D. KASTLER

A C^*-Algebra Approach to Field Theory

Les rencontres physiciens-mathématiciens de Strasbourg - RCP25, 1968, tome 4
« Conférences de R. Stora et textes sur les C*-Algèbres de S. Doplicher, A. Guichardet, D. Kastler et G. Loupias », , exp. n° 4, p. 1-17
<http://www.numdam.org/item?id=RCP25_1968__4__A4_0>

IV

A C^* - ALGEBRA APPROACH TO FIELD THEORY

D. KASTLER

Department of Physics - University of Illinois*

*Present address : Physique théorique - Faculté des Sciences
Place Victor Hugo - Marseille 3ème - France

A C^* - algebra Approach to Field Theory

This talk is a report of a common work with R. Haag which will be published elsewhere in extenso [1]. The main ideas are Haag's, my role consisted in bringing them into contact with the mathematical literature. You know that I.E. Segal was the first to recommend the use of a C* - algebra for quantum mechanics : he proposed to interpret its self adjoint elements as physical observables and its positive forms a physical states [2]. On the other hand C* - algebras appear naturally in all the works concerned with the representation of the canonical commutation relations. A quantum mechanical frame based on a C* - algebra has the appealing feature of being purely algebraic, since one knows that the norm of a C* - algebra is algebraically determined. Our objective in this work is the theory of coupled fields and we offer :

1) an analysis of the concept of physical equivalence of two theories which, drawing upon mathematical results of J.M.G. Fell [3], leads to a purely algebraic setting for general quantum mechanics

2) a purely algebraic approach to field theory whose basic mathematical structure appears to be "the algebra of quasi local observables" faithfully represented in each super selection sector. This approach is obtained by combining 1) with Haag's "principle of locality" for field theory. I shall discuss at the end the relation of the present C* - algebra approach with the theory of local Von Neuman rings [4], [5] , [12]. For self containment we add a mathematical appendix describing Fell's results.

§ 1. Physical equivalence of representations . A purely algebraic setting for Quantum Mechanics . Our aim in this paragraph is to show that two quantum mechanical theories can be physically equivalent (that is , they can convey the same physical information) without being unitarily equivalent. Physical equivalence will be shown to coïncide with "weak equivalence" as defined by J.M.G. Fell [3] . Fell's "equivalence theorem " then implies the possibility of a purely algebraic setting for quantum mechanics.

We start from the usual hypothesis (which we here accept uncritically) that the observables of quantum mechanics are the self adjoint elements of a $*$ - algebra $\mathfrak{A}$ which can be realised as a $*$ - algebra of bounded operators on some Hilbert space. We assume furthermore that $\mathfrak{A}$ is complete with respect to the operator norm , i.e. that $\mathfrak{A}$ is a C^*-algebra (if $\mathfrak{A}$ were not complete we would get a C^*-algebra by the standard process of completion - we would assume in that case that the elements of the completion still correspond to physical observables - and would take the completion for $\mathfrak{A}$ itself). Now we are confronted with two possibilities as to the relevant mathematical object for the description of physics. It can be :

- (1) either $\mathfrak{A}$ as a concrete (norm closed) $*$ - algebra of bounded operators on an Hilbert space $\mathcal{H}$ (up to unitary equivalence)
- (2) or $\mathfrak{A}$ as an abstract C^*-algebra without reference to some particular realization as a norm-closed operator algebra on some Hilbert space.

Traditionally the choice made for the frame of quantum mechanics is that of possibility (1). The pure states of the physical system are described by the vectors of $\mathcal{H}$. The "mixtures" are described by density matrices, i.e. positive operators Φ on $\mathcal{H}$ with finite traces, the expectation value of $A \in \mathfrak{A}$ in the state Φ being given by

$$\Phi(A) = \mathrm{Tr}\,\{\ \Phi A\ \}$$

This frame obviously contains more structure than possibility (2) since it needs not only the specification of the C^*-algebra $\mathfrak{A}$ (as in (2)) but also the specification of its concrete realization on the Hilbert space $\mathcal{H}$ (that is, of a certain faithful $*$ - representation of $\mathfrak{A}$ up to unitary equivalence) . Contrasting with the choice of (2) as a frame for quantum mechanics implies that the specification of a special representation is physically irrelevant, all the physical information being contained in the algebraic structure of the abstract algebra $\mathfrak{A}$ alone.

