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SOME MATHEMATICAL PROBLEMS IN THE RENORMALIZATION OF QUANTUM FIELD THEORIES

by

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§ 1 . INTRODUCTION.

Despite the pompous title of this talk, we are going to speak only of relatively trivial things and for those of you who did listen to my talk in Varenna last Summer, they don't have to blame themselves if, by any chance, they again don't understand me, the content of the talk is completely different.

The purpose of this talk is to show to those of you who have tried to study the Varenna lectures of Glimm and Jaffre, that quite a lot of the problems which appear in considering interesting field theories are already present in the case of the relatively trivial quadratic interactions. For those of you who have no idea of the little games we are playing at, I hope it can serve as an elementary introduction.

The model we shall consider here is the renormalization of a free field, both the mass and the field strength renormalization. Before I go any further, let me state many persons in Geneva have learn playing with these problems and other ones and in particular, that my assistant J.P. Eckmann has done a good part of the work I shall report upon.

To recall the general lines of the program ,one first considers the quasi-local C^*-algebra $\mathfrak{A}$ generated by the free field operators with compact support. The time evolution is then considered abstractly as an 1-parameter abelian group of automorphisms σ_t of $\mathfrak{A}$. If the theory is the theory of a free field, these automorphisms are generated in the Fock representation by unitary operators, the infinitesimal generator of which is H_o , the free Hamiltonian.

Most of the problems of quantum field theory have their root in the fact that whenever we want to have a theory with an non-trivial interaction, the total Hamiltonian $H = H_o+H_1$ is not defined as an operator on the Fock space $\mathcal{F}$. This means in practice that we have to find some other way of generating the automorphisms representing the time evolution. The basic idea

* Talk given November 22,1968 at the 7th meeting between physicists and mathematicians organized by the department of mathematic of the University of Strasbourg.

is to butcher the theory, in introducing as many cut-offs as necessary so that to make the Hamiltonian well defined, to generate the automorphism with that butchered Hamiltonian, and then to remove the various cut-offs, considering the limit of the automorphisms, and not the limit of the Hamiltonian since we know that this last one doesn't exist.

Unfortunately the situation is not quite as simple. One of the main difficulty is the one I did try to make clear in my Varenna lectures, namely that in the case one has to admit an infinite field strength renormalization the algebra $\mathfrak{A}$ in the limit of no cut-offs also changes one has a contraction of the algebra $\mathfrak{A}$ we started with.

In other words if $\sigma_{\Lambda t}$ are the cut-off Λ dependent automorphisms of the time evolution, $\lim_{\Lambda \to \infty} \sigma_{\Lambda t}[a]$ doesn't exists for all elements of $\mathfrak{A}$. In face of this difficulty, one may take at least two attitudes , first one could consider a much smaller C^*-algebra than $\mathfrak{A}$ which would not be generated by local von Neumann algebras, but by smaller local C^*- algebras, or secondly, one could consider the limiting procedure as implying a contraction of the algebra $\mathfrak{A}$. We are not going to discussthe points here this time.

Suppose one is able to overcome all difficulties, that mean that one has ended up with a C^*-algebra $\mathfrak{A}$ and an abelian group of automorphisms on it, which represents time evolution. The main problem there, for the physicists, is to find a vacuum state, that is a positive linear functional f_o on $\mathfrak{A}$, invariant under time-translations and the other groups of invariance which we want to consider. This alone would be very easy to construct, but there is more. With that positive linear functional, we can make a Gelfand construction, we then get an Hilbert space $\mathfrak{H}_{f_o}$ and if everything is O.K., our time evolution is implemented by unitary operators with infinitesimal generators $H_{Phys.}$. This $H_{Phys.}$ must be the physical Hamiltonian, and that means for the physicist that it must have the right spectral properties, in particular, it must be a non-negative self-adjoint operator. This last requirement is difficult to satisfy, as we shall already see it in our simple examples.

We shall be obliged to introduce so called dressing transformation in order to be able to construct our vacuum functionals, but after so many words, to the work.

§ 2 . THE MODEL.

