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Propagation of singularities for non real pseudo-differential operators

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Abstract

The purpose of this work is to prove a theorem of propagation of singularities for a class of non real pseudo-differential operator with multiple characteristics. The main tools are L^2 estimates on the time dependent Schrödinger equation related to P. We extend here the results of [6]; we improve the results announced by the second author in [7].

The second part of this work consists in an extension of the result of [5] to complex valued symbols.

Contents

Part I
The General result.

We start by a general result which could not be optimal in all the cases scanned here. The approach is similar to [6] but we use also time dependent L^2 estimate and some informations on the parametrix constructed in [6]. The main difference with the proof in [7] is that we need to have an analysis of the microlocal structure of the parametrix of the time dependent Schrödinger equation associated with the self-adjoint part of our operator.

1 Introduction and main statement

Let $P(x, \lambda^{-1}D_x, \lambda)$ be a pseudo-differential operator depending on a large parameter λ, defined by the Weyl formula :

$$
\begin{aligned}
P(x,\lambda^{-1}D_x,\lambda)u(x,\lambda) &= op_{1/2}((p(x,\xi/\lambda,\lambda))(u)(x) \\
&= (\frac{\lambda}{2\pi})^n \int p(\frac{x+y}{2},\xi,\lambda)e^{i\lambda(x-y)\xi}u(y,\lambda)\,dyd\xi
\end{aligned}
\tag{1}
$$

We shall write the operator given by formula 1 $(p(x,\xi,\lambda))^{w_\lambda}$.

The full symbol $p(x,\xi,\lambda)$ has an expansion as

$$
p(x,\xi,\lambda) = p_1(x,\xi) + ip_2(x,\xi) + \lambda^{-1}p_0(x,\xi,\lambda) \tag{2}
$$

where p_1 and p_2 are real and $p_2 \geq 0$. $p_0(x,\xi,\lambda)$ is a zero order symbol i.e. satisfies estimates :

$$
\textit{For all multi-indices } \alpha \textit{ and } \beta \quad \left|D_x^\alpha D_\xi^\beta p_0(x,\xi,\lambda)\right| \leq C_{\alpha,\beta}.
$$

It is a consequence of formulas 1 and 2 that

$$
P(x,\lambda^{-1}D_x,\lambda) = P_1(x,\lambda^{-1}D_x,\lambda) + iP_2(x,\lambda^{-1}D_x,\lambda) \tag{3}
$$

where P_1 and P_2 are self-adjoint pseudo-differential operators with symbols respectively

$$
P_1 = (p_1(x,\xi) + \lambda^{-1}Re(p_0(x,\xi,\lambda))^{w_\lambda} \tag{4}
$$

$$
P_2 = (p_2(x,\xi) + \lambda^{-1}Im(p_0(x,\xi,\lambda))^{w_\lambda} \tag{5}
$$

We shall make L^2 estimates on the solutions $u(t)$ of the Schrödinger equation $(D_t + P(x,\lambda^{-1}D_x,\lambda))u(t) = 0$. We shall therefore use some constructions made in [6]. We need to recall the hypotheses of this work.

- $(\mathrm{H})_1$: Let Φ_t be the bicharacteristic flow of p_1 at the time t. Let ρ_0 in $\mathrm{T}^*\mathbf{R}^n$ be a point near which we shall work. Let $h(t) \in o(t)$ when $t \to \infty$, $h \geq 0$ a function, and W a neighborhood of ρ_0 such that any bicharacteristic curve of p_1 with end points lying in

$$
\Lambda(W,h) = \left\{ \begin{array}{c} (\rho_1,\rho_2) \in W^2 \ ; \ \exists t \geq 0 \textit{ such that } \rho_1 = \Phi_t(\rho_2), \\ \textit{and if } 0 \leq s \leq t\, \Phi_s(\rho_2) \in W \textit{ and } |p_1(\rho_1)| \leq \exp(-h(t)) \end{array} \right\} \tag{6}
$$

is N_0 admissible[1] for a function $\varepsilon \in \,]0,1] \to N_0(\varepsilon) \in \mathbf{R}^+$. We refer to [6] pp 468-469 for a definition and for sufficient conditions which

[1]A curve $t \in [0,T] \to \gamma(t) \in W$ is N_0 admissible if for any $\varepsilon > 0$, there exists a partition of $[0,T]$ in intervalls of type $\mathcal{I}$ and $\mathcal{J}$, the number of these intervalls is less than $N_0(\varepsilon)$. An interval of type $\mathcal{I}$ remains at a distance less than ε from a point in the double characteristic set of P_1, an interval of type $\mathcal{J}$ has a lenght less than $N_0(\varepsilon)$.

imply this property. We shall not recall here the details, but we just mention that it is satisfied if the bicharacteristics of p_1 whose lenght is large enough leave a neighborhood of (ρ_0, ρ_0).

Let $N = \{\rho \in T^*(\mathbf{R}^n); p_1(\rho) = dp_1(\rho) = 0\}$ be the set of double characteristics of P_1.

- $(\mathrm{H})_2$: The main assumption is that on N, the dimension of the space spanned by the generalized eigenvectors associated with eigenvalues of positive imaginary part is constant.
- $(\mathrm{H})_3$: On N, $Imp_0(\rho) > 0$. This inequality means that Imp_0 has a positive lower bound with respect to ρ and λ.

Let us define

$$C(\overline{W}) = \bigcap_{h \in \omega} \overline{\Lambda(W, h)} \tag{7}$$

where ω is the set of all non negative increasing functions $h(t) \in o(t)$ when $t \to \infty$.

We consider $C(\overline{W})$ as a relation in $T^*(\mathbf{R}^n)$.

We note by $OF(u)$ the oscillatory front set of a bounded family of tempered distributions $u(x, \lambda)$.

Let us recall that we say that $(x_0, \xi_0) \in (OF(u))^c$ if there are neighborhoods V of x_0 and L of ξ_0 such that for any $\varphi \in C_0^\infty(V)$

$$\textit{For all } N \in \mathbf{N}, \textit{ for } \lambda \geq 1 \ \sup_{\xi \in L} |\widehat{\varphi u}|\,(\lambda \xi, \lambda) \leq C_N \lambda^{-N}.$$

The main result can now be stated.

Theorem 1 *Assume that the assumptions* $(\mathrm{H})_1$ $(\mathrm{H})_2$, $(\mathrm{H})_3$ *are satisfied for a suitable set* W *and for some function* $h_0 \in \omega$. *Let* $u(x, \lambda)$ *be a bounded family of tempered distributions. If* $OF(Pu) \cap \overline{W} = \emptyset$ *and* $C(\overline{W})(\rho) \cap \partial W \cap OF(u) = \emptyset$, *then* $\rho \notin OF(u)$.

We easily deduce :

Corollary 1 *Let* $\rho_0 \in T^*(\mathbf{R}^n) \backslash 0$, $P(x, D_x)$ *a pseudo-differential operator in the usual sense. Assume that* $(\mathrm{H})_1$ $(\mathrm{H})_2$, $(\mathrm{H})_3$ *are satisfied for a neighborhod* W *of* ρ_0 *and a function* $h_0 \in \omega$. *Let* u *be a distribution such that* $WF(Pu) \cap \overline{W} = \emptyset$ *and* $C(\overline{W})(\rho) \cap \partial W \cap WF(u) = \emptyset$, *then* $\rho \notin WF(u)$.

The main difference between the proof of this theorem and the corresponding result in [7] is the presence in the bicharacteristic flow of p_1 of expansive directions. This will make us to use fully the construction of the parametrix of [6] instead of using only microlocalisations in a semi-global L^2 inequality for the solutions of the time dependent Schrödinger equation associated with P_1.

2 Estimates on solutions of the Schrödinger equation

We shall work with a family of solutions of the Schrödinger equation $(D_t + P(x, \lambda^{-1}D_x))u(t) = 0$, where $D_t = (1/i\lambda)\partial/\partial t$.

We need to make a Fourier-Bros-Iagolnitzer transformation (see [9] and [6]). Let

$$Tu(x,\lambda) = (\frac{\lambda}{2i\pi})^{n/2} \int e^{-\lambda/2((x-y)^2 - x^2/2)} u(y,\lambda)\, dy \tag{8}$$

this is a unitary transformation from $L^2(\mathbf{R}^n)$ to the space $H_{\varphi_0}(\mathbf{C}^n)$ of entire functions in $L^2(\mathbf{C}^n, e^{-2\lambda\varphi_0}\mathrm{L}(dx))$, where $\mathrm{L}(dx)$ is the Lebesgue measure in $\mathbf{C}^n$, and $\varphi_0(x) = \frac{1}{4}|x|^2$.