In order to decide between (1) and (2) let us consider the abstract C^*-algebra $\mathfrak{A}$ and two representations R_1 and R_2 of $\mathfrak{A}$ on Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ (for generality, we do not take R_1 and R_2 to be faithful or irreducible). What shall we require from R_1 and R_2 in order for them to be physically equivalent ? We want the results of any finite set of measurements on a physical state to be equally well describable in terms of a density matrix on $\mathcal{H}_1$ or a density matrix on $\mathcal{H}_2$. As measurements are never totally accurate the wording : equally well is to be understood as : to any desired degree of accuracy. We are thus led to the following statement :

R and R' are physically equivalent if for any finite subset A_1, $A_2, \ldots A_n$ of $\mathfrak{A}$, any positive operator with finite trace Φ_1 on $\mathcal{H}_1$ and any $\varepsilon > 0$ there should exist a positive operator with finite trace Φ_2 on $\mathcal{H}_2$ such that

$$|\Phi_1(A_k) - \Phi_2(A_k)| = |\mathrm{Tr}\{ \Phi_1 R_1(A_k) \} - \mathrm{Tr}\{ \Phi_2 R_2 (A_k) | < \varepsilon$$

$$k = 1,2,\ldots n$$

and vice versa . This statement means that the respective sets of positive linear forms on the C^* - algebra $\mathfrak{A}$ defined by the density matrices in the representations R_1 and R_2 should have the same closure as subsets of the dual space $\mathfrak{A}^*$ of $\mathfrak{A}$ equipped with its weak topology (with respect to $\mathfrak{A}$) . This is the situation described by Fell as the "<u>weak equivalence</u>" of the representations R_1 and R_2. Now Fell's "equivalence theorem" asserts that R_1 and R_2 will be weakly equivalent (for us, physically equivalent) if and only if they have the same kernel i.e. if the abstract elements of $\mathfrak{A}$ with zero representations are the same for R_1

and R_2. This is exactly the result needed to conclude our discussion in favour of the choice of (2) for the frame of quantum mechanics : it shows indeed that the physically relevant object is not a concrete realization of $\mathfrak{A}$ but the algebra $\mathfrak{A}$ itself since any two different concrete realizations (= faithful $*$-representations, or representations with zero kernel) will be physically equivalent.

Haag's notion of "physical equivalence" as described above arose in the course of a work on infra-particles [6]. Fell developped the same notion, to which he gave the name of "weak equivalence", in [3]. Fell's "equivalence theorem" characterizing weak equivalence on a purely algebraic way effects the passage from the discussion of physical equivalence, to a purely algebraic frame for quantum mechanics.

In the case of <u>irreducible</u> representations R_1 and R_2 we could have given the same argument replacing mixtures by pure states. (Fell's equivalence theorem can namely be stated for irreducible representations replacing density matrices by vectors of the corresponding spaces).

Note that if $\mathfrak{A}$ is separable, which is natural to assume, the weak topology of $\mathfrak{A}^*$ is metrizable on its unit ball. In that case the substitute Φ_2 in the R_2-description of a Φ_1 in the R_1-description can be chosen out of a Cauchy sequence $\{ \Phi_2^n \}$ of density matrices on $\mathcal{H}_2$ converging weakly towards Φ_1. So one should not feel uncomfortable about the fact that Φ_2 a priori depends on the set A_k and on Φ_1.

The above discussion presents the algebraic frame (2) as resulting from the traditional frame (1) through the recognition that all concrete realization of $\mathfrak{A}$ are physically equivalent. This might be appropriate in order to convince the supporters of frame (1) but is philosophically unduly short ranged. For a direct introduction of frame (2) based on an analysis of the way in which physical states are prepared and monitored, we refer to reference [1] .

§ 2 . The principle of locality. A purely algebraic approach to field theory. Relation to superselection.

We now turn our attention to the quantum theory of coupled fields. In order to provide a description of physics the general frame discussed above must be substanciated by(i) a precise mathematical specification of algebra $\mathfrak{A}$ (ii) a dictionary stating the meaning of each element of $\mathfrak{A}$ in terms of laboratory procedures. Both (i, and (ii) are provided to a certain extent by "Haag's " principle of locality" first put forward in [4] . This principle states that it is meaningful to consider measurements within localized regions and that these measurements correspond to the self adjoint elements of a "local algebra". In the present frame this principle leads to the following axioms (which are the transcription of Haag's Von Neumannring axioms to the (more general) C^* - algebra setting :

To each "region" B (i.e. open space-time domain with compact closure) there corresponds uniquely a C^*-algebra $\mathfrak{A}(B)$ so that one has

I) Isotony : $B_1 \subset B_2$ implies $\mathfrak{A}(B_1) \subset \mathfrak{A}(B_2)$

As a result of this axiom $\bigcup_B \mathfrak{A}(B)$ is an incomplete C^* -algebra whose completion we denote by $\mathfrak{A}$ and call the <u>algebra of quasi local observables</u>.

II) Local commutativity $B_1 \subset B'_2$ (i.e. B_1 and B_2 lie space-like to each other) implies that $\mathfrak{A}(B_1) \subset \mathfrak{A}(B_2)'$ ($\mathfrak{A}(B_2)'$ denotes the commutant of $\mathfrak{A}(B_2)$ in $\mathfrak{A}$).

