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by

Nicolas LERNER

Abstract : The main goal of the present paper is to provide an example of a classical principal type pseudo-differential operator $P = p(x,D_x)$, with an homogeneous principal symbol p of order 1 , satisfying condition (ψ), so that the equation $Pu = f$ has no L^2 solution for most f in L^2 .

1. Introduction

We are interested in local solvability properties for pseudo-differential operators : the operator P is said to be locally solvable if for any smooth f satisfying a finite number of compatibility conditions, there is a distribution u local solution of $Pu = f$ (see definition 26.4.1 in [5] for a precise statement). Most of the research in this domain was oriented toward a characterization of local solvability of a principal type pseudo-differential operator in terms of a geometric property of its principal symbol, the so-called condition (ψ). We briefly recall here some of the basic facts related to this problem.

Let P be a pseudo-differential operator of principal type (i.e. the hamiltonian field H_p of the principal symbol p is independent of the Liouville vector field). If the principal symbol is real-valued, a propagation-of-singularities result is true and implies global existence (see theorem 26.1.9 in [5]). When the principal symbol is complex-valued,the situation is much more complicated ; in 1957, Hans Lewy found a principal type differential operator without solution. His example ,

$$\frac{\partial}{\partial \bar{z}} + i z \frac{\partial}{\partial t} \quad \text{in} \quad \mathbb{C}_z \times \mathbb{R}_t ,$$

is the Cauchy-Riemann operator on the boundary of a strictly pseudo-convex domain. The simple models $M_k = D_t + i t^k D_x$, $k \in \mathbb{N}$, studied by Mizohata [8] for the analytic-hypoellipticity, were the starting point in Nirenberg-Treves [10] : the Hans Lewy operator is equivalent to M_1 and the M_{2k+1} are non-solvable whereas the degenerate Cauchy-Riemann operators M_{2k} are solvable. Local solvability of differential operators is now known to be characterized by condition (P) : the symbol p is said to satisfy condition (P) if the imaginary part Imp does not change sign along the bicharacteristic curves of $\mathbb{R}$ep (see Nirenberg-Treves [11] with an analyticity assumption, Beals-Fefferman [1] in the general case for local solutions, Hörmander's theorem 26.11.3 in [5] for a semi-global existence result).

In the pseudo-differential case, a (quite natural) extension of condition (P) is condition (ψ): the imaginary part Imp does not change sign from - to + along the oriented bicharacteristic curves of $\mathbb{R}$ep (see definition 26.4.6 in [5]). This condition was proven invariant by multiplication by an elliptic factor in [11] (see also lemma 26.4.10 in [5]). The importance of this geometrical condition was stressed by Nirenberg and Treves [11] who conjectured condition (ψ) is equivalent to local solvability and proved it in a number of cases. The necessity of condition (ψ) for local solvability was established for general pseudo-differential equations after the works of Moyer [9] in two dimensions and Hörmander in the general case (Corollary 26.4.8 in [5]). Moreover, the sufficiency in two dimensions is proved in [6], yielding condition (ψ) as an iff condition for solvability in that case . Hörmander's work on subellipticity (theorem 27.1.11 of [5]) showed that if the symbol p satisfies condition (ψ) and a finite type assumption ((27.1.8) in [5]) then the quantization of $\bar{p}$ is hypoelliptic and thus the operator with symbol p is solvable.

It was shown by Nirenberg and Treves [11] that a solvability problem for a principal type operator is equivalent by localisation and canonical transformation to proving an a priori estimate for a first order pseudo-differential operator

$$\frac{1}{i}\frac{\partial}{\partial t} + i\, Q(t,x,D_x) ,$$

where $Q(t,x,D_x)$ is a first-order pseudo-differential operator with real principal symbol q . In that framework, condition (ψ) for the adjoint operator is expressed as

$$(1.1) \qquad q(t, x,\xi) > 0 \text{ and } s > t \qquad \text{imply} \qquad q(s,x,\xi) \geq 0.$$

In this paper, we give first the construction a symbol $q(t, x,\xi)$ in the $S^1_{1,0}$ class, i.e. a smooth function of five real variables such that, for any five-uple of integers $k , \alpha_1 , \alpha_2 , \beta_1 , \beta_2$,

$$(1.2) \qquad \sup_{t,x_1,x_2,\xi_1,\xi_2 \in \mathbb{R}^5} |(D_t^k D_{x_1}^{\alpha_1} D_{x_2}^{\alpha_2} D_{\xi_1}^{\beta_1} D_{\xi_2}^{\beta_2} q)(t,x_1,x_2,\xi_1,\xi_2)| \; [1+|\xi_1|+|\xi_2|]^{-1+|\beta_1|+|\beta_2|} < +\infty$$

such that q satisfies (1.1) and such that no L^2 estimate can be proved for $\frac{1}{i}\frac{\partial}{\partial t} + i\, q(t,x,D_x)$: we have

$$(1.3) \qquad \inf_{u \in C_0^\infty(\Omega) ,\ \| u \|_{L^2(\mathbb{R}^3)} = 1} \left\| \frac{1}{i}\frac{\partial u}{\partial t} + i\, q(t,x,D_x)u \right\|_{L^2(\mathbb{R}^3)} = 0$$

for any Ω neighborhood of $0_{\mathbb{R}^3}$. We thus prove that the equation $\frac{\partial v}{\partial t} + q(t,x,D_x)v = f$ has no L^2 solution for a general right -hand side f in L^2 although the operator $\partial_t + q(t,x,D_x)$ satisfies condition (ψ).

Next, we provide an homogeneous symbol $q(t, x,\xi)$ of order 1, i.e. a smooth function of five real variables t,x_1,x_2,ξ_1,ξ_2 , homogeneous of order 1 with respect to ξ_1, ξ_2 , which satisfies (1.1) on $\{ \xi_2 \geq 0 \}$ and such that (1.3) is satisfied for $\frac{1}{i}\frac{\partial}{\partial t} + i\, Op(q(t,x,\xi)\, \gamma (\tau,\xi))$, where Op(a) stands for the operator with symbol a, $\gamma (\tau,\xi)$ is an homogeneous function of degree 0 supported in the cone $\{ \xi_2 > 0 \}$. This gives an example of a classical homogeneous pseudo-differential operator of order one without local L^2 solution for a general right -hand side in L^2. Let's remark here that we only disprove the L^2 solvability, which is satisfactory for a first order principal type operator . At any rate, this counterexample shows that no L^2 estimate can be proved for a classical homogeneous pseudo-differential operator $\frac{1}{i}\frac{\partial}{\partial t} + i\, q(t,x,D_x)$ under the sole assumption of condition (1.1).

The reader eager for precise statements and proofs may proceed directly to the main body of the text, starting in section 2. However, we wish to devote some space to an informal introduction to the main ideas involved in the construction, as well as outlining the steps of the proof. Let's choose first our notations : we want to discuss the solvability of an evolution operator $\frac{d}{dt} + Q(t)$, where each Q(t) is a selfadjoint operator (unbounded) on a Hilbert space $\mathbb{H}$. This is equivalent to discussing a priori estimates for $\frac{d}{dt} - Q(t) = i\,[\,D_t + i\,Q(t)\,]$. It is quite clear that the above problem is far too general, and so we wish to start our discussion with the simplest non trivial example : instead of dealing with infinite dimensional Hilbert space, let's take $\mathbb{H} = \mathbb{R}^2$, so that Q(t) is a 2 x 2 symmetric matrix, defining a (bounded!) operator on $\mathbb{R}^2$, allowed to depend on large parameters. Since it could be still complicated, let's assume

$$(1.4) \qquad Q(t) = H(-t)\,Q_1 + H(t)\,Q_2\ ,$$

where H is the Heaviside function(characteristic function of $\mathbb{R}^+$), Q_1 and Q_2 (2 x 2) symmetric matrices. There is of course no difficulty solving the equation $\frac{dv}{dt} + Q(t)\,v = f$; however, if we want to get uniform estimates with respect to the size of the coefficients of Q(t), we have to choose carefully our solutions, even in finite dimension . When $Q_1 = Q_2$, the good fundamental solution is given by $H(t)\,E_1^+ \exp -t\,Q_1 - H(-t)\,E_1^- \exp -t\,Q_1$, where E_1^+ and E_1^- are the spectral projections corresponding to the half axes. If we go back to (1.4) with $Q_1 \neq Q_2$, there is a trivial case in which the operator $\frac{d}{dt} + Q(t)$ is uniformly solvable : the monotone increasing situation $Q_1 \leq Q_2$ yielding the estimate

$$\| D_t u + i\,Q(t)u \|_{L^2} \geq \| D_t u \|_{L^2} \quad \text{, where } L^2 \text{ stands for } L^2(\mathbb{R}, \mathbb{H}).$$

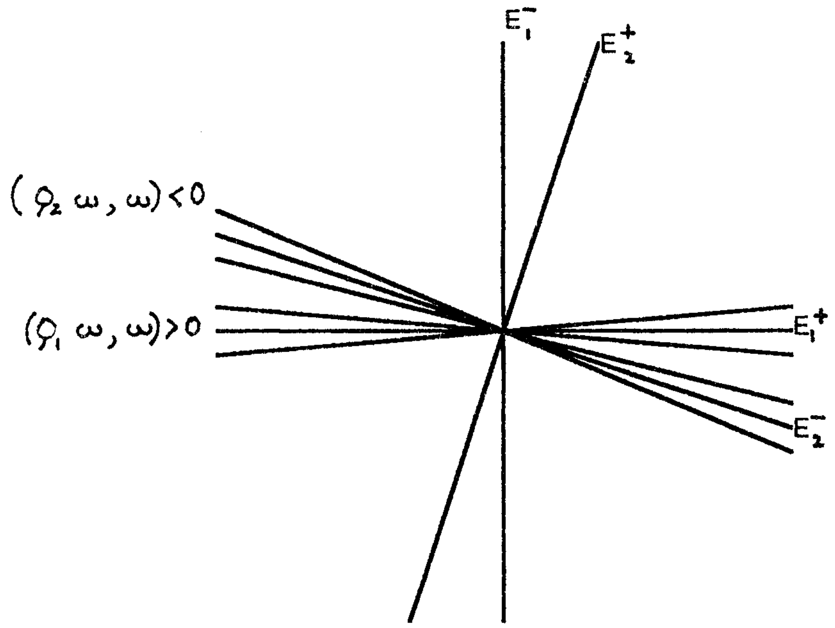
We eventually come to our first point :

is it true that solvability for $\frac{d}{dt} + Q(t)$ implies the same property for $\frac{d}{dt} + \alpha(t)\,Q(t)$,

where α is a non-negative scalar function ? This question is naturally linked with condition (ψ), since whenever $q(t,x,\xi)$ satisfies (1.1) so does $a(t,x,\xi)\,q(t,x,\xi)$ for a non-negative symbol a. We are thus quite naturally led to discuss the uniform solvability of $\frac{d}{dt} + Q(t)$, with

$$(1.5) \quad Q(t) = H(-t)\,Q_1 + H(t-\theta)\,Q_2\ , \quad Q_1 \leq Q_2\ , \text{(2 x 2) symmetric matrices}\ , \theta > 0.$$

The most remarkable fact for the pair of matrices $Q_1 \leq Q_2$ is the " DRIFT " : the best way to explain it is to look at the following picture (E_j^+ will stand also for its range) :



- figure 1 -

The condition $Q_1 \leq Q_2$ implies that the cones $\{\omega, (Q_2\,\omega, \omega) < 0\}$ and $\{\omega, (Q_1\,\omega, \omega) > 0\}$ are disjoint but does not prevent E_1^+ and E_2^- to get very close : let's define the drift for the pair Q_1, Q_2 as the absolute value of the cotangent of the angle between E_1^+ and E_2^-, so that

the drift is zero when $E_2^- \subset E_1^-$ and $E_1^+ \subset E_2^+$,
the drift is infinite when $E_1^+ \cap E_2^-$ is not reduced to zero ,
the drift is unbounded when the distance between the spheres of E_1^+ and E_2^- is zero.