We note by the same letter an operator and its conjugate by T.

We have a Bergman projector from $L^2(\mathbf{C}^n, e^{-2\lambda\varphi_0}\mathrm{L}(dx))$ to $H_{\varphi_0}(\mathbf{C}^n)$ given by the formula

$$Su(x,\lambda) = (\frac{\lambda}{2i\pi})^{n} \int e^{\lambda x\overline{y}/2} u(y,\lambda) e^{-2\lambda\varphi_0(y)}\, \mathrm{L}(dy) \tag{9}$$

see [9] for these formulas. Let us say that the formula 9 is obtained by integrating the formal integral in TT^{-1} along a suitable contour.

In [6] we can find some constructions for an approximate solution $E_t u(x,\lambda)$ of the equation $(D_t + P_1(x, \lambda^{-1}D_x))(E_t u) \equiv 0$; $E_t u|_{t=0} \equiv u$, we shall make this more precise later.

$$E_t u(x,\lambda) = (\frac{\lambda}{2i\pi})^{n} \int e^{i\lambda\varphi(t,x,\overline{y})} e(t,x,\overline{y},\lambda)\chi(t,x,\overline{y}) u(y,\lambda) e^{-2\lambda\varphi_0(y)}\, \mathrm{L}(dy) \tag{10}$$

where $\varphi(t,x,y)$ is a solution of the phase equation with value $\varphi(0,x,\overline{y}) = -ix\overline{y}/2$; $e(t,x,y,\lambda)$ is a solution of transport equations, $\chi(t,x,y)$ is a cut-off function.

Let

$$\Gamma_t(W,h) = \left\{(x,y); (x, \frac{2}{i}\frac{\partial\varphi_0}{\partial x}, y, \frac{2}{i}\frac{\partial\varphi_0}{\partial y}) \in \Lambda_t'(W,h)\right\} \tag{11}$$

where $\Lambda_t'(W,h)$ is the image by the complex canonical transformation generated by $\varphi_0(x)$ of the Lagrangean sub manifold

$$\Lambda_t(W,h) = \left\{ \begin{array}{c} (\rho_1, \rho_2') \in W^2;\ \textit{such that}\ \rho_1 = \exp(tH_{p_1})(\rho_2)\ \textit{if}\ 0 \leq s \leq t \\ \exp(sH_{p_1})(\rho_2) \in W\ \textit{and}\ |p_1(\rho_1)| \leq \exp(-h(t)) \end{array} \right\} \tag{12}$$

where ρ_2' is the antipodal point of ρ_2.

$\Gamma_t(W,h)$ is totally real in $\mathbf{C}^n \times \mathbf{C}^n$. We again refer to [6] for the construction of a convenient projection $\mathbf{C}^n \times \mathbf{C}^n \to \Gamma_t(W,h)$; $z \to m(t,z) \in \Gamma_t(W,h)$; $m(t,z)$ will be defined uniquely by the additional property that $z - m(t,z) \in i\mathrm{T}_{m(t,z)}\Gamma_t(W,h)$.

Therefore functions on $\Gamma_t(W,h)$ give rise to almost analytic extension in $\mathbf{C}^n \times \mathbf{C}^n$. Appropriate controls with respect to t are obtained in [6]. From these controls it follows that all derivatives with respect to (t,x) of these maps are bounded by some $\exp(h(t))$ with a function $h(t) \in o(t)$ when $t \to \infty$. So $\varphi(t,x,y), \ldots, \chi(t,x,y)$ will be almost analytic on Γ_t and there derivatives are bounded by some $\exp(h(t))$. A more precise decay in time for the amplitude $e(t,x,y)$ is obtained and this will be discussed later since this point is essential in our discussion.

Proposition 1 *There are two constants $C > 0$, $\gamma > 0$ such that*

$$\|E_t\| \leq C\lambda^n \exp(-1/2\gamma t) \tag{13}$$

γ being as close as we wish of the lower bound of

$$T^+(\rho) = \sum_{z_j \in Spec(F_{p_1}),\ Re z_j > 0} Re z_j.$$

We shall not use this result here, but it is woth mentionning since it is the key point of the proof in [6]. The proof will be derived from elements of the proof of the following Proposition.

Proposition 2 *There are constants $M > 0$, $C > 0$ such that*

$$\|E_t\| \leq C \tag{14}$$

if $\exp(Mt) \leq \lambda$, the norms are taken in $H_{\varphi_0}(\mathbf{C}^n)$.

Proof. Let us prove Proposition 2. Let E_t^* be the adjoint of E_t in $H_{\varphi_0}(\mathbf{C}^n)$. If we write $d\mu(x) = e^{-2\lambda\varphi_0(x)}\,\mathrm{L}(dx)$ the kernel (with respect to μ) of E_t^* is given by $E_t^*(x,y) = \int S(x,z)\overline{E_t(y,z)}\,d\mu(z)$, the kernel of $E_t^*E_t$ is therefore $E_t^*E_t(x,y) = \int S(x,x_1)\overline{E_t(x_2,x_1)}E_t(x_2,y)\,d\mu(x_1)d\mu(x_2)$. We write this integral

$$E_t^*E_t(x,y) = \int e^{i\lambda H(t,x,y;x_1,x_2)} f(t,x,y;x_1x_2,\lambda)\,d\mu(x_1)d\mu(x_2) \tag{15}$$

In 15 we have

$$H(t,x,y;x_1,x_2) = \psi(x,x_1) - \overline{\varphi(t,x_2,\overline{x}_1)} + \varphi(t,x_2,\overline{y}) + 2i(\varphi_0(x_1)+\varphi_0(x_2)) \tag{16}$$

and

$$f(t,x,y;x_1,x_2,\lambda) = c\lambda^{3n}\overline{e(t,x_2,\overline{x}_1)}e(t,x_2,\overline{y})\overline{\chi(t,x_2,\overline{x}_1)}\chi(t,x_2,\overline{y}) \tag{17}$$

where c is some absolute constant, $\psi(x,x_1) = -ix\overline{x_1}/2$. We first investigate the critical points of H with respect to (x_1,x_2); we estimate ,using [6] page 505 (5.30)

$$\begin{aligned} &ImH(t,x,y;x_1,x_2) + \Phi(x,y) \geq \\ &\quad c(|x-x_1|^2 + |(x_{2,}\overline{x}_1) - m_t(x_2,\overline{x}_1)|^2 + |(x_{2,}\overline{y}) - m_t(x_2,\overline{y})|^2) \end{aligned} \tag{18}$$

with the notation $\Phi(x,y) = \varphi_0(x) + \varphi_0(y)$. We have

$$\begin{aligned} H'_{x_1} &= -\overline{\varphi'_y(t,x_2,\overline{x}_1)} - i/2\overline{x}_1 \\ H'_{\overline{x_1}} &= -i/2x - \overline{\varphi'_{\overline{y}}(t,x_2,\overline{x}_1)} + i/2x_1 \\ H'_{x_2} &= -\overline{\varphi'_{\overline{x}}(t,x_2,\overline{x}_1)} + \varphi'_x(t,x_2,\overline{y}) + i/2\overline{x}_2 \\ H'_{\overline{x_2}} &= -\overline{\varphi'_x(t,x_2,\overline{x}_1)} + \varphi'_{\overline{x}}(t,x_2,\overline{y}) + i/2x_2 \end{aligned} \tag{19}$$

It results from these relations that we have a "real" critical point when $x_1 = x$, $(x_2,\overline{x}_1) \in \Gamma_t$, $(x_2,\overline{y}) \in \Gamma_t$, i.e. when $x = y$ the critical point being $x_1 = x, x_2 = \Phi_t(\overline{x})$. Let $\varepsilon_1 > 0$ a small number to be chosen later.