Let $\phi(x)$ be a scalar neutral free field of mass m defined on its Fock space $\mathcal{H}$. H_o is the free Hamiltonian :

$$H_o = \frac{1}{2}\int \{m^2 :\phi^2:(\underline{x}) + :\dot{\phi}^2:(\underline{x}) + :(\nabla\phi)^2:(\underline{x})\}\, d^s x$$

$$= \int \omega(\underline{k})\, a^*(\underline{k})a(\underline{k})d^s k$$

with the usual rules and notations.

$$\phi(\underline{x}) = \frac{1}{(2\pi)^{s/2}}\int \frac{d^s k}{(2\omega(\underline{k}))^{\frac{1}{2}}}\, \{a(\underline{k})+a^*(-\underline{k})\}\, e^{i\underline{k}\cdot\underline{x}}$$

$$\dot{\phi}(\underline{x}) = \frac{i}{(2\pi)^{s/2}}\int \frac{d^s k}{(2\omega(k))^{\frac{1}{2}}}\, \omega(\underline{k})\, \{a^*(-\underline{k})-a(\underline{k})\}\, e^{i\underline{k}\cdot\underline{x}}$$

$$[a(\underline{k}),a(\underline{k}')]=[a^*(\underline{k}),a^*(\underline{k}')]=0 \qquad [a(\underline{k}),a^*(\underline{k}')] = \delta(\underline{k}-\underline{k}')$$

$$\omega(\underline{k}) = (\underline{k}^2+m^2)^{\frac{1}{2}}$$

We want now to add to this H_o an interaction term H_I of the form

$$H_I = \mu\int :\phi^2:(\underline{x})d\underline{x} + \vartheta\int \Box :\phi^2:(\underline{x})\, d\underline{x}$$

$$= \mu\int \frac{d\underline{k}}{2\omega(\underline{k})}\, \{a(\underline{k})a(-\underline{k})+2a^*(\underline{k})a(\underline{k})+a^*(-\underline{k})a^*(\underline{k})\}$$

$$- 2\vartheta\int d\underline{k}\, \omega(\underline{k})\, \{a(\underline{k})a(-\underline{k})+a^*(\underline{k})a^*(-\underline{k})\} .$$

It does not take long to any body to convince himself that the objects we did write as the parts of H_I are in fact not defined as operators on $\mathfrak{h}$ or rather, that only the zero vector belongs to their domain. What happens in the case of H_0 is that the ambiguities cancel between the different terms, and this precisely because the mass m^2 appearing in front of the first term is exactly the mass m^2 entering the $\omega(\underline{k})$ in the definition of the field.

Nevertheless, in a formal way we imagine that a theory with total Hamiltonian H_0+H_I should correspond to a theory with a mass $m'=(m^2+2\mu)^{\frac{1}{2}}$ and a field amplitude proportional to $(1+4\vartheta)^{\frac{1}{2}}$, instead of being proportional to one. For these reasons, these two terms are respectively called mass and field amplitude (or strength) renormalization .

To follow our general program, the first thing to do is to butcher H_I sufficiently so that to make every thing well defined. We could try to define

$$H_{I\alpha} = \mu \int :\phi^2: (\underline{x})\ f_\alpha(\underline{x})d\underline{x} + \vartheta\int (\Box :\phi^2:(\underline{x}))\ f_\alpha(\underline{x})d\underline{x}$$

but this is not enough, as we can see in applying the first term to the vacuum ; we get

$$\| \int :\phi^2:(\underline{x}) f_\alpha(\underline{x}) d^s x\ |0> \|^2 \sim \int \frac{d^s k_1\ d^s k_2}{\omega(\underline{k}_1)\,\omega(\underline{k}_2)}\ |\tilde{f}_\alpha(\underline{k}_1+\underline{k}_2)|^2$$

which diverges for $s \geq 2$, even if $f_\alpha \in \mathcal{S}$. The same remains true if we take any vector with a finite number of particles.

