QUANSHENG LIU

On the Survival Probability of a Branching Process in a Random Environment

Publications de l'Institut de recherche mathématiques de Rennes, 1993, fascicule 2
« Fascicule de probabilités », , p. 1-9
<http://www.numdam.org/item?id=PSMIR_1993___2_A6_0>

ON THE SURVIVAL PROBABILITY OF A BRANCHING PROCESS IN A RANDOM ENVIRONMENT

Quansheng LIU

Institut de Recherche Mathématique de Rennes, Université de Rennes I

Campus de Beaulieu, 35042 Rennes, France

ABSTRACT. – We determine the decay rate of the survival probability of a branching process in an independent and identically distributed random environment with a countable state space.

RÉSUMÉ. – Nous déterminons la vitesse de décroissance de la probabilité de survie d'un processus de branchement dans un environnement aléatoire indépendant et identiquement distribué avec l'espace d'états dénombrable.

1. Introduction

Let (Z_n) $(n\geq 0)$ be a branching process in a random environment (BPRE), i.e. we are given an environment sequence $(\zeta_0, \zeta_1, \ldots)$ whose realization determines a sequence of generational probability generating functions $f_{\zeta_n}(s)$ $(n\geq 0)$, where $Z_0\equiv 1$ stands for the single initial population number at the 0-th generation, Z_n represents the population size at the n-th generation, and, given ζ_n, all members of this generation reproduce independently each other according to the probability generating function $f_{\zeta_n}(s)$. We assume that the environment stochastic process $\zeta := (\zeta_0, \zeta_1, \ldots)$ is stationnary and ergodic and we shall especially consider the case where it is a sequence of independent and identically distributed random variables. As usual, we assume that $m(\zeta)<+\infty$

AMS 1980 Subject Classifications (1985 revision): Primary 60J80, 60J85; Secondary 60F15.

Key wods and phrases: Branching processes with random environments, survival probability.

almost surely (a.s.), where $m(\zeta) := \mathbb{E}(Z_1|\zeta)$ denotes the conditional expectation of the offspring of the single object at the 0-th generation, given the environment ζ. We note that $m(\zeta) = f'_{\zeta_0}(1)$, where the later denotes the left hand derivative of $f_{\zeta_0}(s)$ at $s = 1$.

The process is called to be of a countable state random environment if the state space of the environment process is countable. It is called subcritical, critical or supercritical according to whether $\mathbb{E}(\text{ Log } m(\zeta))$ is negative, zero or positive, and it is called strongly subcritical if $\mathbb{E}(m(\zeta) \text{ Log } m(\zeta)) \le 0$ and $P[m(\zeta)=1]<1$. Then a strongly subcritical process is obviously subcritical according to Jensen's inequality. We shall exclude the degenerate process with $m(\zeta) = 0$ a.s.

It is well known that $P(Z_n > 0)$ tends to zero as n tends to infinity if and only if (Z_n) is critical or subcritical, see for example [2] or [7]. For an ordinary branching process, the rate at which $P(Z_n>0)$ tends to zero is known: $\lim_{n\to\infty} P(Z_n>0)^{1/n} = \mathbb{E}(Z_1)$ if $\mathbb{E}(Z_1\text{Log}^+Z_1) < +\infty$. In 1987 [3], F.M. Dekking established a similar result for a BPRE in a two state random environment, which was extended in 1988 [4] to the case of finite state random environment, under the assumption $\mathbb{E}(Z_1{}^2) < +\infty$. The study of this problem is interesting not only for its theoretical aspects, but also for its applications, such as to the study of some fractal sets (see for example [5]) or to some percolation problems on trees [3].

