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Abstract. We consider the space Ω of marked Galton-Watson trees. Each $\omega \in \Omega$ corresponds to a tree $T = T(\omega)$. The tree begins with the root $\emptyset \in T(\omega)$ representing the initial ancestor; each node (=individual) $u \in T(\omega)$ gives birth to $N_u = N_u(\omega) \geq 0$ children, denoted by $ui \in \omega, 1 \leq i \leq N_u$, and each child ui is marked with a non-negative number $A_{ui} = A_{ui}(\omega)$. The individus in the same generation behave independently each other, with the same probability law given by the random variable $(N, A_1, ..., A_N) := (N_\emptyset, A_{\emptyset 1}, ..., A_{\emptyset N})$, which is normalized such that $E\sum_{i=1}^N A_i = 1$. Write $X_u = A_{u_1} A_{u_1 u_2} \cdots A_{u_1 ... u_n}$ if $u = u_1...u_n$, and put $Y_n = \sum_{|u|=n} X_u$, where the sum is taken over all nodes $u \in T(\omega)$ of length n. Then $\{Y_n : n \geq 1\}$ forms a martingale, and converges almost surely to a non-negative random variable, Z, as $n \to \infty$. In the case where the limit is non-degenerate ($P(Z = 0) < 1$), we give necessary and sufficient conditions for existence of its moments of given order $p > 1$, obtain an equivalence of the tail probabilities $P(Z > x)$ as $x \to \infty$, and prove that its distribution has a continuous density (with respect to the Lebesgue measure) on $(0, \infty)$ under some moment conditions. The results are of applications in the study of: (a) Mandelbrot's self-similar cascades, (b) invariant measures of some infinite particle systems, (c) branching random walks, (d) flows in trees and (e) exact Housdorff measures in random constructions. The proofs make use of the random difference equation

$X = A_1 X_1 + B_1$, where X_1 is a random variable independent of $(A_1, B_1) \in \mathbb{R}_+^2$ and has the same law as X.

1. Introduction and main results

Let $\mathbb{N}^\star = \{1, 2, \cdots\}$ be the set of positive integers with the discret topology. Put $\mathbb{N} = \{0\} \cup \mathbb{N}^\star$ and write

$$\mathbf{U} = \bigcup_{k=0}^{\infty} (\mathbb{N}^\star)^k$$

for the set of all finite sequences $\mathbf{i} = i_1 i_2 ... i_n$ ($i_k \in \mathbb{N}^\star$), where by convention $\mathbb{N}^{\star 0} = \{\emptyset\}$ contains the null sequence $\emptyset$. Let

$$\mathbf{I} = (\mathbb{N}^\star)^{\mathbb{N}^\star}$$

be the set of all infinite sequences $\mathbf{i} = i_1 i_2 ...$ ($i_k \in \mathbb{N}^\star$) with the product topology. If $\mathbf{i} = i_1 i_2 .. i_n$ ($n \leq \infty$) is a sequence, we write $\mid \mathbf{i} \mid = n$ for its length, and $\mathbf{i} \mid k = i_1 i_2 ... i_k$ ($k \leq n$;) for the curtailment of $\mathbf{i}$ after k terms; conventioally, $|\emptyset| = 0$ and $\mathbf{i}|0 = \emptyset$. If $\mathbf{i} \in \mathbf{U}$ and $\mathbf{j} \in \mathbf{U}$ or $\mathbf{I}$ we write $\mathbf{ij} = (\mathbf{i}, \mathbf{j})$ for the sequence obtained by juxtaposition. In particular $\mathbf{i}\emptyset = \emptyset\mathbf{i} = \mathbf{i}$. We partially order $\mathbf{U}$ by writing $\mathbf{i} < \mathbf{j}$ to mean that for some $\mathbf{i}' \in \mathbf{U}, \mathbf{j} = \mathbf{ii}'$, and we use a similar notation if $\mathbf{i} \in \mathbf{U}$ and $\mathbf{j} \in \mathbf{I}$. If $\mathbf{i}$ and $\mathbf{j}$ are two sequences, we write $\mathbf{i} \wedge \mathbf{j}$ for the common maximal sequence of $\mathbf{i}$ and $\mathbf{j}$, that is, the maximal sequence $\mathbf{q}$ such that $\mathbf{q} < \mathbf{i}$ and $\mathbf{q} < \mathbf{j}$.

A tree T is a subset of $\mathbf{U}$ satisfying three conditions (cf. Neveu (1986)):

i) $\emptyset \in T$;

ii) if $\mathbf{ij} \in T$, then $\mathbf{i} \in T$;

iii) if $\mathbf{i} \in T$ and $j \in \mathbb{N}^\star$, then $\mathbf{i}j \in T$ if and only if $1 \leq j \leq N_{\mathbf{i}}$ for a positive integer $N_{\mathbf{i}}$.

We shall write N for $N_\emptyset$. The boundary of a tree T is defined as

$$\partial T = \{\mathbf{i} \in \mathbf{I} : \quad \mathbf{i}|n \in T \text{ for all } n \in \mathbb{N}\}.$$

As a subset of $\mathbf{I}$, it is a metrical and compact topological space with

$$B(\mathbf{i}) = \{\mathbf{j} \in \partial\omega : \mathbf{i} < \mathbf{j}\}, \qquad \mathbf{i} \in T(\omega),$$

its topological basis; a possible choice of metric is

$$d_c(\mathbf{i}, \mathbf{j}) = c^{-|\mathbf{i} \wedge \mathbf{j}|},$$

where c is a given number in $(1, \infty)$.The set $B(\mathbf{i})$ is then a ball of radius $c^{-|\mathbf{i}|}$.

An element of T is called a node. Each node $u \in T$ is marked with a vector $\eta_u = (A_{u1}, A_{u2}, ...)$ of $\mathbb{R}_+^{\mathbb{N}^*}$, where $\mathbb{R}_+ = [0, \infty)$. If $1 \leq j \leq N_u$, we can imagine that the number A_{uj} is associated with the edge (u, uj) linking the nodes u and uj; the values A_{uj} for $j > N_u$ are of no influence for our purpose, and will be taken as 0 for convenience. The marked tree will be denoted by $(T, (\eta_u, u \in T))$.