III) Lorentz invariance. The inhomogeneous connected Lorentz group is represented in the automorphism group of $\mathfrak{A}$ in such a way that

$$\mathfrak{A}(L\,B) = \mathfrak{A}(B)^L$$

(L being a Lorentz transformation, L B is the region resulting from B by applying L, $A \in \mathfrak{A} \to A^L \in \mathfrak{A}$ being the automorphism of $\mathfrak{A}$ induced by L)

These axioms give a partial answer to (i) and (ii). (ii) is satisfied inasmuch as all experiments on elementary particles ultimately result in geometric measurements (for instance it will be sufficient for calculating cross sections - see [7]). On the other hand it is hoped that a structure theory of axioms I), II) and III) and possibly other axioms to be added will give an answer to requirement (i).

Of course one expects that the correspondance $B \to \mathfrak{A}(B)$ will have an extension to more general domains than bounded "regions" in a way similar to what is done in measure theory. A problem of particular interest is the following. Take two domains D_1 and D_2 space like to each other (D_1 and D_2 are or are not bounded regions). $\mathfrak{A}(D_1)$ and $\mathfrak{A}(D_2)$ are then expected to commute as an extension of property II). Let $\mathfrak{B}$ be the sub C^*-algebra of $\mathfrak{A}$ generated by $\mathfrak{A}(D_1)$ and $\mathfrak{A}(D_2)$. Under which circumstances is $\mathfrak{B}$ the direct product of $\mathfrak{A}(D_1)$ and $\mathfrak{A}(D_2)$ in the sense of Turumaru [8] ? In particular does one have the property $\mathfrak{A} = \mathfrak{A}(D) \otimes \mathfrak{A}(D')$? This would be possible even if $\mathfrak{A}(D)$ and $\mathfrak{A}(D')$ do not give rise to associated factors of type I in certain representations (for this we refer to [9]). A safer conjecture is that $\mathfrak{A}(D_1)$ and $\mathfrak{A}(D_2)$ "combine tensorially" when the causal shadows of D_1 and D_2 have disjoint closures so that "contact effects" are excluded.

It is important to realize that the definition of $\mathfrak{A}$ excludes from it the "global quantities" like the total energy, the total charge etc . Neither are the Lorentz automorphisms $L \to A^L$ implementable by elements of $\mathfrak{A}$ since a Lorentz transformation is a global operation (in other words the Lorentz automorphisms are outer automorphisms). The distinction between local and global quantities is particularly striking in connection with superselection rules. Let R_k be the "superselection sectors" of standard field theory invariant under all operators of the theory. The algebra $\mathfrak{A}$ will have a $*$-representation R_k on each $\mathcal{H}_k$, the direct sum of which is the faithful representation usually considered in field theory. A simple physical argument shows that the representations R_k are all mutually physically equivalent : any density matrix on a given sector can be simulated with arbitrary accuracy by a density matrix in any other preassigned sector by adding to the system which it describes some particles or antiparticles in a remote portion of space-time so as to compensate appropriately the value of the superselecting quantities. We thus come out with the conclusion that all representations R_k are faithful, each of them taken separately being a complete description of physics. The direct sum of the R_k has a umformly closed range since $\mathfrak{A}$ is a C^*-algebra. It is important to realize that it is not weakly closed and that its weak closure contains the global operators : we know that we obtain the weak closure by taking the double commutant. Now the R_k being irreducible and mutually inequivalent the commutant of their sum consists in all bounded linear combination of the projectors P_k on the $\mathcal{H}_k$. So the bicommutant consists in the direct product of all full operator rings $\mathcal{L}(\mathcal{H}_k)$ on the different $\mathcal{H}_k$ and so contains all the operators of the standard theory - but it has no interesting algebraic structure.

Note that the algebra $\mathfrak{A}$ in NGCR in the sense of Glimm [10] since it has many irreducible non equivalent faithful representations. According to R.V.Kadison in [11] the set of those representations has then the power of continuum. What singles out the discreteset of 'superselection sectors ? - Perhaps the requirement that the Lorentz automorphisms should be implementable by unitaries on the representation spaces.