The purpose of this paper is to extend Dekking's results to the case of countable state environment under weaker conditions. We shall prove the following

Theorem 1.1 Let (Z_n) be a branching process in an independent and identically distributed random environment with $\mathbb{E}(Z_1) < +\infty$, then

$$\lim_{n\to\infty} P(Z_n > 0)^{1/n} = \rho := \inf_{t\in[0,1]} \mathbb{E}[m(\zeta)^t], \tag{1.1}$$

where $\rho=1$ if (Z_n) is supercritical or critical, $\rho=\mathbb{E}Z_1$ if (Z_n) is strongly subcritical, and $\rho<\min(1,\mathbb{E}Z_1)$ if (Z_n) is subcritical but not strongly subcritical.

Remark. Dekking (1988) obtained the same conclusion in the case of finite state environment under the additional assumption $\mathbb{E}(Z_1^2) < +\infty$, and then conjectured that the second moment condition would be relaxed to $\mathbb{E}(Z_1 \mathrm{Log}^+ Z_1) < \infty$ [5, Remark 1]. Our result here shows that this is indeed the case, and that it can even be relaxed to $\mathbb{E}(Z_1) < +\infty$. This shows in particular that for a subcritical Galton-Watson process (Z_n) the condition $\mathbb{E}(Z_1 \mathrm{Log}^+ Z_1)<\infty$ is *not* necessary for $\lim_{n\to\infty} P(Z_n>0)^{1/n} = \mathbb{E}(Z_1)$ (but is necessary for $\lim_{n\to\infty} P(Z_n>0)/(\mathbb{E}Z_1)^n = c$ for some constant $c>0$).

2. The upper bound

In this section we give a general upper bound on the survival probability of a BPRE, where the state space of the environment is not necessarily countable. The approach is direct and much simpler than that of Dekking [1987 and 1988].

Theorem 2.1. Let (Z_n) be a BPRE. For all $t\in[0,1]$ and all $n\geq 0$, we have

$$P(Z_n>0) \leq \mathbb{E}[P_n^{\,t}],$$

where $P_n = \prod_{i=0}^{n-1} f_{\zeta_i}'(1)$ $(n\geq 1)$. In particular, if the environment is independent and identically distributed, then for all $n\geq 0$,

$$P(Z_n>0) \leq \rho^n.$$

Proof. Fix $t\in[0,1]$. By Markov's inequality and Jensen's inequality we have

$$P(Z_n> 0) = P(Z_n\geq 1) \leq \mathbb{E}(Z_n^{\,t})$$
$$= \mathbb{E}\Big(\mathbb{E}(Z_n^{\,t}|\zeta)\Big) \leq \mathbb{E}\Big[\Big(\mathbb{E}(Z_n|\zeta)\Big)^t\Big] = \mathbb{E}[P_n^{\,t}],$$

where the last equality holds by the definition of the process. Noting that $\mathbb{E}[P_n^{\,t}] = [\mathbb{E}(m(\zeta)^t)]^n$ if the environment is independent and identically distributed, the conclusions follow. ∎

3. The lower bound

This section is to give a lower bound on $P(Z_n > 0)$ of a BPRE (Z_n) with a countable state independent and identically distributed random environment in a special case.

Theorem 3.1. Let (Z_n) be a branching process in a countable state independent and identically distributied random environment with $P(m(\zeta)=0) = 0$ and $\mathbb{E}(Z_1{}^2) < \infty$, then $\liminf_{n\to\infty} P(Z_n>0)^{1/n} \ge \rho$.

Before giving the proof, we first introduce some notations. With no restrictions, we suppose that the environmental state space is $\mathbb{N}$, the non-negative integers, such that

$$P(\zeta_n=i) = p_i \ , \text{ where } p_i \ge 0 \quad \text{and} \quad \sum_{i=0}^{\infty} p_i = 1.$$

For fixed $r\ge1$ and $n\in\mathbb{N}$, Let $K_{n,r}$ be the set of $(k_0,K_1,\dots,k_{r-1})\in\mathbb{N}^r$ such that $\sum_{i=0}^{r-1} k_i = n$. For $\underline{k}:= (k_0,K_1,\dots,k_{r-1}) \in K_{n,r}$, we denote by $W_{n,r,\underline{k}}$ the subset of $\{0,1,\dots,r-1\}^n$ consisting of those sequences that contain k_i occurences of the symbol $i \in \{ 0,1,\dots,r-1 \}$. The cardinality of $W_{n,r,\underline{k}}$ is then

$$\binom{n}{\underline{k}} := \frac{n!}{k_0!k_1!\dots k_{r-1}!} .$$

In the proof of theorem 3.1, we shall use the following lemma of Dekking (1988), which was based on a result of Agresti (1975, Lemma 2) about the lower bound of the survival probability of a branching process in a varying environment.