Let $\mathbb{T}$ be the set of all trees, and Ω be the set of all marked trees ω (marked as above). An element ω of Ω will be written as $(T(\omega), (\eta_u, u \in T(\omega)))$, where $T(\omega)$ is the underlying tree. We may regard T as the canonical projection from Ω to $\mathbb{T}$. Thus T may stand for a tree or an operator, according to the context. If ω is a marked tree and if $\mathbf{i} \in T(\omega)$, we write $T_{\mathbf{i}}(\omega) = \{\mathbf{j} \in \mathbf{U} : \mathbf{ij} \in T(\omega)\}$ for the shifted tree of $T(\omega)$ at $\mathbf{i}$. Note that $T_{\mathbf{i}}(\omega) \in \mathbb{T}$. Denote by

$$z_k(\omega) = \{\mathbf{i} \in T(\omega) : |\mathbf{i}| = k\}$$

the set of nodes of $T(\omega)$ with length k ($k \in \mathbb{N}$), and consider the filtration

$$\mathcal{F}_k = \sigma\{(N_{\mathbf{i}}, A_{\mathbf{i}1}, A_{\mathbf{i}2}, ...) : \quad \mathbf{i} \in z_{k-1}\}, \qquad k \geq 1.$$

Let $\mathcal{F} := \sigma(\mathcal{F}_k, k \geq 1)$. For simplicity, we write $(N, A_1, A_2, ...)$ for $(N_{\emptyset}(\omega), A_{\emptyset 1}(\omega), A_{\emptyset 2}(\omega), ...)$.

By a result of Neveu (1986), for each probability distribution q on $\mathbb{N} \times \mathbb{R}_+^{\mathbb{N}^*}$, there is a probability law $P = P_q$ on $(\Omega, \mathcal{F})$ such that

(i) the distribution of the random variable $(N, A_1, A_2, ...)$ is q;

(ii) given $\mathcal{F}_k$, the random variables $(N_{\mathbf{i}}, A_{\mathbf{i}1}, A_{\mathbf{i}2} ...)$, $\mathbf{i} \in z_k(\omega)$, are conditionally independent, and their conditional distribution is q.

The property (ii) is referred as the *branching property.*

Assume that the initial distribution is normalized such that

$$E(\sum_{i=1}^{N} A_i) = 1.$$

If $u \notin T(\omega)$, the values $A_u(\omega)$ are of no influence for our problem, and may be non-defined; however, for convenience, we set

$$A_u = 0 \text{ if } u \in \mathbf{U} \setminus T(\omega).$$

In particular, for all $u \in T(\omega), A_{ui} = 0$ if $u > N_u$. Put

$$X_\emptyset = 1,\ X_u = A_{u_1} A_{u_1 u_2} \cdots A_{u_1 \dots u_n} \text{ if } u = u_1 \dots u_n \in \mathbf{U}$$

and

$$Y_n = \sum_{|v|=n, v \in T(\omega)} X_v, \qquad n \geq 1.$$

Then $(Y_n, \mathcal{F}_n)(n \geq 1)$ is a (non-negative) martingale, so the limit

$$Z = \lim_{n \to} Y_n$$

exists almost surely. Similarly, we put

$$Z_u = \lim_{n \to \infty} \sum_{v = v_1 \dots v_n \in T_u(\omega)} A_{uv_1} A_{uv_1 v_2} \cdots A_{uv_1 \dots v_n} \text{ if } u \in T(\omega)$$

and $Z_u = 1$ if $u \in \mathbf{U} \setminus T(\omega)$. Then $Z = Z_\emptyset$, and, by the branching property, given $\mathcal{F}_n$, the random variables $Z_u, u \in z_n(\omega)$, are conditionally independent, and their conditional law is the distribution of Z . It is easily seen that for almost all $\omega \in \Omega$ and all $u \in T(\omega)$,

$$X_u Z_u = \sum_{i=1}^{N_u} X_{ui} Z_{ui}.$$

Therefore for almost all $\omega \in \Omega$, there is a unique Borel measure, μ_ω, defined on $\partial T(\omega)$, such that

$$\mu_\omega(B(u)) = X_u Z_u \text{ for all } u \in T(\omega).$$

We extend this measure as a Borel measure on $\mathbf{I}$ by letting $\mu_\omega(A) = \mu_\omega(A \cap \partial T(\omega))$. Then μ_ω is a random measure on $\mathbf{I}$ with mass $Z(\omega)$.

The preceding identity on Z_u shows that the distribution of Z satisfies the equation

$$Z = \sum_{i=1}^{N} A_i Z_i, \tag{1.1}$$

where the sum is taken to be zero if $N = 0$, and, given $(N, A_1, A_2, \dots)$, the random variables $Z_i (1 \leq i \leq N)$ are conditionally independent, and their conditional distribution is the law of Z. In terms of characteristic functions or Laplace transforms, it reads

$$\phi(t) = E \prod_{i=1}^{N} \phi(A_i t), \tag{1.1'}$$

where $\phi(t) = E(e^{itZ})$ or $E(e^{-tZ}), t \in \mathbb{R}$ or $\mathbb{R}_+$, and the empty product is taken to be 1.

The problem is to study the measure μ_ω and the distribution of Z.

Kahane and Peyrière (1976), and Guivarc'h (1990) studied the problem in the case where N is constant and the A_i $(1 \leq i \leq N)$ are i.i.d. Their works were motivated by questions raised by Mandelbrot related to a model of turbulence of Yaglom. Holley and Liggett (1981) studied the same problem in the case where N is constant and the A_i are fixed multiples of one random variable, and Durrett and Liggett (1983) considered the more general case where N is constant but the A_i have arbitrary joint distribution. Their works were motivated by a number of problems in infinite particle systems. Closely related results are given in Kahane (1987), Ben Nasr (1987), Holley and Waymire (1992), Collet and Koukiou (1992), Chauvin and Rouault (1996), and Liu and Rouault (1996), etc.

If $1 < m = EN < \infty$ and $A_i = 1/m$ $(1 \leq i \leq N)$, then Y_n becomes the well-known martingale $card(z_n)/m^n$ of the Galton-Walton process, where $card(z_n)$ is the population size at $n-th$ generation . Similar martingales arise in age-dependent branching processes or branching random walks. Many authors have contributed to the subject, see for example Harris (1948), Kesten and Stigum (1966), Seneta (1968 and 1969), Athreya (1971), Doney (1972 and 1973), Bingham and Doney (1974 and 1975) and Biggins (1977).

The martingale (Y_n) and the equation (1.1), in its various forms, were also used to study some fractal sets or flows in networks, implicitly or directly by Mauldin and Williams (1986), Falconer (1986 and 1987) and Liu (1993 and 1996).