It is easy to see that a bounded drift is equivalent to the invertibility of the non-negative operator $E_2^+ + E_1^-$ and this provides the "good" definition for the drift in infinite dimension.If we consider for instance the following pair of 2x2 symmetric matrices :

$$(1.6) \qquad Q_{1,\nu} = \begin{bmatrix} \nu^2 & 0 \\ 0 & -\nu^3 \end{bmatrix} \leq e^{-i\alpha_\nu} \begin{bmatrix} \nu^3 & 0 \\ 0 & -\nu \end{bmatrix} e^{i\alpha_\nu} = Q_{2,\nu}$$

where ν is a large positive parameter, $e^{i\alpha_\nu}$ the rotation of angle α_ν , with $\cos^2\alpha_\nu = 2/\nu$, the drift goes to infinity with ν , since

(1.7) the square of the distance between the spheres of E_1^+ and E_2^- is equivalent to $2/\nu$.

We now claim the non-uniform solvability of the operator $\frac{d}{dt} + Q(t)$,with $Q(t)$ given by (1.5), Q_1 and Q_2 by (1.6). We set up,with ω_1 and ω_2 unit vectors respectively in E_1^+ and E_2^- ,

$$
u(t) = \begin{cases} e^{tQ_1}\omega_1 & \text{on } t < 0 \\ \omega_1 + \frac{t}{\theta}(\omega_2 - \omega_1) & \text{on } 0 < t < \theta \\ e^{(t-\theta)Q_2}\omega_2 & \text{on } t > \theta \end{cases} \tag{1.8}
$$

Let's compute now

$$
\left\| \frac{du}{dt} - Q(t)\,u \right\|_{L^2}^2 = \int_0^\theta \frac{|\omega_2 - \omega_1|^2}{\theta^2}\,dt = \theta^{-1}|\omega_2 - \omega_1|^2 \quad , \tag{1.9}
$$

where $|\ |$ stands for the norm on $\mathbb{H}$. On the other hand, since ω_1 , ω_2 are unit vectors, we get

$$
\| u \|_{L^2}^2 \geq \int_0^\theta \left| \omega_1 + \frac{t}{\theta}(\omega_2 - \omega_1) \right|^2 dt \geq \frac{\theta}{2} - \theta\,|\omega_2 - \omega_1|^2 \geq \frac{\theta}{4} \tag{1.10}
$$

if $|\omega_2 - \omega_1|^2 \leq \frac{1}{4}$, an easily satisfied requirement subsequent to(1.7) . Consequently, using (1.9), (1.10), we get

$$
\| u \|_{L^2}^{-2} \left\| \frac{du}{dt} - Q(t)\,u \right\|_{L^2}^2 \leq \theta^{-2}|\omega_2 - \omega_1|^2\, 4 \quad . \tag{1.11}
$$

Since ω_1 and ω_2 can be chosen arbitrarily close and independently of the size of the "hole" θ, we get easily a non-solvable operator on $l^2(\mathbb{N})$ by taking direct sums. Note that (1.11) can be satisfied by a compactly supported u since the eigenvalues corresponding to ω_1 and ω_2 are going to infinity with ν, in such a way that there is no difficulty to multiply u by a cut-off function. In the next sections, we 'll say more about the drift of an ordered pair (Q_1 , Q_2) of selfadjoint operators ; it will turn out that the solvability of $\frac{d}{dt} + Q(t)$ will depend very closely upon the drift of the family $(Q(t))$, and that the solvability of all the operators $\frac{d}{dt} + \alpha(t)Q(t)$, when α is a non-negative scalar function, will require more or less that the drift for the family $Q(t)$ is bounded.

What we've done so far is to get an "abstract" non-solvable operator obtained by change of time-scale from a monotone increasing situation ; the basic device for the construction was the unbounded drift of $Q_1 \leq Q_2$. Since we are interested in pseudo-differential operators, the next question is obviously : is an unbounded drift possible for $Q_1 \leq Q_2$, both of them pseudo-differential ? We 'll see the answer is yes, leading to our counterexample. It is quite interesting to remark that operators satisfying condition (P) do not drift (in particular differential operators satisfying condition (ψ), equivalent to (P) in the differential case), as shown by the Beals-Fefferman reduction : after a non-homogeneous microlocalisation and canonical transformation, their procedure leads to an evolution operator

$$\frac{d}{dt} + Q(t), \qquad \text{with } Q(t) = Q(t,x,D_x) \qquad \text{and } Q(t,x,\xi) = \xi_1\, a(t,x,\xi),$$

where a is a non-negative symbol of order 0 (in a non-homogeneous class). Then, a Nirenberg-Treves commutator argument gives way to an estimate, after multiplication by the sign of ξ_1. Quite noticeable too, the fact that 2-dimensional pseudo-differential operators satisfying condition (ψ) do not drift, since the sign function is monotone matrix on operators whose symbols are defined on a lagrangean manifold ; the last remark led the author to a proof of local solvability in two dimensions [6] and for oblique-derivative type operators [7]. Our example (1.13) below shows that subelliptic operators can drift, but in a bounded way. We'll see that condition (ψ) prevents the drift to become infinite, but allows unbounded drifting. Throughout the paper, our definition of the Fourier transform will be,

$$(1.12) \qquad \hat{u}(\xi) = \int e^{-2i\pi x\xi} u(x)\, dx \quad \text{so that} \quad u = \check{\hat{u}} \quad \text{with } \check{u}(x) = u(-x).$$

Let's first study the very simple case

$$(1.13)\quad Q_1 = D_{x_1} = \frac{1}{2i\pi}\frac{\partial}{\partial x_1} \le Q_2 = D_{x_1} + \Lambda x_1^2 = e^{-2i\pi\frac{\Lambda x_1^3}{3}} D_{x_1} e^{2i\pi\frac{\Lambda x_1^3}{3}},$$

where Λ is a large positive parameter. Consider ω_1 a unit vector in E_1^+, i.e.

$$(1.14) \qquad \omega_1(x) = \int \kappa_1(\xi)\, e^{2i\pi x\xi} d\xi, \quad 1 = \|\kappa_1\|_{L^2}, \quad \operatorname{supp}\kappa_1 \subset \mathbb{R}^+,$$

and ω_2 a unit vector in E_2^-, i.e.

$$(1.15) \qquad \omega_2(x)\, e^{2i\pi\frac{\Lambda x^3}{3}} = \int \kappa_2(-\xi)\, e^{2i\pi x\xi} d\xi, \ 1 = \|\kappa_2\|_{L^2}, \ \operatorname{supp}\kappa_2 \subset \mathbb{R}^+.$$

A convenient way of estimating the drift of the pair (Q_1, Q_2) is to get an upper bound smaller than 1 for $|<\omega_1, \omega_2>_{L^2}|$: this quantity is 0 if the pair is not drifting, is 1 if the drift is infinite. We have

$$<\omega_1, \omega_2> = \int\int\int \kappa_1(\xi)\ e^{2i\pi x(\xi+\eta)}\ \bar{\kappa}_2(\eta)\ e^{2i\pi \frac{\Lambda x^3}{3}}\ dx\ d\eta\ d\xi\ ,$$

and thus,

$$(1.16)\qquad <\omega_1, \omega_2> = \int\int \kappa_1(\Lambda^{1/3}\xi)\Lambda^{1/6}\ \bar{\kappa}_2(\Lambda^{1/3}\eta)\Lambda^{1/6}\ A(\xi+\eta)\ d\eta\ d\xi,$$

where

$$(1.17)\qquad A(\xi) = \int e^{2i\pi \frac{x^3}{3}}\ e^{2i\pi x\xi}\ d\xi \quad \text{is the Airy function.}$$

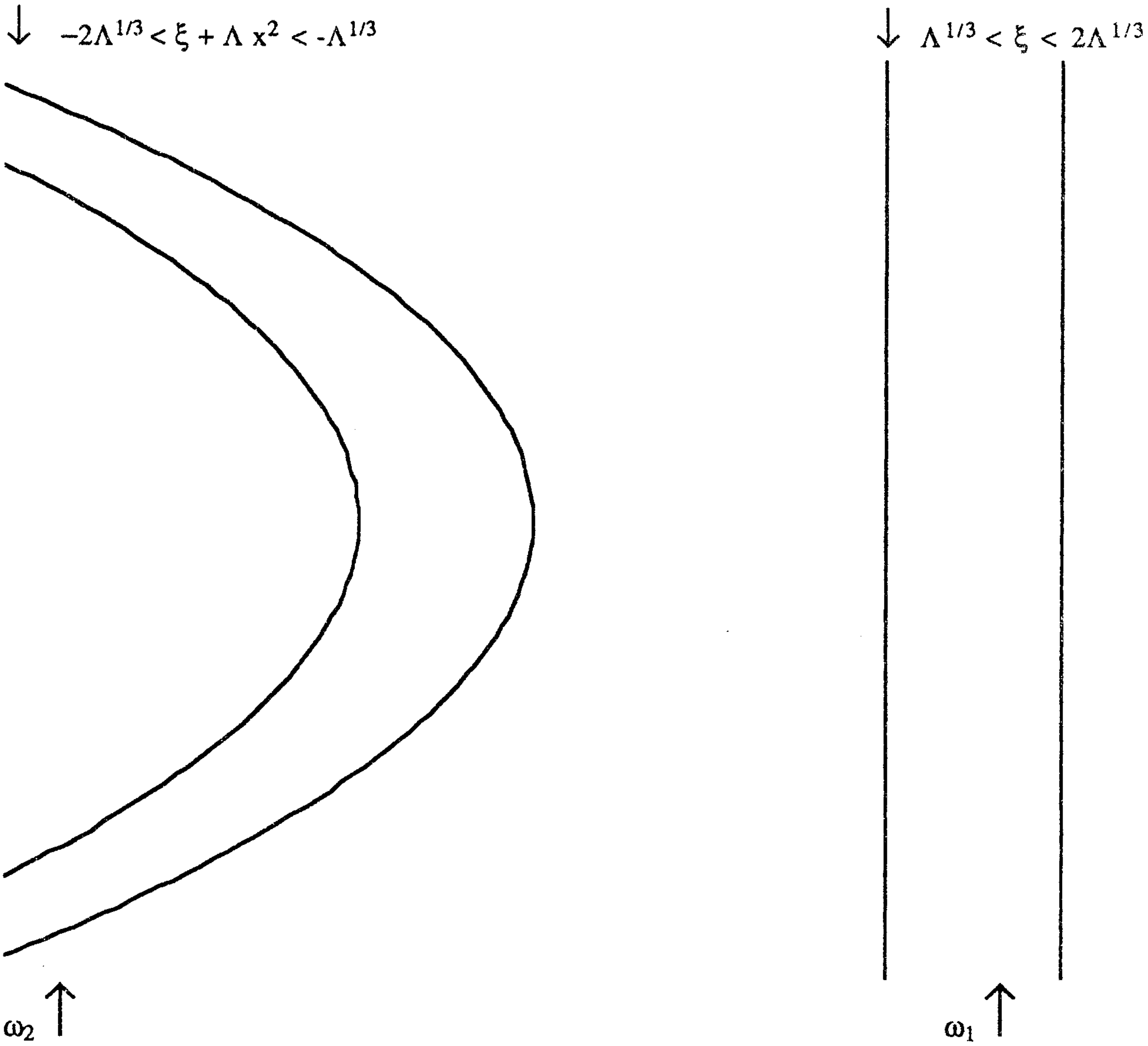
Consequently,

$$(1.18)\qquad \sup_{\omega_1\in E_1^+, \omega_2\in E_2^-, \|\omega_1\|=\|\omega_2\|=1} |<\omega_1, \omega_2>_{L^2}| > 0\ ,$$

since for $\kappa_1(\xi)=\bar{\kappa}_2(\xi)=\kappa(\xi\Lambda^{-1/3})\Lambda^{-1/6}$ with a non-negative κ, supported in the interval [1,2], $1=\|\kappa\|_{L^2}$, (1.16) gives

$$(1.19)\qquad <\omega_1, \omega_2> = \int\int \kappa(\xi)\kappa(\eta)\ A(\xi+\eta)\ d\eta\ d\xi = \int A(\xi)(\kappa*\kappa)(\xi)d\xi\ ,$$

the last term is a positive constant, independent of Λ (the Airy function given by (1.15) is positive on $\mathbb{R}^+$). A picture will be useful for the understanding of these inequalities :



- figure 2 -

this picture in the (x_1,ξ_1) symplectic plane shows that even though ω_1 and ω_2 are "living" in two far away strips, one of which is a curved one, their dot product could be large. If we add one dimension to get an homogeneous version, Λ would be $|\xi_1| + |\xi_2|$, so that the above localisations in the phase space appear as two different second microlocalisations with respect to the hypersurfaces $\{ \xi_1 = 0 \}$ on the one hand, $\{ \xi_1 + |\xi| x_1^2 = 0 \}$ on the other hand (see [2], [3]). These second microlocalisations are somehow incompatible so that the long range interaction between two far away boxes corresponding to two different calculus could be large, as shown by the equality (1.19).