In the integral 15, using 18 we can restrict the integration over the set of (x_1,x_2) such that $|(x_{2,}\overline{x}) - m_t(x_2,\overline{x})|^2 + |(x_{2,}\overline{y}) - m_t(x_2,\overline{y})|^2 \leq \varepsilon_0\lambda^{-1+\varepsilon_1}$, $|x-x_1|^2 \leq \varepsilon_0\lambda^{-1+\varepsilon_1}$ for some ε_0 to be chosen later; we neglect then a term $\mathcal{O}(\lambda^{-\infty})$. We deduce from the relations 19 that a zone

$|\varphi'_x(t,x_2,\overline{x}) - \varphi'_x(t,x_2,\overline{y})| \geq C\varepsilon_0^{1/2}\lambda^{(-1+\varepsilon_1)/2}$ give also a term $\mathcal{O}(\lambda^{-\infty})$. As $m_t(x_2,\overline{x})$ and $m_t(x_2,\overline{y}) \in \Gamma_t$, we have $\Phi_t(m_t(x_2,\overline{x})_y) = m_t(x_2,\overline{x})_x$ and $\Phi_t(m_t(x_2,\overline{y})_y) = m_t(x_2,\overline{y})_x$, Φ_t is a diffeomorphism whose first two derivatives are bounded by some e^{ct}, we have then $|x-y| \leq C\varepsilon_0 e^{ct}\lambda^{(-1+\varepsilon_1)/2}$.

Let us estimate the Hessian of H at real critical points.

We first make a complexification and we write $\tilde{x}_1, \tilde{x}_2$ instead of $\overline{x}_1, \overline{x}_2$, we shall refer to the set $\{\tilde{x}_1 = \overline{x_1}, \tilde{x}_2 = \overline{x}_2\}$ as the real. Let $\varphi_1(t,\tilde{x}_2,x_1)$ be an almost analytic extension of $\overline{\varphi(t,\overline{\tilde{x}_2},\overline{x}_1)}$. Let $\widetilde{H}$ be an almost analytic function on

$$\widetilde{\Gamma}_t = \left\{(x,y,x_1,\tilde{x}_1,x_2,\tilde{x}_2); (\overline{\tilde{x}_2},\overline{x}_1) \in \Gamma_t, (x_2,\overline{y}) \in \Gamma_t\right\}$$

extending H . We compute $\nabla^2\widetilde{H}(t,x,x;x,\overline{x},\Phi_t(\overline{x}),\overline{\Phi_t(\overline{x})})$ as the map

$$\begin{aligned}
&(\delta x_1, \delta\tilde{x}_1, \delta x_2, \delta\tilde{x}_2) \to \\
&\quad (i/2\delta\tilde{x}_1 - \varphi''_{1yy}\delta x_1 - \varphi''_{1xy}\delta\tilde{x}_2, i/2\delta x_1, \varphi''_{xx}\delta x_2 + i/2\delta\tilde{x}_2, \\
&\quad -\varphi''_{1xx}\delta\tilde{x}_2 - \varphi''_{1yx}\delta x_1 + i/2\delta x_2)
\end{aligned} \tag{20}$$

the computation being simplified by the fact that at such particular points $\nabla^2\varphi$ is C-linear. This map is invertible and its inverse will have the same norms as the inverse of

$$(\delta x_2, \delta\tilde{x}_2) \to (\varphi''_{xx}(t,x_2^c,\overline{x})\delta x_2 + i/2\delta\tilde{x}_2, -\overline{\varphi}''_{xx}(t,x_2^c,\overline{x})\delta\tilde{x}_2 + i/2\delta x_2) \tag{21}$$

with $x_2^c = \Phi_t(\overline{x})$. We have then to estimate the inverse of the map $\overline{\varphi}''_{xx}(t,x_2^c,\overline{x})\varphi''_{xx}(t,x_2^c,\overline{x}) - \frac{1}{4}$. We refer to ([6] Proposition 5.2 (5.28)) to obtain

$$\nabla^2\varphi(t,z) = \begin{pmatrix} \frac{1}{4}\overline{b}a^{-1} & \frac{i}{2}{}^t a^{-1} \\ -\frac{i}{2}a^{-1} & \frac{1}{4}{}^t b {}^t a^{-1} \end{pmatrix} \tag{22}$$

where $(t,z) \in \Gamma$, a and b are defined by the relation 23

$$(\delta x - i\delta\xi, \frac{\delta x + i\delta\xi}{2i}) = \begin{pmatrix} a & b \\ \frac{1}{4}\overline{b} & \overline{a} \end{pmatrix}(\delta y - i\delta\eta, \frac{\delta y + i\delta\eta}{2i}) \tag{23}$$

if $(\delta x, \delta\xi, \delta y, \delta\eta)$ is a tangent vector of Λ_t. We refer to [6] section 5, page 490. We have

$$(t,z) \in \Gamma, \ \left\|\varphi''^{-1}_{xy}(t,z)\right\| \leq 2\left\|a(t,z)\right\| \tag{24}$$

Moreover $\overline{\varphi}''_{xx}\varphi''_{xx} - \frac{1}{4} = -\frac{1}{4}a^{-1*}a^{-1}$ as a consequence of the relation $a^*a - \frac{1}{4}{}^t b\overline{b} = I$. So the norm of the inverse of the map 21 is $|\det a(t,z)|^2$. As computed in [6] the module of $e(t,z)$ at a point $(t,z) \in \Gamma$ is precisely $|\det a(t,z)|^{-1/2}$.

This means that in the stationary phase expansion of integral 15 the powers of λ and the exponentials decays in time vanish, it remains only the normal λ^n. See [6] section 6.2, page 509-510, relations (6.4) and (6.5).

The condition $\exp(Mt) \leq \lambda$ will allow us to give sense to the application of stationary phase expansion with a complex phase function (see [4]) uniformly with respect to (t,x,y).

We need to be more specific in the application of the stationary phase method. We check here some steps with uniform controls in t.

In a neighborhood $V = \left\{(x,y);\ |x-y| \leq \varepsilon_0 e^{-M_0 t}\right\}$ of the diagonal we have a C^∞ map $(t,x,y) \to Z_c(t,x,y) \in \mathbb{C}^{4n}$, where $Z = (x_1, \tilde{x}_1, x_2, \tilde{x}_2)$, such that $\partial_Z \tilde{H}(t,x,y; Z_c(t,x,y)) = 0$. The derivatives of this map satisfy some estimates like $\left|D^\alpha_{t,x,y} Z_c(t,x,y)\right| \leq C_\alpha \exp(ct\,|\alpha|)$ for some constants C_α and $c > 0$. Hence we have

$$|Z_c(t,x,y) - Z_c(t,x,x)| \leq C\exp(ct)\,|x-y| \tag{25}$$

We can define a symmetric complex matrix $Q(t,x,y)$ such that

$$\partial^2_{ZZ}\tilde{H}(t,x,y;Z_c(t,x,y)) = iQ^2(t,x,y)$$

which is well defined and smooth since V is connected and simply connected.

We have $\|Q^{-1}(t,x,y)\| \leq C\exp(ct)$ for some constants c and $C > 0$. We have

$$\begin{aligned} Im\tilde{H}(t,x,y;Z) &= Im(\tilde{H}(t,x,y;Z_c(t,x,y)) + \\ &\quad 1/2 Im\partial^2 \tilde{H}_{ZZ}(t,x,y;Z_c(t,x,y))(Z - Z_c(t,x,y))^2 + \\ &\quad O((dist(Z_c(t,x,y), real))^\infty + |Z - Z_c(t,x,y)|^3) \end{aligned} \tag{26}$$

we choose a point Z such that $Z \in real = \{x_1 = \tilde{x}_1; x_2 = \tilde{x}_2\}$, then $Im(\tilde{H}(t,x,y;Z) + \Phi(x,y) \geq 0$, and $dist(Z_c(t,x,y)), real) \leq |Z - Z_c(t,x,y)|$. We use then the estimate (2.6) of [4] and we derive

$$Im\tilde{H}(t,x,y;Z_c(t,x,y)) + \Phi(x,y) \geq Ce^{-at} dist(Z_c(t,x,y), real)^2 \tag{27}$$

This relation shows that the stationary phase expansions are independent of the choice of particular almost analytic extensions but is inadequate to bound the L^2 norm.

