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THE POISSON BOUNDARY OF POLYCYCLIC GROUPS

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ABSTRACT

The Poisson boundary for measures with a finite first moment on polycyclic groups is shown to coincide with the corresponding contracting nilpotent Lie group. The proof is based on using a geometric "strip criterion" of boundary maximality due to the author and a simple estimate of the "neutral component" of the random walk.

0. Introduction

Let μ be an absolutely continuous probability measure on a locally compact group G. Then one can associate with the pair (G,μ) a probability measure space (Γ,ν) endowed with an ergodic action of G which is called the *Poisson boundary*. The *harmonic measure* ν is *μ-stationary* (i.e., $\nu = \int g\nu \, d\mu(g)$), and the *Poisson formula* $f(g) = \langle \widehat{f}, g\nu \rangle$ (a generalization of the classic Poisson formula for bounded harmonic functions in the disk) establishes an isometry between the space of bounded *μ-harmonic functions* on G (those that satisfy the mean value property $f(g) = \int f(gx)\, d\mu(x) \ \forall g \in G$) and the space $L^\infty(\Gamma,\nu)$. The Poisson boundary was first introduced by Furstenberg[4] for semi-simple Lie groups. It can be defined in a number of various equivalent ways and is an important invariant of the pair (G,μ), see Kaimanovich[11].

In terms of the *random walk* (G,μ) the Poisson boundary is the space of *ergodic components of the time shift* in the path space $(G^{\mathbb{Z}_+},\mathbf{P})$. Thus, if the group G is equivariantly mapped to a topological space B, and the image of a.e. sample path $\{y_n\}$ converges to a limit $y_\infty = \pi(\{y_n\}) \in B$, then the space B with the measure $\lambda = \pi(\mathbf{P})$ is a quotient of the Poisson boundary with respect to a certain G-invariant partition. Such quotients are called *μ-boundaries*. The Poisson boundary is then the *maximal* μ-boundary.

The problem of *describing the Poisson boundary* of (G,μ) can be subdivided into two parts: (1) to find (in geometric or combinatorial terms) a μ-boundary; (2) to prove maximality of this μ-boundary. In other words, first one has to exhibit a

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certain system of invariants of stochastically significant behavior of sample paths at infinity, and then to show completeness of this system.

For random walks on general Lie groups the Poisson boundary was described by Raugi[16]; however, his approach to proving maximality strongly depends on the structure theory of Lie groups and can not be applied to discrete groups. A completely different method based on the entropy theory of random walks and volume estimates for conditional walks was proposed by the author in Kaimanovich[6]; it leads to several geometrical criteria of boundary maximality, see Kaimanovich[10].

In the present paper we consider the problem of describing the Poisson boundary of *polycyclic groups*. In a sense, these groups can be considered as "finitely dimensional" discrete solvable groups and are closely connected with solvable Lie groups. Roughly speaking (up to a *semi-simple splitting*), polycyclic groups are just semi-direct products $G = A \leftthreetimes N$ of finitely generated free abelian and torsion free nilpotent groups. This makes their boundary behaviour much more tractable than for general discrete solvable groups (see Kaimanovich[9] for examples) and close to that of solvable Lie groups.

Indeed, let μ be a probability measure on $G = A \leftthreetimes N$ with a finite first moment. Then the barycenter of the projection of the measure μ onto A determines a decomposition of the Lie hull $\mathcal{N}$ of the group N into contracting $\mathcal{N}_-$, neutral $\mathcal{N}_0$ and expanding $\mathcal{N}_+$ subgroups, and one can show (basically in the same way as in the Lie groups case, see Raugi[16]) that the $\mathcal{N}_-$-components converge along a.e. sample path of the random walk, so that $\mathcal{N}_-$ (identified with the G-space $(A \leftthreetimes \mathcal{N})/A\mathcal{N}_0\mathcal{N}_+$) becomes a μ-boundary. However, several reasons, in particular impossibility for subgroups of G (being countable) to act transitively on $\mathcal{N}_-$, make inapplicable the technique used for proving maximality in the case of Lie groups. The method suggested in Kaimanovich[8] (see also Kaimanovich[9]) is based on a global law of large numbers for solvable Lie groups and requires rather lengthy calculations.

In the present paper we show how the *"strip criterion"* from Kaimanovich[10] allows one to solve the problem of describing the Poisson boundary in a rather simple geometric fashion. The strip criterion requires considering simultaneously with the measure μ its *reflected measure* $\check{\mu}(g) = \mu(g^{-1})$. In the same way as for the measure μ we obtain a $\check{\mu}$-boundary $\mathcal{N}_+$ (because the contracting subgroup for the reflected measure $\check{\mu}$ is precisely the expanding subgroup corresponding to μ). Since the points from $\mathcal{N}_-$ (resp., $\mathcal{N}_+$) are identified with cosets of the subgroup $A\mathcal{N}_0\mathcal{N}_+$ (resp., $A\mathcal{N}_0\mathcal{N}_-$) in $A \leftthreetimes \mathcal{N}$, any pair of points from $\mathcal{N}_-$ and $\mathcal{N}_+$ determines (as intersection of the corresponding cosets) a coset of $A\mathcal{N}_0$. After an easy modification this gives an equivariant map assigning to pairs of points from $\mathcal{N}_- \times \mathcal{N}_+$ subsets ("strips") in G. Now, according to the strip criterion, for proving maximality we have to show that these strips are "thin enough", which boils down to an easy estimate of the growth of the neutral component along sample paths of the random walk. The general situation is intermediate between two extreme cases: when

$\mathcal{N}_0 = \{e\}$, and when $\mathcal{N}_0 = \mathcal{N}$. In the first case the strips are just cosets of A, so that the required estimate is trivial; whereas in the second case the strips coincide with G, and proving maximality of $\mathcal{N}_- = \{e\}$ is equivalent to proving triviality of the Poisson boundary of (G, μ).