If E is a set or a statement, we write 1_E or $1\{E\}$ for its indicator function. Let

$$\tilde{N} := \sum_{i=1}^{N} 1_{\{A_i > 0\}}$$

be the number of non-zero terms of A_i, $1 \leq i \leq N$. To simplify the discussion, we suppose throughout the paper that

$$P(\tilde{N} = 0 \text{ or } 1) < 1, \quad P(\forall i \in \{1, ..., N\},\ A_i = 0 \text{ or } 1) < 1, \tag{1.2}$$

$$E\tilde{N} < \infty \text{ and } E\sum_{i=1}^{N} A_i \log^+ A_i < \infty, \tag{1.3}$$

where $\log^+ x = \max(0, \log x)$. For $x \in [0, \infty)$, write

$$S(x) := \sum_{i=1}^{N} A_i^x \text{ and } \gamma(x) := ES(x), \tag{1.4}$$

where (and throughout) the sum is taken over all the i's such that $A_i > 0$. The function γ is defined on $[0, \infty)$ with values in $[0, \infty]$. Then

$$\gamma(x) < \infty \text{ and } \gamma'(x) = E \sum_{i=1}^{N} A_i^x \log A_i < \infty$$

exists for all $x \in (0, 1]$ (and $\gamma'(1)$ denotes the left derivative); and γ is strictly convex on $(0, 1)$.

The following results have been known. (We recall that $Y_1 = \sum_{i=1}^{N} A_i$ by our notations.)

Theorem 0. *(Biggins 1977 and Liu 1997) (a) The following assertions are equivalent: (i) Z is non-degenerate [i.e. $P(Z = 0) < 1$]; (ii) $E(Z|\mathcal{F}_n) = Y_n$ for all $n \geq 1$; (iii)*

$$E[Y_1 \log^+ Y_1] < \infty \text{ and } E \sum_{i=1}^{N} A_i \log A_i < 0. \tag{1.5}$$

(b) Assume only (1.2) and (1.3) [so $\gamma(1)$ is not necessarily equal to 1]. Then the equation (1.1) has a non-trivial solution if and only if for some $\alpha \in (0, 1]$, $\gamma(\alpha) = 1$ and $\gamma'(\alpha) \leq 0$; it has a nontrivial solution with finite mean if and only if $\gamma(1) = 1$ and (1.5) holds. Moreover, there is at most one solution with mean 1.

Here, we only consider solutions of (1.1) in the class of probability laws on $[0, \infty)$. We remark that $Z = 1$ if and only if $Y_1 = 1$ almost surely. Therefore we suppose throughout that Y_1 is not a.s.a constant.

The main aim of this paper is to study the moments, the tail probabilities and the absolutely continuity of the distribution of Z, in the case where Z is non-degenerate . The following results will be shown.

Theorem 1 (Moments). *Assume (1.5). Then For each fixed* $p > 1$,

$$E(Z^p) < \infty$$

if and only if

$$E[Y_1^p] < \infty \text{ and } E[\sum_{i=1}^{N} A_i^p] < 1.$$

We remark that if $\gamma(p) < 1$ for all $p > 1$, then $\| \sup_{1 \leq i \leq N} A_i \|_p \leq 1$ for all $p > 1$, and so $\| \sup_{1 \leq i \leq N} A_i \|_\infty \leq 1$. Therefore, by Theorem 1, *Z has moments of all orders if and only if*

$$P(\forall i \in \{1, ..., N\},\ A_i \leq 1) = 1 \text{ and } E[Y_1^p] < \infty \text{ for all } p > 1. \tag{1.6}$$

The problem will be called *lattice* if for some $h > 0$ and almost all $\omega \in \Omega$, each $\log A_i$ is an integer multiple of h whenever $1 \leq i \leq N$ and $A_i > 0$; the largest such h will be called the span. Otherwise, it is called *non-lattice.*

Theorem 2 (Tail probabilities).

(a) (The case where $P(\exists i \in \{1, ..., N\}, A_i > 1) > 0$.) Suppose that for some $\chi > 1$,

$$E[\sum_{i=1}^{N} A_i^\chi] = 1,\ E[\sum_{i=1}^{N} A_i^\chi \log^+ A_i] < \infty \text{ and } E[(\sum_{i=1}^{N} A_i)^\chi] < \infty.$$

If the problem is non-lattice, then there is a constant $c \in (0, \infty)$ such that

$$\lim_{x \to \infty} x^\chi P(Z > x) = c;$$

If the problem is lattice, then

$$0 < \liminf_{x \to +\infty} x^\chi P(Z > x) \leq \limsup_{x \to +\infty} x^\chi P(Z > x) < \infty.$$

(b)(The case where $P(\forall i \in \{1, ..., N\}, A_i \leq 1) = 1$.) Suppose that $\|N\|_\infty < \infty$ and that for some $x > 0$, $\| \sum_{i=1}^{N} A_i^x \|_\infty \leq 1$. Let ρ be the least solution in $(1, \infty)$ of the equation

$$\| \sum_{i=1}^{N} A_i^\rho \|_\infty = 1.$$

(Such a solution certainly exists under the preceding conditions.) Assume also that for some constants $0 < \delta < 1$, $0 \le a < \infty$, $0 < c < \infty$ and all $0 < x < 1$ sufficiently small,

$$P(\sum_{i=1}^{N} A_i^\rho > 1 - x \text{ and } A_i \le \delta \text{ for all } 1 \le i \le N) \ge cx^a.$$

Then for some constants $0 < c_1 \le c_2 < \infty$ and all $x > 0$ sufficiently large,

$$exp\{-c_2 x^{\rho/(\rho-1)}\} \le P(Z \ge x) \le exp\{-c_1 x^{\rho/(\rho-1)}\}.$$

In the non-lattice case, part (a) of the theorem is due to Guivarc'h (1990) if $N = c \ge 2$ is constant and A_i $(1 \le i \le c)$ are i.i.d. Our proof develops an idea of Guivarc'h, linking the distributional equation (1.1) with the random difference equation (see sections 2 and 3).