We shall compute $H_c(t,x,y) = (t,x,y;Z_c(t,x,y))$ with a Taylor expansion on the diagonal

$$\begin{aligned} H_c(t,x,y) &= H_c(t,x,x) + \nabla_y \widetilde{H}(t,x,x)(y-x) + 1/2\nabla^2_{yy}H(t,x,x)(y-x)^2 \\ &\quad -1/2((\widetilde{H}''_{ZZ})^{-1}\widetilde{H}''_{Zy}(y-x), \widetilde{H}''_{Zy}(y-x))) + \mathcal{O}(e^{at}(x-y)^3) \end{aligned} \tag{28}$$

The second term in 28 is $\varphi'_y(t,x_2^c,\overline{x})(\overline{y}-\overline{x}) = -i/2x(\overline{x}-\overline{y})$. The third term is given by $1/2\varphi''_{yy}(\overline{y}-\overline{x})^2$. $\widetilde{H}''_{Zy}(y-x) = (0,0,\varphi''_{xy}(\overline{y}-\overline{x}),0)$. The inverse $(\widetilde{H}''_{ZZ})^{-1}\delta X = \delta Z$ is given by the relations

$$\overline{\varphi}''_{yx}\delta Z_2 - \overline{\varphi}''_{yy}\delta Z_1 - i/2\delta\widetilde{Z}_1 = \delta X_1,\ i/2\delta Z_1 = \delta\widetilde{X}_1 \tag{29}$$

$$(\delta Z_2, \delta\widetilde{Z}_2) = \begin{pmatrix} -(1/4-\overline{\varphi}''_{xx}\varphi''_{xx})^{-1}\overline{\varphi}''_{xx} & -i/2(1/4-\overline{\varphi}''_{xx}\varphi''_{xx})^{-1} \\ -i/2(1/4-\varphi''_{xx}\overline{\varphi}''_{xx})^{-1} & (1/4-\varphi''_{xx}\overline{\varphi}''_{xx})^{-1}\varphi''_{xx} \end{pmatrix}(\delta X_2, \delta\widetilde{X}_2) \tag{30}$$

We make $\delta X = (0,0,\varphi''_{xy}(\overline{y}-\overline{x}),0)$. The fourth term in 28 is given by $1/2(\varphi''_{xy}(\overline{y}-\overline{x}),(1/4-\varphi''_{xx}\overline{\varphi}''_{xx})^{-1}\overline{\varphi}''_{xx}\varphi''_{xy}(\overline{y}-\overline{x}))$. Adding these two terms we have to compute $1/2(\varphi''_{yy}+\varphi''_{yx}(1/4-\overline{\varphi}''_{xx}\varphi''_{xx})^{-1}\overline{\varphi}''_{xx}\varphi''_{xy})$. We recall that $(1/4-\overline{\varphi}''_{xx}\varphi''_{xx}) = 1/4a^{-1*}a^{-1}$, so $\varphi''_{yy}+\varphi''_{yx}(1/4-\overline{\varphi}''_{xx}\varphi''_{xx})^{-1}\overline{\varphi}''_{xx}\varphi''_{xy} = 1/4^t b^t a^{-1} - 1/4a^{-1}aa^* b\overline{a}^{-1t}a^{-1} = 0$. We have therefore $H_c(t,x,y) = H_c(t,x,x) - i/2x\overline{y} + i/2x\overline{x}$.

We have now to compute $H_c(t,x,x)$, $(\partial/\partial t)H_c(t,x,x) = -\overline{\varphi}'_t(t,x_2^c,\overline{x}) + \varphi'_t(t,x_2^c,\overline{x})$ where $x_2^c = \Phi_t(\overline{x})$; on Γ_t $\varphi'_t(t,x,y) + p_1(x,\varphi'_x(t,x,y)) = 0$ then $Im\varphi'_t(t,x,y) = 0$; so $H_c(t,x,x) = H_c(0,x,x) = \psi(x,x) + 2iIm\psi(x,x) + 4i\varphi_0(x) = -\frac{i}{2}|x|^2$.

We have obtained

$$H_c(t,x,y) = -\frac{i}{2}x\overline{y} + \mathcal{O}(e^{at}|x-y|^3) \tag{31}$$

We deduce from 31 that in V,

$$ImH_c(t,x,y) + \Phi(x,y) \geq C^{-1}|x-y|^2 \tag{32}$$

From the usual estimate of L^2 norms, we obtain the result of Proposition 2. Moreover the relation $e^{Mt}\lambda^{-1} \leq 1$ shows that we have an convergent asymptotic development in term of uniform decay in λ. ♯

Proposition 3 *Let M and t satisfy $e^{Mt}\lambda^{-1} \leq 1$, then $E_t^* E_t$ is a pseudo-differential operator of order zero belonging to a class of $S(1, g_\varepsilon)$ (see [3] Chapter 18) where $g_\varepsilon = \lambda^{2\varepsilon}(|dx|^2 + |d\xi|^2)$, where $\varepsilon > 0$ depends on M and on the properties of the flow of H_{p_1}; when $M \to \infty$, $\varepsilon \to 0$.*

Let us recall that a symbol $a \in S(1, g_\varepsilon)$ satisfies uniform estimates

$$\textit{For all multi-indices } \alpha,\ \beta,\ \left|D_x^\alpha D_\xi^\beta a(x,\xi,\lambda)\right| \leq C_{\alpha,\beta}\lambda^{\varepsilon(|\alpha|+|\beta|)}.$$

Proof.

We shall derive this property from Proposition 2 and from the characterization of pseudo-differential operators due to Beals [1]. Let us estimate the L^2 norm of the first commutators $x^\alpha D_x^\beta E_t^* E_t - E_t^* E_t x^\alpha D_x^\beta$ for $|\alpha| + |\beta| = 1$. Using the computations of the proof above we express

$$\begin{aligned} x^\alpha D_x^\beta E_t^* E_t u(x) &= \int x^\alpha (H'_{xc}(t,x,y))^\beta f(t,x,y) e^{i\lambda H_c(t,x,y)} u(y)\, d\mu(y) \\ &\quad + \mathcal{O}(e^{at}\lambda^{-1}) \end{aligned} \tag{33}$$

in this formula $u \in H_{\varphi_0}(\mathbb{C}^n)$, the notation $\mathcal{O}$ means that the remainder has the same form but the order of the symbol is lowered. We have $E_t^* E_t(x^\alpha D_x^\beta u)(x) = \int f(t,x,y) e^{i\lambda H_c(t,x,y)} y^\alpha D_y^\beta u(y)\, d\mu(y)$, we integrate by part in this formula so

$$\begin{aligned} E_t^* E_t(x^\alpha D_x^\beta u)(x) &= (-1)^{|\beta|} \int f(t,x,y) e^{i\lambda H_c(t,x,y)} y^\alpha \\ &\quad (H'_{yc}(t,x,y) + i\overline{y}/2)^\beta u(y)\, d\mu(y) + \mathcal{O}(e^{at}\lambda^{-1}) \end{aligned} \tag{34}$$

We compare 33 and 34, using 31 we get $H'_{xc}(t,x,y) = -i\overline{y}/2 + \mathcal{O}(e^{at}(x-y)^2)$ and $H'_{yc}(t,x,y) = \mathcal{O}(e^{at}(x-y)^2)$, $x - y + \mathcal{O}(e^{at}(x-y)^2) = H'_{\overline{y}c}(t,x,y) + i/2y$.

We need an extra notation to make these integration by parts (more) rigorous. Let $G = (\lambda^{-1/2} + |x-y|)^{-1} e^{Ct}(|dx|^2 + |dy|^2)$, $M_k = (\lambda^{-1/2} + |x-y|)^{-k}$, let $h = H_c(t,x,y) + 2i\varphi_0(y)$. Assume that an amplitude $f(t,x,y) \in S(M_k, G)$, we note $A(f)$ the integral operator with amplitude f and phase function h. Using an integration by parts with the operator $L = (\left|h'_{\overline{y}}\right|^2 + \lambda^{-1})^{-1}(\overline{h}'_{\overline{y}}\partial/\partial\overline{y} + 1)$ and the fact that $u(y)$ is holomorphic we can replace f by $\lambda^{-N}\, {}^tL^N(f)$; so the same operator is given with an amplitude in $S(M_{k+2N}\lambda^{-N} e^{CNt}, G)$. We have shown before that if $f \in S(M_k, G)$ $ad_X(A(f)) = A(f_1)$ with $f_1 \in S(M_k e^{at} M_{-2}, G)$, where X is either x_j or D_{x_k}. So $ad_{X_1} \ldots ad_{X_k}(A(f)) = A(f_k)$ with $f_k \in S(e^{Ckt}\lambda^{-k}, G)$.♯

We deduce then that $E_t^* E_t$ is a pseudo-differential operator in the class $S(1, g_t)$ where g_t is the metric $g_t = e^{2Ct}(|dx|^2 + |d\xi|^2)$.

