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THE POISSON BOUNDARY OF THE MAPPING CLASS GROUP AND OF TEICHMÜLLER SPACE

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ABSTRACT. A theory of random walks on the mapping class group and Teichmüller space is developed. We prove convergence of sample paths in the Thurston compactification and show that the space of projective measured foliations with the corresponding harmonic measure can be identified with the Poisson boundary of random walks. The methods are based on an analysis of the asymptotic geometry of Teichmüller space and of the contraction properties of the action of the mapping class group on the Thurston boundary. We prove, in particular, that Teichmüller space is roughly isometric to a graph with uniformly bounded vertex degrees. Using our analysis of the mapping class group action on the Thurston boundary we prove that no non-elementary subgroup of the mapping class group can be a lattice in a higher rank semi-simple Lie group. For studying boundary behavior of bounded range invariant Markov operators on Teichmüller space we establish a Harnack inequality, which is then used for discretization of corecurrent operators.

CONTENTS

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0. Introduction

The *mapping class group* $\Gamma = \mathrm{Mod}(g) = \mathrm{Diff}^+(M)/\mathrm{Diff}_0(M)$ of a closed C^∞ surface M of genus $g \geq 2$ consists of isotopy classes of orientation preserving diffeomorphisms of M. This group plays a fundamental role in the topology in dimensions 2 and 3. A study of the dynamics of Γ was initiated by Thurston [Th]. He constructed a space $\mathcal{PMF}$ of *projective measured foliations* of M on which Γ acts, and used this action to classify the elements of Γ into finite order ones, reducible, and pseudo-Anosov. This classification generalizes the classification of elements of $\mathrm{Mod}(1) = SL(2,\mathbb{Z})$ into elliptic, parabolic, and hyperbolic ones. Thurston also showed that topologically $\mathcal{PMF}$ is a sphere of dimension $6g-7$, that it is the boundary of a compactification of *Teichmüller space* T_g of genus g, and that the natural discontinuous action of Γ on T_g by isometries extends to an action on $\mathcal{PMF}$. This picture gives a vast generalization of the genus 1 case, in which T_1 is the hyperbolic plane $\mathbf{H}^2$, and $\mathcal{PMF}$ is its circle at infinity [FLP].

The action of Γ on Teichmüller space has been used to solve the Nielsen realization problem [Ke2] (every finite subgroup of Γ can be realized as a finite subgroup of $\mathrm{Diff}^+(M)$), and the dynamics of the action of Γ on $\mathcal{PMF}$ – to prove algebraic results about Γ. For example, McCarthy [Mc] proved a Tits alternative theorem for Γ, and Ivanov [Iv2] showed that $\mathrm{Out}(\Gamma)$ consists of 2 elements for $g \geq 3$.

Another far reaching generalization of the action of $SL(2,\mathbb{Z})$ on $\mathbf{H}^2$ is provided by the Gromov theory of *hyperbolic spaces and groups* [Gr]. Although for $g \geq 2$ neither Teichmüller space T_g is a Gromov hyperbolic space [MW], nor the mapping class group Γ is a word hyperbolic group (as it contains rank 2 abelian subgroups generated by Dehn twists about disjoint curves), they still share some important global properties with general Gromov hyperbolic spaces and groups as we shall see below (actually, our approach is based on exploiting these properties), and apparently can be considered as prospective examples for a future "semi-hyperbolic" theory.

From a completely different perspective one can often understand groups from studying boundary behavior of the group, more specifically, measure type preserving group actions on boundaries naturally associated with the group. The most spectacular example is the Mostow–Margulis *rigidity theory* [Mar], [Mo], [Zi].

Another example is the Patterson–Sullivan theory of *conformal densities* on the sphere at infinity of Cartan–Hadamard manifolds. In the constant curvature case this notion is non-trivial (conformal density does not belong to the Lebesgue measure type) only for groups with co-infinite volume [Pa], [Su]; however, this is not so if the curvature is non-constant. For example, in the cocompact case the corresponding conformal density is directly connected with the Bowen–Margulis (maximal entropy) invariant measure of the geodesic flow [Ka3]. For a discrete group G of isometries of a Cartan–Hadamard manifold G-invariant conformal densities are obtained by taking weak limits (with respect to the visibility compactification) of the normalized family of measures on the G-orbit of a reference point o with exponential weights proportional to $e^{-sd(o,go)}$, as s tends to the critical exponent of the Poincaré series $\sum_g e^{-sd(o,go)}$.

Generally speaking, one may try to construct "boundary actions" of a discrete group G by taking limits of sequences of probability measures on G tending to infinity. A natural sequence of this kind is the sequence of n-fold *convolutions* μ_n of a given probability measure μ on G. It turns out that one can associate with the pair (G, μ) a probability measure space $(\partial G, \nu)$ endowed with an ergodic action of G which is called the *Poisson boundary*. The *harmonic measure* ν is *μ-stationary* (i.e., $\nu = \sum \mu(g) g\nu$), and the *Poisson formula* $f(g) = \langle \widehat{f}, g\nu \rangle$ (a direct analogue of the classic Poisson formula for bounded harmonic functions in the disk) establishes an isometry between the space of bounded *μ-harmonic functions* on G (those that satisfy the mean value property $f(g) = \sum_x f(gx)\mu(x)\ \forall g \in G$) and the space $L^\infty(\partial G, \nu)$. The Poisson boundary was first introduced by Furstenberg for semi-simple Lie groups [Fu1], and can be defined in a number of various equivalent ways (see [Ka11]).

Although the harmonic measure ν on the Poisson boundary can be in a sense considered as a limit of the sequence μ_n, the Poisson boundary is a purely measure theoretical object and does not require for its definition any *a priori* compactification of G. Using this invariant of the pair (G, μ) Furstenberg proved that a discrete subgroup of a rank one semi-simple Lie group cannot be a lattice in a higher rank semi-simple group, which was one of the first results of rigidity theory [Fu2].

For any measure on an abelian, or more generally, a nilpotent group the Poisson boundary is trivial (i.e., consists of a single point). This is equivalent to saying that there are no non-constant bounded harmonic functions on such groups. For a general amenable group there always exists a measure μ with trivial Poisson boundary (but there may also be measures with a non-trivial boundary). On the other hand, for any (non-degenerate) measure on a non-amenable group the Poisson boundary is non-trivial [KV].

There is also a topological *Martin boundary* associated with the pair (G, μ), which is the boundary of the *Martin compactification* of G and is responsible for integral representation of all positive μ-harmonic functions. Note that the constructions of the Martin and the Thurston boundaries are in a sense parallel. Using the Green kernel (resp., the intersection function) one embeds the group (resp., the set $\mathcal{S}$ of homotopy classes of simple curves) into the space of functions on itself, after which the boundary is obtained by taking the closure in the corresponding projective space (the Martin kernel is precisely the projectivization of the Green kernel). The Martin boundary considered as a measure space with the representing measure of the constant harmonic function $\mathbf{1}$ is isomorphic to the Poisson boundary, so that the Poisson boundary retains only "significant" (up to measure 0) information about the Martin boundary. However, in a sense the Martin boundary is a "less functorial" object than the Poisson boundary, and describing the Martin boundary in intrinsic terms is a much more difficult problem than that of describing the Poisson boundary (see [K5], [K11]).

Due to the fact that the sequence of measures μ_n is obtained by iterative convolutions with the measure μ, these measures can be presented as one-dimensional distributions of a Markov chain (*random walk*) on G with transition probabilities $p(g, g') = \mu(g^{-1}g')$ determined by the measure μ. In other words, if we start at time 0 from the identity of the group, then the position of the random walk at time n is $g_n = \gamma_1\gamma_2\cdots\gamma_n$, where $\gamma_i,\ i \geq 1$ are independent μ-distributed *increments* of the random walk. Now one can not just consider the convolutions μ_n (which describe the position of the random walk at time n), but also look at the individual behavior at infinity of the *sample paths* $\boldsymbol{g} = \{g_n\},\ n \geq 0$ (a.e. with respect to the probability measure $\mathbf{P}$ in the path space $G^{\mathbb{Z}_+}$).

In terms of the path space $(G^{\mathbb{Z}_+}, \mathbf{P})$ the Poisson boundary can be defined as the space of *ergodic components of the time shift*. Thus, if the group G is equivariantly embedded into a topological space B, and $\mathbf{P}$-a.e. sample path $\boldsymbol{g} = \{g_n\}$ converges to a limit $g_\infty = \pi(\boldsymbol{g}) \in B$, then the space B with the corresponding harmonic measure $\lambda = \pi(\mathbf{P})$ on it is necessarily a quotient of the Poisson boundary with respect to a certain G-invariant partition. Such quotients are called *μ-boundaries* (this definition is equivalent to the one given in [Fu3]). The Poisson boundary is then the *maximal* μ-boundary.

The problem of *describing the Poisson boundary* of (G, μ) consists of two parts:

(1) To find (in geometric or combinatorial terms) a μ-boundary (B, λ);
(2) To show that this μ-boundary is maximal.

In other words, first one has to exhibit a certain system of invariants of stochastically significant behavior of sample paths at infinity, and then to show completeness of this system. If a certain compactification of the group G has the property that sample paths of the random walks on G converge a.e. in this compactification (so that it is a μ-boundary), and this μ-boundary is in fact isomorphic to the Poisson boundary of (G, μ), then it means that this compactification is indeed maximal in a measure theoretical sense, i.e., there is no way (up to measure 0) of splitting further the boundary points of this compactification. Note that this property has nothing to do with solvability of the Dirichlet problem with respect to this compactification. For example, the Dirichlet problem is trivially solvable for the one-point compactification; on the other hand, even if the boundary of a certain group compactification can be identified with the Poisson boundary, it does *not* imply in general that the Dirichlet problem is solvable (or even that the support of the harmonic measure is the whole topological boundary).

One can ask the question about identification of the Poisson (resp., Martin) boundary for Markov operators arising in various situations. For example, see [Bi], [MP] for a description of Euclidean domains for which the Poisson boundary can be identified with the topological boundary. For pinched Cartan–Hadamard manifolds the Martin (thereby the Poisson) boundary was shown to coincide with the sphere at infinity [AS], [An1]. In the discrete setup the most general result on the description of the Martin boundary is its identification with the hyperbolic boundary for finite range random walks on hyperbolic graphs satisfying a strong isoperimetric inequality (in particular, for random walks determined by finitely supported measures on word hyperbolic groups) [An2]. Note that the Martin boundary methods usually do not use group invariance. For random walks on general Lie groups the Poisson boundary was described by Raugi [Ra]; however, his approach strongly depends on the structure theory of Lie groups and can not be applied for discrete groups.

A powerful technique for describing the Poisson boundary for random walks on groups is provided by ergodic methods, more specifically, by the entropy theory of random walks [KV], [Ka1], [De2]. It leads to several simple geometric criteria of boundary maximality which allow one to identify the Poisson boundary with "natural" boundaries for word hyperbolic groups, groups with infinitely many ends, discrete subgroups of semi-simple Lie groups, cocompact lattices in rank 1 Cartan–Hadamard manifolds, polycyclic groups under mild conditions on the measure μ (finite first moment is sufficient) [BL1], [Ka1], [Ka7], [Ka10], [Le1]. It is this technique that we are using in this paper.

The main results of the paper are the following.

Theorem 2.2.4. *If μ is a probability measure on the mapping class group Γ such that the group generated by its support is non-elementary, then there exists a unique μ-stationary probability measure ν on the space $\mathcal{PMF}$, which is purely non-atomic and concentrated on the subset $\mathcal{UE} \subset \mathcal{PMF}$ of uniquely ergodic foliations, and the measure space $(\mathcal{UE}, \nu)$ is a μ-boundary. For any $x \in T_g$ and $\mathbf{P}$-a.e. sample path $\boldsymbol{g} = \{g_n\}$ of the random walk (Γ, μ) the sequence $g_n x$ converges in $\mathcal{PMF}$ to a limit $F = F(\boldsymbol{g}) \in \mathcal{UE}$, and the distribution of the limits $F(\boldsymbol{g})$ is given by the measure ν.*

In particular, for any $x \in T_g$ the sequence of measures $\mu_n * \delta_x$ converges weakly to the measure ν.

Theorem 2.3.1. *If, in addition, the measure μ has a finite entropy and finite first logarithmic moment with respect to the Teichmüller distance, then the space $(\mathcal{PMF}, \nu)$ is the Poisson boundary of (G, μ).*

Since Teichmüller space has exponentially bounded growth (Theorem 1.3.2), these conditions are satisfied if the measure μ has a finite first moment $\sum \mu(\gamma) d_T(o, \gamma o)$ with respect to the Teichmüller distance and, in particular, if it has a finite first moment with respect to a word metric on Γ (Theorem 2.3.2). Thus, it turns out that indeed *the Thurston boundary is a maximal boundary of the mapping class group from a measure theoretical point of view.*

Our analysis of the action of Γ on probability measures on $\mathcal{PMF}$ allows us to give a new proof of the fact that Γ is not isomorphic to a lattice in a semi-simple Lie group, a result first proved by Ivanov [Iv1]. Actually, we prove a stronger result (Theorem 2.4.1), that any subgroup of Γ which satisfies a natural non-elementarity condition (NE) (see below) cannot be a lattice in a semi-simple group of rank ≥ 2.

Passing from random walks on Γ to Γ-invariant Markov operators P on Teichmüller space T_g, we prove (under appropriate geometric assumptions) in Theorem 3.3.2 that if the quotient operator $\overline{P}$ on the moduli space $M_g = T_g/\Gamma$ is *recurrent*, then the Poisson boundary of P coincides with the Poisson boundary of Γ with a certain measure μ (depending on P). If, moreover, $\overline{P}$ is *positively recurrent* (i.e., Γ has cofinite volume with respect to the unique Γ-invariant stationary measure of P), then almost all sample paths of the Markov chain determined by P converge to $\mathcal{UE} \subset \mathcal{PMF}$, and $\mathcal{PMF}$ with the corresponding harmonic measure is the Poisson boundary of P (Theorem 3.4.2). In particular, this result applies to the *geodesic random walks* on T_g considered earlier by Masur [Ma4].

This paper is intended to be interdisciplinary, appealing to specialists from different areas. It is inevitable therefore that extra space is needed to explain things that might be self evident to experts in one area.

The paper is organized as follows.

In Section 1 we review the relevant parts of the Thurston theory of measured foliations and of Teichmüller theory. §§1.1 and 1.2 are devoted to elementary properties of the action of the mapping class group Γ on the space $\mathcal{PMF}$ of projective measured foliations. In §1.3 by using recent results of Minsky [Mi] we prove that *the Teichmüller space is roughly isometric to a graph with uniformly bounded vertex degrees* (Theorem 1.3.2), which means that the "volume" of Teichmüller balls grows at most exponentially with the radius (we say that Teichmüller space has *exponentially bounded growth*). In §1.4 we prove several auxiliary results on the Teichmüller geodesic lines in T_g (close analogues of the corresponding properties of Gromov hyperbolic spaces). Finally, in §1.5 we consider contracting properties of the action of the mapping class group on probability measures on $\mathcal{PMF}$, which are a key ingredient of our proof of convergence of random walks on Γ (once again these properties are analogous to those of the action of the isometry group on the hyperbolic boundary of a Gromov hyperbolic space). We say that a sequence $g_n \in \Gamma$ is universally convergent if it tends to infinity in Γ and for any simple closed curve $\alpha \in \mathcal{S}$ on M the sequence $g_n^{-1}\alpha$ converges in $\mathcal{PMF}$. If g_n is such a sequence, then there exists a foliation $F \in \mathcal{PMF}$ such that for any $x \in T_g$ all limit points of the sequence $g_n x$ are contained in the set $\{H : i(F,H) = 0, i(F,\alpha) = 0 \iff i(H,\alpha) = 0 \; \forall \alpha \in \mathcal{S}\}$ (Lemma 1.5.4). Hence, for any probability measure ν on $\mathcal{PMF}$ satisfying a natural non-degeneracy condition all weak limits of the translations $g_n\nu$ are concentrated either on the set $\widetilde{F} = \{H : i(F,H) = 0\} \subset \mathcal{MIN}$ for a certain minimal foliation F, or on the set $Z_F = \{H : i(F,\alpha) = 0 \iff i(H,\alpha) = 0 \; \forall \alpha \in \mathcal{S}\}$ for a certain non-minimal F (Lemma 1.5.6).

Section 2 is devoted to the proofs of the Theorems on the random walks on Γ. We say that a subgroup Γ' of the mapping class group Γ satisfies condition (NE) (is *non-elementary*) if it is not a finite extension of the stabilizer of a set $\widetilde{F}$ or Z_F (which is a direct analogue of the notion of non-elementary groups of isometries of hyperbolic spaces). For proving Theorem 2.2.4 we use the following idea of Furstenberg applied first to the discrete subgroups of $SL(2,\mathbb{R})$ [Fu3]. Take an arbitrary μ-stationary probability measure ν on $\mathcal{PMF}$ which exists by compactness considerations. Then by the Martingale Convergence Theorem the sequence of translations $g_n\nu$ converges weakly to a (random) limit $\lambda(\boldsymbol{g})$ for a.e. sample path $\boldsymbol{g} = \{g_n\}$, and the measure ν is an integral of the limit measures $\lambda(\boldsymbol{g})$. Using weak dissipativity of the action of Γ on $\mathcal{PMF} \setminus \mathcal{MIN}$ (established in §1.2) we show that the measure ν (hence, a.e. limit measure $\lambda(\boldsymbol{g})$) is concentrated on $\mathcal{MIN}$. Further, for proving that the measure ν is concentrated on $\mathcal{UE}$ we use the fact that for any Teichmüller geodesic ray determined by a non-minimal foliation its projection to the moduli space T_g/Γ tends to infinity [Ma3]. Then convergence of sample paths follows from the contraction properties established in §1.5 (cf. with analogous convergence theorems for semi-simple Lie groups [GR] and for hyperbolic groups [CS], [Ka10], [Wo2]). In particular, the Poisson boundary of any probability measure whose support generates a non-elementary subgroup is non-trivial, which implies that any non-elementary subgroup is non-amenable.

For proving maximality of the Thurston boundary (Theorem 2.3.1) we use a geometric "strip criterion" due to Kaimanovich [Ka7], [Ka10]. Under the conditions of finiteness of the entropy of μ and of its first logarithmic moment it requires considering a μ-boundary (B_+,ν_+) simultaneously with a $\check{\mu}$-boundary (B_-,ν_-) for the reflected measure $\check{\mu}(g) = \mu(g^{-1})$. If there exists an equivariant measurable map assigning to a.e. pair of points $(F_-,F_+) \in B_- \times B_+$ a "strip" $S(F_-,F_+) \subset \Gamma$ which is sufficiently "thin" in the sense that intersections of a.e. strip with balls in T_g grow polynomially, then (B_+,ν_+) is the Poisson boundary of (G,μ). These strips are easily constructed by using

Teichmüller geodesic lines determined by any pair of distinct uniquely ergodic foliations. For deducing Theorem 2.3.2 from Theorem 2.3.1 we use the fact (established in §1.3) that Teichmüller space has exponentially bounded growth.

For proving Theorem 2.4.1 we use the following remarkable result of Furstenberg which he used in his rigidity theorem [Fu2], [Fu3]. If G is a lattice in a semi-simple Lie group of rank ≥ 2, then there exist a probability measure μ on G with $\operatorname{supp}\mu = G$ and a number $\varepsilon > 0$ such that for any two μ-harmonic functions f_1 and f_2 on G conditions $0 \leq f_i(g) \leq 1 \; \forall g \in G$ and $f_i(e) \geq \frac{1}{2} - \varepsilon$ imply that $\min\{f_1(g), f_2(g)\}$ does *not* tend to zero as $g \to \infty$. By using our description of the unique μ-stationary probability measure ν on $\mathcal{PMF}$ we are able to construct for any probability measure μ on a non-elementary subgroup $\Gamma' \subset \Gamma$ and any $\varepsilon > 0$ two closed disjoint subsets $\overline{Q}_1, \overline{Q}_2$ contained in $\mathcal{MIN}$ such that $\nu\overline{Q}_i \geq \frac{1}{2} - \varepsilon$, and for any $F \in \mathcal{MIN}$ there is a neighborhood of $\widetilde{F}$ which does not intersect $\overline{Q}_1$ and $\overline{Q}_2$ simultaneously. Then Lemma 1.5.6 implies that the harmonic functions $f_i(g) = g\nu(\overline{Q}_i)$ have the property that $\min\{f_1(g), f_2(g)\} \to 0$ as g tends to infinity in Γ', so that Γ' cannot be a lattice in a higher rank semi-simple group.

In Section 3 we consider *bounded range* Γ-invariant Markov operators on T_g (i.e., such that the step lengths are uniformly bounded). For continuous time diffusion processes the elliptic Harnack inequality automatically follows from boundedness of geometry of the generating operator; under appropriate "bounded geometry" and irreducibility conditions on the transition densities we prove in Theorem 3.2.2 a general *Harnack inequality* (uniform equivalence of the probability measures obtained by the balayage of δ-measures to the complement of sufficiently large balls) for bounded range Markov operators on continuous state spaces (it applies, for example, to geodesic random walks on Riemannian manifolds).

Assuming that a Γ-invariant Markov operator P on T_g is *corecurrent* (i.e., the quotient Markov operator $\overline{P}$ on the moduli space $M_g = T_g/\Gamma$ is recurrent in the sense of Harris, so that its sample paths visit infinitely often any positive measure subset of M_g), we then use the Harnack inequality for a *discretization* of P, which allows one to put a measure μ on the group Γ (identified with an orbit Γo, $o \in T_g$), such that the Poisson boundary of (Γ, μ) is the same as the Poisson boundary of the operator P. More precisely, the restriction of any bounded P-harmonic function to the orbit Γo is μ-harmonic, and, conversely, any bounded μ-harmonic function can be uniquely extended from Γo to a P-harmonic function on T_g (Theorem 3.3.2). This discretization is based on the *balayage method* introduced by Furstenberg [Fu3] and Lyons–Sullivan [LS] (see also [An2], [BL2], [Ka4]). In view of the results from Section 2 it implies that $\mathcal{PMF}$ with a uniquely determined Γ-invariant system of harmonic measures on it is a quotient of the Poisson boundary of the operator P (Theorem 3.4.1). If the quotient operator $\overline{P}$ is positively recurrent, then the measure μ can be chosen to have a finite first moment with respect to the Teichmüller distance, and the sample paths $\{x_n\}$ on T_g can be approximated by the sample paths of the random walk (G, μ) well enough to ensure that $\{x_n\}$ converges a.e. to $\mathcal{UE} \subset \mathcal{PMF}$ (this also gives a new proof of convergence of sample paths for the modified *geodesic random walk* on T_g first proved in [Ma4] by analyzing the train tracks decomposition). Then by the results of Section 2 $\mathcal{PMF}$ with the corresponding harmonic measure is the Poisson boundary of P (Theorem 3.4.2).

In conclusion we formulate several open questions connected with the results of the paper. First, what can one say about the type of the harmonic measure $\nu = \nu(\mu)$ on $\mathcal{PMF}$ determined by a measure μ on Γ? It is known that for the sphere at infinity of the universal cover $\widetilde{M}$ of a compact negatively curved manifold the Lebesgue measure type, the harmonic measure type (corresponding to the Brownian motion on $\widetilde{M}$), and the Patterson–Sullivan measure type are pairwise singular in the general case (see [Le2]). In our situation there is a smooth (Lebesgue) measure type on $\mathcal{PMF}$ (concentrated in fact on $\mathcal{UE}$ [Ma1]). Further, one may define "conformal densities" on $\mathcal{PMF}$ by using the usual limit procedure. In a sense, the mapping class group Γ should be considered as having "cofinite volume" in T_g, so that one may expect that the Patterson–Sullivan measure type would be unique and also concentrated on $\mathcal{UE}$. Apparently, it should be singular with respect to the Lebesgue measure type as the Teichmüller space should be considered as having "non-constant curvature" with respect to the Teichmüller metric. One might expect that the harmonic measures $\nu(\mu)$ are singular with respect to both Lebesgue and Patterson–Sullivan measure types for any finitely supported μ (note that in the Riemannian situation this question is still open).

Another question is connected with invariant measures of the geodesic flow on the moduli space $M_g = T_g/\Gamma$. Recall that the Lebesgue measure types determines a (unique) ergodic invariant measure of the geodesic flow on M_g [Ma1], [Ve]. Is the same true about the harmonic (or about the Patterson–Sullivan) measure type? By a general result on Poisson boundaries Γ acts ergodically on the square of the Poisson boundary of any symmetric measure μ on Γ and on the square of the Poisson boundary of any *reversible* corecurrent invariant Markov operator on T_g [Ka9] (it is also known that the measure μ obtained from discretization of an invariant Markov operator P on T_g by using Theorem 3.3.2 can be chosen symmetric if P is reversible [BL2]). Thus, one might expect to obtain (at least, in some situations) a harmonic invariant measure of the geodesic flow on M_g from the square of the harmonic measure on $\mathcal{PMF}$. Note, however, that all known constructions of the harmonic invariant measure of the geodesic flow for hyperbolic spaces and groups require a rather strong almost multiplicativity property of the Green kernel [An2], [Ka8].

The end of proof is denoted by the sign $\square$. On several occasions we had to subdivide proofs into separate claims, in which case the sign $\triangle$ denotes the end of the proof of each claim.

1. Asymptotic properties of Teichmüller space

Basic references for the material on measured foliations and Teichmüller theory in this section are [FLP], [Be], and [Ga].