1. Polycyclic groups

Definition 1 (e.g., see Segal[17]). A discrete group G is called *polycyclic* if it admits a finite normal series with cyclic factor-groups, i.e., a series of subgroups

$$\{e\} = G_0 \subset G_1 \subset G_2 \subset \cdots \subset G_n = G$$

such that each G_i is a normal subgroup in G_{i+1} and all the quotients G_{i+1}/G_i are cyclic.

In a certain sense, polycyclic groups can be considered as "finite dimensional" discrete solvable groups. Indeed (see Auslander[2], Segal[17]),

- Polycyclic groups can be characterized as solvable groups with finitely generated subgroups, or, even more, as solvable groups with finitely generated abelian subgroups;
- Solvable groups of integer matrices are polycyclic, and, conversely, every polycyclic group has a faithful representation in $GL(n, \mathbb{Z})$.

Let T be an action of a group A by automorphisms of another group N. The set $A \times N$ endowed with the group operation

$$(a_1, n_1)(a_2, n_2) = (a_1 a_2, n_1 \cdot T^{a_1} n_2)$$

is called the *semi-direct product* of the groups A and N and is denoted

$$A \leftthreetimes N = A \overset{T}{\leftthreetimes} N$$

(the action T is often omitted in this notation). The groups A and N are embedded into the semi-direct product $G = A \leftthreetimes N$ by the maps

$$a \mapsto (a, e_N) \ , \qquad n \mapsto (e_A, n) \ ,$$

and

$$(a, n) = (e_A, n)(a, e_N) = na \ ,$$

so that (1) $G = NA$, and (2) $N \cap A = \{e_G\}$. Moreover, (3) N is a normal subgroup of G with $G/N \cong A$. Conversely, any pair of subgroups A, N of a group G satisfying conditions (1) – (3) determines a decomposition of G into a semi-direct product $A \leftthreetimes N$.

If A is a finitely generated abelian group, and N is a finitely generated nilpotent group, then any semi-direct product $A \leftthreetimes N$ is polycyclic. In fact, *all* polycyclic groups can be "essentially" obtained in this way. Before formulating the corresponding result (*semi-simple splitting of polycyclic groups*), recall that for any finitely generated torsion free nilpotent group N there is a uniquely determined simply connected real nilpotent Lie group $\mathcal{N}$ (the *Lie hull* of N) containing N as a uniform ($\equiv$ cocompact) lattice, and any automorphism of N uniquely extends to an automorphism of $\mathcal{N}$, see Auslander[2] and Segal[17]. An automorphsim of the Lie group $\mathcal{N}$ is called *semi-simple* if the tangent automorphism of its Lie algebra $\mathfrak{N}$ is diagonalisable in the complexification $\mathfrak{N}^{\mathbb{C}}$. An action of a group A by automorphisms of N is semi-simple if all corresponding automorphisms of the Lie hull $\mathcal{N}$ are semi-simple.

Definition 2. A discrete group G is an $\mathcal{S}$-*group* if it can be presented as a semi-direct product $G = A \overset{T}{\leftthreetimes} N$, where

(i) The group $A \cong \mathbb{Z}^d$ is finitely generated free abelian;
(ii) The group N is finitely generated torsion free nilpotent;
(iii) The action T of the group A on N is semi-simple.

A polycyclic group G is *splittable* if it is contained in an $\mathcal{S}$-group G'. In this case the embedding $G \subset G'$ is called a *semi-simple splitting* of G.

Proposition 1 (*semi-simple splitting*, see Segal[17], Theorem 7.2). *Every polycyclic group contains a normal splittable polycyclic subgroup of finite index.*

Remark. The definition of semi-simple splitting given in Segal[17] is more restrictive and imposes some additional conditions on the group G' and the embedding $G \subset G'$. However, for our purposes it is sufficient to use the simplified Definition 2.

2. Contracting and expanding components of nilpotent Lie groups

Let $\mathcal{N}$ be a simply connected real nilpotent Lie group. We shall identify $\mathcal{N}$ with its Lie algebra $\mathfrak{N}$ by using the *Baker–Campbell–Hausdorff multiplication formula*

$$x \times y = x + y + \tfrac{1}{2}[x,y] + \dots , \qquad x, y \in \mathfrak{N} . \tag{1}$$