We shall therefore put

$$H_{I\alpha,\beta} = \mu \int f_\alpha(\underline{x})\ \varphi_\beta(\underline{y}) : \ \phi(\underline{x}+\underline{y})\phi(\underline{x}-\underline{y}) : d\underline{x}d\underline{y}$$

$$+ \vartheta \int f_\alpha(\underline{x})\ \varphi_\beta(\underline{y})\, \Box_x : \phi(\underline{x}+\underline{y})\phi(\underline{x}-\underline{y}) : d\underline{x}d\underline{y}$$

$H_{\alpha\beta} = H_o + H_{I\,\alpha,\beta}$, and in order to be completely explicit

$$f_\alpha(\underline{x}) = \prod_{i=1}^{s} e^{-\frac{x_i^2 \alpha_i}{4}} \xrightarrow[\{\alpha_i\}\to 0]{\text{in } \mathcal{S}'} 1$$

$$\varphi_\beta(\underline{x}) = \left(\frac{1}{\pi}\right)^{s/2} \prod_{i=1}^{s} \frac{1}{\sqrt{\beta_i}}\, e^{-\frac{x_i^2}{\beta_i}} \xrightarrow[\{\beta_i\}\to 0]{\text{in } \mathcal{S}'} \prod_{i=1}^{s} \delta(x_i)$$

It is clear that in order to compute emplicitely the automorphism, the only thing we need to know is the

$$a(\mathbf{p},t) = \lim_{\{\alpha i\},\{\beta i\}\to 0} a_{\alpha_i,\beta_i}(\underline{p},t) = \lim e^{iH_{\alpha,\beta}t}\, a(\underline{p})\, e^{-iH_{\alpha,\beta}t}$$

Using methods developped in [1], one shows that the limit exists in the strong topology and that the answer is

$$a(\underline{p},t) = A_1(\underline{p},t)a(\underline{p}) + A_2(\underline{p},t)a^*(-\underline{p})$$

with

$$A_1(\underline{p},t) = \frac{1}{4}\left\{\left(1-\frac{\omega(1+4\vartheta)}{\Omega}\right)\left[1-\frac{\omega}{\Omega}\left(1-4\vartheta+\frac{2\mu}{\omega^2}\right)\right]e^{i\Omega t} + \left(1+\frac{\omega(1+4\vartheta)}{\Omega}\right)\left[1+\frac{\omega}{\Omega}\left(1-4\vartheta+\frac{2\mu}{\omega^2}\right)\right]e^{-i\Omega t}\right\}$$

$$A_2(\underline{p},t) = -\tfrac{1}{4}\left\{\left(1-\frac{\omega(1-4\nu)}{\Omega}\right)\left[1+\frac{\omega}{\Omega}\left(1-4\nu+\frac{2\mu}{\omega^2}\right)\right]e^{i\Omega t}+\left(1-\frac{\omega(1+4\nu)}{\Omega}\right)\left[1-\frac{\omega}{\Omega}\left(1-4\nu+\frac{2\mu}{\omega^2}\right)\right]e^{-i\Omega t}\right\}$$

where $\Omega^2 = \omega^2(1+4\nu)(1-4\nu+\frac{2\mu}{\omega^2})$. We could also write

$$A_1(\underline{p},t) = \cos\Omega t + i\,\frac{\omega}{\Omega}\left(1+\frac{\mu}{\omega^2}\right)\sin\Omega t$$

$$A_2(\underline{p},t) = i\,\frac{\omega}{\Omega}\left(4\nu-\frac{\mu}{\omega^2}\right)\sin\Omega t$$

We note that A_1 and A_2 are entire analytic functions in μ and ν. This is remarkable, since it implies an infinite radius of convergence for our automorphisms; remember that already in the case of a mass renormalization, the Green-functions expansions

$$\frac{1}{p^2-m^2-2\mu+i0} = \frac{1}{p^2-m^2+i0}\sum_{n=0}^{\infty}\frac{(2\mu)^n}{(p^2-m^2+i0)^n}$$

has a zero radius of convergence, as an element of $\mathcal{S}'(\mathbb{R}^4)$. (This remark is due to K. Hepp).

If we now define

$$b(\underline{k}) = \frac{1}{2[\Omega\omega(1+4\nu)]^{\frac{1}{2}}}\left[(\Omega+\omega(1+4\nu))a(\underline{k})+(\Omega-\omega(1+4\nu))a^*(-\underline{k})\right]$$

we can write, as for element of $\mathcal{S}'$

$$\phi(\underline{x},t) = s-\lim_{\{\alpha_i\},\{\beta_i\}\to 0} e^{iH_{\alpha\beta}t}\,\phi(\underline{x})\,e^{-iH_{\alpha\beta}t}$$

$$= \frac{\sqrt{1+4\nu}}{\sqrt{(2\pi)^s}}\int\frac{d^s k}{\sqrt{2\pi}}\,e^{i\underline{k}\cdot\underline{x}}\left\{b(\underline{k})e^{-i\Omega t}+b^*(-\underline{k})e^{i\Omega t}\right\}$$

The automorphism therefore represents a Bogoliubov -Valatin transformation, provided of course, that Ω is real. It is easy to check that $b(\underline{k})$ and $b^*(\underline{k}')$ do satisfy the same commutation relations as the $a(\underline{k})$ and $a^*(\underline{k}')$.