Let

$$\zeta_1 = \inf\{i > 0 : \ A_i > 0\}, \text{ where } \inf \emptyset = \infty,$$

be the first (random) index i for which $A_i > 0$, and put

$$\zeta_{k+1} = \inf\{i > \zeta_k : \ A_i > 0\}, \text{ where } \inf \emptyset = \infty, \quad k \ge 1.$$

Define

$$A_{\zeta_k}(\omega) = \begin{cases} A_i(\omega) & \text{if } \zeta_k(\omega) = i \text{ for some } i \in \mathbb{N}^*, \\ 0 & \text{if } \zeta_k(\omega) = \infty. \end{cases}$$

Then $A_{\zeta_k}(\omega) > 0$ if $\tilde{N}(\omega) \ge k \ge 1$. It is easily seen that the functional equation (1.1') is nothing but

$$\phi(t) = E \prod_{i=1}^{\tilde{N}} \phi(A_{\zeta_i} t).$$

The advantage of the new equation is that $A_{\zeta_i} > 0$ for all $1 \le i \le \tilde{N}$.

Theorem 3 (Absolute continuity). *The distribution of Z is either absolutely continuous or (purely) singularly continuous on $\mathbb{R}_+^* = (0, \infty)$. It is absolutely continuous if one of the following conditions holds:*

(i) $\tilde{N} \geq 2$ a.s. and, for some $\epsilon > 0$,

$$E[A_{\zeta_1}^{-\epsilon} + A_{\zeta_2}^{-\epsilon}] < \infty, \; E\sum_{i=1}^{N} M_i^{-\epsilon} < \infty \text{ and } E\sum_{i=1}^{N} A_i M_i^{-\epsilon} < \infty,$$

where $M_i = \max_{j \neq i} A_j$.

(ii) $\tilde{N} \geq 1$ a.s. and A_{ζ_1} has an absolutely continuous distribution on $\mathbb{R}_+^*$.

Moreover, in case (i), the density function of the distribution of Z is continuous (on $\mathbb{R}_+^*$).

We remark that the condition (i) holds if, for example, $N \geq 2$ almost surely and, given N, the random variables $A_i, 1 \leq i \leq N$, are conditionally independent, and their conditional law is the law of A_1, which satisfies the property that $EA_1^{-\epsilon} < \infty$ for some $\epsilon > 0$.

Informations on the rate of convergence of characteristic functions or Laplace transforms of Z will be given in proofs.

2. The random difference equation

In this section, $(\Omega, \mathcal{F}, P)$ denotes an arbitrary probability space, (A, B) and $(A_n, B_n)(n \geq 1)$ are i.i.d. random variable defined on $(\Omega, \mathcal{F}, P)$, with values in $\mathbb{R}^2$.

Consider the random difference equation

$$X \stackrel{L}{=} AX + B, \tag{2.1}$$

where X is a real random variable independent of (A, B), and $\stackrel{L}{=}$ denotes equality in law; the law of X is unknown. In terms of characteristic functions, the equation reads

$$\phi(t) = E[e^{iAt}\phi(Bt)], \qquad t \in R. \tag{2.1'}$$

A probability law, μ, is said to be a solution of (2.1) if there is a random variable X having μ as its law and satisfying (2.1); when we say that a random variable X is the unique solution of (2.1), we mean that the corresponding law is the unique solution.

Lemma 2.1 *[Grintsevichyus (1974), Th.1 and Prop.1]. If*

$$P(A \neq 0) = 1, \ -\infty < Elog|A| < 0 \text{ and } Elog^{+}|B| < \infty, \qquad (2.2)$$

then (2.1) has a unique solution, and this solution is given by

$$X = B_1 + A_1 B_2 + A_1 A_2 B_3 + \cdots + A_1 ... A_{n-1} B_n + ... \qquad (2.3)$$

the series being convergent a.s.

Lemma 2.2 *[Grintsevichyus (1974), Th.3]. Assume (2.2) and let μ be the unique solution of (2.1). Then there are only three possible cases: (a) μ is absolutely continuous; (b) μ is singularly continuous; (c) μ is concentrated at some point c. The case (c) holds if and only if $P(c = Ac + B) = 1$.*

Lemma 2.3.*Assume (2.2) and let X be the unique solution of (2.1). Let p be any fixed number in $(0, \infty)$. If*

$$E(|A|^p) < 1 \text{ and } E(|B|^p) < \infty, \qquad (2.4)$$

then $E(|X|^p) < \infty$; the converse holds if additionally $A \geq 0$ and $B \geq 0$ a.s. with $P(B > 0) > 0$.

Proof. We denote by $\|.\|_p$ the norm in L^p, $p > 0$. By Lemma 2.1, we can suppose that X is given by (2.3). If $\|A\|_p < 1$ and $\|B\|_p < \infty$, then by (2.3) and the triangular inequality in L^p, we obtain

$$\|X\|_p \leq \|B\|_p + \|A\|_p \|B\|_p + ... + \|A\|_p^{n-1} \|B\|_p + ... = \frac{\|B\|_p}{1 - \|A\|_p} < \infty.$$

Conversely if $A \geq 0$ and $B \geq 0$ a.s. and if X is a solution of (2.1) with $E(|X|^p) < \infty$, then $X \geq 0$ by (2.3), and $E(B^p) \leq E[(AX + B)^p] < \infty$ by (2.1); since $P(B > 0) > 0$, we have also $P(X > 0) > 0$, $0 < E(X^p) < \infty$ and

$$E[(AX)^p] < E[(AX + B)^p] = E(X^p),$$

which implies $E[(A)^p] < 1$ by the independence of A and X.

Lemma 2.4 *[Kesten(1973)- [Grintsevichyus (1975)]. Assume that $P(A > 0) = P(B \geq 0) = 1, P(B > 0) > 0$, and that for some $\lambda \in (0, \infty)$,*

$$E(A^{\lambda}) = 1, E(A^{\lambda} log^{+} A) < \infty \text{ and } E(B^{\lambda}) < \infty.$$

Let X be the unique solution of (2.1) and suppose that X is not a.s. a constant. [That is, there does not exist a constant c such that $P(c = Ac + B) = 1$.] Then

(a) if $\log A$ is not of lattice type, then

$$\lim_{x\to\infty} x^{\lambda} P(X > x) = c, \tag{2.5}$$

where $c \in (0,\infty)$ is a constant.

(b)if $\log A$ is of lattice type with span $h > 0$, then for all real x,

$$\lim_{x\to\infty} e^{(x+nh)\lambda} P(X > e^{(x+nh)\lambda}) = c(x), \tag{2.6}$$

where $c(x) \in (0,\infty), x \in R$, is a strictly positive and h-periodic function on R. In particular,

$$0 < \liminf_{x\to+\infty} x^{\lambda} P(X > x) \leq \limsup_{x\to+\infty} x^{\lambda} P(X > x) < \infty. \tag{2.7}$$

3. The random difference equation satisfied by $xP_Z(dx)$; moments and tails.

For a random variable X, we write P_X or $L(X)$ for its law.

