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A LOCAL LIMIT THEOREM ON THE SEMI-DIRECT PRODUCT OF $\mathbb{R}^{*+}$ AND $\mathbb{R}^d$

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Abstract : Let G be the semi-direct product of $\mathbb{R}^{*+}$ and $\mathbb{R}^d$ and μ a probability measure on G which is absolutely continuous with respect to the right Haar measure. Let μ^{*n} be the n^{th} power of convolution of μ. Under quite general assumptions on μ, one may prove that there exists $\rho \in]0,1]$ such that the sequence of Radon measures $(\frac{n^{3/2}}{\rho^n}\mu^{*n})_{n\geq 1}$ converges weakly to a non-degenerate measure when n goes to $+\infty$.

Résumé : Soit G le groupe produit semi-direct de $\mathbb{R}^{*+}$ et de $\mathbb{R}^d$ et μ une mesure de probabilité sur G absolument continue par rapport à la mesure de Haar à droite. On note μ^{*n} la $n^{ième}$ convolée de μ. Sous des hypothèses assez générales sur μ, on établit l'existence d'un réel $\rho \in]0,1]$ tel que la suite de mesures de Radon $(\frac{n^{3/2}}{\rho^n}\mu^{*n})_{n\geq 1}$ converge vaguement vers une mesure non nulle lorsque n tend vers $+\infty$.

Key words : Random walk, local limit theorem, factorisation, ratio-limit theorem

I Introduction

Fix a norm $\|.\|$ on $\mathbb{R}^d, d \geq 1$, and consider the connected group G of transformations

$$\begin{array}{rrcl} g: & \mathbb{R}^d & \rightarrow & \mathbb{R}^d \\ & x & \mapsto & g.x = ax + b \end{array}$$

where $(a, b) \in \mathbb{R}^{*+} \times \mathbb{R}^d$.
Let a (resp. b) be the projection from G on $\mathbb{R}^{*+}$ (resp. on $\mathbb{R}^d$). Consequently, any transformation $g \in G$ is noted $(a(g), b(g))$ (or $g = (a, b)$ when there is no mistake) ; for example, $e = (1, 0)$ is the unity in G.
The group G is also the semi-direct product of $\mathbb{R}^{*+}$ and $\mathbb{R}^d$ with the composition law

$$\forall g = (a, b), \forall g' = (a', b') \in G \qquad gg' = (aa', ab' + b).$$

Recall that G is a non unimodular solvable group with exponential growth and let m_D be the right Haar measure on $G : m_D(da\ db) = \dfrac{da\ db}{a}$. Note that if $d = 1$, the group G is the affine group of the real line.

Let μ be a probability measure on G, μ^{*n} its n^{th} power of convolution, $\tilde{\mu}$ the image of μ by the application $g = (a, b) \mapsto \tilde{g} = (\dfrac{1}{a}, \dfrac{b}{a})$ and $\overline{\mu}$ the image of μ by the application $g \mapsto g^{-1}$. If λ is a Radon measure on $\mathbb{R}^d$, $\mu * \lambda$ denotes the Radon measure on $\mathbb{R}^d$ defined by $\mu * \lambda(\varphi) = \int_{G \times \mathbb{R}^d} \varphi(g.x)\ \mu(dg)\ \lambda(dx)$ for any continuous function φ with compact support from $\mathbb{R}^d$ into $\mathbb{R}$. Lastly, δ_x is the Dirac measure at the point x.

In the present paper, we prove under suitable hypotheses that the probability μ satisfies a local limit theorem : there exists a sequence $(\alpha_n)_{n \geq 0}$ of positive reals, depending only on the group when μ is centered, such that the sequence $(\alpha_n\ \mu^{*n})_{n \geq 0}$ converges weakly to a non-degenerate measure. This problem has been already tackled by Ph. Bougerol in ([3]) where he established local limit theorems on some solvable groups with exponential growth, typically the groups NA which occur in the Iwasawa decomposition of a semi-simple group. The affine group of the real line is the most simple example of such a group. In this particular case, Ph. Bougerol proved that, for a class R of centered probability measures μ satisfying some invariance properties, the sequence $(n^{3/2} \mu^{*n})_{n \geq 0}$ converges weakly to a non-degenerate measure on G. His method is roughly the following one : if μ satisfies some invariance properties, it can be lifted on the associated semi-simple group in a measure m_μ (not necessary bounded) which is bi-invariant under the action of a maximal and compact sub-group. In a second step, using theory of Guelfand couples, he showed that the measure m_μ satisfies a analogous of the local limit theorem established in ([2]). The aim of the present paper is to obtained such a local limit theorem when the measure μ does not belong to the class R.

This work is connected to the ones of N.T. Varopoulos, L. Saloff-Coste and T. Coulhon ([12]) where there are some estimations for the heat diffusion kernel on a Lie group that is not necessary unimodular. There are also closed relations with the study of the asymptotic behavior of some processes in random environment ([1]) .

We have the

Theorem A
Let μ be a probability measure on G satisfying the conditions

A1) μ has the density ϕ_μ with respect to the Haar measure m_D on G

A2) there exists $\alpha > 0$ such that $\int_G (exp(\alpha |Log\ a(g)|) + \|b(g)\|^\alpha)\ \mu(dg)\ < +\infty$

A3) $\int_G Log\ a(g)\ \mu(dg)\ =\ 0.$

A4) there exist $\beta, q\ \in]1, +\infty[$ such that $\int_0^1 \sqrt[q]{\int_{\mathbb{R}} \phi_\mu^q(a,b)db}\ \frac{da}{a^\beta}\ <\ +\infty.$

Then, the sequence of finite measures $(n^{3/2}\mu^{*n})_{n\geq 0}$ converges weakly to a non-degenerate Radon measure ν_0 on G.

Using L. Elie's results ([5]), one can prove, under additionnal assumptions on μ, that the double equation $\mu * \nu = \nu * \mu = \nu$ has one and only one solution (up to a multiplicative constant) in the space of Radon measure on G ; more, it is possible to obtain the explicit form of this solution. Using a ratio-limit theorem due to Y. Guivarc'h ([9]), the measure ν_0 of theorem A may be identified, up to a multiplicative constant. More precisely, we have the

Theorem B
Suppose that hypotheses of theorem A hold and assume the additionnal conditions

B1) the density ϕ_μ of μ is continuous with compact support

B2)$\phi_\mu(e) > 0$

Then, the measure ν_0 of theorem A may be decomposed as follows

$$\nu_0 = (\delta_1 \otimes \lambda) * \overline{(\frac{da}{a} \otimes \lambda_1)}$$

where λ (resp. λ_1) is, up to a multiplicative constant, the unique Radon measure on $\mathbb{R}^d$ which satisfies the convolution equation $\mu * \lambda = \lambda$ (resp. $\overline{\mu} * \lambda_1 = \lambda_1$).
More, for any positive and continuous function $\varphi, \varphi \not\equiv 0$, with compact support in G, we

have $\nu_0(\varphi) > 0$ *and*

$$\mu^{*n}(\varphi) \sim \frac{\nu_0(\varphi)}{n^{3/2}} \quad \textit{as} \quad n \to +\infty.$$

When the probability measure μ is not centered (that is $Log\ a(g)\ \mu(dg) \neq 0$) we use a classical method which consists to operate by a relativisation in order to bring back the study to the centered case. One can then easily obtain the

Theorem A'

Let μ *be a probability measure on* G *satisfying the conditions*

$A1)$ μ *has the density* ϕ_μ *with respect to the Haar measure* m_D *on* G

$A'2)$ *there exists* $\alpha > 0$ *such that for any* $t \in I\!R$ $\int_G (exp(t|Log(a(g)|) + \|b(g)\|^\alpha)\ \mu(dg) \ < +\infty$

$A'3)$ $\int_G Log\ a(g)\ \mu(dg) \ \neq\ 0, \quad \mu\{g \in G : a(g) < 1\} > 0 \quad \textit{and} \quad \mu\{g \in G : a(g) > 1\} > 0.$

Then, there exists a unique $t_0 \in I\!R$ *and* $\rho = \rho(\mu) \in]0,1[$ *such that*

$$\int_G a(g)^{t_0}\ \mu(dg) = \inf_{t \in I\!R} \int_G a(g)^t\ \mu(dg) = \rho(\mu).$$

More, if there exist $q\ \in]1,+\infty[$ *and* $\beta \in]1-t_0,+\infty[$ *such that* $\int_0^1 \sqrt[q]{\int_{I\!R} \phi^q_\mu(a,b)db}\ \frac{da}{a^\beta}\ <\ +\infty,$ *the sequence* $(\frac{n^{3/2}}{\rho^n}\mu^{*n})_{n\geq 1}$ *converges weakly to a non-degenerate Radon measure* ν_0 *on* G.

Let us now get briefly the ideas of our approach. Fix a probability space $(\Omega, \mathcal{F}, I\!P)$ and let $g_n = (a_n, b_n), n = 1, 2, \cdots$ be G-valued independant and identically distributed random variables of law μ defined on $(\Omega, \mathcal{F}, I\!P)$. Note $\mathcal{F}_n$ the σ-algebra generated by the variables $g_1, g_2, \cdots, g_n$. For any $n \geq 1$, put $G_1^n = g_1 \cdots g_n = (A_1^n, B_1^n)$; a direct calculation gives $A_1^n = a_1 a_2 \cdots a_n$ and $B_1^n = \sum_{k=1}^n a_1 a_2 \cdots a_{k-1} b_k$. Lastly, introduce the variable $M_n = max(0, Log\ A_1^1, Log\ A_1^2 \cdots, Log\ A_1^n)$.
Put $\mathcal{A} = \{g \in G : a(g) > 1\}$ and consider the transition kernel $P_\mathcal{A}$ associated with the couple $(\mu, \mathcal{A})$ and defined, for any Borel set $\mathcal{B} \subset G$, by

$$\forall g \in G \quad P_\mathcal{A}(g, \mathcal{B}) = \int_G 1_{\mathcal{A}^c \cap \mathcal{B}}(gh)\ \mu(dh).$$

In the same way, put $\mathcal{A}' = \{g \in G/a(g) \geq 1\}$ and let $\tilde{P}_{\mathcal{A}'}$ be the operator associated with the couple $(\tilde{\mu}, \mathcal{A}')$. Following Grincevicius's paper, we are led to what we call the **Grincevicius-Spitzer identity** ([8]):

$$\mu^{*n}(\varphi) = \sum_{k=0}^n \int_G \tilde{P}^k_{\mathcal{A}'}(e, dg)\ \int_G P^{n-k}_\mathcal{A}(e, dh) \quad \varphi(\frac{a(h)}{a(g)}, \frac{b(g) + b(h)}{a(g)})$$

for any continuous function φ with compact support in G. This formula allows to bring back the study of the asymptotic behavior of the sequence $(\mu^{*n})_{n\geq 1}$ to the one of the powers of the operators $P_{\mathcal{A}}$ and $\tilde{P}_{\mathcal{A}'}$. It is the first main idea of this paper.