1.1. The space of projective measured foliations.

For a closed surface M of genus $g \geq 2$ let $\mathcal{S}$ be the set of homotopy classes of simple closed curves on M given the discrete topology. The *geometric intersection number* of $\alpha, \beta \in \mathcal{S}$, i.e., the minimal number of intersections of any two their representatives, is denoted by $i(\alpha,\beta)$. Let $\mathbb{R}_+^{\mathcal{S}}$ be the space of non-negative functions on $\mathcal{S}$ given the product topology. The quotient of $\mathbb{R}_+^{\mathcal{S}} \setminus \{0\}$ with respect to the multiplicative action of $\mathbb{R}_+$ is the (compact) projective space $P\mathbb{R}_+^{\mathcal{S}}$.

The map

$$\alpha \mapsto i(\cdot, \alpha)$$

determines an embedding of $\mathcal{S}$ into $\mathbb{R}_+^{\mathcal{S}}$ which projects to an embedding of $\mathcal{S}$ into $P\mathbb{R}_+^{\mathcal{S}}$. The closure of the set $\{r\alpha,\ r \geq 0, \alpha \in \mathcal{S}\}$ in $\mathbb{R}_+^{\mathcal{S}}$ is denoted by $\mathcal{MF}$, and the closure of the embedding of $\mathcal{S}$ into $P\mathbb{R}_+^{\mathcal{S}}$ (i.e., the quotient of $\mathcal{MF}$ with respect to the multiplicative action of $\mathbb{R}_+$) is denoted by $\mathcal{PMF}$.

A *measured foliation* on M is determined by a finite number of points $P_k \in M$ and an atlas of coordinate charts $(x_i, y_i) : U_i \to \mathbb{R}^2$ on the complement $M \setminus \{P_k\}$ such that for any two overlapping charts

$$x_j = f_{ij}(x_i, y_i)\,, \qquad y_j = \pm y_i + C\,.$$

The foliation is defined by the lines $y = Const$, and the *transverse measure* of the foliation is $|dy|$. The foliation has a standard form of a p_k-pronged singularity at each point P_k. For any $\alpha \in \mathcal{S}$,

$$i(F,\alpha) = \inf_{\alpha_0 \sim \alpha} \int_{\alpha_0} |dy|\,.$$

where the infimum is taken over all representatives α_0 of the class α.

Two measured foliations F, G are *equivalent* if $i(F,\alpha) = i(G,\alpha)\ \forall\, \alpha \in \mathcal{S}$. Topologically, it means that there is a finite sequence of homeomorphisms homotopic to the identity (and preserving the transverse measure) and of Whitehead moves or their inverses that take F to G. So, points from $\mathcal{MF}$ can be identified with equivalence classes of measured foliations on M.

There is a natural action of $\mathbb{R}_+$ on the space of measured foliations: $rF,\ r > 0$ is topologically the same measured foliation as F with the transverse measure scaled by r. Thus, points from $\mathcal{PMF}$ are identified with equivalence classes of *projective measured foliations*. Topologically $\mathcal{PMF}$ is a sphere of dimension $6g - 7$ [FLP].

The embeddings $\mathcal{S} \hookrightarrow \mathcal{MF}, \mathcal{S} \hookrightarrow \mathcal{PMF}$ have the following geometric interpretation: any homotopy class $\alpha \in \mathcal{S}$ gives rise to a measured foliation for which all closed regular leaves are homotopic to α and form a cylinder with the transverse measure across the cylinder being 1 [FLP]. In particular, this foliation has the same intersections numbers with curves from $\mathcal{S}$ as α.

The intersection number $i(\cdot,\cdot)$ extends to a continuous function on $\mathcal{MF} \times \mathcal{MF}$, and

$$i(r_1 F_1, r_2 F_2) = r_1 r_2 i(F_1, F_2) \qquad \forall F_1, F_2 \in \mathcal{MF},\ r_1, r_2 \in \mathbb{R}_+ .$$

So, given two projective measured foliations $F_1, F_2 \in \mathcal{PMF}$ we can say whether their "projective intersection number" is zero or non-zero and use the notations $i(F_1, F_2) = 0$ and $i(F_1, F_2) > 0$, respectively.

Below we shall often identify a measured foliation $F \in \mathcal{MF}$ with its projective class $\{rF,\ r > 0\}$ from $\mathcal{PMF}$ and *vice versa*. However, we shall always distinguish between convergence in $\mathcal{MF}$ and projective convergence in $\mathcal{PMF}$; a sequence $F_n \in \mathcal{MF}$ converges to $F \in \mathcal{MF}$ in $P\mathbb{R}_+^{\mathcal{S}}$ (notation: $F_n \xrightarrow{\mathcal{PMF}} F$) if there exists a sequence $r_n > 0$ such that $r_n F_n \to F$ in $\mathbb{R}_+^{\mathcal{S}}$ (notation: $r_n F_n \xrightarrow{\mathcal{MF}} F$), i.e., $r_n i(F_n, \alpha) \to i(F, \alpha)\ \forall\, \alpha \in \mathcal{S}$. We shall say that a sequence $F_n \in \mathcal{MF}$ *tends to infinity* if there exists $H \in \mathcal{MF}$ such that $i(F_n, H) \to \infty$ (notation: $F_n \xrightarrow{\mathcal{MF}} \infty$).

We shall often use decompositions of the surface M into spheres with three holes (*"pairs of pants"*). Every such decomposition is determined by a disjoint system of homotopy classes of $3g-3$ simple closed curves $A = \{\alpha_1, \dots \alpha_{3g-3}\} \subset \mathcal{S}$ (i.e., $i(\alpha_i, \alpha_j) = 0$), and conversely, any disjoint system A consisting of $3g - 3$ simple closed curves determines a pants decomposition and is maximal in the sense that for any $\beta \in \mathcal{S} \setminus A$ there is $\alpha \in A$ with $i(\alpha, \beta) > 0$.

Lemma 1.1.1. *Any distinct sequence $\alpha_n \in \mathcal{S}$ tends to infinity in $\mathcal{MF}$.*

Proof. Take a pants decomposition of M determined by curves $\beta_1, \dots, \beta_{3g}$. If α_n intersects each β_i a bounded number of times, then α_n must "wrap" around some β_i an unbounded number of times, and therefore must intersect some curve crossing β_i an unbounded number of times. □

A foliation $F \in \mathcal{MF}$ is *minimal* if $i(F, \alpha) > 0$ for any $\alpha \in \mathcal{S}$. Topologically, it means that F is equivalent to a foliation all of whose leaves are dense. Denote by $\mathcal{MIN}$ the subset of $\mathcal{PMF}$ which consists of projective classes of minimal foliations. For a foliation $F \in \mathcal{PMF}$ let

$$\widetilde{F} = \{G \in \mathcal{PMF} : i(F, G) = 0\} .$$

If $F \in \mathcal{MIN}$, then all foliations from $\widetilde{F}$ are also minimal, and $\widetilde{F}$ is the set of all foliations topologically equivalent to F, so that

$$F \sim G \iff i(F,G) = 0 \,, \qquad F, G \in \mathcal{MIN} \tag{1.1.1}$$

is an equivalence relation on $\mathcal{MIN}$ [Re]. The equivalence class $\widetilde{F}$, $F \in \mathcal{MIN}$ is closed and has the natural structure of a convex set (of transverse measures). If $\widetilde{F}$ consists of a single point, the foliation F is called *uniquely ergodic*. Denote by $\mathcal{UE} \subset \mathcal{PMF}$ the set of uniquely ergodic projective measured foliations.

Let $\widetilde{\mathcal{MIN}}$ be the quotient of the set $\mathcal{MIN}$ with respect to the equivalence relation $\sim$ (1.1.1).

Lemma 1.1.2. *There exists a countable family $\{B_i\}$ of Borel subsets of $\mathcal{MIN}$ which are unions of classes of the equivalence relation $\sim$ and separate any two such classes, i.e., for any $\widetilde{F}_1 \neq \widetilde{F}_2 \in \widetilde{\mathcal{MIN}}$ there exists a set $B \in \{B_i\}$ such that either $\widetilde{F}_1 \subset B$, $\widetilde{F}_2 \cap B = \varnothing$, or the other way round.*

Proof. The sets B_i are provided by *train tracks* [Pe], [Ke3].

A train track is a 1-dimensional branched submanifold of the surface M. It has a finite number of *switches* which we can assume are *trivalent*, i.e., for every switch there is one *large* branch which forms a C^1 path with each of the other two *small* branches, whereas the small branches form a cusp. A *winged branch* is one which is large at each of the switches at its endpoints. We will assume that the domains complementary to the track have at least 3 cusps each. A train track is *complete* if every complementary component is simply connected and has exactly 3 cusps.

The set of weights $w_i \geq 0$ on the branches of a train track τ which are normalized by requiring $\sum w_i = 1$ and satisfy the *switch condition* (the weight of the large branch equals the sum of the weights of the small branches) is a polyhedron $\Lambda(\tau)$. This polyhedron parametrizes the set of foliations (also denoted $\Lambda(\tau)$) *carried* by τ: one runs groups of leaves along the branches, assigning transverse measures according to the weights; then one fills in each complementary region with p cusps with a p-pronged singularity. If τ is a complete train track, then $\Lambda(\tau)$ is the closure of an open set in $\mathcal{PMF}$. If a track τ carries a minimal foliation, then all complementary domains of τ are simply connected. We use the notation $\tau_1 < \tau_2$ to mean that every foliation carried by τ_1 is carried by τ_2.

There are two basic operations on train tracks: reduction and splitting.

Any face of the polyhedron $\Lambda(\tau)$ is determined by the condition $w_i = 0$ for a certain branch of τ. In this case we can erase that branch, so that any foliation corresponding to this face is carried by a track with one branch and one complementary domain less (the foliation then has a saddle connection). Given a foliation $F \in \Lambda(\tau)$ we can continue this process until all weights lie in the interior of the polyhedron $\Lambda(\widetilde{\tau})$ corresponding to a *reduced* track $\widetilde{\tau}$.

The other operation is the *splitting* of a track τ along a chosen winged branch. The result is two train tracks τ' and τ'' such that $\Lambda(\tau) = \Lambda(\tau') \cup \Lambda(\tau'')$. The intersection $\Lambda(\tau') \cap \Lambda(\tau'')$ has codimension 1 and is the set of foliations carried by both τ' and τ'' with weight 0 assigned to their corresponding branches obtained from splitting the winged branch. Erasing these branches gives a track $\overline{\tau}$ (*degeneration* of τ) such that $\Lambda(\tau') \cap \Lambda(\tau'') = \Lambda(\overline{\tau})$. Every foliation carried by $\overline{\tau}$ has a saddle connection. To make the splitting process well defined, fix, once and for all, a winged branch for each combinatorial type of train tracks.

Now start with a minimal foliation F carried by a complete train track $\tau = \tau_0$. Reducing if necessary, we may assume that F lies in the interior of the polyhedron $\Lambda(\tau)$. Split the track τ, and take τ_1 to be either τ' (resp., τ'') if F is in the interior of $\Lambda(\tau')$ (resp., $\Lambda(\tau'')$), or $\overline{\tau}$ if F is carried by $\overline{\tau}$. Applying the same procedure to τ_1, and so on, we find that every minimal foliation F has an infinite expansion $\tau_0 > \tau_1 > \dots$ by train tracks. This expansion has the property that two minimal foliations are equivalent if and only if they have the same sequence of combinatorial types $[\tau_0], [\tau_1], \dots$.

Since the number of different combinatorial types of tracks is finite, and subsets of $\mathcal{MIN} \subset \mathcal{PMF}$ obtained by fixing any first n combinatorial types $[\tau_0], \dots, [\tau_n]$ are Borel, we are done. $\square$

1.2. The mapping class group.

Let $\mathrm{Diff}_0(M)$ and $\mathrm{Diff}^+(M)$ be the group of all diffeomorphisms of M homotopic to the identity and the group of all orientation preserving diffeomorphisms of M, respectively. The *mapping class group*

$$\Gamma = \mathrm{Mod}(g) = \mathrm{Diff}^+(M)/\mathrm{Diff}_0(M)$$

is finitely generated and naturally acts on $\mathcal{S}$ and $\mathcal{MF}$ (thereby on $\mathcal{PMF}$). The intersection number is Γ-invariant, i.e.,

$$i(gF_1, gF_2) = i(F_1, F_2) \qquad \forall g \in \Gamma,\ F_1, F_2 \in \mathcal{MF}\,.$$

In particular,

$$g\widetilde{F} = \widetilde{(gF)} \qquad \forall g \in \Gamma,\ F \in \mathcal{MIN}\,,$$

so that the group Γ acts on $\widetilde{\mathcal{MIN}}$ as well.

Note that there are two equivalent ways of defining the Γ-action on $\mathcal{MF}$: one geometrical, and the other one using the embedding $F \mapsto i(F, \cdot)$ of $\mathcal{MF}$ into the space of functions on $\mathcal{S}$ and the Γ-action on $\mathcal{S}$, so that the intersection numbers of the foliation gF, $g \in \Gamma, F \in \mathcal{MF}$ are by definition

$$i(gF,\alpha) = i(F, g^{-1}\alpha) , \qquad \alpha \in \mathcal{S} .$$

One can associate with any $\sigma \in \mathcal{S}$ an element $\gamma_\sigma \in \Gamma$ which is called the *Dehn twist* about σ [FLP].

Lemma 1.2.1. *For any $\alpha, \sigma \in \mathcal{S}$ with $i(\alpha,\sigma) > 0$ there exists a sequence $r_n \to 0$ such that*

$$r_n \gamma_\sigma^n \alpha \xrightarrow{\mathcal{MF}} \sigma .$$

Proof. The curves $\gamma_\sigma^n \alpha$ wrap more and more around σ. Consequently, for any two curves $\beta_1, \beta_2 \in \mathcal{S}$ with $i(\sigma,\beta_i) > 0$

$$\frac{i(\gamma_\sigma^n\alpha,\beta_1)}{i(\gamma_\sigma^n\alpha,\beta_2)} \to \frac{i(\sigma,\beta_1)}{i(\sigma,\beta_2)} ,$$

where the denominator and the numerator in the left-hand side tend to infinity. □

Lemma 1.2.2. *For any $\alpha \in \mathcal{S}$ its Γ-orbit in $\mathcal{S}$ is infinite.*

Proof. Take $\sigma \in \mathcal{S}$ such that $i(\alpha,\sigma) > 0$, and let γ_σ be the Dehn twist about σ. By Lemma 1.2.1, if the Γ-orbit of α is finite (so that the set $\{\gamma_\sigma^n \alpha\}$ is also finite), then $i(\sigma,\omega) = 0$ for any $\omega \in \mathcal{S}$, which is impossible. □

Remarks. 1. Lemma 1.2.2 also immediately follows from the more general fact that the action of Γ on $\mathcal{PMF}$ is minimal [FLP].

2. Lemma 1.2.2 means that for any $\alpha \in \mathcal{S}$ its stabilizer $\operatorname{Stab}\alpha \subset \Gamma$ has infinite index in Γ. On the other hand, $\operatorname{Stab}\alpha$ is infinite, because by Lemma 1.2.1 it contains the infinite cyclic subgroup generated by γ_α.

3. The number of Γ-orbits in $\mathcal{S}$ is finite. Indeed, if α_1, α_2 are simple closed curves whose complements are topologically the same, then there is a homeomorphism taking α_1 to α_2. But there are only finitely many possible topologically different complementary regions.

For each $F \in \mathcal{PMF} \setminus \mathcal{MIN}$ let

$$Z_F = \{H \in \mathcal{PMF} \setminus \mathcal{MIN} : i(H,\alpha) = 0 \iff i(F,\alpha) = 0 \; \forall\, \alpha \in \mathcal{S}\} ,$$

so that Z_F is the set of all H which have zero intersection with exactly the same curves from $\mathcal{S}$ as F. It is clear from the definition that any two such Z_F either coincide or are disjoint, and that they partition $\mathcal{PMF} \setminus \mathcal{MIN}$. The next Lemma shows that the action of Γ on the complement $\mathcal{PMF} \setminus \mathcal{MIN}$ is *weakly dissipative* in the following sense:

Lemma 1.2.3. *The partition $\{Z_F\}$ of $\mathcal{PMF} \setminus \mathcal{MIN}$ has the following properties*

(i) *It is countable, i.e., there are countably many distinct sets Z_F;*
(ii) *It is Γ-invariant, i.e., together with any set Z_F it contains all its translations γZ_F, $\gamma \in \Gamma$.*
(iii) *For any set Z_F the number of pairwise disjoint translations γZ_F, $\gamma \in \Gamma$ is infinite.*

Proof. If $F \in \mathcal{PMF}\setminus\mathcal{MIN}$, then the graph (not necessarily connected) $\mathcal{G}_F$ of compact (critical) leaves of the foliation F is non-empty, and $i(F,\alpha) = 0$ iff α is homotopic to a closed loop in $\mathcal{G}_F$. Since any such graph has a bounded number of edges, there are only finitely many combinatorial types of graphs. As Γ is countable, there are therefore only countably many graphs $\mathcal{G}_i$ up to equivalence $\sim$ by homeomorphisms homotopic to the identity and Whitehead moves. Let

$$B_i = \{F \in \mathcal{PMF} \setminus \mathcal{MIN} : \mathcal{G}_F \sim \mathcal{G}_i\} .$$

Since Whitehead moves do not change the homotopy classes of closed loops contained in the graph, each set B_i is contained in some Z_F, and so the partition $\{Z_F\}$ is countable. Clearly,

$$Z_{\gamma F} = \gamma Z_F \qquad \forall \gamma \in \Gamma, F \in \mathcal{PMF} \setminus \mathcal{MIN} ,$$

so that this partition is Γ-invariant.

Given any $F \in \mathcal{PMF}\setminus\mathcal{MIN}$, the surface decomposes into a union of annuli in which every leaf is closed, and minimal domains in which every leaf is dense in the domain. The boundary of each domain is comprised of critical leaves of F [FLP], [St]. Take a boundary curve α and a curve σ that does not lie in a boundary with $i(\alpha,\sigma) > 0$, then $i(F,\alpha) = 0$ and $i(F,\sigma) > 0$. Denote by γ_σ the Dehn twist about σ. Then by Lemma 1.2.1 $\gamma_\sigma^n \alpha \xrightarrow{\mathcal{PMF}} \sigma$, and $i(F,\gamma_\sigma^n \alpha) = i(\gamma_\sigma^{-n} F, \alpha) > 0$ for all n greater than a certain number N. Thus, $\gamma_\sigma^{-n} F \notin Z_F$ and $\gamma_\sigma^{-n} Z_F \cap Z_F = \varnothing$ for all $n \geq N$, so that the sets $\gamma_\sigma^{iN} Z_F$, $i = 0, 1, 2 \dots$ are all pairwise disjoint. □

Remark. We do not know whether the sets $B_i = \{F \in \mathcal{PMF} \setminus \mathcal{MIN} : \mathcal{G}_F \sim \mathcal{G}_i\}$ are in general smaller than the sets Z_F or coincide.

1.3. Teichmüller space.

A *conformal structure* x on the surface M is determined by an atlas of coordinate charts (U_μ, z_μ), where $\{U_\mu\}$ is an open cover of M, and *local uniformizers* $z_\mu : U_\mu \to \mathbb{C}$ have the property that $z_\mu \circ z_\nu^{-1}$ is analytic whenever defined. The *Teichmüller space* T_g is the space of all conformal structures on M endowed with the *Teichmüller metric*

$$d_T(x,y) = \frac{1}{2} \log \inf_h K(h) , \qquad x, y \in T_g ,$$

where the infimum is taken over all quasiconformal maps $h : x \to y$ homotopic to the identity, and $K(h)$ is the maximal dilatation of h. The infimum in the definition of the Teichmüller metric is realized by a unique *Teichmüller map*, and for any two points $x \neq y \in T_g$ there exists a unique *Teichmüller geodesic line* (i.e., an isometric embedding of $\mathbb{R}$ into T_g) passing through x and y.

The group Γ naturally acts on T_g by isometries, and this action is properly discontinuous. The stabilizer $\operatorname{Stab} x \subset \Gamma$ of a point $x \in T_g$ corresponds to a group of conformal self-mappings of the surface, so that $\operatorname{card}\operatorname{Stab} x \leq 84(g-1)\ \forall\, x \in T_g$ by Hurwitz' Theorem [FK, p.242]. For each such group, the set of fixed points is a lower dimensional Teichmüller space [Ga, p.151]. There are only countably many such finite groups. Thus, in the case $g > 2$ points with non-trivial stabilizers lie on a countable union of positive codimension subvarieties in T_g. In the case $g = 2$ the situation is somewhat different as there is the *hyperelliptic involution* $\gamma_0 \in \Gamma$ which fixes every point in T_2, so that $\{e, \gamma_0\}$ is a 2-element normal subgroup of Γ. However, the quotient group $\Gamma' = \Gamma/\{e, \gamma_0\}$ acts on T_2, and points with non-trivial stabilizers in Γ' lie on a countable union of positive codimension subvarieties in T_2.

For any $x \in T_g$ and $\alpha \in \mathcal{S}$ let

$$\operatorname{Ext}_x(\alpha) = \sup_\rho \inf_{\alpha_0 \sim \alpha} \rho(\alpha)^2 ,$$

be the *extremal length* of the homotopy class α with respect to the conformal structure x (here the supremum is taken over all conformal metrics ρ on x with area 1), and let

$$\operatorname{Ext}_x = \inf\{\operatorname{Ext}_x(\alpha) : \alpha \in \mathcal{S}\} .$$

By [Ke1, Theorem 4]

$$e^{2d_T(x,y)} = \sup_{\alpha \in \mathcal{S}} \frac{\operatorname{Ext}_x(\alpha)}{\operatorname{Ext}_y(\alpha)} , \tag{1.3.1}$$

which implies that

$$\frac{\operatorname{Ext}_x}{\operatorname{Ext}_y} \leq e^{2d_T(x,y)} \qquad \forall\, x, y \in T_g . \tag{1.3.2}$$

In particular, the function $x \mapsto \operatorname{Ext}_x$ is continuous on T_g.

Let $\pi : x \mapsto \overline{x}$ be the projection from T_g to the *moduli space* $M_g = T_g/\Gamma$. Denote by $\overline{d}_T$ the distance on M_g induced by the distance d_T on T_g, and by $\overline{x} \mapsto \operatorname{Ext}_{\overline{x}}$ the projection to M_g of the Γ-invariant function $x \mapsto \operatorname{Ext}_x$. By the Mumford compactness theorem [Mu] a subset X of M_g has compact closure if and only if the function $\overline{x} \mapsto \operatorname{Ext}_{\overline{x}}$ is bounded on X from below. Hence, by continuity the values $\operatorname{Ext}_{\overline{x}}$, $\overline{x} \in M_g$ (and consequently, the values Ext_x, $x \in T_g$) are uniformly bounded from above.

The map

$$x \mapsto i(x, \cdot) \in \mathbb{R}_+^{\mathcal{S}},$$

where now $i(x,\alpha)$, $\alpha \in \mathcal{S}$ is the length with respect to the hyperbolic Riemannian structure determined by x of the (unique) geodesic from the class α, defines an embedding of T_g into $\mathbb{R}_+^{\mathcal{S}}$ which projects to an embedding of T_g into $P\mathbb{R}_+^{\mathcal{S}}$ whose boundary is $\mathcal{PMF}$. This embedding is equivariant with respect to the action of Γ, and (see [FLP], [Ke2], [CB]) the intersection number $i(\cdot,\cdot)$ extends continuously from $T_g \times \mathcal{S}$ to $T_g \times \mathcal{MF}$ in such way that

$$i(x, F) > 0 \qquad \forall\, x \in T_g,\ F \in \mathcal{MF}.$$

Comparing the definitions of $i(x,\alpha)$ and $\mathrm{Ext}_x(\alpha)$ gives the inequality

$$i(x,\alpha) \le a_g^{1/2}\mathrm{Ext}_x(\alpha)^{1/2}, \tag{1.3.3}$$

where $a_g = 2\pi(2g-2)$ is the area of the hyperbolic metric on x. Below we shall also use the following well known fact: if $i(\alpha,\beta) > 0$, and $x_n \in T_g$ is a sequence such that $\mathrm{Ext}_{x_n}(\alpha) \to 0$, then $\mathrm{Ext}_{x_n}(\beta) \to \infty$. For, since $\mathrm{Ext}_{x_n}(\alpha) \to 0$, by (1.3.3) the hyperbolic length $i(x_n,\alpha)$ of α tends to 0. As β crosses α, by [Kr, p.570],

$$\sinh\frac{i(x_n,\beta)}{2}\sinh\frac{i(x_n,\alpha)}{2} > 1,$$

which implies that $i(x_n,\beta) \to \infty$, and therefore $\mathrm{Ext}_{x_n}(\beta) \to \infty$. In particular, there exists $\varepsilon > 0$ such that

$$\mathrm{Ext}_x(\alpha) \le \varepsilon \implies \mathrm{Ext}_x(\beta) \ge 4\varepsilon \qquad \forall\, x \in T_g,\ \alpha,\beta \in \mathcal{S} : i(\alpha,\beta) > 0. \tag{1.3.4}$$

Let $A \subset \mathcal{S}$ be a disjoint system of homotopy classes of simple closed curves, i.e., $i(\alpha,\beta) = 0\ \forall \alpha \ne \beta \in A$. Its cardinality does not exceed $3g-3$. Moreover, since there is only a finite number of homotopy types for the complement of such systems, there is a finite collection $\{A_i\}$ of disjoint systems $A_i \subset \mathcal{S}$ with the property that any other disjoint system A has the form $A = \gamma A_i$ for some A_i and $\gamma \in \Gamma$.