For $u \in \mathbf{U}$, we use the notations A_u, X_u and Z_u introduced in section 1. For $(\omega, \mathbf{i}) \in \Omega \times \mathbf{I}$, put

$$\begin{aligned} \tilde{Z}(\omega,\mathbf{i}) &= Z_{\emptyset}(\omega) = Z(\omega), \\ \tilde{A}_1(\omega,\mathbf{i}) &= A_{\mathbf{i}|1}(\omega), \\ \tilde{Z}_1(\omega,\mathbf{i}) &= Z_{\mathbf{i}|1}(\omega). \end{aligned}$$

They are measurable functions on $\Omega \times \mathbf{I}$ associated with the product σ-field of $\mathcal{F}$ and $\mathcal{B}$, $\mathcal{B}$ being the Borel σ-field on $\mathbf{I}$. Let Q be the Peyrière's measure on $\Omega \times \mathbf{I}$, defined by

$$Q(A) = E\int_{\mathbf{I}} 1_A(\omega,\mathbf{i})\mu_\omega(d\mathbf{i}), A \in \mathcal{F}\times\mathcal{B}.$$

As $EZ = 1$, Q is a probability measure. We write $E_Q[f]$ for the integration of f with respect to Q. The following result give the distributions of the random variables $\tilde{Z}, (\tilde{A}_1, \tilde{B}_1)$, and shows that the probability law $xP_Z(dx)$ satisfies a random difference equation.

Lemma 3.1. *For all* $(\omega, \mathbf{i}) \in \Omega \times \mathbf{I}$,

$$\tilde{Z}(\omega, \mathbf{i}) = \tilde{A}_1(\omega, \mathbf{i})\tilde{Z}_1(\omega, \mathbf{i}) + \tilde{B}_1(\omega, \mathbf{i}),$$

where

$$\tilde{B}_1(\omega, \mathbf{i}) = \sum_{i=1}^{N(\omega)} A_i(\omega) Z_i(\omega) 1\{\mathbf{i}|1 \neq i\},$$

$\tilde{Z}_1$ *is independent of* $(\tilde{A}_1, \tilde{B}_1)$, *and has the same distribution as* $\tilde{Z}$. *Moreover, for all non-negative Borel functions* f, g *and* h *(defined on* $\mathbb{R}$ *or* $\mathbb{R}^2$*),*

$$\begin{aligned}
E_Q[f(\tilde{A}_1)] &= E[\sum_{i=1}^{N} f(A_i)A_i], \\
E_Q[f(\tilde{B}_1)] &= E[\sum_{1 \leq k \leq N} A_k f(\sum_{1 \leq i \leq N, i \neq k} A_i Z_i)], \\
E_Q[h(\tilde{A}_1, \tilde{B}_1)] &= E[\sum_{1 \leq k \leq N} A_k h(A_k, \sum_{1 \leq i \leq N, i \neq k} A_i Z_i)], \\
E_Q[g(\tilde{Z}_1)] &= E[g(Z)Z] = E_Q[g(\tilde{Z})].
\end{aligned}$$

In particular,

$$L(\tilde{Z}) = xP_Z(dx).$$

Proof. We have

$$\begin{aligned}
A_{\mathbf{i}|1} Z_{\mathbf{i}|1} &= A_i Z_i \text{ if } \mathbf{i}|1 = i \\
&= \sum_{i=1}^{\infty} A_i Z_i 1\{\mathbf{i}|1 = i\} \\
&= \sum_{i=1}^{N} A_i Z_i 1\{\mathbf{i}|1 = i\}, \\
Z &= \sum_{i=1}^{N} A_i Z_i \\
&= \sum_{i=1}^{N} A_i Z_i [1\{\mathbf{i}|1 = i\} + 1\{\mathbf{i}|1 \neq i\}]
\end{aligned}$$

$$= \quad A_{\mathbf{i}|1} Z_{\mathbf{i}|1} + \sum_{i=1}^{N} A_i Z_i 1\{\mathbf{i}|1 \neq i\}.$$

Therefore

$$\tilde{Z}(\omega, \mathbf{i}) = \tilde{A}_1(\omega, \mathbf{i}) \tilde{Z}_1(\omega, \mathbf{i}) + \tilde{B}_1(\omega, \mathbf{i}).$$

We claim that $\tilde{Z}_1$ is independent of $(\tilde{A}_1, \tilde{B}_1)$, and has the same distribution as $\tilde{Z}$; by the way, we shall give the distribution of $(\tilde{A}_1, \tilde{B}_1)$. In fact, for all non-negative Borel functions g and h, we have

$$\begin{aligned} E_Q[h(\tilde{A}_1, \tilde{B}_1) g(\tilde{Z}_1)] &= E[\sum_{1 \leq k \leq N} h(A_k, \sum_{1 \leq i \leq N, i \neq k} A_i Z_i)\, g(Z_k) A_k Z_k] \\ &= E[\sum_{1 \leq k \leq N} A_k h(A_k, \sum_{1 \leq i \leq N, i \neq k} A_i Z_i)]\, E[g(Z)Z]. \end{aligned}$$

Taking $g = 1$ or $h = 1$ gives the expressions of $E_Q[h(\tilde{A}_1, \tilde{B}_1)]$ and $E_Q[g(\tilde{Z}_1)]$. Consequently

$$E_Q[h(\tilde{A}_1, \tilde{B}_1) g(\tilde{Z}_1)] = E_Q[h(\tilde{A}_1, \tilde{B}_1)]\, E_Q[f(\tilde{Z}_1)].$$

This gives the independence of $(\tilde{A}_1, \tilde{B}_1)$ and $\tilde{Z}_1$. The expressions of $E_Q[f(\tilde{A}_1)]$ and $E_Q[f(\tilde{B}_1)]$ come from $E_Q[h(\tilde{A}_1, \tilde{B}_1)]$ by taking $h(x, y) = f(x)$ or $f(y)$. So the proof is finished.

Lemma 3.2. *For all $p > 1$,*

$$E_Q[(\tilde{B}_1)^{p-1}] \leq E[Z^{p-1}]\, E[(\sum_{k=1}^{N} A_k)^p].$$

Proof. By lemma 1,

$$E_Q[(\tilde{B}_1)^{p-1}] = E[\sum_{1 \leq k \leq N} A_k (\sum_{1 \leq i \leq N, i \neq k} A_i Z_i)^{p-1}].$$

Denote by I the expectation above.