The second main idea relies on the Grenander's conjecture, proved by Grincevicius in ([8]) on a weaker form : if $\int_G Log\ a(g)\ \mu(dg) = 0$ and $d = 1$, the asymptotic distribution of the random variable $|Log\ B_1^n|$ is the same as the asymptotic distribution of M_n. One may thus expect that the asymptotic behavior of the sequence $(g_n)_{n\geq 0}$ is quite similar to the one of the sequence $(A_1^n, exp(M_n))_{n\geq 0}$; we will justify this in section 3.

Section 2 is devoted to the study of the behavior as n goes to $+\infty$ of the sequence $(Log\ A_1^n, M_n)_{n\geq 0}$ and in section 3 we establish theorems A and B.

II A preliminary result

Troughout this section, $X_1, X_2 \cdots$ are independant real valued random variables of law p defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $(S_n)_{n\geq 0}$ be the associated random walk on $\mathbb{R}$ starting from 0 (that is $S_0 = 0$ and $S_n = X_1 + \cdots + X_n$ for $n \geq 1$) ; the law of S_n is the n^{th} power of convolution p^{*n} of the measure p. Note $\mathcal{F}_n$ the σ-algebra generated by the variables $X_1, X_2, \cdots, X_n, n \geq 1$. Lastly, put $M_n = max(0, S_1, \cdots, S_n)$.
The study of the asymptotic behavior of the variable M_n is very interesting since seemingly many problems in applied probability theory may be reformulated as questions concerning this random variable. A few papers have been devoted to this subject ; in the present section we find the behavior as n goes to $+\infty$ of the probability $\mathbb{P}[[M_n \in [\alpha, \beta]] \cap [M_n - S_n \in [\gamma, \delta]]$ where $[\alpha, \beta] \times [\gamma, \delta] \subset \mathbb{R}^+ \times \mathbb{R}^+$. Following Spitzer's approach ([11]), we have to introduce the two following waiting times T_+ and T_- with respect to the filtration $(\mathcal{F}_n)_{n\geq 1}$:

$$T_+ = inf\{n \geq 1 : S_n > 0\} \quad \text{and} \quad T_- = inf\{n \geq 1 : S_n < 0\}.$$

The variable T_+ (resp. T_-) is classicaly called the *first ascending* (resp. *descending*) *ladder epoch* of the random walk $(S_n)_{n\geq 0}$ ([11], [6]). We note p_{T_+} (resp. p_{T_-}) the law of the variable S_{T_+} (resp. S_{T_-}).

In the second part of this section, we show that the study of the sequence $(M_n, M_n - S_n)_{n\geq 1}$ is closely related to the one of the sequences $(\mathbb{E}[[T_+ > n]; \varphi(S_n)])_{n\geq 1}$ and $(\mathbb{E}[[T_- > n]; \varphi(S_n)])_{n\geq 1}$ for a suitable bounded Borel function φ on $\mathbb{R}$. The first part of this section is devoted to the study of these last sequences.

II-a A local limit theorem for a killed random walk on a half line

In this part, we study the behavior as n goes to $+\infty$ of the sequence $(\mathbb{E}[[T_+ > n]; \varphi(S_n)])_{n\geq 1}$; the proof goes along the same lines for the sequence $(\mathbb{E}[[T_- > n]; \varphi(S_n)])_{n\geq 1}$.

Introducing the operator $P_{]0,+\infty[}$ defined by

$$\forall x \in \mathbb{R} \quad P_{]0,+\infty[}\varphi(x) = 1_{]-\infty,0]}(x) \int_{\mathbb{R}} 1_{]-\infty,0]}(x+y)\varphi(x+y)p(dy),$$

we obtain $\forall n \geq 1 \quad \mathbb{E}[[T_+ > n]; \varphi(S_n)] = P^n_{]0,+\infty[}\varphi(0)$. This section is thus devoted to the asymptotic behavior as n goes to $+\infty$ of the n^{th} power of the operator $P_{]0,+\infty[}$.
The following result is already well known ([10]). The proof given here is quite different from the classical one and is based on the following idea : we prove, under suitable hypotheses on p, that the function $z \mapsto \sum_{n=0}^{+\infty} p^{*n}(\varphi)z^n$ may be analytically extended on a certain neighbourhood of the unit complex disc except the pole 1. So the approximation of this function around its singularity may be translated into an approximation of its Taylor coefficients. We have the

Theorem II.1 *Suppose that*

i) the law p of the variables $X_n, n \geq 1$, is absolutely continuous with respect to the Lebesgue measure λ on $\mathbb{R}$.

ii) the characteristic function $t \longrightarrow \hat{p}(t) = \mathbb{E}[e^{itX_1}]$ belongs to $\mathbb{L}_1(\mathbb{R})$

iii) $\sigma^2 = \mathbb{E}[X_1^2] < +\infty$ *and* $\mathbb{E}[X_1] = 0$.

Then, for any continuous function φ with compact support on $\mathbb{R}^-$, we have

$$\lim_{n\to+\infty} n^{3/2} \quad \mathbb{E}[[T_+ > n] \; ; \; \varphi(S_n)] \;=\; \frac{1}{\sigma\sqrt{2\pi}} \int_{\mathbb{R}^-\times\mathbb{R}^-} \varphi(x) \; U_{T_-} * \lambda_-(dx)$$

where λ_- denotes the restriction of the Lebesgue measure on $\mathbb{R}^-$ and U_{T_-} the σ-finite measure $\sum_{n=0}^{+\infty}(p_{T_-})^{*n}$.

Proof - Using relations $P5(a)$ and $P5(c)$ in Spitzer's book, page 181 ([11]) (see also [6], chap. XVIII), one obtains

$$\forall z \in \mathbb{C} \;, |z| < 1, \forall a > 0 \qquad \sum_{n=0}^{+\infty} z^n \mathbb{E}[[T_+ > n] \; ; \; e^{aS_n}] = \sum_{n=0}^{+\infty} \mathbb{E}[z^{T_-} \; exp(aS_{T_-})]^n$$

and so $\sum_{n=0}^{+\infty} \mathbb{E}[[T_+ > n] \; ; \; e^{aS_n}] = \sum_{n=0}^{+\infty} \mathbb{E}[\; exp(aS_{T_-})]^n = \int_{-\infty}^0 e^{ax} \; U_{T_-}(dx)$
(Note that $-\infty < \mathbb{E}[S_{T_-}] < 0$ which ensures that the above series converge ([6],[11]). Consequently, we have

$$\begin{aligned}
\forall a > 0 \qquad \int_{-\infty}^0 e^{ax} \; U_{T_-} * \lambda_-(dx) \;\; &= \;\; \int_{-\infty}^0 \frac{e^{ax}}{a} U_{T_-}(dx) \\
&= \;\; \sum_{n=0}^{+\infty} \mathbb{E}[[T_+ > n]; \frac{e^{aS_n}}{a}].
\end{aligned}$$

Thus, to prove theorem 2.1, it suffices to show that

$$\forall a > 0 \qquad \lim_{n \to +\infty} n^{3/2} \quad \mathbb{E}[[T_+ > n] \; ; \; e^{aS_n}] \;\; = \;\; \frac{1}{\sigma\sqrt{2\pi}} \sum_{n=0}^{+\infty} \mathbb{E}[[T_+ > n] \; ; \; \frac{e^{aS_n}}{a}].$$

Note that $\mathbb{E}[[T_+ > n] \; ; \; e^{aS_n}]$ is the n^{th} Taylor coefficient of the function ϕ_a defined by

$$\forall z \in \mathbf{C} \;, |z| < 1 \qquad \phi_a(s) = \sum_{n=0}^{+\infty} z^n \mathbb{E}[[T_+ > n] \; ; \; e^{aS_n}].$$

Recall the **Spitzer-identity** ([11], $P5(c), p.181$)

$$\forall z \in \mathbf{C} \;, |z| < 1 \quad \phi_a(z) = exp(A(z)) \quad \text{with} \quad A(z) = \sum_{n=1}^{+\infty} \frac{z^n}{n} \mathbb{E}[[S_n \leq 0] \; ; \; e^{aS_n}].$$

By the local limit theorem on $\mathbb{R}$, one can easily see that ϕ_a has a continuous extension on the closed unit complex disc. We will prove that in fact ϕ_a may be analytically extended on a neighbourhood of the unit complex disc containing the unit circle except the point 1. We next translate the behaviour of ϕ_a around the point 1 into an asymptotic equivalent of its n^{th} Taylor coefficient at 0. Since this method is of independant interest, we state it in the following general lemma, inspired by ([?]), and which may be compared to the classical Darboux method, or to the Tauberian theorems, although the conditions of validity differ appreciably.

Lemma II.2 *Let $G(z)$ be analytic in a domain*

$$D_{\rho,\theta} = \{z / z \neq 1, |z| < \rho, |arg(z-1)| \geq \theta \quad with \quad \rho > 1 \quad and \quad 0 < \theta < \frac{\pi}{2}\}.$$

Assume that

i) the function $\sqrt{1-z}\; G(z)$ is bounded on $D_{\rho,\theta}$

ii) there exists $C > 0$ such that $\quad \lim_{\substack{z \to 1 \\ z \in D_{\rho,\theta}}} \sqrt{1-z}\; G(z) \; = C.$

Then, the n^{th} Taylor coefficient g_n of $G(z)$ at the origin admits the asymptotic equivalent

$$g_n \quad \sim \quad \frac{C}{\sqrt{\pi n}} \qquad as \quad n \to +\infty.$$

Hence, to prove theorem 2.1, we have to extend analytically the function ϕ_a on a set $D_{\rho,\theta}$. The Fourier inversion formula leads to

$$\begin{aligned}
\forall z \in \mathbb{C}\ , |z| < 1 \qquad A'(z) &= \sum_{n=0}^{+\infty} z^n \mathbb{E}[[S_{n+1} \leq 0]\ ;\ e^{aS_{n+1}}] \\
&= \sum_{n=0}^{+\infty} z^n \int_{-\infty}^{0} e^{ax} \phi_p^{*(n+1)}(x) dx \\
&= \sum_{n=0}^{+\infty} z^n \int_{-\infty}^{0} e^{ax} (\frac{1}{2\pi} \int_{\mathbb{R}} \hat{p}^{(n+1)}(t)\ e^{-itx} dt)\ dx \\
&= \frac{1}{2\pi} \int_{\mathbb{R}} h_a(-t) \frac{\hat{p}(t)}{1 - z\hat{p}(t)}\ dt
\end{aligned}$$

where h_a denotes the Fourier transform of the function $e^{ax}\ 1_{]-\infty,0]}$ (i.e. $h_a(t) = \dfrac{1}{a+it}$). This new expression of A' will allow us to extend analytically the function ϕ_a. We have the

Lemma II.3 *Let f be an integrable and continuous function on $\mathbb{R}$. Then, there exist $\rho > 1$ and $\theta \in]0, \frac{\pi}{2}[$ such that the function*

$$z \ \mapsto \int_{\mathbb{R}} \frac{f(t)}{1 - z\hat{p}(t)} dt$$

is analytic on the region $D_{\rho,\theta}$ defined in lemma 2.2.