§ 3 - C^*-algebra formalism versus theory of local Von Neuman algebras. We shall now briefly discuss the relation of the present formalism to the theory of local Von Neumann algebras [4] ; [5] , [12] (for brevity we refer to those formalisms respectively as the C^*-theory and the W^*-theory). The W^*-theory was originally stated in terms of a $*$-representation on a Hilbert space. However, nothing prevents from considering the local Von Neumann algebras as abstract algebras : the axioms of the W^*-theory are then obtained by replacing in axioms I), II), III) above the local C^*-algebras $\mathfrak{A}(B)$ by local Von Neumann algebras $\mathfrak{R}(B)$ (writing Von Neumann algebra for C^*-algebra wherever the word occurs). Note that the theory thus obtained is, like the C^*-theory , purely algebraically defined since the strongest topology of a Von Neumn algebra is determined by the algebraic structure alone (its continuous linear forms being differences of normal positive forms). Consider a $*$-representation R of the C^*-theory and put $\mathfrak{R}(B) = \overline{R(\mathfrak{A}(B))}$ where the bar denotes the closure in the strongest topology of operators. If R is such that the Lorentz automorphisms are continuous in the strongest topology of operators, they can be extended to the $\mathfrak{R}(B)$ which will then fulfill the axioms of the W^*-theory. In this case under what conditions will two $*$-representations R and R' of the C^*-theory thus lead to $*$-representations of the same W^*-theory ? If and only if for each region B the $*$-automorphism $R(\mathfrak{A}(B)) \longleftrightarrow R'(\mathfrak{A}(B))$ which they define is extendable to $*$-automorphisms $\overline{R(\mathfrak{A}(B))} \longleftrightarrow \overline{R'(\mathfrak{A}(B))}$ (the bars denote closures in the strongest topologies respectively defined by R and R'). This can be expressed by requiring that R and R' be locally quasi equivalent in the sense that their restriction to all local algebras $\mathfrak{A}(B)$ be quasi equivalent in the sense of Mackey. At the present stage we do not yet understand the role played by local quasi equivalence in field theory. A comparison of the different superselection sectors with respect to charge in free fermion field theory under this angle would be desirable as a first exploration in this connection.

[1] R. Haag and D. Kastler — An algebraic approach to field theory (to be published)

[2] I.E. Segal — Postulates for general quantum mechanics (Ann. of Math 48 p.930 (1947)

[3] J.M.G. Fell — The dual spaces of C* - algebras (Trans. Amer. Math. Soc. 94 , (1960) p.365)

[4] R. Haag — Discussion des "axiomes" et des propriétés asymptotiques d'une théorie des champs locale avec particules composées (Colloques Internationaux du C.N.R.S. - Lille 1957 p.151)

[5] H. Araki — Zürich mimeographed lectures (1962)

A lattice of Von Neumann Algebras associated with the Quantum Theory of a free Bose field (to be published)

Von Neumann algebras of local observables for free scalar field (to be published)

[6] H. Borchers, R. Haag, B. Schroer — The vacuum state in Quantum Field Theory (Nuovo Cimento 29 p.148 (1963))

B. Schroer — Fortschritte der Physik (1963)

[7] R. Haag — Quantum Field theory with composite particules and asymptotic conditions (Phys. Rev. 112 p.669 (1958))

[8] T. Turumaru — On the direct product of operators algebras I and II - (Tôhoku Mathematical Journal 4 (1953) p.242-251, 5 (1953) p. 1-7)

[9] A. Guichardet — to be published

[10] J. Glimm — Type I C* - algebras (Am. Math. 73 (1961) p.572)

[11] R.V. Kadison — States and representations (Transl. Amer. Math. Soc. 103 (1962) p.304)

[12] R. Haag, B. Schroer — Postulates of quantum field theory (Journal Math. Phys. 3 p.248 (1962))

MATHEMATICAL APPENDIX [1]

Preorder relation. A relation $x \propto y$ between the elements of a set $\mathfrak{M}$ is called a preorder relation if

α) $x \propto y$ and $y \propto z$ implies $x \propto z$ for any $x,y,z \in \mathfrak{M}$.

β) $x \propto x$ for all $x \in \mathfrak{M}$

Note that it is not required that the preordering be total, i.e., that given any two $x,y \in \mathfrak{M}$ they be preordered (that $x \propto y$ or $y \propto x$).

Order relation. A preorder relation $\propto$ is called an order relation if one has the additional condition

γ) $x \propto y, y \propto x$ imply that $x = y$ (that x and y be identical).

Given any preorder relation $\propto$ on a set $\mathfrak{M}$ and defining $x \sim y$ to mean that $x \propto y$ and $y \propto x$ one gets an equivalence relation called the equivalence associated with the preordering $\propto$. One sees immediately that the preorder relation $\propto$ induces an order relation on the set $\mathfrak{M}$ of equivalence classes of $\mathfrak{M}$ modulo $\sim$. This is called the quotient ordering of the preordering $\propto$.

Join and Meet. Let $\mathfrak{M}$ be an ordered set (a set equipped with an order relation $\propto$). One says that the element $a \in \mathfrak{M}$ is the join (meet) of a subset $\mathfrak{X} \in \mathfrak{M}$ if

i) $x \propto a$ $(a \propto x)$ for all $x \in \mathfrak{X}$

ii) any $b \in \mathfrak{M}$ with the same property $(x \propto b$ $(b \propto x)$ for all $x \in \mathfrak{X})$ is smaller (greater) than a.

The condition γ) implies the uniqueness of the join (meet) if it exists. The joint (meet) of $\mathfrak{X} \in \mathfrak{M}$ are respectively denoted by $\bigvee_{x \in \mathfrak{X}} x$ $\left(\bigwedge_{x \in \mathfrak{X}} x\right)$.