Lemma 3.2. [5, Lemma 1] Let f_i be some offspring probability generating functions and set $C(r)= \max \{ f_i''(1)/f_i'(1),\ 0\le i\le r-1 \}$. Suppose that $0 < f_i'(1) < \infty$ and $f_i''(1) < \infty$ for all $i = 0,1,\dots,r-1$, then for any $\underline{k} = (k_0,K_1,\dots,k_{r-1}) \in K_{n,r}$, the following holds for at least a fraction $\frac{1}{n}$ of the sequences w in $W_{n,r,\underline{k}}$:

$$P(Z_n{}^w > 0) \ge \frac{\min \{1,\mathbb{E}(Z_n^w)\}}{1 + nC(r)} ,$$

where $w:= w_0 w_1 \dots w_{n-1}$ and Z_n^w denotes the n-th generation of the process (Z_n) conditioned on the environment $[w]:= [\zeta_0=w_0, \zeta_1=w_1, \dots, \zeta_{n-1}=w_{n-1}]$.

Proof of theorem 3.1. Write $F(t)= \mathbb{E}(m(\zeta)^t)$, then $\rho= \inf_{t\in[0,1]} F(t)$. Note that $F(t)$ is convex, $F(0)=1$, $F(1)=\mathbb{E}Z_1$, $F'(0)= \mathbb{E}\log m(\zeta)$ and $F'(1)= \mathbb{E}m(\zeta)\log m(\zeta)$. So it is easily seen that $\rho=1$ if (Z_n) is supercritical or critical, $\rho=\mathbb{E}Z_1$ if (Z_n) is strongly subcritical, and $\rho<\min(1,\mathbb{E}Z_1)$ if (Z_n) is subcritical but not strongly subcritical.

Since $P(m(\zeta)=0)=0$ and $\mathbb{E}(Z_1^2)<\infty$, we can suppose that $0<m_i:= \mathbb{E}(Z_1 | \zeta_0=i) \equiv f_i'(1) < \infty$ and $f_i''(1) < \infty$ for all $i \in \mathbb{N}$. Now

$$P(Z_n>0) = \sum_{w\in\mathbb{N}^n} P(Z_n^w > 0)P([w]) \geq \sum_{w\in\{0,1,\dots,r-1\}^n} P(Z_n^w > 0)P([w])$$

$$= \sum_{\underline{k}\in K_{n,r}} \sum_{w\in W_{n,r,\underline{k}}} P(Z_n^w >0)P([w]) = \sum_{\underline{k}\in K_{n,r}} \sum_{w\in W_{n,r,\underline{k}}} P(Z_n^w >0)\, \underline{p}^{\underline{k}}$$

where $\underline{k}= (k_0,K_1,\dots,k_{r-1})$, $\underline{p}^{\underline{k}} = p_0^{k_0} p_1^{k_1}\dots p_{r-1}^{k_{r-1}}$ and $0<r\in\mathbb{N}$. Noting that $\mathbb{E}(Z_n^w)= m_0^{k_0} m_1^{k_1}\dots m_{r-1}^{k_{r-1}} =: \underline{m}^{\underline{k}}$ if $w \in W_{n,r,\underline{k}}$, by Lemma 3.2 we have then

$$P(Z_n>0) \geq \sum_{\underline{k}\in K_{n,r}} \frac{1}{n} \binom{n}{\underline{k}} \frac{\min(1,\underline{m}^{\underline{k}})}{(1+ nC(r))} \underline{p}^{\underline{k}} .$$

We now divide the proof into two cases according as $\mathbb{E}\, m(\zeta)\log m(\zeta) >0$ or ≤ 0.

