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FLOWS IN NETWORKS AND HAUSDORFF MEASURES ASSOCIATED. APPLICATIONS TO FRACTAL SETS IN EUCLIDIAN SPACE

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Abstract

We consider a ramdomly capacited network $\mathcal{G}$, composed with a tree $\mathcal{T}$ generated by a branchiny process and a capacity $X_\sigma>0$ assigned to each vertex $\sigma\in\mathcal{T}$, where $X_\sigma(\sigma\in\mathcal{T})$ satisfy some natural independent and self-similar properties. The main purpose of this paper is to find an optimal weight function ϕ so that a positive flow is possible through the network with modified capacities $\phi(X_\sigma)$. The problem is translated to a study of some Hausdorff measures associated. The function is found to be of the form $t^\alpha|\log\log\frac{1}{t}|^\beta$ with α and β caculated explicitely. The results answer a question of Falconer(1987) and solve a conjecture of Hawkes(1981). As applications to random constructions of fractal sets in Euclidian space, we generalize and improve the results of Graf, Mauldin and Williams (1988). As a byproduct, we give also a generalization of a result of Kahane and Peyrière (1976).

1.Introduction

Let $\mathcal{T} = \mathcal{T}(\omega)$ be a random tree generated by a branching process with a single founder member and with a family distribution N. The root of $\mathcal{T}$ is identified to the founder member which is represented by the null sequence $\emptyset$. The vertices in the n-th level are represented by a n-terms sequence σ =

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$(\sigma_1,\sigma_2,\ldots,\sigma_n)$ of non-negative integers which correspond to the particles in the n-th generation of the branching process. The edges of $\mathcal{T}$, noted by $(\sigma,\sigma*j)$ $(1\le j\le N_\sigma)$ are formed by joining the vertices $\sigma = (\sigma_1,\sigma_2,\ldots,\sigma_n)$ to their descendants $\sigma*j= (\sigma_1,\sigma_2,\ldots,\sigma_n,j)$, where N_σ denotes the number of descendants of σ in the next generation. $\mathcal{T}$ is then identified to a set of finite sequences of positive integers.

Let $\tilde{\mathcal{T}}=\tilde{\mathcal{T}}(\omega)$ be the boundary set of $\mathcal{T}$, namely the set of infinite sequences $\mathbf{i}$ such that $\mathbf{i}|n\in\mathcal{T}$ for all $n\ge 0$, where $\mathbf{i}|0=\emptyset$ and $\mathbf{i}|n= (i_1,i_2,\ldots,i_n)$ if $\mathbf{i}= (i_1,i_2,\ldots,i_n,\ldots)$. $\tilde{\mathcal{T}}$ is called to be the *branching set* associated to $\mathcal{T}$.

Let $\mathcal{G}=\mathcal{G}(\omega) = (\sigma,X_\sigma)$ $(\sigma\in\mathcal{T})$ be an associated random network formed by the tree $\mathcal{T}=\mathcal{T}(\omega)$ and a capacity $X_\sigma>0$ associated to each $\sigma\in\mathcal{T}$. We suppose always that (X_σ) is *decreasing* in that

$$X_{\sigma*j}\le X_\sigma$$

if $\sigma*j\in\mathcal{T}$, where $\sigma*j= (\sigma_1,\sigma_2,\ldots,\sigma_n,j)$ if $\sigma= (\sigma_1,\sigma_2,\ldots,\sigma_n)$. We suppose also that the network $\mathcal{G}(\omega)$ is *self-similar* in that for each $\sigma\in\mathcal{T}$, the random vectors

$$Z_\sigma:= (N_\sigma;\ T_{\sigma*1}, T_{\sigma*2}, \ldots, T_{\sigma*N_\sigma})$$

are independent and identically distributed, where

$$T_{\sigma*j}= X_{\sigma*j}/X_j$$

$(1\le j\le N_\sigma)$ represent the ratios of the capacites $X_{\sigma*j}$ to X_σ. Thus $\forall\sigma\in\mathcal{T}$

$$0<T_\sigma\le 1 \quad\text{and}\quad X_\sigma= X_\emptyset \prod_{k=1}^{|\sigma|} T_{\sigma|k},$$

where $\sigma|k= (\sigma_1,\sigma_2,\ldots,\sigma_k)$ if $\sigma= (\sigma_1,\sigma_2,\ldots,\sigma_n)$ $(1\le k\le n)$ and $|\sigma|=n$ denotes the length of σ. For convenience, we assume the normalization

$$X_\emptyset=1$$

a.s. Thus $X_\sigma= \prod_{k=1}^{|\sigma|} T_{\sigma|k}$ and $T_i=X_i$ $(1\le i\le N)$. Also, we write $T_\emptyset$ for $X_\emptyset$ and

$$Z = (N;\ T_1, \ldots, T_N)$$

for $Z_\emptyset = (N_\emptyset;\ T_1, \ldots, T_N)$, and we say that $\mathcal{G}= \mathcal{G}(\omega)$ is a self-similar network generated by Z.

A flow or *positive flow* in the network $\mathcal{G}(\omega)$ is a function $f=f_{\omega}$: $\mathcal{T} \longrightarrow [0,\infty)$ such that

$$f(\mathrm{i}) = \Sigma_{\mathrm{i:i*i}\in\mathcal{T}} f(\mathrm{i*i}) \ (\mathrm{i}\in\mathcal{T}), \quad (f1)$$

$$0 \le f(\mathrm{i}) \le X_{\mathrm{i}} \ (\mathrm{i}\in\mathcal{T}), \quad (f2)$$

$$0 < f(\emptyset). \quad (f3)$$

Intuitively f represents the rate of flow of a liquid through the network. Condition ($f1$) reflects the fact that the amount of liquid reacting a vertex of $\mathcal{G}$ equals that leaving it, ($f2$) ensures that the flow through each edge does not exceed the edge capacity, and ($f3$) is the positivity condition, that a positive amount of liquid is able to flow the system from $\emptyset$ to infinity. We shall principally be concerned with conditions under which a positive flow through a network exists.

The most important problem is to know when a flow through a network is possible and, more presicely, how to modify the capacities in a optimal way in some sense such that a flow is possible. The main general result on the existence of flows is the "*max-flow min cut*" theorem of Ford and Fulkerson (1962). Let $\mathcal{G}=(\sigma,X_{\sigma})_{\sigma\in\mathcal{T}}$ be a self-similar network generated by Z= (N; T_1,...T_N), Falconer (1986) proved that a flow through the network is possible with positive probability if $\mathbb{E}(\Sigma_{i=1}^{N} T_i)$ >1 and is a.s. impossible if $\mathbb{E}(\Sigma_{i=1}^{N} T_i)$ <1. Here is to solve the more exact problem as follows:

Given a self-similar network $\mathcal{G}$ = $(\sigma, X_{\sigma})_{\sigma\in\mathcal{T}}$, how to modify the capacities X_{σ} in a homogeneous and optimal way in some sense such that a flow through the network is possible? More exactly, what is the optimal function ϕ: $[0,\infty) \longrightarrow [0,\infty)$ for which a positive flow through the network $(\sigma, \phi(X_{\sigma}))_{\sigma\in\mathcal{T}}$ exists?

To solve this problem, we shall study some Hausdorff measures on the branching set $\tilde{\mathcal{T}}$ associated with the network $\mathcal{G}$ with a metric d defined by

$$d(\mathrm{i},\mathrm{j})= X_{\mathrm{i}\wedge\mathrm{j}} ,$$

$i \wedge j$ (called the *common sequence* of i and j) being the maximal sequence $q=(q_1,...,q_k)$ such that $(i_1,...i_k)= (j_1,...,j_k) =(q_1,...,q_k)$.

First of all, we shall translate the criterion of Ford and Fulkerson (1962) in terms of Hausdorff measures:

Theorem 1. Let $\mathcal{G}=(\sigma,X_\sigma)_{\sigma\in\mathcal{T}}$ *be a self-similar network generated by* $Z= (N; T_1,...T_N)$, *and* ϕ: $[0,\infty) \rightarrow [0,\infty)$ *a non-negative function, increasing and continuous from the right. Then almost surely*

a positive flow through the network $(\sigma,\phi(X_\sigma))$ $(\sigma\in\mathcal{T})$ *is possible*

if and only if

$$\mathcal{H}^{\phi}(\tilde{\mathcal{T}}) > 0$$

where $\mathcal{H}^{\phi}(.)$ *represents the Hausdorff measure on* $\tilde{\mathcal{T}}$ *with respect to the dimension function* ϕ, $\tilde{\mathcal{T}}$ *carrying the metric defined above.*

Our question is then to find an optimal dimension function ϕ to measure the branching set $\tilde{\mathcal{T}}$. As we may expect, our results will be havily dependent of the distribution of

$$S(x):= \Sigma_{i=1}^{N} T_i^x,$$

where $x\in[0,\infty)$. Writing

$$\alpha= \min \{a\in[0,\infty): \mathbb{E}[S(a)] \le 1\}, \quad \text{where } \min \emptyset =+\infty,$$

then $\alpha\in[0,\infty]$ is well defined and $\mathbb{E}[S(\alpha)]\le 1$ if $\alpha<\infty$, as we shall see later. We exclude the case where N=1 a.s. Thus $\alpha=0$ if and only if $E(N)\le 1$, or if and only if the tree teminates a.s. or again, if and only if $\tilde{\mathcal{T}}=\emptyset$ a.s., and $\alpha< \infty$ if and only if there is a $M>0$ sufficiently large such that $\mathbb{E}(S(M)]\le 1$, which hapens quite often. Thus we suppose always that

$$0<\alpha<\infty$$

if it is not specified further. We shall see that α is the least solution of $\mathbb{E}[S(x)]=1$ if there is at least a solution. Moreover, the equation has a unique solution if additionally $1<\mathbb{E}(N)<\infty$. Usually we have $\mathbb{E}(S(\alpha))=1$, but the case $\mathbb{E}(S(\alpha))<1$ may hapen. (see Lemma α in section 3).

In all the theorems stated here, we suppose always the moments conditions

$$\mathbb{E}[S(\alpha)^2]<\infty \quad \text{and} \quad \mathbb{E}(\sum_{i=1}^{\infty} T_i^{\alpha}\log\frac{1}{T_i})<\infty.$$

We shall see that α is in fact the Hausdorff dimension of $\tilde{\mathcal{T}}$.

Theorem 2. (i) $\dim \tilde{\mathcal{T}}(\omega) = \alpha$ *a.s. on* $\tilde{\mathcal{T}}(\omega) \neq \varnothing$. *(ii)* $\mathcal{H}^{\alpha}(\tilde{\mathcal{T}}) <\infty$ *a.s. if* $\alpha<\infty$ *(iii) (a) If* $\mathbb{E}[S(\alpha)]<1$ *then* $\mathcal{H}^{\alpha}(\tilde{\mathcal{T}})=0$ *a.s. (b) If* $\mathbb{E}[S(\alpha)]=1$ *then* $0<\mathcal{H}^{\alpha}(\tilde{\mathcal{T}})<\infty$ *if and only if* $S(\alpha) =1$ *a.s. Consequently,* $\mathcal{H}^{\alpha}(\tilde{\mathcal{T}})=0$ *a.s. if* $S(\alpha)$ *is not a.s. a constant.*

Let us write

$$\phi_b(t) = t^{\alpha}(\log\log \frac{1}{t})^b,$$

where $0\leq b<\infty$, and

$$\beta = \min \{b\in [0,1): S(\frac{\alpha}{1-b}) \leq 1 \text{ a.s.}\}, \quad \text{where } \min \varnothing :=1,$$

we shall see that $\beta\in [0,1]$ is well defined and $\beta<1$ if and only if $S(M)\leq 1$ a.s. for some sufficiently large $M>0$, which happens usually. If $\mathbb{E}(S(\alpha))=1$, then $\beta=0$ if and only if $S(\alpha)=1$ a.s. If the equation

$$\text{ess sup } S(\frac{\alpha}{1-b}) = 1$$

has a solution or some solutions, then β is the least one and certainly $\beta<1$.

For the remainder of this section we suppose always that

$$\mathbb{E}(S(\alpha))=1.$$

If $X\geq 0$ is a random variable, we write $\|X\|=$ ess.sup X for the essential superior of X.

Theorem 3. If $\|N\|_{\infty}<\infty$ *and* $0<\beta<1$, *then* $\mathcal{H}^{\phi_b}(\tilde{\mathcal{T}}) >0$ *a.s. on* $\tilde{\mathcal{T}}\neq\varnothing$ *if and only if* $b\geq\beta$; *If* $\|N\|_{\infty}=\infty$ *or* $\beta=1$, *then* $\mathcal{H}^{\phi_b}(\tilde{\mathcal{T}}) =0$ *a.s. if* $0\leq b<\beta$.

Let us write now

$$W:= \lim_{k\to\infty} \sum_{|\sigma|=k} X_{\sigma}^{\alpha}.$$

Since $\mathbb{E}[S(\alpha)]=1$, the limit exists a.s. with $\mathbb{E}(W)\leq 1$ by the martingale convergence theorem. We shall denote by $r_b=r(W^b)$ the radius of convergence of the moment generating function $\mathbb{E}(e^{tW^b})$ of W^b $(0<b<\infty)$. The following result deals with the critical case where $b=\beta$.

Theorem 4. (i) If $\beta=0$ *then* $0<\mathcal{H}^{\phi_\beta}(\tilde{\mathcal{T}})<\infty$ *a.s. on* $\tilde{\mathcal{T}}\neq\emptyset$, *and, in fact,* $\mathcal{H}^{\phi_\beta}(\tilde{\mathcal{T}})=1$ *a.s. if* $\tilde{\mathcal{T}}\neq\emptyset$. *(ii) If* $\beta>0$, *then* $0<\mathcal{H}^{\phi_\beta}(\tilde{\mathcal{T}})<\infty$ *a.s. on* $\tilde{\mathcal{T}}\neq\emptyset$ *if and only if* $0<r(W^{1/\beta})<\infty$. *If* $\|N\|_\infty<\infty$ *and* $0<\beta<1$, *the condition reduces to* $r(W^{1/\beta})<\infty$, *which holds particularly if there exists* $n>1$ *such that*

$$\prod_{i=1}^{\infty}\frac{\mathbb{E}[S(\alpha)^{n^i}\,|\,N=n]^{1/n^i}}{n^\beta} > 0, \qquad (L_n)$$

or equivalently $\sum_{i=1}^{\infty}\left(1-\frac{\mathbb{E}[S(\alpha)^{n^i}\,|\,N=n]^{1/n^i}}{n^\beta}\right) < \infty$. *Moreover, we have*

$$\mathcal{H}^{\phi_\beta}(\tilde{\mathcal{T}})= (r_{1/\beta})^\beta W \quad a.s.$$

Corollary 1. Let $\mathcal{G}= (\sigma, X_\sigma)_{\sigma\in\mathcal{T}}$ be a self-similar network generated by $Z=(N;T_1,\ldots T_N)$. For $0\le b<\infty$, write $\phi_b(t) = t^\alpha(\log\log \frac{1}{t})^b$. Then the function $\phi_\beta(t) = t^\alpha(\log\log \frac{1}{t})^\beta$ is the optimal weight function for the existence of positive flows through the network in that

(i) if $b<\beta$, a positive flow through the network $(\sigma, \phi_b(X_\sigma))_{\sigma\in\mathcal{T}}$ is a.s. impossible;

(ii) if $b\ge\beta$ a positive flow through the network $(\sigma, \phi_b(X_\sigma))_{\sigma\in\mathcal{T}}$ is a.s. possible on the event that the tree process does not terminate. Here, in the case where $\|N\|_\infty=\infty$ or $\beta=1$, we suppose additionally that $r_{1/\beta}>0$.

(iii). If $r_{1/\beta}<\infty$ (or more paticularly if (L_n) holds) and $\phi(t)\ge0$ is a function smaller than ϕ_β in that $\lim_{t\to 0+}\phi(t)/\phi_\beta(t)=0$, then a positive flow through the network $(\sigma, \phi_b(X_\sigma))_{\sigma\in\mathcal{T}}$ is a.s. impossible.

This answers our original question.

A subset Γ of $\mathcal{T}$ is termed a *cut-set* if $\forall i\in\tilde{\mathcal{T}}$ there exists a unique $n\ge0$ such that $i|n\in\Gamma$. Let ϕ be a non-negative function defined on $[0,\infty)$, we are interested to the limit behavious of cut-set sums $\sum_{\sigma\in\Gamma}\phi(X_\sigma)$. Put

$$\mathcal{M}^\phi(\mathcal{T}):= \underline{\lim}_{k\to\infty}\ \mathcal{M}_k(\mathcal{T}),$$

where

$$\mathcal{M}_k(\mathcal{T}) = \inf\{\Sigma_{\sigma\in\Gamma}\ \phi(X_\sigma): \Gamma \text{ is a cut-set of } \mathcal{T} \text{ and } |\sigma|\geq k\ \forall\sigma\in\Gamma\}.$$

For a self-similar network $\mathcal{G}= (\sigma, X_\sigma)$ $(\sigma\in\mathcal{T})$ generated by $Z=(N; T_1,\ldots,T_N)$, after showing that $\mathcal{M}^{\phi}(\mathcal{T})=0$ a.s. if $\phi(t)=t^{\alpha}$ and $\mathcal{M}^{\phi}(\mathcal{T})=\infty$ a.s. on $\tilde{\mathcal{T}}\neq\emptyset$ if $\phi(t)= t^{\alpha}(\log\frac{1}{t})^{a}$ $(\forall a>1)$ under some conditions on Z, Falconer (1987) demanded what was the exact function ϕ for which $0<\mathcal{M}^{\phi}(\mathcal{T})<\infty$ a.s. on $\tilde{\mathcal{T}}\neq\emptyset$? The following result answers this question:

Corallary 2. Let $\mathcal{G}= (\sigma, X_\sigma)$ $(\sigma\in\mathcal{T})$ be a self-similar network and suppose that $p(S(\alpha)>1)>0$. Then $0<\mathcal{M}^{\phi_\beta}(\mathcal{T})<\infty$ a.s. on $\tilde{\mathcal{T}}\neq\emptyset$ if and only if $0<r(W^{1/\beta})<\infty$. If $\|N\|_\infty$ and $0<\beta<1$, the condition reduces to $r(W^{1/\beta})<\infty$, which holds if there exists $n>1$ such that $\prod_{i=1}^{\infty}\frac{\mathbb{E}[S(\alpha)^{n^i}\,|N=n]^{1/n^i}}{n^\beta} > 0$. Moreover, we have

$$\mathcal{M}^{\phi}(\mathcal{T}) = (r_{1/\beta})^{\beta} W \quad \text{a.s.}$$

If $\tilde{\mathcal{T}}$ carries the metric $d_2(i,j)= 2^{-|i\wedge j|}$ and N is of geometric distribution, Hawkes (1981) proved that $0<\mathcal{H}^{\phi_1}(\tilde{\mathcal{T}})<\infty$ a.s. The author (1992) has recently extended this result to the general case where N is of arbitrary distribution, answering a conjecture of Hawkes (1981). We remark that Theorem 4 applies for $X_i=2^{-|i|}$ $(Z=(N;\frac{1}{2},\ldots,\frac{1}{2}))$, yielding that

Corollary 3. If $\tilde{\mathcal{T}}$ carries the metric $d_2(i,j)= 2^{-|i\wedge j|}$, then $0<\mathcal{H}^{\phi_\beta}(\tilde{\mathcal{T}})<\infty$ a.s. on $\tilde{\mathcal{T}}\neq\emptyset$, where $\phi_\beta(t) = t^{\alpha}(\log\log\frac{1}{t})^{\beta}$ with $\alpha= \log\mathbb{E}(N)/\log 2$ and $\beta= 1-\log\mathbb{E}(N)/\log\|N\|_\infty$ if either of the following conditions holds: (i) $\|N\|_\infty<\infty$ (thus $\beta<1$); (ii) $\|N\|_\infty=\infty$ (thus $\beta=1$) and there exist $t_1>t_2>0$ such that $\mathbb{E}(e^{t_1 N})=\infty$ and $\mathbb{E}(e^{t_1 N})<\infty$. Moreover, $\mathcal{H}^{\phi_\beta}(\tilde{\mathcal{T}})=(r_{1/\beta})^{\beta}W$ a.s.

Our next result generalizes a theorem of Kahane and Peyrière (1976).

Corollary 4. If $\|N\|_\infty<\infty$ and $\beta<1$, then $\lim_{k\to\infty}\frac{\log\mathbb{E}(W^k)}{k\log k} = \beta$; if $\|N\|_\infty=\infty$ or $\beta=1$, then $\lim_{k\to\infty}\inf\frac{\log\mathbb{E}(W^k)}{k\log k} \geq \beta$.

The last application is to study the Hausdorff measures of some fractal sets in $\mathbb{R}^m$. We shall generalize or improve some of the results of S.Graf, R.D.Maudin and Williams (1988) and of Falconer (1986 and 1987) concerning the exact Hausdorff dimension of some self-similar fractals.

Let us indroduce the random construction of Graf, Mauldin and Williams (1986 and 1988). Let J be a nonempty compact subset of $\mathbb{R}^m$ which is equal to the closure of its interior, $\mathcal{T}$ be a tree generated by N and

$$\mathbf{J} = (J_\sigma)(\sigma \in \mathcal{T})$$

be a family of random subsets of $\mathbb{R}^m$ satisfying three properties:

(1) For almost all $\omega \in \Omega$, $J_\varnothing(\omega) = J$ and for every $\sigma \in \mathcal{T}$, $J_\sigma(\omega)$ is geometrically similar to J;

(2) For almost every ω and for every $\sigma \in \mathcal{T}$, $J_{\sigma*1}(\omega), J_{\sigma*2}(\omega), \ldots, J_{\sigma*N_\sigma}(\omega)$ is a sequence of nonoverlapping subsets of $J_\sigma(\omega)$ (A and B nonoverlapping means int A $\cap$ int B = $\varnothing$);

(3) The random vectors $Z_\sigma = (N_\sigma; T_{\sigma*1}, \ldots, T_{\sigma*N_\sigma})$, $\sigma \in \mathcal{T}$, are i.i.d., where $T_{\sigma*n}(\omega)$ equals the ratio of the diameter of $J_{\sigma*n}(\omega)$ to the diameter of $J_\sigma(\omega)$.

Our interest centers on the asymptotic properties of the random set

$$K(\omega) = \bigcap_{n=1}^{\infty} \bigcup_{|\sigma|=n} J_\sigma(\omega)$$

Given a dimension function ϕ, the net measure $v^\phi(K)$ of$K$ with respect to the net **J** is defined in a analogous way as Hausdorff measures but using covers of sets in **J**:

$$v^\phi(K) = \lim_{\delta \to \infty} v^\phi_\delta(K),$$

where $$v^\phi_\delta(K) = \inf\{\Sigma_i \phi(U_i) : K \subseteq \cup_i U_i, \ \mathrm{diam}(U_i) \leq \delta \text{ and } U_i \in \mathbf{J}\}.$$

Net measures and Hausdorff measures are closely related, see Rogers (1970) and Falconer (1986). We write v^a for v^ϕ if $\phi(t)=t^a$ $(a>0)$. The Hausdorff dimension of K with respect to the net **J** (or to the net measure) is by definition

$$\dim_v A = \inf\{a>0 : v^a(A)=0\} = \sup\{a>0 : v^a(A)=\infty\}.$$

Mauldin and Williams (1986) and Falconer (1986) proved that the Hausdorff dimension of K is a.s. α defined above if $K\neq\emptyset$. In the case where N is a constant, Graf, Mauldin and Williams (1988) have calculated some exact dimension functions. We shall establish a very general result for net measures, which prove very powerful to find exact dimension functions. Even in the case where N is bounded, our result is sharp, both in theory and in practice.

Corollary 5. Let K be a random set constructed above. Suppose that for each $\sigma\in\mathcal{T}$, $J_{\sigma*i}$ contains a point of K which is not contained in any $J_{\sigma*j}$ $(i\neq j)$. Using the notations above, we have

(i) $\dim_V(K)=\alpha$ a.s. on $K\neq\emptyset$;

(ii) If $\beta=0$, then $0<v^{\phi_\beta}(K)<\infty$ a.s. on $K\neq\emptyset$. In fact, $v^{\phi_\beta}(K)=1$ a.s. if $K\neq\emptyset$.

(iii) If $\beta>0$, then $0<v^{\phi_\beta}(K)<\infty$ a.s. on $K\neq\emptyset$ if and only if $0<r_{1/\beta}<\infty$. If $\|N\|_\infty<\infty$ and $0<\beta<1$, the condition reduces to $r_{1/\beta}<\infty$, which holds in particular if there exists n>1 such that $\prod_{i=1}^{\infty}\frac{\mathbb{E}[S(\alpha)^{n^i}\,|\,N=n]^{1/n^i}}{n^\beta}>0$. Moreover, we have

$$v^{\phi_\beta}(K)=(r_{1/\beta})^\beta W \text{ a.s.}$$

This result enables us to calculate all the exact dimension functions of almost all the examples of Graf et al.(1988) and Falconer (1986).

The content proceeds as follows:

In section 2, we give some preliminaries containing the notations, definitions of trees and capacited networks and Hausdorff measures associated. We shall also gather some topological properties of the limit set $\tilde{\mathcal{T}}$.

In section 3, we establish some interesting limit theorems on tree processes. We shall calculate the critical value β. A necessary and sufficient condition will be given so that the radius of convergence $r(W^{1/\beta})$ of the momoent generating function $\mathbb{E}(e^{tW^{1/\beta}})$ is positive. This generalizes a result of Graf et al.(1988). As a corollary, we obtain the order of growth of the moments of W, which generalizes a result of Kahane and Peyrière (1976) concerning a martingale of Mandelbrot. A simple sufficient condition will be also given to ensure $r(W^{1/\beta})<\infty$. The argument is mainly based on a distributional equation of the type $W=\sum_{i=1}^{N} A_i W_i$ with some independent properties (see the equation (3.3)).

Section 4 is to give basic estimations on Hausdorff measures of the branching set $\tilde{\mathcal{T}}$ with adduced metric of the network. We establish the results for cut-set sums for convenience, although they hold for Hausdorff measures. We calculate the exact value of the lower limit value of cut-set sums of the type $\sum_{\sigma\in\Gamma} \phi_\beta(X_\sigma)$, and, in particular, we give a necessary and sufficient condition for the lower limit to be zero, positive or finite.

The main results are stated in section 5. Theorem 5.1 is to translate a criterion of Ford and Fulkerson in terms of Hausdorff measures for existence of a positive flow through a network. The result holds in the deterministic case. Theorem 5.2 gives the Hausdorff dimension α and the α-dimensioal measures. In Theorem 5.3, we calculate the exact values of Hausdorff measures $\mathcal{H}^{\phi_b}(\tilde{\mathcal{T}})$ of the limit set $\tilde{\mathcal{T}}$ with respect to the dimension function of the type $\phi_b(t)= t^{\alpha}(\log\log\frac{1}{t})^b$ $(b\geq 0)$. We then establish a criterion for the Hausdorff measures $\mathcal{H}^{\phi_b}(\tilde{\mathcal{T}})$ to be zero, positive or infinite. Theorem 5.4 gives the critical value β and ensures the positivity of Hausdorff measures with respect to the critical function $\phi_\beta(t)=t^{\alpha}(\log\log\frac{1}{t})^{\beta}$. Theorem 5.5 is the most important and fluquently used result. It gives a criterion for ϕ_β to be an exact dimension function. Theorem 5.6 deals with the case where $S(\alpha)$ is too large, such that the function of the form $t^{\alpha}(\log\log\frac{1}{t})^b$ is too small to measure the set. We shall see that Falconer's results (1986 and 1987) will be considerably improved.

Section 6 is to give some applications of the main results to random constructions of fractal sets in Euclidian space. Theorem 6.3 will prove very powerful to find exact dimension functions.

In section 7, we give a series of examples to show how Theorem 6.3 enables us to caculate exact dimension functions of self-similar fractal sets. Examples 7.1 and 7.2 is a generization of the construction of the classical Cantor set. Example 7.3 is a construction of random Von-Koch curves. In example 7.4, we give a quite general construction of a random set of high connectivity. The example is taken from Falconer (1986) where the a.s. dimension is calculated. Here we give an exact dimension function. As a corollary, we obtain the exact dimension functions of Graf et al.(1988) on Mandelbrot's percolation Processes and their modified curdling. Examples 7.5-7.7 give constructions where the number of descendants may be unbounded. Example 5.8 is about the zero set of Brownian bridge. This is taken from Graf et al.(1988) where the exact dimension has been given. We take it to illustrate how the famous function $t^{1/2}(\log\log\frac{1}{t})^{1/2}$ can be obtained very easily by Theorem 6.3. In example 7.9, we give a construction of a fractal for which the functions of the form $t^{\alpha}(\log\log\frac{1}{t})^{b}$ ($\forall b>0$) are too small to be exact dimention functions. In this case, we calculate a critical function of the form $t^{\alpha}(\log\frac{1}{t})^{a}$.

2. Capacited networks and Hausdorff measures associated

2.1 Sequences and trees

Let $\mathbb{N}$ be the set of positive integers, N^k the set of all k term sequences, $T = \bigcup_{k=0}^{\infty}\mathbb{N}^k$ the set of all finite sequences and $\mathbf{I} = \mathbb{N}^{\mathbb{N}}$ the infinite sequences **i** $= (i_1, i_2, \ldots)$. We make the convention that $\mathbb{N}^0$ contains the null sequence $\varnothing$.