---

1) The material contained in this appendix is borrowed from G.Birkhoff, Lattice Theory Amer.Math.Soc.Colloqu.Pub.Chap.I, III and IV and from the above quoted article by J.M.G. Fell. We express Fell's results in a lattice theoretic language and give some variants of his theorems useful for our purposes.

They are sometimes called l.u.b.(g.l.b.) of $\mathfrak{X}$.

<u>Lattice</u> An ordered set $\mathfrak{M}$ is called a <u>lattice</u> if all its finite subsets $\mathfrak{X}$ have a joint and a meet in $\mathfrak{M}$. If this is the case for all subsets $\mathfrak{X}$ without restriction $\mathfrak{M}$ is called a <u>complete lattice</u>.

<u>Lattice theoretic closure operation</u>. Let M be an arbitrary set and $\mathfrak{M}$ be the collection of all the subsets of M. $\mathfrak{M}$ is a complete lattice for the ordering $\subseteq$ defined by the inclusion of subsets, the joint (meet) being the set-theoretic union (intersection). We now define a <u>lattice theoretic closure operation on</u> $\mathfrak{M}$ to be the assignment to each subset $X \subset M$ of another subset $\overline{X} \subset M$, called its (<u>lattice theoretic</u>) <u>closure</u>, in such a way that

1) $X_1, X_2 \subset M$, $X_1 \subseteq X_2$ implies $\overline{X}_1 \subset \overline{X}_2$
2) $X \subseteq \overline{X}$ for each $X \subset M$
3) $\overline{\overline{X}} = \overline{X}$ for each $X \subset M$

A subset $X \subset M$ (element $X \in \mathfrak{M}$) is said to be <u>closed</u> if $X = \overline{X}$. $X \subset M$ is closed if and only if it is the closure $\overline{Y}$ of some $Y \subset M$. The closed subsets of M constitute a subcollection $\overline{\mathfrak{M}}$ of $\mathfrak{M}$ ordered by $\subseteq$ and it is not difficult to show that $\overline{\mathfrak{M}}$ is a complete lattice with the following definition of joins and meets :

$$\bigvee_{X \in \mathfrak{X}} X = \overline{\bigcup_{X \in \mathfrak{X}} X} \quad , \quad \bigwedge_{X \in \mathfrak{X}} X = \bigcap_{X \in \mathfrak{X}} X$$

Given the closure operation $X \to \overline{X}$ on $\mathfrak{M}$, if we define the relation $X \prec Y$ for $X, Y \subset M$ to mean that $\overline{X} \subseteq \overline{Y}$ (or equivalently $X \subseteq \overline{Y}$) we get a preorder relation on $\mathfrak{M}$ whose associated equivalence relation is $\overline{X} = \overline{Y}$. The set of the corresponding equivalence classes of $\mathfrak{M}$, equipped with the quotient ordering of the preordering $\prec$ is then isomorphic (as a complete lattice) with the above considered collection $\overline{\mathfrak{M}}$ of closed subsets of $\mathfrak{M}$.

Examples of lattice theoretic closure operations : the linear or the convex closure in a linear space ; the topological closure in a topological space. A standard way of generating a closure operation on $\mathfrak{M}$ is to start from a symmetric binary relation between the points of M (which we will write $x \leftrightarrow y$), define as the "polar" of a subset $X \subset M$ the collection of points of M which fulfil the relation $\leftrightarrow$ with all the points of X :

$$X' = \{ x \in M \mid x \leftrightarrow y \quad \text{for all} \quad y \in X \}$$

and put $\overline{X} = (X')' = X''$. Examples of physical interest :

a) Take for M the Minkowski space of special relativity and for $x \leftrightarrow y$ the circumstance that x and y be space-like to each other. X' is then the region of M lying space-like to the region X and X'' is the "local closure" of X.

b) Take for M a *-algebra (e.g.,of operators) and define X' as the commutant of the set $X \subset M$. X'' is the bicommutant of X. For the case of a *-algebra X of bounded linear operators on a Hilbert space containing the unit, the closure operation $X \to X''$ is the same as the topological closure in the weak operator topology. The parallelism of examples a) and b) is one of the appealing features of the local-ring approach to field theory.