However, the pair given by (1.12) has a bounded drift, i.e. the quantity (1.18) is bounded above by a number strictly smaller than 1. This implies the solvability of

$$(1.20)\qquad \frac{d}{dt} + \alpha(t,x,D_x)\Big[\ H(-t)D_{x_1} + H(t)\ (\ D_{x_1} + x_1^2\sqrt{D_{x_1}^2 + D_{x_2}^2}\ \Big]\ ,$$

where $\alpha(t,x,\xi)$ is a non-negative symbol of order zero, flat at $t = 0$. Since we are not going to use this result, we leave its proof to the reader with an hint : compute the real parts of

$$< D_t u + iQ(t)u\ ,\ i\,H(t-T)E_2^+ u\ >\ ,\quad < D_t u + iQ(t)u\ ,\ -i\,H(T-t)H(t)E_2^- u\ >,\quad \text{for non-negative } T,$$

$$< D_t u + iQ(t)u\ ,\ -\,i\,H(T-t)E_1^- u\ >\ ,\quad < D_t u + iQ(t)u\ ,\ i\,H(t-T)H(-t)E_1^+ u >,\quad \text{for non-positive } T\,,$$

use the bounded drift , meaning $E_2^+ + E_1^-$ invertible, and the Nirenberg-Treves commutator argument (see e.g. lemma 26.8.2 in [5]).

We now start over our discussion on pseudo-differential operators and study the following case, which turns out to be the generic one, using the microlocalisation procedure of [1] and the Egorov principle of [4] :

$$(1.21)\quad Q_1 = D_{x_1} = \frac{1}{2i\pi}\frac{\partial}{\partial x_1} \ \le\ Q_2 = D_{x_1} + V(x_1) = e^{-2i\pi\phi(x_1)}\, D_{x_1}\, e^{2i\pi\phi(x_1)}\ ,$$

with a non-negative $V = \phi'$. Following the lines of the computations starting at (1.12), the question at hand is to estimate from above

$$(1.22)\qquad <\omega_1\,,\omega_2> = \int\!\!\int\!\!\int \kappa_1(\xi)\ e^{2i\pi x(\xi+\eta)}\ \bar{\kappa}_2(\eta)\, e^{2i\pi\phi(x)}\ dx\, d\eta\, d\xi\ ,$$

with

$$(1.23)\qquad 1 = \|\ \kappa_1\ \|_{L^2} = \|\ \kappa_2\ \|_{L^2}\ ,\quad \text{supp}\ \kappa_1 \subset \mathbb{R}^+\ ,\ \text{supp}\ \kappa_2 \subset \mathbb{R}^+\ .$$

This means estimating from above the $\mathcal{L}(L^2)$norm of the product $\Pi = H(-D_x)\ e^{2i\pi\phi}\ H(D_x)$, where $H(D_x)$ is the Fourier multiplier by the Heaviside function H. If ϕ is $\frac{1}{2}$ H(x), then

$$(1.24)\qquad i\,\Pi = i\,H(-D_x)\,[\ -H(x) + H(-x)\ ]\,H(D_x) = F\,\Omega\,F,$$

where F is the Fourier transform, and Ω the Hardy operator,whose kernel is $H(\xi)H(\eta)/\pi(\xi+\eta)$ (the norm of Ω is obviously ≤ 1 from (1.24)). It is not difficult to see that the norm of the Hardy operator is exactly 1, as shown in section 4. As a consequence, we get an unbounded drift for the pair (D_{x_1} ; $D_{x_1} + \frac{1}{2}\delta(x_1)$), at least in a formal way ; we will approximate the Dirac mass by a sequence of smooth functions $\frac{1}{2}\nu W(\nu x_1)$, where W is non-negative with integral 1, and in order to get a symbol , we'll perform this approximation at the frequencies equivalent to 2^ν. Moreover, we shall choose carefully the size and the regularization of the "hole" θ depending on this frequency. The paper is organised as follows :

2. Statement of the results

It is convenient for our exposition to give first a result for a non-homogeneous $S^1_{1,0}$ symbol :

Theorem 2.1

There exists a real valued symbol $q = q(t,x,\xi)$ in the $S^1_{1,0}$ class on $\mathbb{R}_t \times \mathbb{R}^2_x \times \mathbb{R}^2_\xi$, i.e. a smooth function q satisfying (1.2), for which condition (1.1) is fulfilled, such that there is no neighborhood Ω of the origin in $\mathbb{R}_t \times \mathbb{R}^2_x$, so that the equation

$$(2.1)\qquad \frac{\partial v}{\partial t} + q(t,x,D_x)v = f$$

has an $L^2_{loc}(\Omega)$ solution v for any f in $\mathcal{D}(\Omega)$. There is a sequence u_ν of functions in $C^\infty_0(\mathbb{R}^3)$, with L^2 norm 1 and support $u_\nu \to \{0\}$ such that

$$(2.2)\qquad \left\| \frac{1}{i}\frac{\partial u_\nu}{\partial t} + i\, q(t,x,D_x)u_\nu \right\|_{L^2(\mathbb{R}^3)} \to 0 \quad \text{when} \quad \nu \to +\infty \quad .$$

We state now our main result :

Theorem 2.2

There exists a principal-type classical pseudo-differential operator P of order 1, in three dimensions, with an homogeneous principal symbol p satisfying condition (ψ) such that the equation $Pu = f$ has no L^2 local solutions for f in L^2.

3. A non- homogeneous operator Q(t) in the $S^1_{1,0}$ class

We set , for $x \in \mathbb{R}^2$, $\xi \in \mathbb{R}^2$, $t \in \mathbb{R}$,

$$(3.1)\qquad Q(t,x,\xi) = \sum_{\nu=2}^{+\infty} \psi^2(2^{-2\nu}(\xi_1^2+\xi_2^2))\,\chi^2(\xi_1/\xi_2)\,\alpha_\nu(t)\,q_\nu(t,x_1,\xi_1) \qquad ,$$

so that

$$(3.2)\qquad \psi \in C^\infty_0(\mathbb{R}),\ \text{supp}\,\psi \subset [2^{-1}, 2],\ \psi = 1 \text{ on } [2^{-1/2}, 2^{1/2}],\ 0 \le \psi \le 1 ,$$

$$(3.3)\qquad \chi \in C^\infty_0(\mathbb{R}),\ \text{supp}\,\chi \subset [-1, +1],\ \chi = 1 \text{ on } [-1/2, 1/2] \quad , 0 \le \chi \le 1 ,$$

(3.4) $\alpha_\nu(t) = H(-t)\beta(t/\theta_\nu)\lambda_{2,\nu}^{-1} + H(t-\theta_\nu)\,\alpha(t/\theta_\nu)\,\lambda_{2,\nu}^{-1}$, with

(3.5) $\theta_\nu = \lambda_{3,\nu}^{-1}$, $e^{\lambda_{3,\nu}} \le \lambda_{2,\nu}$, ($\lambda_{3,\nu}$, $\lambda_{2,\nu}$ are positive parameters),

(3.6)
$\beta \in C^\infty(\mathbb{R})$, supp $\beta = (-\infty, 0]$, $\beta > 0$ on $(-\infty, 0)$, $\beta \equiv 1$ on $(-\infty, -2)$,
$\alpha \in C^\infty(\mathbb{R})$, supp $\alpha = [1, +\infty)$, $\alpha > 0$ on $(1, +\infty)$, $\alpha \equiv 1$ on $(2, +\infty)$,
α and β bounded as well as all their derivatives.

Moreover, we set-up

(3.7) $q_\nu(t,x_1,\xi_1) = H(-t)\xi_1 + H(t-\theta_\nu)\,[\,\xi_1 + \frac{1}{2}\lambda_{1,\nu}W(\lambda_{1,\nu}x_1)\,]$, where

(3.8) $2^{\lambda_{1,\nu}} \le \lambda_{0,\nu} = 2^\nu$ and

(3.9) $W \in C_0^\infty(-\frac{1}{2}, +\frac{1}{2})$, $W \ge 0$, $\int W(x)\,dx = 1$.

Lemma 3.1
The function Q defined by (3.1) $\in S^{1}_{1,0}$ *and satisfies (1.1).*

We'll begin proving Q is a smooth function. Let's remark that the open rings

(3.10) $\Delta_\nu = \{\,(\xi_1, \xi_2) \in \mathbb{R}^2\ ,\ 2^{-1} < 2^{-2\nu}(\xi_1^2 + \xi_2^2) < 2\,\}$

are disjoint when ν runs through the integers, and that

(3.11) $2^{\nu-1} < |\xi_2| < 2^{\nu+1/2}$ on $\Delta_\nu \cap$ supp Q .

It is thus enough to check

$$\alpha_\nu(t)\,q_\nu(t,x_1,\xi_1) = \alpha_\nu(t)\left\{H(-t)\xi_1 + H(t-\theta_\nu)\,[\,\xi_1 + \tfrac{1}{2}\lambda_{1,\nu}W(\lambda_{1,\nu}x_1)\,]\right\}$$

which is a smooth function since α_ν is C^∞ and zero on $[0, \theta_\nu]$. To get (1.2), we must verify

(3.12) $$|(D_t^k D_{x_1}^{\alpha_1} D_{x_2}^{\alpha_2} D_{\xi_1}^{\beta_1} D_{\xi_2}^{\beta_2} Q)(t,x_1,x_2,\xi_1,\xi_2)| \leq 2^{\nu(1-|\beta_1|-|\beta_2|)} C_{k\alpha_1\alpha_2\beta_1\beta_2} ,$$

for $(\xi_1, \xi_2) \in \Delta_\nu$, where the constants C do not depend on ν. We may assume $\beta_1 \in \{0,1\}$ and $\beta_2 = 0 = \alpha_2$. Since

$$D_t^k D_{x_1}^{\alpha_1} Q = D_t^k D_{x_1}^{\alpha_1} \left\{ \beta(t/\theta_\nu)\lambda_{2,\nu}^{-1}\, \xi_1 + \alpha(t/\theta_\nu)\lambda_{2,\nu}^{-1} [\, \xi_1 + \tfrac{1}{2}\lambda_{1,\nu} W(\lambda_{1,\nu} x_1)] \right\},$$

we get for $\beta_1 = 0$,

(3.13) $$|D_t^k D_{x_1}^{\alpha_1} Q| \leq 2^{\nu+1/2} \|\beta^{(k)}\|_{L^\infty} \lambda_{3,\nu}^k \lambda_{2,\nu}^{-1} + \|\alpha^{(k)}\|_{L^\infty} \lambda_{3,\nu}^k \lambda_{2,\nu}^{-1} [\, 2^{\nu+1/2} + \tfrac{1}{2} \lambda_{1,\nu}^{1+\alpha_1} \| W^{(\alpha_1)}\|_{L^\infty}] .$$

Moreover, we have for $\beta_1 = 1$,

(3.14) $$|D_t^k D_{x_1}^{\alpha_1} D_{\xi_1} Q| \leq (\|\beta^{(k)}\|_{L^\infty} + \|\alpha^{(k)}\|_{L^\infty}) \lambda_{3,\nu}^k \lambda_{2,\nu}^{-1} ,$$

so that (3.5), (3.8) , (3.13) and (3.14) give (3.12). We need to prove(1.1) : assume $Q(t,x,\xi) > 0$ and $s > t$. Since ξ belongs at most to one Δ_ν , we know $Q(t,x,\xi)$(resp. $Q(s,x,\xi)$) must be the product of a positive quantity by $\alpha_\nu(t)\, q_\nu(t,x_1,\xi_1)$ (resp. $\alpha_\nu(s)\, q_\nu(s,x_1,\xi_1)$). In fact, the function α_ν given by (3.4) is non-negative, and $t \to q_\nu(t, x_1,\xi_1)$ is non-decreasing(from $W \geq 0$ in (3.9)). This concludes the proof of lemma 3.1 .