If $\mathbf{i} = (i_1, i_2, \ldots, i_n)$ $(n \le \infty)$ is a sequence, we write $|\mathbf{i}| = n$ for the length of **i**, and $\mathbf{i}|k = (i_1, i_2, \ldots, i_k)$ $(k \le n;\ \mathbf{i}|0 = \varnothing)$ for the curtailement of **i** after k-terms. If $n < \infty$, we write $\mathbf{i}^* = (i_1, i_2, \ldots, i_n + 1) \in T$ for the sequence obtained by augmenting the n-th component i_n of **i** to $i_n + 1$. If **j**$= (j_1, j_2,$

$\ldots,j_m)$ is another sequence, we write $\mathbf{i.j} = \mathbf{i*j} = (i_1,i_2,\ldots,i_n, j_1,j_2,\ldots,j_m)$ for the sequence obtained by juxtaposition. We partially order T by writing $\sigma<\tau$ or $\tau>\sigma$ to mean that the sequence τ is an extension of σ, that is $\tau = \sigma*\tau'$ for some sequence $\tau'\in T$. We use a similar notation if $\sigma \in T$ and $\tau \in$ I. We remark that the null sequence $\varnothing < \mathbf{i}$ for any sequence $\mathbf{i}$. If $\mathbf{i}$ and $\mathbf{j}$ are two sequence, we write $\mathbf{i}\wedge\mathbf{j}$ for the *common sequence* of $\mathbf{i}$ and $\mathbf{j}$, that is, the maximal sequence $\mathbf{q}$ such that $\mathbf{q}<\mathbf{i}$ and $\mathbf{q}<\mathbf{j}$.

A tree $\mathcal{T}$ is a collection of finite sequences of positive integers satisfing three conditions: (i) $\varnothing\in\mathcal{T}$; (ii) $\mathbf{i}\in\mathcal{T}$ implies $\mathbf{i}'\in\mathcal{T}$ for any $\mathbf{i}'<\mathbf{i}$; (iii) If $\mathbf{i}\in\mathcal{T}$, then $\mathbf{i}*j\in\mathcal{T}$ if and only if $1\le j\le N_{\mathbf{i}}$ for a positive integer $N_{\mathbf{i}}\ge 0$ (We allow the possibility that $N_{\mathbf{i}}=0$, but always assume that $N_{\mathbf{i}}<\infty$). The sequences $\mathbf{i}$ of $\mathcal{T}$ are called the vertices of $\mathcal{T}$, and the couples $(\mathbf{i},\mathbf{i}*i)$ the edges of $\mathcal{T}$, where $\mathbf{i}\in\mathcal{T}$ and $\mathbf{i}*i \in \mathcal{T}$. Thus $N_{\mathbf{i}}\equiv \#\{i\in\mathbb{N}: \mathbf{i}*i \in \mathcal{T}\}$ represents the number of outgoing edges from the vertex $\mathbf{i}$ in the graph of $\mathcal{T}$. We write $\mathcal{T}_k = \{\mathbf{i}\in\mathcal{T}: |\mathbf{i}|=k\}$ for the set of sequences in $\mathcal{T}$ of length k. (cf. Neveu 1986)

Let $\tilde{\mathcal{T}}$ be the set of infinite sequences $\mathbf{j}$ such that $\mathbf{i} \in \mathcal{T}$ for every finite curtailment $\mathbf{i} < \mathbf{j}$. We may regard $\tilde{\mathcal{T}}$ as a topological space in a natural way by taking as a basis of topology $\{B(\mathbf{i})\}_{\mathbf{i}\in\mathcal{T}}$, where

$$B(\mathbf{i})= \{\mathbf{j}\in\tilde{\mathcal{T}}: \mathbf{i}<\mathbf{j}\}. \tag{2.1}$$

The $B(\mathbf{i})(\mathbf{i}\in\mathcal{T})$ will be called the *balls* of $(\tilde{\mathcal{T}},\tau)$. The basis $\{B(\mathbf{i})\}$ $(\mathbf{i}\in\mathcal{T})$ is countable.The topology τ is that induced by the product topology of $\mathbb{N}^{\mathbb{N}}$ when $\tilde{\mathcal{T}}$ is regarded as a subspace of $\mathbb{N}^{\mathbb{N}}$, $\mathbb{N}$ carrying the discrete topology. The space $(\tilde{\mathcal{T}}, \tau)$ is metrizable, and a possible choice of the metric is $d_2(\mathbf{i},\mathbf{j}) = 2^{-|\mathbf{i}\wedge\mathbf{j}|}$. We gather some topological properties of $(\tilde{\mathcal{T}},\tau)$ as follows:

Lemma 2.1. $(\tilde{\mathcal{T}},\tau)$ is a metrizable and compact topological space.

Proof. We only need to show that $(\tilde{\mathcal{T}},\tau)$ is a compact topological space. To see this, we remark that $(\tilde{\mathcal{T}},\tau)$ can be regarded as a closed subspace of the product topological space $E=\prod_{n=1}^{\infty}E_n$, where $E_n=\{1,\ldots,Z_n\}$ carries the discrete topology, Z_n= card$\{i\in\mathcal{T}:\ |i|=n\}$. Since E is compact by Tychonov's theorem, the proof is completed. □

$\tilde{\mathcal{T}}$ will be called the *boundary* of $\mathcal{T}$. If $\mathcal{T} = \mathcal{T}(\omega)$ is a random tree generated by a branching process (that is, the numbers N_i of outgoing edges from the vertexes i form a family of independent and identically distributed random variables), $\tilde{\mathcal{T}}$ is then called the *branching set* associated with $\mathcal{T}$ (Hawkes 1981).

We say that a subset Γ of $\mathcal{T}$ *covers* $\tilde{\mathcal{T}}$ if for every $j\in\tilde{\mathcal{T}}$ there is a sequence $i\in\Gamma$ with $i<j$, or namely $j|n\in\Gamma$ for some $n\geq0$. If Γ covers $\tilde{\mathcal{T}}$, we say also that Γ is a *covering* of $\tilde{\mathcal{T}}$ or Γ is *complete in* $\mathcal{T}$, or again Γ is *maximal* in $\mathcal{T}$. A cover Γ is *minimal* if for every $j\in\tilde{\mathcal{T}}$ there is a unique $i\in\Gamma$ with $i<j$. A minimal cover of $\tilde{\mathcal{T}}$ is also called a *cut-set* of $\mathcal{T}$ or a *maximal antichain* in $\mathcal{T}$. Intuitively a cut-set seperates $\emptyset$ from the "vertices at infinity". Any covering collection of sequences may be reduced to a finite cover using the compactness of $(\tilde{\mathcal{T}},\tau)$. Moreover, any covering collection of sequences may be reduced to a minimal collection by taking $\{i\in\Gamma$: if $i'\in\Gamma$ and $i'<i$ then $i'=i\}$. Let $\mathfrak{C}$ denote the collection of all cut-sets in $\mathcal{T}$. There is an induced partial ordering that makes $\mathfrak{C}$ into a net: For $\Gamma_1,\Gamma_2\in\mathfrak{C}$, we write $\Gamma_1<\Gamma_2$ (or $\Gamma_2>\Gamma_1$, and we say that Γ_2 is a *refinement* of Γ_1) if for every $\sigma\in\Gamma_2$, there exists a unique $\tau\in\Gamma_1$ with $\tau<\sigma$ (in other words Γ_1 seperates Γ_2 from $\emptyset$). Trivially the sets $\mathcal{T}_k$ are themselves cut-sets of $\mathcal{T}$ with $\mathcal{T}_{k_1}<\mathcal{T}_{k_2}$ if $k_1<k_2$.

Sometimes, it is convenient to regard T as a tree with $N_i\equiv\infty$ (thus $\tilde{T}$=I). For subsets Γ of T, we use the same definitions as above, but, to avoid logical difficulties and to ensure that the associated set Γ is countable, we suppose that $\sup\{|i|: i\in\Gamma\}<\infty$. For example, we term a subset Γ of T a *cut-set* or a *minimal covering* of I if for every $i\in I$ there exists a unique sequence $\sigma\in\mathcal{T}$ such that $\sigma<i$, and if there exists k such that $|\sigma|\leq k$ for all $\sigma\in\Gamma$.

2.2. Valuations on trees and cut-set sums. Let $\mathcal{T}$ be a tree. Suppose that a number $X_i>0$ is associated to each $i\in\mathcal{T}$, which may be regarded the capacity of i. We shall always assume that the X_i are decreasing in that

$$X_j \le X_i \text{ if } j > i. \tag{2.2}$$

These assumptions hold in the practical examples encountered so far.

Let $\phi = \phi(t)$ be a positive function defined on $[0,\infty)$, non-decreasing and continuous from the right. We shall be interested in the limiting properties of the cut-set sums of capacities $\phi(X_\sigma)$. Write

$$\mathcal{M}^{\phi}(\mathcal{T}) = \lim_{k\to\infty} \mathcal{M}_k^{\phi}(\mathcal{T}) \tag{2.3}$$

where $\mathcal{M}_k^{\phi}(\mathcal{T}) = \inf\{ \sum_{\sigma\in\Gamma} \phi(X_\sigma)\text{: } \Gamma \text{ is a cut-set of } \mathcal{T} \text{ and } |\sigma|\ge k\ \forall\sigma\in\Gamma\}$. (2.4)

If $\phi(t)=t^a$ $(a\ge0)$ we write $\mathcal{M}^a(\mathcal{T})$ for $\mathcal{M}^{\phi}(\mathcal{T})$ and $\mathcal{M}_k^a(\mathcal{T})$ for $\mathcal{M}_k^{\phi}(\mathcal{T})$. We shall find an exact critical function ϕ such that $0 < \mathcal{M}^{\phi}(\mathcal{T}) < \infty$ under some conditions on $(X_\sigma)_{\sigma\in\mathcal{T}}$.

It will prove convenient to write

$$T_{\sigma*j} = X_{\sigma*j}/X_\sigma \tag{2.5}$$

if $\sigma\in\mathcal{T}$ and $\sigma*j\in\mathcal{T}$. If $\sigma\notin\mathcal{T}$, we shall write $X_\sigma=T_\sigma=0$. Thus X_σ and T_σ are defined for all $\sigma\in T$, and

$$X_\sigma = X_\varnothing \prod_{i=1}^{|\sigma|} T_{\sigma|i} \quad (\forall\sigma\in T). \tag{2.6}$$

The following lemma is to give some alternatives of $\mathcal{M}_k^{\phi}(\mathcal{T})$. The proof is immediate by the compactness of $(\mathcal{T},\tilde{\tau})$ and the remarks in section 2.1.

Lemma 2.2. $\mathcal{M}_k^{\phi}(\mathcal{T}) = \inf\{ \sum_{\sigma\in\Gamma} \phi(X_\sigma)\text{: } \Gamma \text{ is a cut-set of } \mathcal{T} \text{ and } |\sigma|\ge k\ \forall\sigma\in\Gamma\}$

$= \inf\{ \sum_{\sigma\in\Gamma} \phi(X_\sigma)\text{: } \Gamma \text{ is a cut-set of } \mathcal{T},\ \Gamma \text{ is finite and } |\sigma|\ge k\ \forall\sigma\in\Gamma\}$

$= \inf\{ \sum_{\sigma\in\Gamma} \phi(X_\sigma)\text{: } \Gamma \text{ is complete in } \mathcal{T} \text{ and } |\sigma|\ge k\ \forall\sigma\in\Gamma\}$

$= \inf\{ \sum_{\sigma\in\Gamma} \phi(X_\sigma)\text{: } \Gamma \text{ is complete and finite in } \mathcal{T} \text{ and } |\sigma|\ge k\ \forall\sigma\in\Gamma\}$

$= \inf\{ \sum_{\sigma\in\Gamma} \phi(X_\sigma)\text{: } \Gamma \text{ is a cut-set of } T \text{ and } |\sigma|\ge k\ \forall\sigma\in\Gamma\}$, (2.7)

etc., where we make the convention that sums over subsets of T are taken over those σ for which $X_\sigma>0$.

2.3. Networks and Hausdorff measures associated

For our purposes a *network* or *capacited network* $\mathcal{G}$ comprises a tree $\mathcal{T}$ with a capacity $X_i>0$ assigned to each $i \in \mathcal{T}$, where $X_i(i\in\mathcal{T})$ satisfy the decreasing condition (2.2). We note $\mathcal{G} = (\sigma,X_\sigma)_{\sigma\in\mathcal{T}}$. Usually we have

$$X_{i|n}\to 0 \quad \text{if } n\to\infty \tag{2.8}$$

if $i\in\tilde{\mathcal{T}}$. If this is the case, define functions f_n: $\tilde{\mathcal{T}} \to [0,\infty)$ by $f_n(i)=X_{i|n}$, then $\{f_n\}$ is a sequence of continuous function on the compact space $(\tilde{\mathcal{T}},\tau)$. By Dini's theorem convergence is uniform, so given $\delta>0$ there exists $k(\delta)$ such that

$$X_i\le \delta \quad \text{whenever } |i|\ge k(\delta) \text{ and } i\in\mathcal{T}. \tag{2.8)'$$

Define

$$d(i,j) = X_{i\wedge j} \tag{2.9}$$

if $i\in\tilde{\mathcal{T}}$ and $j\in\tilde{\mathcal{T}}$. It can be easily verified that d is a metric on $\tilde{\mathcal{T}}$ and, in fact d is a ultra-metric in that

$$d(i,j)\le \text{Max }\{d(i,k),\ d(k,j)\} \tag{2.10}$$

for all i, j and $k\in\tilde{\mathcal{T}}$. Thus $(\tilde{\mathcal{T}},d)$ is a ultra-metric space. We shall see that the metric topology d is in general weaker than τ.

Proposition 2.3. Suppose that (2.2) and (2.8) hold. Then any d-open ball is a τ-open ball. The converse holds if additionally X_i is strictly decreasing in that

$$X_i>X_j \quad \text{if } i<j \text{ and } i\ne j. \tag{2.11}$$

Proof. (i) Let $B_d(i,r)=\{j\in\tilde{\mathcal{T}}: d(i,j)<r\}$ be a d-open ball in $\tilde{\mathcal{T}}$. We shall prove that $\forall i\in\tilde{\mathcal{T}}$ and $\forall r>0$

$$B_d(i,r)= B(i|k),$$

where $k=\min\{n\ge0: X_{i|n}<r\}$ ($k<\infty$ by (2.8)). In fact, if $j\in B_d(i,r)$, then $X_{i\wedge j}<r$. It follows that $|i\wedge j|\ge k$. Thus $j> i|k$, that is, $j\in B(i|k)$. Hence $B_d(i,r)\subseteq B(i|k)$. Conversely, if $j\in B(i|k)$, then $j> i|k$, $|i\wedge j|\ge k$ and $X_{i\wedge j}\le X_{i|k}<r$ by the definition of k. Namely $d(i,j)<r$. This shows $B(i|k)\subseteq B_d(i,r)$, which ensures

that a d-open ball is a τ-open ball.

(ii) Suppose now (2.11) holds and $B(i|n)$ is an arbitary τ-open ball, where $i \in \tilde{\mathcal{T}}$ and $n \geq 0$. If $n=0$, then $B(i|n)=\tilde{\mathcal{T}}$ is evidently a d-open ball. If $n>1$, by (2.11) we can choose r such that $X_{i|n} < r < X_{i|(n-1)}$ and we can conclude that $B(i|n)= B_d(i,r)$. The proof is completed. □

Corallary 2.3. Suppose that (2.2) and (2.8) hold. Then (i) The metric topology d is weaker than τ in that any d-open set is τ-open. The two topologies coincide if additionally (2.11) holds. (ii) Any τ-compact set is d-compact; (iii) $(\tilde{\mathcal{T}},d)$ is a ultra-metric compact topological space.

Suppose now (E,ρ) is a metric space and that f is a Hausdorff *dimension function* (Rogers 1970) in that $\phi(t)\geq 0$ is a positive function defined on $[0,\infty)$, nondecreasing and continuous from the right. The Hausdoff measure of $A \subseteq E$ with respect to the dimension function f is by definition

$$\mathcal{H}^f(A) = \lim_{\delta\to 0+} \mathcal{H}^f_\delta(A) \tag{2.12}$$

where

$$\mathcal{H}^f_\delta(A) = \inf \{ \Sigma_{i=1}^{\infty} f(|U_i|): A \subset \cup_{i=1}^{\infty} U_i, \ |U_i| \leq \delta \}, \tag{2.13}$$

$|U_i| = \text{diam}\,(U_i)$ representing the diameter of U_i. It is not difficult to see that the quantity $\mathcal{H}^f(A)$ remains the same if in the definition we use covers of just open sets or just closed sets, or again just subsets of A,see for example Rogers (1970). If we use covers of just balls, we obtain the spherical Hausdorff measure:

$$\mu^f(A) = \lim_{\delta\to 0+} \mu^f_\delta(A) \tag{2.12'}$$

where

$$\mu^f_\delta(A)= \inf \{\Sigma_{i=1}^{\infty} f(|U_i|): A \subset \cup_{i=1}^{\infty} U_i, \ |U_i| \leq \delta \text{ and } U_i \text{ are balls}\}. \tag{2.13'}$$

The two measures $\mathcal{H}^f(.)$ and $\mu^f(.)$ are in general not identical (see Besicovitch 1928, chapter 3) but equivalent if $f(2t)\leq cf(t)$ for some $c>0$ (Liu,1992). However, they coincide on a ultra-metric space:

Theorem 2.4. Suppose that (E,ρ) is a ultra-metric space, $f(t)\geq 0$ is a positive function defined on $[0,\infty)$, non-decreasing and continuous on the right. Then $\forall A\subseteq E$ and $\forall\delta>0$

$$\mathcal{H}^f_\delta(A) = \mu^f_\delta(A) \quad \text{and} \quad \mathcal{H}^f(A) = \mu^f(A) \tag{2.14}$$

Proof. Clearly $\mathcal{H}^f_\delta(A) \leq \mu^f_\delta(A)$ since any δ-cover of A by balls is a permissible covering in the definition of $\mathcal{H}^f_\delta(A)$. Also, if $\{U_i\}$ is a δ-cover of A, then so is $\{B_i\}$, where, for each i, B_i is choosen to be some ball containing U_i and of radius $|U_i|\leq\delta$. In fact, any ball $B_i=B(x_i,|U_i|)$ of a center $x_i\in U_i$ (we may suppose that $U_i\neq\emptyset$) and radius $|U_i|$ meets our needs. To see this, it suffices to show that $|B_i|\leq |U_i|$. This is so since for any x and y of B_i, we have $d(x,y)\leq \max\{d(x,x_i),d(x_i,y)\} \leq|U_i|$. Thus

$$\sum f(|B_i|) \leq \sum f(|U_i|)$$

and taking infima gives $\mu^f_\delta(A) \leq \mathcal{H}^f_\delta(A)$. Hence $\mathcal{H}^f_\delta(A)=\mu^f_\delta(A)$. Letting $\delta \to 0$ gives $\mathcal{H}^f(A) = \mu^f(A)$. □

Since $(\tilde{\mathcal{T}},d)$ is a ultra-metric space, we have immidiately

Corollary 2.4. On $(\tilde{\mathcal{T}},d)$, the two measures $\mathcal{H}^f(.)$ and $\mu^f(.)$ coincide.

If $0 < \mathcal{H}^f(A) < \infty$, we say that f is an *exact dimension function* of A, or simply an *exact dimension* of A, or an *exact measure function* of A. If $f(t) = t^a$ $(a>0)$, we write $\mathcal{H}^a(A)$ instead of $\mathcal{H}^f(A)$, and we call it the a-dimensional Hausdorff measure of A. The Hausdorff dimension of A is defined as

$$\dim A = \sup\{ a > 0 \mid \mathcal{H}^a(A) = +\infty \} \equiv \inf\{ a > 0 \mid \mathcal{H}^a(A) = 0 \}.$$

Then $\mathcal{H}^a(A) = +\infty$ if $a < \dim A$ and $\mathcal{H}^a(A) = 0$ if $a > \dim A$.

The following result gives an alternative for $\mu^\phi(.)$ on $(\tilde{\mathcal{T}},d)$.

Proposition 2.5. Soppose that (2.2) and (2.8) hold, and ϕ is a Hausdorff dimension function. Then $\forall A\subseteq\tilde{\mathcal{T}}$

$$\mu^\phi(A) = \lim_{k\to\infty} \mu^\phi_k(A) \tag{2.12"}$$

where

$$\mu^\phi_k(A)= \inf\{\textstyle\sum_{i\in\Gamma}\phi(|B_i|): A\subset\bigcup_{i\in\Gamma}B(i),\ \Gamma\subseteq\mathcal{T} \text{ and } |i|\geq k \text{ if } i\in\Gamma\}. \tag{2.13"}$$

Proof. By (2.8)', $\forall\delta>0$, $\exists$ $k(\delta)$ sufficiently large such that $X_i\le\delta$ whenever $i\in\mathcal{T}$ and $|i|\ge k(\delta)$. Noting that $\mathrm{diam}(B(i))=X_i$, any cover in the definition of $\mu^{\phi}_{k(\delta)}(A)$ is an admissable covering in the definition of $\mu^{\phi}_{\delta}(A)$, so

$$\mu^{\phi}_{\delta}(A) \le \mu^{\phi}_{k(\delta)}, \tag{2.14'}$$

leading to $\mu^{\phi}(A) \le \lim_{k\to\infty} \mu^{\phi}_{k}(A)$.

Conversely, $\forall\delta>0$, suppose that $\{B(i)\}(i\in\Gamma(\delta))$ is a δ-cover of A by balls such that $|B(i)|\le\delta$ $\forall i\in\Gamma(\delta)$ and

$$\mu^{\phi}_{\delta}(A) \le \Sigma_{i\in\Gamma(\delta)}\, \phi(|B_i|) \le \mu^{\phi}_{\delta}(A)+\delta. \tag{2.15}$$

Writing

$$k(\delta)=\min\{|i|:\ i\in\Gamma(\delta)\},$$

then $\forall i\in\Gamma(\delta)$ $|i|\ge k(\delta)$ and consequently

$$\mu^{\phi}_{k(\delta)}(A) \le \Sigma_{i\in\Gamma(\delta)}\, \phi(|B_i|). \tag{2.16}$$

We claim that $k(\delta)$ is not bounded. Otherwise, there would exist some k_0 such that $\sup_{\delta>0} k(\delta)<k_0$. Write $a=\min\{X_i:\ i\in\mathcal{T}$ and $|i|=k_0\}$, then $a>0$. Choose $i(\delta)\in\Gamma(\delta)$ such that $|i(\delta)|=k(\delta)$. Thus

$$0<a=\min\{X_i:\ i\in\mathcal{T} \text{ and } |i|=k_0\} \le \min\{X_i:\ i\in\mathcal{T} \text{ and } |i|=k(\delta)\}\le X_{i(\delta)}\le\delta$$

for all $\delta>0$, which is impossible. Hence $\sup_{\delta>0} k(\delta)=\infty$. Take $\delta_n\to0$ and $k(\delta_n)\to\infty$. From (2.15) and (2.16) we have

$$\mu^{\phi}_{k(\delta_n)}(A) \le \mu^{\phi}_{\delta_n}(A)+\delta_n.$$

Letting $n\to\infty$, it gives $\lim_{k\to\infty} \mu^{\phi}_{k}(A)\le \mu^{\phi}(A)$. The proof is then completed. □

Remarking that any cover $\{B(i)\}_{i\in\Gamma}$ of $\tilde{\mathcal{T}}$ by balls means that Γ is complete in $\mathcal{T}$, and that $|B(i)|=X_i$, (2.4), Lemma 2.2 and (2.13)' give

Lemma 2.6. For any dimension function ϕ and any $k\in\mathbb{N}$, we have

$$\mathcal{M}^{\phi}_{k}(\mathcal{T}) = \mu^{\phi}_{k}(\tilde{\mathcal{T}}) \quad\text{and}\quad \mathcal{M}^{\phi}(\mathcal{T}) = \mu^{\phi}(\tilde{\mathcal{T}}). \tag{2.17}$$

Combining Corollary 2.4, Proposition 2.5 and Lemma 2.6, we obtain

Theorem 2.7. If (2.2) and (2.8) hold, and ϕ is a Hausdorff dimension function, then

$$\mathcal{H}^{\Phi}(\tilde{\mathcal{T}}) = \mathcal{M}^{\Phi}(\mathcal{T}) = \mu^{\Phi}(\tilde{\mathcal{T}}). \tag{2.18}$$

2.4. Self-similar networks and tree processes

We now examine a model for random networks based on a Galton-Watson branching process, see Falconer (1986). Let Ω be a set and let $\mathcal{G}(\omega)$ be a network formed by a tree $\mathcal{T}(\omega)$ and capacities $X_i(\omega)$ ($i \in \mathcal{T}$) for each $\omega \in \Omega$. We obtain an increasing sequence of σ-fields of subsets of Ω. Let $\mathbb{F}_1 = \sigma(N_\varnothing; X_i: 1 \le i \le N_\varnothing)$, and given $\mathbb{F}_k$ define

$$\mathbb{F}_{k+1} = \sigma(\mathbb{F}_k;\ N(i): i \in \mathcal{T}_k;\ X_{i*(i_{k+1})}: 1 \le i_{k+1} \le N_i)$$

Let $\mathbb{F} = \bigcup_{k=1}^{\infty} \mathbb{F}_k$, and assume that p is a probability measure on the sets in $\mathbb{F}$, making $(\Omega, \mathbb{F}, p)$ into a probability space.

We term $\mathcal{G}(\omega)$ a *self-similar network* if for each $i \in \mathcal{T}$ the random elements

$$Z_i = (N_i;\ T_{i*1}, \ldots, T_{i*N_i}),$$

are independent and identically distributed, where

$$T_{i*j} = X_{i*j}/X_i \ (j=1,\ldots,N_i)$$

For convenience we shall always assume the nomalization

$$X_\varnothing = 1,$$

so that

$$X_i = \prod_{j=1}^{|i|} T_{i|j}$$

and in particular $X_i = T_i$ $\forall i \in \mathbb{N}$. We also assume that the decreasing condition (2.8) holds a.s., thus $0 \le T_i \le 1$ a.s. We may regard T_i, rather than X_i, as the definig random variables of the network $\mathcal{G} = \mathcal{G}(\omega)$. Thus the random capacited tree $\mathcal{T}$ is generated by the random element

$$Z \equiv Z_\varnothing = (N;\ T_1,\ldots,T_N),$$

writing $N = N_\varnothing$ as we frequently shall. Note that we do not require that the T_i to be independent each other or to have the same distribution, as occurs in some applications. Let q be the unique quantity in [0,1) satisfying

$$q = \sum_{k=0}^{\infty} p(N=k)q^k.$$

Then q is the extinction probability of the Galton-Watson process underlying the network $\mathcal{G}$, obtained by attaching an individual to the vertices of $\mathcal{T}$.

As in the preceding, it will prove convenient to write

$$X_i = T_i = 0 \text{ if } i \notin \mathcal{T}.$$

Thus X_i and T_i are defined for all $i \in T$, and (X_σ) $(\sigma \in T)$ is a non-negative *self-similar tree prosess* with respect to the σ-fields $(\mathbb{F}_k)(k \in \mathbb{N})$ in that the X_σ are $\mathbb{F}_{|\sigma|}$-measurable, $X_\sigma \geq 0$, and the random vectors

$$(T_{\sigma,1}, T_{\sigma,2}, \ldots)$$

are independent and identically distributed for each $\sigma \in T$ (Falconer 1987).

Suppose $\alpha > 0$ is such that $\mathbb{E}(\Sigma_{i=1}^{\infty} T_i^{\alpha}) \leq 1$, Then (X_σ^α) is a *tree supermartingale* in that

$$\mathbb{E}(\Sigma_{i=1}^{\infty} X_{i*i}^{\alpha} \mid \mathbb{F}_{|i|}) \leq X_i^\alpha \quad (i \in T),$$

and

$$W := \lim_{k\to\infty} \sum_{\sigma \in \mathbb{N}^k} X_\sigma^{\alpha}$$

exists a.s. with $0 \leq \mathbb{E}(W) \leq \mathbb{E}(X_\varnothing)$. The supermartingale becomes a martingale if $\mathbb{E}[\Sigma_{i=1}^{N} T_i^\alpha)=1$.

3. Limit theorems on tree processes

We shall need some limit theorems on self-similar tree processes which themselves are interesting. We suppose that $(X_\sigma)_{\sigma \in T}$ is a self-similar tree process defined as in section 2.4, which is identified to the self-similar network $\mathcal{G} = \mathcal{G}(\omega) = (\sigma, X_\sigma)_{\sigma \in \mathcal{T}}$ generated by $Z = (N; T_1, T_2, \ldots, T_N)$.

Let

$$S(x) := \Sigma_{i=1}^{N} T_i^x, \tag{3.1}$$

where $x \in [0,\infty)$ and $\Sigma_\varnothing := 0$. Thus $S(x)=0$ if $N=0$.

Lemma S. (i) $S(x)$ is a.s. decreasing and continuous on $[0,\infty)$; $S(0)=N$ and $S(x)$ is strictly decreasing if and only if $N>0$ and $\exists 1 \leq i \leq N$ such that $0<T_i<1$.

(ii) E(S(x)) is decreasing and continuous from the right; E(S(0))=E(N).