To apply lemma 2.2, we shall need the

Lemma II.4 *Let f be an integrable, continuous and bounded function on $\mathbb{R}$ and $D_{\rho,\theta}$ the set described in lemma 2.3. We have*

$$\lim_{\substack{z \to 1 \\ z \in D_{\rho,\theta}}} \sqrt{1-z} \int_{\mathbb{R}} \frac{f(t)}{1 - z\hat{p}(t)} dt \ = \frac{\pi\sqrt{2}}{\sigma} f(0).$$

Setting in lemmas 2.3 and 2.4 $f(t) = \frac{1}{2\pi}\ \hat{p}(t)\ h_a(t)$, we obtain

$$\lim_{\substack{z \to 1 \\ z \in D_{\rho,\theta}}} \sqrt{1-z}\ \phi_a'(z) = \frac{1}{\sigma\sqrt{2}} \sum_{n=0}^{+\infty} \mathbb{E}[[T_+ > n]\ ;\ \frac{e^{aS_n}}{a}].$$

Moreover, without loss of generality, one may suppose that the function $z \mapsto \sqrt{1-z}\ \phi'_a(z)$ is bounded on $D_{\rho,\theta}$ (if it is necesssary, one can modify the values of ρ and θ). The claim follows readily from lemma 2.2 and the well-known relation between the Taylor coefficients at 0 of ϕ_a and the ones of ϕ'_a. □

We now prove lemmas 2.2, 2.3 and 2.4.

Proof of lemma 2.2 - The n^{th} Taylor coefficient can be computed using Cauchy's residue theorem as

$$g_n = \frac{1}{2i\pi}\int_\Gamma G(z)\frac{dz}{z^{n+1}}$$

where the contour Γ simply encircles the origin and is inside the domain of analyticity of the function G. For fixed $\alpha > \theta$, $1 < r < \rho$ and $0 < \epsilon < r-1$, we take the specific contour (depicted on Fig. 1) $\Gamma = \Gamma_{\epsilon,\alpha,r} = \Gamma^0_{\epsilon,\alpha} \cup \Gamma^1_{\epsilon,\alpha,r} \cup \Gamma^2_{\alpha,r}$, defined by

$$\begin{aligned}
\Gamma^0_{\epsilon,\alpha} &= \{z/|z-1| = \epsilon, |arg(z-1)| \geq \alpha\}\\
\Gamma^1_{\epsilon,\alpha,r} &= \{z/|z-1| \geq \epsilon, |z| \leq r, |arg(z-1)| = \alpha\}\\
\Gamma^2_{\alpha,r} &= \{z/|z| = r, |arg(z-1)| \geq \alpha\}
\end{aligned}$$

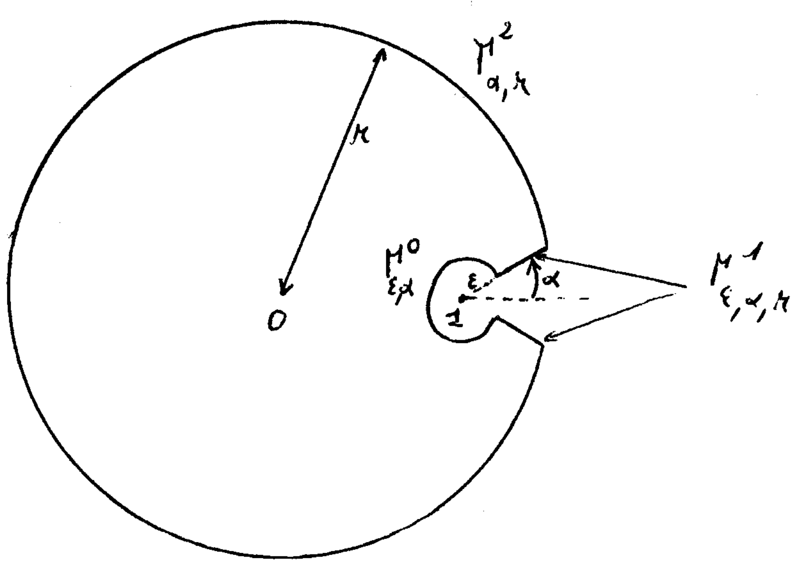

Figure 1 - Diagram showing the contour $\Gamma_{\epsilon,\alpha,r}$

The function $F(z) = G(z) - \dfrac{C}{\sqrt{1-z}}$ is analytic on $D_{\rho,\theta}$; thus, for $|z| < 1$, we have $F(z) = \sum_{n=0}^{+\infty} f_n z^n$ with $f_n = \dfrac{1}{2i\pi}\displaystyle\int_{\Gamma_{\epsilon,\alpha,r}} F(z)\frac{dz}{z^{n+1}}$. Note that this last integral does not depend on $\alpha > \theta, r \in]1,\rho[$ and $\epsilon \in]0, r-1[$. If we put $M = \sup_{z \in D_{\rho,\theta}} |\sqrt{1-z}F(z)|$, we obtain

$$\begin{aligned} |\frac{1}{2i\pi}\int_{\Gamma^0_{\epsilon,\alpha}} F(z)\frac{dz}{z^{n+1}}| &\leq \frac{M\sqrt{\epsilon}}{(1-\epsilon)^{n+1}} \\ |\frac{1}{2i\pi}\int_{\Gamma^2_{\epsilon,\alpha}} F(z)\frac{dz}{z^{n+1}}| &\leq \frac{M}{r^n\sqrt{1-r}}. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} |\frac{1}{2i\pi}\int_{\Gamma^1_{\epsilon,\alpha}} F(z)\frac{dz}{z^{n+1}}| &\leq \frac{1}{\pi}\sup_{z\in\Gamma^1_{\epsilon,\alpha,r}} |\sqrt{1-z}F(z)| \int_0^{+\infty} \frac{dt}{\sqrt{t}|1+tcos\alpha|^{n+1}} \\ &\leq \frac{K}{\sqrt{n}}\sup_{z\in\Gamma^1_{\epsilon,\alpha,r}} |\sqrt{1-z}F(z)|. \end{aligned}$$

Thanks to these inequalities, one can prove that $\lim_{n\to+\infty} \sqrt{n}\ f_n = 0$. By hypothesis ii), for every $\delta > 0$, one can choose $r > 1$ such that $\sup_{\substack{z\in D_{\rho,\theta} \\ |z-1|<r-1}} |\sqrt{1-z}F(z)| < \delta$; we thus have for $n \geq 1$

$$|f_n| \leq \frac{M\sqrt{\epsilon}}{(1-\epsilon)^{n+1}} + \frac{M}{r^n\sqrt{1-r}} + \frac{K\ \delta}{\sqrt{n}}.$$

Let ϵ goes to 0 ; for n large enough, one obtains $|f_n| \leq \frac{2K\ \delta}{\sqrt{n}}$, which gives the expected convergence.

Now, we have $\frac{1}{\sqrt{1-z}} = \sum_{n=0}^{+\infty} a_n z^n$ with $a_n = \frac{2n\ !}{4^n(n!)^2} = \frac{1+\epsilon(n)}{\sqrt{\pi n}}$ with $\lim_{n\to+\infty} \epsilon(n) = 0$. Putting everything together, we have thus shown that

$$g_n = f_n + Ca_n \sim \frac{C}{\sqrt{\pi n}} \quad \text{as} \quad n \to +\infty.$$

□

Proof of lemma 2.3 - The variable X_1 being centered, we have

$$\hat{p}(t) = 1 - \frac{\sigma^2}{2}t^2 + t^2\delta(t)$$

where δ is a bounded and continuous function vanishing at 0. Then $\frac{1}{\hat{p}(t)} = 1 + x(t) + iy(t)$ with $x(t) > 0$ for $t \neq 0$ quite small and $\lim_{t\to 0}\frac{y(t)}{x(t)} = 0$. Thus, there exist $\eta > 0, \rho > 1$ and $\theta \in]0, \frac{\pi}{2}[$ such that

$$\forall z \in D_{\rho,\theta}, \forall t \in]-\eta,\eta[\qquad |1 - z\hat{p}(t)| > 0.$$

It follows immediately that the function $z \mapsto \int_{-\eta}^{\eta} \frac{f(t)}{1 - z\hat{p}(t)} dt$ is analytic on $D_{\rho,\theta}$.
On the other hand, since p is absolutely continuous with respect to Lebesgue measure on $\mathbb{R}$, we have $\sup_{t,|t|\geq\eta} |\hat{p}(t)| < 1$ ([6]), which ensures that the function $z \mapsto \int_{[-\eta,\eta]^c} \frac{f(t)}{1 - z\hat{p}(t)} dt$ is analytic on a complex disc of radius $\rho_\eta > 1$. The proof of lemma 2.3 is complete. □

Proof of lemma 2.4 - Let $D_{\rho,\theta}$ be the set described in lemma 2.2 and recall the local expansion of $\hat{p}(t)$ around the origin :

$$\hat{p}(t) = P(t) + t^2\delta(t)$$

where $P(t) = 1 - \frac{\sigma^2 t^2}{2}$ and δ is a continuous function vanishing in 0. It is well known that δ is bounded by $\frac{3}{2}\sigma^2$ ([4]). To prove lemma 2.4, we need a series of inequalities.

Key inequalities *There exist C_1, C_2, C_3 in $\mathbb{R}^{*+}$ and ϵ_0 in $]0,1[$ such that, for any $\epsilon \in]0, \epsilon_0[, \alpha \in [\theta, 2\pi - \theta]$ and $u \in \mathbb{R}$, we have*

$$i) \left|\frac{1 - (1 + \epsilon e^{i\alpha})P(\sqrt{\epsilon}\, u)}{\epsilon}\right| \geq C_1 u^2 - 1$$

$$ii) \left|\frac{1 - (1 + \epsilon e^{i\alpha})P(\sqrt{\epsilon}\, u)}{\epsilon}\right| \geq C_2 - \epsilon\frac{\sigma^2 u^2}{2}$$

$$iii) \left|\frac{1 - (1 + \epsilon e^{i\alpha})\hat{p}(\sqrt{\epsilon}\, u)}{\epsilon}\right| \geq C_2 - \epsilon\frac{7\ \sigma^2 u^2}{2}$$

iv) If A is a strictly positive real with $\mathbb{P}[|X_1| \leq A] > 0$, one can choose $C_3 > 0$ such that, for any $u \in [-\frac{1}{A\sqrt{\epsilon}}, \frac{1}{A\sqrt{\epsilon}}]$, we have $\left|\frac{1 - (1 + \epsilon e^{i\alpha})\hat{p}(\sqrt{\epsilon}\, u)}{\epsilon}\right| \geq C_3 u^2 - 1$.