This result is then optimal with the restriction that we may not have the best constant C and that we consider here spatially homogeneous metrics.$\sharp$

3 Semi global L^2 estimates.

We shall work as in [7] with L^2 estimates for solutions of the Schrödinger equation $(D_t + P(x, D_x))u(t) = 0$. More precisely

$$\begin{aligned}2\lambda Im \int_T^{T_0} ((D_t + P(x, \lambda^{-1}D_x)u(t), \alpha(t)u(t))\, dt = \\ (\alpha(T)u(T), u(T)) - (\alpha(T_0)u(T_0), u(T_0)) + \int_T^{T_0} (M(t)u(t), u(t))\, dt\end{aligned} \tag{35}$$

with the notation

$$M(t) = (\partial\alpha(t)/\partial t) - i\lambda[P_1, \alpha(t)] + 2\lambda Re(\alpha(t)P_2)) \tag{36}$$

$\alpha(t)(x, \lambda^{-1}D_x)$ is a family of self-adjoint operators to be chosen later, T_0 will depend only on λ. We have to make $(\partial\alpha(t)/\partial t) - i\lambda[P_1, \alpha(t)] + 2\lambda Re(\alpha(t)P_2))$ as large as possible. We choose $\alpha(t) = E_t\beta(t)E_t^*$ where E_t has been constructed in section 2. $\beta(t)$ will be chosen later.

If we note $R(t) = (\partial/\partial t)E_t - i\lambda P_1 E_t$, we obtain

$$\begin{aligned}(\partial\alpha(t)/\partial t) - i\lambda[P_1, \alpha(t)] + 2\lambda Re(\alpha(t)P_2)) = \\ E_t(\partial\beta(t)/\partial t)E_t^* + 2\lambda Re(E_t\beta(t)E_t^* P_2) + 2Re(R(t)\beta(t)E_t^*)\end{aligned} \tag{37}$$

we shall deal later with the last term in 37. We have written in 5 $P_2 = ((p_2(x, \xi) + \lambda^{-1}Im(p_0(x, \xi, \lambda))^w$. Let $\beta(t) = \exp(-2\gamma t)$ where $Im(p_0(x, \xi, \lambda) \geq \gamma$. If $Q = P_2 - \gamma\lambda^{-1}$ the Weyl symbol of Q is non negative. We have to estimate from below the operator $Re(E_t E_t^* Q)$, we have proved in Proposition 3 that $E_t E_t^* \in S(1, g_\varepsilon)$, it is then a consequence of the Fefferman-Phong inequality that $Re(E_t E_t^* Q) \geq -C\lambda^{-2+2\varepsilon}$. So from 37 we obtain

$$(M(t)u(t), u(t)) \;\geq\; -C\lambda^{-1+2\varepsilon}e^{-2\gamma t}|u(t)|^2 + 2(Re(R(t)\beta(t)E_t^*)u(t), u(t)) \tag{38}$$

We shall deal later with the last term in 38. In 35 we shall input $u(t) = u$, so the left hand-side of 35 is an $\mathcal{O}(\lambda^{-\infty})$ uniformly in time.

We make an induction. Let W be an open neighborhood of ρ_0, we say that $u \in H^\sigma(W)$ if for any pseudo-differential operator $\varphi(x, \lambda^{-1}D_x)$ with $supp\varphi \subset W$ u satisfies $|\varphi u| \leq C\lambda^{-\sigma}$. Assume $u \in H^\sigma(W)$.

Let W_1, W_2 be two open sets such that $W_1 \subset\subset W_2 \subset\subset W$. We use the construction of [6] section 6.3 of a cut-off function $\chi(t, x, y)$ defined by taking an almost analytic extension of the restriction to Γ_t of the function

$$\widetilde{\chi}(t, y) = \begin{cases} \zeta_1(y)\exp(-\int_0^t \frac{\psi}{\zeta^2}(\Phi_s(y))ds) & if \int_0^t \frac{\psi}{\zeta^2}(\Phi_s(y))ds < \infty \\ 0 & if \int_0^t \frac{\psi}{\zeta^2}(\Phi_s(y))ds = \infty \end{cases} \tag{39}$$

with the notations $\zeta \in C_0^\infty(W_2)$, $\zeta \equiv 1$ in $\overline{W}_1$; $\psi \in C_0^\infty(W_2\backslash\overline{W}_1), \psi \equiv 1$ in a neighborhood of $\partial\{x; \zeta(x) > 0\}$; $\zeta_1(y) \in C_0^\infty(\{x; \zeta(x) > 0\})$ is one on a neighborhood of $\overline{W}_1$. It was proved in [6] that such constructions give a smooth function whose derivatives are bounded by some $\exp(h_1(t))$ with $h_1 \in o(t)$ when $t \to \infty$. If we add a cut-off function in $\left\{y;\, |p_1(y)| \leq e^{-h(t)}\right\}$ the corresponding $\widetilde{\chi}$ will be supported in

$$\Lambda_{W_2} = \left\{(t, x, y);\, x = \Phi_t(y),\, for\, 0 \leq s \leq t\, \Phi_s(y) \in W_2,\, |p_1(y)| \leq c_2 e^{-h(t)}\right\} \tag{40}$$

with value 1 on Λ_{W_1}.

In view the lower bound

$$Im\varphi(t, x, y) + \Phi(x, y) \geq C^{-1}\left|(x, y) - m_t(x, y)\right|^2 \tag{41}$$

we shall remain as close to Γ_t as we wish.

We make $T = 0$ and $T_0 = 1/M \ln\lambda$. The condition in Proposition 3 is satisfied for $0 \leq t \leq T_0$. $\alpha(0) = E_0E_0^*$ is elliptic in W_1. $\beta(T_0)E_{T_0}E_{T_0}^*$ is a pseudo-differential operator with wave front set contained in W_2, belonging to the class $S(e^{-2\gamma T_0}, g_\varepsilon)$. We have therefore $(\alpha(T_0)u, u) \leq C\lambda^{-2\sigma-2\gamma/M}$ if $u \in H^\sigma(W)$. Using 38 and the fact that ε is close to 0, we shall conclude that $H^{\sigma+\gamma/M}(W)$ after taking care of the last term 35.

The operator $R(t)$ comes from that E_t is not an exact solution of the equation $(D_t + P_1)E_t = 0$, which is due to the presence of the χ. The function χ is itself necessary since we want to localize near $\Lambda(W, h)$ to get our theorem. This analysis has been carried out in [6] section 6.4. That we wish to say is that the assumption $\overline{\Lambda(W, h)}(\rho) \cap \partial W \cap OF(u) = \emptyset$ implies that $|R(t)^*u| = \mathcal{O}(\lambda^{-\infty})$. This kind of troncature are precisely what we need to derive a propagation of singularities theorem from an ordinary L^2 inequality; as we said above this makes all this machineary necessary. ♯

Remark 1 *The condition $(H)_3$ allows to prove a theorem of propagation of singularities with a loss of one derivative, in this sense this condition is sharp.*

Part II

A more precise result in a particular case.

We shall be able to get a sharper result in a symplectic case analogous to the case treated in [5].

4 Construction of the stable manifolds.

In this section we shall use some elements of [10] Appendix A, and [6] Section 4, we prefer to recall all this material in our proof than to use obscure references to these works.

Let $p(x,\xi)$ be an analytic complex function. Let $\rho_0 \in N \cap \mathbf{R}^{2n}$ where

$$N = \left\{(x,\xi) \in \mathbf{C}^{2n};\ p(x,\xi) = dp(x,\xi) = 0\right\} \tag{42}$$

Let $H_p = p'_\xi \partial/\partial x - p'_x \partial/\partial\xi$ be the hamiltonian field, we mean by bicharacteristic of p the integral curves of the real vector field on $\mathbf{C}^{2n}$ $H_p + \overline{H}_p$.

Let

$$\Lambda_t = \left\{(\rho(t,\rho),\rho);\ \rho \in \mathbf{C}^{2n},\ p(\rho) = 0\right\}\ and\ \Lambda_{t,\mathbf{R}} = \left\{(\rho(t,\rho),\rho);\ \rho \in \mathbf{R}^{2n},\ p(\rho) = 0\right\} \tag{43}$$

in 43 $t \to \rho(t,\rho)$ is a bicharacteristic curve starting at ρ. We shall assume

- (H_1) :

$$Imp(\rho) \geq 0\ if\ \rho \in \mathbf{R}^{2n}. \tag{44}$$

The fundamental matrix is $F_p(x,\xi) = dH_p(x,\xi) = \begin{pmatrix} p''_{\xi x} & p''_{\xi\xi} \\ -p''_{xx} & -p''_{x\xi} \end{pmatrix}$.