(iii) If $\mathbb{E}(S(x_0))<\infty$ for some $x_0\geq 0$, then $\mathbb{E}(S(x))$ is dicreasing and continuous on $[x_0,\infty)$. Moreover $\mathbb{E}(S(x))$ is strictly decreasing on $[x_0,\infty)$ if and only if

$$p(T_1=T_2=...=T_N=1 \mid N>0) <1. \tag{3.2}$$

(iv) If $\mathbb{E}(N)<\infty$ then $\mathbb{E}(S(x))$ is dicreasing and continuous on $[0,\infty)$. Moreover $\mathbb{E}(S(x))$ is strictly decreasing on $[0,\infty)$ if and only if (3.2) holds.

(v) The function $\psi(x):=\text{ess sup } S(x)$ is decreasing on $[0,\infty)$.

Proof. (i) is evident. (ii) holds by the monotone convergence theorem. The first conclusion in (iii) follows by the same reason. If $\mathbb{E}(S(x))$ is strictly decreasing on $[x_0,\infty)$, then (3.2) holds immediatly since otherwise $S(x)\equiv N$. Suppose now that (3.2) is satisfied and that $\mathbb{E}(S(x))= \mathbb{E}(S(y))$, where $x\geq y\geq x_0$. Since the function S(.) is decreasing we have S(x)=S(y) a.s., that is, $T_i^x=T_i^y$ $\forall 1\leq i\leq N$ a.s. Choosing $0<T_i<1$ implies x=y. This completes the proof of (iii). (iv) is a particular case of (iii). (v) holds since S(x) is decreasing. □

Remark. It will be useful to note that the condition (3.2) is equivalent to $\mathbb{E}[\Sigma_{i=1}^N T_i^{\alpha}\log\frac{1}{T_i})>0$, or to $\frac{d}{dx}(\mathbb{E}[S(x)])\Big|_{x=\alpha}<0$.

Let us write now

$$\alpha= \inf \{a\in[0,\infty): \mathbb{E}[S(a)] \leq 1\}, \quad \text{where } \inf \varnothing =+\infty. \tag{α)$_0$}$$

Thus $0\leq\alpha\leq\infty$ and, if $\alpha<\infty$, then $\mathbb{E}[S(\alpha)]\leq 1$ since $\mathbb{E}[S(x)]$ is decreasing and continuous from the right. Therefore we can write

$$\alpha= \min \{a\in[0,\infty): \mathbb{E}[S(a)] \leq 1\}, \quad \text{where } \min \varnothing =+\infty. \tag{α}$$

Lemma α. (i) $\alpha=0$ if and only if $\mathbb{E}(N)\leq 1$ if and only if the tree process terminates a.s., or again, if and only if $\tilde{\mathcal{T}}=\varnothing$ a.s.;

(ii) $\alpha<\infty$ if and only if there exists $a\geq 0$ such that $\mathbb{E}(S(a))\leq 1$;

(iii) If $\alpha<\infty$ then $\mathbb{E}(S(\alpha))\leq 1$. If addtionally E(N)>1 then (3.2) holds.

(iv) α is the least solution of the equation

$$\mathbb{E}(S(a))=1 \tag{Eα}$$

$(0\leq a<\infty)$ if there is (at least) a solution.

(v) Suppose that $1<\mathbb{E}(N)<\infty$ and $\mathbb{E}(S(y))\leq 1$ for some $y>0$, then α is the unique solution in $(0,y]$ of the equation $\mathbb{E}(S(x))= 1$.

proof. (i) and (ii) are clear. (iii) $\mathbb{E}(S(\alpha))\leq 1$ by (α). If (3.2) does not hold, then $S(x)\equiv N$, so $E(S(x)) \equiv \mathbb{E}(N)>1$ $\forall x\geq 0$. Consequently $\alpha=\infty$. Hence $\alpha<\infty$ implies (3.2). To prove (iv), write

$$\alpha_e = \inf\{a\in[0,\infty): \mathbb{E}[S(a)] = 1\}. \qquad (\alpha_e)_0$$

If $(E\alpha)$ has a solution then α_e is well defined and $\alpha_e<\infty$. If a_n is a decreasing sequence such that $a_n\to a_e$ and $\mathbb{E}[S(a_n)] = 1$, then $\mathbb{E}[S(a_e)] = 1$ since $\mathbb{E}[S(x)]$ is continuous from the right. Hence the least solution exists and we can write

$$\alpha_e = \min\{a\in[0,\infty): \mathbb{E}[S(a)] = 1\}. \qquad (\alpha_e)$$

We prove now $\alpha=\alpha_0$. Clearly $\alpha\leq\alpha_0$. Conversely, $\forall a<\alpha_0$ $\mathbb{E}[S(a)] \neq 1$ by the definition of α_0. Thus $\mathbb{E}[S(a)] > 1$ since $\mathbb{E}[S(x)]$ is decreasing. Therefore $a<\alpha$ by the definition of α. Letting $a\to\alpha_0$ gives $\alpha_0\leq\alpha$. This ends the proof of (iv).

We now prove (v). Since $1<\mathbb{E}(N)<\infty$ and $\alpha<\infty$, (3.2) holds by (iii) above. Thus $\mathbb{E}(S(x))$ is strictly decreasing. As $\mathbb{E}(N)<\infty$, $\mathbb{E}(S(x))$ is continuous on $[0,\infty)$. Noting that $\mathbb{E}(S(0))>1$ and $\mathbb{E}(S(y))\leq 1$, there exists a unique $\alpha\in(0,y]$ such that $\mathbb{E}[S(\alpha)]=1$. The proof is then completed. □

We shall suppose always that $0<\alpha<\infty$ if it is not specified further.We define

$$\beta = \inf\{b\in[0,1): S(\tfrac{\alpha}{1-b}) \leq 1 \text{ a.s.}\}, \quad \text{where } \inf\varnothing :=1. \qquad (\beta)_0$$

Thus $0\leq\beta\leq 1$. If $\beta<1$, then $S(\frac{\alpha}{1-b}) \leq 1$ a.s. $\forall b>\beta$, so $S(\frac{\alpha}{1-\beta}) \leq 1$ a.s. Hence we can write

$$\beta = \min\{b\in[0,1): S(\tfrac{\alpha}{1-b}) \leq 1 \text{ a.s.}\}, \quad \text{where } \min\varnothing :=1. \qquad (\beta)$$

Lemma β. (i) $\beta<1$ if and only if $S(a)\leq 1$ a.s. for some sufficiently large $a>0$.

(ii) $\beta=0$ if and only if $S(\alpha)\leq 1$ a.s.

If $\mathbb{E}(S(\alpha))=1$, then

$\beta=0$ if and only if $S(\alpha)=1$ a.s.

(iii) Suppose that $p(S(\alpha)>1)>0$. If the equation

$$\text{ess sup } S(\frac{\alpha}{1-b}) = 1 \tag{Eβ}$$

($0 \le b < 1$) has at least a solution, then β is the least one and certainly $\beta < 1$.

Proof. (i) Clearly by the definition of β. (ii) The first conclusion comes directly from the expression (β). The second conclusion holds since, if $\mathbb{E}(S(\alpha))=1$, then $S(\alpha) \le 1$ a.s. if and only if $S(\alpha)=1$ a.s. (iii) Write

$$\beta_e = \inf\{b \in [0,1): \text{ess sup } S(\frac{\alpha}{1-b}) = 1\}. \tag{β_e)$_0$}$$

If (Eβ) has a solution, $\beta_e < 1$ is well defined. If b_n is a decreasing sequenece such that $1 > b_n \to \beta_e$ ($n \to \infty$) and ess sup $S(\frac{\alpha}{1-b_n}) = 1$, then $S(\frac{\alpha}{1-b_n}) \le 1$ a.s. Letting $n \to \infty$ gives $S(\frac{\alpha}{1-\beta_e}) \le 1$ a.s. On the other hand, since ess sup $S(x)$ is a decreasing function of x, we have ess sup $S(\frac{\alpha}{1-\beta_e}) \ge$ ess sup $S(\frac{\alpha}{1-b_n}) = 1$. Thus ess sup $S(\frac{\alpha}{1-\beta_e}) = 1$. So we can write

$$\beta_e = \min\{b \in [0,1): \text{ess sup } S(\frac{\alpha}{1-b}) = 1\}. \tag{β_e}$$

We shall prove that $\beta = \beta_e$. Clearly $\beta \le \beta_e$. Conversely, for each $b < \beta_e$, ess sup $S(\frac{\alpha}{1-b}) \ne 1$ by the definition of β_e. Thus ess sup $S(\frac{\alpha}{1-b}) > 1$ since ess sup $S(x)$ is decreasing. Hence $b < \beta$ by the definition of β. Letting $b \to \beta_e$ gives $\beta_e \le \beta$. This completes the proof of (iii). □

Put

$$Z_k = \sum_{\sigma \in \mathbb{N}^k} X_\sigma^\alpha.$$

Since $\mathbb{E}[S(\alpha)] \le 1$, $(Z_k, \mathbb{F}_k)$ ($\mathbb{F}_k$ is the σ-algebra generated by all the T_i such that $|i| \le k$) is a non-negative supermartingale and

$$W := \lim_{k \to \infty} Z_k$$

exists a.s. with $0 \le W < +\infty$ and $\mathbb{E}(W) \le 1$ by the martingale convegence theorem. It will prove very useful to note that

$$W = \sum_{i=1}^{N} T_i^\alpha W_i \, , \tag{3.3}$$

where

$$W_i = \lim \sum_{|\tau| = k} \prod_{n=1}^{|\tau|} T^\alpha_{i*(\tau|n)}$$

($1 \le i \le N$) are independent of each other and of $(N; T_1,\dots,T_N)$, having the same distribution as W. If $\mathbb{E}(S(\alpha))<1$, then $W=0$ a.s. For the remainder of this section, We suppose always that

$$0<\alpha<\infty \text{ and } \mathbb{E}(S(\alpha))=1.$$

The conclusion (i) of the following lemma was established in Falconer (1987) without proof and condition. But it seems to me that a moment condition such as (3.4) below is necessary, although it may be probably weakened. The 'if' part of the conclusion (ii) of the Lemma was proved by Mauldin and Williams (1986, Th.2.1), but we prefer here to give a simpler proof since their method is very complicated.

Lemma 3.1. Suppose that

$$\mathbb{E}[S(\alpha)^2] < +\infty. \tag{3.4}$$

(i) With probability q we have $X_\sigma = 0$ for all $\sigma \in T$ with $|\sigma| \ge k$ for some $k \in \mathbb{N}$, and with probability 1-q we have $W > 0$. Moreover, $\mathbb{E}(W)=1$ and $\mathbb{E}(W^2)<\infty$.

(ii) For each *integer* $k>1$, $\mathbb{E}(W^k)<\infty$ if and only if $\mathbb{E}(S^k(\alpha))<\infty$. Moreover, for each *real* $p>1$, $\mathbb{E}(W^p)<\infty$ implies $\mathbb{E}(S^p(\alpha))<\infty$.

Proof. (i) We shall see that the martingale $(Z_k, \mathbb{F}_k)$ is L_2-bounded when $\mathbb{E}(S^2) < +\infty$. For simplicity, we write here S for $S(\alpha)$. We have

$$\mathbb{E}(Z^2_{k+1} \mid \mathbb{F}_k) = \mathbb{E}\,[(\sum_{\sigma\in\mathbb{N}^{k+1}} X_\sigma^{\alpha})^2 \mid \mathbb{F}_k] = \mathbb{E}\,[(\sum_{\sigma\in\mathbb{N}^k} X_\sigma^{\alpha} \sum_{i=1}^{\infty} T^{\alpha}_{\sigma * i})^2 \mid \mathbb{F}_k]$$

$$= \sum_{\sigma,\tau\in\mathbb{N}^k} X_\sigma^{\alpha} X_\tau^{\alpha}\, \mathbb{E}\,[(\sum_{i=1}^{\infty} T^{\alpha}_{\sigma*i} \sum_{j=1}^{\infty} T^{\alpha}_{\tau*j}) \mid \mathbb{F}_k]$$

$$= \sum_{\sigma\neq\tau\in\mathbb{N}^k} X_\sigma^{\alpha} X_\tau^{\alpha}\, \mathbb{E}^2(S) + \sum_{\sigma\in\mathbb{N}^k} X_\sigma^{2\alpha}\, \mathbb{E}(S^2)$$

$$= (\sum_{\sigma\in\mathbb{N}^k} X_\sigma^{\alpha})^2 + \sum_{\sigma\in\mathbb{N}^k} X_\sigma^{2\alpha}(\mathbb{E}(S^2)-1)$$

$$= Z_k^2 + (\mathbb{E}(S^2)-1) \sum_{\sigma\in\mathbb{N}^k} X_\sigma^{2\alpha},$$

and consequently

$$\sup_{k\geq 0} \mathbb{E}(Z_k^2) \leq (\mathbb{E}(S^2)-1) \sum_{k=1}^{\infty} \mathbb{E}\Big[\sum_{\sigma\in\mathbb{N}^k} X_\sigma^{2\alpha}\Big] + 1$$

$$= 1 + (\mathbb{E}(S^2)-1)/\ [1- \mathbb{E}(\sum_{i=1}^{\infty} T_i^{2\alpha})] < +\infty$$

since $\mathbb{E}(\sum_{i=1}^{\infty} T_i^{2\alpha}) < 1$ (we recall that $\mathbb{E}[S(x)]$ is strictly decreasing on $[\alpha,\infty)$). Thus the martingale is L^2-bounded and $\mathbb{E}(W) = \lim_{k\to\infty} \mathbb{E}(Z_k) = 1$ by the martingale convergence theorem. It follows that $p(W=0) <1$.

On the other hand, by the recursive relation (3.3) and the fact that $T_i > 0$ a.s., we obtain

$$p(W=0) = \sum_{n=0}^{\infty} p(W=0 \mid N=n)\ p(N=n)$$
$$= p(N=0) + \sum_{n=1}^{\infty} p(\ W_i=0 \text{ for } i=1,\ldots,n)\ p(N=n)$$
$$= \sum_{n=0}^{\infty} p(W=0)^n p(N=n).$$

Therefore $p(W=0) = q$.

Since with probability q the adduced branching process terminates,i.e. the cardinalities $\#(\mathcal{T}_k)$ vanishes for k sufficiently large, and $\sum_{\sigma\in\mathbb{N}^k} X_\sigma \leq \#(\mathcal{T}_k)$, thus $X_\sigma=0$ if $|\sigma|$ is sufficiently large. This ends the proof of (i).

(ii) The proof above shows that $(Z_k,\mathbb{F}_k)$ is an L_2-bounded martingale if $\mathbb{E}(S^2(\alpha))<\infty$. Thus $Z_k=\mathbb{E}(W \mid \mathbb{F}_k)$. Jensen's inequality gives then,for all reals p>1,

$$\mathbb{E}[S^p(\alpha)] = \mathbb{E}[Z_1^p] = \mathbb{E}[\mathbb{E}^p(W \mid \mathbb{F}_1)] \leq \mathbb{E}[\mathbb{E}(W^p \mid \mathbb{F}_1)] = \mathbb{E}(W^p)$$

Thus $\mathbb{E}(W^p)<\infty$ implies $\mathbb{E}(S^p(\alpha))<\infty$. It then suffices to prove that

$$\mathbb{E}[S^k(\alpha)]<\infty \text{ implies } \mathbb{E}(W^k)<\infty$$

for all integers k>1. In fact, by the recursive relation (3.3), we have

$$W^k = \sum_{i=1}^{N} T_i^{k\alpha} W_i^k + \sum_{\substack{k_1+k_2+\ldots+k_N=k \\ 0\leq k_i\leq k-1}} \gamma_{k_1,\ldots,k_N} \prod_{i=1}^{N} (T_i^{k_i\alpha} W_i^{k_i}),$$

where $\gamma_{k_1,\ldots,k_N} = \frac{k!}{k_1!\ldots k_n!}$. Thus

$$\mathbb{E}[W^k \mid \mathcal{F}_1] = \sum_{i=1}^{N} T_i^{k\alpha}\mathbb{E}[W^k] + \sum_{idem} \gamma_{k_1,\dots,k_N} \prod_{i=1}^{N} T_i^{k_i\alpha} \prod_{i=1}^{N} \mathbb{E}[W^{k_i}].$$

Since $\mathbb{E}(W^{k_i}) \le [\mathbb{E}(W^{k-1})]^{k_i/(k-1)}$ for all $0 \le k_i \le k-1$, we have

$$\mathbb{E}[W^k \mid \mathcal{F}_1] \le \sum_{i=1}^{N} T_i^{k\alpha}\mathbb{E}[W^k] + [\sum_{idem} \gamma_{k_1,\dots,k_N} \prod_{i=1}^{N} T_i^{k_i\alpha}]\,(\mathbb{E}[W^{k-1}])^{k/(k-1)}.$$

It follows that

$$\mathbb{E}[W^k] \le \mathbb{E}(\sum_{i=1}^{N} T_i^{k\alpha})\mathbb{E}[W^k] + \mathbb{E}(\sum_{idem} \gamma_{k_1,\dots,k_N} \prod_{i=1}^{N} T_i^{k_i\alpha})\,(\mathbb{E}[W^{k-1}])^{k/(k-1)}.$$

$$= \mathbb{E}(\sum_{i=1}^{N} T_i^{k\alpha})\mathbb{E}[W^k] + \mathbb{E}\left[(\sum_{i=1}^{N} T_i^{\alpha})^k - \sum_{i=1}^{N} T_i^{k\alpha}\right](\mathbb{E}[W^{k-1}])^{k/(k-1)}.$$

Noting that $(\mathbb{E}[W^{k-1}])^{k/(k-1)} \le \mathbb{E}[W^k]$, we obtain eventually

$$(\mathbb{E}[W^k])^{1/k} \le (\mathbb{E}[S^k(\alpha)])^{1/k}(\mathbb{E}[W^{k-1}])^{1/(k-1)}.$$

In particular, $\mathbb{E}[S^k(\alpha)]<\infty$ and $\mathbb{E}[W^{k-1}]<\infty$ imply $\mathbb{E}[W^k]<\infty$. Since $\mathbb{E}(W)<\infty$, by induction on k we know that $\mathbb{E}[S^k(\alpha)]<\infty$ implies $\mathbb{E}[W^k]<\infty$. The proof is then finished. □

As a direct consequence of the fact $\mathbb{E}[S^p(\alpha)] \le \mathbb{E}[W^p]$ (all real p>1), we have

Corollary 3.1. Let $b\in(0,\infty)$. Denote by $r(W^b)$ the radius of convergence of the moment generating function $\mathbb{E}(e^{tW^b})$ of W^b, and $r(S^b)$ that of $S^b(\alpha)$. Then

$$r(W^b) \le r(S^b).$$

In particular, $r(S^b)<\infty$ implies $r(W^b)<\infty$.

This result will prove useful to ensure $r(W^b)<\infty$, in the case where $\|N\|_\infty=\infty$.

The following result generalizes a result of Graf et al.(1988, Theorem 2.5,p.14).

Theorem 3.2. Let $b\in(0,1)$ and denote by $r(W^b)$ the radius of convergence of the moment generating function $\mathbb{E}(e^{tW^b})$ of W^b.

(a) If $\|N\|_\infty<\infty$, then

$$r(W^{1/b})>0 \tag{3.5}$$

if and only if

$$S(\tfrac{\alpha}{1-b}) \le 1 \text{ a.s.} \tag{3.6}$$

(b) If $\|N\|_\infty=\infty$, then (3.5) implies (3.6) or, equivalently, $p(S(\frac{\alpha}{1-b})>1)>0$ implies $r(W^{1/b})=0$.

Proof. (i) We first prove that (3.6) implies (3.5), if $\|N\|_\infty<\infty$. We shall denote by $\mathbb{E}(X|\ N = n)$ the expectation of X conditioned on N = n. By (3.3) we have

$$\mathbb{E}[W^k|\ N=n] = \mathbb{E}(\sum_{i=1}^n T_i^{k\alpha}\ |\ N=n)\mathbb{E}[W^k] + \sum_{\substack{k_1+k_2+\dots+k_n=k\\ 0\le k_i\le k-1}} \frac{k!}{k_1!\dots k_n!}\,\mathbb{E}\Big[\prod_{i=1}^n T_i^{k_i\alpha}\ |\ N=n\Big]\prod_{i=1}^n \mathbb{E}[W^{k_i}] \tag{3.7}$$

Since the function $(y_1,y_2,\dots,y_n)\to\prod_{i=1}^n y_i^{(1-b)k_i}$ with $\sum_{i=1}^n y_i \le 1$ and $y_i\ge 0$ $(1\le i\le n)$ attains its maximum at $(k_1/k, k_2/k, \dots, k_n/k)$, where $k = k_1+ k_2+\dots+ k_n$, we obtain

$$\prod_{i=1}^n T_i^{k_i\alpha} = \prod_{i=1}^n [T_i^{\alpha/(1-b)}]^{(1-b)k_i} \le \prod_{i=1}^n (\tfrac{k_i}{k})^{(1-b)k_i}. \tag{3.8}$$

Thus (3.7) gives

$$\mathbb{E}[W^k|\ N=n] \le \mathbb{E}(\sum_{i=1}^n T_i^{k\alpha}\ |\ N=n)\mathbb{E}[W^k] + \sum_{\substack{k_1+k_2+\dots+k_n=k\\ 0\le k_i\le k-1}} \frac{k!}{k_1!\dots k_n!}\,\mathbb{E}\Big[\prod_{i=1}^n (\tfrac{k_i}{k})^{(1-b)k_i}|\ N=n\Big]\prod_{i=1}^n \mathbb{E}[W^{k_i}]$$

Taking expectation on N implies

$$\mathbb{E}[W^k]\,[1- \mathbb{E}(\sum_{i=1}^N T_i^{k\alpha})] \le \mathbb{E}\Big\{\mathbb{E}\Big[\{\sum_{\substack{k_1+k_2+\dots+k_n=k\\ 0\le k_i\le k-1}} \frac{k!}{k_1!\dots k_n!}\prod_{i=1}^n (\tfrac{k_i}{k})^{(1-b)k_i}\mathbb{E}[W^{k_i}]\}\,|\ N=n\Big]\Big\},$$

namely

$$\frac{\mathbb{E}[W^k]}{k!}\,k^{(1-b)k} \le \frac{1}{1-c_k}\mathbb{E}\Big\{\mathbb{E}\Big[\{\sum_{\substack{k_1+k_2+\dots+k_n=k\\ 0\le k_i\le k-1}} \prod_{i=1}^n \frac{\mathbb{E}[W^{k_i}]}{k_i!}\,k_i^{(1-b)k_i}\}\,|\ N=n\Big]\Big\}, \tag{3.9}$$

where $c_k = \mathbb{E}[\sum_{i=1}^{N} T_i^{k\alpha}]$. Writing

$$t_k = \frac{\mathbb{E}[W^k]}{k!} k^{(1-b)k}$$

From (3.9) we have

$$t_k \le \frac{1}{1-c_k} \mathbb{E}\left[\sum_{\substack{k_1+k_2+\ldots+k_N=k \\ 0\le k_i \le k-1}} \prod_{i=1}^{N} t_{k_i}\right] (\forall k\ge 2), \tag{3.9'}$$

Since $\sum_{idem} \prod_{i=1}^{N} t_{k_i}$ is an increasing function of N, we have, for $n=\|N\|_\infty$,

$$t_k \le c \sum_{\substack{k_1+k_2+\ldots+k_n=k \\ 0\le k_i \le k-1}} \prod_{i=1}^{n} t_{k_i} \quad (\forall k\ge 2), \tag{3.10}$$

where $c = \sup_{k\ge 2} \frac{1}{1-c_k} > 0$. As a consequence of (3.10) we have $\limsup_{k\to\infty} t_k^{1/k} < \infty$

(see Graf, Mauldin and Williams 1988, Lemma 2.6). That is

$$\infty > \limsup_{k\to\infty} \left[\frac{\mathbb{E}[W^k]}{k!} k^{(1-b)k}\right]^{1/k} = \limsup_{k\to\infty} \frac{\mathbb{E}[W^k]^{1/k}}{k/e} k^{(1-b)} = \limsup_{k\to\infty} \frac{e\mathbb{E}[W^k]^{1/k}}{k^{1/b}}.$$

Now for each k>0, choose $K\in\mathbb{N}$ such that $kb\le K<kb+1$, thus

$$\limsup_{k\to\infty} \left(\frac{\mathbb{E}[W^{k/b}]}{k!}\right)^{1/k} \le \limsup_{k\to\infty} \frac{\mathbb{E}[W^K]^{b/K}}{k/e} \le \limsup_{K\to\infty} \frac{\mathbb{E}[W^K]^{b/K}}{(K-1)/(be)}$$

$$= \limsup_{k\to\infty} eb \left(\frac{\mathbb{E}[W^K]^{1/K}}{K^{1/b}}\right)^b < \infty.$$

Namely $r(W^{1/b}) > 0$.

(ii) We now prove that (3.5) implies (3.6) or namely, if

$$p(S(\tfrac{\alpha}{1-b}) > 1) > 0, \tag{3.11}$$

then $r(W^{1/b})=0$. We remark that the latter holds if

$$\liminf_{k\to\infty} s_k^{1/k} = +\infty, \tag{3.12}$$

where

$$s_k = \frac{\mathbb{E}(W^k)}{(k!)^b} \quad (\forall k\ge 0). \tag{3.13}$$

We shall prove that (3.11) implies (3.12), without the assumption $\|N\|_\infty<\infty$. From (3.7) we have

$$\mathbb{E}[W^k][1-\mathbb{E}(\sum_{i=1}^{N} T_i^{k\alpha})]$$

$$=\mathbb{E}\left\{\sum_{\substack{k_1+k_2+\dots+k_n=k \\ 0\le k_i\le k-1}} \frac{k!}{k_1!\dots k_n!}\,\mathbb{E}\left[\prod_{i=1}^{n} T_i^{k_i\alpha}\;\middle|\; N=n\right]\prod_{i=1}^{N}\mathbb{E}[W^{k_i}]\right\}. \tag{3.14}$$

Suppose that $s_j \ge r^j$ for some $r>0$ and all $j<k$, we shall see that $s_k \ge r^k$ if k is sufficiently large. In fact, from (3.14),

$$\mathbb{E}[W^k](1-c_k)\ge \mathbb{E}\left\{\sum_{\substack{k_1+k_2+\dots+k_N=k \\ 0\le k_i\le k-1}} \frac{k!}{k_1!\dots k_N!}\,\mathbb{E}\left[\prod_{i=1}^{n} T_i^{k_i\alpha}\;\middle|\; N=n\right] r^k(k_1!\dots k_N!)^b\right\}$$

$$=\mathbb{E}\left\{\mathbb{E}\left[\sum_{\substack{k_1+k_2+\dots+k_n=k \\ 0\le k_i\le k-1}} \frac{k!}{(k_1!\dots k_n!)^{1-b}}\prod_{i=1}^{n} T_i^{k_i\alpha}\;\middle|\; N=n\right]\right\}r^k$$

$$=\mathbb{E}\left\{\mathbb{E}\left[\sum_{\mathrm{idem}} \left(\frac{k!}{k_1!\dots k_n!}\prod_{i=1}^{n} T_i^{k_i\alpha/(1-b)}\right)^{1-b}\;\middle|\; N=n\right]\right\}r^k k!^b$$

$$\ge \mathbb{E}\left\{\mathbb{E}\left[\left(\sum_{\mathrm{idem}} \frac{k!}{k_1!\dots k_n!}\prod_{i=1}^{n} T_i^{k_i\alpha/(1-b)}\right)^{1-b}\;\middle|\; N=n\right]\right\}r^k k!^b,$$

where the last step holds since $\sum x_i^{1-b} \ge (\sum x_i)^{1-b}$. Now we remark that

$$\sum_{\mathrm{idem}} \frac{k!}{k_1!\dots k_n!}\prod_{i=1}^{n} T_i^{k_i\alpha/(1-b)} = (\sum_{i=1}^{n} T_i^{\alpha/(1-b)})^k - \sum_{i=1}^{n} T_i^{k\alpha/(1-b)}.$$

As $(x-y)^{1-b}\ge x^{1-b}-y^{1-b}$ if $x\ge y\ge 0$, the above equality gives

$$\left(\sum_{\mathrm{idem}} \frac{k!}{k_1!\dots k_n!}\prod_{i=1}^{n} T_i^{k_i\alpha/(1-b)}\right)^{1-b}$$

$$\ge (\sum_{i=1}^{n} T_i^{\alpha/(1-b)})^{k(1-b)} - (\sum_{i=1}^{n} T_i^{k\alpha/(1-b)})^{1-b}.$$

Hence $\mathbb{E}[W^k](1-c_k) \ge \mathbb{E}\left[\left(\sum_{i=1}^{N} T_i^{\alpha/(1-b)}\right)^{k(1-b)} - \left(\sum_{i=1}^{N} T_i^{k\alpha/(1-b)}\right)^{1-b}\right] r^k k!^b,$

or

$$\frac{\mathbb{E}[W^k]}{k!^b}\ge \frac{\mathbb{E}\left([S(\frac{\alpha}{1-b})]^{k(1-b)}\right)-c_k}{1-c_k}\, r^k. \tag{3.15}$$

If (3.6) does not hold, then $p(S(\frac{\alpha}{1-b})>1)>0$ and we can choose $a>1$ such that

$$c:=p(\,S(\tfrac{\alpha}{1-b})>a\,)>0.$$

It follows that

$$\mathbb{E}\left([S(\tfrac{\alpha}{1-b})]^{k(1-b)}\right) \geq ca^{k(1-b)}.$$

Choose k_0 sufficiently large such that $ca^{k(1-b)}>1$ if $k\geq k_0$. Put $r_0=\min_{0\leq i<k_0} s_i^{1/i}$ (>0), then (3.15) implies $s_k^{1/k}\geq r_0$ for all $k\geq k_0$. Thus $\liminf_{k\to\infty} s_k^{1/k}>0$. Namely $\liminf_{k\to\infty}\left(\frac{\mathbb{E}[W^k]}{k!^b}\right)^{1/k}>0$. Since we can choose b'>b such that $p(S(\frac{\alpha}{1-b'})>1)>0$ also, we have in fact $\liminf_{k\to\infty}\left(\frac{\mathbb{E}[W^k]}{k!^b}\right)^{1/k}=\infty$, applying the result for b'. This ends meantime the proof of (a) and (b). □

Corollary 3.2. $p(S(\frac{\alpha}{1-b})>1)>0$ implies

$$\lim_{k\to\infty}\left(\frac{\mathbb{E}[W^k]}{k!^b}\right)^{1/k}=\infty.$$

Proof. This is shown in the proof of Theorem 3.2. □

Theorem 3.3. (i) If $\|N\|_\infty<\infty$ and $0<\beta<1$, then $r(W^{1/b})>0$ if and only if $b\geq\beta$; (ii) If $\|N\|_\infty=\infty$ or $\beta=1$, then $r(W^{1/b})=0\ \forall 0<b<\beta$.