Proof - *i)* We have

$$\begin{aligned} |1 - (1 + \epsilon e^{i\alpha})P(\sqrt{\epsilon}\, u)| &= \epsilon\left| - e^{i\alpha} + \frac{\sigma^2 u^2}{2}(1 + \epsilon e^{i\alpha})\right| \\ &\geq \epsilon\left(\frac{\sigma^2 u^2}{2}(1 - \epsilon) - 1\right) \end{aligned}$$

ii) A similar argument gives

$$\begin{aligned} \frac{|1 - (1 + \epsilon e^{i\alpha})P(\sqrt{\epsilon} u)|}{\epsilon} &\geq \left|\frac{\sigma^2 u^2}{2} - e^{i\alpha}\right| - \epsilon\frac{\sigma^2 u^2}{2} \\ &\geq \begin{cases} |\sin\theta| - \epsilon\frac{\sigma^2 u^2}{2} & \text{if} \quad |\sin\alpha| \geq |\sin\theta| \\ |\cos\theta| - \epsilon\frac{\sigma^2 u^2}{2} & \text{otherwise} \end{cases} \end{aligned}$$

Inequality $iii)$ may be obtained combining assertion $ii)$ and the following local expansion

$$|1 - z\hat{p}(\sqrt{\epsilon}\, u)| = |1 - zP(\sqrt{\epsilon}\, u) - z\epsilon^2 u^2 \delta(\sqrt{\epsilon}\, u)|$$

$iv)$ If one substitute P for $\hat{p}$, one obtains

$$\begin{aligned}
|1 - (1 + \epsilon e^{i\alpha})\hat{p}(\sqrt{\epsilon}\, u)| &\geq |1 - \hat{p}(\sqrt{\epsilon}\, u)| - \epsilon|\hat{p}(\sqrt{\epsilon}\, u)| \\
&\geq \mathbb{E}[1 - cos\sqrt{\epsilon}\, uX_1] - \epsilon \\
&\geq \mathbb{E}[1_{[|X_1|\leq A]}(1 - cos\sqrt{\epsilon}\, uX_1)] - \epsilon \qquad \text{with } \mathbb{P}[|X_1| \leq A] > 0 \\
&\geq k\, \epsilon\, u^2\, \mathbb{E}[1_{[|X_1|\leq A]} X_1^2] - \epsilon \quad \text{where } k \text{ is a positive constant}
\end{aligned}$$

The four equalities are thus established. □

We are now able to give the proof of the lemma 2.4. Fix $A > 0$ such that $\mathbb{P}[|X_1| \leq A] > 0$; since $\inf_{\substack{z\in D_{\rho,\theta} \\ |t|\geq A}} |1 - \hat{p}(t)| > 0$, one may easily see that

$$\lim_{\substack{z\to 1 \\ z\in D_{\rho,\theta}}} \sqrt{1-z} \int_{|t|\geq A} \frac{f(t)}{1 - z\hat{p}(t)} dt = 0.$$

To find the behaviour of $\int_{-A}^{A} \frac{f(t)}{1 - z\hat{p}(t)} dt$ as $z \to 1$, we shall use the following decomposition

$$\int_{-A}^{A} \frac{f(t)}{1 - z\hat{p}(t)} dt = I(z) + J(z)$$

with $I(z) = \int_{-A}^{A} f(t)(\frac{1}{1 - z\hat{p}(t)} - \frac{1}{1 - zP(t)})\, dt$ and $J(z) = \int_{-A}^{A} \frac{f(t)}{1 - zP(t)} dt.$
Setting $t = \sqrt{\epsilon}\, u$ and $z = 1 + \epsilon e^{i\alpha}$ with $\theta \leq \alpha \leq 2\pi - \theta$ and using the local Taylor expansion around 0 of the function $\hat{p}$, we obtain

$$\sqrt{1-z}\; I(z) = e^{i\frac{\alpha+\pi}{2}}(1 + \epsilon e^{i\alpha}) \int_{-A/\sqrt{\epsilon}}^{A/\sqrt{\epsilon}} f(\sqrt{\epsilon}\, u) u^2 \delta(\sqrt{\epsilon} u) \psi_\epsilon(u) du$$

with $\psi_\epsilon(u) = \dfrac{\epsilon^2}{(1 - (1 + \epsilon e^{i\alpha})\hat{p}(\sqrt{\epsilon} u))(1 - (1 + \epsilon e^{i\alpha})P(\sqrt{\epsilon} u))}$.
Let us use the "key inequalities". Put $\eta = max(\sqrt{\frac{2}{C_1}}, \sqrt{\frac{2}{C_3}})$; so, we have

$$\forall u \in \mathbb{R}, |u| \geq \eta \quad C_1 u^2 - 1 \geq 1 \quad \text{and} \quad C_3 u^2 - \geq 1$$

which implies $\forall u \in \mathbb{R}, |u| \geq \eta \quad \psi_\epsilon(u) \leq \dfrac{1}{(C_1 u^2 - 1)(C_3 u^2 - 1)}$.
On the other hand, one can choose ϵ_0 small enough such that

$$\forall \epsilon \leq \epsilon_0, \forall u \in [-\eta, \eta] \quad C_2 - \epsilon\frac{7\sigma^2 u^2}{2} \geq \frac{C_2}{2}.$$

Consequently, since f and δ are bounded and $\lim_{t\to 0}\delta(t)=0$, we obtain, by the dominated convergence theorem

$$lim_{\substack{z\to 1\\ z\in D_{\rho,\theta}}} \sqrt{1-z}\; I(z)=0.$$

To obtain the claim, it suffices to prove

$$\lim_{\substack{z\to 1\\ z\in D_{\rho,\theta}}} \sqrt{1-z}\; J(z) \;=\frac{\pi\sqrt{2}}{\sigma}f(0).$$

Indeed, replacing z with $1+\epsilon e^{i\alpha}$ in the integral $J(z)$ we have

$$\sqrt{1-z}\; J(z)=e^{i\frac{\alpha+\pi}{2}}\int_{-A/\sqrt{\epsilon}}^{A/\sqrt{\epsilon}}\frac{f(\sqrt{\epsilon}\, u)}{\frac{u^2\sigma^2}{2}(1+\epsilon e^{i\alpha})-e^{i\alpha}}du.$$

By the dominated convergence theorem, this last expression converges as ϵ goes to 0 to the limit $f(0)\; e^{i\frac{\alpha+\pi}{2}}\int_{-\infty}^{+\infty}\frac{du}{\frac{u^2\sigma^2}{2}-e^{i\alpha}}$ uniformly on $[\theta,2\pi-\theta]$. A direct calculation gives

$$\int_{-\infty}^{+\infty}\frac{du}{\frac{u^2\sigma^2}{2}-e^{i\alpha}}=\frac{\pi\sqrt{2}}{\sigma e^{i\frac{\alpha+\pi}{2}}}$$

The proof of lemma 2.4 is complete. □

II-b A local limit theorem for the process $(M_n, M_n - S_n)_{n\geq 0}$ on $\mathbb{R}^+\times\mathbb{R}^+$

We are now able to state the following theorem concerning the behaviour as n goes to $+\infty$ of the sequence $(\mathbb{E}[\varphi(M_n, M_n-S_n)])_{n\geq 1}$ where φ is a continuous function with compact support on $\mathbb{R}^+\times\mathbb{R}^+$.

Theorem II.5 *Suppose that*

i) the law p of the variables $X_n, n\geq 1$, is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$

ii) the characteristic function $t\mapsto\hat{p}(t)=\mathbb{E}[e^{itX_1}]$ belongs to $\mathbb{L}_1(\mathbb{R})$.

iii) $\sigma^2=\mathbb{E}[X_1^2]<+\infty$ *and* $\mathbb{E}[X_1]=0$.

Then, for any continuous function φ with compact support on $\mathbb{R}^+ \times \mathbb{R}^+$, we have

$$\lim_{n\to+\infty} n^{3/2} \quad \mathbb{E}[\varphi(M_n, M_n - S_n)] = \frac{1}{\sigma\sqrt{2\pi}} \int_0^{+\infty}\int_0^{+\infty} \varphi(x,y)\ \lambda_+ * U_{T_+}(dx)\ \overline{U}_{T_-}(dy) + \frac{1}{\sigma\sqrt{2\pi}} \int_0^{+\infty}\int_0^{+\infty} \varphi(x,y)\ U_{T_+}(dx)\ \lambda_+ * \overline{U}_{T_-}(dy)$$

where λ_+ is the restriction of the Lebesgue measure on $\mathbb{R}^+$, $U_{T_+} = \sum_{n=0}^{+\infty}(p_{T_+})^{*n}$ and $\overline{U}_{T_-}$ the image by the map $x \mapsto -x$ of the potential $U_{T_-} = \sum_{n=0}^{+\infty}(p_{T_-})^{*n}$.

Proof - In his book, F. Spitzer introduces the variable T_n denoting the first time at which the random walk $(S_n)_{n\geq 0}$ reaches its maximun M_n during the first n steps. Recall that T_n is not a stopping time with respect to the filtration $(\mathcal{F}_n)_{n\geq 1}$; nevertheless, it plays a crucial role in order to obtain the following identities
$\forall \in \mathbb{C}$, $|z| < 1, \forall a, b > 0$

$$\sum_{n=0}^{+\infty} z^n \mathbb{E}[e^{-aM_n}\ e^{-b(M_n - S_n)}] = (\sum_{n=0}^{+\infty} z^n \mathbb{E}[[T_- > n]\ ;\ e^{-aS_n}])(\sum_{n=0}^{+\infty} z^n \mathbb{E}[[T_+ > n]\ ;\ e^{bS_n}])$$

$$\sum_{n=0}^{+\infty} z^n \mathbb{E}[[T_- > n]\ ;\ e^{-aS_n}] = \sum_{n=0}^{+\infty} z^n \mathbb{E}[e^{-aS_{T_+}}]^n$$

$$\sum_{n=0}^{+\infty} z^n \mathbb{E}[[T_+ > n]\ ;\ e^{bS_n}] = \sum_{n=0}^{+\infty} z^n \mathbb{E}[e^{bS_{T_-}}]^n$$

So $\displaystyle\sum_{n=0}^{+\infty} \mathbb{E}[e^{-aM_n}\ e^{-b(M_n - S_n)}] = \int_0^{+\infty}\int_0^{+\infty} e^{-ax}e^{-by}\ U_{T_+}(dx)\ \overline{U}_{T_-}(dy).$
Consequently, we have

$$\int_0^{+\infty}\int_0^{+\infty} e^{-ax}e^{-by}\ \lambda_+ * U_{T_+}(dx)\ \overline{U}_{T_-}(dy) = \frac{1}{a}\sum_{n=0}^{+\infty} \mathbb{E}[e^{-aM_n}e^{-b(M_n - S_n)}]$$

and $\displaystyle\int_0^{+\infty}\int_0^{+\infty} e^{-ax}e^{-by}U_{T_+}(dx)\ \lambda_+ * \overline{U}_{T_-}(dy) = \frac{1}{b}\sum_{n=0}^{+\infty} \mathbb{E}[e^{-aM_n}e^{-b(M_n - S_n)}].$
To prove theorem 2.5, it thus suffices to show that

$$\lim_{n\to+\infty} n^{3/2}\mathbb{E}[e^{-aM_n}\ e^{-b(M_n - S_n)}] = \frac{1}{\sigma\sqrt{2\pi}}(\frac{1}{a} + \frac{1}{b})\sum_{n=0}^{+\infty} \mathbb{E}[e^{-aM_n}\ e^{-b(M_n - S_n)}].$$

Using the same method than the one in section 2.1, one introduces the function $\phi_{a,b}$ defined by

$$\forall z \in \mathbb{C}\ , |z| < 1 \qquad \phi_{a,b}(z) = \sum_{n=0}^{+\infty} z^n\ \mathbb{E}[e^{-aM_n}\ e^{-b(M_n - S_n)}].$$

We have $\phi_{a,b}(z) = exp(A(z) + B(z))$ (and so $\phi'_{a,b}(z) = (A'(z) + B'(z))exp[A(z) + B(z)]$) with $A(z) = \sum_{n=1}^{+\infty} \frac{z^n}{n} \mathbb{E}[[S_n > 0] \; ; \; e^{-aS_n}]$ and $B(z) = \sum_{n=1}^{+\infty} \frac{z^n}{n} \mathbb{E}[[S_n < 0] \; ; \; e^{bS_n}]$.