§ 3 . THE VACUUM

We have now an algebra, on which the time evolution is given by a group of automorphisms; we want to have a vacuum functional. If we take the vacuum vector $|0\rangle$ of our original Fock-space ,we would get for our 2-points function

$$W_0^2 \equiv \langle 0|\, \phi(\underline{x},t)\phi(\underline{y},s)\, |0\rangle = \frac{(1+4\nu)}{(2\pi)^s} \cdot \frac{1}{4} \int \frac{d^s k}{2\pi(\underline{k})} \cdot \frac{1}{\Omega\omega(1+4\nu)} \cdot$$

$$\cdot \{ [\Omega^2-\omega^2(1+4\nu)^2]\, e^{-i\Omega(t+s)} e^{i\underline{k}\cdot(\underline{x}+\underline{y})}$$

$$+[\Omega+\omega(1+4\nu)]^2\, e^{-i\Omega(t-s)}\, e^{i\underline{k}\cdot(\underline{x}-\underline{y})}$$

$$+[\Omega-\omega(1+4\nu)]^2\, e^{+i\Omega(t-s)}\, e^{-i\underline{k}\cdot(\underline{x}-\underline{y})}$$

$$+[\Omega^2-\omega^2(1+4\nu)^2]\, e^{+i\Omega(t+s)}\, e^{-i\underline{k}\cdot(\underline{x}+\underline{y})} \}$$

This is obviously not the right answer, it is not even invariant under time translation. Making a big translation of the Borcher's type in a space-like direction gives nothing more, either we stay in the plane t = cte., and nothing happens, or we have a small time like component and the expression oscillates.

The next thing to try is some kind of ergodic mean. We could define

$$W^2(\underline{x},t\ ;\ \underline{y},s) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{+T} W_0^2(\underline{x},t+\tau\ ;\ \underline{y},\ s+\tau)\ d\tau$$

and this simply make the first and fourth terms to drop. In this may one does indeed construct a linear invariant positive functional over the algebra $\mathcal{O}\!\mathcal{l}$ and it is then possible to make the Gelfand construction.

The answer is, however, not satisfactory, because in the Hilbert space defined by the Gelfand construction, the energy will not be positive, as is apparent from the structure of the 2-points functions :

$$W^2(\underline{x},t\ ;\ \underline{y},s) = \frac{(1+4\vartheta)}{(2\pi)^s} \circ \frac{1}{4}\int \frac{d^s k}{2\Omega(\underline{k})} \cdot \frac{1}{\Omega\omega(1+4\vartheta)}$$

$$\{[\Omega+\omega(1+4\vartheta)]^2 e^{-i\Omega(t-s)}\ e^{i\underline{k}(\underline{x}-\underline{y})} + [\Omega-\omega(1+4\vartheta)]^2\ e^{i\Omega(t-s)} e^{-i\underline{k}(\underline{x}-\underline{y})}\}$$

which apart from functions of k which are not the right ones, contains both positive and negative frequencies. Of course, the right answer would be

$$W^2(\underline{x},t\ ;\ \underline{y},s) = \frac{(1+4\vartheta)}{(2\pi)^s}\int \frac{d^s k}{2\Omega(\underline{k})}\quad e^{-i\Omega(\underline{k})(t-s)}\ e^{i\underline{k}(\underline{x}-\underline{y})}$$

There exist of course the possibility of defining abstractly a vacuum functional, using fixed point theorems of the Markov-Kakutari type for instance. In practice this is of no use, since it is difficult to ensure that the functional will lead to a representation with positive energy.