For each disjoint system $A \subset \mathcal{S}$ there is a *boundary Teichmüller space* T_A obtained by *pinching* or *degenerating* along the curves $\alpha \in A$. The space T_A consists of *noded* or *punctured* Riemann surfaces, for which the curves in A have been assigned zero hyperbolic length. We can think of T_g as corresponding to the empty set A. If $A' \subseteq A$ we say that T_A is a *deformation* of $T_{A'}$; every curve that is assigned 0 length in $T_{A'}$ is also assigned 0 length in T_A. Then T_A is a (trivial) deformation of itself, and each T_A is a deformation of T_g. Denote by $\Gamma^A = \{\gamma \in \Gamma : \gamma A = A\}$ the stabilizer of the set A, and by Γ_0^A the normal subgroup of Γ^A that is isotopic to the identity on each component of the complement of curves from A. Then Γ^A/Γ_0^A is the mapping class group of T_A.

Lemma 1.3.1. *Given a constant $L > 0$ there exists $N > 0$ such that for any $x \in T_g$ there are at most N curves $\alpha \in \mathcal{S}$ with $\mathrm{Ext}_x(\alpha) \le L$.*

Proof. By the invariance of Ext_x under the Γ-action it is enough to consider the projections $\overline{x} \in M_g$ of points $x \in T_g$. For any compact $\overline{X} \subset M_g$ and $x \in \pi^{-1}\overline{X}$, we may find a pants decomposition of the surface x in which the pants curves have hyperbolic length bounded from below, and the distance between the curves in each pair of pants is also bounded from below (with constants depending on X only) [FLP]. Then for each L there is $N = N(L, X)$ such that the number of curves of hyperbolic length at most L on x does not exceed N. By the formula (1.3.3) this implies the result for the extremal lengths as well.

Thus, we need to find bounds in the case when $\overline{x}$ lies in a neighborhood of infinity in M_g. First we describe a neighborhood basis at infinity.

Let $M_{A_i} = T_{A_i}/(\Gamma^{A_i}/\Gamma_0^{A_i})$ be the moduli space of T_{A_i}. There are various equivalent ways of defining a topology on

$$\overline{M}_g = M_g \cup \bigcup_i M_{A_i}$$

in a neighborhood of $\bigcup_i M_{A_i}$ which compactifies M_g. We indicate one of them. Suppose $\overline{y} \in M_{A_i}$. Let V be a union of disjoint neighborhoods of the punctures of y, and let $\rho > 0$. Denote by $\mathcal{N}(y, V, \rho)$ the set of all $\overline{x} \in \overline{M}_g$ such that

(i) If $\overline{x} \in M_{A_j}$, then T_{A_i} is a deformation of T_{A_j}, i.e., $A_j \subset A_i$.
(ii) There are disjoint open sets U_α on x, one for each $\alpha \in A_i$, and such that U_α is an annular neighborhood of the geodesic α on x if $\alpha \in A_i \setminus A_j$, and U_α is a neighborhood of the punctures on x corresponding to α if $\alpha \in A_j$. There is a homeomorphism from $x \setminus \bigcup_\alpha U_\alpha$ to $y \setminus V$.
(iii) The homeomorphism from (ii) is a $(1+\rho)$-distortion of hyperbolic metrics.

The neighborhoods $\mathcal{N}(y, V, \rho)$, $\rho < 1$ form a basis for a topology at infinity on $\overline{M}_g$ compatible with the topology on M_g in the sense that intersections of these neighborhoods with M_g are open in M_g. With this topology $\overline{M}_g$ is compact [Ab].

Now fix $\rho < 1$. For each puncture of $y \in \overline{M}_g \setminus M_g$ we may take two disjoint curves homotopic to the puncture such that any arc crossing the annulus bounded by these two curves has hyperbolic length at least L. Let V be the neighborhood of the punctures of y whose boundary consists of the "inner" curves. Then any geodesic arc with endpoints on the boundary, not homotopic to an arc on the boundary, must cross the annuli twice and thus has length at least $2L$. The neighborhoods $\mathcal{N}(y, V, \rho)$ obtained in this way form a cover of the compact $\overline{M}_g \setminus M_g$. Take a finite subcover $\{\mathcal{N}(y_k, V_k, \rho_k)\}$, then its complement is compact in M_g. For each $y_k \in A_i$ there are at most $N = N(y_k, L)$ homotopy classes of curves with hyperbolic length $\leq 2L$ on y_k.

Suppose now $x \in \mathcal{N}(y_k, V_k, \rho_k) \cap M_g$, and β is a closed geodesic in x of length at most L. We claim that if $\beta \notin A_i$, then β cannot intersect $\alpha \in A_i$, or indeed even enter one of the annular neighborhoods U_α of α described in (ii). For if it did, then there would be an arc of β lying in $x \setminus \bigcup_\alpha U_\alpha$ whose endpoints lie on the boundary of $\bigcup_\alpha U_\alpha$, and which is not homotopic to an arc on this boundary. Via the homeomorphism of (ii) this arc maps to an arc in y with endpoints on the boundary of V, not homotopic to an arc on the boundary of V. This arc in y has length at least $2L$. By the distortion property (iii), the arc of β has length at least $2L/(1+\rho) > L$, and therefore β has length greater than L, contrary to assumption, proving the claim.

Now by the claim and the distortion property (iii), the closed geodesic β maps to a closed curve on y of length at most $L(1+\rho) \leq 2L$. There are at most N homotopy classes of geodesics on y with length at most $2L$, and this means that there are at most N geodesics $\beta \notin A_i$ on x of length at most L, and therefore a total of at most $N + |A_i| \leq N + 3g - 3$ geodesics of length at most L on x. Since there are a finite number of such y_k, and the complement of $\bigcup_k \mathcal{N}(y_k, V_k, \rho_k)$ is compact in M_g, we are done. □

Recall that a map f from one metric space (X_1, d_1) to another (X_2, d_2) is called a *rough isometry* [Kan] if there exists a constant $C > 0$ such that

$$\frac{1}{C} d_1(x,y) - C \leq d_2\big(f(x), f(y)\big) \leq C d_1(x,y) + C \qquad \forall\, x, y \in X_1$$

(some authors use in this situation the term "quasi-isometry", e.g., see [CDP], [GH], [Gr]). Two spaces (X_1, d_1) and (X_2, d_2) are *roughly isometric* if there exist rough isometries $f_1 : X_1 \to X_2$ and $f_2 : X_2 \to X_1$. We shall say that a metric space has *exponentially bounded growth* if it is roughly isometric to a graph with uniformly bounded vertex degrees.

Theorem 1.3.2. *The Teichmüller space T_g with Teichmüller metric d_T has exponentially bounded growth.*

Proof. We shall construct a graph $\mathcal{G}$ with uniformly bounded vertex degrees whose vertex set $X = \{x_i\}$ is a subset of T_g, and the embedding $\mathcal{G} \hookrightarrow T_g$ is a rough isometry (then necessarily this embedding is discrete). If in addition

$$\sup_{x \in T_g} d_T(x, X) < \infty ,$$

then the map from T_g to $\mathcal{G}$ assigning to any point $x \in T_g$ the nearest point from X (or, if it is not unique, the nearest point x_i with minimal index i) is also a rough isometry, so that T_g and $\mathcal{G}$ are roughly isometric.

Fix an $\varepsilon > 0$ satisfying the property (1.3.4), and let

$$\Omega = \{x \in T_g : \mathrm{Ext}_x > \tfrac{1}{4}\varepsilon\}$$
$$\overline{\Omega} = \{\overline{x} \in M_g : \mathrm{Ext}_{\overline{x}} > \tfrac{1}{4}\varepsilon\} = \pi(\Omega) ,$$

so that $\overline{\Omega}$ has a compact closure in M_g by the Mumford theorem.

For any disjoint system $A \subset \mathcal{S}$ and $\varepsilon > 0$ put

$$\begin{aligned} X_A(\varepsilon) &= \{x \in T_g : \mathrm{Ext}_x(\alpha) < \varepsilon,\ \alpha \in A;\ \mathrm{Ext}_x(\alpha) > \tfrac{1}{2}\varepsilon,\ \alpha \notin A\} , \\ \overline{X}_A(\varepsilon) &= \pi(X_A) \subset M_g \end{aligned} \tag{1.3.5}$$

Claim 1. The space T_g is covered by Ω and the sets X_A, and this cover has non-zero Lebesgue number σ.

We wish to find a constant $\sigma > 0$ such that for each $x \in T_g$ there is a ball of radius σ about x contained in either Ω or a single set X_A.

If $\mathrm{Ext}_x \geq \frac{3}{4}\varepsilon$, then $x \in \Omega$, and by (1.3.1) any point $y \in T_g \setminus \Omega$ (i.e, such that $\mathrm{Ext}_y \leq \frac{1}{4}\varepsilon$) has the property that

$$d_T(x,y) \geq \frac{1}{2}\log\frac{\frac{3}{4}}{\frac{1}{4}} .$$

Suppose then that $\mathrm{Ext}_x < \frac{3}{4}\varepsilon$. Let

$$A = A(x, \tfrac{3}{4}\varepsilon) = \{\alpha \in \mathcal{S} : \mathrm{Ext}_x(\alpha) < \tfrac{3}{4}\varepsilon\} .$$

It follows from (1.3.4) that the system A is disjoint, and quite clearly $x \in X_A$. Suppose $y \in T_g \setminus X_A$. Then either $\mathrm{Ext}_y(\alpha) \geq \varepsilon$ for some $\alpha \in A$, or $\mathrm{Ext}_y(\beta) \leq \frac{1}{2}\varepsilon$ for some $\beta \notin A$. On the other hand, by the definition of the set A we have $\mathrm{Ext}_x(\alpha) < \frac{3}{4}\varepsilon$ for $\alpha \in A$ and $\mathrm{Ext}_x(\beta) \geq \frac{3}{4}\varepsilon$ for $\beta \notin A$. Hence, in the first case by (1.3.1),

$$d_T(x,y) > \frac{1}{2}\log\frac{1}{\frac{3}{4}} ,$$

and in the second case

$$d_T(x,y) \geq \frac{1}{2}\log\frac{\frac{3}{4}}{\frac{1}{2}} .$$

Thus,

$$d_T(x,y) > \frac{1}{2}\log\tfrac{4}{3} ,$$

so that finally we can take $\sigma = \frac{1}{2}\log\frac{4}{3}$. △

Denote by $\mathbf{H}^p$ the product of p copies of the hyperbolic 2-space $\mathbf{H}$ with the product topology and sup-metric, and put the sup-metric d on $\mathbf{H}^p \times T_A$, where p is the cardinality of a disjoint system $A \subset \mathcal{S}$. By [Mi] there exists a constant $C = C(\varepsilon)$ and for each disjoint system $A \subset \mathcal{S}$ a continuous map

$$F = (F_1, F_2) : X_A \to \mathbf{H}^p \times T_A$$

such that

$$\Big| d\big(F(x), F(y)\big) - d_T(x,y) \Big| \le C \qquad \forall\, x, y \in X_A\ , \tag{1.3.6}$$

and

$$\mathrm{Ext}_{F_2(x)} \ge C\varepsilon \qquad \forall\, x \in X_A\ . \tag{1.3.7}$$

The Mumford compactness theorem and (1.3.7) imply that $F_2(X_A)$ has a compact quotient by Γ^A/Γ^A_0, and therefore so does the $(C+\sigma)/2$-neighborhood $F_2'(X_A)$ of $F_2(X_A)$. Take covers of $\mathbf{H}^p$ and of $F_2'(X_A)$ by balls of radius σ with a positive Lebesgue number, and with the property that each ball intersects a bounded number of others, and consider the product cover $\{V_i\}$ of $\mathbf{H}^p \times F_2'(X_A)$. Then the sets $F^{-1}(V_i)$ cover X_A. By (1.3.6) each of them is contained in a ball of radius $C + \sigma$.

Take a cover of $\overline{\Omega}$ by a finite number of balls $B(\overline{x}, \sigma)$, and denote by $\{W_i\}$ the cover of Ω by the balls of radius $\sigma + C$ centered at all points from the Γ-orbits of the points $x \mapsto \overline{x}$. Clearly, the cover $\{W_i\}$ has a positive Lebesgue number and the property that each element of this cover intersects a bounded number of other elements of the cover. The sets W_i together with the sets $F^{-1}(V_i)$ (taken for all disjoint systems $A \subset \mathcal{S}$) form a cover of T_g which we denote by $\{U_i\}$. Any set U_i is contained in a Teichmüller ball of radius $R = C + \sigma$.

Claim 2. There is a constant K such that each set U_i intersects at most K other sets U_j.

As the sets U_i have uniformly bounded diameter, any U_i intersects a bounded number of the sets W_j. Further, if a set U_i intersects a set $F^{-1}(V_j)$, then U_i lies within a bounded distance from the corresponding set X_A, so that by (1.3.1) $\mathrm{Ext}_x(\alpha)$ is uniformly bounded from above on U_i for all $\alpha \in A$. If there were no universal bound to the number of intersections of U_i with the sets $F^{-1}(V_j)$, then there would not be a universal bound for the number of curves whose extremal length is bounded from above with a certain constant. This contradicts Lemma 1.3.1. △

Claim 3. There is a constant L such that any two points $y, z \in T_g$ with $d_T(y,z) \le \sigma$ can be joined by a chain of at most L sets U_i.

By the choice of σ, the σ-ball around y is contained either in Ω or in a certain set X_A.

In the first case the geodesic joining y and z is also contained in Ω. Since the cover $\{W_i\}$ of Ω has a positive Lebesgue number, the points y, z can be joined by a bounded number of W_i along this geodesic.

In the second case $y, z \in X_A$, and $d\big(F(y), F(z)\big) \leq C + \sigma$ by (1.3.6), so that the geodesic joining $F(y)$ and $F(z)$ is contained in $\mathbf{H}^p \times F_2'(X_A)$. Since the cover $\{V_i\}$ has a positive Lebesgue number, the same argument as in the first case shows that the points $F(y)$ and $F(z)$ can be joined by a bounded number of V_i. $\triangle$

Now choose in every set U_i a point x_i such that U_i is contained in the ball of radius $R = C + \sigma$ centered at x_i, and consider the graph $\mathcal{G}$ with the vertex set $\{x_i\}$ such that two vertices x_i and x_j are joined with an edge iff the sets U_i and U_j intersect. By Claim 2 each vertex has at most K neighbours. Let d be the graph distance in $\mathcal{G}$. Since any U_i is contained in the R-ball centered at x_i, the Teichmüller distance between any two neighboring points in the graph $\mathcal{G}$ does not exceed $2R$, and

$$d_T(x_i, x_j) \leq 2R\, d(x_i, x_j) \qquad \forall\, x_i, x_j \in \mathcal{G}\ .$$

Conversely, by Claim 3 above

$$d(x_i, x_j) \leq L[d_T(x_i, x_j)/\sigma + 1]\ .$$

Thus, the identity map $f_1 : \mathcal{G} \to T_g$ and the map $f_2 : T_g \to \mathcal{G}$ assigning to any point $x \in T_g$ the nearest among the points x_i are rough isometries, so that T_g and $\mathcal{G}$ are roughly isometric. $\square$

Remarks. 1. The fact that any (not necessarily covering) Riemannian manifold with bounded geometry (bounded curvature and injectivity radius) has exponentially bounded growth is rather straightforward [Kan]. Milnor [Mil] proved that for a regular cover of a compact Riemannian manifold the natural embedding of the Cayley graph of the deck group into the cover obtained by identifying the group with its orbit is a rough isometry (Claim 3 from the proof of Theorem 1.3.2 basically uses the same argument). In general, this is not true for covers of non-compact manifolds, the simplest counterexample being the action of the group $SL(2, \mathbb{Z})$ on the hyperbolic plane. However, by a recent result of Lubotzki, Mozes and Raghunathan [LMR] this embedding is a rough isometry for the action of lattices in higher rank semi-simple groups on the corresponding symmetric spaces. Apparently, one should be able to prove that this embedding is *not* a rough isometry for free orbits of the mapping class group Γ in Teichmüller space T_g.

2. It is not immediately clear whether the graph $\mathcal{G}$ in Theorem 1.3.2 can be made Γ-invariant (which depends on Γ-invariance of the cover $\{F^{-1}(V_i)\}$).

Corollary 1. *There exist constants $D, R > 0$ such that any r-ball in T_g can be covered by not more than De^{Dr} balls of radius R.*

Proof. By Theorem 1.3.2 there is a countable set $X = \{x_i\} \subset T_g$ and a constant R such that

$$T_g = \bigcup_i B(x_i, R) .$$

Then for any $x \in T_g$

$$B(x,r) \subset \bigcup_{x_i : d_T(x,x_i) \le r+R} B(x_i, R) .$$

Further, X can be given a graph structure in such way that any point has at most K neighbours, and the graph distance d on X satisfies the inequality

$$d(x_i, x_j) \le C d_T(x_i, x_j) + C$$

for an absolute constant C. Hence, if $x_0 \in X$ is such that $d_T(x, x_0) \le R$, then

$$\begin{aligned} &\operatorname{card}\{x_i \in X : d_T(x, x_i) \le r + R\} \\ &\le \operatorname{card}\{x_i \in X : d_T(x_0, x_i) \le r + 2R\} \\ &\le \operatorname{card}\{x_i \in X : d(x_0, x_i) \le C(r + 2R) + C\} \le (K+1)^{C(r+2R)+C} . \end{aligned}$$

□

Corollary 2. *For any point $y \in T_g$ there is a constant D_y such that*

$$\operatorname{card}\{g \in \Gamma : d_T(x, gy) \le r\} \le D_y e^{D_y r} \qquad \forall\, x \in T_g,\ r > 0 .$$

Proof. Since the orbit of y in T_g is discrete, and the stabilizer $\operatorname{Stab} y \subset \Gamma$ is finite, the number

$$N_y = \operatorname{card}\{g \in \Gamma : d_T(y, gy) \le 2R\}$$

is finite (here R is the constant from Corollary 1). Hence,

$$\operatorname{card}\{g \in \Gamma : d_T(x, gy) \le R\} \le N_y \qquad \forall\, x \in T_g ,$$

and the statement follows from Corollary 1. □

Remark. The numbers N_y (hence, D_y) are not uniformly bounded for $y \in T_g$. Indeed, take $y \in T_g$ and $\alpha \in \mathcal{S}$ with $\operatorname{Ext}_y(\alpha)$ very small. Then the Dehn twist about α is like a parabolic element – it moves x very little, so that one can iterate it many times and still stay within a bounded distance.

1.4. Teichmüller geodesic lines and quadratic differentials.

The Teichmüller maps and the Teichmüller geodesic lines are described in terms of quadratic differentials. A holomorphic *quadratic differential* $\varphi(z)dz^2$ on $x \in T_g$ associates to each uniformizing parameter z_μ a holomorphic function $\varphi_\mu(z_\mu)$ in such way that

$$\varphi_\nu(z_\nu)\left(\frac{dz_\nu}{dz_\mu}\right)^2 = \varphi_\mu(z_\mu)$$

if the coordinate charts of z_ν and z_μ overlap. Any non-zero quadratic differential φ has a finite number of zeroes on M. Other points are called *regular points* of φ.

In a neighbourhood of any regular point $p \in M$ of φ there exists a *natural* uniformizing parameter w such that

$$dw^2 = \varphi(z)dz^2 .$$

If w_1 and w_2 are two overlapping natural coordinates, then $w_2 = \pm w_1 + c$, so that each quadratic differential φ determines by the formula $|dw| = |\varphi(z)^{1/2}dz|$ a flat metric on the complement of the finite set of zeroes of φ, where this metric has cone type singularities. The set of quadratic differentials Q^x on x is a Banach space of complex dimension $3g-3$, the norm $\|\varphi\| = \int |\varphi dz^2|$ being the area of M with respect to the flat metric $|\varphi(z)dz^2|$. Denote by S^x the unit sphere in Q^x.

For a geodesic β of the metric $|\varphi(z)dz^2|$ denote by

$$\begin{aligned} |\beta|_\varphi &= \int_\beta |\varphi(z)^{1/2}dz| = \int_\beta |dw| , \\ h_\varphi(\beta) &= \int_\beta |\Re\varphi(z)^{1/2}dz| = \int_\beta |\Re dw| , \\ v_\varphi(\beta) &= \int_\beta |\Im\varphi(z)^{1/2}dz| = \int_\beta |\Im dw| \end{aligned}$$

its length, *horizontal length* and *vertical length*, respectively. Clearly,

$$h_\varphi(\beta), v_\varphi(\beta) \le |\beta|_\varphi , \tag{1.4.1}$$

and

$$|\beta|_\varphi \le \mathrm{Ext}_x(\beta)^{1/2} \qquad \forall \varphi \in S^x,\ \beta \in \mathcal{S} , \tag{1.4.2}$$

where $|\beta|_\varphi$ is the length of the geodesic from the class $\beta \in \mathcal{S}$ with respect to the flat metric determined by φ.

The *horizontal* (resp., *vertical*) trajectories of a quadratic differential φ are curves $z(t)$ such that $\varphi(z(t))z'(t)^2 > 0$ (resp., < 0), i.e., $\Im w(t) = Const$ (resp., $\Re w(t) = Const$) for any natural parameter w. The horizontal (resp., vertical) trajectories of φ given the transverse measure $|\Im\varphi(z)^{1/2}dz| = |\Im dw|$ (resp., $|\Re\varphi(z)^{1/2}dz| = |\Re dw|$) define the *horizontal* H_φ (resp., *vertical* V_φ) *measured foliation* of φ. The foliations H_φ and V_φ are *transverse*, which is equivalent to saying that $i(H_\varphi, V_\varphi) > 0$ and for any $\alpha \in \mathcal{S}$ either $i(H_\varphi, \alpha) > 0$ or $i(V_\varphi, \alpha) > 0$ (the foliations H_φ and V_φ *fill*). Conversely, any two transverse measured foliations $F_1, F_2 \in \mathcal{MF}$ (in particular, any two topologically non-equivalent minimal foliations) uniquely determine a point $x \in T_g$ and a quadratic differential $\varphi \in Q^x$ such that $F_1 = H_\varphi$ and $F_2 = V_\varphi$. The intersection numbers of the vertical foliation of a quadratic differential φ and the horizontal length with respect to φ are connected by the formula

$$h_\varphi(\alpha) = i(V_\varphi, \alpha) , \tag{1.4.3}$$

where in the left-hand side $h_\varphi(\alpha)$ is the horizontal length of the geodesic from a class $\alpha \in \mathcal{S}$ with respect to the flat metric determined by φ. This formula follows from the fact that this geodesic is quasi-transverse to V_φ, so that it realizes the minimum in the definition of $i(V_\varphi, \beta)$ [HM]. Another proof of (1.4.3) using Jenkins–Strebel differentials is given in [Ma4, Lemma 2.2].

Given $x, y \in T_g$ the extremal quasiconformal or *Teichmüller map*, from x to y is defined by an *initial* quadratic differential $\varphi \in S^x$ and a number $K > 1$. There is a *terminal* quadratic differential $\psi \in S^y$. The Teichmüller map sends zeroes of φ to zeroes of ψ of the same order. Away from the zeroes, in terms of the natural parameters $w = u + iv$ for x and $\zeta = \xi + i\eta$ for y, the Teichmüller map is given by the formulas

$$\begin{aligned} \xi &= K^{1/2}u , \\ \eta &= K^{-1/2}v \end{aligned}$$

with $d_T(x, y) = 1/2 \log K$. Equivalently, the Teichmüller map sends horizontal (resp., vertical) trajectories of φ to horizontal (resp., vertical) trajectories of ψ stretching by a factor of $K^{1/2}$ (resp., $K^{-1/2}$), so that

$$\begin{aligned} h_\psi(\beta) &= K^{1/2}h_\varphi(\beta) , \\ v_\psi(\beta) &= K^{-1/2}v_\varphi(\beta) , \end{aligned}$$

and

$$\begin{aligned} H_\psi &= K^{-1/2}H_\varphi , \\ V_\psi &= K^{1/2}V_\varphi . \end{aligned} \tag{1.4.4}$$

The *Teichmüller geodesic line* determined by φ consists of the set of image points y of Teichmüller maps as K varies, $0 < K < \infty$. Thus, any Teichmüller geodesic line l determines a pair of transverse projective measured foliations: the projective classes of the horizontal and vertical foliations of quadratic differentials along l. Conversely, any two transverse foliations $F_-, F_+ \in \mathcal{PMF}$ uniquely determine a Teichmüller geodesic line $[F_-, F_+] = l$. Clearly, exchanging F_- and F_+ corresponds to changing the direction of l. As we shall see now, the vertical foliation F_+ (resp., horizontal F_-) can be in some situations considered as the endpoint of the geodesic $[F_-, F_+]$ at $+\infty$ (resp., $-\infty$) with respect to the compactification $T_g \hookrightarrow \mathcal{PMF}$.