Following the proof of theorem 2.1, one may readily see that it is possible to extend analitically the functions A' and B' on a domain

$$D_{\rho,\theta} = \{z/z \neq 1, |z| < \rho, |arg(z-1)| \geq \theta \quad \text{with} \quad \rho > 1 \quad \text{and} \quad 0 < \theta < \frac{\pi}{2}\}$$

and that we have

$$\lim_{\substack{z \to 1 \\ z \in D_{\rho,\theta}}} \sqrt{1-z}\; \phi'_{a,b}(z) \;=\; \frac{1}{\sigma\sqrt{2}}\left(\frac{1}{a} + \frac{1}{b}\right)\left(\sum_{n=0}^{+\infty} \mathbb{E}[[T_+ > n] \; ; \; e^{bS_n}]\right)\left(\sum_{n=0}^{+\infty} \mathbb{E}[[T_- > n] \; ; \; e^{-aS_n}]\right).$$

The proof is now complete. □

III Proofs of theorems A and B

III-a The Spitzer-Grincevicius factorisation

Let us first recall and precise some notations. Let $g_n = (a_n, b_n), n = 1, 2, \cdots$ be independant and identically distributed random variables of law μ. Note $\mathcal{F}_n$ the σ-algebra generated by the variables $g_1, g_2, \cdots, g_n, n \geq 1$. For any $n \geq 1$, put $G_1^n = g_1 \cdots g_n = (A_1^n, B_1^n)$; we have $A_1^n = a_1 \cdots a_n$ and $B_1^n = \sum_{k=1}^{n} a_1 \cdots a_{k-1} b_k$. More generally, for any $1 \leq n < m$, we will put $A_n^m = a_n \cdots a_m$ and $B_n^m = \sum_{k=n}^{m} a_n \cdots a_{k-1} b_k$. We also introduce the variables S_n, M_n and T_n defined by $S_n = Log A_1^n$ and $S_0 = 0$, $M_n = max(S_0, S_1, \cdots, S_n)$ and $T_n = inf\{0 \leq k \leq n / S_k = M_n\}$.

In the same way, let $\tilde{\mu}$ be the image of μ by the application $g \mapsto (\frac{1}{a(g)}, \frac{b(g)}{a(g)})$; if $\tilde{g}_n = (\tilde{a}_n, \tilde{b}_n), n = 1, 2, \cdots$ are independant and identically distributed random variables of law $\tilde{\mu}$ on G, we will put $\tilde{G}_1^n = \tilde{g}_1 \cdots \tilde{g}_n, \tilde{A}_1^n = \tilde{a}_1 \cdots \tilde{a}_n, \tilde{B}_1^n = \sum_{k=1}^{n} \tilde{a}_1 \cdots \tilde{a}_{k-1} \tilde{b}_k, \tilde{S}_n = Log \tilde{A}_n$ with $\tilde{S}_0 = 0$ and $\tilde{M}_n = max(\tilde{S}_0, \tilde{S}_1, \cdots, \tilde{S}_n)$. Note $\tilde{\mathcal{F}}_n$ the σ-algebra generated by the variables $\tilde{g}_1, \tilde{g}_2, \cdots, \tilde{g}_n, n \geq 1$.

Fix two positive functions φ and ψ, with compact support, defined respectively on $\mathbb{R}^{*+}$ and $\mathbb{R}^d$. For technical reasons, we suppose, without loss of generality, that ψ is continuously differentiable on $\mathbb{R}^d$. We are interested with the behaviour of the sequence

$(\mathbb{E}[\varphi(A_n)\psi(B_n)])_{n\geq 1}$ as n goes to $+\infty$; following ([8]), we have

$$\begin{aligned}
\mathbb{E}[\varphi(A_1^n)\psi(B_1^n)] &= \sum_{k=0}^{n} \mathbb{E}[[T_n = k]; \varphi(A_1^n)\psi(B_1^n)] \\
&= \sum_{k=0}^{n} \mathbb{E}[[A_1^k > 1] \cap [A_2^k > 1] \cap \cdots \cap [A_k^k > 1] \\
&\qquad \cap [A_{k+1}^{k+1} \leq 1] \cap [A_{k+1}^{k+2} \leq 1] \cdots \cap [A_{k+1}^n \leq 1] \; ; \; \varphi(A_1^n)\psi(B_1^n)]
\end{aligned}$$

The last expectation can be simplified as it is clear that the terms $A_1^k, A_2^k, \cdots, A_k^k$ are independant of the terms $A_{k+1}^{k+1}, A_{k+1}^{k+2} \cdots, A_{k+1}^n$; from the equality $B_1^n = A_1^k(\sum_{j=1}^{k} \frac{b_j}{A_j^k} + \sum_{j=k+1}^{n} A_{k+1}^{j-1} b_j)$ and by a duality argument, one obtains

$$\begin{aligned}
\mathbb{E}[\varphi(A_1^n)\psi(B_1^n)] &= \sum_{k=0}^{n} \mathbb{E}[[\tilde{A}_1^1 < 1] \cap [\tilde{A}_1^2 < 1] \cap \cdots \cap [\tilde{A}_1^k < 1] \\
&\qquad \cap [A_{k+1}^{k+1} \leq 1] \cap [A_{k+1}^{k+2} \leq 1] \cdots \cap [A_{k+1}^n \leq 1] \; ; \\
&\qquad \varphi(\frac{A_{k+1}^n}{\tilde{A}_1^k})\psi(\frac{1}{\tilde{A}_1^k}(\sum_{j=1}^{k} \tilde{A}_1^{j-1}\tilde{b}_j + \sum_{j=k+1}^{n} A_{k+1}^{j-1} b_j))].
\end{aligned}$$

Put $\mathcal{A} = \{g \in G : a(g) > 1\}$ and consider the transition kernel $P_{\mathcal{A}}$ associated with the couple $(\mu, \mathcal{A})$ and defined, for any Borel set $\mathcal{B} \subset G$, by

$$\forall g \in G \quad P_{\mathcal{A}}(g, \mathcal{B}) = \int_G 1_{\mathcal{A}^c \cap \mathcal{B}}(gh) \; \mu(dh).$$

Let us give the probabilistic interpretation of $P_{\mathcal{A}}$. Put $T_{\mathcal{A}} = inf\{n \geq 1 : G_1^n \in \mathcal{A}\}$; the random variable $T_{\mathcal{A}}$ is a waiting time with respect to the filtration $(\mathcal{F}_n)_{n\geq 1}$ and we have the following identity

$$\forall n \geq 1 \quad P_{\mathcal{A}}^n(e, B) = \mathbb{P}[[T_{\mathcal{A}} > n] \cap [G_1^n \in B]].$$

In the same way, put $\mathcal{A}' = \{g \in G / a(g) \geq 1\}$ and let $\tilde{P}_{\mathcal{A}'}$ be the operator associated with the couple $(\tilde{\mu}, \mathcal{A}')$. Denoting $\tilde{T}_{\mathcal{A}'}$ the waiting time with respect to the filtration $(\tilde{\mathcal{F}}_n))_{n\geq 1}$ defined by $\tilde{T}_{\mathcal{A}'} = inf\{n \geq 1 : \tilde{G}_1^n \in \mathcal{A}'\}$, we have

$$\forall n \geq 1 \quad \tilde{P}_{\mathcal{A}'}^n(e, B) = \mathbb{P}[[\tilde{T}_{\mathcal{A}'} > n] \cap [\tilde{G}_1^n \in B]].$$

From the previous expression of $\mathbb{E}[\varphi(A_1^n)\psi(B_1^n)]$, we obtain the **Spitzer-Grincevicius factorisation :**

$$\mathbb{E}[\varphi(A_1^n)\psi(B_1^n)] = \sum_{k=0}^{n} I_{k,n}(\varphi, \psi)$$

where

$$I_{k,n}(\varphi, \psi) = \int_{G\times G} \varphi(\frac{a(h)}{a(g)})\psi(\frac{b(g) + b(h)}{a(g)})\tilde{P}_{\mathcal{A}'}^k(e, dg)P_{\mathcal{A}}^{n-k}(e, dh).$$

III-b Proof of theorem A

The starting point of the proof is the Spitzer-Grincevicius factorisation. First, thanks to the two following lemmas, we are going to control the sum $\sum_{k=i+1}^{n-j} I_{k,n}(\varphi,\psi)$ for fixed large enough integers i and j.

Lemma III.1 *There exists $\lambda > 0$ and $C = C(\lambda, \varphi, \psi) > 0$ such that for any $g \in G$ and any $l > 0$, we have*

$$\int_G \varphi(\frac{a(h)}{a(g)})\psi(\frac{b(g)+b(h)}{a(g)})P^l_{\mathcal{A}}(e,dh) \quad \leq \quad \frac{C}{l^{3/2}}\, a(g)^\lambda$$

By theorem 2.1, the sequence $(k^{3/2}\int_G a(g)^\lambda \tilde{P}^k_{\mathcal{A}'}(e,dg))_{k\geq 0}$ is bounded since

$$\int_G a(g)^\lambda \tilde{P}^k_{\mathcal{A}'}(e,dg) = I\!E[[\tilde{T}_{\mathcal{A}'} > k]\ ; exp(\lambda \tilde{S}_k)].$$

Hence, using lemma 3.1, we obtain for any $0 < k < n$

$$\int_{G\times G} \varphi(\frac{a(h)}{a(g)})\psi(\frac{b(g)+b(h)}{a(g)})\tilde{P}^k_{\mathcal{A}'}(e,dg)P^{n-k}_{\mathcal{A}}(e,dh) \quad \leq \quad \frac{C_1}{k^{3/2}(n-k)^{3/2}}.$$

Using lemma 3.2, one can thus choose two integers i and $j, 0 < i < n-j < n$ such that the sum $n^{3/2}\sum_{k=i+1}^{n-j} I_{k,n}$ is quite small as wanted.