In the Jordan decomposition of $F_p(\rho)$, we note $W_+(\rho) = \oplus_{Re\lambda>0,\ \lambda\in Spec(F_p)} V_\lambda$, $W_-(\rho) = \oplus_{Re\lambda<0,\ \lambda\in Spec(F_p)} V_\lambda$. V_λ are the generalized eigenspaces.

- (H_2) : We assume that

$$\mathbf{C}^{2n} = W_+(\rho) \oplus W_-(\rho) \oplus W_0(\rho) \oplus Ker F_p \tag{45}$$

and the dimensions of these three spaces are constant along N.

The assuption (H_2) means there are no non zero eigenvalue in $i\mathbf{R}$, that there is also no generalized eigenspace relative to zero and that $\dim W_+(\rho) = r_+$, $\dim W_-(\rho) = r_-$ are constant. We note respectively by $P_+(\rho)$, $P_-(\rho)$ and $P_0(\rho)$ the corresponding projectors, it follows from our assumption that these maps are analytic.

As $\sigma(V_\lambda, V_\mu) = 0$ if $\lambda+\mu \neq 0$, $W_+(\rho) \oplus W_0(\rho) \subseteq W_+(\rho)^{\perp_\sigma}$, then $r_+ \leq r_-$, then

$$r_+ = r_- = r \text{ and } W_\pm(\rho)^{\perp_\sigma} = W_\pm(\rho) \oplus W_0(\rho) \tag{46}$$

- (H_3) : We also assume that in a neighborhod of ρ_0, there is a constant C_0 such

$$|H_p(\rho)| \leq C_0 \, |(I - P_0(\rho_0)) H_p(\rho)| \tag{47}$$

it is a consequence of the assumptions of constant ranks that 47 is independent of ρ_0.

- (H_4) : We shall assume that on $N' = \{(x, \xi) \in \mathbf{R}^{2n}; Rep(x,\xi) = dRep(x,\xi) = 0\}$, we have $\mathbf{C}^{2n} = W'_+ \oplus W'_- \oplus W'_{+i} \oplus W'_{-i} \oplus Ker F_{Rep}$ where $W'_\pm$ are the correponding spaces for F_{Rep} and $W'_{\pm i} = \oplus_{i\lambda \in Spec(F_{Rep}), \pm\lambda > 0} V_\lambda$. We suppose also that the quadratic form $[v, \bar{v}] = \frac{1}{i}\sigma(v, \bar{v}) \leq 0$ on W'_{+i}. This means in fact simply that $V'_0 = Ker F_{Rep}$ and that there are no no difference of harmonic oscillators in a spectral decomposition of F_{Rep}. In addition we assume that N' is a smooth manifold and that $Ker F_{Rep}(\rho) = \mathrm{T}_\rho N'$.

The first step is to construct stable manifolds for the complex symbol p. Let (x, y) be coordinates such that $x \in W_+(\rho_0) \oplus W_0(\rho_0)$, $y \in W_-(\rho_0)$, we split again $x = (x', z)$ where $x' \in W_+(\rho_0)$, $z \in W_0(\rho_0)$. We note $W^+(\rho) = W_+(\rho) \oplus W_0(\rho)$. Let us note again by $P_+(\rho)$ an analytic extension of this function away from N.

When we split $\mathbf{C}^{2n} = W^+(\rho) \oplus W_-(\rho)$, we have a decomposition of $F_p(\rho) = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}(\rho)$. We shall split further $\alpha = \begin{pmatrix} \alpha_0 & \alpha_1 \\ \alpha_2 & \alpha_3 \end{pmatrix}$ along $W_+(\rho) \oplus W_0(\rho)$.

It is a consequence of the assumptions that in a close neighborhood of N,

$$\begin{array}{c} Spec(\alpha_0) \subseteq \{z \in \mathbf{C},\ Re z \geq c\},\ Spec(\delta) \subseteq \{z \in \mathbf{C},\ Re z \leq -c\}, \\ \|\beta\| \leq \varepsilon,\ \|\gamma\| \leq \varepsilon\, \|\alpha_i\| \leq \varepsilon \ for\ 1 \leq i \leq 3;\ c > 0,\ \varepsilon \ is\ small. \end{array} \tag{48}$$

Let ρ be a point close to ρ_0 and $t \to \rho(t, \rho)$ the bicharacteristic issued from ρ. The evolution of tangent vectors is given by the linear differential equation $\frac{d}{dt} v_t = F_p(\rho(t, \rho)) v_t,\ v_{t|_{t=0}} = v_0$.

We find a linear map $\varphi_t(\rho)$ from $W^+(\rho_0)$ to $W_-(\rho_0)$ such that the evolution of the space $W^+(\rho_0)$ along the flow is given by $W_t^+(\rho) = \{(\delta x, \varphi_t(\rho)\delta x), \delta x \in W^+(\rho_0)\}$. This is achieved as in [6] by solving the equation

$$\dot{\varphi}_t + \varphi_t \alpha - \delta \varphi_t + \varphi_t \beta \varphi_t - \gamma = 0, \varphi|_{t=0} = 0. \tag{49}$$

In view of the relations 48, which imply that $Spec(\alpha) \subseteq \{z \in \mathbf{C},\ Re z \geq -\varepsilon\}$ we know that the equation 49 can be solved for $t \geq 0$ and we have $\|\varphi_t\| \leq C\varepsilon$ for some small ε.

We define a suitable norm to construct regions stable under the flow. Let $\alpha_0 = \alpha(\rho_0)$ and

$$C_0 = \int_{-\infty}^{0} \exp(t\alpha_0^*) \exp(t\alpha_0)\, dt,\ C_0 > 0,\ C_0 \alpha_0^* + \alpha_0 C_0 = Id_{W_+(\rho_0)} \tag{50}$$

The restriction of $F_p(\rho_0)$ to $W_(\rho_0)$ is expressed by $\delta y \to \delta_0(\delta y)$, $Spec(\delta_0) \subseteq \{z \in \mathbf{C},\ Re z < -c\}$. We define $D_0 > 0, D_0 \delta_0^* + \delta_0 D_0 = -Id_{W-(\rho_0)}$.

We note $\|v_{x'}\|^* = (C_0 v_{x'}, v_{x'})^{1/2}$, $\|v_x\|^2 = \|v_{x'}\|^{*2} + |v_z|^2$, $\|v_y\|_* = (D_0 v_y, v_y)^{1/2}$ and $\|v\|^2 = \|v_{x'}\|^{*2} + \|v_y\|_*^2 + |v_z|^2$.

We expand
$\frac{d}{dt}(\rho_t - \rho_0) = F_p(\rho_0)(\rho_t - \rho_0) + \mathcal{O}((\rho_t - \rho_0)^2)$ so

$$\frac{d}{dt} \|\rho_t - \rho_0\|^2 = |(\rho_t - \rho_0)_{x'}|^2 - |(\rho_t - \rho_0)_y|^2 + \mathcal{O}((\rho_t - \rho_0)^3) \tag{51}$$

Proposition 4 *Let $0 \leq f(x) \in C_0^\infty(\mathbf{R}^+)$ be a function $f \leq \eta$. Let*

$$B(\rho_0, f) = \left\{x \in W^+(\rho_0);\ \|x'\|^* < f(|z|^2)\right\} \tag{52}$$

$E(\rho_0, T, f)$ be the region

$$E(\rho_0, T, f) = \left\{ \begin{array}{c} \rho; \rho - \rho_0 \in W^+(\rho_0);\ \rho_t - \rho_0 \in B(\rho_0, f) \\ and\ |\rho_t - \rho_0| < \varepsilon\ for\ 0 \leq t \leq T \end{array} \right\} \tag{53}$$

There exist a bounded set of analytic functions $x \in B(\rho_0, f) \to \lambda(t,x) \in W_-(\rho_0)$ such that $E(\rho_0,T,f)$ can be identified with the set

$$E'(\rho_0,T,f) = \left\{ \begin{array}{c} \rho;\ \rho-\rho_0 \in W^+(\rho_0),\ \text{for } 0 \le t \le T\ |\rho_t-\rho_0| < \varepsilon \text{ and} \\ \exists x_t \in W^+(\rho_0),\ x_t \in B(\rho_0,f) \text{ such that } \rho_t - \rho_0 = (x_t, \lambda(t,x_t)) \end{array} \right\} \tag{54}$$

This is proved as in [6] by induction on T. Let us sketch the proof.