Proof. (a) If $\|N\|_\infty<\infty$ and $1>b\geq\beta$, then $S(\frac{\alpha}{1-b})\leq 1$ a.s. Theorem 3.2. shows then $r(W^{1/b})>0$ for all $b\in[\beta,1)$ and then for all $b\geq\beta$. (b) If $0<b<\beta$, then $p(S(\frac{\alpha}{1-b})>1)>0$. Theorem 3.2 applies again, showing that $r(W^{1/b})=0$. This ends at the meanwhile the proof of (i) and (ii). □

The following Theorem generalizes a result of Kahane and Peyrière (1976,Théorème 3). In Kahane and Peyrière's case, $N=c\geq 2$ is a constant, and T_i $(1\leq i\leq c)$ are independent and identically distributed.

Theorem 3.4. (i) If $\|N\|_\infty<\infty$ and $\beta<1$, then

$$\lim_{k\to+\infty}\frac{\mathrm{Log}\ \mathbb{E}(W^k)}{k\ \mathrm{Log}\ k}=\beta;$$

(ii) If $\|N\|_\infty=\infty$ or $\beta=1$, then

$$\liminf_{k\to+\infty}\frac{\mathrm{Log}\ \mathbb{E}(W^k)}{k\ \mathrm{Log}\ k}\geq\beta.$$

Proof. If $\|N\|_\infty<\infty$ and $1>b>\beta$, then $S(\frac{\alpha}{1-b})\leq 1$ a.s. Thus $r(W^{1/b})>0$ by Theorem (3.2). That is $\limsup_{k\to\infty}\left(\frac{\mathbb{E}[W^{k/b}]}{k!}\right)^{1/k}<\infty$. Stirling's Theorem gives $\lim_{k\to\infty}$

$\sup \frac{(\mathbb{E}[W^{k/b}])^{1/k}}{k/e} < \infty$. Consequently there exists a M>0 such that $\frac{(\mathbb{E}[W^{k/b}])^{b/k}}{(k/e)^b} \leq M$ for all k>0. Noting that $[\mathbb{E}(W^x)]^{1/x}$ is an increasing function of x and k/b >k, we obtain $\frac{(\mathbb{E}[W^k])^k}{(k/e)^b} \leq M$ ($\forall$k>1), which gives $\limsup_{k\to+\infty} \frac{\text{Log } \mathbb{E}(W^k)}{k \text{ Log } k} \leq b$. Letting b→β, we see that

$$\limsup_{k\to+\infty} \frac{\text{Log } \mathbb{E}(W^k)}{k \text{ Log } k} \leq \beta.$$

If β=0, The proof is then finished. Suppose that β>0. Then $\forall 0<b<\beta$ we have

$$p(S(\frac{\alpha}{1-b}) > 1) > 0.$$

Thus

$$\lim_{k\to\infty} \left(\frac{\mathbb{E}[W^k]}{k!^b}\right)^{1/k} = \infty$$

by Corollary 3.2. Using stirling's Theorem again,it gives $\liminf_{k\to+\infty} \frac{\text{Log } \mathbb{E}(W^k)}{k \text{ Log } k} \geq b$. Letting b→β gives

$$\liminf_{k\to+\infty} \frac{\text{Log } \mathbb{E}(W^k)}{k \text{ Log } k} \geq \beta,$$

which ends in the meantime the proof of (i) and (ii). □

To ensure $r(W^{1/\beta}) < \infty$, in the case where N is a constant, Graf et al.(1988) have given a "corner" condition and some conditions associated (see their Theorem 2.11 and corollaries 2.12-14, pp.30-37). But it seems to me that their conditions are not ideal. Here, in the genaral setting, we give a simple result which covers almost all the examples of Graf et al.(1988) and of Falconer (1986).

Theorem 3.5. Suppose that β>0. If there exists n>1 such that

$$\prod_{i=1}^{\infty} \frac{\mathbb{E}[S(\alpha)^{n^i} \mid N=n]^{1/n^i}}{n^\beta} > 0, \tag{3.16}$$

or, equivalently,

$$\sum_{i=1}^{\infty} \left(1 - \frac{\mathbb{E}[S(\alpha)^{n^i} \mid N=n]^{1/n^i}}{n^\beta}\right) < \infty, \tag{3.16'}$$

then $r(W^{1/\beta}) < \infty$.

Proof. Again from the recursive relation (3.3), we have

$$\mathbb{E}[W^k \mid N=n] = \mathbb{E}(\sum_{i=1}^n T_i^{k\alpha} \mid N=n)\mathbb{E}[W^k]$$
$$+ \sum_{\substack{k_1+k_2+\ldots+k_n=k\\ 0\le k_i\le k-1}} \frac{k!}{k_1!\ldots k_n!}\,\mathbb{E}\Big[\prod_{i=1}^n T_i^{k_i\alpha} \mid N=n\Big]\prod_{i=1}^n \mathbb{E}[W^{k_i}]$$
$$\ge \mathbb{E}(\sum_{i=1}^n T_i^{k\alpha})\,\mathbb{E}[W^k]$$
$$+ \inf \prod_{i=1}^n \mathbb{E}[W^{k_i}] \sum_{\substack{k_1+k_2+\ldots+k_n=k\\ 0\le k_i\le k-1}} \frac{k!}{k_1!\ldots k_n!}\,\mathbb{E}\Big[\prod_{i=1}^n T_i^{k_i\alpha} \mid N=n\Big],$$

where the inferior is taken over all the $(k_1,k_2,\ldots,k_n)$ such that $k_1+k_2+\ldots+k_n=k$ and that $0 \le k_i \le k-1$. If $k = n\tilde{k}$, it is $(\mathbb{E}[W^{\tilde{k}}])^n$. Hence

$$\mathbb{E}[W^{n\tilde{k}} \mid N=n] \ge \mathbb{E}(\sum_{i=1}^n T_i^{k\alpha} \mid N=n)\mathbb{E}[W^{n\tilde{k}}]$$
$$+(\mathbb{E}[W^{\tilde{k}}])^n\Big\{\mathbb{E}\Big[(\sum_{i=1}^n T_i^{\alpha})^k \mid N=n\Big]-\mathbb{E}\Big[\sum_{i=1}^n T_i^{k\alpha} \mid N=n\Big]\Big\}$$
$$\ge(\mathbb{E}[W^{\tilde{k}}])^n\mathbb{E}\Big[(\sum_{i=1}^n T_i^{\alpha})^{n\tilde{k}} \mid N=n\Big].$$

Consequently

$$\mathbb{E}[W^{n\tilde{k}}] \ge p_n(\mathbb{E}[W^{\tilde{k}}])^n\mathbb{E}[(S(\alpha))^{n\tilde{k}} \mid N=n],$$

where $p_n=p(N=n)>0$. Therefore

$$\frac{1}{n\tilde{k}} \operatorname{Log} \mathbb{E}[W^{n\tilde{k}}] \ge \frac{1}{n\tilde{k}} \operatorname{Log} \{p_n\mathbb{E}[S(\alpha)^{n\tilde{k}} \mid N=n]\} + \frac{1}{\tilde{k}} \operatorname{Log} \mathbb{E}[W^{\tilde{k}}]$$
$$= \frac{1}{n\tilde{k}} \operatorname{Log} p_n + \frac{1}{n\tilde{k}} \operatorname{Log} \mathbb{E}[S(\alpha)^{n\tilde{k}} \mid N=n]\} + \frac{1}{\tilde{k}} \operatorname{Log} \mathbb{E}[W^{\tilde{k}}].$$

Choosing $\tilde{k} = n^{r-1}$ $(0<r\in\mathbb{N})$ and using this inequality repeatedly, we see that

$$\frac{1}{n^r} \operatorname{Log} \mathbb{E}[W^{n^r}] \ge (\operatorname{Log} p_n)\sum_{j=1}^r \frac{1}{n^j} + \sum_{j=1}^r \frac{1}{n^j} \operatorname{Log} \mathbb{E}[S(\alpha)^{n^j} \mid N=n]$$

For $k \in \mathbb{N}$ sufficiently large, choose $r \in \mathbb{N}$ such that

$n^r \le k/\beta < n^{r+1}$. Using Stirling's formular gives then

$$\limsup_{k\to\infty} \left(\frac{\mathbb{E}(W^{k/\beta})}{k!}\right)^{1/k} \ge \limsup_{k\to\infty} \frac{\mathbb{E}[W^{n^r}])^{1/(n^r\beta)}}{k/e}$$

$$\geq \limsup_{r\to\infty} \frac{e^{C(n)/\beta}}{n^{r+1}/e} \exp\{ \sum_{j=1}^{r} \frac{1}{n^j \beta} \operatorname{Log} \mathbb{E}[S(\alpha)^{n^j} \mid N=n]\},$$

where $C(n)= (\operatorname{Log} p_n) \sum_{j=1}^{\infty} \frac{1}{n^j} >-\infty$. Thus

$$\limsup_{k\to\infty} \left(\frac{\mathbb{E}(W^{k/\beta})}{k!}\right)^{1/k} \geq \limsup_{r\to\infty} \frac{e^{C(n)/\beta}}{n^{r+1}/e} \prod_{j=1}^{r} \mathbb{E}[S(\alpha)^{n^j} \mid N=n]\}^{1/(n^j\beta)}$$

$$= \limsup_{r\to\infty} \frac{e^{C(n)/\beta}}{n/e} \left(\prod_{j=1}^{r} \frac{\mathbb{E}[S(\alpha)^{n^j} \mid N=n]\}^{1/n^j}}{n^{\beta}} \right)^{1/\beta}.$$

Thus (3.16) implies that $r^{1/\beta}<\infty$. Since the equivalence of (3.16) and (3.16)' is evident, the proof is completed. □

Let us now investigate the condition (3.16). Since $\lim_{k\to\infty} \{\mathbb{E}[S(\alpha)^k \mid N=n]\}^{1/k} =\|S(\alpha)1_{N=n}\|_{\infty}$, where $\|X\|_{\infty}:=$ ess sup X, a necessary condition for (3.16) to hold is $\beta=\beta_n$, where

$$\beta_n = \log\|S(\alpha)1_{N=n}\|_{\infty}/ \log n \quad (n>1). \tag{β_n}$$

Proposition 3.6. A necessary and sufficient condition for (3.16) to hold is that there exists n>1 such that $\beta=\beta_n$ and

$$\sum_{i=1}^{\infty} \left(1-\frac{\mathbb{E}[S(\alpha)^{n^i} \mid N=n]^{1/n^i}}{\|S(\alpha)1_{N=n}\|_{\infty}}\right) < \infty. \tag{3.16"}$$

The proof is simple, thus omitted. □

Remark 3.6. (3.16)" holds usually. It holds for example if

$$p(S(\alpha)=\|S(\alpha)1_{N=n}\|_{\infty};\ N=n)>0,$$

or more particularilly, if conditioned on N=n, $T_i(1\leq i\leq n)$ take only finitely many values.

In fact, if $c:=p(S(\alpha)=\|S(\alpha)1_{N=n}\|_{\infty};\ N=n)>0$, then

$$\mathbb{E}[S^k(\alpha) \mid N=n] \geq c\|S(\alpha)1_{N=n}\|_{\infty}^{k}, \quad \frac{\mathbb{E}[S(\alpha)^{n^i} \mid N=n]^{1/n^i}}{\|S(\alpha)1_{N=n}\|_{\infty}} \geq c^{1/n^i},$$

and consequently $-\log \prod_{i=1}^{\infty} \frac{\mathbb{E}[S(\alpha)^{n^i} \mid N=n]^{1/n^i}}{\|S(\alpha)1_{N=n}\|_{\infty}} <\infty$, which implies (3.16)".

The result holds also in most cases where $S(\alpha)$ is of continuous distribution, as we shall see later.

Practical examples show that if $\|N\|_\infty<\infty$, we have often $\beta=\beta_n$, where $n=\|N\|_\infty$. This is so for almost all the examples of Graf, Mauldin and Williams (1988) and Falconer (1986). In general, we have

Proposition 3.7. $$\beta \geq \underline{\beta}:= \sup_{n\geq 0} \beta_n,$$

where $\beta_0=\beta_1=0$ and $\beta_n = \log\|S(\alpha)1_{N=n}\|_\infty / \log n \ (n\geq 2)$. (β_n)

Moreover $\underline{\beta}=$ ess sup $[\log S(\alpha)/\log N]$, where $\log S(\alpha)/\log N :=0$ if $N=0$ or 1.

Lemma 3.8. If $\sum_{i=1}^n t_i^{\alpha/(1-b)}\leq 1$, where $n\in\mathbb{N}$, $0\leq\alpha<\infty$, $0\leq b<1$ and $t_i\geq 0$ $\forall i$, then $b \geq \log(\sum_{i=1}^n t_i^\alpha) /\log n$, where $\log(\sum_{i=1}^n t_i^\alpha) /\log n :=0$ if $n=0$ or 1.

Proof. We suppose that $n>1$. Write $c=\sum_{i=1}^n t_i^\alpha$ and consider the function

$$f(x_1,\dots,x_n) =\sum_{i=1}^n x_i^{1/(1-b)},$$

where $x_1,\dots,x_n\geq 0$ and $\sum_{i=1}^n x_i= c$. The minimum of f (with the constraint) is attained at $x_1=x_2=\dots=x_n=$ c/n. Thus $1\geq f(t_1^\alpha,\dots,t_n^\alpha)\geq$ $f(c/n,\dots,c/n)$ $=n(c/n)^{1/(1-b)}$, giving the result desired. .□

Remark. The proof above shows that if $\sum_{i=1}^n t_i^{\alpha/(1-b)}=1$, then $b \geq \log(\sum_{i=1}^n t_i^\alpha)/\log n$ and the equality holds if and only if $t_1=\dots=t_n$ $(n>1)$.

Proof of Proposition 3.7. Since $\sum_{i=1}^N T_i^{\alpha/(1-\beta)}\leq 1$ a.s. by the definition of β, Lemma 3.8 gives immediately $\beta\geq$ ess sup $[\log S(\alpha)/\log N]$ with the convension in the proposition. Since it is easily seen that $\underline{\beta}=$ ess sup $[\log S(\alpha)/\log N]$, the proof is completed. □

Corollary 3.10. If $N=n$ $(n\geq 2)$ is a.s.a constant and $\beta=\beta_n:= \log\|S(\alpha)\|_\infty/ \log n$, then a suffisiant condition for $r(W^{1/\beta})<\infty$ is

$$\prod_{i=1}^\infty \|S(\alpha)\|_i/ \|S(\alpha)\|_\infty>0$$

or, equivalently, $\sum_{i=1}^{\infty} (1-\|S(\alpha)\|_{n^i} / \|S(\alpha)\|_{\infty}) < \infty,$ (3.17)

where $\|X\|_p = [\mathbb{E}(|X|^p)]^{1/p}$ (p>0).

This Corollary enables us to calculate the exact dimension functions of almost all the examples of Graf, Mauldin and Williams (1988), in a very simple way, see section 7 below. Besides, the condition (3.17) is in some way sharp. To see this, let us take the counter-example of Graf et al.(1988, p.104, example 6.10). In this example, n=2, $T_1=T_2=\frac{1}{2}(U,U)$, where U has distribution μ on [0,1] and $\mathbb{E}(U^k) \sim e^{-kd/(\log k)^{\gamma}}$. It is easily seen that (3.17) holds if and only if $\gamma>1$. In fact, we have $r(W^{1/\beta})<\infty$ if $\gamma>1$ and $r(W^{1/\beta})=\infty$ if $0<\gamma<1$, as was shown by Graf et al.(1988).

Remarks. (i) The condition (3.16)" means that, conditioned on N=n, the rate of convergence of the L_p norm $\|S(\alpha)\|_{p,n}$ to the L_∞ norm $\|S(\alpha)\|_{\infty,n}$ $(p\to\infty)$ is sufficiently large, where $\|S(\alpha)\|_{p,n} := (\mathbb{E}[(S(\alpha))^p| \ N=n])^{1/p}$. As we shall see in section 7, the condition is usually satisfied and easily verified. It holds for almost all the examples of Graf, Mauldin and Williams (1988). It seems to me that this kind of condition is more natural then the "corner" condition of Graf et al. (1988).

(ii) All the results in this section are based on the equation (3.3) with the independent properties cited therein, where the distribution of W is unknown and the T_i's and N are given. Thus the conclusions hold whenever the equation is satisfied. Kahane and Peyrière (1976) have considered such a equation in a special case for a study of a martingale of Mandelbrot. Many interesting results concerning this equation with N=n a constant and T_i i.i.d. may be found in Yves GUIVARC'H (1990).

4. Estimations on cut-set sums

4.1. Construction of a random measure μ_ω on $I(\equiv\mathbb{N}^{\mathbb{N}})$.

Let $\mathcal{G}(\omega)=(\sigma,X_\sigma)_{\sigma\in\mathcal{T}}$ be a self-similar network, and $(X_\sigma)_{\sigma\in T}$ be the associated tree process, where $X_\sigma=0$ if $\sigma\notin\mathcal{T}$. We shall regard the ratios $(T_\sigma)_{\sigma\in T}$ as defining elements, where $T_\sigma=0$ if $\sigma\notin\mathcal{T}$ (see section 2).

Let α be difined as in the preceding. Throughout this section, we suppose always that

$$0<\alpha<\infty,\ \mathbb{E}(S(\alpha))=1,\ \mathbb{E}(W)=1$$

and

$$p(T_1=...=T_N=1 \mid N>0) <1. \tag{4.0}$$

For $\sigma\in\mathcal{T}$, define

$$W_\sigma=\lim \sum_{|\tau|=k} \prod_{n=1}^{|\tau|} T^\alpha_{\sigma*(\tau|n)} \tag{4.1}$$

where $T_\emptyset=X_\emptyset=1$ and $T_{\sigma*i}=X_{\sigma*i}/X_\sigma$ by our notations. Then $W_\emptyset\equiv W$ and each W_σ is of the same distribution as W. Moreover W_σ is independent of W_τ if neither $\sigma<\tau$ nor $\tau<\sigma$, and W_σ is independent of X_σ and of X_τ unless $\sigma<\tau$. If $\sigma\in T\setminus\mathcal{T}$, we choose W_σ as an independent copy of W such that W_σ ($\sigma\in T\setminus\mathcal{T}$) are independent each other and, as a family, independent of $W_\sigma(\sigma\in\mathcal{T})$.

Given $\sigma\in T$, let $\mathbb{F}_\sigma$ denote the σ-field generated by $\{(T_{(\sigma|i)*1},T_{(\sigma|i)*2},...);\ 0\le i\le|\sigma|-1\}$:

$$\mathbb{F}_\sigma=\sigma\left((T_{(\sigma|i)*1},T_{(\sigma|i)*2},...);\ 0\le i\le|\sigma|-1\right).$$

Then W_τ is independent of $\mathbb{F}_\sigma$ unless $\tau<\sigma$. It is easily verified that

$$X^\alpha_\sigma W_\sigma = \sum_{i=1}^{\infty} X^\alpha_{\sigma*i} W_{\sigma*i} \tag{4.2}$$

almost surely (note that $X_\sigma=\prod_{n=1}^{|\sigma|} T_{\sigma|n}$). So if $\Gamma\in\mathcal{T}$ is a cut-set then

$$W_\sigma=\lim \sum_{\tau\in\Gamma} \prod_{n=1}^{|\tau|} T^\alpha_{\sigma*(\tau|n)} W_{\sigma*\tau} \tag{4.3}$$

a.s.for each $\sigma\in T$. Let

$$B(\sigma)=\{\eta\in\mathbb{N}^{\mathbb{N}}:\ \eta>\sigma\} \tag{4.4}$$

($\sigma\in T$) be a *ball* in $\mathbb{N}^{\mathbb{N}}$ associated with σ and define

$$\mu_\omega(B(\sigma)) = X^\alpha_\sigma(\omega)\ W_\sigma(\omega). \tag{4.5}$$

By (4.2) μ_ω is well defined. It can be uniquely extended to a Borel measure on $\mathbb{N}^{\mathbb{N}}$ which will be called μ_ω again.

Proposition 4.1 (i) If $\mathbb{E}(e^{tW^{1/b}}) < \infty$, then with probability 1

$$\limsup_{n\to\infty} \frac{tW(i\,|\,n)^{1/b}}{\text{Log} n} \le 1$$

and

$$\limsup_{n\to\infty} \frac{tW(i\,|\,n)^{1/b}}{\text{Loglog}\frac{1}{X_{(i\,|\,n)}}} \le 1$$

for μ_ω almost all $i \in I$.

(ii) If $\mathbb{E}(W^{1+1/b}) < \infty$ for some $0<b<\infty$, then $\forall\varepsilon>0$ we have, with probability 1

$$\limsup_{n\to\infty} \frac{W(i\,|\,n)^{1/b}}{n^{1+\varepsilon}} \le 1$$

and

$$\limsup_{n\to\infty} \frac{W(i\,|\,n)^{1/b}}{(\log \frac{1}{X_{(i\,|\,n)}})^{1+\varepsilon}} \le \frac{1}{\mathbb{E}(\Sigma\ T_i^\alpha \log\frac{1}{T_i})} < \infty$$

for μ_ω- a.e. $i\in I$.

Proof. (i) Let (Ω,p) denote the underlying probability space and consider the product space $I \times \Omega$ with the product σ-field with probability law Q defined by

$$Q(A) = \mathbb{E} \int 1_A(\omega,i)\ d\mu_\omega(i). \tag{4.6}$$

Then $\forall\ \varepsilon > 0\ \forall t'<t$

$$Q(e^{t'(W(i\,|\,n))^{1/b}} \ge n^{1+\varepsilon}) \le \frac{1}{n^{1+\varepsilon}}\ \mathbb{E}_Q(e^{t'W(i\,|\,n)^{1/b}}) = \frac{1}{n^{1+\varepsilon}}\ \mathbb{E}(We^{t'W^{1/b}})$$

The Borel-Cantelli lemma ensures that

$$\limsup_{n\to\infty} \frac{W(i\,|\,n)^{1/b}}{\text{Log } n} \le (1+\varepsilon)\ \frac{1}{t'},$$

Q-almost surely. Hence the first inequlity follows. Note that the random variables

$$\hat{T}_k(\eta,\omega) := T_{\eta\,|\,k}(\omega) \tag{4.7}$$

on $I \times \Omega$ are independent and identically distributed, the theorem of large numbers gives

$$\lim_{n\to\infty} (\log \frac{1}{X_{i|n}})/n \equiv \lim_{n\to\infty} [\log 1/ \prod_{k=1}^{n} T_{i|k}]/n$$

$$=\mathbb{E}_Q[\log (1/ \hat{T})] = \mathbb{E}(\Sigma_{i=1}^{\infty} T_i^{\alpha} \log \frac{1}{T_i}) >0, \tag{4.8}$$

where the last step holds since $p(T_1=...=T_N=1 \mid N>0)<1$. Consequently the second inequlity follows from the first.

(ii) The approach is almost the same as above by means of the Borel-Cantelli Lemma, noting that

$$Q((W(i|n)^{1/b})\geq n^{1+\varepsilon}) \leq \frac{1}{n^{1+\varepsilon}} \mathbb{E}_Q(W(i|n^{1/b}) = \frac{1}{n^{1+\varepsilon}} \mathbb{E}(W\ W^{1/b}). \qquad \square$$

Remark 4.1. The same idea can be applied to prove the following:

If $\mathbb{E}(W^{1+1/b}) < \infty$, then with probability 1

$$\limsup_{n\to\infty} \frac{W(i|n)^{1/b}}{h(n)} \leq 1 \quad \text{and} \quad \limsup_{n\to\infty} \frac{W(i|n)^{1/b}}{h(\log\frac{1}{X(i|n)})} \leq 1$$

for μ_ω almost all $i \in I$ ($\equiv\mathbb{N}^{\mathbb{N}}$), where $h(t) = t^{1+\varepsilon}$, $t(\log t)^{1+\varepsilon}$, $t(\log t)(\log\log t)^{1+\varepsilon}$, etc. ($\forall\varepsilon>0$).

4.2. The lower bound.

Proposition 4.2 For $0<\beta<\infty$, let

$$\phi_\beta(t) = t^{\alpha}(\mathrm{LogLog}\ \frac{1}{t})^{\beta}. \tag{4.9}$$

(i) If $\mathbb{E}(e^{rW^{1/\beta}}) < \infty$ for some $r > 0$, then

$$\mathcal{M}^{\phi_\beta}(\mathcal{T}) \geq r^{\beta} W \tag{4.10}$$

almost surely.

(ii) If $\mathbb{E}(W^{1+1/\beta}) < \infty$, then

$$\mathcal{M}^{\psi}(\mathcal{T}) = +\infty \quad \text{a.s. on } W>0,$$

where

$$\psi(t)=t^{\alpha}(\log\tfrac{1}{t})^{\beta+\varepsilon}, t^{\alpha}(\log\tfrac{1}{t})^{\beta}(\log\log\tfrac{1}{t})^{\beta+\varepsilon}, t^{\alpha}(\log\tfrac{1}{t})^{\beta}(\log\log\tfrac{1}{t})^{\beta}(\log\log\log\tfrac{1}{t})^{\beta+\varepsilon},$$

etc., $\forall\varepsilon>0$.

Proof. (i) We first note that $\mu_\omega(\tilde{\mathcal{T}}) = W(\omega)$. By Proposition 4.1, for each $\varepsilon > 0$ we can choose a compact subset $\tilde{\mathcal{T}}'$ of $\tilde{\mathcal{T}}$ such that $\mu(\tilde{\mathcal{T}}') \geq W\text{-}\varepsilon$ and

$$W_{i|n} \leq (1+\varepsilon)\ (\ \frac{1}{r}\ \mathrm{LogLog}\ \frac{1}{X_{i|n}}\ )^{\beta}$$

for all $i \in \tilde{\mathcal{T}}'$ and all $n \geq N_0 \equiv N_0(\omega)$ (Proposition 4.1(i) ensures that this can be done almost surely).

For each $\sigma \in \mathcal{T}$ with $|\sigma| \geq N_0$, let us consider

$$U_\sigma = B(\sigma) \cap \tilde{\mathcal{T}}'$$

such that $U_\sigma \neq \emptyset$. Take an arbitrary $i \in U_\sigma$, then $i \in \tilde{\mathcal{T}}'$ and $i|(|\sigma|) = \sigma$. Thus

$$U_\sigma = B(i|(|\sigma|)) \cap \tilde{\mathcal{T}} \subseteq B(i|(|\sigma|))$$

and consequently

$$\begin{aligned}\mu_\omega(U_\sigma) &\leq \mu_\omega(\ B(i|(|\sigma|))\ ) = X^\alpha_{i|(|\sigma|)} W_{i|(|\sigma|)}\\ &\leq X^\alpha_{i|(|\sigma|)} (1+\varepsilon) \left(\frac{1}{r}\mathrm{loglog}\frac{1}{X_{i|(|\sigma|)}}\right)^{\beta}\\ &= (1+\varepsilon)\phi_\beta(X_{i|(|\sigma|)})/r^\beta = (1+\varepsilon)\phi_\beta(X_\sigma)/r^\beta.\end{aligned}$$

Thus $\qquad \mu_\omega(U_\sigma) \leq (1+\varepsilon)\phi_\beta(X_\sigma)/r^\beta$ whenever $\sigma \in \mathcal{T}$ and $|\sigma| \geq N_0$.

It is evident that it holds also if $U_\sigma = \emptyset$.