Lemma III.2 *There exists a constant $C > 0$ such that, for any $n \in I\!N^*$ and $0 < i < n-j < n$, we have*

$$n^{3/2}\sum_{k=i+1}^{n-j}\frac{1}{k^{3/2}(n-k)^{3/2}} \ \leq\ C(\frac{1}{\sqrt{i}}+\frac{1}{\sqrt{j}}).$$

Next, we look at the behaviour of the integral $\int_G \varphi(\frac{a(h)}{a(g)})\psi(\frac{b(g)+b(h)}{a(g)})P^l_{\mathcal{A}}(e,dh)$ as l goes to $+\infty$.

Lemma III.3 *For any $g \in G$, the sequence*

$$(l^{3/2}\int_G \varphi(\frac{a(h)}{a(g)})\psi(\frac{b(g)+b(h)}{a(g)})P^l_{\mathcal{A}}(e,dh))_{l\geq 0}$$

converges to a finite limit as l goes to $+\infty$.

Hence, for any $i \in \mathbb{N}$ and any compact $K \subset \mathbb{R}^{*+} \times \mathbb{R}$, the dominated convergence theorem yields the existence of a finite limit as n goes to $+\infty$ for the sequence $(n^{3/2} \sum_{k=0}^{i} I_{k,n}(\varphi, \psi, K))_{n \geq 0}$ where

$$I_{k,n}(\varphi, \psi, K) = \int_G 1_K(g) \quad (\int_G \varphi(\frac{a(h)}{a(g)}) \psi(\frac{b(g)+b(h)}{a(g)}) P_{\mathcal{A}}^{n-k}(e, dh)) \quad \tilde{P}_{\mathcal{A}'}^k(e, dg)).$$

The proof of the existence of a finite limit for the sequence

$$(n^{3/2} \sum_{k=n-j}^{n} I_{k,n}(\varphi, \psi, K))_{n \geq 0}$$

goes along the same line. The only thing we have now to verify is that the indicator function 1_K does not disturb too much the behaviour of the above integrals.
Fix $0 < \delta < 1$; according to lemma 3.1, we have

$$\begin{aligned}
&\sum_{k=1}^{i} \int_{\{g \in G: a(g) \leq \delta\}} \quad (\int_G \varphi(\frac{a(h)}{a(g)}) \psi(\frac{b(g)+b(h)}{a(g)}) P_{\mathcal{A}}^{n-k}(e, dh)) \quad \tilde{P}_{\mathcal{A}'}^k(e, dg) \\
\leq \quad & C(\lambda, \varphi, \psi) \sum_{k=1}^{i} \frac{1}{(n-k)^{3/2}} \mathbb{E}[[\tilde{T}_{\mathcal{A}'} > k] \cap [\tilde{S}_k \leq Log\delta] \; ; \; exp(\lambda \tilde{S}_k)] \\
\leq \quad & C(\lambda, \varphi, \psi) \; \delta^{\lambda/2} \sum_{k=1}^{i} \frac{1}{(n-k)^{3/2}} \quad E[[\tilde{T}_{\mathcal{A}'} > k] \; ; \; exp(\frac{\lambda}{2} \tilde{S}_k)] \\
\leq \quad & C_1 \delta^{\lambda/2} \sum_{k=1}^{i} \frac{1}{(n-k)^{3/2} k^{3/2}}.
\end{aligned}$$

On the other hand, fix $B > 0$; according to lemma 3.1, we have

$$\begin{aligned}
&\sum_{k=1}^{i} \int_{\{g \in G: b(g) \geq B\}} \quad (\int_G \varphi(\frac{a(h)}{a(g)}) \psi(\frac{b(g)+b(h)}{a(g)}) P_{\mathcal{A}}^{n-k}(e, dh)) \quad \tilde{P}_{\mathcal{A}'}^k(e, dg) \\
\leq \quad & C(\lambda, \varphi, \psi) \sum_{k=1}^{i} \frac{1}{(n-k)^{3/2}} \mathbb{E}[[\tilde{T}_{\mathcal{A}'} > k] \cap [\|\tilde{B}_k\| \geq B] \; ; \; exp(\lambda \tilde{S}_k)] \\
\leq \quad & \frac{C(\lambda, \varphi, \psi)}{B^{\lambda/2}} \sum_{k=1}^{i} \frac{1}{(n-k)^{3/2}} \; E[[\tilde{T}_{\mathcal{A}'} > k] \; ; \; exp(\lambda \tilde{S}_k) \; \|\tilde{B}_k\|^{\lambda/2}] \\
\leq \quad & \frac{C_1}{B^{\lambda/2}} \sum_{k=1}^{i} \frac{1}{(n-k)^{3/2} k^{3/2}}.
\end{aligned}$$

where the last inequality is guaranteed by the following

Lemma III.4 *There exists $\epsilon_0 > 0$ such that for any $0 < \epsilon < \epsilon_0$*

$$\sup_{l \geq 1} \quad l^{3/2} \mathbb{E}[[\tilde{T}_{\mathcal{A}'} > l] \; ; \; exp(2\epsilon \tilde{S}_l) \; \|\tilde{B}_l\|^{\epsilon}] \quad < \quad +\infty.$$

Finally, using the Spitzer-Grincevicius factorisation, we have prove that, for any $\eta > 0$, one can find $i, j \in \mathbb{N}$ and a compact $K \subset G$ such that for any $n > i + j$ we have

$$|n^{3/2}\mathbb{E}[\varphi(A_1^n)\psi(B_1^n)] - n^{3/2}\sum_{k=0}^{i} I_{k,n}(\varphi,\psi,K) - n^{3/2}\sum_{k=n-j}^{n} I_{k,n}(\varphi,\psi,K)| \leq \eta.$$

On the other hand, we have also proved that the sequence

$$(n^{3/2}\sum_{k=0}^{i} I_{k,n}(\varphi,\psi,K) + n^{3/2}\sum_{k=n-j}^{n} I_{k,n}(\varphi,\psi,K))_{n\geq 0}$$

converges as n goes to $+\infty$. So $(n^{3/2}\mathbb{E}[\varphi(A_1^n)\psi(B_1^n)])_{n\geq 0}$ converges to a finite limit as n goes to $+\infty$. Thus, by a standart argument in Radon measure theory, it readily follows that the sequence of measures $(n^{3/2}\mu^{*n})_{n\geq 1}$ weakly converges to a Radon measure ν_0 ; the fact that ν_0 is not degenerated follows from the

Lemma III.5 *There exist an integer n_0 and a compact set $K_0 \subset G$ such that*

$$\inf_{n\geq n_0} n^{3/2}\mathbb{E}[g_1\cdots g_n \in K_0] \ > \ 0.$$

The proof of theorem A is now complete ; it just remains to establish the different lemmas.

Proof of lemma 3.1. - Fix $p > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$. For any $g \in G$ and $l \geq 1$, we have

$$\begin{aligned}
&\int_G \varphi(\frac{a(h)}{a(g)})\psi(\frac{b(g)+b(h)}{a(g)})P_A^l(e,dh)\\
&= \int_{]0,1]\times\mathbb{R}} \mathbb{E}[[aA_2^2 \leq 1]\cap\cdots\cap[aA_2^l \leq 1] \ ; \ \varphi(\frac{aA_2^l}{a(g)}) \ \psi(\frac{b(g)+\sum_{i=2}^{l} aA_2^{i-1}b_i + b}{a(g)})] \ \phi_\mu(a,b) \ \frac{dadb}{a}\\
&\leq a(g)^{\frac{1}{p}} \ \|\psi\|_p \int_0^1 \sqrt[q]{\int_{\mathbb{R}} \phi_\mu^q(a,b)db} \ \mathbb{E}[[aA_2^2 \leq 1]\cap\cdots\cap[aA_2^l \leq 1] \ ; \ \varphi(\frac{aA_2^l}{a(g)})] \ \frac{da}{a}\\
&\leq a(g)^{\frac{1}{p}} \ \|\psi\|_p \int_0^1 \sqrt[q]{\int_{\mathbb{R}} \phi_\mu^q(a,b)db} \ \mathbb{E}[[exp(M_{l-1}) \leq \frac{1}{a}] \ ; \ \varphi(\frac{aA_1^{l-1}}{a(g)})] \ \frac{da}{a}\\
&\leq a(g)^{\frac{1}{p}} \ \|\psi\|_p \int_0^1 \sqrt[q]{\int_{\mathbb{R}} \phi_\mu^q(a,b)db} \ \frac{1}{a^{2\epsilon}}\mathbb{E}[exp(-2\epsilon M_{l-1}) \ \varphi(\frac{aA_1^{l-1}}{a(g)})] \ \frac{da}{a} \quad \text{for any} \quad \epsilon > 0.
\end{aligned}$$

Since the support of φ is compact in $]0,+\infty[$, there exists $K = K(\epsilon,\varphi) > 0$ such that $\forall a > 0 \quad |\varphi(a)| \leq K \ a^\epsilon$; so

$$\begin{aligned}
&\int_G \varphi(\frac{a(h)}{a(g)})\psi(\frac{b(g)+b(h)}{a(g)})P_A^l(e,dh)\\
&\qquad \leq K \ a(g)^{\frac{1}{p}-\epsilon} \ \|\psi\|_p \int_0^1 \sqrt[q]{\int_{\mathbb{R}} \phi_\mu^q(a,b)db} \ \mathbb{E}[[exp(-\epsilon(M_{l-1}-S_{l-1})) \ exp(-\epsilon M_{l-1})]\frac{da}{a^{1+\epsilon}}.
\end{aligned}$$

Choosing ϵ quite small such that $\frac{1}{p} - \epsilon > 0$ and $1 + \epsilon < \beta$, one thus obtains by theorem 2.5

$$\int_G \varphi(\frac{a(h)}{a(g)})\psi(\frac{b(g)+b(h)}{a(g)})P_A^l(e,dh) \leq \frac{C}{l^{3/2}}\ a(g)^{\frac{1}{p}-\epsilon}.$$

The proof is complete. □

Proof of lemma 3.2. - We have

$$\begin{aligned} n^{3/2}\sum_{k=i+1}^{n-j}\frac{1}{k^{3/2}(n-k)^{3/2}} &= \frac{1}{n^{3/2}}\sum_{k=i+1}^{n-j}\frac{1}{(\frac{k}{n})^{3/2}(1-\frac{k}{n})^{3/2}} \\ &\sim \frac{1}{\sqrt{n}}\int_{\frac{i}{n}}^{1-\frac{j}{n}}\frac{dx}{x^{3/2}(1-x)^{3/2}} \quad \text{as} \quad n \to +\infty. \end{aligned}$$

The lemma follows from an elementary overestimation of this last integral. □

Proof of lemma 3.3. - Without loss of generality, one may suppose $g = e$. For any $n \in \mathbb{N}^*$, set

$$\nu_n(\varphi,\psi) = n^{3/2}\mathbb{E}[[T_A > n]\ ;\ \varphi(A_1^n)\psi(B_1^n)].$$

Fix $i, j \in \mathbb{N}$ such that $1 \leq i < n-j \leq n$ and consider

$$\nu_n(\varphi,\psi,i,j) = n^{3/2}\mathbb{E}[[T_A > n]\ ;\ \varphi(A_1^n)\psi(B_1^i + A_1^{n-j}B_{n-j+1}^n)].$$

To obtain the claim, it suffices to prove that

a) $\limsup_{i,j\to+\infty}\ \limsup_{n\to+\infty}|\nu_n(\varphi,\psi) - \nu_n(\varphi,\psi,i,j)| = 0$

b) for any fixed $i, j \in \mathbb{N}$, the sequence $(\nu_n(\varphi,\psi,i,j))_{n\geq 1}$ converges to a finite limit.