The preceding method for generating closure operations on M can be somewhat generalized by considering a binary relation $x \leftrightarrow \varphi$ between the $x \in M$ and the elements φ of some other set N. The respective "polars" of the subsets $X \subset M$ and $\Phi \subset N$ are defined as the subsets $X^{*} \subset N$ and $\Phi^{+} \subset M$ given by

$$X^{*} = \{ \varphi \in N \mid x \leftrightarrow \varphi \quad \text{for all} \quad x \in X \}$$

$$\Phi^{+} = \{ x \in M \mid x \leftrightarrow \varphi \quad \text{for all} \quad \varphi \in \Phi \}$$

One easily sees that $X \subseteq X_1$ and $\Phi \subseteq \Phi_1$ imply respectively $X_1^{*} \subset X^{*}$ and $\Phi_1^{+} \subset \Phi^{+}$ and that one has $((X^{*})^{+})^{*} = X^{*}$, $((\Phi^{+})^{*})^{+} = \Phi^{+}$ for arbitrary $X \subset M$, $\Phi \subset N$. It results that the operation $X \to (X^{*})^{+}$ consisting of taking the bipolar of $X \subset M$ is a lattice theoretic closure operation on $\mathfrak{M}$. Example : let N be a Banach space, M its topological dual space and define $x \leftrightarrow \varphi$ to mean

$$\operatorname{Re} \varphi(x) \leq 1$$

It is a well known theorem of Mackey that $(\underset{\sim}{X}^{*})^{+}$ coincides with the topological closure in the weak topology of $\underset{\sim}{M}$ (with respect to $\underset{\sim}{N}$) of the convex hull of $\underset{\sim}{X} \cup \{0\}$. This result is at the origin of the equivalence theorem discussed in the next section, on which hinges the notion of physical equivalence of representations.

Weak containment and weak equivalence of representations.

Let $\mathcal{A}$ be a C^{*}-algebra with or without unit and let us denote by $\mathrm{Rep}(\mathcal{A})$ the collection of all its (continuous *) representations. We shall define on the subsets of $\mathrm{Rep}(\mathcal{A})$ (the sets of representations of $\mathcal{A}$) a preorder relation characterizing their being altogether more or less faithful. Let us first, for a single representation S of $\mathcal{A}$, denote by $\mathcal{H}(S)$ its representation space, by $\mathrm{Ker}(S)$ its kernel (i.e., the set of all elements of $\mathcal{A}$ with vanishing representatives in S) and by $\omega(S)$ the collection of all expectation values ω_{ψ} for all the vectors $\psi \in \mathcal{H}(S)$ (considered as positive linear forms on $\mathcal{A}$, thus elements of the dual space $\mathcal{A}^{*}$ of $\mathcal{A}$). Next, considering a set $\mathcal{S} \in \mathrm{Rep}(\mathcal{A})$ of representations of $\mathcal{A}$, we define its <u>kernel</u> and denote by $\mathrm{Ker}(\mathcal{S})$ the intersection of the kernels of all $S \in \mathcal{S}$ and call <u>hull of its kernel</u> and denote $\mathrm{HK}(\mathcal{S})$ the set of all representations of $\mathcal{A}$ whose kernels contain $\mathrm{Ker}(\mathcal{S})$. Thus $\mathrm{Ker}(\mathcal{S})$ is the set of elements of $\mathcal{A}$ with vanishing representatives in all $S \in \mathcal{S}$ and $\mathrm{HK}(\mathcal{S})$ is the set of representations of $\mathcal{A}$ which send to zero all elements of $\mathcal{A}$ sent to zero by a all $S \in \mathcal{S}$. It is easily verified that $\mathrm{Ker}(\mathcal{S})$ coincides with the kernel (in the usual sense) of the representation $\sum^{\oplus}_{S\in\mathcal{S}} S$ direct sum of all $S \in \mathcal{S}$ and that the operation $\mathcal{S} \to \mathrm{HK}(\mathcal{S})$ is a lattice-theoretic closure operation on the subsets of $\mathrm{Rep}(\mathcal{A})$ as described above . Consequently we get a preorder relation on those subsets by setting the

<u>Definition</u> Given two sets of representations $\mathcal{S}, \mathcal{T} \in \mathrm{Rep}(\mathcal{A})$ we call $\mathcal{S}$ <u>weakly contained</u> in $\mathcal{T}$ and write $\mathcal{S} \prec \mathcal{T}$ if $\mathrm{HK}(\mathcal{S}) \subseteq \mathrm{HK}(\mathcal{T})$. $\mathcal{S}$ and $\mathcal{T}$ are said to be <u>weakly</u> or <u>physically equivalent</u> if $\mathrm{HK}(\mathcal{S}) = \mathrm{HK}(\mathcal{T})$.

It should be obvious that requiring $\mathcal{S} \subseteq \mathrm{HK}(\mathcal{T})$, or $\mathrm{Ker}(\mathcal{S}) \supseteq \mathrm{Ker}(\mathcal{T})$, or $\mathrm{Ker}(\sum^{\oplus}_{S\in\mathcal{S}} S) \supseteq \mathrm{Ker} \sum^{\oplus}_{T\in\mathcal{T}} T$ give alternative definitions of the relation $\mathcal{S} \propto \mathcal{T}$. This relation means that the elements of $\mathfrak{A}$ with vanishing representatives in all representations $T \in \mathcal{T}$ have a fortiori vanishing representatives in all representations $S \in \mathcal{S}$. We can thus express it by saying that taken all together the representations of $\mathcal{S}$ are less faithful than those of $\mathcal{T}$ -- or that $\mathfrak{A}$ is better separated by the $T \in \mathcal{T}$ than by the $S \in \mathcal{S}$ in the sense of the separation of its elements by their values in some representations.