Denote by

$$\mathfrak{N}_1 = \mathfrak{N} \supset \mathfrak{N}_2 = [\mathfrak{N}, \mathfrak{N}] \supset \mathfrak{N}_3 = [\mathfrak{N}, \mathfrak{N}_2] \supset \cdots \supset \mathfrak{N}_{r+1} = \{0\}$$

the *lower central series* of the Lie algebra $\mathfrak{N}$, where r is the *nilpotency class* of $\mathfrak{N}$, and by

$$\deg x = \max\{l : x \in \mathfrak{N}_l\}$$

the corresponding graduation on $\mathfrak{N}$. By $\deg P$ we shall denote the degree of a polynomial P on $\mathfrak{N}$ with respect to the graduation deg. It is well known (e.g., see Goodman[5]) that the group multiplication given by Eq. 1 is polynomial, and, moreover, it is *linear in principal terms* with respect to the graduation deg in the following sense: if $\{e_i\}$ is a linear basis in $\mathfrak{N}$ *adapted* to the filtration $\{\mathfrak{N}_l\}$ (i.e., it contains precisely $\dim \mathfrak{N}_l$ vectors from $\mathfrak{N}_l$ for any $l = 1, 2, \dots, r$), then

$$(x \times y)_i = x_i + y_i + P_i(x, y) ,$$

where P_i is a polynomial with $\deg P_i \le \deg e_i$ and with partial degrees with respect to x and y strictly less than $\deg e_i$.

If T is an automorphism of the Lie algebra $\mathfrak{N}$, then $T(x \times y) = (Tx \times Ty)$ by Eq. 1, so that the map $T : \mathfrak{N} \to \mathfrak{N}$ is also an automorphism of the Lie group $\mathcal{N} = (\mathfrak{N}, \times)$. Conversely, any automorphism of the Lie group $\mathcal{N}$ coincides (as a map of $\mathfrak{N}$ onto itself) with the corresponding tangent automorphism of the Lie algebra $\mathfrak{N}$.

Let T be a semi-simple action on $\mathcal{N} \cong \mathfrak{N}$ of a free abelian group $A \cong \mathbb{Z}^d$, and $\Lambda \subset \mathrm{Hom}(A, \mathbb{C}^*)$ be the set of *weights* of the representation T of the group A in the complexification $\mathfrak{N}^{\mathbb{C}}$. By

$$\mathfrak{N}^{\mathbb{C}}_\lambda = \{x \in \mathfrak{N}^{\mathbb{C}} : T^a x = \lambda(a) x \quad \forall a \in A\} \subset \mathfrak{N}^{\mathbb{C}} , \qquad \lambda \in \Lambda$$

denote the corresponding *weight subspaces*. Since T is semi-simple,

$$\mathfrak{N}^{\mathbb{C}} = \bigoplus_{\lambda \in \Lambda} \mathfrak{N}^{\mathbb{C}}_\lambda .$$

Denote by $\mathbf{A} \cong \mathbb{R}^d$ the linear space containing $A \cong \mathbb{Z}^d$ as a lattice. Then the functions $\log |\lambda|$, $\lambda \in \Lambda$ are uniquely extendable to homomorphisms from $\mathbf{A}$ to the additive group $\mathbb{R}$.

For a vector $\mathbf{a} \in \mathbf{A}$ let

$$\begin{aligned}
\Lambda_- &= \Lambda_-(\mathbf{a}) = \{\lambda \in \Lambda : \log |\lambda|(\mathbf{a}) < 0\} , \\
\Lambda_0 &= \Lambda_0(\mathbf{a}) = \{\lambda \in \Lambda : \log |\lambda|(\mathbf{a}) = 0\} , \\
\Lambda_+ &= \Lambda_+(\mathbf{a}) = \{\lambda \in \Lambda : \log |\lambda|(\mathbf{a}) > 0\} ,
\end{aligned}$$

be the sets of *contracting*, *neutral* and *expanding* weights (with respect to $\mathbf{a}$), and let

$$\begin{aligned}
\mathfrak{N}^{\mathbb{C}}_- &= \mathfrak{N}^{\mathbb{C}}_-(\mathbf{a}) = \bigoplus_{\lambda \in \Lambda_-} \mathfrak{N}^{\mathbb{C}}_\lambda , \\
\mathfrak{N}^{\mathbb{C}}_0 &= \mathfrak{N}^{\mathbb{C}}_0(\mathbf{a}) = \bigoplus_{\lambda \in \Lambda_0} \mathfrak{N}^{\mathbb{C}}_\lambda , \\
\mathfrak{N}^{\mathbb{C}}_+ &= \mathfrak{N}^{\mathbb{C}}_+(\mathbf{a}) = \bigoplus_{\lambda \in \Lambda_+} \mathfrak{N}^{\mathbb{C}}_\lambda ,
\end{aligned}$$

be the corresponding (contracting, neutral and expanding) subspaces of $\mathfrak{N}^{\mathbb{C}}$. *Below we shall usually omit the (fixed) vector* $\mathbf{a} \in \mathbf{A}$ *from our notations.* The meaning of the terms "contracting", "neutral" and "expanding" should be clear from the property (ii) below.

Proposition 2. *Let T be a semi-simple action of a free abelian group A on a nilpotent Lie algebra $\mathfrak{N}$. Then*

(i) *The subspaces $\mathfrak{N}_-^{\mathbb{C}}, \mathfrak{N}_0^{\mathbb{C}}, \mathfrak{N}_+^{\mathbb{C}} \subset \mathfrak{N}^{\mathbb{C}}$ are complexifications of subspaces $\mathfrak{N}_-, \mathfrak{N}_0, \mathfrak{N}_+ \subset \mathfrak{N}$, and $\mathfrak{N} = \mathfrak{N}_- \oplus \mathfrak{N}_0 \oplus \mathfrak{N}_+$.*