To prove assumption a), we use the identity $B_1^n = B_1^i + A_1^i B_{i+1}^{n-j} + A_1^{n-j}B_{n-j+1}^n$; since the support of ψ is compact and ψ is continuously differentiable, we have for some $0 < \epsilon < 1$

$$\begin{aligned} |\nu_n(\varphi,\psi) - \nu_n(\varphi,\psi,i,j)| &\leq C_1\ n^{3/2}\ \mathbb{E}[[T_A > n]\ ;\ \varphi(A_1^n)(A_1^i)^\epsilon\|B_{i+1}^{n-j}\|^\epsilon] \\ &\leq C_1\ n^{3/2}\sum_{k=i+1}^{n-j}\mathbb{E}[[T_A > n]\ ;\ \varphi(A_1^n)(A_1^{k-1})^\epsilon\|b_k\|^\epsilon] \\ &\leq C_1\|\varphi\|_\infty\ n^{3/2}\sum_{k=i+1}^{n-j}\mathbb{E}[[T_A > k-1]\ ;\ (A_1^{k-1})^\epsilon\|b_k\|^\epsilon] \\ &\leq C_1\|\varphi\|_\infty\mathbb{E}[\|b_1\|^\epsilon]\ n^{3/2}\sum_{k=i+1}^{n-j}\mathbb{E}[[T_A > k-1]\ ;\ (A_1^{k-1})^\epsilon] \\ &\leq C_2(\frac{1}{\sqrt{i}} + \frac{1}{\sqrt{j}}) \end{aligned}$$

the last inequality being guaranteed by theorem 2.1 and lemma 3.2. Let i and j go to $+\infty$; we obtain the claim a).

Next, we prove b). Fix two integers i and j ; we have the following equality

$$\nu_n(\varphi,\psi,i,j) = \int_{G^{j+1}} E_n(\varphi,\psi,g,h_1,h_2,\cdots,h_j) P_{\mathcal{A}}^i(e,dg)\mu(dh_1)\mu(dh_2)\cdots\mu(dh_j)$$

with

$$\begin{aligned} E_n(\varphi,\psi,g,h_1,h_2,\cdots,h_j) \; = \; & I\!E[[max(A_{i+1}^{i+1},\cdots,A_{i+1}^{n-j}) \leq \frac{1}{a(g)}] \cap \\ & [A_{i+1}^{n-j} \leq min(\frac{1}{a(g)},\frac{1}{a(g)a(h_1)},\cdots,\frac{1}{a(g)a(h_1)\cdots a(h_j)})]; \\ & \varphi(a(g)A_{i+1}^{n-j}a(h_1)\cdots a(h_j))\psi(b(g)+a(g)A_{i+1}^{n-j}b(h_1\cdots h_j))] \end{aligned}$$

Using theorem 2.5, one may see that, for any $g,h_1,\cdots,h_j \in G$, the sequence $(n^{3/2}E_n(\varphi,\psi,g,h_1,h_2,\cdots,h_j))_{n\geq 1}$ converges to a finite limit. To obtain the claim b), we have to use Lebesgue dominated convergence theorem and so, we have to obtain an appropriate overestimation of $n^{3/2}E_n(\varphi,\psi,g,h_1,h_2,\cdots,h_j)$. Since the support of φ is compact in $]0,+\infty[$, for any $\epsilon > 0$, there exists $K = K(\epsilon,\varphi) > 0$ such that $\forall a > 0\ |\varphi(a)| \leq Ka^\epsilon$. On the other hand, we have the inclusions

$$[max(A_{i+1}^{i+1},\cdots,A_{i+1}^{n-j}) \leq \frac{1}{a(g)}] \subset [max(1,A_{i+1}^{i+1},\cdots,A_{i+1}^{n-j}) \leq \frac{1}{a(g)}] \quad \text{because} \quad a(g)\leq 1$$

and

$$[A_{i+1}^{n-j} \leq min(\frac{1}{a(g)},\frac{1}{a(g)a(h_1)},\cdots,\frac{1}{a(g)a(h_1)\cdots a(h_j)}] \subset [A_{i+1}^{n-j} \leq \frac{1}{a(g)}].$$

Consequently

$$\begin{aligned} & n^{3/2}E_n(\varphi,\psi,g,h_1,h_2,\cdots,h_j) \\ & \leq C\|\psi\|_\infty n^{3/2} I\!E[a(g)^\epsilon (A_{i+1}^{n-j})^\epsilon a(h_1)^\epsilon\cdots a(h_j)^\epsilon \\ & \qquad \times \frac{1}{a(g)^{2\epsilon} max(1,A_{i+1}^{i+1},\cdots,A_{i+1}^{n-j})^{2\epsilon}} \frac{1}{(A_{i+1}^{n-j})^{\epsilon/2} a(g)^{\epsilon/2}}] \\ & \leq C\|\psi\|_\infty a(g)^{-3\epsilon/2} a(h_1)^\epsilon \cdots a(h_j)^\epsilon\ n^{3/2} I\!E[(A_{i+1}^{n-j})^{\epsilon/2} max(1,A_{i+1}^{i+1},\cdots,A_{i+1}^{n-j})^{-2\epsilon}] \\ & \leq C\|\psi\|_\infty a(g)^{-3\epsilon/2}\ a(h_1)^\epsilon \cdots a(h_j)^\epsilon n^{3/2} I\!E[(A_{i+1}^{n-j})^{\epsilon/2} max(1,A_{i+1}^{i+1},\cdots,A_{i+1}^{n-j})^{-2\epsilon}] \quad \text{since } a(g)\leq 1 \\ & \leq C_1 a(g)^{-3\epsilon/2}\ a(h_1)^\epsilon\cdots a(h_j)^\epsilon \end{aligned}$$

the last inequality being guaranteed by theorem 2.5. Then, by hypothesis $A2$, for ϵ quite small, one may use Lebesgue dominated convergence theorem. The proof of claim b) is now complete. □

Proof of lemma 3.4. - By a duality argument, it suffices to prove that, for some $\epsilon > 0$

$$\sup_{n\geq 1} \quad n^{3/2} I\!E[[T_{\mathcal{A}} > n]\ ;\ (A_1^n)^{2\epsilon}\ \|B_1^n\|^\epsilon] \quad < \quad +\infty.$$

Using the identity $B_1^n = \sum_{k=1}^{n} A_1^{k-1} b_k$, we obtain

$$\mathbb{E}[[T_{\mathcal{A}} > n] \; ; \; (A_1^n)^{2\epsilon} \; \|B_1^n\|^\epsilon] \leq \sum_{k=1}^{n} \mathbb{E}[[T_{\mathcal{A}} > n] \; ; \; (A_1^{k-1})^{3\epsilon} a_k^{2\epsilon} \|b_k\|^\epsilon (A_{k+1}^n)^{2\epsilon}].$$

From the definition of the waiting time $T_{\mathcal{A}}$, we have

$$\begin{aligned}
&\mathbb{E}[[T_{\mathcal{A}} > n] \; ; \; (A_1^{k-1})^{3\epsilon} a_k^{2\epsilon} \|b_k\|^\epsilon (A_{k+1}^n)^{2\epsilon}] \\
&\leq \; \mathbb{E}[[A_1^1 \leq 1] \cap \cdots [A_1^{k-1} \leq 1] \cap [a_k \leq \frac{1}{A_1^{k-1}}] \cap [A_{k+1}^{k+1} \leq \frac{1}{A_1^{k-1} a_k}] \cap \cdots [A_{k+1}^n \leq \frac{1}{A_1^{k-1} a_k}] \; ; \\
&\qquad\qquad (A_1^{k-1})^{3\epsilon} a_k^{2\epsilon} \|b_k\|^\epsilon (A_{k+1}^n)^{2\epsilon}] \\
&\leq \; \int_G a(g)^{3\epsilon} \quad [\int_{\{h\in G : a(g)a(h) \leq 1\}} a(h)^{2\epsilon} \|b(h)\|^\epsilon K_{k,n}(g,h) \mu(dh)] \quad P_{\mathcal{A}}^{k-1}(e, dg)
\end{aligned}$$

with

$$\begin{aligned}
K_{k,n}(g,h) &= \mathbb{E}[[A_{k+1}^{k+1} \leq \frac{1}{a(g)a(h)}] \cap \cdots \cap [A_{k+1}^n) \leq \frac{1}{a(g)a(h)}] \; ; \; (A_{k+1}^n)^{2\epsilon}] \\
&= \mathbb{E}[[A_1^1 \leq \frac{1}{a(g)a(h)}] \cap \cdots \cap [A_1^{n-k} \leq \frac{1}{a(g)a(h)}] \; ; \; (A_1^{n-k})^{2\epsilon}] \\
&\leq \mathbb{E}[[max(1, A_1^1, \cdots, A_1^{n-k}) \leq \frac{1}{a(g)a(h)}] \; ; \; (A_1^{n-k})^{2\epsilon}] \quad \text{since } a(g)a(h) \leq 1 \\
&\leq \frac{1}{a(g)^{5\epsilon/2} a(h)^{5\epsilon/2}} \mathbb{E}[exp(-\frac{\epsilon}{2} M_{n-k}) \quad exp(-2\epsilon(M_{n-k} - S_{n-k}))] \\
&\leq \frac{1}{a(g)^{5\epsilon/2} a(h)^{25\epsilon/2}} \frac{C_1}{(n-k)^{3/2}}
\end{aligned}$$

Hence, we have

$$\mathbb{E}[[T_{\mathcal{A}} > n] \; ; \; (A_1^n)^{2\epsilon} \; \|B_1^n\|^\epsilon] \leq \frac{C_1}{(n-k)^{3/2}} \mathbb{E}[\frac{\|b_1\|^\epsilon}{a_1^{\epsilon/2}}] \int_G a(g)^{\epsilon/2} P_{\mathcal{A}}^{k-1}(e, dg)$$

One readily concludes, using hypothesis $A2$ and the fact that the sequence $(n^{3/2} \sum_{k=1}^{n-1} \frac{1}{k^{3/2}(n-k)^{3/2}})_n$ is bounded. □

Proof of lemma 3.5- By theorem 2.1, there exist $n_0 \in \mathbb{N}$, $C_0 > 0$ and $[\alpha, \beta] \subset \mathbb{R}^{*+}$ such that

$$\forall n \geq n_0 \quad n^{3/2} \mathbb{E}[[T_{\mathcal{A}} > n] \cap [\alpha \leq A_1^n \leq \beta]] \quad \geq \quad C_0.$$