Note that if $\mathcal{S}$, resp. $\mathcal{T}$, each consists of one single representation S, resp. T, $\mathcal{S} \propto \mathcal{T}$ simply means $\mathrm{Ker}(S) \subseteq \mathrm{Ker}(T)$, i.E., that S is less faithful than T. In this case we write $S \propto T$, $\propto$ being now a preordering of the (single) representations.

Now let us shift our attention from the representations $S \in \mathrm{Rep}(\mathfrak{A})$ to the corresponding subsets $\omega(S)$ of $\mathfrak{A}^*$. For an arbitrary subset $X \in \mathfrak{A}^*$ we denote by $\overline{\mathrm{conv}}\{X\}$ (resp. $\overline{\mathrm{lin}}\{X\}$) the closure in the weak topology of $\mathfrak{A}^*$ of the convex hull (resp the linear hull) of X. It is immediate that $X \to \overline{\mathrm{conv}}\{X\}$ and $Y \to \overline{\mathrm{lin}}\{Y\}$ define lattice-theoretic closures of the subsets of $\mathfrak{A}^*$. Fell's "<u>equivalence theorem</u>", of which we will quote several variants, displays a parallelism between those closure operations performed on the subsets $\omega(S)$ of the dual $\mathfrak{A}^*$ of $\mathfrak{A}$ and the hull-kernel closure operation mentioned before. Precisely one has the

<u>Theorem 1</u> . For any two sets $\mathcal{S}, \mathcal{T}$ of representations of $\mathfrak{A}$ the following are equivalent (we recall that Σ and $\overset{\circ}{\Sigma}$ denote, respectively the unit ball and the unit sphere of $\mathfrak{A}^*$)

α) $\mathcal{S} \propto \mathcal{T}$

β) for each $S \in \mathcal{S}$ $\omega(S) \subseteq \overline{\mathrm{lin}}\left\{\bigcup_{T \in \mathcal{T}} \omega(T)\right\}$

γ) for each $S \in \mathcal{S}$ $\omega(S) \subseteq \overline{\mathrm{conv}}\left\{\bigcup_{T \in \mathcal{T}} \omega(T)\right\}$

δ) for each $S \in \mathcal{S}$ $\omega(S) \cap \Sigma = \overline{\mathrm{conv}}\left\{\bigcup_{T \in \mathcal{T}} \omega(T) \cap \Sigma\right\}$

ε) for each $S \in \mathcal{S}$ $\omega(S) \cap \overset{\circ}{\Sigma} = \overline{\mathrm{conv}}\left\{\bigcup_{T \in \mathcal{T}} \omega(T) \cap \overset{\circ}{\Sigma}\right\}$

β') $\overline{\mathrm{lin}}\left\{\bigcup_{S \in \mathcal{S}} \omega(S)\right\} = \overline{\mathrm{lin}}\left\{\bigcup_{T \in \mathcal{T}} \omega(T)\right\}$

$$\gamma') \quad \overline{\mathrm{conv}}\Big\{\bigcup_{S\in\mathcal{S}} \omega(S)\Big\} \subseteq \overline{\mathrm{conv}}\Big\{\bigcup_{T\in\mathcal{T}} \omega(T)\Big\}$$

$$\delta') \quad \overline{\mathrm{conv}}\Big\{\bigcup_{S\in\mathcal{S}} \omega(S)\cap\Sigma\Big\} \subseteq \overline{\mathrm{conv}}\Big\{\bigcup_{T\in\mathcal{T}} \omega(T)\cap\Sigma\Big\}$$

$$\varepsilon') \quad \overline{\mathrm{conv}}\Big\{\bigcup_{S\in\mathcal{S}} \omega(S)\cap\mathring{\Sigma}\Big\} \subseteq \overline{\mathrm{conv}}\Big\{\bigcup_{T\in\mathcal{T}} \omega(T)\cap\mathring{\Sigma}\Big\}$$

The equivalence of the primed and unprimed statements results from property 3) of the closure operations $\overline{\mathrm{conv}}$, $\overline{\mathrm{lim}}$. The primed statements manifestly define a preordering. To show the equivalence of $\alpha)$, $\beta),\gamma),\delta)$ one can first reduce them to the case of the single representations

$$S_1 = \sum_{S\in\mathcal{S}}^{\oplus} S \quad \text{and} \quad T_1 = \sum_{T\in\mathcal{T}}^{\oplus} T, \text{viz :}$$

$\alpha_1)$ $S_1 \propto T_1$ or $\mathrm{Ker}(S_1) \supseteq \mathrm{Ker}(T_1)$

$\beta_1)$ $\omega(S_1) \subseteq \overline{\mathrm{lim}}\,\{\omega(T_1)\}$

$\gamma_1)$ $\omega(S_1) \subseteq \overline{\mathrm{conv}}\{\omega(T_1)\}$

$\delta_1)$ $\omega(S_1)\cap\Sigma \subseteq \overline{\mathrm{conv}}\,\{\omega(T_1)\cap\Sigma\}$