4. Drift of operators

Following the heuristic discussion in section 1 about the drift ,we consider first the Hardy operator .

Lemma 4.1

If $\mathcal{H}$ is the Hilbert transform, i.e. the convolution with $pv(1/i\pi\xi)$, E_+ (resp E_-) the projector defined by $(E_+\kappa)(\xi) = H(\xi)\kappa(\xi)$ (resp$(E_-\kappa)(\xi) = H(-\xi)\kappa(\xi)$), where H is the characteristic function of $\mathbb{R}^+$, the Hardy operator Ω is $i\,E_+\mathcal{H}\,E_-\,C$, where $(C\kappa)(\xi) = \kappa(-\xi)$.The $\mathcal{L}(L^2)$norm of Ω is 1 and its kernel is $H(\xi)H(\eta)/\pi(\xi+\eta)$. *Set-up, for $\varepsilon > 0$,*

(4.1) $\kappa_\varepsilon(\xi) = \Gamma(\varepsilon)^{-1/2} e^{-\xi/2} \xi^{(-1+\varepsilon)/2} H(\xi)$ *, where Γ stands for the gamma function. We have*

(4.2) $1 > (\Omega\kappa_\varepsilon, \kappa_\varepsilon)_{L^2} > 1 - \varepsilon$.

Proof. It is pure routine to check that the kernel of $\Omega = i\,E_+\mathcal{H}\,E_-C$ is $H(\xi)H(\eta)/\pi(\xi+\eta)$ and this factorisation implies readily that the $\mathcal{L}(L^2)$norm of Ω is smaller than 1.It is thus enough to prove (4.2): from the change of variables $t = (\xi+\eta)/2$, $t\sin\theta = (\xi-\eta)/2$, we get

(4.3) $$(\Omega\kappa_\varepsilon, \kappa_\varepsilon)_{L^2} = \frac{2}{\pi}\int_0^{\pi/2} \cos^\varepsilon\theta\ d\theta \quad ,$$

which satisfies (4.2) for $\varepsilon > 0$.

We consider now

(4.4) $$\phi(x) = \frac{1}{2}\int_{-\infty}^{x} W(t)\,dt \quad , \text{ with W as in (3.9) .}$$

We set, with κ_ε given by (4.1) , $J = [\delta, 1]$, 1_J the characteristic function of J ,

(4.5) $$\Omega_W(\varepsilon,\delta) = \iiint e^{2i\pi x(\xi+\eta)}\, e^{2i\pi\phi(x)}\, \kappa_\varepsilon(\xi)\, 1_J(\xi) i\, \kappa_\varepsilon(\eta)\, 1_J(\eta) dx\, d\eta\, d\xi \ .$$

Lemma 4.2

There exists a constant C_o, such that for all positive numbers ε, δ satisfying $0 < \varepsilon \le 1/2$, $0 < \delta \le \varepsilon^{2\varepsilon^{-1}}$ and any function W as in (3.9), $\mathbb{R}e\ \Omega_W(\varepsilon,\delta) \ge (1 - C_o\varepsilon)$.

Proof. Noting first that $e^{2i\pi\phi} = -\,\text{sign} + (\,\text{sign} + e^{2i\pi\phi}\,)\,1_{(-1/2,1/2)}$, we get, using (1.12) and lemma 4.1, (4.6) $\Omega_W(\varepsilon,\delta) = (\,\Omega\, 1_J\, \kappa_\varepsilon\,,\, 1_J\, \kappa_\varepsilon\,)_{L^2} +$

$$\iint \kappa_\varepsilon(\xi)\, 1_J(\xi)\; i\; \kappa_\varepsilon(\eta)\, 1_J(\eta) \int_{-1/2}^{1/2} (\,\text{sign}\, x + e^{2i\pi\phi(x)}\,)\, e^{2i\pi x(\xi+\eta)} dx \;\; d\xi d\eta\ .$$

To evaluate the first term in the right-hand side of (4.6), we remark

(4.7) $(\,\Omega\, 1_J\, \kappa_\varepsilon\,,\, 1_J\, \kappa_\varepsilon\,)_{L^2} = (\,\Omega\kappa_\varepsilon\,,\, \kappa_\varepsilon\,)_{L^2} + R(\varepsilon, \delta)$ with

(4.8) $R(\varepsilon, \delta) = \dfrac{-1}{\pi\Gamma(\varepsilon)} \displaystyle\iint_A (\xi+\eta)^{-1}\, e^{-(\xi+\eta)/2}\, \xi^{(\varepsilon-1)/2}\, \eta^{(\varepsilon-1)/2}\, d\xi d\eta$, where

(4.9) $A = A_1 \cup A_2 \cup A_3 \cup A_4$ with

(4.10)
$$A_1 = \{\ 0 < \xi < \delta\ ,\ 0 < \eta\ \} \quad A_2 = \{\ 1 < \xi\ ,\ 0 < \eta\ \}$$
$$A_3 = \{\ \delta < \xi < 1,\ \eta < \delta\ \} \quad A_4 = \{\delta < \xi < 1\ ,\ 1 < \eta\,\}\ .$$

We estimate

(4.11) $$R_1(\varepsilon, \delta) = \frac{1}{\pi\Gamma(\varepsilon)} \iint_{A_1} (\xi+\eta)^{-1}\, e^{-(\xi+\eta)/2}\, \xi^{(\varepsilon-1)/2}\, \eta^{(\varepsilon-1)/2}\, d\xi d\eta\ .$$

We set $R_1(\varepsilon, \delta) = R_{11} + R_{12}$ with $(\, x = \xi+\eta\ ,\ \ y = (\xi-\eta)/2\ )$,

(4.12) $$R_{11} = \frac{1}{\pi\Gamma(\varepsilon)} \int_0^\delta x^{-1} e^{-x/2}\, 2 \int_0^{x/2} [\,\frac{x^2}{4} - y^2\,]^{(\varepsilon-1)/2}\, dy\, dx \ \ \le\ \ \delta^\varepsilon / (\Gamma(1+\varepsilon)\, 2^\varepsilon)\ ,$$

(4.13) $$R_{12} = \frac{1}{\pi\Gamma(\varepsilon)} \int_\delta^{+\infty} x^{-1} e^{-x/2} \int_{-x/2}^{\delta - x/2} [\,\frac{x^2}{4} - y^2\,]^{(\varepsilon-1)/2}\, dy\, dx\ .$$

The inequality (4.12) is obtained as (4.3) setting $y = (x/2)\sin\theta$. We have moreover

(4.14) $$R_{12} \le \frac{1}{\pi\Gamma(\varepsilon)} \int_{\delta}^{+\infty} t^{-1} e^{-t} \; [\mathrm{Arcsin}(y/t)]_{y=t-\delta}^{y=t} \; (t^2 - (t-\delta)^2)^{\varepsilon/2} \, dt$$

$$+ \frac{1}{\pi\Gamma(\varepsilon)} \int_{\delta/2}^{\delta} t^{\varepsilon-1} \; [\mathrm{Arcsin}(y/t)]_{y=t-\delta}^{y=t} \, dt \quad .$$

To prove (4.14), we first change the variables $x = 2t$, $y = -y'$ in (4.13), drop the ' later on, and estimate from above $(t^2 - y^2)^{\varepsilon/2}$: this quantity can be estimated from above by $(t^2 - (t-\delta)^2)^{\varepsilon/2}$ whenever $t \ge y \ge t - \delta \ge 0$ and by t^{ε} if $y \in (t-\delta, t)$ and $t \in (\delta/2, \delta)$. Eventually, one gets from (4.14)

(4.15) $$R_{12} \le \frac{1}{\pi\Gamma(\varepsilon)} \{\delta^{\varepsilon/2} \, \Gamma(\varepsilon/2) \, 2^{\varepsilon/2} + \varepsilon^{-1} \delta^{\varepsilon} \} \frac{\pi}{2} \le C_1 \delta^{\varepsilon/2} \quad ,$$

where C_1 is an absolute constant . Consequently, we have from (4.15), (4.12)(see (4.11)),

(4.16) $$R_1(\varepsilon, \delta) \le C_2 \; \delta^{\varepsilon/2}$$, where C_2 is an absolute constant.

We set , with A_2 defined in (4.10),

(4.17) $$R_2(\varepsilon, \delta) = \frac{1}{\pi\Gamma(\varepsilon)} \iint_{A_2} (\xi+\eta)^{-1} \; e^{-(\xi+\eta)/2} \; \xi^{(\varepsilon-1)/2} \, \eta^{(\varepsilon-1)/2} \, d\xi d\eta \, .$$

We have on A_2 , $(\xi+\eta)^{-1} \le \xi^{-1}$, $e^{-(\xi+\eta)/2} \le e^{-\eta/2}$, so we get

(4.18) $$R_2(\varepsilon, \delta) \le C_3 \; \frac{\varepsilon}{1-\varepsilon} \quad ,$$ where C_3 is an absolute constant.

Consequently, the inequalities (4.16), (4.18) and their analogues for the integrals in (4.8) over A_3 (smaller than over A_1) and A_4 (smaller than over A_2) give from (4.8) the existence of an absolute constant C_4 , such that, for any $\varepsilon \in (0, 1/2]$, $\delta \in (0, \varepsilon^{2/\varepsilon}]$,

(4.19) $$| R(\varepsilon, \delta) | \le C_4 \, \varepsilon \quad .$$

We need now to check the second term in the right - hand side of (4.6), namely

(4.20) $S(\varepsilon, \delta) = \int\int \kappa_\varepsilon(\xi)\, 1_J(\xi) i\, \kappa_\varepsilon(\eta)\, 1_J(\eta) \int_{-1/2}^{1/2} (\operatorname{sign} x + e^{2i\pi\phi(x)})\, e^{2i\pi x(\xi+\eta)} dx d\xi d\eta.$

We obtain, from (4.1),

(4.21) $| S(\varepsilon, \delta) | \leq \Gamma(\varepsilon)^{-1}\, \Gamma((\varepsilon + 1)/2)^2\, 2^{\varepsilon+2} \leq C_5\, \varepsilon$, where C_5 is an absolute constant.

Finally, we obtain the result of the lemma 4.2, collecting the inequalities (4.21), (4.19),(4.2), and the equalities (4.6),(4.7),(4.20).