Lemma 1.4.1. *Let $[F_-, F_+]$ be the Teichmüller geodesic line determined by transverse projective measured foliations F_-, F_+. If F_+ is minimal, then for any $M > 0$ and any $x_0 \in [F_-, F_+]$ all limit points of the M-neighbourhood of the geodesic ray $[x_0, F_+]$ belong to the equivalence class $\widetilde{F}_+ \in \widetilde{\mathcal{MIN}}$.*

Proof. The choice of x_0 determines a parameterization $l(t)$, $t \in \mathbb{R}$ on $[F_-, F_+]$ with $l(0) = x_0$. Let y_n be a sequence from the M-neighbourhood of the ray $\{l(t),\ t \geq 0\}$ such that $y_n \xrightarrow{\mathcal{PMF}} H \in \mathcal{PMF}$. As $y_n \to \infty$, there is $\alpha \in \mathcal{S}$ with $i(y_n, \alpha) \to \infty$, i.e., $r_n y_n \xrightarrow{\mathcal{MF}} H$ for a certain sequence $r_n \to 0$ [FLP]. There exists a sequence $t_n \to \infty$ such that $d_T(x_n, y_n) \leq M$ for $x_n = l(t_n)$. Since the values Ext_x, $x \in T_g$ are uniformly bounded from above, there exist $C > 0$ and a sequence $\beta_n \in \mathcal{S}$ such that

$$\mathrm{Ext}_{x_n}(\beta_n) \leq C \qquad \forall n \geq 1 .$$

Then by formulas (1.4.1) and (1.4.2)

$$h_{\varphi_n}(\beta_n) \leq |\beta_n|_{\varphi_n} \leq \mathrm{Ext}_{x_n}(\beta_n)^{1/2} \leq C^{1/2} .$$

On the other hand, denote by $\varphi \in S^{x_0}$ and $\varphi_n \in S^{x_n}$ the initial and terminal quadratic differentials of the Teichmüller map from x_0 to x_n (rescaling F_+ we may assume that $V_\varphi = F_+$). Then by the formulas (1.4.3) and (1.4.4)

$$h_{\varphi_n}(\beta_n) = i(V_{\varphi_n}, \beta) = e^{t_n} i(F_+, \beta_n) ,$$

so that

$$i(F_+, \beta_n) \underset{n\to\infty}{\longrightarrow} 0 .$$

Since the foliation F_+ is minimal, β_n are all distinct for sufficiently large n. Choose a subsequence again labeled β_n which is convergent in $\mathcal{PMF}$ to a foliation H', i.e., $s_n \beta_n \xrightarrow{\mathcal{MF}} H'$ for a sequence $s_n \to 0$ (Lemma 1.1.1). Then

$$i(F_+, H') = \lim s_n i(F_+, \beta_n) = 0 .$$

Since $d_T(x_n, y_n) \leq M$ and $\mathrm{Ext}_{x_n}(\beta_n) \leq C$, Kerckhoff's formula (1.3.1) says that $\mathrm{Ext}_{y_n}(\beta_n)$ is bounded. Hence by (1.3.3) the sequence $i(y_n, \beta_n)$ is also bounded, but then

$$i(H, H') = \lim i(r_n y_n, s_n \beta_n) = \lim r_n s_n i(y_n, \beta_n) = 0 ,$$

thereby $i(F_+, H) = 0$. □

Lemma 1.4.2. *Suppose that $x_n \in T_g$ is a sequence such that $x_n \to F \in \mathcal{UE}$. If*

$$d_T(x_0, x_n) - d_T(x_n, y_n) \to \infty$$

for a sequence $y_n \in T_g$, then $y_n \to F$.

Proof. By [Ma2] convergence of the sequence x_n to F is equivalent to convergence

$$V_{\varphi_n} \xrightarrow{\mathcal{PMF}} F\,,$$

where $\varphi_n \in S^{x_0}$ is the initial quadratic differential of the Teichmüller map from x_0 to x_n. Denote by $\theta_n \in S^{x_0}$ and $\psi_n \in S^{y_n}$ the initial and terminal quadratic differentials of the Teichmüller map from x_0 to y_n. We have to show that

$$V_{\theta_n} \xrightarrow{\mathcal{PMF}} F\,.$$

As in the last Lemma we choose a sequence β_n such that $\mathrm{Ext}_{x_n}(\beta_n)$ is bounded, which implies that

$$i(V_{\varphi_n}, \beta_n) \to 0\,.$$

Since $F \in \mathcal{UE}$, we have

$$\beta_n \xrightarrow{\mathcal{PMF}} F\,.$$

Now by (1.3.1),

$$\mathrm{Ext}_{y_n}(\beta_n) = O(e^{2d_T(x_n, y_n)})\,,$$

which by (1.4.1) and (1.4.2) in turn gives

$$h_{\psi_n}(\beta_n) = O(e^{d_T(x_n, y_n)})\,,$$

and since $d_T(x_0, y_n) - d_T(x_n, y_n) \to \infty$, (1.4.3) and (1.4.4) give

$$i(V_{\theta_n}, \beta_n) = h_{\theta_n}(\beta_n) \to 0\,.$$

Again, since $\beta_n \xrightarrow{\mathcal{PMF}} F$, we have $V_{\theta_n} \xrightarrow{\mathcal{PMF}} F$. □

Remark. Lemma 1.4.2 is an immediate analogue of the corresponding property of Gromov hyperbolic spaces (in particular, Cartan–Hadamard manifolds with pinched curvature; see [CDP], [GH], [Gr]). If X is a Gromov hyperbolic space with the hyperbolic boundary ∂X, and $x_n \in X$ is a sequence convergent to a point $\omega \in \partial X$, then any sequence y_n such that $d(x_n, x_0) - d(x_n, y_n) \to \infty$ converges to the same limit $\omega \in \partial X$. The reason is straightforward: the Gromov product

$$(x_n|y_n)_{x_0} = \frac{1}{2}\big[d(x_n, x_0) + d(y_n, x_0) - d(x_n, y_n)\big]$$

tends to infinity.

Lemma 1.4.3. *For any reference point $o \in T_g$ the function*

$$f : (F,G) \mapsto d_T(o,[F,G])$$

assigning to a pair of transverse foliations (F,G) the distance from o to the Teichmüller geodesic line $[F,G]$ is continuous on the subset of $\mathcal{PMF} \times \mathcal{PMF}$ where it is defined.

Proof. Suppose $(F_n, G_n) \xrightarrow{\mathcal{PMF}} (F_0, G_0)$. Let z_0 be the point on the geodesic $[F_0, G_0]$ realizing $f(F_0, G_0)$. On z_0 there is a quadratic differential φ_0 with horizontal and vertical foliations projectively equivalent to F_0 and G_0, respectively. By rescaling, we may assume that $H_{\varphi_0} = F_0$ and $V_{\varphi_0} = G_0$. Find representatives again denoted F_n, G_n in the corresponding projective classes such that $F_n \xrightarrow{\mathcal{MF}} F_0$ and $G_n \xrightarrow{\mathcal{MF}} G_0$. Then the quadratic differentials φ_n with F_n and G_n as horizontal and vertical foliations converge to φ_0, and the corresponding points z_n converge to z_0, so that

$$\limsup f(F_n, G_n) \le \lim d_T(o, z_n) = d_T(o, z_0) = f(F_0, G_0) .$$

For the opposite inequality take a subsequence (F_{n_k}, G_{n_k}) such that $\lim f(F_{n_k}, G_{n_k}) = \liminf f(F_n, G_n)$, and denote by $w_k \in [F_{n_k}, G_{n_k}]$ the point realizing $f(F_{n_k}, G_{n_k})$. Since w_k remains in a bounded subset of T_g, passing again to a subsequence we may assume that w_k converges to a point w_0. The points w_k carry normalized quadratic differentials ψ_k with horizontal and vertical foliations from the projective classes of F_{n_k} and G_{n_k}, respectively. Passing to a subsequence we may assume that $\psi_k \to \psi_0$, a quadratic differential on w_0. The horizontal and vertical foliations of ψ_k converge to the horizontal and vertical foliations of ψ_0. Since $(F_{n_k}, G_{n_k}) \xrightarrow{\mathcal{PMF}} (F_0, G_0)$, the horizontal and vertical foliations of ψ_0 are represented by the projective classes of F_0 and G_0, respectively, so that $w_0 \in [F_0, G_0]$. Then

$$\liminf f(F_n, G_n) = \lim f(F_{n_k}, G_{n_k}) = \lim d_T(o, w_k) = d_T(o, w_0) \ge f(F_0, G_0) .$$

$\square$

Denote by

$$\mathbf{Graph}(\sim) = \{(F,G) \in \mathcal{MIN} \times \mathcal{MIN} : F \sim G\}$$

the set of all pairs of equivalent minimal foliations.

Lemma 1.4.4. *For any reference point $o \in T_g$ the function*

$$\Phi(F_0, G_0) = \widetilde{\Phi}(\widetilde{F}_0, \widetilde{G}_0) = \sup\{d_T(o,[F,G]) : F \sim F_0,\ G \sim G_0\}$$

on $\mathcal{MIN} \times \mathcal{MIN} \setminus$ **Graph**$(\sim)$ *is upper semicontinuous.*

Proof. We have to show that for any sequence $(F_n, G_n) \to (F_0, G_0)$ there is a subsequence (F_{n_k}, G_{n_k}) such that $\lim \Phi(F_{n_k}, G_{n_k}) \le \Phi(F_0, G_0)$.

Since the equivalence classes $\widetilde{F} \in \widetilde{\mathcal{MIN}}$ are closed in $\mathcal{PMF}$, by Lemma 1.4.3 the sup in the definition of the function Φ is attained, and for any $(F_0, G_0) \in \mathcal{MIN} \times \mathcal{MIN} \setminus$ **Graph**$(\sim)$ there exist $F \sim F_0$, $G \sim G_0$ such that

$$\Phi(F_0, G_0) = d_T\big(o, [F, G]\big) .$$

Now take a sequence $(F_n, G_n) \to (F_0, G_0)$. Since $\widetilde{F}_0$ and $\widetilde{G}_0$ are two disjoint closed sets, $F_n \not\sim G_n$ for all sufficiently large n. Let $[F'_n, G'_n]$, $F'_n \sim F_n, G'_n \sim G_n$ be the geodesic realizing $\Phi(F_n, G_n)$. Take a subsequence (F'_{n_k}, G'_{n_k}) of the sequence (F'_n, G'_n) convergent to (F'_0, G'_0). Since $i(F_n, F'_n) = 0$, the continuity of $i(\cdot,\cdot)$ implies that $i(F_0, F'_0) = 0$, and similarly $i(G_0, G'_0) = 0$.

Thus,

$$\Phi(F_0, G_0) \ge d_T\big(o, [F'_0, G'_0]\big) = \lim d_T\big(o, [F'_{n_k}, G'_{n_k}]\big) = \lim \Phi(F'_{n_k}, G'_{n_k}),$$

proving the claim. □

1.5. Action of the mapping class group on $\mathcal{PMF}$.

We shall say that a sequence $g_n \in \Gamma$ is *universally convergent* if it tends to infinity in Γ, and for any $\alpha \in \mathcal{S}$ there exists a limit

$$\lim g_n^{-1}\alpha = F_\alpha \in \mathcal{PMF} ,$$

i.e., for any $\alpha \in \mathcal{S}$ there exists a sequence s_n^α such that

$$s_n^\alpha g_n^{-1}\alpha \xrightarrow{\mathcal{MF}} F_\alpha .$$

As it follows from Lemma 1.1.1, any sequence s_n^α is bounded, and it tends to zero iff $g_n^{-1}\alpha \xrightarrow{\mathcal{MF}} \infty$. Clearly, any unbounded sequence in Γ contains a universally convergent subsequence, and any subsequence of a universally convergent sequence is also universally convergent with the same limits F_α.

By Lemma 1.1.1, if g_n is a universally convergent sequence, then for any $\alpha \in \mathcal{S}$ either $g_n^{-1}\alpha \xrightarrow{\mathcal{MF}} \infty$, or there is $\beta = \beta(\alpha) \in \mathcal{S}$ with the property that $g_n^{-1}\alpha = \beta$ for infinitely many values of n. In the former case put $N_\alpha^0 = \varnothing$ and $N_\alpha^\infty = \{1, 2, \dots\}$. In the latter case such β is clearly unique, and we put

$$N_\alpha^0 = \{n : g_n^{-1}\alpha = \beta\}$$
$$N_\alpha^\infty = \{n : g_n^{-1}\alpha \ne \beta\} .$$

Again by Lemma 1.1.1, if the set N_α^∞ is infinite, then $g_n^{-1}\alpha \xrightarrow{\mathcal{MF}} \infty$ when n goes to infinity along the set N_α^∞. Note that the set N_α^∞ may well be empty (see Remark 2 after Lemma 1.2.2).

Lemma 1.5.1. *For any universally convergent sequence g_n there exists $\alpha \in \mathcal{S}$ with infinite set N_α^∞.*

Proof. Take a disjoint system $A = \{\alpha_1, \dots, \alpha_{3g-3}\} \subset \mathcal{S}$ of curves which determines a pants decomposition of the surface. Suppose that the sets $N_{\alpha_i}^\infty$ are all finite, then the sequences $g_n^{-1}\alpha_i$ all stabilize. Denote by $\beta_i \in \mathcal{S}$ the corresponding limits, so that $g_n^{-1}\alpha_i = \beta_i$ for all sufficiently large n. Then clearly the curves β_i are all distinct and pairwise disjoint, hence they also form a pants decomposition. Since any map that preserves the curves of a pants decomposition is a product of the commuting Dehn twists about these curves [FLP], any g_n^{-1} with sufficiently large n must have the form

$$g_n^{-1} = \gamma_1^{p_1} \cdots \gamma_{3g-3}^{p_{3g-3}} g \,, \qquad p_i = p_i(n) \,,$$

where $g \in \Gamma$ is some fixed map such that $g\alpha_i = \beta_i$, $i = 1, \dots, 3g-3$, and $\gamma_i = \gamma_{\beta_i}$ are the Dehn twists around the curves β_i. As g_n goes to infinity in Γ,

$$\sum_i |p_i(n)| \to \infty \,,$$

so that there is a curve β_i and a subsequence g_{n_k} such that $|p_i(n_k)| \to \infty$, which means that $g_{n_k}^{-1}(\omega)$ wraps more and more around this β_i for any curve ω that crosses α_i, and hence goes to infinity. □

We shall say that a sequence $g_n \in \Gamma$ is *strongly universally convergent* if it is universally convergent, and in addition there is $\alpha \in \mathcal{S}$ such that $g_n^{-1}\alpha \xrightarrow{\mathcal{MF}} \infty$. By Lemma 1.5.1 any unbounded sequence in Γ contains a strongly universally convergent subsequence.

Remark. In fact, one can show that any universally convergent sequence is strongly universally convergent. However, the weaker (and easier to prove) statement of Lemma 1.5.1 is sufficient for our purposes.

Lemma 1.5.2. *Let g_n be a universally convergent sequence. If for a certain $\alpha \in \mathcal{S}$ the set N_α^∞ is infinite, then $i(F_\alpha, F_\beta) = 0$ for all $\beta \in \mathcal{S}$.*

Proof. Pass to the subsequence of g_n (again denoted g_n) with indices $n \in N_\alpha^\infty$. Then $g_n^{-1}\alpha \xrightarrow{\mathcal{MF}} \infty$, so that $s_n^\alpha \to 0$. Since for any $\beta \in \mathcal{S}$ the sequence s_n^β is bounded, by continuity and Γ-invariance of the intersection number we have

$$i(F_\alpha, F_\beta) = \lim i(s_n^\alpha g_n^{-1}\alpha, s_n^\beta g_n^{-1}\beta) = \lim s_n^\alpha s_n^\beta i(\alpha, \beta) = 0 \,.$$

□

For a given universally convergent sequence g_n let

$$X = \bigcup_{\alpha \in \mathcal{S}} \widetilde{F}_\alpha \subset \mathcal{PMF}\,, \tag{1.5.1}$$

Note that as it follows from Lemma 1.5.2, the intersection of the set X with $\mathcal{MIN}$ consists of at most one equivalence class $\widetilde{F}$, $F \in \mathcal{MIN}$.

Lemma 1.5.3. *If g_n is a strongly universally convergent sequence, then $g_n F \xrightarrow{\mathcal{MF}} \infty$ for any $F \in T_g \cup \mathcal{PMF} \setminus X$.*

Proof. Take $\alpha \in S$ such that $g_n^{-1}\alpha \xrightarrow{\mathcal{MF}} \infty$, i.e., $s_n^\alpha \to 0$. Since

$$s_n^\alpha i(g_n F, \alpha) = s_n^\alpha i(F, g_n^{-1}\alpha) = i(F, s_n^\alpha g_n^{-1}\alpha) \to i(F, F_\alpha) \neq 0\,,$$

we have that $i(g_n F, \alpha) \to \infty$. □

Lemma 1.5.4. *Let g_n be a strongly universally convergent sequence, and foliations $F, F' \in T_g \cup \mathcal{PMF} \setminus X$ be such that $g_n F \xrightarrow{\mathcal{PMF}} H \in \mathcal{PMF}$ and $g_n F' \xrightarrow{\mathcal{PMF}} H' \in \mathcal{PMF}$. Then*

(i) $i(H, H') = 0$;
(ii) $i(H, \alpha) = 0 \iff i(H', \alpha) = 0 \quad \forall \alpha \in \mathcal{S}$.

Proof. (i). By Lemma 1.5.3 there exist sequences $t_n, t'_n \to 0$ such that $t_n g_n F \xrightarrow{\mathcal{MF}} H$ and $t'_n g_n F' \xrightarrow{\mathcal{MF}} H'$. Then

$$i(H, H') = \lim t_n t'_n i(g_n F, g_n F') = \lim t_n t'_n i(F, F') = 0\,.$$

(ii). For any $\alpha \in \mathcal{S}$ we have

$$\frac{i(H,\alpha)}{i(F,F_\alpha)} = \frac{\lim t_n i(g_n F, \alpha)}{\lim s_n^\alpha i(F, g_n^{-1}\alpha)} = \lim \frac{t_n}{s_n^\alpha}\,, \tag{1.5.2}$$

and in the same way

$$\frac{i(H',\alpha)}{i(F',F_\alpha)} = \lim \frac{t'_n}{s_n^\alpha}\,. \tag{1.5.3}$$

Now take $\omega \in \mathcal{S}$ with $i(H,\omega), i(H',\omega) > 0$. Such ω exists, for, otherwise, for any $\alpha \in \mathcal{S}$ either $i(H,\alpha) = 0$, or $i(H',\alpha) = 0$, hence, since $\mathcal{S}$ is dense in $\mathcal{PMF}$, for any $F \in \mathcal{PMF}$ either $i(H,F) = 0$, or $i(H',F) = 0$, which is impossible, because $i(H,F), i(H',F) > 0$ for any $F \in \mathcal{UE} \setminus \{H, H'\}$.

Then by (1.5.2) and (1.5.3) the limits $\lim t_n/s_n^\omega$ and $\lim t'_n/s_n^\omega$ are both non-zero, so that there exists a non-zero limit $\lim t_n/t'_n$. Comparing again formulas (1.5.2) and (1.5.3) yields the desired statement. □

Corollary. *If g_n is a strongly universally convergent sequence, then there exists $H \in \mathcal{PMF}$ such that all limit points of the sequences g_nF, $F \in T_g \cup \mathcal{PMF} \setminus X$ are contained in the set determined by the conditions* (i) *and* (ii).

Now we shall study the limit points of translations of measures in $\mathcal{PMF}$. We begin with the following elementary statement.

Lemma 1.5.5. *Let ν be a Borel probability measure on $\mathcal{PMF}$, and g_n – a sequence in Γ such that $g_n\nu$ converges weakly to a measure λ on $\mathcal{PMF}$. If there is a set $E \subset \mathcal{PMF}$ with $\nu E = 0$ and a G_δ-set $\Omega \subset \mathcal{PMF}$ such that Ω contains all limit points of sequences g_nF, $F \in \mathcal{PMF} \setminus E$, then the measure λ is supported on Ω.*

Proof. Let U be an arbitrary open neighbourhood of Ω. Then for any $F \notin E$ there is a finite number $n(U, F)$ such that $g_nF \in U$ for all $n \geq n(U, F)$. Since $\nu E = 0$, for any $\varepsilon > 0$ there is $N > 0$ such that $\nu\{F : n(U, F) \leq N\} \geq 1 - \varepsilon$. Hence, $g_n\nu(U) \geq 1 - \varepsilon$ for all $n \geq N$, and $\lambda U \geq 1 - \varepsilon$. Since ε is arbitrary, $\lambda U = 1$. Being G_δ, the set Ω is a countable intersection of its open neighbourhoods, hence $\lambda\Omega = 1$. □

Lemma 1.5.6. *Let ν be a Borel probability measure on $\mathcal{PMF}$ such that $\nu(\widetilde{F}) = 0$ for any $F \in \mathcal{PMF}$, and g_n – an unbounded sequence in Γ such that $g_n\nu$ converges weakly to a measure λ on $\mathcal{PMF}$. Then*

(i) *Either the measure λ is concentrated on the set*

$$\widetilde{H} = \{F : i(F, H) = 0\} \subset \mathcal{MIN} \tag{1.5.4}$$

for a certain $H \in \mathcal{MIN}$, or it is concentrated on the set

$$Z_H = \{F : i(F, \alpha) = 0 \iff i(H, \alpha) = 0 \;\forall\, \alpha \in \mathcal{S}\} \subset \mathcal{PMF} \setminus \mathcal{MIN} \tag{1.5.5}$$

for a certain $H \in \mathcal{PMF} \setminus \mathcal{MIN}$.

(ii) *In the first case all limit points of the sequences g_nx, $x \in T_g$ are contained in the set $\widetilde{H}$, and in the second case – in the set Z_H.*

Proof. (i). By passing to a subsequence we may assume that the sequence g_n is strongly universally convergent. Let $X \subset \mathcal{PMF}$ be the corresponding set (1.5.1). Take a foliation $F \in \mathcal{PMF} \setminus X$. Passing again to a subsequence we may assume that $g_nF \xrightarrow{\mathcal{PMF}} H \in \mathcal{PMF}$. Now we have two possibilities: either $H \in \mathcal{MIN}$, or $H \in \mathcal{PMF} \setminus \mathcal{MIN}$. In the first case denote by $\Omega = \Omega(H)$ the corresponding set $\widetilde{H}$ (1.5.4), and in the second case – the corresponding set Z_H (1.5.5).

Any set $\widetilde{H}$ is the countable intersection of open sets

$$\{F \in \mathcal{PMF} : i(F,H) < \tfrac{1}{n} i(F,\alpha)\} , \qquad \alpha \in \mathcal{S}, n > 0 ,$$

so that it is G_δ. Any set Z_H is also G_δ, as it is the countable intersection

$$Z_H = \left[\bigcap_{\alpha : i(H,\alpha)=0} \{F : i(F,\alpha) = 0\} \right] \cap \left[\bigcap_{\alpha : i(H,\alpha)>0} \{F : i(F,\alpha) > 0\} \right]$$

of the G_δ-sets $\{F \in \mathcal{PMF} : i(F,\alpha) = 0\}$ and open sets $\{F \in \mathcal{PMF} : i(F,\alpha) > 0\}$.

Thus, the set Ω is G_δ. Now by Lemma 1.5.4 all limit points of sequences $g_n F'$, $F' \in \mathcal{PMF} \setminus X$ belong to Ω. Since $\nu X = 0$, by Lemma 1.5.5 the measure λ is supported on Ω.

(ii). This immediately follows from Lemma 1.5.4. □

Remark. As it follows from the proof of Lemma 1.5.6, the set $\Omega(H)$ is completely determined by the sequence g_n, so that if ν' is another probability measure on $\mathcal{PMF}$ such that $\nu'(\widetilde{F}) = 0 \ \forall F \in \mathcal{PMF}$, and $g_n \nu' \to \lambda'$, then λ' is concentrated on the same set $\Omega(H)$ as the measure λ.

2. Random walks on the mapping class group

2.1. The Poisson boundary of random walks on groups.

Let G be a countable group, and μ – a probability measure on G. We shall denote by $\operatorname{sgr}(\mu)$ (resp., $\operatorname{gr}(\mu)$) the semigroup (resp., the group) generated by the support of the measure μ. The *random walk* on G determined by the measure μ is the Markov chain on G with the transition probabilities

$$p(g,h) = \mu(g^{-1}h)$$

invariant with respect to the left action of the group G on itself. Thus, the position g_n of the random walk at time n is obtained from its position g_0 at time 0 by multiplying by independent μ-distributed increments γ_i:

$$g_n = g_0 \gamma_1 \gamma_2 \cdots \gamma_n ,$$

and the set of all points in G attained by the random walk from the identity e is the semigroup $\operatorname{sgr}(\mu)$.

Denote by $\mathbf{P}$ the probability measure in the space $G^{\mathbb{Z}_+}$ of the *sample paths* $\boldsymbol{g} = \{g_n\}$, $n \geq 0$ which corresponds to the initial distribution concentrated at the identity (i.e., $g_0 = e$). The *one-dimensional distribution* of $\mathbf{P}$ at time n (i.e., the distribution of g_n) is the n-fold *convolution* μ_n of the measure μ. The *Markov operator* P_μ of the random walk (G, μ) (i.e., the operator of averaging with respect to the transition probabilities of the random walk) is

$$P_\mu f(g) = \sum_h p(g,h) f(h) = \sum_\gamma \mu(\gamma) f(g\gamma) .$$

A function f is called *μ-harmonic* if $P_\mu f = f$. By $H^\infty(G, \mu)$ we denote the Banach space of bounded *μ-harmonic* functions on $\operatorname{sgr}(\mu)$ with the sup-norm.

Suppose for a moment that the group G is embedded into a topological G-space B, and $\mathbf{P}$-a.e. sample path $\boldsymbol{g} = \{g_n\}$ converges to a limit $g_\infty = \pi(\boldsymbol{g}) \in B$. Then the *harmonic measure* $\lambda = \pi(\mathbf{P})$ is *μ-stationary* in the sense that

$$\mu\lambda = \sum \mu(g) g\lambda = \lambda ,$$

and the *Poisson formula*

$$f(g) = \langle \widehat{f}, g\lambda \rangle \tag{2.1.1}$$

determines an isometric embedding $\widehat{f} \mapsto f$ of the space $L^\infty(B, \lambda)$ into $H^\infty(G, \mu)$. When is this embedding a bijection? That is, when can every bounded harmonic function be represented as a Poisson integral (2.1.1) over the space (B, λ)?

