \leq T \leq 1$. [See Dixmier [9] 2.5.1].
Let a group G act by automorphisms on $\mathfrak{A}$ and ρ, f be G-invariant (See Section 3) then the uniqueness of T yields $T \in U(G)'$.

A.8. A variant of the theorem of Dunford-Pettis.

Let m be a measure on the compact set M such that $L^1(m)$ is separable. Let $\mathfrak{A}$ be any Banach space and $\mathfrak{A}'$ its strong dual. For any continuous linear mapping $F. : L^1(m) \longrightarrow \mathfrak{A}'$ there is a function $f_\bullet : M \longrightarrow \mathfrak{A}'$ with $\sup_{x \in M} \| f_x \| \leq \| F \|$ such that for every $A \in \mathfrak{A}$, $f_\bullet(A)$ is m-measurable and, for every $\psi \in L^1(m)$,

$$\int m(dx)\, \psi(x)\, f_x(A) = F_\psi (A)$$

[See Bourbaki [5] § 2, Exercise 19*)]

A.9. Theorem of Milman.

Let M be a set in a locally convex space. If the closed convex hull of M is compact, its extremal points lie in the closure of M . [See Köthe [24] § 25, 1, (7)].

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*) I am indebted to A. Grothendieck for explaining a solution of this exercise to me.

Appendix B. *)

B.1. Proposition. Let $\mathcal{J}$ be a closed two-sided ideal of the C^*-algebra $\mathfrak{A}$. Every state ρ' on $\mathcal{J}$ has a unique extension to a state ρ on $\mathfrak{A}$; if $(\mathfrak{H}, \pi, \Omega)$ is the canonical cyclic representation associated with ρ, $\pi(\mathcal{J})$ is strongly dense in $\pi(\mathfrak{A})$.

This follows from Dixmier [9] Proposition 2.10.4.

B.2. Proposition. Let the C^*-algebra $\mathfrak{A}$ have an identity and $\mathfrak{A}_o$ be a separable sub-C^*-algebra of $\mathfrak{A}$.

(a) The set $\mathcal{F}_o$ of states on $\mathfrak{A}$ which have a restriction of norm 1 to $\mathfrak{A}_o$ is a Baire subset of the set E of all states on $\mathfrak{A}$.

(b) If a measure μ on E has resultant $\rho \in \mathcal{F}_o$, then μ is carried by $\mathcal{F}_o$

Let (A_n) be a dense sequence in the self-adjoint part of the unit ball of $\mathfrak{A}_o$; (a) results from

$$\mathcal{F}_o = \{\sigma \in E : \sup_n \sigma(A_n) = 1\} = \cap_{m>o} V_m$$

$$V_m = \cup_n \{\sigma \in E : \sigma(A_n) > 1 - \frac{1}{m}\}$$

To prove (b) suppose that $\mu = \mu' + \mu''$ where μ' and μ'' are carried respectively by V_m and its complement.

We have

$$\rho(A_n) = \mu'(\hat{A}_n) + \mu''(\hat{A}_n) \leq \| \mu' \| + \| \mu'' \| (1 - \frac{1}{m})$$

$$= 1 - \frac{1}{m} \| \mu'' \|$$

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*) The main ideas of this appendix come from [35], see also [36] Ch. 6.

Since $\sup_n |\rho(A_n)| = 1$, we find $\| \mu'' \| = 0$

B.3. Proposition. We use the notation of Section 2 and assume that condition S is satisfied.

(a) If $\sigma \in \mathcal{F}$, then $\mathfrak{h}_\sigma$ is separable and the sequence $\pi_\sigma(A_i)\, \Omega_\sigma$ is dense in $\mathfrak{h}_\sigma$.

(b) $\mathcal{F}$ is a Baire subset of E .

(c) μ is carried by $\mathcal{F}$.

Part (a) results from Proposition B.1, parts (b) and (c) result from Proposition B.2.

Acknowledgements.

I am indebted to D. Kastler for advanced communication of his results on integral representations of states. I want also to thank H. Araki for critical reading of the manuscript, and J. Dixmier for useful references.

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