We consider now, with the notations of lemmas 4.1 and 4.2, for positive μ and λ,

(4.22) $\omega_1(x_1) = \int e^{2i\pi x_1 \xi}\, \kappa_\varepsilon(\xi/\mu)\, 1_J(\xi/\mu) \mu^{-1/2} d\xi$,

(4.23) $\omega_2(x_1) = -i\, e^{-2i\pi\phi(\lambda x_1)} \int e^{2i\pi x_1 \eta}\, \kappa_\varepsilon(-\eta/\mu)\, 1_J(-\eta/\mu) \mu^{-1/2} d\eta$, so that

(4.24) $(\omega_1, \omega_2)_{L^2} =$

$$\int\int\int e^{2i\pi x_1 \xi} \kappa_\varepsilon(\xi/\mu)\, 1_J(\xi/\mu) \mu^{-1/2}\, i\, e^{2i\pi\phi(\lambda x_1)} e^{2i\pi x_1 \eta}\, \kappa_\varepsilon(\eta/\mu)\, 1_J(\eta/\mu) \mu^{-1/2} d\xi d\eta\, dx_1$$

$$= \int\int\int e^{2i\pi x_1(\xi+\eta)}\, e^{2i\pi\phi(\lambda\mu^{-1} x_1)} \kappa_\varepsilon(\xi)\, 1_J(\xi)\; i\kappa_\varepsilon(\eta)\, 1_J(\eta)\, d\xi d\eta\, dx_1$$

$= \Omega_{W_{\lambda\mu^{-1}}}(\varepsilon,\delta)$, as given by (4.5), with $W_{\lambda\mu^{-1}}(x) = \lambda\mu^{-1}\, W(\lambda\mu^{-1} x)$ and W given by (3.9).

We obtain the following

Lemma 4.3

There exists a constant C_o, such that for all positive numbers ε, δ satisfying $0 < \varepsilon \leq 1/2$, $0 < \delta \leq \varepsilon^{2\varepsilon^{-1}}$, all functions W and ϕ as in (4.4), all positive numbers λ, μ so that $\lambda\mu^{-1} \geq 1$, all functions ω_1 and ω_2 given by (4.22) and (4.23),

(4.26) $\mathbb{R}e\, (\omega_{10}, \omega_{20})_{L^2} \geq 1 - C_o\, \varepsilon$,

where $\omega_{j0} = \omega_j / \|\omega_j\|_{L^2}$, $j = 1, 2$.

Proof. The equality (4.24) , $\lambda\mu^{-1} \geq 1$, lemma 4.2 and $\|\omega_j\|_{L^2} \leq 1$ give the result.

Let

$$(4.27)\quad \left|\begin{array}{l} \lambda_{o,\nu} = 2^{\nu} \ , \ \lambda_{1,\nu} = \nu \ , \ \lambda_{2,\nu} = \mathrm{Log}\nu \ , \ \lambda_{3,\nu} = (\mathrm{Log}(\mathrm{Log}\nu))^{1/8} \\ \varepsilon_\nu = (\mathrm{Log}(\mathrm{Log}\nu))^{-1/2} \ , \delta = (\mathrm{Log}\nu)^{-1} , \ \theta_\nu = (\mathrm{Log}(\mathrm{Log}\nu))^{-1/8} . \end{array}\right.$$

be a choice of parameters satisfying (3.5), (3.8), and the conditions in lemma 4.2 .We consider the operator

$$(4.28)\qquad Q_\nu(t) = \beta(t/\theta_\nu)\,(\mathrm{Log}\nu)^{-1}\ H(-t)D_1 + \alpha(t/\theta_\nu)\,(\mathrm{Log}\nu)^{-1}\ H(t-\theta_\nu)[\ D_1 + \tfrac{1}{2}\nu\ W(\nu\ x_1)]$$

where α and β satisfy (3.6), $D_1 = \frac{1}{2i\pi}\frac{\partial}{\partial x_1}$, the function W satisfies (3.9). We have

$$(4.29)\qquad Q_\nu(t) = \alpha_\nu(t)\ (\ H(-t)\ Q_1 + H(t-\theta_\nu)\ Q_2^{(\nu)}\ ) \ , \qquad \text{with}$$

$$(4.30)\qquad \alpha_\nu(t) = \beta(t/\theta_\nu)\,(\mathrm{Log}\nu)^{-1} H(-t) + \alpha(t/\theta_\nu)\,(\mathrm{Log}\nu)^{-1}\ H(t-\theta_\nu) \ ,$$

$$(4.31)\qquad Q_1 = D_1 \qquad , \qquad Q_2^{(\nu)} = D_1 + \tfrac{1}{2}\nu\ W(\nu\ x_1) \ .$$

We define

$$(4.32)\qquad \omega_1^{(\nu)}(x_1) = \int e^{2i\pi x_1\xi}\,\kappa_\varepsilon(\xi/\mu)\,1_J(\xi/\mu)\mu^{-1/2}d\xi \quad \|\kappa_\varepsilon 1_J\|_{L^2}^{-1} \qquad , \text{with}$$

$$(4.33)\qquad \mu = \nu \qquad , \qquad J = [\delta\ ,\ 1] \qquad ,$$

$$(4.34)\qquad \omega_2^{(\nu)}(x_1) = -i\,e^{-2i\pi\phi(\nu x_1)} \int e^{2i\pi x_1\eta}\,\kappa_\varepsilon(-\eta/\mu)\,1_J(-\eta/\mu)\mu^{-1/2}d\eta \ \|\kappa_\varepsilon 1_J\|_{L^2}^{-1} \quad ,$$

with ϕ defined in (4.4).We can state now

Lemma 4.4

There exist a constant C_o and an integer ν_o such that, if ν is larger than ν_o , and μ , ε , J , δ , $\omega_1^{(\nu)}$, $\omega_2^{(\nu)}$ are given by (4.32-34), (4.27), we have

(4.35) $\qquad \mathrm{Re}\,(\ \omega_1^{(\nu)}, \omega_2^{(\nu)}\ )_{L^2} \ \geq\ 1 - \varepsilon_\nu C_o$ *, and $\omega_1^{(\nu)}$, $\omega_2^{(\nu)}$ are unit vectors in L^2 .*

Moreover, these functions have the following spectral properties :

$$(4.36)\qquad \left|\begin{array}{l} spec\ \omega_1^{(v)} \subset [\, v\,(Log v)^{-1}\ ,\ v\ ], \text{with respect to } Q_1 \\ spec\ \omega_2^{(v)} \subset [\,-v\ ,\ -v\,(Log v)^{-1}\ ], \text{with respect to } Q_2^{(v)} \end{array}\right. .$$

Finally,

$$(4.37)\qquad \varepsilon_v \theta_v^{-2} = \varepsilon_v^{1/2}$$

Proof. The inequality (4.35) is a reformulation of lemma 4.3, the spectral properties (4.36) are obvious on formulas (4.32) and (4.34) and (4.37)follows from (4.27) .

We define

$$(4.38)\qquad \Omega_1^{(v)}(x_1, x_2) = \omega_1^{(v)}(x_1)\,\rho_v(x_2) \quad , \qquad \Omega_2^{(v)}(x_1, x_2) = \omega_2^{(v)}(x_1)\rho_v(x_2)\ ,$$

where ρ_v is a function with norm 1 in $L^2(\mathbb{R})$ such that

$$(4.39)\qquad \text{support } \hat{\rho}_v \subset [\ 2^{v-1/8}\ ,\ 2^{v+1/8}\ ]\quad .$$

We set-up , with α_v , Q_1 , $Q_2^{(v)}$, $\Omega_1^{(v)}$, $\Omega_2^{(v)}$,θ_v as above in (4.30), (4.31), (4.38), (4.27), $\chi_v = \chi\,(\,2^{-v+2}\,D_1)$, χ given in (3.3),

(4.39)' σ is a function in $C^\infty(\mathbb{R},[\,0,\,1]\,)$, $\sigma = 0$ on $(-\infty,\ 1/3\,)$, $\sigma = 1$ on $(\,2/3\ ,+\infty)$, $\| \sigma' \|_{L^2}^2 \le 4$,

$$(4.40)\qquad u_v(t) = \begin{cases} \exp\left[\displaystyle\int_0^t \alpha_v(s)ds\ Q_1\right] \Omega_1^{(v)} & , \text{on}\quad t < 0\ , \\ \Omega_1^{(v)} + \sigma(\tfrac{t}{\theta_v})\,(\,\chi_v\,\Omega_2^{(v)} - \Omega_1^{(v)}\,) & , \text{on}\quad 0 < t < \theta_v\ , \\ \chi_v \exp\left[\displaystyle\int_{\theta_v}^t \alpha_v(s)ds\ Q_2^{(v)}\right] \Omega_2^{(v)} & , \text{on}\quad \theta_v < t\ . \end{cases}$$

Lemma 4.5
The function $u_v(t)$ defined in (4.40) belongs to $L^2(\mathbb{R}^2_{x_1,x_2})$

$$(4.41)\quad spectrum(u_v(t)) \subset \tilde{\Delta}_v = \{(\xi_1\,,\,\xi_2) \in \mathbb{R}^2\ ,\ 2^{2v-1/2} \le \xi_1^2 + \xi_2^2 \le 2^{2v+1/2} \text{ and } 2\,|\xi_1| \le |\xi_2|\,\}$$

(the spectrum is the support of the Fourier transform in $\mathbb{R}^2_{\xi_1,\xi_2}$). Moreover, with $|\ |$ standing for the $L^2(\mathbb{R}^2_{x_1x_2})$ norm, we have, with α, β given in (3.6),

(4.42) $$|u_\nu(t)|^2 \leq \exp\{-\int_{t/\theta_\nu}^{0} 2\beta(s)ds\ \theta_\nu\, \nu\, (\mathrm{Log}\nu)^{-2}\}\ , \qquad \text{if } t<0\ ,$$

(4.43) $$1-2\varepsilon_\nu C_0 \leq |u_\nu(t)|^2 \leq 1\ , \text{ if } 0<t<\theta_\nu\ ,$$ where C_0 is given in lemma 4.4,

(4.44) $$\left|\exp\left[\int_{\theta_\nu}^{t}\alpha_\nu(s)ds\ Q_2^{(\nu)}\right]\Omega_2^{(\nu)}\right|^2 \leq \exp\{-\int_1^{t/\theta_\nu} 2\alpha(s)ds\,\theta_\nu\, \nu\, (\mathrm{Log}\nu)^{-2}\},\quad \text{if } t>\theta_\nu .$$

Moreover, if $a_\nu = a_\nu(\xi)$belongs uniformly to $S(1, |d\xi|^2 2^{-2\nu})$ (using Hörmander's notation (18.4.6) in [5]), the commutator $[\,Op(a_\nu), Q_2^{(\nu)}\,]$ is L^2 bounded(Op(a) stands for the operator with symbol a) and, for $\nu > \nu_0$,

(4.45) $$\|\,[\,Op(a_\nu), Q_2^{(\nu)}\,]\,\|_{\mathcal{L}(L^2(\mathbb{R}))} \leq 2^{-\nu/2}\quad .$$

Proof. From (4.38), (4.39) and (4.36) we get that the Fourier transform of $\Omega_1^{(\nu)}$ is supported in the rectangle

$$\nu\,(\mathrm{Log}\nu)^{-1} \leq \xi_1 \leq \nu \qquad .\ 2^{\nu-1/8} \leq \xi_2 \leq 2^{\nu+1/8} .$$

Consequently, on $t<0$, from (4.31), (4.30), (4.36) and (4.40), we obtain that the square of the modulus of the Fourier transform of $u_\nu(t)$ is smaller than

$$\exp\{-\int_t^0 2\beta(s/\theta_\nu)(\mathrm{Log}\nu)^{-1}ds\ \nu\,(\mathrm{Log}\nu)^{-1}\}\ |\mathcal{F}(\Omega_1^{(\nu)})(\xi_1,\xi_2)|^2\ ,$$

where $\mathcal{F}$ stands for the Fourier transform.On the support of the unit vector $\mathcal{F}(\Omega_1^{(\nu)})$, for $\nu > \nu_0$, we have

$$2^{2\nu-1/2} \leq 2^{2\nu-1/4} \leq \xi_1^2+\xi_2^2 \leq \nu^2 + 2^{2\nu+1/4} \leq 2^{2\nu+1/2} \quad \text{and} \quad 2|\xi_1| \leq |\xi_2|\ .$$