Assume that we have constructed the function $\lambda(t,x)$ for $t = T_0$. We shall prove that it can be extended for some amount in time. The curve $s \to \rho(s)$ defined by $\rho(s) = \exp(-T_0H_p)(\rho_0 + (x_{T_0} + s\delta x, \lambda(T_0, x_{T_0} + s\delta x))$, is a curve in $\rho_0 + W^+(\rho_0)$. It follows from the definition of φ_t that $(\partial/\partial x)\lambda(T_0,x) = \varphi_{T_0}(\rho(0))$, therefore $\|(\partial/\partial x)\lambda(T_0,x)\| \le C\varepsilon$. We have also $\lambda(T_0,0) = 0$. In view of the analyticity we derive further controls on all the derivatives.

We note $g(x) = \|x'\|^* - f(|z|^2)$, and $\tilde{f}(z) = f(|z|^2)$ with $f(0) > 0$.

We define $\psi_{t,T_0}(x) = P^+(\rho_0)(\exp((t-T_0)H_p))(\rho_0 + (x, \lambda(T_0,x)) - \rho_0), \tilde{x} = \psi_{t,T_0}(x)$. The map ψ_{t,T_0} is close to the identity when t is close to T_0. We have

$$|\psi_{\tau,T_0}(x) - x| \le C\,|\tau - T_0|\,|x| \tag{55}$$

We want to prove that $\psi_{t,T_0}(x) \in B(\rho_0,f)$ implies $x \in B(\rho_0,f)$. We assume first that $|x'| \ge C^{-1}|x|$.

$$\begin{aligned} \frac{d}{d\tau} g(\psi_{\tau,T_0}(x)) &= \|\psi_{\tau,T_0}(x)_{x'}\|^{*-1} < \psi_{\tau,T_0}(x)_{x'}, (\frac{d}{d\tau}\psi_{\tau,T_0}(x))_{x'} > \\ &\quad -\nabla\tilde{f}((\psi_{\tau,T_0}(x))_z).(\frac{d}{d\tau}\psi_{\tau,T_0}(x))_z \end{aligned} \tag{56}$$

In view of relation 55 we can replace $\psi_{\tau,T_0}(x)$ by x in the first term of 56 modulo $\mathcal{O}((\tau - T_0))$.

We compute $\frac{d}{d\tau}\psi_{\tau,T_0}(x) = P^+(\rho_0)H_p(\mu(\tau,T_0,x))$ where $\mu(\tau,T_0,x) = \exp((\tau-T_0)H_p))(\rho_0 + (x, \lambda(T_0,x)))$.

Using the estimate $\|\mu(\tau,T_0,x) - (\rho_0 + (x,\lambda(T_0,x))\| \le C\,|\tau - T_0|\,|x|$, we obtain $H_p(\mu(\tau,T_0,x)) = F_p(\rho_0)(x,\lambda(T_0,x))) + \mathcal{O}((\tau-T_0)\,|x| + |x|^2)$. Therefore $H_p(\mu(\tau,T_0,x)) = (\alpha_0(\rho_0)x', 0, \delta(\rho_0)\lambda(T_0,x)) + \mathcal{O}((\tau-T_0)\,|x| + |x|^2)$. Hence

$$\|\psi_{\tau,T_0}(x)_{x'}\|^{*-1} < \psi_{\tau,T_0}(x)_{x'}, (\frac{d}{d\tau}\psi_{\tau,T_0}(x))_{x'} > \ge C^{-1}|x'| \tag{57}$$

We want now to estimate the second term in 56, $(\frac{d}{d\tau}\psi_{\tau,T_0}(x))_z = \mathcal{O}((\tau - T_0)\,|x| + |x|^2)$, $\nabla\tilde{f}(z) = \mathcal{O}(|z|\,|\nabla f|)$. Therefore $\frac{d}{d\tau}g(\psi_{\tau,T_0}(x)) \ge C^{-1}|x'|$ when $|x'| \ge C^{-1}|x|$. Then $g(\psi_{\tau,T_0}(x)) \ge g(x)$, so $x \in B(\rho_0,f)$.

If on the contrary we have $|z| \geq C\,\|x'\|^*$, then the point x is interior to $B(\rho_0, f)$.♯

We prove now :

Proposition 5 *There exist an involutive manifold $E(\rho_0,\infty)$ of codimension r, stable under H_p, contained in $p^{-1}(0)$, such that $\lim_{t\to\infty}\rho(-t,\rho)$ exist and belongs to $N = \{\rho;\, p(\rho) = H_p(\rho) = 0\}$ for any bicharacteristic curve issued from a point $\rho \in E(\rho_0,\infty)$.*

By the Ascoli's theorem, we know that there is a sequence $t_j \to \infty$, such that the functions $\lambda(t_j, x) \to \lambda(\infty, x)$.

Let $E^t(\rho_0, f) = \exp(tH_p)(E(\rho_0, t, f))$.

Let $t \to \rho(t,\rho)$ be a bicharacteristic curve such that $\rho_t \notin E^t(\rho_0, f)$, let $\gamma_t \in E^t(\rho_0, f)$ such that $\rho_t - \gamma_t \in (\mathrm{T}_{\gamma_t}E^t(\rho_0, f))^\perp$, the orthogonality being relative to the $\|\ \|$ norm, the length of $\rho_t - \gamma_t$ measures the distance $d(\rho_t, E^t(\rho_0, f))$.

We compute $\frac{d}{dt}(\rho_t - \gamma_t) = H_p(\rho_t) - \frac{d}{dt}\gamma_t$. Let us write $\gamma_t = \rho_0 + (x_t, \lambda(t, x_t))$; and $\gamma_t = \exp(tH_p)(\delta_t)$, $\delta_t \in E(\rho_0, t, f)$, so $\frac{d}{dt}\gamma_t = H_p(\gamma_t) + d(\exp(tH_p)(\delta_t))\dot\delta_t$, $\dot\delta_t \in W^+(\rho_0)$. Then $d(\exp(tH_p))(\delta_t))\dot\delta_t$, $\dot\delta_t = (\zeta_t, \varphi_t(\delta_t)\zeta_t)$ for some $\zeta_t \in W_+(\rho_0)$. We have proved above that $\varphi_t(\delta_t) = (\partial/\partial x)\lambda(t, y_t)$ where $\exp(-tH_p)((\rho_0 + (y_t, \lambda(t, y_t))) = \delta_t$, so $y_t = x_t$ and $(\zeta_t, \varphi_t(\delta_t)\zeta_t) \in \mathrm{T}_{\gamma_t}E^t(\rho_0, f)$.

Therefore $\frac{d}{dt}\,\|\rho_t - \gamma_t\|^2 = < H_p(\rho_t) - H_p(\gamma_t), \rho_t - \gamma_t >$, where $<,>$ is the scalar product for $\|\ \|$.

$H_p(\rho_t) - H_p(\gamma_t) = F_p(\rho_0)(\rho_t - \gamma_t) + \mathcal{O}(|(\rho_t - \gamma_t)|^2 + |(\rho_t - \gamma_t)|\,|\gamma_t - \rho_0|)$.

As $(\rho_t - \gamma_t) \in (\mathrm{T}_{\gamma_t}E^t(\rho_0, f))^\perp$, we have the relation

$$\begin{pmatrix} C_0 & 0 \\ 0 & I \end{pmatrix}(\rho_t - \gamma_t)_x + \varphi_t(\delta_t)^* D_0(\rho_t - \gamma_t)_y = 0.$$

We deduce that $|(\rho_t - \gamma_t)_x| \leq C\varepsilon\,|(\rho_t - \gamma_t)_y|$.

$< F_p(\rho_0)(\rho_t - \gamma_t), \rho_t - \gamma_t > = |(\rho_t - \gamma_t)_x|^2 - |(\rho_t - \gamma_t)_y|^2$, therefore

$$d(\rho_t, E^t(\rho_0, f)) \leq C\exp(-C^{-1}t) \tag{58}$$

Let $\rho_t = (x, \lambda(t,x)) \in E^t(\rho_0, f)$, if $s \leq t$ we write $\rho_t = \rho_s(\rho_{t-s})$, then there exist $y \in W^+(\rho_0)$ such that $|(y, \lambda(s,y)) - (x, \lambda(t,x))| \leq C\exp(-C^{-1}s)$, so $|\lambda(t,x) - \lambda(s,x)| \leq C\varepsilon\,|x - y| + |\lambda(s,y)) - \lambda(t,x)| \leq 2C\exp(-C^{-1}s)$. We have therefore proved that $\lambda(t,x) \to \lambda(\infty, x)$ in the space of holomorphic functions.