Let Γ be any cut-set of $\mathcal{T}$ with $\min\{|\sigma| : \sigma \in \Gamma\} \geq N_0$. Then

$$\tilde{\mathcal{T}} \subseteq \cup_{\sigma\in\Gamma} B(\sigma),\ \tilde{\mathcal{T}}' \subseteq \cup_{\sigma\in\Gamma} (B(\sigma) \cap \tilde{\mathcal{T}}'),$$

$$\mu(\tilde{\mathcal{T}}') \leq \Sigma_{\sigma\in\Gamma}\ \mu(U_\sigma),$$

thus

$$W\text{-}\varepsilon \leq \mu(\tilde{\mathcal{T}}') \leq (1+\varepsilon) \sum_{\sigma\in\Gamma} \phi_\beta(X_\sigma)/r^\beta.$$

This implies that

$$\mathcal{M}^{\phi_\beta}_{N_0}(\mathcal{T}) \geq \frac{1}{1+\varepsilon}\ r^\beta(W\text{-}\varepsilon)$$

almost surely. Letting $\varepsilon \to 0$, it gives $\mathcal{M}^{\phi_\beta}_{N_0}(\mathcal{T}) \geq r^\beta W$, and then the result desired.

(ii) The same argument as above. Take for example $\psi(t) = \psi_{\beta+\varepsilon}(t) = t^\alpha(\log\frac{1}{t})^{\beta+\varepsilon}$. $\forall\eta>0$, choose a compact subset $\tilde{\mathcal{T}}'$ of $\tilde{\mathcal{T}}$ such that $\mu(\tilde{\mathcal{T}}') \geq W\text{-}\eta$ and

$$W_{i|n} \leq (\mathrm{Log}\frac{1}{X_{i|n}})^{\beta+\varepsilon} \text{ for all } i\in\tilde{\mathcal{T}}' \text{ and } n\geq N_0,$$

using Proposition 4.2 (ii). Thus

$$\mu_\omega(B(\sigma)\cap\tilde{\mathcal{T}}) \leq \psi(X_\sigma) \quad \text{whenever} \quad \sigma\in\mathcal{T} \text{ and } |\sigma|\geq N_0.$$

Hence

$$W\text{-}\eta \leq \mu(\tilde{\mathcal{T}}') \leq \sum_{\sigma\in\Gamma}\psi(X_\sigma),$$

Γ being any cover set in $\mathcal{T}$ with $\min\{|\sigma| : \sigma\in\Gamma\}\geq N_0$. Thus $W\text{-}\eta \leq \mathcal{M}^{\psi}_{N_0}(\mathcal{T})$ and so $W \leq \mathcal{M}^{\psi}(\mathcal{T})$, $\mathcal{M}^{\psi}(\mathcal{T})>0$ a.s. on $W>0$, where $\psi=\psi_{\beta+\varepsilon}$. Since $\mathcal{M}^{\psi_{\beta+\varepsilon/2}}(\mathcal{T}) >0$ also a.s. on $W>0$, we have $\mathcal{M}^{\psi_{\beta+\varepsilon}}(\mathcal{T}) = \infty$ a.s. on $W>0$. For the function $\psi(t) = t^{\alpha}(\log\frac{1}{t})^{\beta}(\log\log\frac{1}{t})^{\beta+\varepsilon}$, etc. the proof is similar, using Remark 4.1. □

4.3. The upper bound.

Proposition 4.3. (i) Suppose that $\mathbb{E}[\sum_{i=1}^{\infty} T_i^{\alpha}\log\frac{1}{T_i}] < \infty$ and $\mathbb{E}[e^{rW^{1/\beta}}] = +\infty$ for some $(\beta,r) \in (0,\infty)^2$. Then

$$\mathcal{M}^{\phi_\beta}(\mathcal{T})<\infty$$

almost surely. In fact $\mathbb{E}(\mathcal{M}^{\phi_\beta}(\mathcal{T})) \leq r^{\beta}$.

(ii) If $\mathbb{E}[W^{1/\beta}] = +\infty$ for some $\beta\in(0,\infty)$, then

$$\mathcal{M}^{\psi_{\beta-\varepsilon}}(\mathcal{T}) = 0 \text{ a.s. } \forall\varepsilon>0,$$

where $\psi_{\beta-\varepsilon}(t) = t^{\alpha}(\log\frac{1}{t})^{\beta-\varepsilon}$. Moreover

$$\mathcal{M}^{\psi_\beta}(\mathcal{T}) < \infty \text{ a.s.}$$

if

$$\limsup_{k\to\infty}\left\{\sum_{\nu=[\mathrm{Log}k]}^{k} p\left[W^{1/\beta}\geq\nu\right] - \beta\,\mathrm{Log}k\right\} > -\infty.$$

The proof relies on the following

Lemma 4.4. (i) Suppose that $\mathbb{E}[e^{rW^{1/\beta}}] = +\infty$ for some $(\beta,r) \in (0,\infty)^2$. For $t > 0$, write

$$B_k^* \equiv B_k^*(\beta,t) = \Big\{\sigma \in \mathbb{N}^k \mid W_{(\sigma|\nu)*}(\omega) < \big(\tfrac{1}{t}\,\mathrm{LogLog}\,\tfrac{1}{X_{\sigma|\nu}}\big)^{\beta} \text{ for all } \nu = [\mathrm{Log}k],[\mathrm{Log}k]+1,\ldots,k.\Big\}, \tag{4.11}$$

where

$$\eta^* = (\eta_1,\eta_2,\ldots,\eta_{n-1},\eta_n+1) \text{ if } \eta = (\eta_1,\eta_2,\ldots,\eta_n), \tag{4.12}$$

and

$$I_k^* \equiv I_k^*(\beta,t) = \int_\Omega \sum_{\sigma\in B_k^*} X_\sigma^\alpha \,(\mathrm{LogLog}\,\tfrac{1}{X(\sigma)})^{\beta} dp\ . \tag{4.13}$$

Then for all $t > r$ we have

$$\liminf_{k\to\infty} I_k^* = 0 \tag{4.14}$$

if

$$\mathbb{E}[\Sigma_{i=1}^{\infty}\ T_i^\alpha \log \tfrac{1}{T_i}] < \infty. \tag{4.15}$$

(ii) For $\beta \in (0,\infty)$, write

$$\mathring{B}_k^* \equiv \mathring{B}_k^*(\beta) = \Big\{\sigma \in \mathbb{N}^k \mid W_{(\sigma|\nu)*}(\omega) < (\mathrm{Log}\,\tfrac{1}{X(\sigma|\nu)})^{\beta} \text{ for all } \nu = [\mathrm{Log}k],[\mathrm{Log}k]+1,\ldots,k.\Big\}, \tag{4.11'}$$

and

$$\mathring{I}_k^* \equiv \mathring{I}_k^*(\beta) = \int_\Omega \sum_{\sigma\in \mathring{B}_k^*} X_\sigma^\alpha \,(\mathrm{Log}\,\tfrac{1}{X(\sigma)})^{\beta} dp\ . \tag{4.13'}$$

Then

$$\limsup_{k\to\infty} \Big\{ \sum_{\nu=[\mathrm{Log}k]}^{k} p\Big[W^{1/\beta} \geq \nu\Big] - \beta\,\mathrm{Log}k \Big\} > -\infty \tag{4.16}$$

implies

$$\liminf_{k\to\infty} \mathring{I}_k^*(\beta) < \infty. \tag{4.17}$$

In particular we have

$$\liminf_{k\to\infty} \mathring{I}_k^*(\beta-\varepsilon) = 0 \quad \forall\varepsilon>0 \tag{4.18}$$

if $\mathbb{E}(W^{1/\beta}) = \infty$.

Remark 4.4. The condition (4.15) is implied by

$$\mathbb{E}(N) < \infty.$$

In fact, $\forall 0<\varepsilon<\alpha$, $\exists C>0$ sufficiently large such that $\log x \le Cx^{\varepsilon}$ ($\forall x\ge 1$). Thus

$$\mathbb{E}[\Sigma_{i=1}^{\infty} T_i^{\alpha}\log \frac{1}{T_i}] \le \mathbb{E}[\Sigma_{i=1}^{\infty} T_i^{\alpha}\, C\, (\frac{1}{T_i})^{\varepsilon}] = C\,\mathbb{E}[\Sigma_{i=1}^{\infty} T_i^{\alpha-\varepsilon}] \le C\,\mathbb{E}(N).$$

For the proof of Lemma 4.4, we shall need the following simple result of analyse in Liu [1992, Lemma 4.3] :

Lemma 4.5. Suppose that a function $g: \mathbb{R} \to [0,1]$ is non-increasing such that $\int_0^{\infty} g(t)dt = +\infty$ and that $j: \mathbb{N} \to \mathbb{R}$ is a function satisfing

$$\limsup_{k\to\infty} \frac{j(k)}{k} < 1$$

then $\forall\ \varepsilon > 0$,

$$\limsup_{k\to\infty} \int_{j(k)^{1/(1+\varepsilon)}}^{k^{1/(1+\varepsilon)}} g(t)t^{\varepsilon}\, dt - k^{\bar{\varepsilon}/(1+\varepsilon)} = +\infty$$

for each $\bar{\varepsilon}$ with $0 < \bar{\varepsilon} < \varepsilon$.

Proof of lemma 4.4. (i) Since $W_{(\sigma|v)*}$ (v = [Logk],...,k.) are independent each other and as a family independent of F_σ (the σ-algebra generated by $(T_{(\sigma|i)*1}, T_{(\sigma|i)*2},\ldots), 0 \le i \le |\sigma|-1.$), where $|\sigma|=k$, we have

$$I_k^* = \sum_{\sigma\in\mathbb{N}^k} \int_{\bigcap_{v=[\mathrm{Log}k]}^{k} \left\{W_{(\sigma|v)*}< \left(\frac{1}{t}\ \mathrm{LogLog}\ \frac{1}{X_{(\sigma|v)}}\right)^{\beta}\right\}} X_\sigma^{\alpha}\left(\mathrm{LogLog}\frac{1}{X_\sigma}\right)^{\beta} dp$$

(conditioned on F_σ firstly)

$$= \sum_{\sigma\in\mathbb{N}^k} \int_{\Omega} X_\sigma^{\alpha}\left(\mathrm{LogLog}\frac{1}{X_\sigma}\right)^{\beta} dp \prod_{v=[\mathrm{Log}k]}^{k} p\left\{ W < \left(\frac{1}{t}\ \mathrm{LogLog}\ \frac{1}{X_{(\sigma|v)}}\right)^{\beta} \right\}$$

$$= I_k^1 + I_k^2 ,$$

where

$$I_k^1 = \sum_{\sigma\in\mathbb{N}^k} \int_{\Omega_1} X_\sigma^{\alpha}\left(\mathrm{LogLog}\frac{1}{X_\sigma}\right)^{\beta} dp \prod_{v=[\mathrm{Log}k]}^{k} p\left\{W < \left(\frac{1}{t}\ \mathrm{LogLog}\ \frac{1}{X_{(\sigma|v)}}\right)^{\beta}\right\}, \tag{4.19}$$

$$I_k^2 = \sum_{\sigma\in\mathbb{N}^k} \int_{\Omega_2} X_\sigma^{\alpha}\left(\mathrm{LogLog}\frac{1}{X_\sigma}\right)^{\beta} dp \prod_{v=[\mathrm{Log}k]}^{k} p\left\{W < \left(\frac{1}{t}\mathrm{LogLog}\frac{1}{X_{(\sigma|v)}}\right)^{\beta}\right\}, \tag{4.20}$$

with

$$\Omega_1 \equiv \Omega_1(\sigma,k) = \bigcap_{\nu=[\log k]}^{k} [\omega : X_{(\sigma|\nu)} \geq c^{\nu}], \tag{4.21}$$

$$\Omega_2 \equiv \Omega_2(\sigma,k) = \bigcup_{\nu=[\log k]}^{k} [\omega : X_{(\sigma|\nu)} < c^{\nu}], \tag{4.22}$$

where $c>0$ is arbitrary at the moment. For

$$c > \max\{ \mathbb{E}(\Sigma_{i=1}^{\infty} T_i^{\alpha+1}),\ 1/e\}, \tag{4.23}$$

we shall see that

$$\liminf_{k\to\infty} I_k^1 = 0, \tag{4.24}$$

and

$$\lim_{k\to\infty} I_k^2 = 0. \tag{4.25}$$

In fact,

$$I_k^1 \leq \sum_{\sigma\in\mathbb{N}^k} \int_{\Omega_1} X_\sigma^\alpha \left(\operatorname{LogLog} \frac{1}{c^k} \right)^{\beta} dp \prod_{\nu=[\operatorname{Log}k]}^{k} p\left\{ W < \left(\frac{1}{t} \operatorname{LogLog} \frac{1}{c^{\nu}}\right)^{\beta} \right\},$$

$$\leq (\log k)^{\beta}\ \mathbb{E}\left(\sum_{\sigma\in\mathbb{N}^k} X_\sigma^\alpha \right) \prod_{\nu=[\operatorname{Log}k]}^{k} p\left\{ W < \left(\frac{1}{t} \operatorname{Log} \nu\right)^{\beta} \right\}$$

if $c > 1/e$. Thus

$$I_k^1 \leq (\log k)^{\beta} \exp\left\{ - \sum_{\nu=[\operatorname{Log}k]}^{k} p\left[W \geq \left(\frac{1}{t} \operatorname{Log} \nu\right)^{\beta} \right] \right\}$$

$$= \exp\left\{ - \sum_{\nu=[\operatorname{Log}k]}^{k} p\left[e^{tW^{1/\beta}} \geq \nu \right] + \beta \operatorname{LogLog} k \right\}. \tag{4.26}$$

We then note that

$$\sum_{\nu=[\operatorname{Log}k]}^{k} p\left[e^{tW^{1/\beta}} \geq \nu \right] \geq \sum_{\nu=[\operatorname{Log}k]}^{k-1} \int_{\nu}^{\nu+1} p\left[e^{tW^{1/\beta}} \geq x \right] dx$$

$$= \int_{[\operatorname{Log}k]}^{k} P\left[e^{tW^{1/\beta}} \geq x \right] dx$$

$$= \frac{1}{1+\varepsilon} \int_{[\operatorname{Log}k]^{1/(1+\varepsilon)}}^{k^{1/(1+\varepsilon)}} P\left[e^{\frac{t}{1+\varepsilon} W^{1/\beta}} \geq y \right] y^{\varepsilon} d\varepsilon \tag{4.27}$$

($x=y^{1+\varepsilon}$), where $\varepsilon > 0$ is chosen such that $t/(1+\varepsilon) > r$. Write

$$f(y) = P\left[e^{\frac{t}{1+\varepsilon} W^{1/\beta}} \geq y \right], \tag{4.28}$$

then

$$\int_0^{\infty} f(y)dy = \mathbb{E}\left[e^{\frac{t}{1+\varepsilon} W^{1/\beta}} \right] = +\infty$$

by the hypothesis. Lemma 4.5 gives then

$$\limsup_{k\to\infty} \int_{[\mathrm{Log}k]^{1/(1+\varepsilon)}}^{k^{1/(1+\varepsilon)}} p\left[e^{\frac{t}{1+\varepsilon}W^{1/\beta}} \geq y\right] y^{\varepsilon} d\varepsilon - k^{\bar{\varepsilon}/(1+\varepsilon)} = +\infty.$$

Hence

$$\limsup_{k\to\infty} \left\{ \sum_{\nu=[\mathrm{Log}k]}^{k} p\left[e^{tW^{1/\beta}} \geq \nu\right] - \beta \ \mathrm{Log}\ \mathrm{Log}k \right\} = +\infty, \tag{4.29}$$

and consequently $\liminf\limits_{k\to\infty} I_k^1 = 0$.

We now prove that $\lim\limits_{k\to\infty} I_k^2 = 0$. We shall use the random variables

$$\hat{T}_k(i,\omega) = T_{i|k}(\omega), \quad \hat{X}_k(i,\omega) = \prod_{\nu=1}^{k} \hat{T}_\nu(i,\omega) \equiv X_{i|k}(\omega)$$

on $I \times \Omega$. For each $k>0$ $\hat{T}_k$ are Q-independent and identically distributed. We note the common distribution by $\hat{T}(i,\omega)$.

Let p and p' be two positive numbers such that

$$p > \max(1,1/\beta) \quad \text{and} \quad 1/p + 1/p' = 1.$$

Using Hölder's inequality we have

$$I_k^2 \leq \Sigma_{\nu=[\log k]}^{k} \int (\log\log \frac{1}{\hat{X}_k})^{\beta}\, 1_{(\hat{X}_\nu < c^\nu)}\, dQ$$

$$\leq \Sigma_{\nu=[\log k]}^{k} \left(\int (\log\log \frac{1}{\hat{X}_k})^{\beta p} dQ\right)^{1/p} \left(Q(\hat{X}_\nu < c^\nu)\right)^{1/p'}. \tag{4.30}$$

Since the function $(\log x)^{\beta p}$ is concave for x sufficiently large, Jensen's inequality gives

$$\int (\log\log \frac{1}{\hat{X}_k})^{\beta p} dQ = \mathbb{E}_Q\left[(\log \log \frac{1}{\hat{X}_k})^{\beta p}\right]$$

$$\leq \left(\log \mathbb{E}_Q(\log \frac{1}{\hat{X}_k})\right)^{\beta p} + C = \left(\log (k\, \mathbb{E}_Q[\log \frac{1}{\hat{T}}]\right)^{\beta p} + C, \tag{4.31}$$

where $C > 0$ is a constant independent of k and E_Q denotes the expectation with respect to Q. On the other hand, by Markov's inequality we have

$$Q(\hat{X}_\nu < c^\nu) \leq \mathbb{E}_Q(\hat{X}_\nu)/ c^\nu = (\mathbb{E}_Q(\hat{T})/ c)^\nu.$$

Therefore

$$I_k^2 \leq C' \Sigma_{\nu=[\log k]}^{k} \left(\log (k\, \mathbb{E}_Q[\log \frac{1}{\hat{T}}]\right)^{\beta} (\mathbb{E}_Q(\hat{T})/ c)^{\nu/p'}$$

$$\leq C' \left(\log (k\, \mathbb{E}_Q[\log \frac{1}{\hat{T}}]\right)^{\beta} (\mathbb{E}_Q(\hat{T})/ c)^{[\log k]/p'} / (1- \frac{1}{c}\, \mathbb{E}_Q(\hat{T})) \tag{4.32}$$

for some constant C'>0 independent of k, where

$$\mathbb{E}_Q[\log \frac{1}{\hat{T}}] = \mathbb{E}(\Sigma_{i=1}^{\infty} T_i^{\alpha} \log \frac{1}{T_i}) < \infty \quad \text{and} \quad \mathbb{E}_Q(\hat{T}) = \mathbb{E}(\Sigma_{i=1}^{\infty} T_i^{\alpha+1}) < c, \tag{4.33}$$

using (4.15) and (4.23). Thus $\lim_{k\to\infty} I_k^2 = 0$, and consequently $\lim_{k\to\infty} \inf I_k^* = 0$, which ends the proof of the first part of the lemma.

(ii) A similar argument as in (i) shows that

$$I_k^*(\beta) = I_k^1 + I_k^2 ,$$

where

$$I_k^1 = \sum_{\sigma\in\mathbb{N}^k} \int_{\Omega_1} X_\sigma^\alpha \Big(\text{Log}\frac{1}{X_\sigma} \Big)^\beta dp \prod_{v=[\text{Logk}]}^{k} p\Big\{W < \Big(\text{Log}\, \frac{1}{X_{(\sigma|v)}}\Big)^\beta\Big\}, \tag{4.33}$$

$$I_k^2 = \sum_{\sigma\in\mathbb{N}^k} \int_{\Omega_2} X_\sigma^\alpha \Big(\text{Log}\frac{1}{X_\sigma} \Big)^\beta dp \prod_{v=[\text{Logk}]}^{k} p\Big\{W < \Big(\frac{1}{t}\text{Log}\frac{1}{X_{(\sigma|v)}}\Big)^\beta\Big\}, \tag{4.34}$$

with Ω_1 and Ω_2 defined in (i): $\Omega_1 = \bigcap_{v=[\log k]}^{k} [X_{(\sigma|v)} \geq c^v], \Omega_2 = \bigcup_{v=[\log k]}^{k} [X_{(\sigma|v)} < c^v]$.

Instead of (4.26), (4.30), (4.31) and (4.32) we have respectively

$$I_k^1 \leq \exp\Big\{ -\Sigma_{v=[\text{Logk}]}^{k} p\Big[W^{1/\beta} \geq v\Big] + \beta \text{ Log } k\Big\}, \tag{4.26'}$$

$$I_k^2 \leq \Sigma_{v=[\log k]}^{k} \int (\log \frac{1}{\hat{X}_k})^\beta 1_{(\hat{X}_v < c^v)} dQ$$

$$\leq \Sigma_{v=[\log k]}^{k} \Big(\int (\log \frac{1}{\hat{X}_k})^{\beta p} dQ\Big)^{1/p} \Big(Q(\hat{X}_v < c^v)\Big)^{1/p'}, \tag{4.30'}$$

$$\int(\log\frac{1}{\hat{X}_k})^{\beta p} dQ = \alpha^{-\beta p}\mathbb{E}_Q\Big[(\log\frac{1}{\hat{X}_k^{\alpha}})^{\beta p}\Big] \leq \Big(\log\, \mathbb{E}_Q(\frac{1}{\hat{X}_k^{\alpha}})\Big)^{\beta p}\alpha^{-\beta p} + C$$

$$= \alpha^{-\beta p}\Big(\log\,(k\, \mathbb{E}_Q(\frac{1}{\hat{T}^{\alpha}})\Big)^{\beta p} + C = [\alpha^{-1}\log(k\mathbb{E}N)]^{\beta p} + C, \tag{4.31'}$$

$$I_k^2 \leq C'\Sigma_{v=[\log k]}^{k} \Big(\alpha^{-1}\log(k\mathbb{E}N)]\Big)^\beta (\mathbb{E}_Q(\hat{T})/ c)^{v/p'}$$

$$\leq C'\Big(\alpha^{-1}\log(k\mathbb{E}N)]\Big)^\beta (\mathbb{E}_Q(\hat{T})/ c)^{[\log k]/p'}/(1-\mathbb{E}_Q(\hat{T})/c) \tag{4.32'}$$

where $\beta p \geq 1$, $1/p+1/p'=1$, $c > \max(1/e, \mathbb{E}_Q(\hat{T}))$, C and C' are some positive constants independent of k. Here we have used the fact that

$$\mathbb{E}_Q(\hat{T}^\xi) = \mathbb{E}(\Sigma_{i=1}^{\infty} T_i^{\alpha+\xi}),\ \alpha+\xi\geq 0.$$

Hence by (4.32)' we have $\lim_{k\to\infty} I_k^2 = 0$, and by (4.26)' we see that (4.16) implies $\lim_{k\to\infty} \inf I_k^1 < +\infty$. Hence (4.16) implies $\lim_{k\to\infty} \inf I_k^* < +\infty$ and the proof is then finished if we note that (4.16) holds with β replaced by β-ε ($\forall\varepsilon>0$) if $\mathbb{E}(W^{1/\beta}) = \infty$. (see liu 1992) □

We are now in a position to prove Proposition 4.3.

Proof of proposition 4.3. (i) Let $t > r$ and B_k^* be defined as in lemmma 4.4:

$$B_k^* = \left\{ \sigma \in \mathbb{N}^k \middle| \ W_{(\sigma|\nu)*}(\omega) < \left(\frac{1}{t} \operatorname{LogLog} \frac{1}{X(\sigma|\nu)} \right)^\beta \right.$$
$$\left. \text{for all } \nu = [\operatorname{Log}k],[\operatorname{Log}k]+1,\dots,k. \right\}.$$

For $\sigma \in \mathbb{N}^k\text{-}B_k^*$, let $k(\sigma)$ be the smallest $\nu \geq [\operatorname{Log}k]$ such that

$$W_{(\sigma|\nu)*}(\omega) \geq \left(\frac{1}{t} \operatorname{LogLog} \frac{1}{X(\sigma|\nu)} \right)^\beta. \tag{4.47}$$

Then $[\log k] \leq k(\sigma) \leq k$. Set

$$\tilde{\Gamma}(k)= \left\{ \sigma|k(\sigma): \sigma \in \mathbb{N}^k\text{-}B_k^* \right\}. \tag{4.48}$$

It is easy to check that $\tilde{\Gamma}(k)$ is an antichain with $\tilde{\Gamma}(k) < \mathbb{N}^k\text{-}\ B_k^*$. Since $\tilde{\Gamma}(k) \cup B_k^*$ is complete in *T*, we can choose a cut-set $\Gamma(k)$ of *T* (that is $\Gamma(k)$ is a maximal antichain in *T*) with

$$\tilde{\Gamma}(k) \subseteq \Gamma(k) \subseteq \tilde{\Gamma}(k) \cup B_k^*. \tag{4.49}$$

By the definition of B_k^* and $\tilde{\Gamma}(k)$, we obtain

$$\sum_{\sigma\in\Gamma(k)} X_\sigma^\alpha \left|\log\log\frac{1}{X_\sigma}\right|^\beta = \sum_{\sigma\in\tilde{\Gamma}(k)} X_\sigma^\alpha \left|\log\log\frac{1}{X_\sigma}\right|^\beta + \sum_{\sigma\in B_k^*} X_\sigma^\alpha \left|\log\log\frac{1}{X_\sigma}\right|^\beta$$
$$\leq t^\beta \sum_{\sigma\in\tilde{\Gamma}(k)} X_\sigma^\alpha W_{\sigma*}(\omega) + \sum_{\sigma\in B_k^*} X_\sigma^\alpha \left|\log\log\frac{1}{X_\sigma}\right|^\beta. \tag{4.50}$$

By Lemma 4.4, we can choose a sequence (k_i) of integers increasing to ∞ such that $I_{k_i}^* \to 0$ $(i\to\infty)$. Hence

$$\mathbb{E}\left[\liminf_{k\to\infty} \sum_{\sigma\in\Gamma(k)} X_\sigma^\alpha \left|\log\log\frac{1}{X_\sigma}\right|^\beta \right]$$
$$\leq \liminf_{k\to\infty} \mathbb{E}\left[\sum_{\sigma\in\Gamma(k)} X_\sigma^\alpha \left|\log\log\frac{1}{X_\sigma}\right|^\beta \right]$$
$$\leq t^\beta \liminf_{i\to\infty} \mathbb{E}\left[\sum_{\sigma\in\tilde{\Gamma}(k_i)} X_\sigma^\alpha W_{\sigma*}(\omega) \right] + \lim_{i\to\infty} I_{k_i}^*$$

$$= t^{\beta} \liminf_{i\to\infty} \mathbb{E}\Big[\sum_{\sigma\in\tilde{\Gamma}(k_i)} X_\sigma^\alpha \, W_{\sigma*}(\omega) \Big] \tag{4.51}$$

First conditioned on $\mathbb{F}_{k_i}$, the σ-algebra generated by $X_\sigma(|\sigma| \le k_i)$, we obtain that

$$\mathbb{E}\Big[\sum_{\sigma\in\tilde{\Gamma}(k_i)} X_\sigma^\alpha \, W_{\sigma*}(\omega) \Big] = \mathbb{E}\Big[\sum_{\sigma\in\tilde{\Gamma}(k_i)} X_\sigma^\alpha \Big] = \mathbb{E}\Big[\sum_{\sigma\in\tilde{\Gamma}(k_i)} X_\sigma^\alpha \, W_\sigma(\omega) \Big] \tag{4.52}$$

since $\mathbb{E}(W_\sigma)=1$ for all $\sigma\in T$. Thus

$$\mathbb{E}\Big[\liminf_{k\to\infty} \sum_{\sigma\in\Gamma(k)} X_\sigma^\alpha \Big|\log\log \frac{1}{X_\sigma}\Big|^{\beta} \Big]$$

$$\le t^{\beta} \liminf_{i\to\infty} \mathbb{E}\Big[\sum_{\sigma\in\tilde{\Gamma}(k_i)} X_\sigma^\alpha \, W_\sigma(\omega) \Big]$$

$$\le t^{\beta} \liminf_{i\to\infty} \mathbb{E}\Big[\sum_{\sigma\in\Gamma(k_i)} X_\sigma^\alpha \, W_\sigma(\omega) \Big] = t^{\beta} \liminf_{i\to\infty} \mathbb{E}(W), \tag{4.53}$$

where the last step holds as

$$W = \sum_{\sigma\in\Gamma} X_\sigma^\alpha W_\sigma$$

for any maximal antichain Γ. Consequently

$$\mathbb{E}\Big[\liminf_{k\to\infty} \sum_{\sigma\in\Gamma(k)} X_\sigma^\alpha \Big|\log\log \frac{1}{X_\sigma}\Big|^{\beta} \Big] \le t^{\beta} \tag{4.54}$$

for all $t > r$. Letting $t\to r$ gives the result desired, if we note that

$$\mathcal{M}_k^{\phi_\beta}(\mathcal{T}) \le \sum_{\sigma\in\Gamma(k)} \phi_\beta(X_\sigma) \text{ and } \mathcal{M}^{\phi_\beta}(\mathcal{T}) = \lim_{k\to\infty} \mathcal{M}_k^{\phi_\beta}(\mathcal{T}) = \lim_{k\to\infty}\inf \mathcal{M}_k^{\phi_\beta}(\mathcal{T}).$$

(ii) By Lemma 4.4(ii), it suffices to prove that (4.16) implies $\mathcal{M}^{\phi_\beta}(\mathcal{T}) < \infty$ a.s. Since $\underline{\lim}_{k\to\infty} \inf I_k^* < \infty$, the proof is very similar to the above: we replace B_k^* and I_k^* by B_k^* and I_k^* respectively. □

4.4. The fundemental results

Combining propositions 4.2 and 4.3, we obtain the exact value of the lower limit of cut-set sums:

Theorem 4.6. (*The fundamental theorem*) Let $(\sigma,X_\sigma)_{\sigma\in\mathcal{T}}$ be a self similar network generated by $(N; T_1,\dots,T_N)$ with

$$\mathbb{E}(S(\alpha)^2)<\infty \tag{L^2}$$

and $$\mathbb{E}(\sum_{i=1}^{N} T_i^{\alpha} \log\frac{1}{T_i})<\infty. \qquad (T\log\frac{1}{T})$$

For $b\in(0,\infty)$, we write $\phi_b(t) = t^{\alpha}(\log\log\frac{1}{t})^b$ and denote by $r_b=r(W^b)$ the radius of convergence of the moment generating function $\mathbb{E}(e^{tW^b})$ of W^b. Then

$$\mathcal{M}^{\phi_b}(\mathcal{T})= (r_{1/b})^b W \quad \text{a.s.} \tag{4.55}$$

where we make the convention that $\infty.0=0$ if $r_{1/b}=\infty$ and $W=0$. Consequently

$\mathcal{H}^{\phi_b}(\tilde{\mathcal{T}})$ is zero, positive and finite, or infinite

almost surely on $W > 0$ if and only if

$r_{1/b}$ is zero, positive and finite, or infinite.

respectively.