Topology on the space B is, in fact, irrelevant, and the only thing one needs from a measure preserving map $\pi : (G^{\mathbb{Z}_+}, \mathbf{P}) \to (B, \lambda)$ in order to have the embedding (2.1.1) is its measurability with respect to the equivalence relation

$$\boldsymbol{g} \sim \boldsymbol{g}' \iff \exists\, k, k' \geq 0 : g_{k+n} = g'_{k'+n}\ \forall n \geq 0 .$$

In other words,

$$\boldsymbol{g} \sim \boldsymbol{g}' \iff \exists\, k, k' \geq 0 : T^k \boldsymbol{g} = T^{k'} \boldsymbol{g}' , \tag{2.1.2}$$

where $(T\boldsymbol{g})_n = g_{n+1}$ is the *time shift* in the path space $G^{\mathbb{Z}_+}$, i.e., the equivalence relation $\sim$ is the *trajectory equivalence relation* of the shift T. Note that the shift T does *not* preserve the measure $\mathbf{P}$, nor its type. However, the measure $\mathbf{P}_\theta$ corresponding to an initial distribution θ with $\operatorname{supp}\theta = G$ is quasi-invariant with respect to T.

The quotient measure space $(\partial P_\mu, \nu)$ of the path space $(G^{\mathbb{Z}_+}, \mathbf{P})$ with respect to the measurable envelope of the equivalence relation $\sim$ (i.e., the *space of ergodic components* of the shift T) is called the *Poisson boundary* of the pair (G, μ). The Poisson boundary is endowed with an action of the group G, and the *harmonic measure* ν is μ-stationary with respect to this action. The Poisson formula (2.1.1) is an isometric isomorphism of the spaces $H^\infty(G, \mu)$ and $L^\infty(\Gamma, \nu)$ [Ka5].

Triviality of the Poisson boundary is equivalent to absence of non-constant bounded μ-harmonic functions on the semigroup $\operatorname{sgr}(\mu)$, or, equivalently, on the group $\operatorname{gr}(\mu)$, (the *Liouville property*), which is the case for all measures on abelian and nilpotent groups. If the group G is *amenable*, then there always exists a measure μ with trivial Poisson boundary (but there may also be measures with a non-trivial boundary). On the other hand, the Poisson boundary is non-trivial for all measures on a non-amenable group G [KV].

Any G-space which is a $\sim$-measurable image of the path space is the quotient of the Poisson boundary with respect to a certain G-invariant measurable partition. Such quotients are called *μ-boundaries* [Fu3], [Ka10]. By definition, the Poisson boundary is the maximal μ-boundary. Thus, the problem of describing the Poisson boundary of a random walk (G, μ) consists of two parts:

(1) To find (in geometric or combinatorial terms) a μ-boundary (B, λ);
(2) To show that this μ-boundary is maximal.

In other words, first one has to exhibit a certain system of invariants of stochastically significant behavior of sample paths at infinity, and then to show completeness of this system.

Note that in the same way as for random walks on groups one can define the notions of harmonic functions and the Poisson boundary (and ask the question about its identification) for an arbitrary Markov operator [Ka5].

2.2. Convergence in the Thurston compactification.

Lemma 2.2.1. *Let μ be a probability measure on a countable group G, and X – a compact G-space. Then there exists a μ-stationary probability measure on X.*

Proof. Let ν be a Borel probability measure on X. Compactness of X means that the space $M(X)$ of Borel probability measures on X is compact in the weak topology. The Cesaro averages

$$\nu_n = \frac{1}{n+1}(\nu + \mu * \nu + \mu_2 * \nu + \ldots + \mu_n * \nu)$$

have the property that

$$\|\mu * \nu_n - \nu_n\| = \frac{1}{n+1}\|\mu_{n+1} * \nu - \nu\| \le \frac{2}{n+1} \xrightarrow[n\to\infty]{} 0$$

(here $\|\lambda\|$ is the total variation of a measure λ). Hence any weak limit point of the sequence ν_n is a μ-stationary measure. □

Lemma 2.2.2 (cf. [Ba], [Wo1]). *Let μ be a probability measure on a countable group G, and ν – a μ-stationary probability measure on a G-space X. Suppose $E \subset X$ is a measurable subset such that for all $g \in \operatorname{gr}(\mu)$ either $gE = E$ or $gE \cap E = \varnothing$, and there is an infinite number of pairwise disjoint translations gE, $g \in \operatorname{gr}(\mu)$. Then $\nu(E) = 0$.*

Proof. Without loss of generality we may assume $G = \operatorname{gr}(\mu)$; if not, just consider instead of G the subgroup $\operatorname{gr}(\mu)$. If $\nu(E) > 0$, then there is a $g_0 \in G$ which maximizes $\nu(gE)$, i.e.,

$$\nu(gE) \le \nu(g_0 E) \qquad \forall\, g \in G\,.$$

Put $E' = g_0 E$. Since the measure ν is μ-stationary,

$$\nu(E') = \sum_g \mu(g)\, g\nu(E') = \sum_g \mu(g)\, \nu(g^{-1}E') \le \sum_g \mu(g)\, \nu(E') = \nu(E')\,.$$

Therefore $\nu(g^{-1}E') = \nu(E')$ for all $g \in \operatorname{supp}\mu$. Applying the same argument to convolutions of the measure μ, we see that

$$\nu(g^{-1}E') = \nu(E') \qquad \forall\, g \in \operatorname{sgr}(\mu)\,,$$

which is only possible if the set of pairwise disjoint translations $g^{-1}E'$, $g \in \operatorname{sgr}(\mu)$ (i.e., the $\operatorname{sgr}(\mu)^{-1}$-orbit of the set E' in the space of subsets of X) is finite. The latter by a standard argument implies that for the group G generated by the semigroup $\operatorname{sgr}(\mu)^{-1}$ the orbit of E' ($\equiv$ the orbit of E) is also finite, which gives a contradiction. $\square$

Lemma 2.2.3. *Let μ be a probability measure on a countable group G, and ν – a μ-stationary probability measure on a compact separable space X. Then for $\mathbf{P}$-a.e. sample path $\boldsymbol{g} = \{g_n\}$ of the random walk (G, μ) the translations $g_n\nu$ converge weakly to a (random) limit $\lambda = \lambda(\boldsymbol{g})$, and*

$$\nu = \int \lambda(\boldsymbol{g})\, d\mathbf{P}(\boldsymbol{g})\,. \tag{2.2.1}$$

Proof. The measure ν being μ-stationary, for any continuous function $\widehat{f} : X \to \mathbb{R}$ the *Poisson integral*

$$f(g) = \langle \widehat{f}, g\nu \rangle = \int_X \widehat{f}(x)\, d\nu(g^{-1}x)$$

is a bounded μ-harmonic function, so that by the Martingale Convergence Theorem the sequence $f(g_n) = \langle \widehat{f}, g_n\nu \rangle$ converges for a.e. sample path $\{g_n\}$. The space X is separable, hence taking $\widehat{f}$ from a dense countable subset of $C(X)$ we obtain that **P**-a.e. sequence of measures $g_n\nu_0$ converges weakly (see [Fu3, Corollary 3.1]). Moreover, passing to the limit on n in the identity

$$\nu = \mu_n * \nu = \int g_n \nu \, d\mathbf{P}(\boldsymbol{g})$$

gives the decomposition (2.2.1). □

We shall say that *a subgroup* Γ' *of the mapping class group* Γ *satisfies condition* (NE) *if it does not fix any finite union of the sets*

$$\widetilde{H} = \{F \in \mathcal{PMF} : i(F, H) = 0\} , \qquad H \in \mathcal{MIN} ,$$

or

$$Z_H = \{F \in \mathcal{PMF} : i(F, \alpha) = 0 \iff i(H, \alpha) = 0 \, \forall \, \alpha \in \mathcal{S}\} , \qquad H \in \mathcal{PMF} \setminus \mathcal{MIN} .$$

Equivalently, $\Gamma' \subset \Gamma$ *satisfies condition* (NE) *if it is not a finite extension of the stabilizer of a set* $\widetilde{H}$ *or* Z_H. This notion is a direct analogue of that of *non-elementary* groups of isometries of hyperbolic spaces [Gr]. Note, however, that unlike in the hyperbolic case, a subgroup of Γ may not satisfy (NE) and still be non-amenable. For example, the subgroup generated by Dehn twists about two intersecting curves that are each disjoint from a third curve α is non-amenable, but fixes the set Z_α. As it follows from minimality of the Γ-action on $\mathcal{PMF}$ and Lemma 1.2.3, the group Γ itself satisfies condition (NE).

Theorem 2.2.4. *Let* μ *be a probability measure on the mapping class group* Γ *such that the group* $\mathrm{gr}(\mu)$ *satisfies condition* (NE). *Then*

(i) *There exists a unique* μ*-stationary probability measure* ν *on the space* $\mathcal{PMF}$, *which is purely non-atomic and concentrated on* $\mathcal{UE}$, *and the measure space* $(\mathcal{UE}, \nu)$ *is a* μ*-boundary;*

(ii) *For* **P**-*a.e. sample path* $\boldsymbol{g} = \{g_n\}$ *of the random walk* (Γ, μ) *and any* $x \in T_g$ *the sequence* $g_n x$ *converges in* $\mathcal{PMF}$ *to a limit* $F = F(\boldsymbol{g}) \in \mathcal{UE}$, *and the distribution of the limits* $F(\boldsymbol{g})$ *is given by the measure* ν.

Proof. (i). Let ν be a μ-stationary probability measure on $\mathcal{PMF}$ which exists by Lemma 2.2.1. Since $\mathcal{PMF}$ is a Polish topological space (complete, metrizable, separable), and ν is a Borel measure, the measure space $(\mathcal{PMF}, \nu)$ is a *Lebesgue space*, so that we can use the standard language of measurable partitions [CFS].

By condition (NE) the $\mathrm{gr}(\mu)$-orbit of any set Z_F, $F \subset \mathcal{PMF} \setminus \mathcal{MIN}$ is infinite, hence by Lemma 2.2.2 the measure ν is concentrated on $\mathcal{MIN}$. By Lemma 1.1.2 the partition of the measure space $(\mathcal{MIN}, \nu)$ into equivalence classes of the relation $\sim$ is measurable, so that there exists a quotient Lebesgue measure space $(\widetilde{\mathcal{MIN}}, \widetilde{\nu})$ whose elements are equivalence classes $\widetilde{F}$, $F \in \mathcal{MIN}$. By Lemma 2.2.2 and condition (NE) the measure $\widetilde{\nu}$ is purely non-atomic.

By Lemma 2.2.3 for $\mathbf{P}$-a.e. sample path $\boldsymbol{g}$ of the random walk (Γ, μ) there exists the weak limit $\lambda(\boldsymbol{g}) = \lim g_n \nu$. Since the measure ν is supported on $\mathcal{MIN}$, the decomposition (2.2.1) implies that a.e. limit measure $\lambda(\boldsymbol{g})$ is also supported on $\mathcal{MIN}$. Hence, by Lemma 1.5.6 a.e. measure $\lambda(\boldsymbol{g})$ is concentrated on a single equivalence class of the relation $\sim$ in $\mathcal{MIN}$, which means that we have a measurable map from the path space to $\widetilde{\mathcal{MIN}}$. Clearly, this map is measurable with respect to the trajectory equivalence relation (2.1.2) in the path space and Γ-equivariant. As it follows from formula (2.2.1), the image of the measure $\mathbf{P}$ in the path space under this map is $\widetilde{\nu}$, so that the quotient measure space $(\widetilde{\mathcal{MIN}}, \widetilde{\nu})$ is a non-trivial μ-boundary. It implies that the Poisson boundary of the pair (Γ, μ) is non-trivial. In particular, the random walk (Γ, μ) is transient, i.e., $g_n \to \infty$ for $\mathbf{P}$-a.e. sample path $\boldsymbol{g} = \{g_n\}$ [KV].

Now we shall show that in fact the measure ν is concentrated on $\mathcal{UE}$, so that the measure spaces $(\widetilde{\mathcal{MIN}}, \widetilde{\nu})$ and $(\mathcal{PMF}, \nu)$ coincide, and $(\mathcal{PMF}, \nu)$ is a μ-boundary.

Consider the measure space $(\Gamma^{\mathbb{Z}}, \overline{\mathbf{P}})$ of *bilateral paths* $\overline{\boldsymbol{g}} = \{g_n,\ n \in \mathbb{Z}\}$ corresponding to bilateral sequences of independent μ-distributed increments $\boldsymbol{\gamma} = \{\gamma_n\}$ by the formula

$$g_n = g_{n-1}\gamma_n, \qquad g_0 = e\ . \tag{2.2.2}$$

Clearly, the formula (2.2.2) states a one-two-one correspondence between bilateral paths $\overline{\boldsymbol{g}} = \{g_n\}$ in Γ passing through e at time 0 and their increments $\gamma_n = g_{n-1}^{-1} g_n$. For negative indices n the formula (2.2.2) can be rewritten as

$$g_{-n} = g_{-n+1}\gamma_{-n+1}^{-1}\ , \qquad n \ge 0\ ,$$

so that

$$\check{g}_n = g_{-n} = \gamma_0^{-1}\gamma_{-1}^{-1}\cdots\gamma_{-n+1}^{-1}\ , \qquad n \ge 0$$

is a sample path of the random walk on Γ governed by the *reflected measure* $\check{\mu}(\gamma) = \mu(\gamma^{-1})$. The unilateral paths $\boldsymbol{g} = \{g_n\}$, $n \ge 0$ and $\check{\boldsymbol{g}} = \{\check{g}_n\} = \{g_{-n}\}$, $n \ge 0$ are independent, or, in other words, the map $\overline{\boldsymbol{g}} \mapsto (\boldsymbol{g}, \check{\boldsymbol{g}})$ is an isomorphism of the measure spaces $(\Gamma^{\mathbb{Z}}, \overline{\mathbf{P}})$ and $(\Gamma^{\mathbb{Z}_+}, \mathbf{P}) \times (\Gamma^{\mathbb{Z}_+}, \check{\mathbf{P}})$, where $\check{\mathbf{P}}$ is the measure in the space of unilateral sample paths of the random walk $(\Gamma, \check{\mu})$.

Denote by U the measure preserving transformation of the space of bilateral paths $(\Gamma^{\mathbb{Z}}, \overline{\mathbf{P}})$ induced by the *Bernoulli shift* (also denoted by U) in the space of increments $\boldsymbol{\gamma} = \{\gamma_n\}$, $n \ge 0$, i.e.,

$$(U\boldsymbol{\gamma})_n = \gamma_{n+1} \qquad \forall n \in \mathbb{Z}\ .$$

Hence, if $\overline{\boldsymbol{g}} = \{g_n\}$ is the bilateral path corresponding to the sequence of increments γ, then

$$(U\overline{\boldsymbol{g}})_{n+1} = (U\overline{\boldsymbol{g}})_n (U\gamma)_{n+1} = (U\overline{\boldsymbol{g}})_n \gamma_{n+2} \qquad \forall n \in \mathbb{Z}\,,$$

so that

$$(U\overline{\boldsymbol{g}})_n = \gamma_1^{-1} g_{n+1} = g_1^{-1} g_{n+1} \qquad \forall n \in \mathbb{Z}\,,$$

or, more generally, for any $k \in \mathbb{Z}$

$$(U^k\overline{\boldsymbol{g}})_n = g_k^{-1} g_{n+k} \qquad \forall n \in \mathbb{Z}\,, \tag{2.2.3}$$

i.e., the path $U^k\overline{\boldsymbol{g}}$ is obtained from the path $\overline{\boldsymbol{g}}$ by translating it both in time (by k) and in space (by multiplying by g_k^{-1} on the left in order to satisfy the condition $(U^k\overline{\boldsymbol{g}})_0 = e$). In terms of the unilateral paths $\boldsymbol{g}$ and $\check{\boldsymbol{g}}$ it means that (for $k > 0$) one cancels first k factors $g_k = \gamma_1\gamma_2\cdots\gamma_k$ from the products $g_n = \gamma_1\gamma_2\cdots\gamma_k\cdots\gamma_n$, $n > 0$ and adds (on the left) k factors $g_k^{-1} = \gamma_k^{-1}\cdots\gamma_2^{-1}\gamma_1^{-1}$ to the products $\check{g}_n = g_{-n} = \gamma_0^{-1}\gamma_{-1}^{-1}\cdots\gamma_{-n+1}^{-1}$:

$$\rlap{$\underbrace{\phantom{\cdots,\gamma_{-1},\gamma_0,\gamma_1,\cdots,\gamma_{k-1},\gamma_k,}}\underbrace{\phantom{\gamma_{k+1},\cdots}}$}\overbrace{\cdots,\gamma_{-1},\gamma_0,}\overbrace{\gamma_1,\cdots,\gamma_{k-1},\gamma_k,\gamma_{k+1},\cdots}$$

By the argument above applied to the measure $\check{\mu}$ there exists a purely non-atomic $\check{\mu}$-stationary measure $\widetilde{\nu}_-$ on $\widetilde{\mathcal{MIN}}$ such that the space $(\widetilde{\mathcal{MIN}}, \widetilde{\nu}_-)$ is a $\check{\mu}$-boundary. For symmetry we shall use the notation $\widetilde{\nu}_+$ for the measure $\widetilde{\nu}$ for the rest of the proof. Denote the boundaries $(\widetilde{\mathcal{MIN}}, \widetilde{\nu}_+)$ and $(\widetilde{\mathcal{MIN}}, \widetilde{\nu}_-)$ by B_+ and B_-, respectively, and let $\mathbf{bnd}_+(\overline{\boldsymbol{g}}) = \mathbf{bnd}(\boldsymbol{g}) \in B_+$ and $\mathbf{bnd}_-(\overline{\boldsymbol{g}}) = \mathbf{bnd}(\check{\boldsymbol{g}}) \in B_-$ be the corresponding boundary points of the unilateral paths $\boldsymbol{g}$ and $\check{\boldsymbol{g}}$.

Independence of $\boldsymbol{g}$ and $\check{\boldsymbol{g}}$ implies that the image of the measure $\overline{\mathbf{P}}$ under the map

$$\pi : \overline{\boldsymbol{g}} \mapsto \big(\mathbf{bnd}_-(\overline{\boldsymbol{g}}), \mathbf{bnd}_+(\overline{\boldsymbol{g}})\big)$$

(i.e., the joint distribution of $\mathbf{bnd}_-(\overline{\boldsymbol{g}})$ and $\mathbf{bnd}_+(\overline{\boldsymbol{g}})$) is $\widetilde{\nu}_- \otimes \widetilde{\nu}_+$. By the formula (2.2.3)

$$\begin{aligned} \mathbf{bnd}_+(U^k\overline{\boldsymbol{g}}) &= g_k^{-1}\mathbf{bnd}_+(\overline{\boldsymbol{g}})\,, \\ \mathbf{bnd}_-(U^k\overline{\boldsymbol{g}}) &= g_k^{-1}\mathbf{bnd}_-(\overline{\boldsymbol{g}})\,. \end{aligned} \tag{2.2.4}$$

Take a reference point $o \in T_g$, and let

$$\begin{aligned} \Psi(\overline{\boldsymbol{g}}) &= \sup\big\{d_T\big(o, [F_-, F_+]\big) : F_- \in \mathbf{bnd}_-(\overline{\boldsymbol{g}}), F_+ \in \mathbf{bnd}_+(\overline{\boldsymbol{g}})\big) \\ &= \widetilde{\Phi}\big(\mathbf{bnd}_-(\overline{\boldsymbol{g}}), \mathbf{bnd}_+(\overline{\boldsymbol{g}})\big) \end{aligned}$$

be the pullback of the function $\widetilde{\Phi}$ defined in Lemma 1.4.4 from $\widetilde{\mathcal{MIN}} \times \widetilde{\mathcal{MIN}}$ to the space of bilateral paths $\Gamma^{\mathbb{Z}}$. Since the measure $\widetilde{\nu}_+$ is purely non-atomic, the function Ψ is a.e. defined, and by Lemma 1.4.4 it is measurable.

Then by the formula (2.2.4) for any $k \in \mathbb{Z}$

$$\begin{aligned}
&\sup\{d_T(g_k o, [F_-, F_+]) : F_- \in \mathbf{bnd}_-(\overline{g}), F_+ \in \mathbf{bnd}_+(\overline{g}))\\
=&\sup\{d_T(o, g_k^{-1}[F_-, F_+]) : F_- \in \mathbf{bnd}_-(\overline{g}), F_+ \in \mathbf{bnd}_+(\overline{g}))\\
=&\sup\{d_T(o, [F_-, F_+]) : F_- \in g_k^{-1}\mathbf{bnd}_-(\overline{g}), F_+ \in g_k^{-1}\mathbf{bnd}_+(\overline{g}))\\
=&\sup\{d_T(o, [F_-, F_+]) : F_- \in \mathbf{bnd}_-(U^k\overline{g}), F_+ \in \mathbf{bnd}_+(U^k\overline{g}))\\
=&\Psi(U^k\overline{g})\ .
\end{aligned}$$

As the function Ψ is a.e. finite and measurable, we can choose a number M such that

$$\overline{\mathbf{P}}\big[\Psi(\overline{g}) \le M\big] = p > 1/2\ ,$$

then by the Ergodic Theorem applied to the transformation U for a.e. bilateral path $\overline{g}$ the density of times $k \ge 0$ such that

$$\sup\{d_T(g_k o, [F_-, F_+]) : F_- \in \mathbf{bnd}_-(\overline{g}), F_+ \in \mathbf{bnd}_+(\overline{g})\} \le M$$

equals p.

Since the unilateral parts g and $\check{g}$ of the bilateral path $\overline{g}$ are independent, it means that for $\mathbf{P}$-a.e. unilateral path g and $\widetilde{\nu}_-$-a.e. class $\widetilde{H}_- \in \widetilde{\mathcal{MIN}}$ the density of times $k \ge 0$ such that

$$\sup\{d_T(g_k o, [F_-, F_+]) : F_- \in \widetilde{H}_-, F_+ \in \mathbf{bnd}(g)\} \le M$$

equals p. The measure $\widetilde{\nu}_-$ being non-atomic, there exist distinct $\widetilde{H}^1_-, \widetilde{H}^2_- \in \widetilde{\mathcal{MIN}}$ with this property. As $p > 1/2$, it means that for $\mathbf{P}$-a.e. unilateral path g there exist an infinite sequence of times n_k and $H^1_- \not\sim H^2_- \in \mathcal{MIN}$ such that

$$\sup\{d_T(g_{n_k} o, [H^i_-, F_+]) : F_+ \in \mathbf{bnd}(g)\} \le M\ , \qquad \forall k \ge 0,\ i = 1,2\ .$$

Suppose $\mathbf{bnd}(g) \in \widetilde{\mathcal{MIN}} \setminus \mathcal{UE}$, and take $F_+ \in \mathbf{bnd}(g)$, then

$$d_T(g_{n_k} o, [H^i_-, F_+]) \le M\ , \qquad \forall k \ge 0,\ i = 1,2\ . \tag{2.2.5}$$

Choose parametrizations $l_i(t)$, $t \in \mathbb{R}$ on the Teichmüller geodesic lines $[H^i_-, F_+]$. Since $F_+ \in \mathcal{MIN} \setminus \mathcal{UE}$, by [Ma3, Theorem 1]

$$d_T(l_i(t), \Gamma o) \underset{t\to+\infty}{\longrightarrow} \infty\ .$$

On the other hand, by Lemma 1.4.1, the intersection of the M-neighbourhoods of the negative rays $\{l_i(t),\ t \le 0\}$ is compact. Thus, there must exist only a finite number of distinct g_{n_k} satisfying the condition (2.2.5), which is impossible because g_n (hence, g_{n_k}) tends to infinity by transience of the random walk (Γ, μ).

Thus, we have shown that the measure $\nu = \nu_+$ is concentrated on $\mathcal{UE}$, and by the decomposition (2.2.1) from Lemma 2.2.3 almost all limit measures $\lambda(\boldsymbol{g}) = \lim g_n \nu$ are δ-measures corresponding to points from $\mathcal{UE}$. If ν' is another μ-stationary probability measure on $\mathcal{PMF}$, then by the Remark after Lemma 1.5.6 it has the same limit measures $\lim g_n \nu = \lim g_n \nu'$, so that in view of the decomposition (2.2.1) $\nu' = \nu$, which means that ν is the unique μ-stationary measure on $\mathcal{PMF}$. Since ν is concentrated on $\mathcal{UE}$, the factorization map $(\mathcal{MIN}, \nu) \mapsto (\widetilde{\mathcal{MIN}}, \widetilde{\nu})$ is an isomorphism of measure spaces. As we have already shown that $(\widetilde{\mathcal{MIN}}, \widetilde{\nu})$ is a μ-boundary, we have that the measure space $(\mathcal{MIN}, \nu) \cong (\widetilde{\mathcal{MIN}}, \widetilde{\nu})$ is a μ-boundary.