Case I: $\mathbb{E}\, m(\zeta)\log m(\zeta)>0$. That is $\sum_{i=0}^{\infty} p_i m_i \log m_i >0$. This contains exactly the supercritical processes, critical processes with $P(m(\zeta)=1)<1$, and subcritical but not strongly subcritical processes. In this case we use the estimation

$$P(Z_n>0) \geq \sum_{\underline{k}\in K_{n,r}\,:\,\underline{m}^{\underline{k}} \geq 1} \frac{1}{n} \binom{n}{\underline{k}} \frac{1}{(1+ nC(r))} \underline{p}^{\underline{k}}$$

$$= \frac{1}{n(1+ nC(r))} (p(r))^{-n} \sum_{\underline{k}\in K_{n,r}\,:\,\underline{m}^{\underline{k}} \geq 1} \binom{n}{\underline{k}} \underline{p}^{\underline{k}} (p(r))^n$$

$$= \frac{(p(r))^{-n}}{n(1+ nC(r))} P(\sum_{i=1}^{n} X_i(r) \geq 0)$$

where $X_1(r)$, $X_2(r)$, ... is a sequence of independent and identically distributed random variables defined by

$$P(X_1(r) = \log m_i) = p(r)p_i \quad (i = 0,1,\dots,r-1),$$

$p(r) = (\sum_{i=0}^{r-1} p_i)^{-1}$ and r is a positive integer large enough such that $\sum_{i=0}^{r-1} p_i>0$. By Chernoff-Cramér theorem, we have

$$\lim_{n\to\infty} P(\sum_{i=1}^{n} X_i(r) \geq 0)^{1/n} = \inf_{t\geq 0} \mathbb{E}[e^{tX_1(r)}].$$

(For the form that we need here, see for example [6,p.129].) Thus

$$\liminf_{n\to\infty} P(Z_n > 0)^{1/n} \geq \frac{1}{p(r)} \inf_{t\geq 0} \mathbb{E}[e^{tX_1(r)}]$$

$$= \inf_{t\geq 0} \sum_{i=0}^{r-1} p_i m_i^t = \inf_{t\in[0,1]} F_r(t),$$

where $F_r(t) := \sum_{i=0}^{r-1} p_i m_i^t$ and r>0 is chosen sufficiently large such that $F_r'(1) := \sum_{i=0}^{r-1} p_i m_i \log m_i > 0$. The last step holds since the function $F_r(t)$ is convex in $[0,\infty)$. Since $F_r(t)$ converges monotonically to $F(t)$ $(=\sum_{i=0}^{\infty} p_i m_i^t)$, the convergence is uniform for $t \in [0,1]$ by Dini's theorem. So for all $\varepsilon>0$, there exists $r_0=r_0(\varepsilon)$ sufficiently large such that $F(t) \geq F_r(t) \geq F(t)-\varepsilon$ for all $r\geq r_0$ and all $t\in[0,1]$. Thus $\forall r\geq r_0$

$$\rho = \inf_{t\in[0,1]} F(t) \geq \inf_{t\in[0,1]} F_r(t) \geq \inf_{t\in[0,1]} F(t) - \varepsilon = \rho-\varepsilon.$$

Therefore $\lim_{r\to\infty} \inf_{t\in[0,1]} F_r(t) = \rho$. The proof is then finished in the present case.