Proof. If $r_{1/b}= 0$, then $\mathbb{E}(e^{rW^{1/b}})=\infty$ for all $r > 0$. Proposition 4.3 shows that $\mathbb{E}[\mathcal{M}^{\phi_b}(\mathcal{T})] \le r^b$ ($\forall r>0$). Thus $\mathbb{E}(\mathcal{M}^{\phi_b}(\mathcal{T}))) = 0$, and so $\mathcal{M}^{\phi_b}(\mathcal{T})=0$ a.s.

If $0 < r_{1/b}<\infty$, Propositions 4.2 and 4.3 ensure that $\mathcal{M}^{\phi_b}(\mathcal{T}) \ge (r_{1/b})^b W$ and $\mathbb{E}[\mathcal{M}^{\phi_b}(\mathcal{T})] \le (r_{1/b})^b$. Thus $\mathbb{E}[\mathcal{M}^{\phi_b}(\mathcal{T})-(r_{1/b})^b]=0$ and $\mathcal{M}^{\phi_b}(\mathcal{T})] = (r_{1/b})^b W$ a.s., noting that $\mathbb{E}(W)=1$.

If $r(W^{1/b}) = \infty$, Proposition 4.2 implies $\mathcal{M}^{\phi_b}(\mathcal{T}) \ge r^b W$ for all $r > 0$. Thus $\mathcal{M}^{\phi_b}(\mathcal{T}) = +\infty$ if $W > 0$. This shows that (4.55) holds a.s. on $W>0$. On the other hand, by Lemma 3.1, we have $X_\sigma=0$ if $|\sigma|$ is suffciently large, almost surely on $W=0$. Thus $\mathcal{M}^{\phi_b}(\mathcal{T})=0$ a.s. if $W=0$. So (4.55) holds also a.s. on $W=0$ by the convention. □

Theorem 4.7. Let $(\sigma,X_\sigma)_{\sigma\in\mathcal{T}}$ be a self similar network generated by $(N; T_1,\dots,T_N)$. For any $\beta\in(0,\infty)$, we have

(i) If $\mathbb{E}[W^{1+1/\beta}] < \infty$, then $\mathcal{M}^{\psi}(\mathcal{T}) = +\infty$ a.s. on $W>0$, where $\psi(t)=t^{\alpha}(\log\frac{1}{t})^{\beta+\varepsilon}$, $t^{\alpha}(\log\frac{1}{t})^{\beta}(\log\log\frac{1}{t})^{\beta+\varepsilon}$, $t^{\alpha}(\log\frac{1}{t})^{\beta}(\log\log\frac{1}{t})^{\beta}(\log\log\log\frac{1}{t})^{\beta+\varepsilon}$, ... ($\forall\varepsilon>0$).

(ii) If $\mathbb{E}[W^{1/\beta}] =\infty$, then $\mathcal{M}^{\psi_{\beta-\varepsilon}}(\mathcal{T}) = 0$ a.s. where $\psi_{\beta-\varepsilon}(t) = t^{\alpha}(\log\frac{1}{t})^{\beta-\varepsilon}$.

Moreover $\mathcal{M}^{\psi_\beta}(\mathcal{T}) < \infty$ a.s.if

$$\limsup_{k\to\infty} \left\{ \sum_{\nu=[\mathrm{Log}k]}^{k} P\left[W^{1/\beta}\ge \nu\right] - \beta\,\mathrm{Log}k \right\} > -\infty.$$

Proof. This is a mere combination of propositions 4.2(ii) and 4.3(ii). □

5. The main results

5.1. flows from self-similar networks and Hausdorff measures associated

As introduced in section 2.2, for our purposes a *network* or *capacited network* $\mathcal{G}$ comprises a tree $\mathcal{T}$ with a capacity $X_i>0$ assigned to each $i \in \mathcal{T}$. We regard X_i as the maximum allowable flow through the edge of the directed gragh $\mathcal{T}$ joining the vertices $i_1 i_2 \ldots i_{k-1}$ and $i_1 \ldots i_{k-1} i_k = i$. We recall that A *flow* or *positive flow* in the network $\mathcal{G}$ is a function f: $\mathcal{T} \longrightarrow [0,\infty)$ such that

$$f(i) = \Sigma_{\,i:i^*i \in \mathcal{T}} f(i^*i) \ (i \in \mathcal{T}), \qquad (f1)$$

$$0 \leq f(i) \leq X_i \ (i \in \mathcal{T}), \qquad (f2)$$

$$0 < f(\varnothing). \qquad (f3)$$

Intuitively f represents the rate of flow of a liquid through the network. Condition (f1) reflects the fact that the amount of liquid reacting a vertex of $\mathcal{G}$ equals that leaving it, (f2) ensures that the flow through each edge does not exceed the edge capacity, and (f3) is the positivity condition, that a positive amount of liquid is able to flow through the system from $\varnothing$ to infinity. We shall principally be concerned with conditions under which a positive flow through a network exists.

The main general result on the existence of flows is the "*max-flow min cut*" theorem of Ford and Fulkerson (1962). Here, this stimulates that the maximum value of $f(\varnothing)$, given that f satisfies (f1)-(f3), is

$$\mathcal{M}_0(\mathcal{G}) := \inf_{\Gamma} \{ \Sigma_{i\in\Gamma} X_i : \Gamma \text{ is a cut-set of } \mathcal{T} \}.$$

The obvious criterion for the existence of a positive flow is

$$\mathcal{M}_0(\mathcal{G}) > 0.$$

It was remarked by Falconer (1986) that this criterion is equivalent to

$$\mathcal{M}(\mathcal{T}) > 0,$$

where

$$\mathcal{M}(\mathcal{T}) := \lim_{k\to\infty} \mathcal{M}_k(\mathcal{T}),$$

$$\mathcal{M}_k(\mathcal{T}) = \inf_{\Gamma} \{\Sigma_{i\in\Gamma} X_i : \Gamma \text{ is a cut-set of } \mathcal{T} \text{ and } |i| \geq k \ \forall i \in \Gamma\}.$$

We recall that $\tilde{\mathcal{T}}$ denotes the boundary of the tree $\mathcal{T}$ (i.e., all the infinite descendants of the members of $\mathcal{T}$), and A subset Γ of $\mathcal{T}$ is termed a cut-set if $\forall i \in \mathcal{T}$ there exists a unique $n \geq 0$ such that $i|n \in \Gamma$.

Let $\mathcal{G}=(\sigma, X_\sigma)_{\sigma \in \mathcal{T}}$ be a self-similar network generated by $Z=(N;\ T_1,...,T_N)$. Falconer (1986) proved that a flow through the network is possible with positive probability if $\mathbb{E}(\sum_i T_i) > 1$ and is a.s. impossible if $\mathbb{E}(\sum_i T_i) < 1$. Here is a more precise problem:

Given a self-similar network $\mathcal{G} = (\sigma,\ X_\sigma)_{\sigma \in \mathcal{T}}$, how to modify the capacities X_σ in a homogeneous and optimal way in some sense such that a positive flow through the network is possible? More exactly, what is the optimal (in a way) weight function ϕ: $[0,\infty) \to [0,\infty)$ for which a positive flow through the network $(\sigma,\ \phi(X_\sigma))_{\sigma \in \mathcal{T}}$ exists?

To solve this problem, we study some Hausdorff measures on the branching set $\tilde{\mathcal{T}}$ associated with the network $\mathcal{G}$. We recall that $\tilde{\mathcal{T}}$ carries a metric d defined by

$$d(i,j)= X_{i \wedge j}.$$

Let $S(x)$, α, β and W be difined as before, that is

$$S(x)= \sum_{i=1}^N T_i^x\ , \quad \text{where } \textstyle\sum_{\emptyset}:=0 \text{ and } x \in [0,\infty), \tag{S}$$

$$\alpha=\min\{a \in [0,\infty):\ \mathbb{E}[S(a)] \leq 1\},\ \text{where } \min \emptyset:=\infty, \tag{α}$$

$$\beta=\min\{b \in [0,1):\ S(\tfrac{\alpha}{1-b}) \leq 1 \text{ a.s.}\},\ \text{where } \min \emptyset:=1, \tag{β}$$

$$W:= \lim_{k \to \infty} \sum_{|\sigma|=k} X_\sigma^\alpha. \tag{W}$$

Since $\mathbb{E}(N) \leq 1 \Leftrightarrow \tilde{\mathcal{T}}=\emptyset$ a.s. $\Leftrightarrow \alpha=0$ (we exclude the degenerate case where N=1 a.s.), the only interesting case is $\mathbb{E}(N)>1$ or equvalently, $\alpha>0$. We shall always assume that

$$N<\infty \text{ a.s.,} \quad 0<\alpha<\infty \tag{N}$$

and

$$\mathbb{E}[S^2(\alpha)]<\infty \tag{L^2}$$

if it is not specified further. Then

$$p(W=0) =p(\tilde{\mathcal{T}}=\emptyset) =q,$$

q being the extinction probability of the associated branching process, which is the unique solution in [0,1) of the equation

$$\mathbb{E}[q^N] = 1.$$

Moreover

$$p(T_1 = \ldots = T_N = 1 \mid N>0) < 1, \tag{p}$$

$\mathbb{E}[S(\alpha)] \le 1$, $\mathbb{E}[S(x)]$ is continuous and strictly decreasing on $[\alpha,\infty)$, and the same holds on $[0,\infty)$ if additionally $\mathbb{E}(\mathbb{N})<\infty$ (Lemma S and Lemma α in section 3).

We shall now collect our main results. For a Hausdorff dimension function ϕ (that is, ϕ is defined on $[0,\infty)$, non-negative, increasing and continuous on the right), we denote by $\mathcal{H}^{\phi}(\tilde{\mathcal{T}})$, $\mu^{\phi}(\tilde{\mathcal{T}})$, and $\mathcal{M}^{\phi}(\mathcal{T})$ the Hausdorff measures, the sherical Hausdorff measures and the lower limit of cut-set sums of $\tilde{\mathcal{T}}$ respectively.

First of all, we translate the criterion of Ford and Fulkerson (1962) in terms of Hausdorff measures on $\tilde{\mathcal{T}}$ (see Theorem 1 in the introduction):

Theorem 5.1. Let $\mathcal{G}=(\sigma,X_\sigma)_{\sigma\in\mathcal{T}}$ *be a self-similar network generated by* $Z=(N; T_1,\ldots T_N)$, *and* ϕ: $[0,\infty) \longrightarrow [0,\infty)$ *a non-negative function, increasing and continuous from the right. Then almost surely*

a positive flow through the network $(\sigma,\phi(X_\sigma))$ $(\sigma\in\mathcal{T})$ *is possible*

if and only if

$$\mathcal{H}^{\phi}(\tilde{\mathcal{T}}) > 0$$

where $\mathcal{H}^{\phi}(.)$ *represents the Hausdorff measure on* $\tilde{\mathcal{T}}$ *associated with the dimension function* ϕ, $\tilde{\mathcal{T}}$ *carrying the metric defined above.*

Proof. Falconer (1986, Lemma 3.1) observed that a positive flow through $(\sigma,\phi(X_\sigma))$ exists if and only if $\mathcal{M}^{\phi}(\tilde{\mathcal{T}})>0$. By Theorem 2.7, we see that

$$\mathcal{H}^{\phi}(\tilde{\mathcal{T}})=\mu^{\phi}(\tilde{\mathcal{T}})= \mathcal{M}^{\phi}(\mathcal{T})$$

if $X_i \to 0$ as $|i|\to\infty$ $(\forall i\in\mathcal{T})$. It then suffices to show that

$$X_i \to 0 \text{ as } |i|\to\infty \ (\forall i\in\mathcal{T}) \text{ almost surely.}$$

Since $\mathbb{E}[S(\alpha)]\le 1$ and $\mathbb{E}[S(x)]$ is strictly decreasing on $[\alpha,\infty)$ we can choose $t>\alpha$

such that $\mathbb{E}[S(t)]<1$. Writing

$$M_k(t)= \sum_{|i|=k} X_i^t,$$

we see that $\mathbb{E}[M_k(t)|\ \mathbb{F}_{k-1}] = \mathbb{E}[S(t)]M_{k-1}(t)$. Thus $\{(\mathbb{E}[S(t)])^{-k}M_k(t),\ \mathbb{F}_k\}$ is a positive supermartingale. The martingale convergence theorem applies, yielding that $M_k \to 0$ a.s. and then the result desired. □

Corollary 5.1. If (N) holds, then

(i) $X_i \to 0$ a.s. whenever $|i| \to \infty$.

(ii) $\mathcal{H}^{\phi}(\tilde{\mathcal{T}})=\mu^{\phi}(\tilde{\mathcal{T}})= \mathcal{M}^{\phi}(\mathcal{T})$ a.s.

Proof. This is shown in the proof above. □

Remark 5.1. We have proved in fact that the result holds in the deterministic case. That is, if $G= (\sigma,C_\sigma)(\sigma\in\mathcal{T})$ is a network with $C_\sigma \to 0$ ($|\sigma|\to\infty$), then a positive flow is possible through the network $\phi(G):= (\sigma,\phi(C_\sigma))(\sigma\in\mathcal{T})$ if and only if the Hausdorff measure $\mathcal{H}^{\phi}(\tilde{\mathcal{T}})$ of the limit set $\tilde{\mathcal{T}}$ is positive. Thus in particular, a positive flow is possible through the network $G= (\sigma,(C_\sigma)(\sigma\in\mathcal{T})$ if and only if the linear Hausdorff measure $\mathcal{H}^1(\tilde{\mathcal{T}})$ of the limit set $\tilde{\mathcal{T}}$ is positive.

Our question is then to find a best dimension function ϕ to measure the branching set $\tilde{\mathcal{T}}$. As we may expect, our results will be havily dependent of the distribution of $S(x)$ defined above.

We shall see that α is in fact the Hausdorff dimension of $\tilde{\mathcal{T}}$. The following result is established as Theorem 2 in section 1.

Theorem 5.2. Suppose that $\mathbb{E}(S(\alpha)^2) < \infty$, *then*

(i) $\dim \tilde{\mathcal{T}}(\omega) = \alpha$ *a.s. on* $\tilde{\mathcal{T}}(\omega) \neq \emptyset$. *(ii)* $\mathcal{H}^{\alpha}(\tilde{\mathcal{T}}) <\infty$ *a.s. if* $0<\alpha<\infty$.

(iii) (a) If $\mathbb{E}[S(\alpha)]<1$ *then* $\mathcal{H}^{\alpha}(\tilde{\mathcal{T}})=0$ *a.s. (b) If* $\mathbb{E}[S(\alpha)]=1$ *then* $0<\mathcal{H}^{\alpha}(\tilde{\mathcal{T}})<\infty$ *if and only if* $S(\alpha)=1$ *a.s. Consequently,* $\mathcal{H}^{\alpha}(\tilde{\mathcal{T}})=0$ *a.s. if* $S(\alpha)$ *is not a.s. a constant.*

Proof. Since $\mathbb{E}[S(\alpha)]\leq 1$ and $\mathbb{E}[S^2(\alpha)]<\infty$, Theorem 4.7 applies, yielding that $\mathcal{M}^{\psi_b}(\mathcal{T})=\infty$ a.s. on $W>0$, where $\psi_b=t^{\alpha}(\log\frac{1}{t})^b$ $\forall b>1$. Thus $\mathcal{H}^a(\tilde{\mathcal{T}})= \mathcal{M}^a(\mathcal{T}) =\infty$

a.s. on $\tilde{\mathcal{T}}\neq\emptyset$ $\forall a>\alpha$, giving that dim $\tilde{\mathcal{T}}(\omega)\leq\alpha$ a.s. Conversely, as is shown in the proof of Theorem 5.1, $\{(\mathbb{E}[S(\alpha)])^{-k}M_k(\alpha), \mathbb{F}_k\}$ is a non-negative supermartingale, where $M_k(\alpha)=\sum_{|i|=k} X_i^\alpha$. Since $\mathbb{E}[S(\alpha)]\leq 1$, $M_k(\alpha)$ is a.s. bounded by martingale convergence theorem. Note that $\mathcal{M}_k^\alpha(\mathcal{T})\leq M_k(\alpha)$, letting $k\to\infty$ gives $\mathcal{M}^\alpha(\mathcal{T})<\infty$ a.s. Then $\mathcal{H}^\alpha(\tilde{\mathcal{T}})<\infty$ and dim $\tilde{\mathcal{T}}\geq\alpha$ a.s. This ends the proof of (i) and (ii).

We now prove (iii). If $\mathbb{E}(S(\alpha))<1$, then $M_k(\alpha)=\sum_{|i|=k} X_i^\alpha\to 0$ since $\mathbb{E}[S(\alpha)])^{-k}M_k(\alpha)$ converges. This gives that $\mathcal{H}^\alpha(\tilde{\mathcal{T}})=0$ a.s. If $\mathbb{E}(S(\alpha))=1$, then $(X_\sigma)_{\sigma\in T}$ is a tree martigale. By Falconer's lemma (1987, p.342, Lemma 4.4), $\mathcal{M}^\alpha(\mathcal{T})=0$ a.s. if $S(\alpha)$ is not a.s. a constant. So $\mathcal{H}^\alpha(\tilde{\mathcal{T}})=0$ a.s. Thus $0<\mathcal{H}^\alpha(\tilde{\mathcal{T}})<\infty$ implies $S(\alpha)=1$ a.s. Conversly, if $S(\alpha)=1$ a.s., then it is easy to verify that $\mathcal{M}^\alpha(\mathcal{T})=1$ a.s. on $\tilde{\mathcal{T}}\neq\emptyset$. Thus $\mathcal{H}^\alpha(\tilde{\mathcal{T}})=1$ a.s. on $\tilde{\mathcal{T}}\neq\emptyset$. □

Remark 5.2. The dimension result dim $\tilde{\mathcal{T}}=\alpha$ holds even if $\alpha=0$ or ∞. In fact, if $\alpha=0$, then $\tilde{\mathcal{T}}(\omega)=\emptyset$ a.s., the result is evident. If $\alpha=\infty$, then $\forall a>0$ $\mathbb{E}[S(a)]>1$. Thus a positive flow through the network $(\sigma, X_\sigma^a)_{\sigma\in\mathcal{T}}$ exists a.s. on $\tilde{\mathcal{T}}(\omega)\neq\emptyset$ by Falconer's result (1986, p.568, Theorem 6.3). Hence $\mathcal{H}^a(\tilde{\mathcal{T}})>0$ a.s. on $\tilde{\mathcal{T}}(\omega)\neq\emptyset$ by Theorem 5.1. Thus dim $\tilde{\mathcal{T}}(\omega)\geq a$ a.s. on $\tilde{\mathcal{T}}(\omega)\neq\emptyset$ $\forall a>0$. Therefore dim $\tilde{\mathcal{T}}(\omega)=\infty$ a.s. on $\tilde{\mathcal{T}}(\omega)\neq\emptyset$.

For $b\in[0,\infty]$, let us write

$$\phi_b(t) = t^\alpha(\log\log\tfrac{1}{t})^b, \text{ where } \phi_\infty(t):=+\infty. \qquad (\phi_b)$$

We shall denote by $r_b=r(W^b)$ the radius of convergence of the moment generating function $\mathbb{E}(e^{tW^b})$ of W^b $(0<b<\infty)$. Since now, we suppose always that

$$\mathbb{E}(S(\alpha))=1.$$

Theorem 5.3. (*The fundamental theorem: a necessary and sufficient condition for $\mathcal{H}^{\phi_b}(\tilde{\mathcal{T}})$ to be zero, positive and finite or infinite*) Let $(\sigma,X_\sigma)_{\sigma\in\mathcal{T}}$ be a self similar network generated by $(N; T_1,\ldots,T_N)$ with

$$\mathbb{E}(\textstyle\sum_{i=1}^N T_i^\alpha\log\frac{1}{T_i})<\infty. \qquad (T\log\tfrac{1}{T})$$

Then for all $b\in(0,\infty)$, we have

$$\mathcal{H}^{\phi_b}(\tilde{\mathcal{T}})= \mathcal{M}^{\phi_b}(\mathcal{T})= (r_{1/b})^b W \quad \text{a.s.} \tag{$\mathcal{H}$}$$

where, if $r_{1/b}=\infty$, we make the convention that $\infty.0=0$ and suppose addtionally

$$\mathbb{E}(S(\alpha)^2)<\infty. \tag{L^2}$$

Consequently, whenever $(T\log\frac{1}{T})$ holds,

$\mathcal{H}^{\phi_b}(\tilde{\mathcal{T}})$ is zero, positive(and finite) or infinite

almost surely on $W>0$ if and only if

$r_{1/b}$ is zero, positive(and finite) or infinite

respectively.

Proof. We first note that $\mathcal{H}^{\phi}(\tilde{\mathcal{T}})=\mu^{\phi}(\tilde{\mathcal{T}})= \mathcal{M}^{\phi}(\mathcal{T})$ a.s. by Corollary 5.1. The result then follows by Theorem 4.6 and Lemma 3.1. □

Remark 5.3. The condition $(T\log\frac{1}{T})$ is implied by $\mathbb{E}(S(x_0))<\infty$ for some $0\le x_0<\alpha$. In particular, it holds if $\mathbb{E}(N)<\infty$.

As a direct consequence of Theorem 5.3 we have

Corollary 5.3. (*critical value of* β) Suppose that the conditions (L^2) and $(T\log\frac{1}{T})$ hold, and put

$$\beta^*= \sup\{b>0:\ r(W^{1/b})=0\} \equiv \inf\{b>0:\ r(W^{1/b})=+\infty\}^1.$$

Then $0 \le \beta^* \le +\infty$ and

$$\mathcal{H}^{\phi_b}(\tilde{\mathcal{T}})= \begin{cases} 0 & \text{if } b<\beta^* \\ +\infty & \text{if } b>\beta^* \end{cases}$$

almost surely on the event that the tree process does not terminate. Moreover

$$0 < \mathcal{H}^{\phi_b}(\tilde{\mathcal{T}}) < \infty \text{ a.s. on } \tilde{\mathcal{T}} \neq \emptyset$$

if and only if $0 < r(W^{1/\beta^*}) < \infty$, provided that $0<\beta^*<\infty$.

Proof. By Theorem 5.3, it suffices to prove that almost surely $W>0$ if and

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[1] It can be easily verified that

$$\sup\{\beta>0:\ r(W^{1/\beta})=0\} \equiv \inf\{\beta>0:\ r(W^{1/\beta})=+\infty\}.$$

By convention we write $\sup \emptyset = 0$ and $\inf \emptyset = +\infty$.

only if $\tilde{\mathcal{T}} \neq \emptyset$. But this is so by Lemma 3.1 under the condition (L^2). □

Using Theorem 3.3, we see that $\beta^*=\beta$ if $\|N\|_\infty<\infty$ and $\beta<1$, and $\beta^*\geq\beta$ if $\|N\|_\infty=\infty$ or $\beta=1$. The following result is stated in the introduction (Theorem 3).

Theorem 5.4. Suppose that (L^2) and $(T\log\frac{1}{T})$ hold.

(i) If $\beta=0$, then $\mathcal{H}^{\phi_\beta}(\tilde{\mathcal{T}}) \equiv \mathcal{H}^{\alpha}(\tilde{\mathcal{T}}) =1$ a.s. on $\tilde{\mathcal{T}}\neq\emptyset$;

(ii) If $\|N\|_\infty<\infty$ and $0<\beta<1$, then $\mathcal{H}^{\phi_b}(\tilde{\mathcal{T}}) >0$ a.s. on $\tilde{\mathcal{T}}\neq\emptyset$ if and only if $b\geq\beta$.

(iii) If $\|N\|_\infty=\infty$ or $\beta=1$, then $\mathcal{H}^{\phi_b}(\tilde{\mathcal{T}}) =0$ a.s. on $\tilde{\mathcal{T}}\neq\emptyset$ if $0\leq b<\beta$.

Proof. The results follow directly from theorems 3.3, 5.2 and 5.3, noting that almost surely $W>0$ if and only if $\tilde{\mathcal{T}} \neq \emptyset$ under the assumption $\mathbb{E}(S(\alpha)^2) < \infty$ (Lemma 3.1). □

The Theorem below will prove very powerful to find exact dimension functions of random fractal sets in Eclidian space. The result has been stated as Theorem 4 in the introduction.

Theorem 5.5. Suppose that $\mathbb{E}(S(\alpha)^2) < \infty$.

(i) If $\beta=0$ then $\mathcal{H}^{\phi_\beta}(\tilde{\mathcal{T}}) =1$ a.s. on $\tilde{\mathcal{T}}\neq\emptyset$;

(ii) If $\beta>0$ and $(T\log\frac{1}{T})$ hold, then $0< \mathcal{H}^{\phi_\beta}(\tilde{\mathcal{T}}) <\infty$ a.s. on $\tilde{\mathcal{T}}\neq\emptyset$ if and only if $0<r(W^{1/\beta})<\infty$. If additionally $\|N\|_\infty<\infty$ and $0<\beta<1$, the condition reduces to $r(W^{1/\beta})<\infty$, which holds if there exists $n>1$ such that

$$\prod_{i=1}^{\infty} \frac{\mathbb{E}[S(\alpha)^{n^i} \mid N=n]^{1/n^i}}{n^\beta} > 0, \qquad (L_n)$$

or equivalently

$$\sum_{i=1}^{\infty} \left(1-\frac{\mathbb{E}[S(\alpha)^{n^i} \mid N=n]^{1/n^i}}{n^\beta}\right) < \infty. \qquad (L_n)'$$

Proof. This is a combination of Theorem 5.3, Lemma 3.1, Theorem 3.3 and Theorem 3.5. □

Remark 5.5. To calculate α and β, we note that α is the least solution of the equation $\mathbb{E}(S(a))=1$ and β is that of ess sup $S(\frac{\alpha}{1-b})=1$ if there are solutions.

We give now a series of corollaries which answer some questions of several authors, improve or generalize some of their results.

Corollary 5.5.1. Let $\mathcal{G}= (\sigma, X_\sigma)_{\sigma\in\mathcal{T}}$ be a self-similar network generated by $Z= (N;T_1,...T_N)$. Put $\phi_b(t) = t^\alpha(\log\log \frac{1}{t})^b$, where $0\le b<\infty$, and suppose that the conditions (L^2) and $(T\log\frac{1}{T})$ hold. Then the function $\phi_\beta(t) = t^\alpha(\log\log \frac{1}{t})^\beta$ is the optimal weight function for the existence of positive flows through the network in that

(i) if $b<\beta$, a positive flow through the network $(\sigma, \phi_b(X_\sigma))_{\sigma\in\mathcal{T}}$ is a.s. impossible;

(ii) if $b\ge\beta$ a positive flow through the network $(\sigma, \phi_b(X_\sigma))_{\sigma\in\mathcal{T}}$ is a.s. possible on the event that the tree process does not terminate. Here, in the case where $\|N\|_\infty=\infty$ or $\beta=1$, we suppose additionally that $r_{1/\beta}>0$;

(iii) If $\phi(t)\ge0$ is a function smaller then ϕ_β in that $\lim_{t\to0} \phi(t)/\phi_\beta(t)=0$, then a positive flow through the network $(\sigma,\phi(X_\sigma))$ $(\sigma\in\mathcal{T})$ is a.s.impossible, if additionally $r_{1/\beta}<\infty$, or more particularly, if (L_n) holds.

Proof. The first two parts comes directly from Theorems 5.1 and 5.4. The last part comes from Theorem 5.5, noting that if $\mathcal{H}^{\phi_\beta}(\tilde{\mathcal{T}}) <\infty$ then $\mathcal{H}^{\phi}(\tilde{\mathcal{T}}) =0$ for all ϕ such that $\lim_{t\to0} \phi(t)/\phi_\beta(t)=0$. □

This answers our original question.

If ϕ is a non-negative function defined on $[0,\infty)$, we shall study limit behavious of cut-set sums $\Sigma_{\sigma\in\Gamma}\ \phi(X_\sigma)$. We recall that

$$\mathcal{M}^\phi(\mathcal{T}):= \lim_{k\to\infty} \mathcal{M}_k(\mathcal{T}),$$

where $\mathcal{M}_k(\mathcal{T})= \inf\ \{\Sigma_{\sigma\in\Gamma}\ \phi(X_\sigma): \Gamma$ is a cut-set of $\mathcal{T}$ and $|\sigma|\ge k\ \forall\sigma\in\Gamma\}$.