On the other hand, we have

$$n^{3/2} \mathbb{E}[[T_{\mathcal{A}} > n] \cap [\alpha \leq A_1^n \leq \beta] \cap [\|B_1^n\| \geq B]] \; \leq \; \frac{n^{3/2}}{B^\epsilon} \mathbb{E}[[T_{\mathcal{A}} > n] \cap [\alpha \leq A_1^n \leq \beta] \; ; \; \|B_1^n\|^\epsilon].$$

By lemma 3.4, we have $\sup_{n\geq 1} n^{3/2}\mathbb{E}[[T_A > n] \cap [\alpha \leq A_1^n \leq \beta] \; ; \; \|B_1^n\|^\epsilon] \; < \; +\infty$; so, one can choose $B > 0$ such that

$$\forall n \geq n_0 \quad n^{3/2}\mathbb{E}[[T_A > n] \cap [\alpha \leq A_1^n \leq \beta] \cap [\|B_1^n\| \leq B]] \quad \geq \quad \frac{C_0}{2}.$$

The lemma readily follows from the inequality

$$n^{3/2}\mathbb{E}[[\alpha \leq A_1^n \leq \beta] \cap [\|B_1^n\| \leq B]] \quad \geq \quad n^{3/2}\mathbb{E}[[T_A > n] \cap [\alpha \leq A_1^n \leq \beta] \cap [\|B_1^n\| \leq B]].$$

□

III-c Proof of theorem B : identification of the limit measure ν_0

We are not always able to explicit the form of the limit measure ν_0 ; nevertheless, if one assumes some additionnal hypotheses on μ, it is possible to identify ν_0, up to a multiplicative constant. In this section, we suppose that μ satisfies hypotheses $A1 - A4$ and the two additionnal conditions

B1) the density ϕ_μ of μ is continuous with compact support

$B2)$ $\phi_\mu(e) > 0$

Remark- Note that under these conditions, the semi-group generated by the support S_μ of μ is dense in G. More, there exist $\gamma > 0$ such that

$$\mu * \mu \geq \gamma\mu.$$

The proof of theorem B is based on ([9]) and may be broken down into two steps ; first, we prove that the random walk of law μ on G satisfies a ratio-limit theorem and secondly we show that the double equation $\mu * \nu = \nu * \mu = \nu$ has a unique solution $\nu_0 \not\equiv 0$ (to within a constant multiple) in the class of Radon measures on G. Let $CK^+(G)$ be the space of positive continuous functions with compact support on G ; we have the

Lemma III.6 *Under hypotheses $B1$ and $B2$, we have*

$$\forall \varphi \in CK^+(G), \; \forall g \in G \quad \lim_{n\to+\infty} (\delta_g * \mu^{*n}(\varphi))^{1/n} = 1.$$

Since there exist $\gamma > 0$ such that $\mu * \mu \geq \gamma\mu$ (see the above remark), we thus may apply the Proposition 3 in ([9]) and the previous lemma implies

$$\lim_{n\to+\infty} \frac{\delta_g * \mu^{*(n+1)}(\varphi)}{\delta_g * \mu^{*n}(\varphi)} = 1,$$

for any $g \in G$ and any function $\varphi \in CK^+(G), \varphi \not\equiv 0$.

Now, let us recall the result established in ([9]) :

Suppose that the semi-group generated by the support of μ is dense in G and that, for any $\varphi \in CK^+(G)$, the sequence $(\frac{\mu^{*(n+1)}(\varphi)}{\mu^{*n}(\varphi)})_{n\geq 1}$ converges to a constant c_0 which does not depend on φ. Then, if the equation $\nu * \mu = \mu * \nu = c_0\nu$ has a unique (to within a constant multiple) solution $\nu_0 \not\equiv 0$ in the class of Radon measure on G, we have for any φ and $\psi \in CK^+(G)$ such that $\nu_0(\psi) > 0$

$$\lim_{n\to+\infty} \frac{\mu^{*n}(\varphi)}{\mu^{*n}(\psi)} = \frac{\nu_0(\varphi)}{\nu_0(\psi)}.$$

In the present case, we have $c_0 = 1$; to prove the theorem, it suffices to establish the

Lemma III.7 *Under hypotheses of theorem B, the equation $\nu*\mu = \mu*\nu = \nu$ has one and only one (to within a constant multiple) solution $\nu_0 \not\equiv 0$ in the class of Radon measure on G. Moreover, this solution may be decomposed as follows*

$$\nu_0 = (\delta_1 \otimes \lambda) * \overline{(\frac{da}{a} \otimes \delta_0)} * (\delta_1 \otimes \overline{\lambda_1})$$

where λ (respectively λ_1) is, up to a multiplicative constant, the unique Radon measure on $\mathbb{R}^d$ which satisfies the convolution equation $\mu * \lambda = \lambda$ (resp. $\overline{\mu} * \lambda_1 = \lambda_1$).

By theorem A one can choose $\psi_0 \in CK^+(G)$ such that the sequence $(n^{3/2}\mu^{*n}(\psi))_{n\geq 0}$ converges to 1 ; so, for any $\phi \in CK^+(G)$ we have

$$\lim_{n\to+\infty} n^{3/2}\mu^{*n}(\varphi) = \frac{\nu_0(\varphi)}{\nu_0(\psi)}.$$

This achieves the proof of theorem B ; it remains to establish the two lemmas.

Proof of lemma 3.6- Fix a function $\varphi \in CK^+(G)$ and consider, for any integer $n \geq 1$, the set $K_n(\varphi) = \{gh^{-1}/g \in \textit{Support}(\varphi) \textit{ and } h \in \textit{Support}(\mu^{*n})\}$. The sets $K_n(\varphi), n \geq 1$, are compact and we have $K_n(\varphi) \subset K_{n+1}(\varphi)$ and $\bigcup_{n=1}^{+\infty} K_n(\varphi) = G$. Then, there exits n_0 such that the compact K_0 of the lemma 3.5 is included in the interior of $K_{n_0}(\varphi)$. Consequently, the continuous function $g \mapsto \int_G \varphi(gh)\mu^{*n_0}(dh)$ is strictly positive on K_0 and so, there exists a constant $C > 0$ such that

$$\forall g \in G \quad \int_G \varphi(gh)\mu^{*n_0}(dh) \geq C\ 1_{K_0}(g).$$

It follows that, for any $n \geq 1$

$$\begin{aligned}
\delta_g * \mu^{*(n_0+n)}(\varphi) &\geq C\ \mu^{*n}(K_0) \\
&\geq \frac{C_1}{n^{3/2}} \qquad \text{with } C_1 > 0 \text{ by the lemma 3.5.}
\end{aligned}$$

It readily follows that for any $g \in G$ we have $\liminf_{n\to+\infty}(\delta_g * \mu^{*n}(\varphi))^{1/n} \geq 1$. On the other hand, we have $\forall n \geq 1 \quad \delta_g * \mu^{*n}(\varphi) \leq \|\varphi\|_\infty$ which implies $\limsup_{n\to+\infty}(\delta_g * \mu^{*n}(\varphi))^{1/n} \geq 1$. The proof is now complete. □

Proof of lemma 3.7- First, let us describe the solutions of the equation $\nu * \mu = \nu$ where ν is a Radon measure on G. From ([5]), it is known that there exists a unique (up to a multiplicative constant) Radon measure λ_1 on $\mathbb{R}^d$ such that $\overline{\mu} * \lambda_1 = \lambda_1$ and that the extremal points of the cone $\mathcal{H}_\mu$ of Radon measure ν such that $\nu * \mu = \nu$ are proportional, either to the right Haar measure m_D or to the measures $\delta_{(1,z)} * \overline{(\frac{da}{a} \otimes \lambda_1)}$, where $z \in \mathbb{R}^d$. Under the hypotheses $B1$ and $B2$, the base of $\mathcal{H}_\mu$ is compact ; thus, by Choquet's theorem, there exit $C \in \mathbb{R}^+$ and a positive measure m on $\mathbb{R}$ such that

$$\nu = C\ m_D + \int_{\mathbb{R}} \delta_{(1,z)} * \overline{(\frac{da}{a} \otimes \lambda_1)}\ m(dz) \qquad\qquad (*).$$

Now, let us find the measures ν satisfying the decomposition $(*)$ and such that $\mu * \nu = \nu$. If $\mu * \nu = \nu$, then $\overline{\nu} * \overline{\mu} = \overline{\nu}$. Since $\overline{m}_D(dadb) = \dfrac{da\ db}{a^2}$, it readily follows that, for any positive Borel function φ on G, we have

$$\begin{aligned}
\overline{m}_D * \overline{\mu}(\varphi) &= \int_{G\times G} \varphi(aa_1, ab_1 + b)\frac{da\ db}{a^2}\ \overline{\mu}(dg_1) \\
&= \int_{G\times G} \frac{a_1}{A^2}\varphi(A,B)\ dA\ dB\ \overline{\mu}(dg_1) \\
&= \overline{m}_D(\varphi) \times \int_G a_1\ \overline{\mu}(dg_1).
\end{aligned}$$

Since $\int_G a_1\ \overline{\mu}(dg_1) > 1$, one obtains $C = 0$ in the decomposition $(*)$. On the other hand, we have

$$\begin{aligned}
&\int_{\mathbb{R}} ((\frac{da}{a} \otimes \lambda_1) * \delta_{(1,-z)} * \overline{\mu})\ (\varphi)\ m(dz) \\
&= \int_{\mathbb{R}}\int_{G\times G} \varphi((a, -az + b)g_1)\ \frac{da}{a}\lambda_1(db)\ \overline{\mu}(dg_1)\ m(dz) \\
&= \int_{\mathbb{R}}\int_{G\times G} \varphi(\frac{a}{a_1}, -a(z + \frac{b_1}{a_1}) + b)\ \frac{da}{a}\ \lambda_1(db)\ \mu(dg_1)\ m(dz) \\
&= \int_{\mathbb{R}}\int_{G\times G} \varphi(A, -A(a_1z + b_1) + b)\ \ m(dz)\ \mu(dg_1)\ \frac{dA}{A}\ \lambda_1(db)
\end{aligned}$$

and $\int_{\boldsymbol{R}}((\frac{da}{a}\otimes\lambda_1)*\delta_{(1,-z)})\ (\varphi)\ m(dz)=\int_{\boldsymbol{R}}\varphi(A,-Az+b)\ m(dz)\ \frac{dA}{A}\ \lambda_1(db).$
Then, the equality

$$\int_{\boldsymbol{R}}((\frac{da}{a}\otimes\lambda_1)*\delta_{(1,-z)}*\overline{\mu})\ (\varphi)\ m(dz)=\int_{\boldsymbol{R}}((\frac{da}{a}\otimes\lambda_1)*\delta_{(1,-z)})\ (\varphi)\ m(dz)$$

is satisfied for any positive Borel function φ if and only if $\mu * m = m$; by ([5]), this equation has one and only one solution λ (up to a constant multiple).
The proof of lemma 3.7 is now complete. □

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