(ii). We have shown that for $\mathbf{P}$-a.e. sample path $\boldsymbol{g}$ there exists a point $F = F(\boldsymbol{g}) \in \mathcal{UE}$ such that $g_n \nu \to \delta_F$ weakly. By Lemma 1.5.6 it implies that $g_n x \to F\ \forall x \in T_g$. In particular, the distribution of the limits $\lim g_n x$ is the same as the distribution of $F(\boldsymbol{g})$ which has been shown to coincide with ν. □

Corollary 1. *For any $x \in T_g$ the sequence of measures $\mu_n * \delta_x$ on T_g converges weakly to the unique μ-stationary measure ν on $\mathcal{PMF}$.*

Proof. By definition,

$$\mu_n * \delta_x = \int g_n \delta_x \, d\mathbf{P}(\boldsymbol{g}) .$$

Since a.e. $g_n x \to \mathbf{bnd}(\boldsymbol{g}) \in \mathcal{UE}$, and the distribution of the limit points $\mathbf{bnd}(\boldsymbol{g})$ is ν, passing to the limit (in the same way as in Lemma 2.2.3) yields the result. □

Corollary 2. *Any subgroup Γ' of Γ satisfying the condition* (NE) *is non-amenable.*

Proof. By Theorem 2.2.4 for any probability measure μ on Γ' with $\mathrm{gr}(\mu) = \Gamma'$ the Poisson boundary is non-trivial. By [KV] (see also [Ro]) this implies that Γ' is non-amenable. □

2.3. Identification of the Poisson boundary.

Theorem 2.3.1. *Let μ be a probability measure on the mapping class group Γ such that the group $\mathrm{gr}(\mu) \subset \Gamma$ satisfies condition* (NE). *If*

(i) *The measure μ has a finite logarithmic moment with respect to the Teichmüller distance*
$$\sum_\gamma \mu(\gamma) \log_+ d_T(o, \gamma o) < \infty \, ;$$

(ii) *The measure μ has a finite entropy*
$$H(\mu) = \sum_\gamma -\mu(\gamma) \log \mu(\gamma) < \infty \, ;$$

then the measure space $(\mathcal{PMF}, \nu)$, where ν is the unique μ-stationary probability measure on $\mathcal{PMF}$, is the Poisson boundary of the pair (Γ, μ).

Proof. We shall use the "strip criterion" from [Ka10] (see also [Ka7]). It requires considering a μ-boundary (B_+, ν_+) simultaneously with a $\check\mu$-boundary (B_-, ν_-). If there exists an equivariant measurable map assigning to a.e. pair of points $(F_-, F_+) \in B_- \times B_+$ a "strip" $S(F_-, F_+) \subset \Gamma \cong \Gamma o$ which is sufficiently "thin" in the sense that intersections of a.e. strip with balls in T_g grow polynomially, then (B_+, ν_+) (resp., (B_-, ν_-)) is the Poisson boundary of the measure μ (resp., $\check\mu$).

By Theorem 2.2.4 there exist unique μ- and $\check\mu$-stationary measures ν_+ and ν_- concentrated on $\mathcal{UE}$ such that the spaces $(\mathcal{UE}, \nu_-)$ and $(\mathcal{UE}, \nu_+)$ are a $\check\mu$-boundary and a μ-boundary, respectively. Since the measures $\nu_\pm$ are purely non-atomic, for $\nu_- \otimes \nu_+$-a.e. pair (F_-, F_+) there exists a unique Teichmüller geodesic line $[F_-, F_+]$. Fix a reference point o, then the function $[F_-, F_+] \mapsto d_T\big(o, [F_-, F_+]\big)$ is a.e. defined and measurable (see Lemma 1.4.3), and there exists $M > 0$ satisfying the condition
$$\nu_- \otimes \nu_+ \big\{ (F_-, F_+) : d_T\big(o, [F_-, F_+]\big) \le M \big\} > 0 \, .$$
Let
$$S(F_-, F_+) = \big\{ \gamma \in \Gamma : d_T\big(\gamma o, [F_-, F_+]\big) \le M \big\}$$
be the "strip" in Γ associated with the pair of points F_-, F_+. By the definition of M the set of pairs (F_-, F_+) with non-empty set $S(F_-, F_+)$ has positive $\nu_- \otimes \nu_+$ measure, so that the ergodicity of the action of $\mathrm{gr}(\mu)$ on the product of the boundaries $(\mathcal{UE}, \nu_-)$ and $(\mathcal{UE}, \nu_+)$ [Ka6] (which follows from ergodicity of the Bernoulli shift U in the space of increments – cf. the proof of Theorem 2.2.4) implies that the sets $S(F_-, F_+)$ are a.e. non-empty.

Let
$$B_n = \big\{ \gamma \in \Gamma : d_T(o, \gamma o) \le n \big\} \, .$$
Since the group Γ acts on the space T_g properly discontinuously, for any $F_- \ne F_+ \in \mathcal{UE}$ the intersections $S(F_-, F_+) \cap B_n$ grow at most linearly with respect to n, so that conditions of the strip criterion are satisfied. □

Theorem 2.3.2. *Let μ be a probability measure on the mapping class group Γ such that the group* $\mathrm{gr}(\mu) \subset \Gamma$ *satisfies condition* (NE). *If the measure μ has a finite first moment with respect to the Teichmüller distance*

$$\sum_\gamma \mu(\gamma) d_T(o, \gamma o) < \infty ,$$

then the measure space $(\mathcal{PMF}, \nu)$, where ν is the unique μ-stationary probability measure on $\mathcal{PMF}$, is the Poisson boundary of the pair (Γ, μ).

Proof. Theorem 2.3.2 follows from Theorem 2.3.1. Clearly, finiteness of the first moment implies finiteness of the first logarithmic moment, so that condition (i) of Theorem 2.3.1 is satisfied. We have to check that finiteness of the first moment implies finiteness of the entropy of the measure μ (condition (ii) of Theorem 2.3.1) as well. This follows at once from the fact that for any reference point $o \in T_g$ the number of elements $\gamma \in \Gamma$ such that $d_T(o, \gamma o) \le R$ grows exponentially as a function of R (Corollary 2 of Theorem 1.3.2). For the sake of completeness, we shall give here the corresponding standard argument (e.g., cf. [De2]).

Let

$$D_k = \{\gamma \in \Gamma : k - 1 \le d_T(o, \gamma o) < k\} , \qquad k = 1, 2, \dots ,$$

so that Γ is the disjoint union of the sets D_k, and let $\pi_k = \mu(D_k)$. Denote by α_k the normalized restrictions of the measure μ onto the sets D_k, so that $\mu = \sum \pi_k \alpha_k$. Then

$$H(\mu) = H(\pi) + \sum_k \pi_k H(\alpha_k) ,$$

where

$$H(p) = -\sum p_i \log p_i$$

is the entropy of a discrete probability distribution $p = (p_i)$. The sets D_k grow at most exponentially, i.e., there is a constant $C > 0$ such that $\log \operatorname{card} D_k \le Ck$. Then by standard properties of the entropy

$$\sum_k \pi_k H(\alpha_k) \le \sum_k \pi_k \log \operatorname{card} D_k \le C \sum_k k\pi_k \le C \sum_\gamma [d_T(o, \gamma o) + 1] \mu(\gamma) < \infty .$$

On the other hand, monotonicity of the function $t \mapsto -t \log t$ on the interval $[0, e^{-1}]$ implies that

$$H(\pi) = \sum_k (-\log \pi_k) \pi_k \le \sum_k \max\{k, -\log \pi_k\} \pi_k \le \sum_k k\pi_k + \sum_k k e^{-k} < \infty .$$

$\square$

Corollary. *Let μ be a probability measure on the mapping class group Γ such that the group* $\mathrm{gr}(\mu) \subset \Gamma$ *satisfies condition* (NE). *If the measure μ has a finite first moment with respect to a word length in Γ, then the measure space $(\mathcal{PMF}, \nu)$ is the Poisson boundary of the pair (Γ, μ).*

2.4. The mapping class group and lattices in semi-simple Lie groups.

Furstenberg in [Fu2] (see also [Fu3]) proved the following remarkable result on lattices in semi-simple groups of rank ≥ 2. If G is such a lattice, then there exist a probability measure μ on G with $\mathrm{supp}\,\mu = G$ and a number $\varepsilon > 0$ such that for any two μ-harmonic functions f_1 and f_2 on G conditions

(i) $0 \leq f_i(g) \leq 1 \;\forall g \in G,\; i = 1,2$,
(ii) $f_i(e) \geq \frac{1}{2} - \varepsilon,\; i = 1,2$

imply that $\min\{f_1(g), f_2(g)\}$ does not tend to zero as $g \to \infty$ [Furstenberg considered the group of real unimodular matrices only, but his argument *verbatim* carries over to general real semi-simple Lie groups]. Using this result of Furstenberg we shall now prove the following theorem. Note that the question about non-arithmeticity of the mapping class group Γ (answered positively by Ivanov [Iv1]) was first asked by Harvey [Ha].

Theorem 2.4.1. *Any subgroup Γ' of the mapping class group satisfying condition* (NE) *is not isomorphic to a lattice in a semi-simple Lie group of rank ≥ 2. The mapping class group itself is also not isomorphic to a lattice in a rank 1 semi-simple group.*

The fact that Γ is not isomorphic to a lattice in a semi-simple Lie group was proved in [Iv1]. Our theorem includes a new proof of that result. Note that a subgroup of Γ satisfying (NE) may however be a lattice in a rank 1 group. For an example take the subgroup generated by the Dehn twists about two curves that fill the surface. This means that every component of the complement of the two curves is simply connected. Such a subgroup is a finite index subgroup of $SL(2, Z)$. On the other hand, since it contains pseudo-Anosov elements [FLP] which have attracting and repelling fixed points, it is easily seen to satisfy (NE).

Proof. First recall that Γ can not be isomorphic to a lattice in a rank 1 semi-simple group for the following reason (this argument was suggested by Ivanov): it contains an element whose centralizer is non-amenable (two Dehn twists commute iff the corresponding curves do not intersect; otherwise they generate a non-amenable group), whereas the fundamental group of a finite volume negatively curved pinched Riemannian manifold can not have this property [BGS].

Thus, we only have to prove that a subgroup $\Gamma' \subset \Gamma$ satisfying (NE) is not isomorphic to a lattice in a semi-simple Lie group of rank ≥ 2.

Let μ be an arbitrary probability measure on Γ' with $\operatorname{supp}\mu = \Gamma'$, and ν – the unique μ-stationary probability measure on $\mathcal{PMF}$ (Theorem 2.2.4). We claim that for any $\varepsilon > 0$

(i) There are two sets $Q_1, Q_2 \subset \mathcal{PMF}$ such that $\nu Q_i \geq \frac{1}{2} - \varepsilon/2$ and for any point $F \in \mathcal{MIN}$ there is a neighborhood U of $\widetilde{F}$ which does not intersect Q_1 and Q_2 simultaneously;

(ii) There is a neighborhood V of $\mathcal{PMF} \setminus \mathcal{MIN}$ such that $\nu V < \varepsilon/2$.

As we have shown in Lemma 1.1.2, every minimal foliation $F \in \mathcal{MIN}$ determines an infinite expansion $[\tau_0] > [\tau_1] > \dots$ by train tracks, and any two minimal foliations are equivalent if and only if they have the same sequence of $[\tau_0], [\tau_1], \dots$. Denote by ξ_n the partition of $\mathcal{MIN}$ into the sets $X_i^n = X([\tau_0], [\tau_1], \dots, [\tau_n])$ obtained by fixing the first $n+1$ terms in the train tracks expansion. Any set X_i^n is the open interior of a polyhedron in $\mathcal{PMF}$. As it follows from Lemma 1.1.2 and the fact that the measure ν is concentrated on $\mathcal{UE}$ (Theorem 2.2.4), the measurable intersection of the increasing sequence of partitions ξ_n is the point partition of the measure space $(\mathcal{UE}, \nu)$. Hence,

$$\max_i \nu X_i^n \underset{n\to\infty}{\longrightarrow} 0 .$$

Thus, for a sufficiently large n there are two disjoint sets Q_1' and Q_2' which are finite unions of the sets X_i^n, and $\nu Q_1', \nu Q_2' \geq \frac{1}{2} - \varepsilon/4$. Let N be the maximal number of the sets X_i^n in these unions. For any X_i^n one can take a closed subset $Y_i \subset X_i^n$ such that $\nu X_i^n - \nu Y_i \leq \varepsilon/4N$, so that replacing the sets Q_1', Q_2' with the corresponding unions of the sets Y_i we obtain two disjoint closed sets Q_1, Q_2 with $\nu Q_1, \nu Q_2 \geq \frac{1}{2} - \varepsilon/2$. Moreover, since each equivalence class $\widetilde{F}$ is closed, there is an open neighborhood of $\widetilde{F}$ which does not intersect Q_1 and Q_2 simultaneously.

As for (ii), the complement $\mathcal{PMF} \setminus \mathcal{MIN}$ is a countable union of the sets

$$E_\alpha = \{F \in \mathcal{PMF} : i(F, \alpha) = 0\} , \qquad \alpha \in \mathcal{S} .$$

Take an ordering $\alpha_1, \alpha_2, \dots$ in $\mathcal{S}$. By Theorem 2.2.4 $\nu(\mathcal{PMF} \setminus \mathcal{MIN}) = 0$, so that $\nu E_{\alpha_i} = 0 \;\forall\, i$. Since the sets E_α are G_δ (see the proof of Lemma 1.5.6), for any i there is an open neighbourhood V_i of E_{α_i} with $\nu V_i < \varepsilon/2^{i+1}$. Then the set $V = \bigcup_i V_i$ satisfies the condition (ii).

Now take the sets $\overline{Q}_i = Q_i \setminus V$, and consider the μ-harmonic functions

$$f_i(g) = g\nu(\overline{Q}_i) , \qquad i = 1, 2 .$$

Then clearly

$$0 \leq f_i(g) \leq 1 \qquad \forall\, g \in \Gamma ,$$

and

$$f_i(e) = \nu\overline{Q}_i \geq \nu Q_i - \nu V > \tfrac{1}{2} - \varepsilon \,.$$

We claim that

$$\min\{f_1(g), f_2(g)\} \underset{g\to\infty}{\longrightarrow} 0 \,,$$

which by Furstenberg's theorem would imply that Γ' cannot be isomorphic to a lattice in a semi-simple Lie group of rank ≥ 2.

Since any sequence in Γ' which tends to infinity contains a strongly universally convergent subsequence, we may assume that we are given a strongly universally convergent sequence g_n. Moreover, by compactness we may also assume that the sequence of translations $g_n\nu$ weakly converges to a measure λ. Under these assumptions we have to show that $\min\{f_1(g_n), f_2(g_n)\} \to 0$. By Lemma 1.5.6 λ is concentrated either on a set $\widetilde{H}$, $H \in \mathcal{MIN}$, or on $\mathcal{PMF} \setminus \mathcal{MIN}$. In the first case eventually an arbitrarily large part of the measures $g_n\nu$ is concentrated on a small neighborhood of $\widetilde{H}$, and we are done by (i). Suppose λ is concentrated on $\mathcal{PMF} \setminus \mathcal{MIN}$. Then $g_n\nu(V) \to 1$, and so $g_n\nu(\overline{Q}_i) = f_i(g_n) \to 0$. □

3. The Poisson boundary of invariant Markov operators on Teichmüller space

3.1. Invariant Markov operators on Teichmüller space.

Suppose that one has assigned in a measurable way a probability measure π_x to any point $x \in T_g$. Then the family of measures π_x, $x \in T_g$ determines a *Markov chain* on T_g with π_x being the distribution of points where one can get from x in one step. Denote by $\mathbf{P}_x$ the probability measure in the space $T_g^{\mathbb{Z}_+}$ of sample paths $\boldsymbol{x} = \{x_0, x_1, \dots\}$ of this chain corresponding to the initial distribution δ_x, $x \in T_g$ (i.e., the measure $\mathbf{P}_x$ is concentrated on sample paths which start from the point $x_0 = x$ at time 0). For an arbitrary σ-finite initial distribution θ (not necessarily a probability one!) put $\mathbf{P}_\theta = \int \mathbf{P}_x \, d\theta(x)$.

Fix a smooth reference Γ-invariant Radon measure m on T_g (i.e., $m(K) < \infty$ for all compact sets $K \subset T_g$), and suppose that all transition probabilities π_x are absolutely continuous with respect to m with densities $p(x,\cdot)$. We shall always assume that *the transition probabilities π_x are Γ-invariant* (more precisely, Γ-equivariant), i.e.,

$$\pi_{\gamma x} = \gamma \pi_x \qquad \forall \gamma \in \Gamma,\ x \in T_g \,, \tag{3.1.1}$$

which by Γ-invariance of the measure m is equivalent to Γ-invariance of the transition densities $p(\cdot,\cdot)$. Then the *transition Markov operator*

$$Pf(x) = \langle f, \pi_x \rangle = \int f(y)\, p(x,y)\, dm(y) \tag{3.1.2}$$

in the space $L^\infty(T_g, m)$ is Γ-invariant (i.e., commutes with the action of Γ). Since Γ acts on T_g properly discontinuously, and the measure m of points from T_g with non-trivial stabilizers is zero, P is a *covering Markov operator* in the sense of [Ka9] with the deck transformations group Γ, i.e., there exists a measurable "fundamental domain" $X \subset T_g$ such that all its Γ-translations are pairwise disjoint, and the complement of $\bigcup_\gamma \gamma X$ in T_g has zero measure m (in the case $g = 2$ instead of Γ one has to take its quotient Γ' with respect to the two-element normal subgroup generated by the hyperelliptic involution).

Denote by Q the *adjoint operator* of P acting in the space of measures on T_g. In probabilistic terms,

$$Q\theta(E) = \mathbf{P}_\theta[x_1 \in E] , \qquad E \subset T_g ,$$

i.e., Q assigns to an initial distribution θ the distribution of the position of the Markov chain at time 1. Since P has absolutely continuous transition probabilities, the operator Q preserves the type of the measure m (moreover, $Q\theta \prec m$ for *any* measure θ on T_g) and acts in the space of densities $\varphi = d\theta/dm$ by the formula

$$Q\varphi(y) = \frac{dQ\theta}{dm}(y) = \int p(x,y)d\theta(x) = \int \varphi(x)p(x,y)dm(x) . \tag{3.1.3}$$

A measure θ is called *P-stationary* (or, *P-invariant*), if $Q\theta = \theta$.

In the same way as for random walks on groups, one can define P-harmonic functions and the Poisson boundary of the operator P (see Section 2.1). A function f on T_g is called *P-harmonic* if $Pf = f$. Denote by ∂P the *Poisson boundary* of the operator P, i.e., the space of *ergodic components* of the shift in the unilateral path space $(T_g^{\mathbb{Z}_+}, \mathbf{P}_m)$, and by $\mathbf{bnd}$ the corresponding projection $T^{\mathbb{Z}_+} \to \partial P$. By $[\nu]$ denote the *harmonic measure class* on ∂P, i.e., the class of measures $\nu_\theta = \mathbf{bnd}(\mathbf{P}_\theta)$, where θ is a probability measure equivalent to m. The Poisson boundary is endowed with a natural Γ-action induced by the action of Γ on the path space by coordinate-wise translations, and the harmonic measure type $[\nu]$ is invariant with respect to this action. For any point $x \in T_g$ the *harmonic measure* $\nu_x = \mathbf{bnd}(\mathbf{P}_x)$ is absolutely continuous with respect to the type $[\nu]$, and the *Poisson formula*

$$f(x) = \langle \widehat{f}, \nu_x \rangle$$

is an isometric isomorphism between the space of $H^\infty(P) = \{f \in L^\infty(T_g, m) : Pf = f\}$ of bounded measurable P-harmonic functions and the space $L^\infty(\partial P, [\nu])$ [Ka5], [Ka9].

3.2. Balayage and the Harnack inequality.

For a measurable set V with $m(V), m(\complement V) \neq 0$ (here $\complement V = T_g \setminus V$ is the complement of V) denote by $\Lambda = \Lambda_V$ the *balayage operator* of the set V which assigns to an initial distribution θ the distribution of the first exit point of the Markov chain determined by the operator P from V, i.e.,

$$\Lambda\theta(E) = \mathbf{P}_\theta[x_\tau \in E] ,$$

where

$$\tau(\boldsymbol{x}) = \tau_{\complement V}(\boldsymbol{x}) = \min\{n \geq 0 : x_n \in \complement V\}$$

is the time of the first exit from V. The measure $\Lambda\theta$ is called the *balayage* of the measure θ. Note that we define balayage for *all* measures on T_g, not only for those supported on V; if $\theta(V) = 0$, then by definition $\Lambda\theta = \theta$. In general, the total mass $\|\Lambda\theta\|$ of the measure $\Lambda\theta$ can be less than the total mass of θ (if the measure $\mathbf{P}_\theta$ of those sample paths which never leave V is non-zero). However, if the set $\complement V$ is *recurrent* in the sense that $\mathbf{P}_m$-a.e. sample path eventually hits $\complement V$, then $\|\Lambda\theta\| = \|\theta\|$ for any measure θ on T.

Denote by P_V the *sub-Markov operator* with the state space V obtained by restricting P to V, so that it has the transition densities

$$p_V(x,y) = \begin{cases} p(x,y) , & \text{if } x,y \in V ; \\ 0 , & \text{otherwise} . \end{cases}$$

Then $P - P_V$ is also a sub-Markov operator. In terms of the adjoint operators Q and Q_V the result of applying the balayage operator Λ to a measure θ supported on V can be expressed as

$$\Lambda\theta = \sum_{n=0}^{\infty}(Q - Q_V)Q_V^n\theta = (Q - Q_V)\sum_{n=0}^{\infty} Q_V^n\theta$$

(each term $(Q - Q_V)Q_V^n\theta$ in this sum corresponds to staying in V for the first n steps and exiting to $\complement V$ at the time $n+1$). Hence, we have

Lemma 3.2.1. *If there is a constant H such that*

$$\sum_{n=0}^{\infty} Q_V^n\theta_1 \leq H \sum_{n=0}^{\infty} Q_V^n\theta_2$$

for two measures θ_1, θ_2 on V, then

$$\Lambda\theta_1 \leq H\Lambda\theta_2 .$$

If $\|\Lambda\theta\| = 1$ for a probability measure θ on T_g, i.e., $\mathbf{P}_\theta$-a.e. sample path eventually leaves the set V, then the harmonic measures ν_θ and $\nu_{\Lambda\theta}$ on the Poisson boundary coincide (this is so because $\Lambda\theta$ is the distribution of the first exit point $x(\tau_{\complement V})$ determined by the *Markov stopping time* $\tau_{\complement V}$ – see [Ka5]). In particular, if the set $\complement V$ is recurrent, then the values on V of any bounded P-harmonic function can be recovered from its values on $\complement V$ by the formula

$$f(x) = \langle \widehat{f}, \nu_x \rangle = \langle \widehat{f}, \nu_{\Lambda\delta_x} \rangle = \langle f, \Lambda\delta_x \rangle \, . \tag{3.2.1}$$

Let

$$\begin{aligned} \tau_1(\boldsymbol{x}) &= \min\{n > 0 : x_n \in \complement V\} \, , \\ \tau_{k+1}(\boldsymbol{x}) &= \min\{n > \tau_k : x_n \in \complement V\} \end{aligned}$$

be the times when a sample path $\boldsymbol{x} = \{x_n\}$ hits the set $\complement V$, then $\{x_{\tau_k}\}$ are sample paths of the *induced chain* on $\complement V$ corresponding to the operator $P^{\complement V}$ with transition probabilities $\Lambda\pi_x$, $x \in \complement V$. The Poisson boundaries of the operators P and $P^{\complement V}$ are isomorphic; for any bounded P-harmonic function its restriction to $\complement V$ is $P^{\complement V}$-harmonic, and, conversely, any bounded $P^{\complement V}$-harmonic function uniquely extends to a P-harmonic function on T by the formula (3.2.1) [Ka5].

Below we shall impose on the measure m and the transition densities $p(\cdot,\cdot)$ the following additional conditions.

(P_1) There exist $\varepsilon, \delta > 0$ such that $mB(x,\varepsilon) > \delta \; \forall \, x \in T_g$.

(P_2) There exists a constant C such that $p(x,y) \le C \; \forall \, x, y \in T_g$.

(P_3) There exists $R > 0$ such that $p(x,y) = 0$ whenever $d_T(x,y) > R$.

(P_4) There exist $c > 0$ and r_1, r_2 with $0 \le r_1 < r_2 \le R$, $r_2 - r_1 \ge 4\varepsilon$ such that $p(x,y) \ge c$ whenever $r_1 \le d_T(x,y) \le r_2$.

Here (P_1) and (P_2) can be considered as *"bounded geometry"* conditions, (P_3) is a *bounded range* condition, and (P_4) is an *irreducibility* condition. Note that condition (P_4) implies that $mB(x,\varepsilon) \le 1/c \; \forall \, x \in T_g$. An analysis of the arguments below shows that condition (P_4) could be significantly relaxed. Moreover, Theorem 3.2.2 uses conditions (P_1) – (P_4) only locally, so that for proving Theorem 3.3.2 it is just sufficient to have conditions (P_1) – (P_4) satisfied on a big compact subset (and its translations) in T_g only. However the bounded range condition (P_3) has to be satisfied for *all* points $x \in T_g$ for proving the moment estimates in Theorem 3.4.2. The constants $\varepsilon, \delta, c, C, r_1, r_2, R$ from conditions (P_1) – (P_4) will be used through the rest of this Section without further notice.