If $p-1 \le 1$, then

$$\begin{aligned}
I &\le E\{\sum_{1\le k\le N} A_k[\sum_{1\le i\le N, i\ne k} (A_i Z_i)^{p-1}]\} \\
&\quad \text{(by the inequality } (\sum x_k)^{p-1} \le \sum x_k^{p-1}) \\
&= E[Z^{p-1}]\, E[\sum_{1\le k\le N} A_k(\sum_{1\le i\le N, i\ne k} A_i^{p-1})] \\
&\quad \text{(by the branching property)} \\
&\le E[Z^{p-1}]\, E[\sum_{k=1}^{\infty} A_k \sum_{k=1}^{\infty} A_k^{p-1}] \\
&\le E[Z^{p-1}]\, \|\sum_{k=1}^{\infty} A_k\|_p \, \|\sum_{k=1}^{\infty} A_k^{p-1}\|_{p/(p-1)} \\
&\quad \text{(by Hölder's inequality)} \\
&\le E[Z^{p-1}]\, \|\sum_{k=1}^{\infty} A_k\|_p \sum_{k=1}^{\infty} \|A_k^{p-1}\|_{p/(p-1)} \\
&\quad \text{(by the triangular inequality)} \\
&= E[Z^{p-1}]\, \|\sum_{k=1}^{\infty} A_k\|_p \, (\sum_{k=1}^{\infty} EA_k^p)^{(p-1)/p} \\
&= E[Z^{p-1}]\, \|\sum_{k=1}^{\infty} A_k\|_p;
\end{aligned}$$

Since $\|\sum_{k=1}^{\infty} A_k\|_p \ge \|\sum_{k=1}^{\infty} A_k\|_1 = 1$, we have $\|\sum_{k=1}^{\infty} A_k\|_p \le E[(\sum_{k=1}^{\infty} A_k)^p]$, so the desired conclusion follows in the case where $p-1 \le 1$.

If $p-1 \ge 1$, using the inequality

$$(\sum a_i z_i)^{p-1} \le \sum a_i z_i^{p-1}, \quad a_i \ge 0, \sum a_i = 1,\ z_i \ge 0$$

(the convexity of the function $z \mapsto z^{p-1}$) for $a_i = A_i / \sum_{j\ne k} A_j$ and $z_i = Z_i, i \ne k$, we obtain

$$(\sum_{i\ne k} A_i Z_i)^{p-1} \le (\sum_{i\ne k} A_i)^{p-2} (\sum_{i\ne k} A_i Z_i^{p-1})$$

(if $\sum_{j\ne k} A_j = 0$, the inequality is evident). Consequently

$$I \;\le\; E(Z^{p-1})\, E[\sum_{k=1}^{N} A_k (\sum_{i\ne k} A_i)^{p-2} (\sum_{i\ne k} A_i)]$$

$$\leq \; E(Z^{p-1})\, E[(\sum_{i=1}^{N} A_i)^p].$$

Proof of Theorem 1. By Lemma 3.1,

$$E_Q[\tilde{Z}_1^{p-1}] = E[Z^p] \text{ and } E_Q[\tilde{A}_1^{p-1}] = E[\sum_{i=1}^{N} A_i^p].$$

So by lemmas 2.3 and 3.2, we see that for all $p > 1$, if

$$E[Z^{p-1}] < \infty, E[(\sum_{k=1}^{N} A_k)^p] < \infty \text{ and } E[\sum_{k=1}^{N} A_k^p] < 1,$$

then $E[Z^p] < \infty$. Noting that $EZ < \infty$, an easy induction argument on n shows that, for all $n = 2, 3, ...$, if

$$p \in (n-1, n], E[(\sum_{k=1}^{N} A_k)^p] < \infty \text{ and } E[\sum_{k=1}^{N} A_k^p] < 1,$$

then $E[Z^p] < \infty$. This gives the sufficiency of the conditions. The necessity is given in Liu (1997).

Proof of Theorem 2. By lemmas 3.1 and 3.2, we have

$$(i) \qquad E_Q[\tilde{A}_1^{\chi-1}] = E[\sum_{i=1}^{N} A_i^{\chi}] = 1,$$

$$(ii) \qquad E_Q[\tilde{A}_1^{\chi-1} \log^+ \tilde{A}_1] = E[\sum_{i=1}^{N} A_i^{\chi} \log^+ A_i] < \infty,$$

and

$$(iii) \qquad E_Q[\tilde{B}_1^{\chi-1}] \leq E[Z^{\chi-1}]\, E[(\sum_{i=1}^{N} A_i)^{\chi}].$$

If the problem is non-lattice, then by Lemma 2.4, the limit

$$\lim_{t\to+\infty} t^{-(\chi-1)} \int_t^{\infty} x P_Z(dx) = \lim_{t\to+\infty} t^{-(\chi-1)} Q(\tilde{Z} > t)$$

exists and is strictly positive and finite. This implies that the limit

$$\lim_{t\to+\infty} t^{-\chi} P(Z > t)$$

exists and is strictly positive. In the lattice case, the corresponding conclusions are

$$0 < \liminf_{t\to+\infty} t^{-(\chi-1)} Q(\tilde{Z} > t) \leq \limsup_{t\to+\infty} t^{-(\chi-1)} Q(\tilde{Z} > t) < \infty$$

and

$$0 < \liminf_{t\to+\infty} t^{\chi} P(Z > t) \leq \limsup_{t\to+\infty} t^{\chi} P(Z > t) < \infty.$$

This gives part (a) of the theorem. Part (b) has been proved in Liu (1996).

4. The absolute continuity

The discussion will be heavily based on the functional equation (1.1'). We recall that $\tilde{N} = \sum_{i=1}^{N} 1\{A_i > 0\}$ is the number of non-zero terms of $\{A_i : 1 \leq i \leq N\}$. Let

$$f(x) := \sum_{i=1}^{N} P(\tilde{N} = i) x^i, \; x \in [0, \infty)$$

be its probability generating function.

Lemma 4.1. *Assume $\tilde{N} \geq 1$ a.s. Let Z be any solution of (1.1), and let $\phi(t) = E(e^{itZ})(t \in R)$ be its characteristic function. Then*

$$\limsup_{|t|\to\infty} |\phi(t)| = 0 \text{ or } 1.$$

Proof. By (1.1'),

$$|\phi(t)| \leq E \prod_{i=1}^{N} |\phi(A_i t)|, \qquad t \in R.$$

Letting $|t| \to \infty$ and using Fatou's lemma gives

$$l \leq f(l),$$

where $l := \limsup_{|t|\to\infty} |\phi(t)|$. Therefore $l = 0$ or 1, noting that $f(x) < x$ if $0 < x < 1$.