This proves (4.42) and (4.41) for $t<0$. Analogously, the inequality (4.44) is a consequence of the spectral location of $\omega_2^{(\nu)}$ with respect to $Q_2^{(\nu)}$ in (4.36) and of (4.30) . Moreover, on $\theta_\nu < t$, the Fourier transform of $u_\nu(t)$ is supported in the rectangle

$$|\xi_1| \leq 2^{\nu-2}\ ,\ 2^{\nu-1/8} \leq \xi_2 \leq 2^{\nu+1/8}$$

which is included in

$$2^{2\nu-1/2} \leq \xi_1^2+\xi_2^2 \leq 2^{2\nu-4} + 2^{2\nu+1/4} \leq 2^{2\nu+1/2} \quad \text{and} \quad 2|\xi_1| \leq 2^{\nu-1} \leq |\xi_2| \ ,$$

proving (4.41) on $\theta_\nu < t$. Similarly , collecting the information on the support of the Fourier transform of $\Omega_1^{(\nu)}$ and $\chi_\nu \Omega_2^{(\nu)}$,we get (4.41) on $0 < t < \theta_\nu$. Moreover , for $t \in (0,\theta_\nu)$, we have, from (4.40),

$$|u_\nu(t)|^2 \geq |\Omega_1^{(\nu)}|^2 + 2\sigma(\frac{t}{\theta_\nu})\mathrm{Re} < \Omega_1^{(\nu)}, \chi_\nu \Omega_2^{(\nu)} - \Omega_1^{(\nu)} >.$$

Since $\Omega_1^{(\nu)}$ is a unit vector, and

$$(4.46) \qquad < \Omega_1^{(\nu)}, \chi_\nu \Omega_2^{(\nu)} > = < \chi_\nu \Omega_1^{(\nu)}, \Omega_2^{(\nu)} > = < \Omega_1^{(\nu)}, \Omega_2^{(\nu)} >,$$

from the fact that the Fourier multiplier χ_ν is 1 on the support of the Fourier transform of $\omega_1^{(\nu)}$, we get , for $t \in (0,\theta_\nu)$, from (4.38) and (4.35),

$$|u_\nu(t)|^2 \geq 1 + 2\sigma(\frac{t}{\theta_\nu})(1 - \varepsilon_\nu C_o - 1) \geq 1 - 2\varepsilon_\nu C_o .$$

The other inequality in (4.43) is just the convexity of the unit ball ($\chi_\nu \Omega_2^{(\nu)}$ has a norm smaller than 1 since $\Omega_2^{(\nu)}$ is a unit vector and χ_ν is a Fourier multiplier valued in [0, 1]).We inspect now the commutator

$$[\, Op(a_\nu(\xi)), Q_2^{(\nu)} \,] = [\, Op(a_\nu(\xi)) , \tfrac{1}{2}\nu W(\nu x_1) \,] \ .$$

We have obviously

$$a_\nu(\xi) \in S(1, |dx|^2\nu^2 + |d\xi|^2 2^{-2\nu}) \ \text{and} \ \tfrac{1}{2}\nu W(\nu x_1) \in S(\nu , |dx|^2\nu^2 + |d\xi|^2 2^{-2\nu}) ,$$

with semi-norms independent of ν (here we use Hörmander's notation (18.4.6) in [5]), and thus, we get , using theorems 18.5.4 and 18.6.3 in [5], that the commutator

$$[\, Op(a_\nu(\xi)), Q_2^{(\nu)} \,] \quad \in \quad Op(S(\nu 2^{-\nu}\nu , |dx|^2\nu^2 + |d\xi|^2 2^{-2\nu})).$$

The estimate (4.45) is then a consequence of theorem 18.6.3 in [5]. Note that a reader not conversant with these sources will easily prove (4.45) directly. The proof of lemma 4.5 is complete.

5. Construction of a null solution for an $S^1_{1,0}$ operator

We shall follow the lines of the computations (1.8 – 11) and use a modification of the operator $Q(t, x, D_x)$ defined in (3.1) : we set

(5.1) $$\mathbb{Q}(t) = \sum_{\nu > \nu_0} \Psi_\nu \; \mathbb{Q}_\nu(t) \, \Psi_\nu \quad , \qquad \text{with } \mathbb{Q}_\nu(t) \text{ given in (4.28),}$$

(5.2) $$\Psi_\nu = Op(\psi_\nu) = Op(\psi(2^{-2\nu}(\xi_1^2 + \xi_2^2)) \; \chi(\xi_1/\xi_2)) \; ,$$

where ψ and χ are given by (3.2), (3.3), and Op(a) stands for the operator with symbol a. The operator $\mathbb{Q}(t)$ has a symbol in the $S^1_{1,0}$ class since if Q is given by (3.1), $\mathbb{Q}(t) - Op(\, Q(t,x,\xi)\,)$ has a symbol in the $S^0_{1,0}$ class. As a matter of fact, we have

(5.3) $$\Psi_\nu \mathbb{Q}_\nu(t) \Psi_\nu = \alpha_\nu(t) \; D_1 \Psi_\nu^2 + (\mathrm{Log}\nu)^{-1} \alpha(t/\theta_\nu) \, \Psi_\nu \, \tfrac{1}{2} \nu \, W(\nu \, x_1) \Psi_\nu \quad .$$

From (4.30), (3.6),(3.5), (4.27), we see that $\alpha_\nu(t)$ and $(\mathrm{Log}\nu)^{-1}\alpha(t/\theta_\nu)$ are smooth functions bounded as well as all their derivatives independently of ν. Moreover, since the symbols ψ_ν defined in (5.2) satisfy

(5.4) $$\psi_\nu \in \; S(1, |\, dx \,|^2 + |\, d\xi \,|^2 2^{-2\nu}) \qquad \text{and}$$

(5.5) $$\nu \, W(\nu \, x_1) \; \in \; S(\nu, \; |\, dx_1 |^2 \nu^2 + |\, d\xi_1 |^2 2^{-2\nu}) \quad , \quad \text{uniformly in } \nu \, ,$$

we obtain , using theorem 18.5.4 in [5] that the composition

(5.6) $$\Psi_\nu \, \nu \, W(\nu \, x_1) \Psi_\nu = Op\big(\psi^2(2^{-2\nu}(\xi_1^2 + \xi_2^2)) \; \chi^2(\xi_1/\xi_2) \, \nu \, W(\nu \, x_1)\big) + Op(r_\nu),$$

where $r_\nu \in S(\, \nu^2 \, 2^{-\nu}, \; |\, dx \,|^2 \nu^2 + |\, d\xi|^2 2^{-2\nu})$ uniformly in ν . In fact, using theorem 18.6.3 in [5], we get that

(5.6)' $$\| \; Op(r_\nu) \, \|_{\mathcal{L}(L^2)} \leq 2^{-\nu/2} \quad ,$$

which is a better estimate than the one subsequent to

(5.5)' $$\nu \, W(\nu \, x_1) \; \in \; S(\, 2^\nu, \; |\, dx \,|^2 + |\, d\xi \,|^2 2^{-2\nu}) \quad , \text{uniformly in } \nu \, .$$

We set

(5.7) $$L = \frac{d}{dt} - \mathbb{Q}(t) \qquad , \quad \text{with } \mathbb{Q} \text{ given by (5.1),}$$

and we calculate Lu_v , with u_v given by (4.40) .The inclusion (4.41) in lemma 4.5, the fact that

(5.8) $\tilde{\Delta}_v \subset \{ \psi_v = 1 \} \subset \text{support } \psi_v \subset \Delta_v$,

where Δ_v are the disjoint sets defined in (3.10), imply ,with $Q_v(t)$ given in (4.28),

(5.9) $$Lu_v = \frac{du_v}{dt} - \Psi_v Q_v(t)\Psi_v u_v = \frac{du_v}{dt} - \Psi_v Q_v(t)\, u_v =$$

$$= \frac{du_v}{dt} - Q_v(t)\, u_v - [\Psi_v , Q_v(t)]u_v =$$

$$= \frac{du_v}{dt} - Q_v(t)\, u_v - H(t-\theta_v)\, \alpha_v(t) [\, Op(\psi_v), \tfrac{1}{2} v\, W(v\, x_1)]\; u_v \;,$$

and thus, from (5.9), (4.45), we get

(5.10) $$Lu_v = \frac{du_v}{dt} - Q_v(t)\, u_v + R_v(t)\, u_v(t) \quad , \quad \text{with } \| R_v(t) \|_{\mathcal{L}(L^2)} \leq 2^{-v/2} \;.$$

We set

(5.11) $$L_v = \frac{d}{dt} - Q_v(t)\,, \text{ where } Q_v(t) \text{ is given in (4.28).}$$

Lemma 5.1

Let L_v be the operator defined in (5.11) and u_v given by (4.40). With

(5.12)
| *$\| \;\|$ standing for the $L^2(\mathbb{R}_t, \mathbb{H}=L^2(\mathbb{R}^2_{x_1,x_2})\,) = L^2(\mathbb{R}^3_{t,x_1,x_2})$ norm,*
| *$|\;|$ for the norm in $\mathbb{H}$*

we have, using the notations of (4.27) and of lemma 4.4,

(5.13) $$\|L_v u_v\|^2 \leq 9\, C_o \varepsilon_v\, \theta_v^{-1} \qquad \text{and} \qquad \|u_v\|^2 \geq \tfrac{1}{2}\,\theta_v \;.$$

Moreover, for

(5.14) $$A_v = (LogLog\, v\,)^{1/16} \quad \text{and} \quad \tfrac{1}{2} A_v\theta_v \leq |\,t\,| \leq A_v\theta_v\,, \qquad \text{we have}$$

(5.15) $$|\,u_v(t)\,|^2 \leq A_v\, \varepsilon_v\,.$$

Proof .We note that, from (4.40) and (5.11), we have

$$(5.16)\qquad L_\nu u_\nu = 0 \quad \text{on} \quad t < 0 \quad ,$$

$$(5.17)\qquad L_\nu u_\nu = \frac{1}{\theta_\nu}\ \sigma'(\frac{t}{\theta_\nu})\ (\chi_\nu \Omega_2^{(\nu)} - \Omega_1^{(\nu)}) \qquad , \qquad \text{on} \quad 0 < t < \theta_\nu \quad ,$$

$$(5.18)\qquad L_\nu u_\nu = -\alpha_\nu(t)\,[\ \tfrac{1}{2}\nu W(\nu x_1),\ \chi_\nu]\ \exp\left[\int_{\theta_\nu}^{t} \alpha_\nu(s)ds\ Q_2^{(\nu)}\right] \Omega_2^{(\nu)} \quad \text{on} \quad \theta_\nu < t \quad .$$

Consequently, applying (4.44), (4.45) we get , on $t > \theta_\nu$,

$$(5.19)\ |(L_\nu u_\nu)(t)|^2 \le 2^{-\nu}\ \exp\{-\int_1^{t/\theta_\nu} 2\alpha(s)ds\,\theta_\nu\,\nu\,(\mathrm{Log}\nu)^{-2}\},$$

and using (3.6), we obtain

$$(5.20)\qquad \int_{\theta_\nu}^{+\infty} |(L_\nu u_\nu)(t)|^2\,dt \ \le\ 2^{-\nu}\,\theta_\nu\ (\ 1 + \frac{1}{2\theta_\nu \nu\,(\mathrm{Log}\nu)^{-2}}\ ) \le \varepsilon_\nu\,\theta_\nu^{-1} \quad ,$$

where the last inequality comes from (4.27) for $\nu > \nu_o$. Moreover, we have from (4.40),(4.46),(4.38) and (4.35)

$$(5.21)\qquad \int_0^{\theta_\nu} |(L_\nu u_\nu)(t)|^2\,dt \ \le\ \theta_\nu\,\theta_\nu^{-2}\ \ |\chi_\nu \Omega_2^{(\nu)} - \Omega_1^{(\nu)}|^2\ \|\sigma'\|_{L^2}^2 =$$

$$\theta_\nu^{-1}\ (\ |\chi_\nu \Omega_2^{(\nu)}|^2 + |\Omega_1^{(\nu)}|^2\ - 2\,\mathrm{Re}\langle \Omega_1^{(\nu)}, \Omega_2^{(\nu)}\rangle\ )\ \|\sigma'\|_{L^2}^2$$

$$\le \theta_\nu^{-1}\ (2 - 2(1 - \varepsilon_\nu C_o))\ \|\sigma'\|_{L^2}^2\ = \varepsilon_\nu \theta_\nu^{-1}\ 2C_o\ \|\sigma'\|_{L^2}^2\ \le \varepsilon_\nu \theta_\nu^{-1}\ 8C_o\,.$$