(ii) *The subspaces $\mathfrak{N}_-, \mathfrak{N}_0, \mathfrak{N}_+$ and $\mathfrak{N}_-^* = \mathfrak{N}_- \oplus \mathfrak{N}_0, \mathfrak{N}_+^* = \mathfrak{N}_+ \oplus \mathfrak{N}_0$ can be characterized in the following way: if $(a_k) \subset A$ is a certain ($\equiv$ any) sequence in A such that $a_k/k \to \mathbf{a}$, then*

$$
\begin{aligned}
x \in \mathfrak{N}_- \setminus \{0\} &\iff \lim \tfrac{1}{k} \log \|T^{a_k} x\| < 0 \,, \\
x \in \mathfrak{N}_-^* \setminus \{0\} &\iff \lim \tfrac{1}{k} \log \|T^{a_k} x\| \le 0 \,, \\
x \in \mathfrak{N}_0 \setminus \{0\} &\iff \lim \tfrac{1}{k} \log \|T^{a_k} x\| = 0 \,, \qquad \lim \tfrac{1}{k} \log \|T^{-a_k} x\| = 0 \,, \\
x \in \mathfrak{N}_+^* \setminus \{0\} &\iff \lim \tfrac{1}{k} \log \|T^{-a_k} x\| < 0 \,, \\
x \in \mathfrak{N}_-^* \setminus \{0\} &\iff \lim \tfrac{1}{k} \log \|T^{-a_k} x\| \le 0 \,,
\end{aligned}
$$

where $\|\cdot\|$ is a certain ($\equiv$ any) norm in $\mathfrak{N}$.

(iii) *The subspaces $\mathfrak{N}_-, \mathfrak{N}_-^*, \mathfrak{N}_0, \mathfrak{N}_+, \mathfrak{N}_+^*$ are T-invariant Lie subalgebras of $\mathfrak{N}$.*

(iiii) *Let $\mathcal{N}_-, \mathcal{N}_-^*, \mathcal{N}_0, \mathcal{N}_+, \mathcal{N}_+^*$ be the simply connected subgroups of $\mathcal{N}$ corresponding to the subalgebras $\mathfrak{N}_-, \mathfrak{N}_-^*, \mathfrak{N}_0, \mathfrak{N}_+, \mathfrak{N}_+^*$, respectively, and identified with the corresponding subalgebras by Eq. 1. Then all these groups are T-invariant, and any element $n \in \mathfrak{N}$ can be uniquely decomposed as*

$$
n = n_- \times n_0 \times n_+ \,, \qquad n_- \in \mathfrak{N}_-, n_0 \in \mathfrak{N}_0, n_+ \in \mathfrak{N}_+ \,. \tag{2}
$$

The map $n \mapsto (n_-, n_0, n_+)$, $\mathfrak{N} \to \mathfrak{N}_- \times \mathfrak{N}_0 \times \mathfrak{N}_+$ is polynomial and linear in principal terms.

Proof. Since the weight subsets $\Lambda_-, \Lambda_0, \Lambda_+$ are invariant with respect to the complex conjugation, (i) and (ii) are obvious. Further, the description (ii) implies (iii). Finally, since T preserves the lower central series filtration $\{\mathfrak{N}_l\}$, property (i) applied to any $\mathfrak{N}_l$ implies that

$$
\mathfrak{N}_l = (\mathfrak{N}_l \cap \mathfrak{N}_-) \oplus (\mathfrak{N}_l \cap \mathfrak{N}_0) \oplus (\mathfrak{N}_l \cap \mathfrak{N}_+) \,.
$$

Thus, the polynomial map $(n_-, n_0, n_+) \mapsto n_- \times n_0 \times n_+$ is linear in principal terms, which implies that it is invertible, and its inverse map is also polynomial and linear in principal terms. □

For a given vector $\mathbf{a} \in \mathbf{A}$ we shall denote by π_-, π_0, π_+ the projections from $\mathcal{N}$ to the groups $\mathcal{N}_-, \mathcal{N}_0, \mathcal{N}_+$, respectively, determined by the decomposition Eq. 2. Let $S = A \overset{T}{\leftthreetimes} \mathcal{N}$ be the semi-direct product determined by the action T. By $\alpha : (a, n) \mapsto a \in A$ and $\Pi : (a, n) \mapsto n$ denote the corresponding coordinate projections, and put also $\Pi_-(a, n) = \pi_-(n)$, etc.

The decomposition

$$S = \mathcal{N}A = \mathcal{N}_-\mathcal{N}_0\mathcal{N}_+A = \mathcal{N}_-\mathcal{N}_+^*A$$

implies that the homogeneous space $S/\mathcal{N}_+^*A$ can be identified with the set $\mathcal{N}_- \cong \mathfrak{N}_-$, and the action of a group element $(a,n) = na \in S$ on $\mathfrak{N}_-$ has the form

$$(a,n).x = na.x = \pi_-(n \times T^a x) \ .$$

In particular,

$$(a,n).\mathbf{0} = \pi_-(n) \ , \tag{3}$$

where $\mathbf{0}$ is the zero vector in $\mathfrak{N}$. Since the algebra $\mathfrak{N}_-$ is T-invariant (Proposition 2),

$$a.x = \pi_-(T^a x) = T^a x \qquad \forall\, a \in A,\ x \in \mathfrak{N}_- \ . \tag{4}$$

Denote by $\mathcal{P}$ the vector space of complex polynomials on $\mathfrak{N}_-$. For any $P \in \mathcal{P}$, $g \in S$ let $P.g(x) = P(g.x)$. One can easily verify (see also Raugi[16], Lemme 3.5) that

1) If $a \in A$, then $P.a \in \mathcal{P}$, and $\deg P.a = \deg P$ [because the action T in $\mathfrak{N}$ preserves the filtration $\{\mathfrak{N}_l\}$];
2) If $n \in \mathcal{N}$, then $P.n \in \mathcal{P}$ with $\deg P.n = \deg P$, and $\deg(P - P.n) < \deg P$ [because the multiplication $\times$ in $\mathfrak{N}$ and the decomposition $n = n_- \times n_0 \times n_+$ are linear in principal terms].