Lemma 4.2. *Write*

$$M = \max_{1 \le i \le N} A_i$$

and suppose that

$$E|\log M| < \infty.$$

Let ϕ be a solution of (1.1') in the class of characteristic functions, which is not of the form e^{itc} for some constant c. Then

$$\limsup_{|t| \to \infty} |\phi(t)| = 0.$$

Proof. By Lemma 4.1, it suffices to prove that $\limsup_{|t| \to \infty} |\phi(t)| < 1$.

(a) We first prove that for all $t \neq 0$, $|\phi(t)| < 1$. Otherwise, by Lemma 4 of Chap.IV.1 of Feller, there is some $h > 0$ such that $|\phi(h)| = 1$ and $|\phi(t)| < 1$ if $0 < t < h$. By the equation (1.1'),

$$1 = |\phi(h)| \le E \prod_{i=1}^{N} |\phi(A_i h|.$$

Therefore, a.s.

$$|\phi(A_i h)| = 1 \text{ for all } i = 1, ..., N.$$

Since $P(0 < M < 1) > 0$ (this is necessary for the equation (1.1) to have a non-trivial solution), it follows that for some $0 < a < 1$, $|\phi(ah)| = 1$, which is a contradiction with the definition of h.

(b) We then prove that $\limsup_{|t| \to \infty} |\phi(t)| < 1$. By the functional equation, we have

$$|\phi(t)| \le E|\phi(Mt)|.$$

We now use ideas of Grintsevichyus (1974) on random walks. Let $M_i (i \ge 1)$ be independent copies of M, then

$$|\phi(t)| \le E|\phi(M_1 ... M_n t)|, \qquad n \ge 1.$$

Fix $t \neq 0$, and write

$$\phi_0 = |\phi(t)|, \quad \phi_n = |\phi(M_1 ... M_n t)|, \quad n \ge 1.$$

Then $\{\phi_n : n \ge 0\}$ is a sub-martingale associated with the natural filtration of σ-fields generated by $\{M_i : 0 \le i \le n\}, n \ge 0$, where by convention $M_0 = 1$. Put

$$S_0 = 1 \text{ and } S_n = \log M_1 + ... + \log M_n \text{ for } n \ge 1.$$

If I is an interval, we write

$$\tau(I) = \tau_1(I) = \inf\{n > 0 : S_n \in I\}, \text{ where } \inf \emptyset = +\infty,$$

for the first time that the random walk (S_n) hits I, and put

$$\tau_{k+1}(I) = \inf\{n > \tau_k : S_n \in I\}, \text{ where } \inf \emptyset = +\infty, \quad k \geq 1,$$

$$U(I) = E(\sum_{n=0}^{\infty} 1\{S_n \in I\}).$$

Then

$$P[\tau_k(I) < \infty] = P[\sum_{n=0}^{\infty} 1\{S_n \in I\} \geq k], \qquad k \geq 1,$$

is the probability that there are at least k hits in I, U(I) is the expected number of hits in I, and

$$U(I) = \sum_{k=1}^{\infty} P[\tau_k(I) < \infty].$$

By the strong Markov's property, it is easily verified that for all $-\infty < a < b < \infty$,

$$P[\tau_{k+1}([a,b]) < \infty] \leq P[\tau([a,b]) < \infty] P[\tau_k([a-b, b-a]) < \infty], \quad k \geq 1.$$

Summing for k=1,2,..., we obtain

$$U([a,b]) \leq P[\tau([a,b]) < \infty][1 + U([a-b, b-a])].$$

Therefore, for all $h > 0$,

$$U([-h,0] - \log|t|) \leq P[\tau([-h,0] - \log|t|) < \infty][1 + U([-h,h])], \quad k \geq 1.$$

On the other hand, by the renewal theorem (cf. Feller Chap. 11.9), there is $h > 0$ such that

$$\lim_{|t| \to \infty} U([-h,0) - \log|t|) = \frac{h}{-E \log M} > 0.$$

(We remark that $E[\log M] < 0$ because $E[\log M] = \alpha^{-1} E[\log M^{\alpha}] < \alpha^{-1}[\log \gamma(\alpha)] = 0$, where α is defined in Theorem 0.) So there are numbers $\delta > 0$ and $t_0 > 0$ such that

$$U([-h,0) - \log|t|) \geq \delta \quad \text{for all } |t| > t_0.$$

It follows that for all $|t| > t_0$,

$$P[\tau([-h,0] - \log|t| \geq \frac{U([-h,0) - \log|t|}{1+U([-h,h])} \geq \frac{\delta}{1+U([-h,h])} = \delta_1 > 0.$$

Therefore, writing $\tau = \tau([-h,0] - \log|t|)$ and

$$\gamma = \max_{|x|\in[e^{-h},1]} |\phi(x)|$$

($\gamma < 1$ by part (a) above), and using the stopped martingale theorem, we obtain that, for all $|t| > t_0$,

$$\begin{aligned} |\phi(t)| &\leq \limsup_{n\to\infty} E|\phi(e^{S_{n\wedge\tau}t})| \qquad (n\wedge\tau = \min(n,\tau)) \\ &\leq E[|\phi(e^{S_\tau t})|1\{\tau<\infty\}] + P(\tau=\infty) \\ &\leq \gamma P(\tau<\infty) + (1 - P(\tau<\infty)) \\ &\leq 1-(1-\gamma)\delta_1. \end{aligned}$$

So $\limsup_{|t|\to\infty} |\phi(t)| < 1$, as desired.

Lemma 4.3. *If $\tilde{N} \geq 2$ almost surely and*

$$E[A_{\zeta_1}^{-a} + A_{\zeta_2}^{-a}] < \infty$$

for some $a > 0$, then

$$\phi(t) = O(|t|^{-a}), \qquad |t| \to \infty,$$

where $\phi(t) = E[e^{itZ}]$, $t \in \mathbb{R}$.