<u>Case II</u>: $\mathbb{E}\, m(\zeta)\log m(\zeta) \leq 0$. That is $\sum_{i=0}^{\infty} p_i m_i \log m_i \leq 0$. This contains exactly the critical processes with $P(m(\zeta)=1)=1$, and the strongly subcritical processes. We have then $\rho=\mathbb{E}(Z_1)$ and, either $m_i=1$ for all i with $p_i>0$, or $\sum_{i=0}^{\infty} p_i \log m_i < 0$. In this case we use the estimation

$$P(Z_n>0) \geq \sum_{\underline{k}\in K_{n,r}:\ \underline{m}^{\underline{k}} \leq 1} \frac{1}{n} \binom{n}{\underline{k}} \frac{1}{(1+ nC(r))} \underline{m}^{\underline{k}} \underline{p}^{\underline{k}}.$$

Define a sequence of independent and identically distributed random variables $Y_i(r)$ $(i \geq 1)$ by

$$P(Y_1(r) = -\log m_i) = l(r)p_i m_i/\mathbb{E}(Z_1) \quad (i = 0,1,\ldots,r-1),$$

where $l(r) = \mathbb{E}(Z_1)(\sum_{i=0}^{r-1} p_i m_i)^{-1}$ and r is a positive integer large enough such that $\sum_{i=0}^{r-1} p_i m_i > 0$, we have then

$$P\Big(\sum_{i=1}^{n} Y_i(r) \geq 0\Big) = \sum_{\underline{k}\in K_{n,r}:\ \underline{m}^{\underline{k}}\leq 1} \binom{n}{\underline{k}} \underline{m}^{\underline{k}} \underline{p}^{\underline{k}} \, (l(r))^n (\mathbb{E}(Z_1))^{-n}$$

and consequently

$$P(Z_n > 0) \geq \frac{1}{n(1+nC(r))} \left(\frac{\mathbb{E}Z_1}{l(r)}\right)^n P\Big(\sum_{i=1}^{n} Y_i(r) \geq 0\Big)$$

for sufficiently large r. The Chernoff-Cramer theorem applies again, yielding that
$$\liminf_{n\to\infty} P(Z_n > 0)^{1/n} \geq \inf_{t\geq 0} \sum_{i=0}^{r-1} p_i m_i^{1-t} .$$

If $m_i=1$ for all i with $p_i>0$, this means $\liminf_{n\to\infty} P(Z_n > 0)^{1/n} \geq \sum_{i=0}^{r-1} p_i$ and the proof is then terminated by letting $r\to\infty$ since $\rho=\mathbb{E}Z_1=1$; otherwise we have $\sum_{i=0}^{\infty} p_i \log m_i<0$ (subcritical). Then for all sufficiently $r>0$, $\sum_{i=0}^{r-1} p_i \log m_i<0$. Since $(\sum_{i=0}^{r-1} p_i m_i^{1-t})'(1) = -\sum_{i=0}^{r-1} p_i \log m_i>0$, we have

$$\inf_{t\geq 0} \sum_{i=0}^{r-1} p_i m_i^{1-t} = \inf_{t\in[0,1]} \sum_{i=0}^{r-1} p_i m_i^{1-t} = \inf_{t\in[0,1]} \sum_{i=0}^{r-1} p_i m_i^{t} \equiv \inf_{t\in[0,1]} F_r(t).$$

This shows that $\liminf_{n\to\infty} P(Z_n > 0)^{1/n} \geq \inf_{t\in[0,1]} F_r(t)$. As in case I, letting $r\to\infty$ gives $\liminf_{n\to\infty} P(Z_n > 0)^{1/n} \geq \rho$, the result desired. This ends the proof of the theorem. ∎

4. Proof of theorem 1.1.

The result has nearly been proved. This section is to show how theorem 1.1 can be deduced from theorems 2.1 and 3.1.

Proof of theorem 1.1. From the remarks about the value of ρ at the begining of the proof of Theorem 3.1, it suffices to prove the formula (1.1).