Thus, for any integer l the group S acts by linear transformations on the finite dimensional space $\mathcal{P}_l$ of polynomials of nilpotent degree $\leq l$. For our purposes it is sufficient to consider only the space $\mathcal{P}_r$, where r is the nilpotency class of $\mathfrak{N}$. As it follows from Proposition 2 and its proof, any basis $\{e_i\}$ in $\mathfrak{N}_-^{\mathbb{C}}$ consisting of weight vectors of the action T is adapted to the lower central series filtration of $\mathfrak{N}_-$. Denote by $\lambda_i \in \Lambda$ the weight of the vector e_i, and by $\varphi_i \in \mathcal{P}_r$ its coordinate function (so that $\deg \varphi_i = \deg e_i$). Monomials $\varphi^l = \prod \varphi_i^{l_i}$ corresponding to multi-indices $l = (l_i)$, $l_i \geq 0$ with $\deg \varphi^l = \sum l_i \deg \varphi_i \leq r$ constitute then a basis in $\mathcal{P}_r$, and this basis contains all coordinate functions φ_i. We shall order these monomials according to their degree, so that the zero degree monomial $\mathbf{1} = \varphi^0$ comes the first, and then any monomial of a lower degree always comes before all monomials of a higher degree.

Denote by $M(g)$, $g \in S$ the matrix representing the transformation $P \mapsto P.g$ in this basis. Then $M(g_1 g_2) = M(g_2)M(g_1)$, so that $g \mapsto M(g)$ is an antirepresentation of the group S in the vector space $\mathcal{P}_r$. The matrices $M(a), a \in A$ are diagonal with entries $\lambda^l(a) = \prod \lambda_i^{l_i}(a)$ as it follows from Eq. 4. In particular, the entry at the top of the diagonal corresponding to the zero multi-index $(0, \ldots, 0)$ is always 1. The matrices $M(n)$, $n \in \mathcal{N}$ are upper triangular with 1's on the diagonal because of the property Eq. 2 above, and Eq. 3 implies that the first row of the matrix $M(n)$ consists of the entries $\prod \varphi_i^{l_i}(\pi_-(n))$. Thus, we have

Proposition 3. *For an arbitrary element $g = (a,n) = na \in S$ the matrix $M(g) = M(a)M(n)$ of the action $P \mapsto P.g$ in the space $\mathcal{P}_r$ has the form*

$$M(g) = \begin{pmatrix} 1 & m(g) \\ 0 & M'(g) \end{pmatrix} ,$$

where $m(g) = m(\pi_-(n))$ is the $(\dim \mathcal{P}_r - 1)$-dimensional vector with components $\varphi^l(\pi_-(n))$ (where $1 \le \sum l_i \deg e_i \le r$), and the $(\dim \mathbf{P}_r - 1) \times (\dim \mathbf{P}_r - 1)$ matrix $M'(g)$ is upper triangular with diagonal entries $\lambda^l(a)$, $1 \le \sum l_i \deg e_i \le r$.

Remark. Analogously, Proposition 3 (with obvious notational modifications) is also true for the action of the group $S = A \ltimes \mathcal{N}$ on the homogeneous space $S/\mathcal{N}_+ A \cong \mathfrak{N}_-^*$.

Below for estimating norms of the matrices $M(g)$ we shall need the following elementary result.

Proposition 4 (see Raugi[16], Lemme 9.4). *Let (M_k) be a sequence of non-degenerate upper triangular matrices of the same order d. If their diagonal elements M_k^{ii}, $1 \le i \le d$ have the property that the limits*

$$\mathbf{m}^i = \lim_{k\to\infty} \frac{1}{k} \sum_{j=1}^{k} \log |M_j^{ii}| , \qquad 1 \le j \le d$$

exist, and

$$\limsup_{k\to\infty} \frac{1}{k} \log \|M_k\| \le 0 ,$$

then

$$\lim_{k\to\infty} \frac{1}{k} \log \|M_k M_{k-1} \dots M_1\| = \max_i \mathbf{m}^i .$$

3. The Poisson boundary of splittable groups

Recall that a probability measure μ on a finitely generated group G has a finite *first moment* if $\sum |g|_K \mu(g) < \infty$, where $|\cdot|_K$ is the *word length* corresponding to a certain ($\equiv$) any finite generating set K. By Kaimanovich[9], Lemma 2.3, if μ is a probability measure with a finite first moment on a finitely generated group G, and $G' \subset G$ – a normal subgroup of finite index, then the group G' is also finitely generated, and there exists a probability measure μ' on G' with finite first moment in G' such that the Poisson boundaries $\Gamma(G,\mu)$ and $\Gamma(G',\mu')$ are naturally isomorphic. Thus, by Proposition 1 the problem of describing the Poisson boundary of a probability measure with a finite first moment on a polycyclic group G is reduced to considering only the case when G is splittable.