We obtain the first inequality in (5.13) from (5.21), (5.20) and (5.16). To get the last one,we remark that (4.43) implies

$$(5.22)\qquad \|u_\nu\|^2 \ge \int_0^{\theta_\nu} |u_\nu(t)|^2\,dt\ \ge \theta_\nu\,(1 - 2\varepsilon_\nu C_o) \ge \tfrac{1}{2}\theta_\nu \quad , \qquad \text{if}\ \nu > \nu_o\,.$$

To check (5.15), we use lemma 4.5 : if $\frac{1}{2} A_\nu\theta_\nu$, $\le |t| \le A_\nu\theta_\nu$, (4.42) and (4.44) give

$$(5.23)\ |u_\nu(t)|^2 \le \mathrm{MAX}\Big[\exp\{-\int_{-(1/2)A_\nu}^{0} 2\beta(s)ds\,\theta_\nu\,\nu\,(\mathrm{Log}\nu)^{-2}\}\ ,\exp\{-\int_0^{(1/2)A_\nu} 2\alpha(s)ds\,\theta_\nu\,\nu\,(\mathrm{Log}\nu)^{-2}\}\Big]$$

$$\leq \mathrm{MAX}\Big[\exp\{-\int_{-1}^{0} 2\beta(s)ds\,\theta_\nu\,\nu\,(\mathrm{Log}\nu)^{-2}\},\exp\{-\int_{0}^{1} 2\alpha(s)ds\,\theta_\nu\,\nu\,(\mathrm{Log}\nu)^{-2}\}\Big]$$

$$\leq e^{-\nu^{1/2}} \leq A_\nu \varepsilon_\nu .$$

The proof of lemma 5.1 is complete.

We consider now , with χ defined in (3.3), u_ν in (4.40), the t-compactly supported function

(5.24) $$v_\nu(t) = \chi(t / A_\nu\theta_\nu) u_\nu(t) \quad .$$

We calculate Lv_ν , where L is given by (5.7) : we obtain, using the notations of (5.10), (5.11),

(5.25) $$Lv_\nu = \chi(t / A_\nu\theta_\nu) Lu_\nu + (A_\nu\theta_\nu)^{-1}\chi'(t / A_\nu\theta_\nu) u_\nu =$$

$$= \chi(t / A_\nu\theta_\nu) L_\nu u_\nu + \chi(t / A_\nu\theta_\nu) R_\nu u_\nu + (A_\nu\theta_\nu)^{-1}\chi'(t / A_\nu\theta_\nu) u_\nu \ .$$

Consequently, from (5.25), (5.13) and χ valued in [0,1], (5.10) and $|u_\nu(t)| \leq 1$(from lemma 4.5), $\chi'(s) = 0$ outside $1/2 \leq |s| \leq 1$ and (5.15), we get

(5.26) $$\|L v_\nu\|^2 \leq 3^3 C_0\varepsilon_\nu \theta_\nu^{-1} + 3\, 2^{-\nu} A_\nu\theta_\nu \|\chi\|^2_{L^2(\mathbb{R})} + 3(A_\nu\theta_\nu)^{-1}\|\chi'\|^2_{L^2(\mathbb{R})} A_\nu \varepsilon_\nu$$

$$\leq \varepsilon_\nu \theta_\nu^{-1} C_1 ,$$

where C_1 is an absolute constant. Moreover, from (5.24), (3.3) and (5.13), we get

(5.27) $$\|v_\nu\|^2 \geq \int_0^{\theta_\nu} |v_\nu(t)|^2 dt = \int_0^{\theta_\nu} |u_\nu(t)|^2 dt \geq \tfrac{1}{2}\theta_\nu \quad ,$$

so that the ratio of (5.26) and (5.27) satisfies the following estimate, using (4.37),

(5.28) $$\|L v_\nu\|^2 \ \|v_\nu\|^{-2} \leq \varepsilon_\nu \theta_\nu^{-2}\, 2C_1 = \varepsilon_\nu^{1/2}\, 2C_1 \quad .$$

Thanks to (5.6) and (5.6)', the estimate (5.28) is valid also with L replaced by $D_t + i\,Q(t ,x, D_x)$, where Q is given by (3.1).

Finally we must discuss cut-off functions in the x_1, x_2 space ; to perform a localisation in the x_2 variable is an easy task : if we inspect the requirement on the function ρ_v expressed in (4.39) , we can take

$$(5.29) \qquad \rho_v(x_2) = \overset{\chi}{g}(2^v x_2)\, 2^{v/2} \ ,$$

where g is a "fixed" function in $C_0^\infty(2^{-1/8}, 2^{1/8})$ with L^2 norm 1.
Consequently, we have, with χ given by (3.3), Ψ_v by (5.2), $\lambda_v > 0$ to be chosen later,

$$(5.30) \qquad \| L\chi(\lambda_v x_2) v_v \| = \| L\chi(\lambda_v x_2)\Psi_v\, v_v \| \leq \varepsilon_v^{1/4} \sqrt{2C_1}\, \|v_v\| + \| [\, L,\chi(\lambda_v x_2)\, \Psi_v\,]\, v_v\|.$$

The commutation relations between

$$(5.31) \qquad Q(t) \in Op\,(\, S(<\xi>/\,\mathrm{LogLog}<\xi>\, , |\,dx\,|^2 + |\,d\xi|^2 <\xi>^{-2})\,) \text{ and}$$

$$(5.32) \qquad e^{-\lambda_v}\chi(\lambda_v x_2)\, \Psi_v \in Op\,(\, S(\,1,\, |\,dx\,|^2 + |\,d\xi|^2 <\xi>^{-2})\,)\ ,$$

where $<\xi> = (e^e + |\xi|^2)^{1/2}$, coming up if we examine the commutator for each fixed t , show that the bracket in (5.30) is estimated by

$$(5.33) \qquad \| v_v \|\, (\mathrm{Log} v\,)^{-1} e^{\lambda_v} \leq \| v_v \|\ \ \varepsilon_v^{1/4}\ , \text{ if}$$

$$(5.34) \qquad e^{\lambda_v} \leq \varepsilon_v^{1/4}\, \mathrm{Log} v \quad : \text{we choose } \ \lambda_v = \tfrac{1}{2}\mathrm{LogLog} v\,.$$

As a matter of fact, the confinement estimates of [3] (theorem 2.2.1) are convenient to show that the bracket $[\, L,\chi(\lambda_v x_2)\Psi_v\,]$ enjoys some confinement properties in the rings Δ_v , and thus behaves essentially as if it were supported in that ring, at least as far as L^2 estimates are at stake. We thus obtain, from (5.30), (5.33) and for λ_v satisfying (5.34),

$$(5.35) \qquad \| L\chi(\lambda_v x_2) v_v\| \leq C_2\ \ \varepsilon_v^{1/4}\, \| v_v\|\ ,$$

where C_2 is an absolute constant.Moreover, we have

$$(5.36) \qquad \| v_v \|^2 = \| \chi(\lambda_v x_2) v_v\|^2 + \| \gamma\, v_v \|^2 \quad ,$$

where γ is a function of x_2 supported in $|\,\lambda_v x_2\,| > 1/2$. Now, we compute, using (5.29)

$L^2(\mathbb{R}, \mathbb{H})$, $L^2(\mathbb{R}, \mathbb{H})$ inequality with a large constant proportional to the reciprocal of the t - diameter of u : if $u(t) = 0$ on $|t| \geq T_o/2 > 0$, we have

$$(5.38)' \quad 2\left[\int \left|\frac{du}{dt}(t) - Q(t)\,u(t)\right|_{\mathbb{H}}^2 dt\right]^{1/2} T_o^{1/2} + p(W)C(d)\left[\int |u(t)|_{\mathbb{H}}^2\, dt\right]^{1/2} T_o^{1/2} \geq$$

$$2\int \left|\frac{du}{dt}(t) - Q(t)\,u(t)\right|_{\mathbb{H}} dt + p(W)C(d)\int |u(t)|_{\mathbb{H}}\, dt \geq \sup_{t\in\mathbb{R}} |u(t)|_{\mathbb{H}} \geq$$

$$\left[\int |u(t)|_{\mathbb{H}}^2\, dt\right]^{1/2} T_o^{-1/2}.$$

It is possible to modify Beals-Fefferman's arguments of [1] to prove an estimate of the form (5.38) for general operators satisfying the (P) condition. Let's now prove the lemma : we compute for T real parameter,

$$(5.39) \quad A(T) = 2\,\mathrm{Re} \langle D_t u + i\,Q(t)u,\ i\,H(D_1)H(t-T)u \rangle_{L^2(\mathbb{R},\mathbb{H})}, \text{ so that}$$

$$(5.40) \quad A(T) = \sum_{\nu>\nu_o} A_\nu(T) \quad ,$$

where the general term of this (later shown) absolutely converging series is

(5.41)

$$A_\nu(T) = 2\mathrm{Re}\langle D_t u + i\Psi_\nu\left(\alpha_\nu(t)D_1 + \alpha_\nu(t)H(t-\theta_\nu)\tfrac{1}{2}\nu W(\nu x_1)\right)\Psi_\nu u,\ i\,H(D_1)H(t-T)u\rangle_{L^2(\mathbb{R},\mathbb{H})},$$

where Ψ_ν is defined in (5.2), α_ν in (4.30), W in (3.9). We know from the assumption on the support of u in the lemma 5.3 and (3.9) that

$$(5.42) \quad \nu W(\nu x_1)\Psi_\nu u = [\,\nu W(\nu x_1), \Psi_\nu]\,u \quad , \quad \text{if } \nu > \frac{1}{2d} \quad .$$

We show below that the operator

$$(5.44) \quad R = \sum_{\nu>\frac{1}{2d}} [\,\nu W(\nu x_1), \Psi_\nu] + \sum_{\frac{1}{2d}\geq\nu>\nu_o} \nu W(\nu x_1)\Psi_\nu$$

is $L^2(\mathbb{R}^2_{x_1,x_2})$ bounded and

$$(5.45) \quad \|R\|_{\mathcal{L}(L^2)} \leq C_1\left(\left(\frac{1}{2d}\right)^{3/2} + 1\right)q(W) \quad ,$$

where C_1 is an absolute constant, q(W) a semi-norm of the function W in $\mathcal{S}$. As a matter of fact, each term of the first sum in (5.44) is easily handled by (4.45) but theorem 4.2.2(b) in [3] is convenient to estimate the sum : nonetheless $[\nu W(\nu x_1),\Psi_\nu]$ is uniformly L^2 bounded as shown by (4.45), but its symbol , though not supported in $\{ |\nu x_1| \leq 1/2\} \times \Delta_\nu$ (see (3.9)) , is "confined" in this set in the sense of definition 2.1.1 in [3] . The second sum in (5.44) is estimated by $(\frac{1}{2d})^{3/2} \|W\|_{L^\infty}$ since

$$\sum_{\frac{1}{2d}\geq\nu>\nu_o} |\nu W(\nu x_1)\Psi_\nu\omega|_{L^2} \leq \frac{1}{2d}\|W\|_{L^\infty} \sum_{\frac{1}{2d}\geq\nu>\nu_o} |\Psi_\nu\omega|_{L^2} \leq (\frac{1}{2d}\|W\|_{L^\infty})(\frac{1}{2d})^{1/2}|\omega|_{L^2} .$$

Thus we obtain , from (5.39), α_ν non-negative and (5.44)

(5.46)

$$\left(2\int_T^{+\infty}\left|\frac{du}{dt}(t) - \mathbb{Q}(t)\,u(t)\right|_{\mathbb{H}} dt + p(W)C_1(d)\int_T^{+\infty}|u(t)|_{\mathbb{H}}\,dt\right)\sup_{t\geq T}|H(D_1)u(t)|_{\mathbb{H}} \geq$$

$$|H(D_1)u(T)|^2_{\mathbb{H}} \quad ,$$

where p is a semi-norm of W in $\mathcal{S}$, and $C_1(d)$ a function of d.