(note that $\overline{\mathrm{conv}}\,\{\bigcup_{S\in\mathcal{S}} \omega(S)\cap\Sigma\} = \overline{\mathrm{conv}}\,\{\omega(\sum_{S\in\mathcal{S}}^{\oplus} S)\cap\Sigma\}$, $\omega(\sum_{S\in\mathcal{S}}^{\oplus} S)$ being even contained in the uniform closure of the convex hull of the $\omega(S)$). The $\mathring{\Sigma}$-version of Theorem 1 (statements ε) and ε')) on the other hand results from the fact that if K is a cone in $\mathfrak{A}^{*+}$, $y \in \overline{\Sigma\cap K}$ and $\varphi\in\mathring{\Sigma}$ imply $\varphi \in \mathring{\Sigma}\cap K$. Finally the equivalence of $\alpha_1)\dots\,\delta_1)$, using the fact that the range of the representation T_1 is itself a representation, reduces to the basic:

Theorem. For a concrete C^*-algebra R of operators on a Hilbert space the convex hull of $\omega(R)$ (resp. $\omega(R)\cap\Sigma$) is dense in R^{*+} (resp. $R^{*+}\cap\Sigma$). This theorem directly results by applying Mackey's theorem mentionned at the end of the last paragraph to the subset $X = \omega(R) \cap \Sigma_R$ of R^* equipped with the weak R-topology (the bipolar of X is easily seen to be $R^{*+} \cap \Sigma_R$) and noting that for any cone $K \subset \mathcal{A}^{*+}$, $\Sigma \cap \mathrm{conv}\{K\} = \mathrm{conv}\{\Sigma \cap K\}$ and $\mathring{\Sigma} \cap \mathrm{conv}\{K\} = \mathrm{conv}\{\mathring{\Sigma} \cap K\}$

The general equivalence theorem discussed so far is, from the physical point of view, a result on density matrices. The following specializations apply directly to state vectors :

<u>Theorem 2</u>. Let $S \in \mathrm{Rep}(\mathcal{A})$ be cyclic with cyclic vector ψ and let $\mathcal{T} \subset \mathrm{Rep}(\mathcal{A})$. $S \prec \mathcal{T}$ (i.e., the set consisting of the single representation S is weakly contained in $\mathcal{T}$) if and only if

$$\omega_\psi \in \overline{\mathrm{conv}} \{ \bigcup_{T \in \mathcal{T}} \omega(T) \} \quad (\text{or } \overline{\lim} \{ \bigcup_{T \in \mathcal{T}} \omega(T) \})$$

This theorem results from Theorem 1 and the two following facts : the set $\overline{\mathrm{conv}}\{ \bigcup_{T \in \mathcal{T}} (T) \}$ is invariant by the multiplications t_A^* (those being the transposed of the left multiplications t_A in $\mathcal{A}$: $t_A B = AB$) and the set of all $t_A^* \omega_\psi$, $A \in \mathcal{A}$, is uniformly (so weakly) dense in $\omega(T)$.

For irreducible representations, for which all vectors are cyclic we have the further specialization:

<u>Theorem 3</u>. Let S and the $T \in \mathcal{T}$ be irreducible representations of $\mathcal{A}$. $S \prec \mathcal{T}$ is equivalent to

$$\omega(S) \cap \mathring{\Sigma} \subseteq \overline{\bigcup_{T \in \mathcal{T}} \omega(T) \cap \mathring{\Sigma}}$$

This theorem is a consequence of the fact that any weakly closed subset of $\mathring{\Sigma}$ contains its external points.

Let us call $\hat{\underset{\sim}{\mathfrak{A}}}$ the set of equivalence classes of irreducible representations. On $\hat{\underset{\sim}{\mathfrak{A}}}$ the HK lattice-theoretic closure operation is a topological closure. The two following results of Fell are of physical interest :

<u>Theorem 4</u>. For any $\underset{\sim}{\mathcal{S}} \in \mathrm{Rep}(\underset{\sim}{\mathfrak{A}})$ there exist a unique closed subset of $\hat{\underset{\sim}{\mathfrak{A}}}$ which is weakly equivalent to $\underset{\sim}{\mathcal{S}}$. It consists of all $T \in \mathrm{Rep}(\underset{\sim}{\mathfrak{A}})$ such that $T \prec \underset{\sim}{\mathcal{S}}$.

<u>Theorem 5.</u> It $S = \sum_t^{\oplus} S^{(t)}$ is a direct integral of representations $S^{(t)}$ of $\underset{\sim}{\mathfrak{A}}$, defined topologically, S is weakly equivalent to the set of all $S^{(t)}$.