Let now μ be a probability measure with a finite first moment on a splittable polycyclic group G. Being splittable, G is contained in an $\mathcal{S}$-group, so that we may assume without loss of generality that the group G itself is an S-group $G = A \ltimes N$. Denote by $\mathcal{N} \supset N$ the Lie hull of the group N, and by $\mathfrak{N}$ the Lie algebra of the group $\mathcal{N}$. We shall keep the notations from the previous Section.

Theorem 1. *Let μ be a probability measure with a finite first moment on an $\mathcal{S}$-group $G = A \leftthreetimes N$. Denote by μ_A the projection of the measure μ onto A, and by*

$$\overline{\mu}_A = \sum_{a \in A} \mu_A(a) a \in \mathbf{A}$$

the barycenter of the measure μ_A. Let $\mathcal{N} = \mathcal{N}_-(\overline{\mu}_A)\mathcal{N}_+^*(\overline{\mu}_A)$, $n = n_- n_+^*$ be the decomposition Eq. 2 of the group $\mathcal{N}$ determined by the vector $\overline{\mu}_A$, and $\Pi_- : (a,n) \mapsto n_-$ be the corresponding map from G to the G-space $S/\mathcal{N}_+^*(\overline{\mu}_A)A \cong \mathfrak{N}_-(\overline{\mu}_A)$. Then for a.e. sample path $\{y_k\}$ of the random walk (G,μ) there exists the limit

$$\lim_{k\to\infty} \Pi_-(y_k) \in \mathfrak{N}_-(\overline{\mu}_A) ,$$

and $\mathfrak{N}_-(\overline{\mu}_A)$ with the corresponding limit distribution coincides with the Poisson boundary of the pair (G,μ).

Proof. I. Convergence. Since the measure μ has a finite first moment in G, its projection $\mu_A = \alpha(\mu)$ onto A also has a finite first moment, and

$$\int \log^+ \|M(g)\| \, \delta\mu(g) < \infty , \tag{5}$$

where $M(g)$ are the matrices from Proposition 3.

Denote by

$$M_k = M(a_k, n_k) = \begin{pmatrix} 1 & m_k \\ 0 & M_k' \end{pmatrix} ,$$

the matrices corresponding to the increments (a_k, n_k) of the random walk. By Eq. 5, a.e. $\log^+ \|M_k\| = o(k)$, so that a.e. the matrices (M_k') satisfy conditions of Proposition 4 with diagonal limits

$$\mathbf{m}_l = \sum l_i \log \lambda_i(\mathbf{a}) < 0 .$$

Since $y_k = (a_1, n_1) \dots (a_k, n_k)$, the matrix

$$M(y_k) = \begin{pmatrix} 1 & m(y_k) \\ 0 & M'(y_k) \end{pmatrix} = M_k \cdots M_1$$

has the entries

$$M'(y_k) = M_k' \cdots M_1'$$

and

$$m'(y_k) = m_1 + m_2 M_1' + \cdots + m_k M_{k-1}' \cdots M_1' . \tag{6}$$

Since

$$\log^+ \|m_k\| \le \log^+ \|M_k\| = o(k) ,$$

and by Proposition 4

$$\lim_{k\to\infty} \log \|M'(y_k)\|/k < 0 ,$$

the sum in Eq. 6 a.e. converges. It implies a.e. convergence of the sequence $\Pi_-(y_k)$, because the vector $m(g) = m(\Pi_-(g))$ contains all coordinates of $\Pi_-(g)$.

II. Maximality. We shall use (in a slightly modified form) the "strip criterion" from Kaimanovich[10]:

> Let μ be a probability measure with a finite first moment on a finitely generated group G, and let (B_-, λ_-) and (B_+, λ_+) be $\check{\mu}$- and μ-boundaries, respectively (here $\check{\mu}(g) = \mu(g^{-1})$ is the reflected measure of μ). Let $\mathcal{G}_1 \subset \mathcal{G}_2 \subset \cdots \subset G$ be an increasing family of subsets exhausting G, and let $|g| = \min\{k : g \in \mathcal{G}_k\}$ be the corresponding "gauge" on G (sets $\mathcal{G}_k$ are not necessarily finite, and the gauge $|\cdot|$ does not have to be symmetric or subadditive in general). If there exists a measurable G-equivariant map S assigning to pairs of points $(b_-, b_+) \in B_- \times B_+$ non-empty "strips" $S(b_-, b_+) \subset G$ such that for all $g \in G$ and $\lambda_- \otimes \lambda_+$-a.e. $(b_-, b_+) \in B_- \times B_+$
>
> $$\frac{1}{n} \log \operatorname{card} \left[S(b_-, b_+) g \cap \mathcal{G}_{|y_n|} \right] \underset{n\to\infty}{\longrightarrow} 0$$
>
> in probability with respect to the measure $\mathbf{P}$ in the space of sample paths $\{y_n\}$ of the random walk (G, μ), then the boundary (B_+, λ_+) is maximal.