For a self-similar network $\mathcal{G}= (\sigma,X_\sigma)$ $(\sigma\in\mathcal{T})$, after showing that $\mathcal{M}^\phi(\mathcal{T})=0$ a.s. if $\phi(t)=t^\alpha$ and $\mathcal{M}^\phi(\mathcal{T})=\infty$ a.s. on $\tilde{\mathcal{T}}\neq\varnothing$ if $\phi(t)= t^\alpha(\log\frac{1}{t})^a$ $(\forall a>1)$ under some conditions on Z (see Remark 5.5.2 below), Falconer (1987) suggested the question that what is the exact function ϕ for which $0<\mathcal{M}^\phi(\mathcal{T})<\infty$ a.s. on $\tilde{\mathcal{T}}\neq\varnothing$? The following result answers this question:

Corallary 5.5.2. Let $\mathcal{G}=(\sigma,X_\sigma)$ $(\sigma\in\mathcal{T})$ be a self-similar network. Then $0<\mathcal{M}^{\phi_\beta}(\mathcal{T})<\infty$ a.s. on $\tilde{\mathcal{T}}\neq\emptyset$ if and only if $0<r(W^{1/\beta})<\infty$. If $\|N\|_\infty<\infty$ and $\beta<1$, the condition reduces to $r(W^{1/\beta})<\infty$, which holds if there exists n>1 such that $\prod_{i=1}^{\infty}\frac{\mathbb{E}[S(\alpha)^{n^i}\,|\,N=n]^{1/n^i}}{n^\beta}>0$. Moreover, we have $\mathcal{M}^{\phi_\beta}(\mathcal{T})=(r_{1/\beta})^\beta W$ a.s.

Proof. Directly from Theorems 5.1 and 5.5. □

Remark 5.5.2. Falconer (1987, Corallary 5.3) has proved that if $S(\alpha)$ is not a.s. a constant, $\mathbb{E}[(S(\alpha))^2]<\infty$ and

$$T_i\leq\bar{\gamma}\text{ for some }\bar{\gamma}<1,\ \forall 1\leq i\leq N, \tag{5.1}$$

then
$$\mathcal{M}^{\psi_a}(\mathcal{T})=\infty\text{ a.s. on }\tilde{\mathcal{T}}\neq\emptyset \tag{5.2}$$

for all a>1, where $\psi_a(t)=t^\alpha(\log\frac{1}{t})^a$. Corollary 5.5.2 implies in fact that (5.2) holds for all a>0 whenever

$$S(M)\leq 1\text{ a.s. for some }M>0 \tag{5.3}$$

and $S(\alpha)$ is not a.s. a constant, since in this case $0<\beta<1$ and $\mathcal{M}^{\phi_\beta}(\mathcal{T})>0$ a.s. on $\tilde{\mathcal{T}}\neq\emptyset$.

If $\tilde{\mathcal{T}}$ carries the metric $d_2(i,j)=2^{-|i\wedge j|}$ and N is of geometric distribution, Hawkes (1981) proved that $\mathcal{H}^{\phi_1}(\tilde{\mathcal{T}})=W$ a.s. The author (1992) has recently extended this result to the general case where N is of arbitrary distribution, solving a conjecture of Hawkes. We remark that Theorem 4 applies for $X_i=2^{-|i|}$ $(Z=(N;\frac{1}{2},\ldots,\frac{1}{2})$ yielding that

Corollary 5.5.3. If $\tilde{\mathcal{T}}$ carries the metric $d_2(i,j)=2^{-|i\wedge j|}$, then $0<\mathcal{H}^{\phi_\beta}(\tilde{\mathcal{T}})<\infty$ a.s. $\tilde{\mathcal{T}}\neq\emptyset$, where $\phi_\beta(t)=t^\alpha(\log\log\frac{1}{t})^\beta$ with $\alpha=\log\mathbb{E}(N)/\log 2$ and $\beta=1-\log\mathbb{E}(N)/\log\|N\|_\infty$ if either of the following conditions holds:

(i) $\|N\|_\infty<\infty$ (thus $\beta<1$);

(ii) $\|N\|_\infty=\infty$ (thus $\beta=1$), $\mathbb{E}(e^{tN})<\infty$ for sufficiently small t>0 and $\mathbb{E}(e^{tN})=\infty$ for sufficiently large t>0.

Moreover, we have $\mathcal{H}^{\phi_\beta}(\tilde{\mathcal{T}})=(r_{1/\beta})^\beta W$ a.s.

Proof. (i) The equations $\mathbb{E}(S(\alpha))=1$ and ess sup $S(\frac{\alpha}{1-\beta})=1$ give

$$\alpha=\log\mathbb{E}(N)/\log 2\text{ and }\beta=1-\log\mathbb{E}(N)/\log\|N\|_\infty.$$

It then suffices to prove that (L_n) holds with $n=\|N\|_\infty$. In fact $\mathbb{E}(S(\alpha)^k \mid N=n)=$ $(\frac{1}{2})^{\alpha k}n^k$, $[\mathbb{E}(S(\alpha)^k \mid N=n)]^{1/k}=(\frac{1}{2})^{\alpha}n=n^\beta$, thus (L_n) holds evidently.

(ii) The caculation of α is the same as above. Since $\|S(x)\|_\infty=\infty$ $\forall x\geq 0$, then $\beta=1$. The proof will be completed if $0<r(W)<\infty$. But this is the case under the given condition on N by Lemma 3.1 in LIU (1992). □

If W is very large, $\mathbb{E}(W^p) = \infty$ for $p>0$ sufficiently large, say, then $r(W^{1/b}) = 0$ $\forall b>0$, so $\mathcal{H}^{\phi_b}(\tilde{\mathcal{T}}) = 0$ a.s. by Theorem 5.5. Hence the function ϕ_b is too small to meet our needs. The following result is to deal with this case. The result improves also that of Falconer (1987).

Theorem 5.6. Suppose that the condition (L^2) holds. Put

$$\psi_a(t) = t^{\alpha}(\log\frac{1}{t})^a \tag{5.4}$$

($\forall a>0$) and

$$\lambda_s = \sup\{p>0:\ \mathbb{E}(S^p(\alpha))<\infty\},\quad \lambda = \sup\{p>0:\ \mathbb{E}W^p<\infty\} \tag{5.5}$$

then $2\leq [\lambda_s]\leq\lambda\leq\lambda_s \leq\infty$ and

(i) $\mathcal{H}^{\psi_a}(\tilde{\mathcal{T}})= 0$ a.s. if $a< 1/\lambda$;

(ii) $\mathcal{H}^{\psi_a}(\tilde{\mathcal{T}})=\infty$ a.s. on $\tilde{\mathcal{T}} \neq \emptyset$ if $a>1/(\lambda-1)$;

(iii) Suppose that $0<\lambda<\infty$, then $\mathcal{H}^{\psi_{1/\lambda}}(\tilde{\mathcal{T}})<\infty$ a.s. if

$$\limsup_{k\to\infty} \left\{ \sum_{\nu=[\mathrm{Log}k]}^{k} P\left[W^{\lambda}\geq \nu\right] - \frac{1}{\lambda}\,\mathrm{Log}k \right\} > -\infty. \tag{5.6}$$

Proof. First of all, by Lemma 3.1(ii), we have $2\leq [\lambda_s]\leq\lambda\leq\lambda_s$. If $0<\lambda<\infty$, the results come from Proposition 4.7. If $\lambda=\infty$, the results mean $\mathcal{H}^{\psi_a}(\tilde{\mathcal{T}}) =0$ $\forall a<0$. This is immediate since $\mathcal{H}^{\alpha}(\tilde{\mathcal{T}})<\infty$ a.s. □

Remark 5.6. We have in fact that $\mathcal{H}^{\psi}(\tilde{\mathcal{T}})=\infty$ a.s. on $\tilde{\mathcal{T}}\neq\emptyset$, where $\psi(t)=$ $t^{\alpha}|\log t|^a$, $t^{\alpha}|\log t|^{1/\lambda}(\log|(\log|t|)|)^a$, etc $\forall a>1/\lambda$. The proof is similar as in the above, using Proposition 4.2(ii).

The following corollary shows that Falconer's result (5.2) can be extended to $a>1/2$ even if the assumption (5.1) is completly removed.

Corollary 5.6. (i) If $\mathbb{E}(S(\alpha)^2)<\infty$, then $\mathcal{H}^{\psi_a}(\tilde{\mathcal{T}}) = \infty$ a.s. on $\tilde{\mathcal{T}} \neq \emptyset$ $\forall a>1/2$.

(ii) If $\mathbb{E}(S(\alpha)^k)<\infty$ for all $k\in\mathbb{N}$, then $\mathcal{H}^{\psi_a}(\tilde{\mathcal{T}}) = \infty$ a.s. on $\tilde{\mathcal{T}} \neq \emptyset$ $\forall a>0$.

The next application is to the study of Hausdorff measures of some fractal sets in Euclidian space, which we state in the following section.

6. Application to a random construction of fractal sets in $\mathbb{R}^m$.

The results above will prove powerful to find exact dimension functions of some fractal sets in Euclidian space.

6.1. *Net fractals*

This section describes the construction and basic properties of a class of fractals (not as yet random) obtained by generizing the classical construction of the 'middle third' Cantor set. Such sets, which occur frequently in theory and in practice, will be termed *net fractals*, following Falconer (1986, §7). It is high time that this class of sets had a name, and this has been choosen because of the closely related net measures (see Falconer (1985, chapter 5) or Rogers (1970, §2.7)). The model is quite similar to that of Maudin and Williams (1986), see also Graf, Mauldin and Williams (1988), but here we emphasize the net measures.

Let $\mathcal{T}$ be a tree and let $\tilde{\mathcal{T}}$ be the associated set of infinite sequences (see section 2). Let

$$J= \{I_i: i\in\mathcal{T}\}$$

be a collection of compact subsets of $\mathbb{R}^m$, partially ordered by inclusion and indexed by $\mathcal{T}$, so that $I_{i'}\subset I_i$ whenever $i<i'$. In particular, there is a set $I_\emptyset$ with $I_i\subset I_\emptyset$ for all $i\in\mathcal{T}$. (We do not at this juncture demand that I_i and $I_{i'}$ be disjoint if i and i' are incomparable under <.)

Write $|\ |$ for the diameter of subsets of $\mathbb{R}^m$. We always assume that $|I_i|>0$ if $i\in\mathcal{T}$. Usually we have

$$|I_{i_1,i_2,\ldots,i_r}| \longrightarrow 0 \quad \text{as} \quad r \longrightarrow \infty \quad \text{if} \quad i=i_1,i_2,\ldots,i_r,\ldots \in \tilde{\mathcal{T}} \tag{6.1}$$

If this is the case, the convergence is uniform for **i** by Dini's theorem. So given $\delta>0$ there exists $k(\delta)$ such that

$$|I_{\mathbf{i}}| \le \delta \quad \text{whenever} \quad |\mathbf{i}| \ge k(\delta). \tag{6.2}$$

Moreover $\cap_{r=1}^{\infty} I_{\mathbf{i}|r}$ is a single point of $\mathbb{R}^m$ for each $\mathbf{i}\in\tilde{\mathcal{T}}$ and the mapping

$$g: \tilde{\mathcal{T}} \to \mathbb{R}^m; \quad g(\mathbf{i}) = \cap_{r=1}^{\infty} I_{\mathbf{i}|r}$$

is continuous on $(\tilde{\mathcal{T}},\tau)$. The compact set

$$K := g(\tilde{\mathcal{T}}) \equiv \cap_{k=0}^{\infty} \cup_{\mathbf{i}\in\mathcal{T}_k} I_{\mathbf{i}}$$

is termed a *net fractal* constructed on the tree $\mathcal{T}$ with the sets $\{I_{\mathbf{i}}: \mathbf{i}\in\mathcal{T}\}$. (For the identity in the above, see Falconer 1986, Lemma 7.1). In general such a set will be a fractal by any reasonable definition. Observe that a given net fractal K may be constructed on many different trees and with many different collections of sets $\{I_{\mathbf{i}}\}$. Net fractals are almost invariably obtained from such nest of sets.

We say that $\mathcal{T}$ and $\{I_{\mathbf{i}}: \mathbf{i}\in\mathcal{T}\}$ provide a *proper* construction for the net fractal K if $K \subset \cup_{\mathbf{i}\in\Gamma}^{\infty} I_{\mathbf{i}}$ implies that Γ is a complete collection of sequences (Falconer 1986). Equivalently, this is the case if for any minimal set (cut-set) of sequences Γ, each set of the collection $\{I_{\mathbf{i}}: \mathbf{i}\in\Gamma\}$ contains a point of K that lies in no other set of the collection, or again equivalently, if this holds for $\Gamma=\mathcal{T}_k$, $k=1,2,\ldots$ We shall say that K is a *proper net fractal,* without reference to the underlying sets, when it is clear what construction is being used.

Given a net **J** and a Hausdorff dimension function ϕ, the *net measure* $\nu^{\phi}(F)$ of a subset $F\subset K$ is defined by

$$\nu^{\phi}(F) = \lim_{\delta\to 0} \nu^{\phi}_{\delta}(F), \tag{6.3}$$

where

$$\nu^{\phi}_{\delta}(F) = \inf_{\Gamma} \{ \sum_{\mathbf{i}\in\Gamma} \phi(|I_{\mathbf{i}}|): F\subseteq \bigcup_{\mathbf{i}\in\Gamma} I_{\mathbf{i}}, \text{ and } |I_{\mathbf{i}}|\le\delta \text{ if } \mathbf{i}\in\Gamma\}. \tag{6.4}$$

By a similar proof as in Proposition2.5, if (6.1) holds, we have

$$v^{\phi}(F) = \lim_{k\to\infty} v_k^{\phi}(F), \tag{6.3)'$$

where

$$v_k^{\phi}(F) = \inf_{\Gamma} \{ \sum_{i\in\Gamma} \phi(|I_i|): F\subseteq \bigcup_{i\in\Gamma} I_i, \text{ and } |i|\geq k \text{ if } i\in\Gamma\}. \tag{6.4)'$$

Then v^{ϕ} is an out measure of Hausdorff type on subsets of K and the Borel sets are v^{ϕ} measurable, see for example Rogers (1970). If $\phi(t)=t^s$ $(s\geq 0)$, we write v^s for v^{ϕ} and $v^s(F)$ for $v^{\phi}(F)$. We call $v^s(F)$ the s-dimensional net measure of F. The Hausdorff dimension of F *with respect to the net* J is by definition the quantity

$$\dim_v F = \inf \{s\geq 0: v^s(F)=0\} = \sup \{s\geq 0: v^s(F)=0\}. \tag{6.5}$$

Recall that the ordinary Hausdorff measure $\mathcal{H}^{\phi}(F)$ of F with respect to the measure function ϕ is defined by

$$\mathcal{H}^{\phi}(F) = \lim_{\delta\to 0} \mathcal{H}_{\delta}^{\phi}(F), \tag{6.3)H$$

where

$$\mathcal{H}_{\delta}^{\phi}(F) = \inf \{ \sum_{i=1}^{\infty} \phi(|U_i|): F\subseteq \bigcup_{i=1}^{\infty} U_i,\ |U_i|\leq \delta\ \forall i\geq 0\}, \tag{6.4)H$$

and the Hausdorff dimension dim F is given by

$$\dim F = \inf \{s\geq 0: \mathcal{H}^s(F)=0\} = \sup \{s\geq 0: \mathcal{H}^s(F)=0\}, \tag{6.5)H$$

$\mathcal{H}^s(F)$ being the s-dimensional Hausdorff measure of F.

By (6.2) any covering in the definition of $v_{k(\delta)}^{\phi}(F)$ is an admissable covering in the definition of $\mathcal{H}_{\delta}^{\phi}(F)$, so

$$\mathcal{H}_{\delta}^{\phi}(F) \leq v_{k(\delta)}^{\phi}(F),$$

leading to

$$\mathcal{H}^{\phi}(F) \leq v^{\phi}(F) \tag{6.6}$$

for $F\subset K$. Thus

$$\dim F \leq \dim_v F. \tag{6.7}$$

We shall need some assuptions to allow inequalities (6.6) and (6.7) to be reserved. First, Assume that

$$I_i = \overline{\text{int } I_i} \quad (i\in \mathcal{J}) \tag{6.8}$$

(the bar denoting closure in $\mathbb{R}^n$), and also that the open sets

$$\{\text{int } I_i: i\in\mathcal{T}\} \text{ form a net,} \tag{6.9}$$

that is int $I_i \supset$ int $I_{i'}$, if $i<i'$, but int $I_i \cap$ int $I_{i'} = \emptyset$ if neither $i<i'$ nor $i'<i$. Next we require a condition to ensure that I_i do not become small too rapidly. Assume that there is a constant $a>0$ such that

$$a|I_i| \le |I_{i,i}| \le |I_i| \quad \text{if } i,i\in\mathcal{T}. \tag{6.10}$$

Finally assume that there are $\eta>0$ and $\chi\ge1$, independent of i, such that

$$\text{inradius }(I_i) \ge \eta|I_i|^{\chi} \ (i\in\mathcal{T}). \tag{6.11}$$

Very often we can take $\chi=1$, so that

$$\text{inradius }(I_i) \ge \eta|I_i| \quad (i\in\mathcal{T}), \tag{6.12}$$

but it is useful to allow the possibility of $\chi>1$ enabling estimates of dimension to be made in 'non-linear' cases.

If K is a net fractal constructed from sets $\{I_i: i\in\mathcal{T}\}$ satisfying (6.1) and (6.8)-(6.11), we say that conditions (CN_χ) hold, following Falconer (1986). It was proved by Falconer (1986) that under the conditions (CN_χ) there is a constant $c>0$ such that

$$cv^{s+(\chi-1)m}(F) \le \mathcal{H}^s(F) \le v^s(F) \tag{6.13}$$

for any $F\subset K$, and so

$$\dim_v K-(\nu-1)m \le \dim K \le \dim_v K. \tag{6.14}$$

A slight change in the proof of Falconer can imply the following

Lemma 6.1. Let K be a net fractal constructed from a collection of sets satisfying (CN_χ), and $L=L(t)$ be a non-decreasing positive function, then there is a constant $c>0$ such that

$$cv^h(F) \le \mathcal{H}^\psi(F) \le v^h(F) \tag{6.15}$$

for any $F\subset K$, where $h=h(t)=t^{s+(\chi-1)m}L(t)$ and $\psi(t)=t^sL(t)$.

Let $\mathcal{G}$ be the network formed by the tree $\mathcal{T}$ and capacities $X_i=|I_i|$, $i\in\mathcal{T}$. Note that if K is proper, then (6.4)' gives

$$v_k^\phi(K) = \inf_\Gamma \{\sum_{i\in\Gamma}\phi(|I_i|): K\subseteq \bigcup_{i\in\Gamma} I_i, \text{ and } |i|\ge k \text{ if } i\in\Gamma\}.$$

$$= \inf_\Gamma \{\sum_{i\in\Gamma}\phi(|I_i|): \Gamma \text{ is complete in } \mathcal{T} \text{ and } |i|\ge k \text{ if } i\in\Gamma\}. \tag{6.4"}$$

Therefore $\nu^{\phi}_k(K)=\mathcal{M}^{\phi}_k(\mathcal{T})$ by Lemma 2.2, and consequently

Lemma 6.2. Let K is a proper net fractal constructed from sets $\{I_i;i\in\mathcal{T}\}$ and $\mathcal{G}=(i,|I_i|)$ $(i\in\mathcal{T})$ be the associated network. If (6.1) holds, then

$$\nu^{\phi}(K)=\mathcal{M}^{\phi}(\mathcal{T})=\mathcal{H}^{\phi}(\bar{\mathcal{T}}). \tag{6.16}$$

Thus the results on Hausdorff measures of the limit set $\bar{\mathcal{T}}$ apply well for the proper net fractal K.

6.2. Random construction

We now randomize the construction. Let $\mathcal{T}=\mathcal{T}(\omega)$ $(\omega\in\Omega)$ be a random tree generated by N. Fix $I_{\varnothing}$. Suppose $J=\{I_i(\omega):i\in\mathcal{T}(\omega)\}$ provide a proper construction for a net fractal $K(\omega)$ for each $\omega\in\Omega$. We obtain then a natural random network $\mathcal{G}(\omega)$ formed by the random tree $\mathcal{T}$ and the capacities $X_i=|I_i|$, $i\in\mathcal{T}$. The construction is termed *self-similar* if so is the corresponding network.

By Lemma 6.2, all the results in section 5 on Hausdorff measures of the branching set $\bar{\mathcal{T}}$ apply well for net measures of proper net fractals formed by the construction above. For example, using Theorem 5.5 and Lemma 6.1 we have

Theorem 6.3. Let $K(\omega)$ be a proper fractal generated by a self-similar construction $\{I_i: i\in\mathcal{T}\}$ and write

$$Z_\sigma=(N_\sigma;\ T_{\sigma*1},\dots,T_{\sigma*N_\sigma}),\ \text{ where }\ T_{\sigma*i}=|I_{\sigma*i}|/|I_\sigma|\ (1\le i\le N_\sigma),$$

the defining elements of the assosiated network. We write Z for $Z_{\varnothing}$.

Let S(x), α, β, W and ϕ_b be difined as before, that is

$$S(x)=\textstyle\sum_{i=1}^{N} T_i^x\ ,\ \text{ where } \sum_{\varnothing}:=0 \text{ and } x\in[0,\infty), \tag{S}$$

$$\alpha=\min\{a\in[0,\infty): \mathbb{E}[S(a)]\le 1\},\ \text{where } \min\varnothing:=\infty, \tag{α}$$

$$\beta=\min\{b\in[0,1): S(\tfrac{\alpha}{1-b})\le 1 \text{ a.s.}\},\ \text{where } \min\varnothing:=1, \tag{β}$$

$$W:=\lim_{k\to\infty}\textstyle\sum_{|\sigma|=k}\prod_{i=1}^{|\sigma|} T^{\alpha}_{\sigma|i}. \tag{W}$$

$$\phi_b(t)=t^{\alpha}(\log\log\tfrac{1}{t})^b. \tag{ϕ_b}$$

(i) If $\mathbb{E}(S(\alpha)^2)<\infty$, then $\dim_V K(\omega)=\alpha$ a.s. on $K\ne\varnothing$.

(ii) If additionally $0<\alpha<\infty$ and $\mathbb{E}(\sum_{i=1}^{N} T_i^{\alpha}\log\frac{1}{T_i})<\infty$, then

(a) $\nu^{\phi_b}(K)>0$ a.s. on $K\neq\emptyset$ if and only if $b\geq\beta$, provided that $\|N\|_{\infty}<\infty$ and $0<\beta<1$. In the case where $\|N\|_{\infty}=\infty$ or $\beta=1$, we have $\nu^{\phi_b}(K)=0$ a.s. $\forall 0\leq b<\beta$.

(b) If $\beta=0$, then $0<\nu^{\phi_\beta}(K)<\infty$ a.s.on $K\neq\emptyset$ if and only if $\mathbb{E}(S(\alpha))=1$; In the case where $\mathbb{E}(S(\alpha))=1$, $\nu^{\phi_\beta}(K)\equiv\nu^{\alpha}(K)=1$ a.s.on $K\neq\emptyset$.

(c) If $\beta>0$ and $\mathbb{E}(S(\alpha))=1$, then $0<\nu^{\phi_\beta}(K)<\infty$ a.s.on $K\neq\emptyset$ if and only if $0<r(W^{1/\beta})<\infty$. When $\|N\|_{\infty}<\infty$ and $0<\beta<1$, the condition reduces to $r(W^{1/\beta})<\infty$, which holds if there exists $n>1$ such that

$$\prod_{i=1}^{\infty}\frac{\mathbb{E}[S(\alpha)^{n^i}\,|\,N=n]^{1/n^i}}{n^{\beta}} > 0, \qquad (L_n)$$

or equivalently

$$\sum_{i=1}^{\infty}\left(1-\frac{\mathbb{E}[S(\alpha)^{n^i}\,|\,N=n]^{1/n^i}}{n^{\beta}}\right) < \infty. \qquad (L_n)'$$

Moreover, we have

$$\nu^{\phi_\beta}(K) = (r_{1/\beta})^{\beta}W \quad \text{a.s.}$$

(iii) If the construction satisfies the conditions (6.8)-(6.10) and (6.12) for all i and for all realizations of the process, then all the conclusions above hold for the the ordinary Hausdorff measures $\mathcal{H}^{\phi}(K)$.

Remark 6.3. (i) To calculate α and β, we note that α is the least solution of the equation $\mathbb{E}(S(a))=1$ and β that of ess sup $S(\frac{\alpha}{1-b})=1$, if there are solutions.

(ii) In practice, the conditions on α and β (such as $0<\alpha<\infty$ and $\beta<1$) can be verified automatically in the calculation.

(iii) If the construction satisfies the conditions of the model of Mauldin and Williams (1986), that is, if in addition $\overline{\mathrm{int}(I_\emptyset)}=I_\emptyset$, I_σ is geometrically similar to $I_\emptyset$, $\mathrm{Int}(I_{\sigma*i})\cap\mathrm{Int}(I_{\sigma*j})=\emptyset$ ($i\neq j,\sigma\in\mathcal{T}$) for all realisations of the process, then summing volums we have $S(m)\leq1$ a.s., so $0<\alpha<m$ and $0<\beta\leq1-\frac{\alpha}{m}$ if $1<\mathbb{E}(N)<\infty$ and $S(\alpha)$ is not a.s. a constant. This proves Corollary 5 in the introduction.

(iv) The condition (L_n) holds if $\beta=\beta_n:=\log\|S(\alpha)\|_\infty/\log n$ and

$$\prod_{i=1}^{\infty}\frac{\mathbb{E}[S(\alpha)^{n^i}\,|\,N=n]^{1/n^i}}{\|S(\alpha)1_{N=n}\|_\infty}>0,$$

which is satisfied usually. The last condition means that, conditioned on N=n, the rate of convergence of the L_k norm of $S(\alpha)$ to the L_∞ norm is sufficiently large. It holds for example if $p(S(\alpha)=\|S(\alpha)1_{N=n}\|_\infty;\ N=n)>0$, or more particularly if on N=n, T_i takes only finitely many values, see Remark 3.6.

(v) In the spacial case where $N=n\geq 2$ a constant, the condition (L_n) becomes very simple:

$$\prod_{i=1}^{\infty}\frac{\mathbb{E}[S(\alpha)^{n^i}]^{1/n^i}}{\|S(\alpha)\|_\infty}>0.$$

In the next section we shall see that Theorem 6.3 can be applied to find the exact Hausdorff dimension functions of the most classical constructions of self-similar random fractals. The following theorem is to deal with the case where $\|N\|_\infty=\infty$ and $S(\alpha)$ is very large. It comes from Theorem 5.6.

Theorem 6.4. Let $K(\omega)$ be a proper fractal formed by a self-similar construction $\{I_i\colon i\in\mathcal{T}\}$ and $\mathcal{G}=(\sigma,X_\sigma)_{\sigma\in\mathcal{T}}$ the corresponding network generated by

$$Z=(N;\ T_1,\dots,T_N),\quad\text{where } T_i=|I_i|/|I_\varnothing|\ (1\leq i\leq N).$$

Suppose that $0<\alpha<\infty$, $\mathbb{E}(S^2(\alpha))<\infty$ and put

$$\lambda_s=\sup\{p>0\colon \mathbb{E}(S^p(\alpha))<\infty\},\quad \lambda=\sup\{p>0\colon \mathbb{E}(W^p)<\infty\}.$$

Then $2\leq[\lambda_s]\leq\lambda\leq\lambda_s\leq\infty$ and

$$\dim K=\alpha$$

$$\nu^{\psi_a}(K)=0 \text{ if } a<1/\lambda \quad\text{and}\quad \nu^{\psi_a}(K)=\infty \text{ if } a>1/(\lambda-1)$$

a.s. on $K\neq\varnothing$, where $\psi_a(t)=t^\alpha(\log\frac{1}{t})^a$ $(\forall a>0)$. Moreover,

$$\nu^{\psi_{1/\lambda}}(K)<\infty \quad\text{a.s.}$$

if $\mathbb{E}(W^\lambda)=\infty$ and $\varliminf_{k\to\infty}\sup\{\sum_{i=[\log k]}^{k}p[W\geq i^{1/\lambda}]-\frac{1}{\lambda}\log k\}>-\infty$.

The conclusions hold also for ordinary Hausdorff measures $\mu^{\psi_a}(K)$ if the conditions (CN_1) hold.

7. Examples

In this section, we give a series of examples to show how Theorem 6.3 enables us to caculate exact dimension functions of self-similar fractal sets. Examples 7.1 and 7.2 are generizations of the construction of classical Cantor sets. Example 7.3 is a construction of random Von-Koch curves. In example 7.4, we give a quite general construction of a random set of high connectivity. The example is taken from Falconer (1986) where the a.s. dimension is calculated. Here we give a exact dimension function. As a corollary, we obtain the exact dimension functions of Graf et al.(1988) on Mandelbrot's percolation Processes and their modified curdling. Examples 7.5-7.7 give constructions where the number of descendants may be unbounded. Example 7.8 is about the zero set of Brownian bridge. This is taken from Graf et al.(1988) where the exact dimension has been given. We take it to illustrate how the famous function $t^{1/2}(\log\log\frac{1}{t})^{1/2}$ can be obtained very easily by Theorem 6.3. In example 7.9, we give a fractal for which the functions of the type $t^{\alpha}(\log\log\frac{1}{t})^{b}$ ($\forall b>0$) are too small to be exact dimension functions. In this case, Theorem 6.4 applies, and we calculate a critical function of the type $t^{\alpha}(\log\frac{1}{t})^{a}$.