Theorem 3.2.2 (Harnack inequality). *Let P be a Γ-invariant Markov operator on T_g satisfying the conditions* (P_1) - (P_4). *For a point $o \in T_g$ and a number $M > 0$ denote by Λ the balayage operator of the ball $V = B(o, M+R)$ of radius $M+R$ centered at o. Then there exists a constant $H > 0$ depending on M and the constants from conditions* (P_1) - (P_4) *such that*

$$\Lambda\delta_x \le H\Lambda\delta_y \qquad \forall\, x, y \in B(o, M) \ .$$

For convenience we shall first prove the following auxiliary statements.

Lemma 3.2.3. *There is a number $N = N(M+R)$ such that for any two points $z, \bar{z} \in B(o, M+R)$ there exists a chain of points $z = z_0, z_1, \dots, z_N = \bar{z}$ in T_g with the property that*

$$d_T(z_i, z_{i+1}) = (r_1 + r_2)/2 = r \qquad \forall\, i = 0, 1, \dots, N-1$$

and

$$d_T(z_i, o) \le M + R - \varepsilon \qquad \forall\, i = 1, 2, \dots, N-1 \ . \tag{3.2.2}$$

Proof. It takes at most $[(M+R)/r]$ steps of length r to attain a point $z' \in B(o, r)$ by moving from z to o along the Teichmüller geodesic segment $[z, o]$ (here $[\cdot]$ is the integer part). Then by continuity of the function $x \mapsto d_T(z', x)$ on the sphere $S(o, r)$ it takes at most 2 steps to attain o from z'. Concatenating the chains joining z and $\bar{z}$ with o we obtain that one can get from z to $\bar{z}$ in not more than $2[(M+R)/r] + 4$ steps. As we want all chains to have the same length, we can further add to any such chain 2 segments $[o, x], [x, o]$ or 3 segments $[o, x], [x, x'], [x', x]$ with $x, x' \in S(o, r)$, $d_T(x, x') = r$ several times until we get a chain of length $N = 2[(M+R)/r] + 6$. The chain obtained in this way satisfies condition (3.2.2), because $\varepsilon < r < R - \varepsilon$ by (P_4). □

Lemma 3.2.4. *If $d(x_0, y_0) = r$, and θ is a measure on T_g such that $d\theta/dm \ge 1$ on $B(x_0, \varepsilon)$, then $dQ\theta/dm \ge c\delta$ on $B(y_0, \varepsilon)$.*

Proof. Since

$$|d_T(x, y) - r| \le 2\varepsilon \qquad \forall\, x \in B(x_0, \varepsilon),\ y \in B(y_0, \varepsilon) \ ,$$

by formula (3.1.3) and by conditions (P_1), (P_4) for any $y \in B(y_0, \varepsilon)$

$$\frac{dQ\theta}{dm}(y) = \int p(x, y) \frac{d\theta}{dm}(x) dm(x) \ge c\, m B(x_0, \varepsilon) \ge c\delta \ .$$

□

Proof of Theorem 3.2.2. First notice that by condition (P_3) the measure $Q\delta_x$ is supported on V for any $x \in B(o, M)$, so that $\Lambda\delta_x = \Lambda Q\delta_x$. Now, by Lemma 3.2.1 it is sufficient to prove that for any $x \in B(o, M)$ the density of the measure $(\sum Q_V^n)Q\delta_x$ with respect to the restriction m_V of the measure m to V is uniformly bounded from below and from above.

By condition (P_4) there exists an ε-ball in V such that the density of the measure $Q\delta_x$ is at least c on this ball. Take the number N from Lemma 3.2.3, then Lemma 3.2.4 applied to the operator Q_V implies that $c(c\delta)^N$ is a lower bound of the density of the measure $Q_V^N Q\delta_x$ with respect to the measure m_V.

For obtaining an upper bound note that the operator Q_V acts in the space $L^\infty(m_V)$, and

$$\|Q_V\|_\infty \le \operatorname{ess\,sup} p_V(x,y) m(V) \le Cm(V) \,. \tag{3.2.3}$$

The transition densities p_V^n of the operators Q_V^n satisfy the relation

$$p_V^{n+k}(x,y) = \int p_V^k(x,z) p_V^n(z,y)\, dm(z) \qquad \forall n,k \ge 1 \,,$$

so that

$$\operatorname{ess\,sup} p_V^{n+k}(x,y) \le \operatorname{ess\,sup} p_V^n(x,y) \qquad \forall n,k \ge 1 \,. \tag{3.2.4}$$

Moreover, there exists a constant k_0 and a number $\alpha > 0$ such that

$$\int p_V^{k_0}(x,z)\, dm(z) \le 1 - \alpha \qquad \forall x \in V$$

(cf. the proofs of Lemmas 3.2.3 and 3.2.4). Hence,

$$\operatorname{ess\,sup} p_V^{n+k_0}(x,y) \le (1-\alpha) \operatorname{ess\,sup} p_V^n(x,y) \qquad \forall n \ge 1 \,. \tag{3.2.5}$$

Formulas (3.2.4) and (3.2.5) imply that $\operatorname{ess\,sup} p_V^n(x,y)$ decays exponentially on n. Thus, by (3.2.3) $\|Q_V^n\|_\infty$ also decays exponentially, hence

$$\left\| \frac{d(\sum Q_V^n) Q\delta_x}{dm} \right\|_\infty \le \sum \|Q_V^n\|_\infty \left\| \frac{dQ\delta_x}{dm} \right\|_\infty \le C \sum \|Q_V^n\|_\infty < \infty \,.$$

Note that the convergence $\|Q_V^n\|_\infty \to 0$ implies that the set $\complement V$ is recurrent. $\square$

Remark. In fact, Theorem 3.2.2 holds for an arbitrary metric space satisfying the property formulated in Lemma 3.2.3. In particular, it applies to geodesic random walks on Riemannian manifolds with bounded geometry.

3.3. Discretization of corecurrent Markov operators.

By $\overline{m}$ denote the measure on the moduli space $M_g = T_g/\Gamma$ which is the image of the restriction of the measure m to any fundamental domain in T_g under the projection $x \mapsto \overline{x}$ from T_g to M_g. Since the transition densities of the operator P are Γ-invariant, the projection $\{\overline{x}_0, \overline{x}_1, \dots\}$ of the Markov chain $\{x_0, x_1, \dots\}$ from T_g to M_g is also a Markov chain with transition densities with respect to the measure $\overline{m}$

$$\overline{p}(\overline{x}, \overline{y}) = \sum_{\gamma \in \Gamma} p(x, \gamma y) ,$$

where $x, y \in T_g$ are inverse images of the points $\overline{x}, \overline{y} \in M_g$. The corresponding *quotient Markov operator* $\overline{P}$ on $L^\infty(M_g, \overline{m})$ can be identified with the restriction of the operator P to the subspace of Γ-invariant functions in $L^\infty(T_g, m)$ [Ka9]. By $\overline{\mathbf{P}}_{\overline{x}}$ denote the probability measure in the path space of the quotient Markov chain $\{\overline{x}_0, \overline{x}_1, \dots\}$ with the initial distribution concentrated at a point $\overline{x}$.

By Lemmas 3.2.3 and 3.2.4 the operator $\overline{P}$ is *irreducible*, i.e., there are no non-trivial sets $E \subset M_g$ such that the characteristic function $\mathbf{1}_E \in L^\infty(M_g, \overline{m})$ is $\overline{P}$-harmonic. Thus, either for any measurable set $E \subset M_g$ with $0 < \overline{m}(E) < \infty$ and any $\overline{x} \in M_g$ the probability $\overline{\mathbf{P}}_{\overline{x}}$ of visiting the set E is strictly less than 1, or any measurable set $E \subset M_g$ with $\overline{m}(E) > 0$ is recurrent (*Hopf's dichotomy*). In the latter case the operator $\overline{P}$ and the corresponding Markov chain are called *Harris recurrent* [Fo], [Kre], [Rev] (recall that $\overline{P}$ is always assumed to have absolutely continuous with respect to $\overline{m}$ transition probabilities). In this situation we shall say that the covering operator P (and the corresponding Markov chain) is *corecurrent*. If $\overline{P}$ is Harris recurrent, then there exists a unique (up to a constant) $\overline{P}$-stationary measure $\overline{\lambda}$ on M_g absolutely continuous with respect to $\overline{m}$. If the measure $\overline{\lambda}$ is finite, then the operator $\overline{P}$ is called *positively* Harris recurrent. Denote by λ the (P-stationary) lift of the measure $\overline{\lambda}$ to T_g.

Lemma 3.3.1. *Under conditions* (P_1) – (P_4), *if the quotient operator* $\overline{P}$ *is Harris recurrent, then the* $\overline{P}$*-stationary measure* $\overline{\lambda}$ *on* M_g *is a Radon measure.*

Proof. As it follows from Lemma 3.2.3 and conditions (P_1), (P_4) (see the proof of Lemma 3.2.4), for any $R > 0$ there exists a number $\alpha = \alpha(R)$ such that

$$\frac{d\lambda}{dm}(x) \geq \alpha\, \lambda B(y, \varepsilon) \qquad \forall\, o \in T_g,\ x, y \in B(o, R) .$$

In particular, the derivative $d\lambda/dm$ is positive m-a.e. Since $d\lambda/dm$ is a.e. finite, the above inequality also implies that $\lambda B(y, \varepsilon) < \infty$ for any $y \in T_g$, so that λ is a Radon measure (because all balls in T_g are compact). □

Theorem 3.3.2 (cf. [Fu3], [LS], [Ka4]) . *Let P be a Γ-invariant Markov operator on T_g satisfying conditions* (P_1) - (P_4). *If the quotient operator $\overline{P}$ is Harris recurrent, then for any point $o \in T_g$ with trivial stabilizer in Γ there exists a probability measure μ on Γ such that the Poisson boundary ∂P of the operator P with the harmonic measure ν_o is isomorphic as a measure Γ-space to the Poisson boundary ∂P_μ of the random walk (Γ, μ). For any bounded P-harmonic function on T_g its restriction to the orbit $\Gamma o \cong \Gamma$ is a bounded μ-harmonic function, and, conversely, any bounded μ-harmonic function can be uniquely extended from the orbit Γo to a bounded P-harmonic function on T_g.*

Proof. The proof will consist of several steps. First we describe a construction of the measure μ, and then show coincidence of the Poisson boundaries ∂P and ∂P_μ.

1. Construction of the measure μ.

We begin by choosing a constant M and a measurable set $E \subset B(o, M)$ such that $m(E) > 0$, and all translations γE, $\gamma \in \Gamma$ are pairwise disjoint. Let $x \mapsto \gamma(x)$ be the map from ΓE to Γ uniquely determined by the condition $x \in \gamma(x) E$. Now for every point $x \in T_g$ we shall construct a probability measure μ^x on the group Γ in such a way that the harmonic measure ν_x on ∂P satisfies the relation

$$\nu_x = \sum_{\gamma \in \Gamma} \mu^x(\gamma) \nu_{\gamma o} = \sum_{\gamma \in \Gamma} \mu^x(\gamma) \gamma \nu_o \,. \tag{3.3.1}$$

We do it by an iterative construction described below. Namely, we construct a sequence $\theta_k = \theta_k^x$ of measures on T_g and a sequence $\varkappa_k = \varkappa_k^x$ of measures on $\Gamma o \cong \Gamma$ such that

(i) $\nu_{\theta_0} = \nu_x$;
(ii) $\nu_{\theta_k} = \nu_{\theta_{k+1}} + \nu_{\varkappa_{k+1}} \; \forall\, k \geq 0$;
(iii) $\|\theta_k\| \to 0$.

Thus, we begin with the harmonic measure $\nu_x = \nu_{\theta_0}$, and at each step we single out a part of it which can be replaced with the harmonic measure of a distribution (denoted $\varkappa_{k+1}$) concentrated on $\Gamma o \cong \Gamma$. Condition (iii) says that finally the whole measure ν_x will be exhausted. The resulting measure

$$\mu^x = \sum_{k \geq 1} \varkappa_k^x$$

then clearly has the property (3.3.1).

Denote by Λ the balayage operator of the set $V = B(o, M + R)$, and let $\omega = \Lambda \delta_o$. Put

$$\theta_0 = \begin{cases} \delta_x, & x \notin \Gamma o \,; \\ \gamma\omega, & x = \gamma o \,. \end{cases}$$

In other words, if $x = \gamma o$ belongs to the orbit Γo, then θ_0 is the balayage of the measure $\delta_x = \gamma\delta_o$ to $\complement\gamma V$; otherwise, $\theta_0 = \delta_x$. Clearly, this choice of θ_0 satisfies condition (i) above. The reason why if $x = \gamma o$ we take $\theta_0 = \gamma\omega$ rather than $\theta_0 = \delta_x$ should become clear in the course of the proof.

Now we define the iterative procedure. Since $m(E) > 0$ and the quotient operator $\overline{P}$ is Harris recurrent, the set ΓE is recurrent for the operator P. Hence, we can balayage the measure θ_k to ΓE. Denote the resulting measure on ΓE by ξ_{k+1}, and denote by ξ^γ_{k+1}, $\gamma \in \Gamma$ the restrictions of ξ_{k+1} to the translations γE, $\gamma \in \Gamma$, so that $\xi_{k+1} = \sum_\gamma \xi^\gamma_{k+1}$, and

$$\nu_{\theta_k} = \nu_{\xi_{k+1}} = \sum_{\gamma\in\Gamma} \nu_{\xi^\gamma_{k+1}} .$$

Let $\zeta^\gamma_{k+1} = \gamma\Lambda\gamma^{-1}\xi^\gamma_{k+1}$ be the balayage of the measure ξ^γ_{k+1} to the set $\complement\gamma V$, then

$$\nu_{\zeta^\gamma_{k+1}} = \nu_{\xi^\gamma_{k+1}} \qquad \forall\gamma \in \Gamma ,$$

and

$$\nu_{\theta_k} = \sum_{\gamma\in\Gamma} \nu_{\zeta^\gamma_{k+1}} .$$

As it follows from Theorem 3.2.2, for any $\gamma \in \Gamma$

$$\frac{\zeta^\gamma_{k+1}}{\|\zeta^\gamma_{k+1}\|} \geq \frac{1}{H}\gamma\omega ,$$

thus if we put

$$\varkappa_{k+1}(\gamma) = \frac{\|\zeta^\gamma_{k+1}\|}{H} = \frac{\|\xi^\gamma_{k+1}\|}{H} ,$$

then all measures

$$\theta^\gamma_{k+1} = \zeta^\gamma_{k+1} - \varkappa_{k+1}(\gamma)\gamma\omega$$

are non-negative. Let

$$\theta_{k+1} = \sum_{\gamma\in\Gamma} \theta^\gamma_{k+1} ,$$

then, since $\nu_\omega = \nu_o$ by definition of the measure ω (so that $\nu_{\gamma\omega} = \nu_{\gamma o} = \gamma\nu_o$ for any $\gamma \in \Gamma$), we have

$$\begin{aligned}
\nu_{\theta_{k+1}} &= \sum_\gamma \nu_{\zeta^\gamma_{k+1}} - \sum_\gamma \varkappa_{k+1}(\gamma)\gamma\nu_\omega \\
&= \sum_\gamma \nu_{\xi^\gamma_{k+1}} - \sum_\gamma \varkappa_{k+1}(\gamma)\gamma\nu_o \\
&= \nu_{\theta_k} - \nu_{\varkappa_{k+1}} ,
\end{aligned}$$

and condition (ii) is satisfied. Finally, by the construction

$$\|\theta_{k+1}\| = (1 - 1/H)\|\theta_k\| \ ,$$

so that the total masses $\|\theta_k\|$ decay exponentially, and condition (iii) is also satisfied.

Clearly, all measures μ^x are supported on the whole group Γ. Note also that the construction is Γ-equivariant, so that

$$\mu^{\gamma x} = \gamma \mu^x \qquad \forall \gamma \in \Gamma, \ x \in T_g \ .$$

2. Coincidence of the Poisson boundaries.

The fact that the restriction of any bounded P-harmonic function from T_g to the orbit $\Gamma o \cong \Gamma$ is a μ-harmonic function for the measure $\mu = \mu^o$ follows from formula (3.3.1). Indeed, if f is a bounded P-harmonic function, then by the Poisson formula there exists a bounded measurable function $\widehat{f}$ on the Poisson boundary ∂P such that

$$f(x) = \langle \widehat{f}, \nu_x \rangle \qquad \forall x \in T_g \ .$$

By (3.3.1) the measure $\nu = \nu_o$ is μ-stationary, i.e., $\nu = \sum_\gamma \mu(\gamma)\gamma\nu$. Thus, for any $\gamma \in \Gamma$

$$f(\gamma o) = \langle \widehat{f}, \gamma\nu \rangle = \sum_{\gamma'} \mu(\gamma') \langle \widehat{f}, \gamma\gamma'\nu \rangle = \sum_{\gamma'} \mu(\gamma') f(\gamma\gamma' o) \ .$$

Note that this statement is in fact equivalent to saying that the harmonic measure ν_o of the point o on the Poisson boundary of the operator P is μ-stationary, so that for any other point $o' \neq o$ the restriction of any bounded P-harmonic function to the orbit $\Gamma o'$ is μ-harmonic (under the identification $\gamma \leftrightarrow \gamma o'$) iff the harmonic measure $\nu_{o'}$ is μ-stationary for the *same* measure μ. Except for some special situations [Ka9], there is no reason for this to be true (although, of course, the measure $\nu_{o'}$ is μ'-stationary for the corresponding measure μ' obtained by taking o' as the reference point in the above construction). Even if o' belongs to the Γ-orbit of o, the measure $\nu_{o'}$ does *not* have to be μ-stationary. Indeed, if $o' = go$, $g \in \Gamma$, then μ-stationarity of ν_o means that

$$\nu_{o'} = g\nu_o = g \sum_\gamma \mu(\gamma)\gamma\nu = \sum_\gamma \mu(g) g\gamma g^{-1} \nu_{o'} \ ,$$

i.e., that $\nu_{o'}$ is stationary with respect to the measure $\mu' = g\mu g^{-1}$ obtained from μ conjugating it by g (in fact, μ' is exactly the measure obtained from the above discretization construction for the reference point o' instead of o).

Now we want to show that, conversely, for any bounded μ-harmonic function f on the orbit $\Gamma o \cong \Gamma$ its extension to T_g by the formula

$$f(x) = \sum_\gamma \mu^x(\gamma) f(\gamma o)$$

is P-harmonic.

We shall prove this statement by constructing a sequence of Markov operators connecting the operator P with the operator P_μ of the random walk (Γ, μ), and such that the Poisson boundaries of any two consecutive operators in this sequence coincide (this argument will also give another proof of the fact that the restriction of any bounded P-harmonic function to Γo is μ-harmonic).

First we have to reformulate the iterative process from the first part of the proof in terms of *Markov stopping times*. This process includes balayages, for which such reformulation is straightforward, and subtracting from the measures ζ^γ_{k+1} the measures $(\|\zeta^\gamma_{k+1}\|/H)\gamma\omega$. The latter operation can be realized by introducing a new random variable α uniformly distributed on $[0,1]$ and independent of all the rest, and stopping the process when α does not exceed the Radon–Nikodym derivative of the measure $(\|\zeta^\gamma_{k+1}\|/H)\gamma\omega$ with respect to the measure ζ^γ_{k+1}.

For a sample path $\boldsymbol{x} = \{x_n\}$ let

$$S_0(\boldsymbol{x}) = \begin{cases} 0, & \boldsymbol{x} \notin \Gamma o\,; \\ \min\{n > 0 : x_n \in \complement\gamma V\}, & x = \gamma o\,, \end{cases}$$

and we define inductively

$$\begin{aligned} R_{k+1}(\boldsymbol{x}) &= \min\{n \ge S_k(\boldsymbol{x}) : x_n \in \Gamma E\}\,, \\ \gamma_{k+1}(\boldsymbol{x}) &= \gamma(x_{R_{k+1}}) \in \Gamma\,, \\ S_{k+1}(\boldsymbol{x}) &= \min\{n > R_{k+1}(\boldsymbol{x}) : x_n \in \complement\gamma_{k+1} V\}\,. \end{aligned}$$

For a pair of points $y \in E,\ z \in \complement V$ let

$$\Delta(y, z) = \frac{d\omega}{d\Lambda\delta_y}(z)$$

be the Radon–Nikodym derivative of the measure $\omega = \Lambda\delta_o$ with respect to the measure $\Lambda\delta_y$ evaluated at the point z, and let

$$\Delta(\gamma; y, z) = \Delta(\gamma^{-1}y, \gamma^{-1}z) = \frac{d\gamma\omega}{d\gamma\Lambda\gamma^{-1}\delta_y}(z)\,, \qquad \gamma \in \Gamma, y \in \gamma E, z \in \complement\gamma V$$

be the Radon–Nikodym derivative at the point z of the measure $\gamma\omega$ with respect to the balayage of the measure δ_y to $\complement\gamma V$.

Take a sequence of i.i.d. random variables $\boldsymbol{\alpha} = \{\alpha_n\}_{n\ge 0}$ which are independent of the chain $\{x_n\}$ and have Lebesgue measure ρ on the interval $[0,1]$ as their common distribution. Formally, it means that from now on we pass from the original path space $(T_g^{\mathbb{Z}_+}, \mathbf{P}_x)$ to its product with the measure space $([0,1]^{\mathbb{Z}_+}, \rho^{\mathbb{Z}_+})$, where $\rho^{\mathbb{Z}_+}$ is the infinite product of measures ρ indexed by the set $\mathbb{Z}_+$. Denote the product measure $\mathbf{P}_x \otimes \rho^{\mathbb{Z}_+}$ on $(T_g \times [0,1])^{\mathbb{Z}_+}$ by $\widetilde{\mathbf{P}}_x$. The measure $\widetilde{\mathbf{P}}_x$ corresponds to the initial distribution $\delta_x \otimes \rho$ in the space of the sample paths $(\boldsymbol{x}, \boldsymbol{\alpha})$ of the Markov operator

$$\widetilde{P} f(x, \alpha) = \iint f(y, \beta)\, d\pi_x(y)\, d\rho(\beta)$$

whose transition probabilities $\pi_x \otimes \rho$ do not depend on the $[0,1]$-component.

Define

$$T_0(\boldsymbol{x}, \boldsymbol{\alpha}) = 0,$$

and for $m \geq 0$ by induction

$$T_{m+1}(\boldsymbol{x}, \boldsymbol{\alpha}) = \min\{k > T_m : \alpha_{S_k} < \frac{1}{H} \Delta(\gamma_k, x_{R_k}, x_{S_k})\}$$

(as $\Delta(\gamma_k, x_{R_k}, x_{S_k}) \geq 1/H$ by Theorem 3.2.2, the times T_m are a.e. finite).

Claim 1. The measures μ^x constructed in the first part of the proof can be presented as

$$\mu^x(\gamma) = \widetilde{\mathbf{P}}_x[\gamma_{T_1} = \gamma]\,. \tag{3.3.2}$$

Indeed, by definition of the stopping time S_0 the distribution of x_{S_0} coincides with the measure θ_0. Now we shall prove by induction that θ_k is the distribution of x_{S_k} restricted to the set $[T_1 > k]$ (by "restricted" we mean here that θ_k is the image under the map $(\boldsymbol{x}, \boldsymbol{\alpha}) \mapsto x_{S_k}$ of the restriction of the measure $\widetilde{\mathbf{P}}_x$ to the set $[T_1 > k]$), and $\varkappa_k$ is the distribution of γ_k restricted to the set $[T_1 = k]$. As $\mu^x = \sum \varkappa_k$, the latter will imply (3.3.2).

Suppose we have already proved this assertion for θ_k and $\varkappa_k$. Then the stopping time R_{k+1} corresponds to the balayage of the measure θ_k to the set ΓE, so that the distribution of $x_{R_{k+1}}$ restricted to the set $[T_1 > k]$ is the measure $\xi_{k+1} = \sum_\gamma \xi^\gamma_{k+1}$. After that the stopping time S_{k+1} corresponds to the balayage of each of the measures ξ^γ_{k+1} from the corresponding set γE to the set $\complement\gamma V$, so that the distribution of $x_{S_{k+1}}$ restricted to the set $[T_1 > k]$ is $\sum_\gamma \zeta^\gamma_{k+1}$. Now the definition of the stopping times T_m means that given $x_{R_{k+1}}$ and $x_{S_{k+1}}$, we have $T_1 = k+1$ with the conditional probability $\Delta(\gamma_{k+1}, x_{R_{k+1}}, x_{S_{k+1}})/H$ which is the Radon–Nikodym derivative of the measure $\gamma_{k+1}\omega/H$ with respect to the measure $\gamma_{k+1}\Lambda\gamma^{-1}_{k+1}\delta_{x_{R_{k+1}}}$ evaluated at $x_{S_{k+1}}$.