Analogously, we get

(5.47)

$$\left(2\int_{-\infty}^{T}\left|\frac{du}{dt}(t) - \mathbb{Q}(t)\,u(t)\right|_{\mathbb{H}} dt + p(W)C_1(d)\int_{-\infty}^{T}|u(t)|_{\mathbb{H}}\,dt\right)\sup_{t\leq T}|H(-D_1)u(t)|_{\mathbb{H}} \geq$$

$$|H(-D_1)u(T)|^2_{\mathbb{H}} \quad .$$

The inequalities (5.46) and (5.47) yield lemma 5.2.

Let's now conclude and prove theorem 2.1. If L given by (5.7) were L^2 solvable near the origin in $\mathbb{R}^3$, we would have, using the above notations, and for $F_o(x_1)$ compactly supported and identically 1 near 0, $F_o^2+F_1^2=1$, the following inequalities :

(5.48) $$\|\chi(\lambda_\nu x_2)v_\nu\|^2 = \|F_o(x_1)\,\chi(\lambda_\nu x_2)v_\nu(t,x)\|^2 + \|F_1(x_1)\chi(\lambda_\nu x_2)v_\nu(t,x)\|^2 \leq$$

$$C \| LF_0(x_1)\, \chi(\lambda_\nu x_2)v_\nu(t,x) \|^2 + C \| LF_1(x_1)\chi(\lambda_\nu x_2)v_\nu(t,x) \|^2 \leq$$

$$2C \| [\, L\, , F_0(x_1)\,]\, \chi(\lambda_\nu x_2)\Psi_\nu v_\nu(t,x) \|^2 + 2C \| [\, L\, , F_1(x_1)\,]\, \chi(\lambda_\nu x_2)\Psi_\nu v_\nu(t,x)\|^2$$

$$+ (1/2)\| \chi(\lambda_\nu x_2)v_\nu\|^2.$$

The first inequality follows , for the term with F_0 from the assumed solvability and for the term with F_1 from lemma 5.3. The second one is a consequence of (5.28) for $\chi(\lambda_\nu x_2)v_\nu$, proved above. Moreover, the commutators

$$[\, \mathbb{Q}(t)\, , F_0(x_1)\,]\, \chi(\lambda_\nu x_2)\Psi_\nu$$

are L^2 bounded operators with norm $\leq C\ \varepsilon_\nu^{1/4}$, for each t (see 5.34) . We would obtain, using (5.37),

(5.49) $\chi(\lambda_\nu x_2)v_\nu = 0$,

and from (5.36) and (5.37)'

(5.50) $v_\nu = 0$, which contradicts (5.27).

The proof of theorem 2.1 is complete.

6. Proof of theorem 2.2 : the homogeneous case .

We define with $t \in \mathbb{R}$, $x = (x_1, x_2) \in \mathbb{R}^2$, $\tau \in \mathbb{R}$, $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$, a function on $\mathbb{R}^6$

(6.1) $p(t, x, \tau, \xi) = p_1 + i\, p_2$,

(6.2) $p_1 = \tau$, $p_2 = \sum_{\nu \in \mathcal{N}} p_2^{[\nu]}$,

(6.3) $p_2^{[\nu]} = (\mathrm{Log}\nu)^{-1}\chi\big((x_2 - y_\nu)\frac{1}{4}\ \mathrm{Log}(\mathrm{Log}\nu)\big)\Big[\beta(t/\theta_\nu)\, \xi_1 + \alpha(t/\theta_\nu)[\, \xi_1 + \frac{1}{2}\ \nu\, W(\nu\, x_1)\, 2^{-\nu}\, \xi_2\,]\, \Big]$

where $\mathcal{N}$ is a subset of $\mathbb{N}$ so that

$$(6.4)\qquad \sum_{\nu\in\mathcal{N}} (\mathrm{Log}\nu)^{-1/2} < +\infty \ , \ (\text{e.g. } \mathcal{N} = \{\, 2^{k^3} \,\}_{k\in\mathbb{N},\, k\geq k_0}$$

χ is given in (3.3), y_ν is a sequence converging to zero when ν goes to infinity so that if $\mathcal{N} = \{\nu_k\}_{k\in\mathbb{N}}$,

$$(6.5)\quad \nu_{k+1} \geq e^{(\mathrm{Log}\nu_k)^{16}} \ , \quad y_{\nu_k} = (\mathrm{Log}(\mathrm{Log}\nu_k))^{-1/2} \ , \text{ e.g. we can take } \nu_k = 3^{3^{3^{4k}}} \ ,$$

β and α are defined in (3.6), $\theta_\nu = (\mathrm{Log}(\mathrm{Log}\nu))^{-1/8}$, W is given by (3.9).

Lemma 6.1

The function p defined by (6.1-4) is homogeneous with respect to (τ, ξ_1, ξ_2), is C^∞ over $\mathbb{R}^6$ and p is of principal type. Moreover, the complex conjugate $\bar{p}$ satisfies condition (ψ) on the cone $\{\xi_2 \geq 0\}$.

Proof. The mth derivative of the smooth function $p_2^{[\nu]}$ brings out a product of L^∞ norms of the fixed functions χ, α, β and their derivatives up to the order m, mth-power of Log(Logν) multiplied by $(\mathrm{Log}\nu)^{-1}$, mth-power of ν multiplied by $2^{-\nu}$:it is bounded from above by $(\mathrm{Log}\nu)^{-1/2}$ and the condition (6.4) implies the smoothness for p_2. The symbol p is obviously homogeneous of degree one and of principal type from dp_1. We need to check condition (ψ) on $\bar{p}$, i.e. (1.1) on p_2: Let's assume $p_2(t,x,\xi) > 0$. If ξ_1 is non-negative, then $p_2(s,x,\xi) \geq 0$ as a sum of non-negative terms ; we remark also that the supports of $\chi\big((x_2 - y_\nu)\frac{1}{2}\ \mathrm{Log}(\mathrm{Log}\nu)\big)$ are disjoint, i.e. the sets

$\{x_2 \ , y_\nu - 4/\mathrm{Log}(\mathrm{Log}\nu) \leq x_2 \leq y_\nu + 4/\mathrm{Log}(\mathrm{Log}\nu)\ \}$ do not intersect for different ν in $\mathcal{N}$: as a matter of fact, we obtain from (6.5),

$$(6.6)\qquad y_{\nu_{k+1}} + \frac{4}{\mathrm{Log}(\mathrm{Log}\nu_{k+1})} \leq y_{\nu_k} - \frac{4}{\mathrm{Log}(\mathrm{Log}\nu_k)} \ ,$$

since

$$(6.7)\quad y_{\nu_{k+1}} + \frac{4}{\mathrm{Log}(\mathrm{Log}\nu_{k+1})} \leq 2(\mathrm{Log}(\mathrm{Log}\nu_{k+1}))^{-1/2} \leq 2^{-1}(\mathrm{Log}(\mathrm{Log}\nu_k))^{-1/2} \leq y_{\nu_k} - \frac{4}{\mathrm{Log}(\mathrm{Log}\nu_k)}$$

where the first and the third inequality are due to $y_{\nu_k} = (\mathrm{Log}(\mathrm{Log}\nu_k))^{-1/2}$, whereas the second one is a consequence of $\mathrm{Log}\nu_{k+1} \geq (\mathrm{Log}\nu_k)^{16}$. If we go back to our assumptions $p_2(t,x,\xi) > 0$ and $\xi_1 < 0$, we see from the above discussion that all $p_2^{[\nu]}$ but one are zero, and for this one t must be larger than θ_ν. Consequently, if $s > t$, the non-negativity of $\xi_2 W$ implies that (1.1) is satisfied. The proof of lemma (6.1) is complete.

We shall now consider , with u_ν defined in (4.40), χ in (3.3), A_ν in (5.14), θ_ν in (4.27), y_ν in (6.5), the function

(6.8) $$V_\nu(t, x_1, x_2) = \chi(t / A_\nu\theta_\nu)\, \chi\big((x_2 - y_\nu)\tfrac{1}{2}\ \mathrm{Log}(\mathrm{Log}\nu)\big)\ u_\nu(t, x_1, x_2 - y_\nu),$$

and the operator

(6.9) $$P = D_t + i \sum_{\nu\in\mathcal{N}} \tilde{\mathrm{O}}\mathrm{p}(p_2^{[\nu]}) ,$$

where $p_2^{[\nu]}$ is defined in (6.3) and $\tilde{\mathrm{O}}\mathrm{p}$ stands for the "adjoint "quantization :

(6.10) $$\tilde{\mathrm{O}}\mathrm{p}(a)u(x) = \int e^{2i\pi<x-x',\xi>}\ a(x', \xi)\, u(x')\, dx'd\xi .$$

Lemma 6.2
The function V_ν defined in (6.8) satisfies

(6.11) $$\| V_\nu\|^2_{L^2(\mathbb{R}^3)} \geq \theta_\nu/4 \quad , \quad \| PV_\nu\|^2_{L^2(\mathbb{R}^3)} \leq C\, \varepsilon_\nu\, \theta_\nu^{-1} \quad ,$$

for P defined in (6.9) and an absolute constant C.
Proof. The first inequality in (6.11) follows from (5.27) and (5.37)'. The second one is a consequence of (5.26) and (5.37)' since

(6.11) $$iPV_\nu = \frac{\partial V_\nu}{\partial t} - \sum_{\mu\in\mathcal{N}} \tilde{\mathrm{O}}\mathrm{p}(p_2^{[\mu]})V_\nu = \frac{\partial V_\nu}{\partial t} - \tilde{\mathrm{O}}\mathrm{p}(p_2^{[\nu]})V_\nu \quad ,$$

where the last equality follows from (6.6)(i.e. the supports of the cut-off functions are disjoint and the function $\chi((x_2 - y_\nu)\frac{1}{4}\ \mathrm{Log}(\mathrm{Log}\nu))$ is 1 on the support of V_ν) and the "adjoint" quantization (6.10). Moreover, with g defined in (5.29), we have

(6.12) $$\xi_2 2^{-\nu}\, g(\xi_2 2^{-\nu})2^{-\nu/2} = (\xi_2 2^{-\nu} - 1)g(\xi_2 2^{-\nu})2^{-\nu/2} + g(\xi_2 2^{-\nu})2^{-\nu/2} .$$

We can choose

(6.13) $$g(\eta) = h((\eta - 1)\nu^2)\nu ,$$

where h is a function with L^2 norm 1 and support in (-1,+1). The equality (6.12) gives

$$(6.14) \quad \xi_2 2^{-\nu} g(\xi_2 2^{-\nu})2^{-\nu/2} = \nu^{-2}(\xi_2 2^{-\nu} - 1)\nu^2 h((\xi_2 2^{-\nu} - 1)\nu^2)\nu\, 2^{-\nu/2} + g(\xi_2 2^{-\nu})2^{-\nu/2}.$$

Since the estimate (5.37) is preserved up to the harmless multiplication of 2^{ν} by ν^{-2}, we get that the contribution of the first term in the right-hand side of(6.14) is $O(2^{-\nu})$. The proof of lemma 6.2 is complete.

We remark then that the operator P satisfies condition (P) on the open set $\{ x_1 \neq 0 \}$ and is elliptic on the cone $\{\tau \neq 0 \}$ and is thus microlocally solvable there. Since V_ν is the product of cut-off functions with a function whose Fourier transform is supported in $\{ 2\,|\xi_1| \leq \xi_2 \}$, we get the result of theorem 2.2 by reductio ad absurdum as in (5.48-50).

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