7.1. *Random Cantor set*

Let $2\le M$ and $0\le N\le M$ be integers with some random distribution. Divide the unit interval into M equal intervals and select N of these. Repeat this independently for each of the selected squares and continue, to get a random fractal *K*. The probability of the process becoming extinct is determined by the distribution of N. If $\mathbb{E}(N)\le 1$, then $K=\emptyset$ a.s. Suppose that $1<\mathbb{E}(N)<\infty$. We have

$$Z=(N;\ \frac{1}{M},\dots,\frac{1}{M}),\ S(x)=NM^{-x}.$$

$\alpha\in(0,1]$ is the unique solution of

$$\mathbb{E}(NM^{-\alpha})=1.$$

Suppose that M is independent of N and $\|M\|<\infty$. Then $\|S(x)\|_{\infty}=\|N\|_{\infty}\|M^{-1}\|^{x}$.

$\beta\in[0,1)$ is found to be

$$\beta=1+\ \alpha\log\|M^{-1}\|_\infty/\log\|N\|_\infty=\beta_n,\quad \text{where } n=\|N\|_\infty.$$

If $\beta=0$, the exact dimension function is $\phi_0(t)=t^\alpha$ since $\mathbb{E}(S(\alpha))=1$. If $\beta>0$, we note that $T_i=1/M$ takes only finitely many values, by Remark 6.3, the condition (L_n) in Theorem 6.3 holds with $n=\|N\|_\infty$. Thus in any case, Theorem 6.3 implies

$$0 < \nu^{\phi_\beta}(K) < \infty \text{ a.s. on } K\neq\emptyset.$$

Besides, the conditions (6.8), (6.9) and (6.12) hold evidently (we note that $T_i=1/M \geq 1/\|M\|_\infty$). Hence we have also

$$0 < \mathcal{H}^{\phi_\beta}(K) < \infty \text{ a.s. on } K\neq\emptyset.$$

where $\phi_\beta(t) = t^\alpha(\log\log\frac{1}{t})^\beta$, with α and β defined in the above.

The classical Cantor set corresponds to the case where $N=2$ and $M=3$ a.s. Thus $\alpha=\log2/\log3$ and $\beta=0$.

Here is a more explicite example of Falconer (1986, example 11.2): divide the unit interval into three equal parts and retain each part independently with probability p. Repeat this with the parts that remain, and so on. In this case, we have

$$M=3 \text{ a.s};\ p(N=k)=\binom{n}{k}p^k(1-p)^{n-k},\ k=0,1,2,3.$$

The extinction probability of the branching process is 1 if $p\leq1/3$ and is the solution of $(1-p+pu)^3=u$ lying between 0 and 1 if $p>1/3$; this is the probability that $K=\emptyset$. Otherwise the exact dimension function for net measures and Hausdorff measures is the function ϕ_β defined above, with

$$\alpha=\log(3p)/\log3 \equiv 1+\log p/\log3 \quad\text{and}\quad \beta=1+\alpha\log(1/3)/\log3=1-\alpha.$$

The number α has been calculated by Falconer (1986).

7.2. Remove from the unit interval a central portion so that the remaining parts have lengths $1/3 < T_1 = T_2 < 1/2$ distributed according to the probability density function *f*. From each of this parts remove a propotion distrbuted in the same way, etc. This time extinction cannot occur. Since

$$\mathbb{E}(T_1^\alpha+T_2^\alpha) = 2\int_{1/3}^{1/2} u^\alpha f(u)du,$$

equating this to 1 gives the almost dimension of the resulting fractal *K*. In the case where f(u) is uniformly distributed over (1/3, 1/2), α is the solution of

$$\alpha+1 = 12[2^{-(\alpha+1)}-3^{-(\alpha+1)}]$$

$0<\alpha<1$, and $\beta=1-\alpha$ $(=\beta_2)$ since $\|S(x)\|_\infty=2^{1-x}$. We now verify the condition (L_n) with n=2. We have

$$\mathbb{E}(S(\alpha)^k)= \int_{1/3}^{1/2} (2u^{\alpha})^k 6du = \frac{2^k 6}{\alpha k+1}(2^{-(\alpha k+1)}-3^{-(\alpha k+1)}) \geq \frac{3}{2(\alpha k+1)}2^{(1-\alpha)k}$$

Thus $$\frac{[\mathbb{E}(S(\alpha)^k)]^{1/k}}{2^{\beta}} \geq \left(\frac{3}{2(\alpha k+1)}\right)^{1/k}, \quad -\log \prod_{\substack{k=2^i \\ i=1}}^{\infty} \frac{[\mathbb{E}(S(\alpha)^k)]^{1/k}}{2^{\beta}} < \infty,$$

so the condition (L_n) holds with n=2. Theorem 6.3 gives then

$$0 < \nu^{\phi}(K) < \infty \quad \text{a.s.}$$

where $\phi(t) = t^{\alpha}(\log\log\frac{1}{t})^{1-\alpha}$. As the conditions (CN_1) hold evidently, we have also

$$0 < \mathcal{H}^{\phi}(K) < \infty \quad \text{a.s.}$$

7.3. *Random von Koch curve*

Let T be a random variable taking valus in $(\frac{1}{3}, \frac{1}{2})$ with probability density function *f*. Let F be a line segment in $\mathbb{R}^2$ and $I_\emptyset$ be the equilateral triangle based on F. Let F_1 and F_4 be (random) subintervals of lengths T obtained by removing a central portion F_c from F, and let F_2 and F_3 be the other two sides of the equilateral triangle based on F_c, always on the same side of F. Define I_i as the equilateral triangles based on F_i $(1\leq i\leq 4)$. I_i may be regarded as images of $I_\emptyset$ with respect to the similarities S_i that map F onto F_i. Repeat this process on each segment $F_i(1\leq i\leq 4)$, we obtain a random von Koch curve *K*. The polygonal curves $P_k := \bigcup_{|\sigma|=k} F_\sigma$ converge to a realization of the random fractal *K* in the Hausdorff metric and *K* is a.s. an unrectifiable Jordan curve (see Falconer 1986,p.581.). In this case N=4 a.s., $T_1=T_4=T$ and $T_2=T_3=1-2T$. The number α is determined by

$$\mathbb{E}(S(\alpha)) \equiv \int_{1/3}^{1/2} f(u)(2u^{\alpha}+2(1-2u)^{\alpha})du = 1.$$

If T is uniformly distributed on $(\frac{1}{3}, \frac{1}{2})$, then $\alpha \cong 1.444$ is determined by

$$\frac{6}{\alpha+1}(2^{-\alpha}-3^{-(\alpha+1)})=1.$$

Since $\|S(x)\|_{\infty}=2\sup\{t^{x}+(1-2t)^{x}:\ t\in(\frac{1}{3}, \frac{1}{2})\}=4(\frac{1}{3})^{x}$, we have $\beta= 1-\alpha\ \log3/\log4 =\beta_4$.

As $\mathbb{E}(S(\alpha)^k)= \int_{1/3}^{1/2} 6[(2u^{\alpha}+2(1-2u)^{\alpha})]^k du \geq \int_{1/3}^{1/2} 6[4(1-2u)^{\alpha}]^k du =(\frac{4}{3^{\alpha}})^k/(\alpha k+1)$,

$$\frac{[\mathbb{E}(S(\alpha)^k)]^{1/k}}{4^{\beta}}= \frac{[\mathbb{E}(S(\alpha)^k)]^{1/k}}{\|S(\alpha)\|_{\infty}} \geq \left(\frac{1}{\alpha k+1}\right)^{1/k},$$

it is clear that the condition (L_n) holds with n=4 (see the calculation in 7.2 above). As the conditions (CN_1) hold evidently, Theorem 6.3 shows that

$$0 < v^{\phi}(K) < \infty \quad \text{and} \quad 0 < \mathcal{H}^{\phi}(K) < \infty \quad \text{a.s.}$$

where $\alpha \cong 1.444$ and $\phi(t) = t^{\alpha}(\log\log\frac{1}{t})^{1-\alpha\log3/\log4}$.

7.4. *A random set of high connectivity*

Let $2\leq M$ and $0\leq N\leq M^2$ be integers with some random distribution. Divide the unit square into M^2 equal sqares and select N of these. Repeat this independently for each of the selected squares and continue, to get a random fractal *K*. The probability of the process becoming extinct is determined by the distribution of N. By selecting squares in an approprite way we can arrange for the sequeneces of homology groups of the sets $\bigcup_{i\in\mathcal{T}_k} I_i$ to be strictly increasing, so that the limiting set *K* has infinite connectivity. (see Falconer 1986, p.581.)

In this case, we have

$$Z=(N; \frac{1}{M},\dots,\frac{1}{M}),\ S(x)= NM^{-x}.$$

Thus the calculation is exactly the same as in section 7.1, and we obtain that, if M is independent of N, then $\alpha\in(0,2]$ is the unique solution of

$$\mathbb{E}(NM^{-\alpha})=1,$$

$$\beta=1+ \alpha\log\|M^{-1}\|_{\infty}/\log\|N\|_{\infty}=\beta_n, \quad \text{where } n=\|N\|_{\infty},$$

$\beta\in[0,1)$, and

$$0 < v^{\phi_\beta}(K) < \infty \quad \text{and} \quad 0 < \mathcal{H}^{\phi_\beta}(K) < \infty$$

a.s. on $K\neq\emptyset$, where $\phi_\beta(t) = t^{\alpha}(\log\log\frac{1}{t})^{\beta}$ with α and β defined above.

As special cases of the model, we obtain the exact dimension functions of Graf, Mauldin and Williams (1988) for Mandelbrot's Percolation Process and Modified Curdling. We note that they were obtained quite difficultly.

Special case 1: *Mandelbrot's Percolation Process* (Graf et al. 1988, example 6.2) In 1974, Mandelbrot introduced a process in $[0,1]^2$ which he called "canonical curdling". Fix a positive integer n and a positive p<1. Partition the unit square into n^2 congruent subsquares: $B_{i,j} = [(i-1)/n,\ i/n] \times [(j-1)/n,\ j/n]$; $1 \le i,\ j \le n$. Each subsquare $B_{i,j}$ "survives" independent of the others with probability p. For each subsquare which survives, rescal and apply the same procedure.

The construction is a special case of the example above with

$$M=n \quad \text{and} \quad p(N=k)= \binom{n^2}{k}\, p^k\, (1-p)^{n^2-k}.$$

Clearly $\mathbb{E}(N) = n^2p$, so $K(\omega)$ is non-empty with positive probability if and only if $p>1/n^2$. Otherwise, using the result above, the a.s. dimension α of K is determined by $\mathbb{E}(Nn^{-\alpha})=1$, that is

$$\alpha = \log\mathbb{E}(N)/\log n = \log(n^2p)/\log n = 2+\log p/\log n,$$

and the number β is determined by

$$\beta = 1+\alpha\log\|M^{-1}\|_\infty/\log\|N\|_\infty = 1+\alpha\log(n^{-1})/\log(n^2) = 1-\alpha/2.$$

Our conclusion above ensures that

$$0 < \nu^\phi(K) < \infty \quad \text{and} \quad 0 < \mathcal{H}^\phi(K) < \infty \quad \text{a.s. on } K\neq\emptyset,$$

where $\phi(t)= t^\alpha(\log\log\frac{1}{t})^{1-\alpha/2}$ and $\alpha = 2+\log p/\log n$.

Special case 2: *Modified curdling* (Graf et al. 1988, example 6.12)

Fix a positive integer n and a probability measure τ on the power set of $\{1, ..., n^2\}$. Let $J_1, ..., J_{n^2}$ be a labelling of the partition of $[0,1] \times [0,1]$ into congruent subsquares. Let $I_\emptyset=[0,1] \times [0,1]$. If the square I_σ has been constructed, then choose $A\subset \{1, ..., n^2\}$ according to τ and let $I_{\sigma*i}$, $i\in A$ be the subsquares of I_σ obtained by scaling J_i, $i\in A$ into I_σ via the natural map.

Let £ be the cardinality map from the power set $\mathcal{P}:=\mathcal{P}(\{1,...,n^2\})$ into $\{1,...,n^2\}$: £(A)= card(A) if $A\subseteq \mathcal{P}(\{1,...,n^2\})$. Example 7.4 applies with M=n and N the random variable distributed according to the image of ν by £, namely

$$p(N=k)= \nu\{A\in \mathcal{P}: £(A)=k\}.$$

Thus, using the preceding result we have

$$\alpha = \log\mathbb{E}(N)/\log n,$$

$$\beta = 1 + \alpha\log\|M^{-1}\|_\infty/\log\|N\|_\infty = 1 + \alpha\log(n^{-1})/\log\|N\|_\infty$$
$$= 1- \alpha\log n/\log\|N\|_\infty = 1- \log\mathbb{E}(N)/\log\|N\|_\infty,$$

and the exact dimension function for net measures and Hausdorff measures is $t^\alpha(\log\log\frac{1}{t})^\beta$, with α and β defined above. We remak that Graf et al.(1988) had to use their rather complicated result (their Theorem 5.2, p.78. See also pp.117-118) to obtain this function.

7.5. Let $N\geq 1$ be a random variable taking values in $\mathbb{N}$, $p(N=k)=p_k$ $0\leq p_k$, $p_1<1$ and $\sum_{k=1}^\infty p_k=1$. Let $I_\emptyset=[0,1]$ and $a\geq 1$. If N=k, we choose k equal intervals I_i such that $T_i:=|I_i|/|I_\emptyset|= 1/k^a$ and $Int(I_i)\cap Int(I_j)=\emptyset$ if $i\neq j$. Repeat this independently for each of the selected intervals, and continue, to get a fractal *K*. In this case $S(x)= NN^{-ax}=N^{1-ax}$. For $x=1/a$, $\mathbb{E}(S(x))=1$. Thus $\alpha=1/a$ is the a.s. dimension of *K*. Since $S(\alpha)=1$ a.s., $\beta=0$. The exact dimension function for net measures is then $t^{1/a}$, that is

$$0 < \nu^{1/a}(K) < \infty \quad \text{a.s.}$$

The same result holds for the ordinary Hausdorff measure $\mathcal{H}^{1/a}(K)$ if $\|N\|_\infty<\infty$ since $T_i\geq 1/\|N\|_\infty$.

7.6. Let N be an integer with some random distribution, $p(N=k)=p_k$, $p_k\geq 0$ and $\sum_{k=0}^\infty p_k=1$. Let $I_\emptyset=[0,1]$ and $0<a\leq e^{-1/e}$ =0.6922... (The last condition is to ensure that $\sup_{k\geq 0} ka^k\leq 1$). If N=k, we choose k equal intervals I_i such that $T_i:=|I_i|/|I_\emptyset|= a^k$ and $Int(I_i)\cap Int(I_j)=\emptyset$ if $i\neq j$. Repeat this independently for

each of the selected intervals, and continue, to get a fractal K. We have then $S(x)= Na^{Nx}$. If $\Sigma_{k=0}^{\infty} kp_k \leq 1$, then $K=\emptyset$ a.s. Otherwise α is the unique solution in $(0,1]$ of the eqution

$$\Sigma_{k=1}^{\infty} ka^{k\alpha}p_k = 1,$$

that is
$$\alpha = \log r/\log a,$$

where r is the unique solution in (0,1) of the equation

$$P'(r)= \frac{1}{r}, \quad \text{with} \quad P(x)= \Sigma_{i=0}^{\infty} p_i x^i,$$

and in fact $e^{-1/e}= 0.6922...<r<1$. α is the a.s. dimension of K if $K\neq\emptyset$. Besides, since $\|S(x)\|_{\infty}=\sup\{ka^{kx}: p_k>0\}$, then

$$\beta=1-\alpha\log a/\log s =1-\log r/\log s,$$

where $0<s<r$ is the solution of the equation

$$\sup\{ks^k: p_k>0\} =1.$$

A simple study on the function $y \longrightarrow ys^y$ shows that s is the solution of the equation

$$\max\{ [\tfrac{1}{\log 1/s}]s^{[\frac{1}{\log 1/s}]}, [\tfrac{1}{\log 1/s}+1]s^{[\frac{1}{\log 1/s}+1]} \} =1$$

if $p_k>0$, where $k=[\frac{1}{\log 1/s}]$, $[\frac{1}{\log 1/s}]+1$. A numerical calculation gives

$$s=0.69336... \text{ if } p_3>0.$$

Theorem 6.3 implies

$$\mathcal{H}^{\phi_b}(K)= \nu^{\phi_b}(K)=0 \text{ if } b<\beta$$

a.s., where $\phi_b(t)=t^{\alpha}(\log\log\frac{1}{t})^b$, with α and β defined above. If $\|N\|_{\infty}<\infty$, then, a.s. on $K\neq\emptyset$,

$$\mathcal{H}^{\phi_b}(K)>0 \text{ and } \nu^{\phi_b}(K)>0 \text{ if } b\geq\beta,$$

since $T_i\geq a^n$, where $n=\|N\|_{\infty}$. We remark an interesting fact that $\beta=1-\log r/\log s$ depends only on the distribution of N, but not on a.

Let us take for example the case where N is of geometric distribution: $p_k=p^{k-1}(1-p)$, $k\geq1$. Then $P(x)=\Sigma_{i=1}^{\infty} x^i p^{i-1}(1-p) = (1-p)x/(1-px)$ and

$$r=\frac{1+p-\sqrt{1+2p-3p^2}}{2p^2},\ s=0.69336...$$

($s<r<1$). A numerical calculation shows that if $a=p=1/2$, then $r=0.7639...$, $\alpha=0.3885...$, $\beta=0.2646$ and, if $a=1/2$ and $p=1/3$, then $r=0.8038...$, $\alpha=0.3151...$ and $\beta=0.4036...$, etc.

7.7. We take a construction similar to the above, but, if N=k, we choose k intervals $I_1,...,I_k$ in $I_\varnothing$ such that $T_i:=|I_i|/|T_\varnothing|=a^i$, where $0<a\leq 1/2$ is a given number. If $\mathbb{E}(N)\leq 1$, then $K=\varnothing$ a.s. Suppose that $\mathbb{E}(N)>1$. We have

$$S(x)=\Sigma_{i=1}^{N}a^{ix},\ S(0)=N \text{ and } S(x)=a^x(1-a^{Nx})/(1-a^x) \text{ if } x>0.$$

α is then given by

$$\alpha=\log r/\log a,$$

where r is the unique solution in (0,1) of the equation

$$\mathbb{E}(r^N)=2-1/r.$$

Since $\|S(x)\|_\infty=t^x(1-t^{nx})/(1-t^x)$, where $n=\|N\|_\infty\leq\infty$, we have

$$\beta=1-\alpha\log a/\log s=1-\log r/\log s,$$

where s is the unique solution in (0,1) of the equation

$$s+s^2+...+s^n=1, \quad \text{where } n=\|N\|_\infty\leq\infty.$$

If $n=\infty$, then $s=1/2$. Theorem 6.3 shows again

$$\mathcal{H}^{\phi_b}(K)=\nu^{\phi_b}(K)=0 \text{ if } b<\beta,$$

and, if $\|N\|_\infty<\infty$, then

$$\mathcal{H}^{\phi_b}(K)>0 \text{ and } \nu^{\phi_b}(K)>0 \text{ if } b\geq\beta$$

a.s. on $K\neq\varnothing$, where $\phi_b(t)=t^\alpha(\log\log\frac{1}{t})^b$, with α and β defined above. The conslution holds for $\mathcal{H}^{\phi_b}(K)$ since $T_i\geq a^n$ ($\forall i$), where $n=\|N\|_\infty$. Again, we note that $\beta=1-\log r/\log s$ depends only on the distribution of N.

If $p(N=k)=p^{k-1}(1-p)$ ($k\geq 1$ and $0<p<1$), then $r=1/(1+p)$, $s=1/2$,

$$\alpha=\log(1+p)/\log(1/a) \quad \text{and} \quad \beta=1-\log(1+p)/\log 2.$$

7.8. *The zero set of Brownian bridge*

Graf, Mauldin and Williams (1988, example 6.1) calculate the Hausdorff dimension function of the zero set of Brownian bridge, here we shall see that how we can easily calculate the exact dimension function with respect to the net measures. Let $(B_t)_{t\geq 0}$ be one -dimensional Brownian motion starting at zero. Let $B^0_t = B_t - tB_1$. Then $(B^0_t)_{0\leq t\leq 1}$ is called the Brownian bridge. Define

$$\tau_1 = \sup \{t\leq 1/2: B^0_t = 0\}$$

and

$$\tau_2 = \inf \{t\geq 1/2: B^0_t = 0\}.$$

Set $I_\varnothing = [0,1]$, $I_1=[0,\tau_1]$ and $I_2=[\tau_2,1]$. Continue this process by rescaling to each of the intervals already obtained. Due to the scaling and invariance properties of Brownian bridge the random set *K* obtained by this recursive construction is the zero-set of Brownian bridge.

Note that we have $T_1=\tau_1$ and $T_2=1-\tau_2$. As is shown by Graf, Mouldin and Williams (1988), it is not difficult to know that the distribution of (T_1,T_2) has the density function

$$\rho(v,t) = \frac{1}{2\pi} 1_{[0,1/2]\times[0,1/2]}(v,t)[vt(1-v-t)^3]^{-1/2}$$

and $\mathbb{E}(S(1/2))=1$. Thus $\alpha=1/2$. Besides

$$\|S(x)\|_\infty = (\tfrac{1}{2})^x+(\tfrac{1}{2})^x = 2^{1-x},$$

so $\beta= 1-\alpha = 1/2$ $(=\beta_2)$. Moreover

$$\mathbb{E}(S(\alpha)^k)= \int_{[0,1/2]^2} (v^\alpha+t^\alpha)^k\rho(v,t)dvdt \geq \frac{1}{2\pi}\int_{[0,1/2]^2} (v^\alpha+t^\alpha)^k(vt)^{-1/2}dvdt$$

$$\geq\frac{1}{\pi}\int_{[0,1/2]^2\cap[v\geq t]} (2t^\alpha)^k(vt)^{-1/2}dvdt = \frac{1}{2\pi(\alpha k+1/2)(\alpha k+1)}2^{(1-\alpha)k},$$

so

$$\frac{[\mathbb{E}(S(\alpha)^k)]^{1/k}}{2^\beta} = \frac{[\mathbb{E}(S(\alpha)^k)]^{1/k}}{\|S(\alpha)\|_\infty} \geq \left(\frac{1}{2\pi(\alpha k+1/2)(\alpha k+1)}\right)^{1/k}.$$

It is then clear that the condition (L_n) holds with n=2, that is

$$\prod_{\substack{k=2\\ i=1}}^{\infty} \frac{[\mathbb{E}(S(\alpha)^k)]^{1/k}}{2^{\beta_i}} >0.$$

Theorem 6.3 applies, yielding that

$$0< \nu^\phi(K) <\infty \quad \text{a.s.}$$

where $\phi(t) = t^{1/2}(\log\log\frac{1}{t})^{1/2}$. Graf et al. (1988) calculate this function for ordinary Hausdorff measures.

7.9. Let $N\geq 1$ be a random variable taking values in $\mathbb{N}$, $p(N=k)=p_k$ $0\leq p_k$, $p_1<1$ and $\sum_{k=1}^{\infty}p_k=1$. If $N=k$, divide the unit interval into k equal parts, and, in each part, remove independently from the right a subinterval of propotion according to the uniform distribution on [0,1]. For each part that remains, repeat independently the precedure and continnue, to get a random fractal *K*.

In this case, $T_i= U_i/N$ (i=1,...,N), $\{U_i\}_{i=1}^{N}$ are independent and identically distributed random variables, each having the uniform distribution U on [0,1]. Thus $S(x)= \sum_{i=1}^{N}N^{-x}U_i^x$. The a.s. dimension α of *K* is determined by

$$\mathbb{E}(N^{1-\alpha})=1+\alpha,$$

where $0<\alpha<1$. Since $\|S(x)\|_{\infty}=\|N\|_{\infty}^{1-x}$ if $0<x<1$, we have

$$\beta=1-\alpha \text{ if } \|N\|_{\infty}<\infty \quad\text{and}\quad \beta=1 \text{ if } \|N\|_{\infty}=\infty.$$

For all $n\in\mathbb{N}$ with $p_n>0$,

$$\mathbb{E}[S^k(\alpha)\mid N=n]= \mathbb{E}[n^{-\alpha k}(\textstyle\sum_{i=1}^{n}U_i^{\alpha})^k] \geq \mathbb{E}[n^{-\alpha k}(n(\prod_{i=1}^{n}U_i^{\alpha})^{1/n})^k]$$

since $\frac{1}{n}\sum_{i=1}^{n}x_i \geq (\prod_{i=1}^{n}x_i)^{1/n}$. It follows that

$$\mathbb{E}[S^k(\alpha)\mid N=n] \geq n^{(1-\alpha)k}\,[\mathbb{E}(U^{\alpha k/n})]^n = n^{(1-\alpha)k}[(\alpha k/n)+1]^{-n}.$$

So

$$-\log \prod_{\substack{k=n^i\\ i=1}}^{\infty} \frac{[\mathbb{E}(S^k(\alpha))\mid N=n]^{1/k}}{n^{1-\alpha}} \leq \sum_{i=1}^{\infty} \frac{n}{n^i}\log(\alpha n^{i-1}+1) <\infty$$

whenever $n>1$ and $p_n>0$. Therefore, if $\|N\|_{\infty}<\infty$, then the condition (L_n) holds for all $n>1$ with $p_n>0$. Since $T_i\geq 1/\|N\|_{\infty}$, the conditions (CN_1) hold. Thus Theorem 6.3 gives

$$0< v^{\phi}(K) <\infty \quad\text{and}\quad 0< \mu^{\phi}(K) <\infty$$

a.s.whenever $\|N\|_{\infty}<\infty$, where $\phi(t)= t^{\alpha}(\log\log\frac{1}{t})^{1-\alpha}$, $0<\alpha<1$ satisfing $\mathbb{E}(N^{1-\alpha})=1+\alpha$.

Suppose now that

$$p(N=2^k)=p^{k-1}(1-p),\ k=1,2,3,\dots\ 0<p<1.$$

Thus $\mathbb{E}(S(x)) = \frac{1}{1+x}\mathbb{E}(N^{1-x})= \dfrac{(1-p)2^{1-x}}{(1+x)(1-p2^{1-x})}$ if $x> 1-\log\frac{1}{p}/\log 2$ and $\mathbb{E}(S(x))=\infty$

if $x \le 1-\log\frac{1}{p}/\log 2$. The number α is the solution in (0,1) of $\frac{(1-p)2^{1-\alpha}}{(1+\alpha)(1-p2^{1-\alpha})}=1$, that is

$$2^{1-\alpha}(1+\alpha p)=1+\alpha, \quad 0<\alpha<1.$$

If $p=3/\sqrt{2}-2= 0.1213...$ then $\alpha=1/2$; if $p>3/\sqrt{2}-2$, then $\alpha>1/2$, etc. Besides

$$\mathbb{E}[S^k(\alpha)]= \mathbb{E}[(\Sigma_{i=1}^{N} N^{-\alpha} U_i^{\alpha})^k] \le \mathbb{E}[N^{(1-\alpha)k}] =\Sigma_{j=1}^{\infty} 2^{(1-\alpha)kj} p^{1-j}(1-p) <\infty$$

if $k< \frac{1}{1-\alpha}\log\frac{1}{p}/\log 2$, and

$$\mathbb{E}[S^k(\alpha)]= \Sigma_{n=0}^{\infty}\mathbb{E}[S^k(\alpha)\,|\,N=n]]p_n \ge \Sigma_{n=1}^{\infty} n^{(1-\alpha)k}[(\alpha k/n)+1]^{-n}p_n$$

$$=\Sigma_{j=1}^{\infty} 2^{(1-\alpha)kj}\,[(\alpha k/2^j)+1]^{-2^j}\, p^{1-j}(1-p) =\infty$$

if $k\ge\frac{1}{1-\alpha}\log\frac{1}{p}/\log 2$. In particular,

$$\mathbb{E}[S^2(\alpha)]<\infty \quad \text{if} \quad 3/\sqrt{2}-2\le p\le 1/2.$$

Let

$$\lambda= \sup\{p>0: \mathbb{E}(W^p)<\infty\},$$

then $[\frac{1}{1-\alpha}\log\frac{1}{p}/\log 2] \le \lambda \le \frac{1}{1-\alpha}\log\frac{1}{p}/\log 2$, where $[x]$ denotes the integral part of x. If $3/\sqrt{2}-2\le p\le 1/2$, then $2\le\lambda$ and, by Theorem 6.4,

$$\nu^{\psi_a}(K)=0 \text{ if } a<1/\lambda \quad \text{and} \quad \nu^{\psi_a}(K)=\infty \text{ if } a>1/(\lambda-1)$$

a.s., where $\psi_a(t)=t^{\alpha}(\log\frac{1}{t})^a$.

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