For the reflected measure $\check{\mu}$ its projection onto A is the reflected measure of the projection μ_A. Thus, $\overline{\check{\mu}_A} = -\overline{\mu}_A$. By definition, $\mathcal{N}_-(\mathbf{a}) = \mathcal{N}_+(-\mathbf{a}), \mathcal{N}_0(\mathbf{a}) = \mathcal{N}_0(-\mathbf{a})\ \forall\, \mathbf{a} \in \mathbf{A}$, so that by the first part of the proof applied to the measure $\check{\mu}$ the homogeneous space $\mathcal{N}_+(\overline{\mu}_A) = S/Ng_-^*(\overline{\mu}_A)A$ is a $\check{\mu}$-boundary.

Now we have to construct G-equivariant strips $S(n_-, n_+) \subset G$, $n_- \in \mathfrak{N}_-, n_+ \in \mathfrak{N}_+$. The decomposition Eq. 2 implies that for any $n_- \in \mathfrak{N}_-, n_+ \in \mathfrak{N}_+$

$$n_-\mathcal{N}_+^* \cap n_+\mathcal{N}_-^* = n_+(n_+^{-1}n_-\mathcal{N}_+^* \cap \mathcal{N}_-^*) = n_+(n'_-\mathcal{N}_+^* \cap \mathcal{N}_-^*) = n_+n'_-\mathcal{N}_0 = \widetilde{n}\mathcal{N}_0 \ , \quad (7)$$

where $n'_- = n_+^{-1}.n_- \in \mathfrak{N}_-$ and $\widetilde{n} = n_+n'_-$. So, the intersection of any two $\mathcal{N}_+^*$ and $\mathcal{N}_-^*$ cosets in $\mathcal{N}$ is a $\mathcal{N}_0$-coset (one can easily see that in fact any $\mathcal{N}_0$-coset can be uniquely presented in this way).

The group N is cocompact in $\mathcal{N}$, so that there exists a compact set $K \subset \mathcal{N}$ such that for any translation nK, $n \in \mathcal{N}$ the set of N-points $nK \cap N$ in nK is non-empty. Then

$$S(n_-, n_+) = \left[(n_-\mathcal{N}_+^* \cap n_+\mathcal{N}_-^*)K \cap N \right] A \subset G \qquad (8)$$

is a G-invariant map assigning to pairs of points from $\mathfrak{N}_- \times \mathfrak{N}_+$ non-empty subsets of G. Clearly, S is measurable, and, as it follows from Eq. 7, all strips have the form

$$S(n_-, n_+) = (\widetilde{n}\mathcal{N}_0 K \cap N)A = S(\widetilde{n})$$

for a certain $\widetilde{n} = \widetilde{n}(n_-, n_+) \in \mathcal{N}$.

Fix linear norms in $\mathbf{A} \supset A$ and in $\mathfrak{N}$ and let

$$\mathcal{G}_k = \{(a,n) \in G : \|a\|, \|\pi_0(n)\| \le e^k\} . \tag{9}$$

We shall now verify that the strips defined by Eq. 8 and the sets $\mathcal{G}_k$ from Eq. 9 satisfy conditions of the strip criterion.

First note that any strip $S(\widetilde{n})$ has at most exponential growth with respect to the gauge $|\cdot|$ determined by Eq. 9 (although the gauge sets $\mathcal{G}_k$ are themselves infinite). Indeed, let

$$g = (a,n) = na = n_-n_0n_+a \in n\mathcal{N}_0KA \cap \mathcal{G}_k ,$$

i.e.,

$$\|a\|, \|n_0\| \le e^k ,$$

and

$$n_-n_0n_+ \in \widetilde{n}\mathcal{N}_0K .$$

The latter formula means that there exists $n' \in K$ such that

$$\widetilde{n}^{-1}n_-n_0n_+n'^{-1} \in \mathcal{N}_0 .$$

On the other hand, since the group multiplication in $\mathcal{N} \cong \mathfrak{N}$ is polynomial and linear in principal terms, for any $n_0 \in \mathcal{N}_0$ and $n_1, n_2 \in \mathcal{N}$ there exist uniquely determined $n_- \in \mathcal{N}_-$ and $n_+ \in \mathcal{N}_+$ such that the product $n_1n_-n_0n_+n_2$ belongs to $\mathcal{N}_0$, and the map $(n_0, n_1, n_2) \mapsto n_-n_0n_+$ is polynomial. The set K is compact, and $\widetilde{n}$ is fixed, so that there is a constant $C = C(\widetilde{n}, K)$ such that $\|n_-n_0n_+\| \le C\|n_0\|^r$. Thus, $\|n\| \le Ce^{kr}$. Since the groups $\mathbf{A}$ and $\mathcal{N}$ have polynomial growth with respect to the Haar measures $\mathrm{vol}_{\mathbf{A}}$ and $\mathrm{vol}_{\mathcal{N}}$, i.e.,

$$\mathrm{vol}_{\mathcal{N}}\{n \in \mathfrak{N} : \|n\| \le t\},\ \mathrm{vol}_{\mathbf{A}}\{\mathbf{a} \in \mathbf{A} : \|\mathbf{a}\| \le t\} \le C't^d$$

for constants $C', d > 0$, and the embeddings $A \subset \mathbf{A}$, $N \subset \mathcal{N}$ are discrete, we finally obtain that

$$\limsup_{k \to \infty} \frac{1}{k} \log \mathrm{card}\left[S(\widetilde{n}) \cap \mathcal{G}_k\right] < \infty \tag{10}$$

for any $\widetilde{n} \in \mathcal{N}$.