Proof. Write

$$\psi(t) = \sup_{|x|\geq t} |\phi(t)|, \qquad t > 0.$$

Then $\psi(t)$ is non-increasing, $\psi(0) = 1$, and $\lim_{t\to\infty}\psi(t) = 0$ by Lemma 4.2. (We remark that the conditions of lemma imply that $E[M^{-a}] \leq E[A_{\zeta_1}^{-a}] < \infty$, so that $E[|\log M|] < \infty$.) By the functional equation (1.1'), it is easily seen that

$$\psi(t) \leq E\prod_{i=1}^{N}\psi(A_i t) \leq E[(\psi(At))^2], \tag{$*$}$$

where $A = \min(A_{\zeta_1}, A_{\zeta_2})$, and the last inequality holds since $\tilde{N} \geq 2$ almost surely. As $P(A \leq x) \leq E[A^{-a}]x^a \leq E[A_{\zeta_1}^{-a} + A_{\zeta_2}^{-a}]x^a$ by Markov's inequality, it follows that for $K = E[A_{\zeta_1}^{-a} + A_{\zeta_2}^{-a}]$ and all $0 \leq u \leq 1$ and $t > 0$,

$$\begin{aligned} \psi(t) &\leq P[A \leq u] + (\psi(ut))^2 \\ &\leq Ku^a + (\psi(ut))^2 \\ &\leq [(Ku)^{a/2} + \psi(ut)]^2. \end{aligned}$$

Therefore, by Lemma 1.4.1.a of Barral (1997), for all $b < a/2$,

$$\psi(t) = O(t^{-b}), \qquad t \to \infty.$$

Fix $b < a/2$. So for some constant $C > 0$ and all $t > 0$, $\psi(t) \leq Ct^{-a/2}$. So $\psi(At) \leq C(At)^{-b}$ and, by the inequality (*), for all $t > 0$,

$$\psi(t) \leq C^2 EA^{-2b}(t^{-2b}).$$

The same argument applies for $b_1 = 2b$ and $C_1 = C^2 EA^{-2b}$, yielding that for all $t > 0$,

$$\psi(t) \leq C_1^2 EA^{-2b_1}(t^{-2b_1}).$$

Taking $b = a/4$ gives the result desired.

Lemma 4.4. *Assume $\tilde{N} \geq 2$ almost surely and put $M_i = \max_{j \neq i} A_j$. Let ϕ be a non-trivial solution of (1.1'). If for some $\epsilon > 0$, $\phi(t) = O(|t|^{-\epsilon}), |t| \geq 1$,*

$$E\sum_{i=1}^{N} M_i^{-\epsilon} < \infty \text{ and } E\sum_{i=1}^{N} A_i M_i^{-\epsilon} < \infty,$$

then for some $\delta > 0$,

$$\phi'(t) = O(|t|^{-(1+\delta)}), \qquad |t| \to \infty.$$

Proof. From the functional equation (1.1'), we obtain

$$\phi'(t) = E[\sum_{i=1}^{N} A_i \phi'(A_i t) \prod_{j \neq i} \phi(A_j t)].$$

Since $|\phi(t)| \leq 1$, it follows that $|\prod_{j \neq i} \phi(A_j t)| \leq \phi(M_i t)$ and

$$|\phi'(t)| \leq E[\sum_{i=1}^{N} A_i |\phi'(A_i t)| \; |\phi(M_i t)|]. \qquad (**)$$

Because $|\phi'(t)| \le 1$ and

$$|\phi(t)| \le K|t|^{-\epsilon}$$

for some constant $K > 0$ and all $t \in \mathbb{R}^*$, the preceding inequality implies that

$$|\phi'(t)| \le E[\sum_{i=1}^{N} A_i K|M_i t|^{-\epsilon}] \le K_1 |t|^{-\epsilon},$$

where $K_1 = KE[\sum_{i=1}^{N} A_i M_i^{-\epsilon}] < \infty$.

If $\epsilon \le 1$, using the inequalities $|\phi'(A_i t)| \le K_1 |A_i t|^{-\epsilon}, \phi(M_i t) \le K|M_i t|^{-\epsilon}$ and (**), we see that

$$|\phi'(t)| \le E[\sum_{i=1}^{N} A_i K_1 |A_i t|^{-\epsilon} K|M_i t|^{-\epsilon}] = K_2 |t|^{-2\epsilon}, \quad t \in \mathbb{R}^*$$

where $K_2 = KK_1 E[\sum_{i=1}^{N} A_i^{1-\epsilon} M_i^{-\epsilon}] < \infty$. (We remark that the moment conditions in the lemma implies that $E[\sum_{i=1}^{N} A_i^x M_i^{-\epsilon}] < \infty$ for all $x \in [0, 1]$.)

If $2\epsilon \le 1$, we continue the procedure, and so on. If $n = \max\{k \ge 0 : k\epsilon \le 1\}$, then $n\epsilon \le 1$ but $(n+1)\epsilon > 1$, and the argument as above shows that for some constant $0 < K_{n+1} < \infty$ and all $t \in \mathbb{R}^*$,

$$|\phi'(t)| \le K_{n+1} |t|^{-(n+1)\epsilon}.$$

Proof of Theorem 3. By Lemma 2.2, $L(\tilde{Z})$ is either absolutely continuous or singularly continuous on $(0, \infty)$. Since $L(\tilde{Z}) = xP_Z(dx)$, the same is true for the distribution of Z.

Let ϕ be the characteristic function of Z.

If (i) holds, then by lemmas 4.3 and 4.4,

$$\phi'(t) = O(|t|^{-(1+\delta)}), \qquad |t| \to \infty$$

for some $\delta > 0$. Since $-i\phi'(t)$ is the characteristic function of the probability measure $xP_Z(dx)$, by the inverse formula of Fourier transform, the measure $xP_Z(dx)$ has a continuous density on $\mathbb{R}$, so $P_Z(dx)$ has a continuous density on $(0, \infty)$.

Assume that (ii) holds. Without loss of generality, we can suppose that $N \ge 1$ and $A_1 > 0$ if $1 \le i \le N$ a.s. (Otherwise, we consider $(\tilde{N}, A_{\zeta_1}, ..., A_{\zeta_{\tilde{N}}})$ instead of $(N, A_1, ..., A_N)$; see the discussion before the statement of Theorem

3.) Therefore $A_1 = A_{\zeta_1}$ is absolutely continuous on $\mathbb{R}$ and $P(Z_1 = 0) = 0$. So by the lemma at page 166 of Grintsevichyus (1974), $L(Z) = L(A_1 Z_1 + B)$ is absolutely continuous on $\mathbb{R}$, where $B = \sum_{2 \leq i \leq N} A_i$.(The sum is taken to be zero if $N < 2$.)

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