Still an other possibility, is to look at what we shall call "formal vacuum states". One can prove that no vector of the original Fock space can generate a vacuum state, nor any density matrix. But we could try to find an "object" $|\Omega\rangle$ such that $b(\underline{k})|\Omega\rangle = 0$. If we try to write down $|\Omega\rangle$ in the form $|\Omega\rangle = \sum_{n=0}^{\infty} c_n |\psi_n\rangle$, where the $|\psi_n\rangle$ are n-particles states of the Fock space, one obtains, that $\sum_{n=0}^{\infty} |c_n|^2 \langle\psi_n|\psi_n\rangle = \infty$, but there exist an answer. Now it is not convenient to work with infinite series, one more easily works with compact expressions. It turns out that one can write formally.

$$|\Omega\rangle = \exp\left(\int f(\underline{k}) a^*(\underline{k}) a^*(-\underline{k})\, d\underline{k}\right) |0\rangle \equiv T_0 |0\rangle$$

where

$$f(\underline{k}) = -\frac{1}{2} \cdot \frac{\Omega - \omega(1+4\nu)}{\Omega + \omega(1+4\nu)}$$

This expression is of course ill-defined, but here and in everything which follows, it is not difficult to make something rigorous out of it, so that in order to save time, we shall keep working with formal, but convenient, expressions .

It is easy to check that

$$W^2(\underline{x}, t ; \underline{y}, s) = \langle 0|T_0^* \phi(\underline{x}, t)\, \phi(\underline{y}, s)\, T_0|0\rangle \cdot \langle 0|T_0\, T|0\rangle^{-1}$$

yields the correct two points function. This kind of formula remembers us of many other similar ones often encountered in field theory.

As a linear functional leading to a representation with positive energy, W^2 is unique. But this doesn't imply that T_0 is unique. In fact, one shows that

$$T|0\rangle = \exp\left(\int f_1(\underline{k})a^*(\underline{k})a^*(-\underline{k}) + \int f_2(\underline{k})\right)$$

with $T = \exp\left(\frac{1}{2}\int \gamma_1(\underline{k})a^*(\underline{k})a^*(-\underline{k}) + \int \gamma_2(\underline{k})a^*(\underline{k})a(\underline{k}) + \frac{1}{2}\int \gamma_3(\underline{k})a(\underline{k})a(-\underline{k})\right)$.

The solution is given by

$$1 = \int_0^{f_1} \frac{dx}{\frac{\gamma_1}{2} + 2\gamma_2 x + 2\gamma_3 x^2} \text{ , } f_2(\underline{k}) = 2\int f_1(\underline{p})\,\delta(\underline{p}+\underline{k})\,\delta(\underline{p}+\underline{k})\,\gamma_3(\underline{k})\,d\underline{p}$$

Our solution T_0 implies that $H\,T_0|0\rangle = 0$. This relation is in fact a much weaker requirement then the one of the existence of an inter-twinning operator $\mathcal{U}$

$$H\,\mathcal{U} = \mathcal{U}H_0$$

If we take T as ansatz for $\mathcal{U}$, we get the answer

$$\mathcal{U} = \exp\left(\int \frac{\lambda(k)}{2}\, a^*(\underline{k})a^*(-\underline{k}) - \int \frac{\lambda(\underline{k})}{2}\, a(\underline{k})a(-\underline{k})\right)$$

with $$\frac{1}{2}\lambda(k) = \mathrm{tgh}\,\frac{1}{2}\,\frac{\Omega-\omega(1+4\lambda\varphi)}{\Omega+\omega(1+4\varphi)}$$

which means that $\mathcal{U}$ is formally unitary.

What is interesting to notice , is that since all Fock spaces are characterized by the field complitude and the mass, the "dressing" transformations of the form of $\mathcal{U}$ map any Fock space onto any other one.

§ 4 . CONCLUSIONS.

We did present here only the spin 0 neutral cas, we also have treated the spin $\frac{1}{2}$ and spin 1 cases, with quite similar, albeit more complicated results. One of the most interesting thing to study in this model is the behaviour of the quasi-local algebra when the field amplitude renormalization becomes infinite, that is round $\nu = -\frac{1}{4}$. An other interesting thing is also to study these renormalizations for $s = 4$ or 5 , this is being studied presently by K. Hepp .

REFERENCES

[1] M. GUENIN - G. VELO : Helv. Phys. Acta 41 (1968) 362 .

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