By using the same argument as in the first part of this proof and considering the G-action on the space of polynomials on $\mathfrak{N}_-^*$ (see Remark after Proposition 3), one shows that a.e.

$$\frac{1}{k} \log^+ \|\Pi_0(y_k)\| = o(k) . \tag{11}$$

The A-component $\alpha(y_k)$ of y_k performs the random walk on A determined by the measure μ_A with a finite first moment, so that a.e.

$$\lim_{k\to\infty} \frac{1}{k}\|\alpha(y_k)\| = \|\overline{\mu}_A\| < \infty \tag{12}$$

Eqs. 11 and 12 imply that a.e. $|y_k| = o(k)$. In combination with Eq. 10 it means that the conditions of the strip criterion are satisfied, which finishes the proof. □

Corollary 1. *If μ is a symmetric measure with a finite first moment on a polycyclic group G, then the Poisson boundary $\Gamma(G,\mu)$ is trivial.*

Corollary 2. *If the Poisson boundary is non-trivial for a certain symmetric probability measure μ with a finite first moment on a finitely generated solvable group G, then G contains an infinitely generated subgroup.*

Remarks. 1. The proof of convergence in Theorem 1 is very close to the proof of an analogous statement in a more general setup of real Lie groups, see Raugi[16]. However, dealing with discrete groups requires some modifications. On the other hand, considering only semi-direct products allows one to avoid a number of technical difficulties connected with general Lie groups.

2. Another (more complicated technically) proof of Theorem 1 can be obtained by using the "global law of large numbers" for solvable Lie groups, see Kaimanovich[8], Kaimanovich[9].

3. Yet another way of obtaining a description of the Poisson boundary of a polycyclic group G consists in embedding G into the matrix group $GL(d,\mathbb{Z})$ and using the description of the Poisson boundary for this group, see Kaimanovich[6], Ledrappier[13]. In this approach the Poisson boundary is identified with (a subset of) a certain flag space in $\mathbb{R}^d$.

4. Since any *poly-(cyclic-or-finite)* group (i.e., a group admitting a finite normal series with finite or cyclic quotients) is a finite extension of a polycyclic group, see Segal[17], Theorem 1 also gives a description of the Poisson boundary for measures with a finite first moment on such groups.

5. The automorphisms $T^a : \mathfrak{N} \to \mathfrak{N}$, $a \in A$ preserve the cocompact group N, hence $|\det T^a| \equiv 1$, and the subalgebras $\mathfrak{N}_-(\mathbf{a})$ and $\mathfrak{N}_+(\mathbf{a})$ are trivial or non-trivial simultaneously in perfect keeping with the fact that the Poisson boundaries $\Gamma(G,\mu)$ and $\Gamma(G,\check{\mu})$ are trivial or non-trivial simultaneously (see Kaimanovich and Vershik[12]).

6. The proof of maximality in Theorem 1 in a sense is a combination of proofs in two important particular cases when the neutral subgroup $\mathcal{N}_0$ is either trivial or coincides with the whole group $\mathcal{N}$. In the first case the strips in G have the form $S(\widetilde{n}) = (\widetilde{n}K \cap N)A$, and the proof of maximality becomes trivial (modulo the strip criterion). In the second case (in particular, if the measure μ is symmetric, or, more generally, if $\overline{\mu}_A = 0$) Theorem 1 reduces to showing that the Poisson boundary of the measure μ is trivial. This can be done by a direct estimate of the rate of escape

of the random walk (G,μ). If (a_k,n_k) are the increments of the random walk, then its position at time k is

$$y_k = (a_1 + \dots a_k, n_1 \times T^{a_1} n_2 \times \dots T^{a_1 + \dots a_{k-1}} n_k) \ .$$

If $|\cdot|$ is a word length on N, then $\log^+ \|n_k\| = o(k)$ (provided the measure μ has a finite first moment). Since $a_1 + \cdots + a_k = o(k)$, it implies that $\log^+ \|T^{a_1 + \dots a_{k-1}} n_k\| = o(k)$, so that $\log^+ \|\pi(y_k)\| = o(k)$. Thus, the entropy of the random walk is zero, because the nilpotent group G has polynomial growth (see Kaimanovich[7], Kaimanovich[8] and expositions of this proof in Ancona[1] and Lyons[14]).

7. As one could expect, the boundary theory for polycyclic groups is parallel to that for solvable Lie groups (although the methods are quite different). The description of the Poisson boundary for polycyclic groups obtained in Theorem 1 is essentially the same as for solvable Lie groups (cf. Azencott[3], Raugi[16]). However, note that the strip criterion in a generalized form is applicable to general locally compact groups, in particular to solvable Lie groups (we shall return to this question in a further publication).

8. Let M be a compact manifold with a polycyclic fundamental group G, and $\mathcal{D}$ – a second order elliptic differential operator on M which nullifies constants. Let $\widetilde{M}$ be the universal covering space of M, and $\widetilde{\mathcal{D}}$ – the lift of the operator $\mathcal{D}$. In view of the technique from Kaimanovich[6], the same methods as in this Section give a Poisson formula for bounded $\widetilde{\mathcal{D}}$-harmonic functions on $\widetilde{M}$ which answers a question raised by Lyons and Sullivan[15]. In particular, if $\mathcal{D}$ is the Laplace–Beltrami operator of a Riemannian metric on M, then the corresponding space of bounded harmonic functions on $\widetilde{M}$ is trivial, see Kaimanovich[6].

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