In order to find the unconditional probability of the event $[T_1 = k+1]$ we have to integrate these conditional probabilities with respect to the conditions. Here we condition by $x_{R_{k+1}}$ and $x_{S_{k+1}}$, so that we have to integrate first with respect to the conditional distribution of $x_{S_{k+1}}$ conditioned by $x_{R_{k+1}}$, and then with respect to the unconditional distribution of $x_{R_{k+1}}$. As the measure $\gamma_{k+1}\Lambda\gamma^{-1}_{k+1}\delta_{x_{R_{k+1}}}$ is precisely the conditional distribution of $x_{S_{k+1}}$ provided $x_{R_{k+1}}$ is fixed, the result of the first integration is the measure $\gamma_{k+1}\omega/H$ for any $x_{R_{k+1}}$ (i.e., it depend only on $\gamma_{k+1} = \gamma(x_{R_{k+1}})$). Thus, the second integration with respect to $x_{R_{k+1}}$ just reduces to multiplying the measures $\gamma\omega/H$ by the probability that γ_{k+1} takes a given value γ, the latter being exactly $\|\xi^{\gamma}_{k+1}\|$. So, we obtain that the distribution of γ_{k+1} (resp., $x_{S_{k+1}}$) restricted to the set $[T_1 = k+1]$ is $\varkappa_{k+1}$ (resp., $\sum \varkappa_{k+1}(\gamma)\gamma\omega$), and that the remaining measure θ_{k+1} is the distribution of S_{k+1} restricted to the set $[T_1 > k+1]$. $\triangle$

In defining the measures μ^x we had to use two stopping times R_k and S_k, both of which are Markov. However, the definition of S_k includes the position x_{R_k} of the Markov chain $\{x_n\}$ at the time R_k. If we want to use the formula (3.2.1) for proving coincidence of the Poisson boundaries, we have to extend the original Markov chain $\{x_n\}$ by adding a component x'_n which keeps track of the positions x_{R_k} until the moment S_k is attained. The second component x_n of the *extended chain* $\{(x'_n, x_n)\}$ on $T_g \times T_g$ coincides with the original chain on T_g, whereas the first component x'_n once in ΓE remains unchanged until the second component leaves the set $\gamma(x'_n)V$; otherwise $x'_n = x_n$. In other words, paths $\{x_n\}$ of the original chain determine paths $\{(x'_n, x_n)\}$ of the extended chain by the formula

$$x'_n = \begin{cases} x_{R_k} , & R_k < n \le S_k ; \\ x_n , & \text{otherwise} . \end{cases} \tag{3.3.3}$$

The transition probabilities of the extended chain are

$$\begin{cases} \delta_{x'} \otimes \pi_x , & \text{if } x' \in \Gamma E, x \in \gamma(x')V ; \\ \delta_x \otimes \pi_x , & \text{if } x' \in \Gamma E, x \in \complement\gamma(x')V \cap \Gamma E ; \\ \operatorname{diag} \pi_x , & \text{otherwise} . \end{cases}$$

Claim 2. All sequences of random variables in the succession

$$\begin{aligned} \{x_n\} \xrightarrow{1} \{(x'_n, x_n)\} \xrightarrow{2} \{(x'_n, x_n, \alpha_n)\} \\ \xrightarrow{3} \{(x_{R_{T_m}}, x_{S_{T_m}}, \alpha_{S_{T_m}})\} \xrightarrow{4} \{x_{S_{T_m}}\} \xrightarrow{5} \{\gamma_{T_m}\} . \end{aligned} \tag{3.3.4}$$

are Markov chains, and all these Markov chains have the same (in a natural sense to be specified in each case) bounded harmonic functions, hence, the same Poisson boundary.

We shall consider transformations in (3.3.4) step by step.

1. As we have just seen, the chain $\{(x'_n, x_n)\}$ is Markov. As the set V is relatively compact, $\complement\Gamma V$ is a recurrent set for the chain $\{x_n\}$, and $\operatorname{diag}\complement\Gamma V$ is a recurrent set for the chain $\{(x'_n, x_n)\}$. By definition of $\{(x'_n, x_n)\}$ the corresponding induced chains on $\complement\Gamma V$ and $\operatorname{diag}\complement\Gamma V$ are isomorphic, so that by formula (3.2.1) the chains $\{x_n\}$ and $\{(x'_n, x_n)\}$ also have the same Poisson boundary. In particular, all harmonic functions of the chain $\{(x'_n, x_n)\}$ depend on the second component only.

Another explanation of why $\{x_n\}$ and $\{(x'_n, x_n)\}$ have the same Poisson boundary can be obtained by using directly the definition of the Poisson boundary as the space of ergodic components of the time shift in the path space. Indeed, formula (3.3.3) states an isomorphism of the measure spaces of sample paths of the chains $\{x_n\}$ and $\{(x'_n, x_n)\}$. As the chain $\{x_n\}$ is a quotient of the chain $\{(x'_n, x_n)\}$, its Poisson boundary must be a quotient of the Poisson boundary of $\{(x'_n, x_n)\}$. On the other hand, if two paths $\{x_n\}$ and $\{y_n\}$ are trajectory equivalent for the shift in the path space, i.e., if there exist integers n_1, n_2 such that $x_{n_1+n} = y_{n_2+n}$ $\forall n \geq 0$, then the corresponding paths $\{(x'_n, x_n)\}$ and $\{(y'_n, y_n)\}$ are also equivalent (for, with probability 1, there exists an arbitrarily large N such that $x_N \in \complement \Gamma V$, and for any such N all stopping times $R_k, S_k > N$ are determined by the positions x_n, $n \geq N$ only).

2. This transformation consists in adding a sequence of i.i.d. random variables $\{\alpha_n\}$ independent of $\{x_n\}$ (hence, of $\{(x'_n, x_n)\}$). As the transition probabilities of the chain $\{(x'_n, x_n, \alpha_n)\}$ do not depend on the α-component, the chain $\{(x'_n, x_n, \alpha_n)\}$ has the same harmonic functions and the same Poisson boundary as the chain $\{(x'_n, x_n)\}$.

3. This step consists in passing to the induced chain on the recurrent subset

$$A = \left\{(x', x, \alpha) : x' \in \Gamma E,\ x \in \complement \gamma(x')V,\ \alpha < \frac{1}{H}\Delta\big(\gamma(x'), x', x\big)\right\} \tag{3.3.5}$$

which again does not change the Poisson boundary by (3.2.1).

4. The transition probabilities $\pi'_{x',x,\alpha}$ of the chain obtained on step 3 depend only on the component x of the triple (x', x, α); denote their projections to the x-component by π'_x. Then clearly $\{x_{S_{T_m}}\}$ is a Markov chain with transition probabilities π'_x, and all $(\pi'_{x',x,\alpha})$-harmonic functions F have the form $F(x', x, \alpha) = f(x)$, where f is a (π'_x)-harmonic function on T_g.

5. As we have shown in the proof of Claim 1, the transition probabilities π'_x are convex combinations of translations of the measure ω:

$$\pi'_x = \sum_\gamma \mu^x(\gamma)\gamma\omega \tag{3.3.6}$$

On the other hand,

$$\mu = \mu^o = \int \mu^x \, d\omega(x)$$

because we construct μ by the balayage beginning with ω while μ^x is constructed from δ_x for all $x \in T_g \setminus \Gamma o$, and the balayage of ω is the integral with respect to ω of the balayée measures of δ_x (clearly, $\omega(\Gamma o) = 0$). [The latter formula is the reason why in the definition of the measures μ^x we had to treat the points from the orbit Γo differently.] Then

$$\int \pi'_x \, d\omega(x) = \sum_\gamma \mu(\gamma)\gamma\omega ,$$

i.e., the result of applying the transition probabilities π'_x to the measure ω is a sum of translations $\gamma\omega$ with weights $\mu(\gamma)$. This fact alone is sufficient to show that there is a natural isomorphism between the spaces of bounded (π'_x)-harmonic functions and bounded μ-harmonic functions on Γ. However, for our further purposes we shall also exhibit explicitly the corresponding random walk (namely, the sequence $\{\gamma_{T_m}\}$).

A one-to-one correspondence between bounded (π'_x)-harmonic functions f on T_g and bounded μ-harmonic functions $\overline{f}$ on Γ is given by the formula

$$\overline{f}(\gamma) = \langle f, \gamma\omega \rangle .$$

Indeed, let f be (π'_x)-harmonic. Then for any x

$$f(x) = \langle f, \pi'_x \rangle = \sum_\gamma \mu^x(\gamma)\langle f, \gamma\omega\rangle = \sum_\gamma \mu^x(\gamma)\overline{f}(\gamma),$$

whence integrating by x with respect to the measure ω we get that $\overline{f}$ is μ-harmonic at the identity e. By Γ-invariance of the Markov operators involved it implies that $\overline{f}$ is μ-harmonic. Conversely, let φ be a μ-harmonic function on Γ, and define

$$f(x) = \sum_\gamma \mu^x(\gamma)\varphi(\gamma) .$$

Then we have

$$\overline{f}(e) = \langle f, \omega\rangle = \sum_\gamma \varphi(\gamma) \int \mu^x(\gamma)\, d\omega(x) = \sum_\gamma \varphi(\gamma)\mu(\gamma) = \varphi(e) ,$$

so that once again by Γ-invariance

$$\overline{f}(\gamma) = \langle f, \gamma\omega\rangle = \varphi(\gamma) \qquad \forall \gamma \in \Gamma .$$

Returning to the definition of f yields

$$f(x) = \sum_\gamma \mu^x(\gamma)\varphi(\gamma) = \sum_\gamma \mu^x(\gamma)\overline{f}(\gamma) = \sum_\gamma \mu^x(\gamma)\langle f, \gamma\omega\rangle$$

which means that f is harmonic with respect to the transition probabilities $\pi'_x = \sum \mu^x(\gamma)\gamma\omega$.

In terms of the stopping times R_{T_m} and S_{T_m} we have that provided $x_{R_{T_m}}$ is fixed, the conditional distribution of $x_{S_{T_m}}$ is $\gamma_{T_m}\omega$ and depends on $\gamma_{T_m} = \gamma(x_{R_{T_m}})$ only (see the proof of Claim 1), hence, the distribution of $x_{S_{T_m}}$ conditioned by γ_{T_m} is $\gamma_{T_m}\omega$. Thus, again by Claim 1, for a given γ_{T_m} the conditional distribution of $\gamma_{T_{m+1}} = \gamma(x_{R_{T_{m+1}}})$ is

$$\int \gamma_{T_m}\mu^x \, d\omega(x) = \gamma_{T_m}\mu \, .$$

It implies that $\{\gamma_{T_m}\}$ is the random walk on Γ governed by the measure μ, i.e., the increments $\gamma_{T_m}^{-1}\gamma_{T_{m+1}}$ are independent and μ-distributed. Since γ_{T_1} has distribution μ^x, the distribution of γ_{T_m}, $m > 1$ is $\mu^x\mu_{m-1}$, where μ_{m-1} is the $(m-1)$-fold convolution of the measure μ. In particular, if we start from the point $x = o$, then $e, \gamma_{T_1}, \gamma_{T_2}, \dots$ is the random walk governed by the measure μ and starting from the identity of Γ.

As the conditional distribution of $x_{S_{T_m}}$ conditioned by γ_{T_m} is $\gamma_{T_m}\omega$, we also have that the chain $\{(x'_n, x_n, \alpha_n)\}$ is up to a group translation renewed at times S_{T_m}, i.e., its further behavior depends on γ_{T_m} only. As the transition probabilities $\pi'_{x',x,\alpha}$ are Γ-invariant, it implies, in particular, that the differences between stopping times $S_{T_2} - S_{T_1}, S_{T_3} - S_{T_2}, \dots$ are i.i.d. random variables (in the case $x_0 \in \Gamma o$ we can also add to this sequence the difference $S_{T_1} - S_0$). △

Going backwards along the sequence (3.3.4) we see that a bounded function f on T_g is P-harmonic if and only if it is (π'_x)-harmonic. Finally, since ω is the balayage of the measure δ_o to $\complement V$, we have

$$f(\gamma o) = \langle f, \gamma\omega \rangle = \overline{f}(\gamma) \qquad \forall \gamma \in \Gamma \, .$$

□

Remarks. 1. As we have already mentioned, in the case $g = 2$ the hyperelliptic involution $\gamma_0 \in \Gamma$ fixes every point in T_2, and P is a covering Markov operator with the deck group $\Gamma' = \Gamma/\{e, \gamma_0\}$. Thus, in this situation Theorem 3.3.2 will provide a measure μ' on Γ' such that the Poisson boundary of the pair (Γ, μ') is isomorphic to the Poisson boundary of the operator P. If μ is any lift of the measure μ' to Γ, then the pair (Γ, μ) has the same Poisson boundary as (Γ', μ'), because Γ' is the quotient of Γ with respect to a finite normal subgroup [Ka9].

2. In fact, Theorem 3.3.2 (with the same proof) holds for an arbitrary covering Markov operator satisfying a Harnack inequality. In particular, it is also applicable to diffusion processes on the Teichmüller space and to geodesic random walks on covering Riemannian manifolds (see Remark after Theorem 3.2.2).

3.4. Convergence and the Poisson boundary for corecurrent Markov operators on T_g.

Now we shall apply Theorem 3.3.2 to proving convergence and identificating the Poisson boundary for corecurrent Markov operators on T_g.

Theorem 3.4.1. *Let P be a Γ-invariant Markov operator on T_g satisfying conditions* (P_1) - (P_4). *If the quotient operator $\overline{P}$ is Harris recurrent, then there exists a unique family of probability measures λ_x, $x \in T_g$ on $\mathcal{PMF}$ such that*

$$\lambda_{\gamma x} = \gamma \lambda_x \qquad \forall \gamma \in \Gamma, x \in T_g$$

and

$$\lambda_x = \int \lambda_y \, d\pi_x(y) \qquad \forall x \in T_g \,. \tag{3.4.1}$$

The measures λ_x are pairwise equivalent and concentrated on $\mathcal{UE}$.

Proof. First note that such a system of measures $\{\lambda_x\}$, $x \in T_g$ is uniquely determined just by the measure λ_o (and its translations $\lambda_{\gamma o} = \gamma \lambda_o$). Indeed, the stationarity property (3.4.1) implies that for any continuous function $\widehat{f}$ on $\mathcal{PMF}$ the integrals

$$f(x) = \langle \widehat{f}, \lambda_x \rangle$$

give a P-harmonic function f on T_g. By Theorem 3.3.2 it is uniquely determined by its values on the orbit Γo. In other words, it means that for any given point $x \in T_g$ the integral $\langle \widehat{f}, \lambda_x \rangle$ is uniquely determined by the integrals $\langle \widehat{f}, \gamma \lambda_o \rangle$, i.e., the measure λ_x is uniquely determined by the measure λ_o.

Let now μ^x, $x \in T_g$ be probability measures on Γ constructed in Theorem 3.3.2, $\mu = \mu^o$, and ν be the unique μ-stationary measure on $\mathcal{PMF}$. Put

$$\lambda_x = \sum_{\gamma \in \Gamma} \mu^x(\gamma) \gamma \nu \,. \tag{3.4.2}$$

Then for any function $\widehat{f} \in C(\mathcal{PMF})$ the Poisson integral

$$f(\gamma o) = \langle \widehat{f}, \gamma \nu \rangle$$

is a μ-harmonic function on $\Gamma o \cong \Gamma$, which by Theorem 3.3.2 extends to a P-harmonic function by the formula

$$f(x) = \sum_{\gamma} \mu^x(\gamma) f(\gamma o) \,.$$

Thus, for any function $\widehat{f} \in C(\mathcal{PMF})$ we have

$$\langle \widehat{f}, \lambda_x \rangle = f(x) = \int f(y)\, d\pi_x(y) = \int \langle \widehat{f}, \lambda_y \rangle d\pi_x(y) ,$$

so that the system of measures (3.4.2) has the stationarity property (3.4.1).

Conversely, condition (3.4.1) implies that for any function $\widehat{f} \in C(\mathcal{PMF})$ the Poisson integral

$$f(x) = \langle \widehat{f}, \lambda_x \rangle$$

is a P-harmonic function. Again by Theorem 3.3.2 the restriction of f to the orbit Γo, $o \in T_g$ is a μ-harmonic function, so that for any $\widehat{f}$

$$\langle \widehat{f}, \lambda_o \rangle = \sum_{\gamma} \mu(\gamma) \langle \widehat{f}, \gamma\lambda_o \rangle ,$$

which implies that $\lambda_o = \sum_\gamma \mu(\gamma)\gamma\lambda_o$, i.e., λ_o must coincide with the unique μ-stationary measure ν on $\mathcal{PMF}$ by Theorem 2.2.4. □

By Theorems 2.2.4 and 3.3.2 the space $\mathcal{PMF}$ endowed with the system of measures λ_x from Theorem 3.4.1 is a quotient of the Poisson boundary of the operator P (the latter being isomorphic to the Poisson boundary of the pair (Γ, μ)) with respect to a certain Γ-invariant partition. Note that, however, this alone does not necessarily mean that the sample paths of the chain on T_g converge a.e. in the Thurston compactification. An explicit description of the map assigning to a sample path $\{x_n\}$ the corresponding point in $\mathcal{UE}$ so far has to be based on the constructions from Theorem 3.3.2.

Theorem 3.4.2. *Let P be a Γ-invariant Markov operator on T_g satisfying conditions* (P_1) – (P_4). *If the quotient operator $\overline{P}$ is positively Harris recurrent, then*

(i) *The measure μ constructed in Theorem 3.3.2 has a finite first moment $\sum_\gamma d_T(o, \gamma o)$ in T_g.*

(ii) *For any point $x \in T_g$ $\mathbf{P}_x$-a.e. sample path of the Markov chain determined by the operator P converges to $\mathcal{UE}$ in the topology of the Thurston compactification of T_g, and the corresponding limit distribution coincides with the measure λ_x from Theorem 3.4.1.*

(iii) *The space $\mathcal{PMF}$ with the system of probability measures λ_x is isomorphic to the Poisson boundary of the operator P.*

Proof. We shall use notations from Theorem 3.3.2. All Markov operators in the sequence (3.3.4) are covering Markov operators with the deck group Γ. Since the quotient operator $\overline{P}$ of the operator P is positively Harris recurrent, all other quotient operators are also positively recurrent and have uniquely determined (up to a constant) stationary measures. Denote by λ' the stationary measure of the chain $\{x'_n, x_n, \alpha_n\}$, and by $\overline{\lambda}'$ the stationary measure of the corresponding quotient chain. Let $\overline{\lambda}'_A$ be the restriction of the measure $\overline{\lambda}'$ to the projection $\overline{A}$ of the Γ-invariant set A (3.3.5) to $T_g \times T_g \times [0,1]/\Gamma$. By multiplying the measure λ' by a constant, we may assume that $\|\overline{\lambda}'_A\| = 1$. Then, by (3.3.6), the projection of $\overline{\lambda}'_A$ onto the second component in $T_g \times T_g \times [0,1]/\Gamma$ is the projection $\overline{\omega}$ of the measure ω to $M_g = T_g/\Gamma$. Since the transition probabilities of the induced chain on A (and of its quotient on $\overline{A}$) depend only on the second component, we obtain that $\widetilde{\mathbf{E}}_o(S_{T_1} - S_0) = \widetilde{\mathbf{E}}_o(S_{T_{m+1}} - S_{T_m})$ coincides with the average of the first return times to $\overline{A}$ with respect to the measure $\overline{\lambda}'_A$. By the Kac formula [CFS] the latter quantity coincides with $\|\overline{\lambda}'\|$ and is finite. Thus, we have shown that if $\overline{P}$ is positively Harris recurrent, then the i.i.d. random variables $S_{T_1} - S_0, S_{T_2} - S_{T_1}, S_{T_3} - S_{T_2}, \dots$ have a finite first moment with respect to the measure $\widetilde{\mathbf{P}}_o$.

(i). The measure μ is the $\widetilde{\mathbf{P}}_o$-distribution of $\gamma_{T_1} = \gamma(x_{R_{T_1}})$. Thus, we have to check that

$$\widetilde{\mathbf{E}}_o d_T(o, \gamma(x_{R_{T_1}})o) < \infty .$$

By the triangle inequality

$$d_T(o, \gamma(x_{R_{T_1}})o) \le d_T(o, x_{S_0}) + d_T(x_{S_0}, x_{S_{T_1}}) + d_T(x_{S_{T_1}}, \gamma(x_{R_{T_1}})o) .$$

The first and the third terms in the right-hand side are uniformly bounded, whereas finiteness of the middle term follows from finiteness of $\widetilde{\mathbf{E}}_o(S_{T_1} - S_0)$ and the bounded range condition (P_3).

(ii). Since the measure μ has a finite first moment, by the Kingman subadditive ergodic theorem (e.g., see [Del]) there exists a finite number l (the *linear rate of escape*) such that for $\mathbf{P}$-a.e. sample path $\boldsymbol{g} = \{g_n\}$ of the random walk (Γ, μ) there exists the limit

$$\lim_{n\to\infty} \frac{d_T(o, g_n o)}{n} = l .$$

The number l is strictly positive, for, otherwise, the random walk (Γ, μ) would have had the zero entropy

$$h(G, \mu) = \lim_{n\to\infty} \frac{H(\mu_n)}{n} ,$$

[Ka10], hence, trivial Poisson boundary [KV] in contradiction with Theorem 2.2.4.

By Claim 2 from the proof of Theorem 3.3.2, γ_{T_m} performs the random walk (Γ, μ), so that by Theorem 2.2.4 $\gamma_{T_m} o$ converges to $\mathcal{UE}$ in the Thurston compactification of T_g. Since $S_{T_1} - S_0, S_{T_2} - S_{T_1}, \dots$ is a sequence of i.i.d. random variables (see the proof of Theorem 3.3.2) with a finite first moment, there exists a.e. a finite limit (the mean stopping time)

$$t = \lim_{m\to\infty} \frac{S_{T_m}}{m} = \widetilde{\mathbf{E}}_o(S_{T_1} - S_0) .$$

Hence, we have that a.e.

$$n - S_{T_{m(n)}} = o(n) ,$$

where

$$m(n) = \max\{m : S_{T_m} \le n\} .$$

Since the operator P has bounded range, it implies that a.e.

$$d_T\big(x_n, x_{S_{T_{m(n)}}}\big) = o(n) .$$

As $d_T(x_{S_{T_{m(n)}}}, \gamma_{T_{m(n)}} o)$ is uniformly bounded, we have that a.e.

$$d_T\big(x_n, \gamma_{T_{m(n)}} o\big) = o(n) .$$

On the other hand, the sequence $\gamma_{T_{m(n)}} o$ converges to $\mathcal{UE}$, and the distance from $\gamma_{T_{m(n)}} o$ to o grows linearly on n. Thus by Lemma 1.4.2 the sequence x_n also converges to the same limit point from $\mathcal{UE}$.

The corresponding limit distributions ν_x on $\mathcal{UE}$ coincide with the measures λ_x from Theorem 3.4.1, because the measures ν_x obviously satisfy the stationarity relations (3.4.1).

(iii). By Theorem 3.3.2 the Poisson boundary of the operator P is isomorphic to the Poisson boundary of the random walk (Γ, μ). As the measure μ has a finite first moment in T_g, by Theorem 2.3.2 the latter is the space $\mathcal{PMF}$ with the measure λ_o, and we are done. □

Remark. Note that in fact Theorems 3.4.1 and 3.4.2 also hold for invariant Markov operators corresponding to diffusion processes on T_g. For Theorem 3.4.1 one needs bounded geometry of the generating operator (which would guarantee the Harnack inequality), and for Theorem 3.4.2 it is sufficient to demand uniform boundedness of the first moments $\int d_T(x,y)\, d\pi_x(y)$ (which would imply existence of a finite rate of escape $\lim d_T(x_0, x_n)/n$ [Ka2]) – cf. [Ka4]. For diffusion processes one can also prove Theorem 3.4.2 in a more direct way (without using the discretization procedure) by using the methods from [Ka2].

Masur in [Ma4] considered a *geodesic random walk* on T_g. Its transition probabilities π_x are defined in the following way. Fix a positive number L. Then from a point $x \in T_g$ we move along the Teichmüller geodesic line with a random direction (whose distribution is the normalized Lebesgue measure on the sphere of the tangent space at x) to a new point x' such that the random distance $d_T(x, x')$ is uniformly distributed between L and $L+1$. By analyzing the train tracks decomposition along the sample paths he proved the following result.

Theorem 3.4.3 [Ma4]. *For sufficiently large L almost all sample paths of the geodesic random walk converge in the Thurston compactification of Teichmüller space T_g, and the corresponding limit distributions λ_x are concentrated on $\mathcal{UE} \subset \mathcal{PMF}$. Moreover, there exists a compact set $\Omega \subset T_g$, such that for all points $x \in \Gamma\Omega$ the expected first return times to $\Gamma\Omega$ are uniformly bounded.*

Choose a smooth Γ-invariant Radon measure m on T_g satisfying condition (P_1). The geodesic random walk clearly satisfies condition (P_3), however, conditions (P_2) and (P_4) (directly connected with the differentiability of the Teichmüller "exponential" map) are not known to be true. Masur [Ma4] showed that for any $\alpha > 0$ one can define the *modified* transition probabilities π_x^{mod} in such way that

1) $\|\pi_x - \pi_x^{mod}\| \le \alpha \ \forall x \in T_g$;
2) The probabilities π_x^{mod} are Γ-invariant and satisfy conditions (P_1) – (P_4);
3) Theorem 3.4.3 still holds for the modified geodesic random walk determined by the transition probabilities π_x^{mod}.

Then Lemma 3.3.1 (which guarantees that $m(\Omega) < \infty$) in combination with uniform boundedness of first return times to $\Gamma\Omega$ implies that the Γ-quotient of the modified geodesic random walk is positively Harris recurrent. Thus, by Theorem 3.4.2 we get

Theorem 3.4.4. *The Thurston boundary $\mathcal{PMF}$ with the family of measures λ_x is the Poisson boundary of the modified geodesic random walk.*

Remarks. 1. Theorem 3.4.2 gives a different proof than in [Ma4] for the convergence of the sample paths in the Thurston compactification.

2. We had to modify the transition probabilities of the geodesic random walk in order to be able to construct a probability measure on Γ with the same Poisson boundary and to use our description of the Poisson boundary of random walks on Γ. However, it seems feasible to apply the entropy technique directly to the geodesic random walk for proving coincidence of the Poisson boundary with $\mathcal{PMF}$ in the spirit of [Ka2] (see also [Ka5]). We shall return to this problem elsewhere.

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