We first consider the case where $\mathbb{E}(Z_1^2) < \infty$. If $P(m(\zeta)=0) = 0$, the result is immediate by theorems 2.1 and 3.1. Otherwise we use a technique of reduction as was remarked by Dekking [5,remark 2]. In fact, let $E = (i \in \mathbb{N} \mid m_i = 0)$ and $p_E = \sum_{i \in E} p_i$, then

$$P(Z_n>0) = P(Z_n>0 \mid \zeta_j \notin E,\ 0 \le j \le n-1)(1-p_E)^n = P(Z_n'>0)(1-p_E)^n,$$

where (Z_n') is a new BPRE with the same offspring distributions as (Z_n) but

$$P(\zeta_0'=i) = \begin{cases} p_i/(1-p_E), & \text{if } i \notin E, \\ 0, & \text{otherwise.} \end{cases}$$

An application of the result to the process (Z_n') gives

$$\lim_{n\to\infty} P(Z_n'>0)^{1/n} = \inf_{t\in[0,1]} \mathbb{E}(m(\zeta')^t) = \frac{1}{1-p_E} \inf_{t\in[0,1]} \mathbb{E}(m(\zeta)^t).$$

Thus the result follows.

We then consider the general case where $\mathbb{E}(Z_1) < \infty$. To this end, we use a *truncated comparison method*. Let X_{ζ_n} $(n\ge0)$ be the random variable whose probability generating function is f_{ζ_n} (the offspring distributions of (Z_n) given the environment sequence $(\zeta_n)(n \ge 0)$). Associated with (Z_n), we define a BPRE (Z_n^*) with the same environment sequence as (Z_n), but with offspring distributions $X^*_{\zeta_n} = X_{\zeta_n} 1_{[0,N]}(X_{\zeta_n})$, where $N > 0$ is a given positive integer and $1_A(\cdot)$ denotes the indicate function of the set A. Then $Z_n \ge Z_n^*$ a.s. and $\mathbb{E}[(Z_1^*)^2] < \infty$. Hence

$$\liminf_{n\to\infty} P(Z_n > 0)^{1/n} \ge \liminf_{n\to\infty} P(Z_n^* > 0)^{1/n} = \inf_{t\in[0,1]} I_N(t),$$

where

$$I_N(t) = \mathbb{E}\left[\left(\int_{[0,N]} x \, dG(\zeta_0,x)\right)^t\right],$$

$G(\zeta_0,\cdot)$ being the distribution of X_{ζ_0}. Now $\lim_{N\to\infty} I_N(t) = \mathbb{E}\,[m(\zeta)^t] = F(t)$ by the monotone convergence theorem. Moreover, the convergence is uniform for $t \in [0,1]$ by Dini's theorem. Thus for all $\varepsilon>0$, there exists $N_0=N_0(\varepsilon)>0$ sufficiently large such that for all $N\ge N_0$ and all $t\in[0,1]$, $F(t)\ge I_N(t) \ge F(t)-\varepsilon$. Taking inferiors shows that

$$\lim_{N\to\infty} \inf_{t\in[0,1]} I_N(t)) = \inf_{t\in[0,1]} F(t)=\rho.$$

Thus $\liminf_{n\to\infty} P(Z_n > 0)^{1/n} \ge \rho$. Combining with Theorem 2.1, this ends the proof of Theorem 1.1. ■

References

[1] A.Agresti, On the extinction times of varing and random environment branching processes, J. Appl. Prob. 12 (1975) 39-46.

[2] K.B.Athreya and S.Karlin, Branching processes with random environments, I & II, Ann. Math. Statist. 42 (1971) 1499-1520 and 1843-1858.

[3] F.M.Dekking, Subcritical branching processes in a two state random environment, and a percolation problem on trees, J. Appl. Prob. 24 (1987) 798-808.

[4] F.M.Dekking, On the survival probability of a branching process in a finite state i.i.d.environment, Stoch. Proc. and Their Appl. 27 (1988) 151-157.

[5] F.M.Dekking and G.R.Grimmett, Superbranching processes and projections of Cantor sets, Proba. Th. & Rel. Fields, 78 (1988) 335-355.

[6] R.Lyons and R.Pemantle, Random walk in a random environment and first-passage percolation on trees. Ann. of Proba. 20 (1992) 125-136.

[7] W.L.Smith and W.E.Wilkinson, On branching processes in random environments, Ann. Math. Statist. 40 (1969) 814-827.