We define

$$E(\rho_0, \infty) = \{\rho; \rho = \rho_0 + (x, \lambda(\infty, x)) \, for\, some\, x \in B(\rho_0, f)\} \tag{59}$$

A proof similar shows that

$$d(\rho_t, E(\rho_0, \infty)) \le C \exp(-C^{-1}t). \tag{60}$$

Starting from a point $\rho \in E(\rho_0, \infty)$, we prove then that $\lim_{t\to\infty} \rho(-t, \rho)$ exist.

We prove first that $E(\rho_0, \infty)$ is H_p invariant. The tangent space $\mathrm{T}_\rho E(\rho_0, \infty) = \left\{(\delta x, (\frac{\partial}{\partial x})\lambda(\infty, x)\delta x)\right\}$ is the limit of the spaces $\left\{(\delta x, (\frac{\partial}{\partial x})\lambda(t, x)\delta x)\right\}$ when $t \to \infty$. For a point $\rho = \rho_0 + (x, \lambda(\infty, x))$ we note $y_t(x) \in B(\rho_0, f)$ the point defined by $\rho_0 + (x, \lambda(t, x)) = \exp(tH_p)(\rho_0 + (y_t(x), 0)$. $(\frac{\partial}{\partial x})\lambda(t, x) = \varphi_t(\rho_0 + (y_t(x), 0))$, $H_p(\rho) = m(t, \rho_0 + (y_t(x), 0))v_t$, where $v_t = H_p(\rho_0 + (y_t(x), 0))$, so $w_t = m(t, \rho_0 + (y_t(x), 0))((v_t)_x, 0) = ((w_t)_x, \varphi_t(\rho_0 + (y_t(x), 0)(w_t)_x)) \in \mathrm{T}_\rho E^t(\rho_0, t)$, the evolution of $(v_t)_y$ by m_t is an $\mathcal{O}(\exp(-C^{-1}t))$, therefore the distance from $H_p(\rho)$ to the space $\left\{(\delta x, (\frac{\partial}{\partial x})\lambda(t, x)\delta x)\right\}$ is also an $\mathcal{O}(\exp(-C^{-1}t))$. So $H_p(\rho) \in \mathrm{T}_\rho E(\rho_0, \infty)$.

$$\frac{d}{dt}\rho(-t, \rho) = -H_p(\rho(-t, \rho)) = -(H_p(\rho(-t, \rho))_x, (\frac{\partial}{\partial x})\lambda(\infty, x_t)H_p(\rho(-t, \rho))_x) \tag{61}$$

where $\rho(-t, \rho) = \rho_0 + (x_t, \lambda(\infty, x_t))$. We bound the $H_p(\rho(-t, \rho))_x$ component of H_p by $C_0(|H_p(\rho(-t, \rho))_{x'}| + |H_p(\rho(-t, \rho))_y|) \le C_0' \, |H_p(\rho(-t, \rho))_{x'}|$, using the assumption 47 and 61. In the backward evolution, the x' directions are contractive so

$$|H_p(\rho(-t, \rho))_{x'}| \le C \exp(-C^{-1}t) \, |H_p(\rho)| \tag{62}$$

this means that $|H_p(\rho(-t, \rho))| \le C \exp(-C^{-1}t) \, |H_p(\rho)|$, then $\tilde{\rho} = \lim_{t\to\infty} \rho(-t, \rho)$ exist and belongs to $N' = \{\rho; H_p(\rho) = 0\}$. On the connected component of ρ_0 of N', $p = 0$, then $\tilde{\rho} \in N$. But $p(\rho) = p(\rho(-t, \rho)) = p(\tilde{\rho}) = 0$, hence $p|_{E(\rho_0, \infty)} = 0$.

$E(\rho_0, \infty)$ is a smooth manifold of codimension r.

We shall prove that $\mathrm{T}_\rho(E(\rho_0, \infty))^{\perp_\sigma} \subsetneq \mathrm{T}_\rho(E(\rho_0, \infty))¿$ $W_-(\rho_0)^{\perp_\sigma} = W_-(\rho_0) \oplus W_0(\rho_0)$.

$$W_0(\rho_0)^{\perp_\sigma} = ImF_p(\rho_0) = W_+(\rho_0) \oplus W_-(\rho_0), \; so\; \sigma|_{W_0(\rho_0)} \; is\; non\; degenerate. \tag{63}$$

Let $v_0 \in \mathrm{T}_\rho(E(\rho_0,\infty))^{\perp_\sigma}$, $v_t = m(t,\rho)v_0$, $v_t \in (\mathrm{T}_{\rho_t}(E(\rho_0,\infty))^{\perp_\sigma}$. Then for all $\delta x \in W^+(\rho_0)$

$$\sigma(\delta x', (v_t)_y) + \sigma(\delta z, (v_t)_z) + \sigma((\frac{\partial}{\partial x})\lambda(\infty)\delta x, (v_t)_{x'}) = 0 \tag{64}$$

so $|(v_t)_y| + |(v_t)_z| \leq C\varepsilon|(v_t)_{x'}|$, therefore $|v_t| = \mathcal{O}(\exp(C^{-1}t))$ when $t \to -\infty$. Therefore $v_0 \in \mathrm{T}_\rho(E(\rho_0,\infty))$. $E(\rho_0,\infty)$ is an involutive manifold.♯.

We shall prove now L^2 estimates. This is done by working on the real line only since we look at C^∞ singularities, let us note by $\{f,g\}$ the usual Poisson bracket.

We can now state the main result of this section.

Theorem 2 *Let $P(x,\lambda^{-1}D_x,\lambda) = (p(x,\xi) + \lambda^{-1}p_1(x,\xi,\lambda))^{w_\lambda}$ be a pseudo-differential operator such that $p(x,\xi)$ satisfies the assumptions H_1, …, H_4. Let λ_j be the eigenvalues of F_p with $Re\lambda_j > 0$ at the points of $N_{\mathbf{R}} = \{\rho \in \mathbf{R}^{2n};\ p(\rho) = dp(\rho) = 0\}$, we assume that*

$$ip_1(x,\xi,\lambda) + \sum_j (\alpha_j + 1/2)\lambda_j \neq 0,\ for\ all\ \alpha_j \in \mathbf{N}. \tag{65}$$

Let $\gamma(\rho)$ and $c(\rho) \geq 0$ be smooth functions such that $\{p,\gamma\} + cp \geq 0$. If ω is a small neighborhood of $\rho_0 \in N_{\mathbf{R}}$, suppose that $\gamma(\rho_0) > 0$ and $\{\gamma > 0\}\cap\omega\cap OF(Pu) = \emptyset$ and $\{\gamma > 0\}\cap\partial\omega\cap OF(u) = \emptyset$, then $\rho_0 \notin OF(u)$.

Remark 2 *It is possible to make a less technical statement in the particular case where $W'_{\pm i} = \{0\}$. In this case N' is a smooth symplectic manifold of codimension $2r'$. The involutive manifolds $E'_\pm$ have a foliation, we note by $F_-(\mu)$ the leaf of E'_- throught $\mu \in N'$. Then the geometric statement of Theorem 2 is : if $F_-(\rho_0)\cap\omega\cap OF(Pu) = \emptyset$ and $F_-(\rho_0)\backslash\{\rho_0\}\cap\omega\cap OF(u) = \emptyset$ then $\rho_0 \notin OF(u)$.*

This remark will be justified below when we will construct appropriate functions γ. Moreover the presence of the function $c(\rho)$ is needed to have a statement invariant by multipication of P by an operator with a positive symbol.

5 The energy estimate.

5.1 The basic L^2 inequality.

The basic L^2 estimate will be described in the case $Imp_1 > 0$. We shall use microlocal weighted estimates. Let $\gamma(x,\xi) \in \mathrm{C}^\infty(\mathbf{R}^{2n})$ a bounded real valued function, we note $e_\gamma = (\lambda^{\gamma(x,\xi)})^{w_\lambda}$, we write $\mu = \ln\lambda/\lambda$. Let $e'_{-\gamma}$ be a parametrix of e_γ. If $A = (a(x,\xi)$ is a pseudo-differential operator with Weyl symbol $a(x,\xi) \in S(1,g)$, then $A_\gamma = e'_\gamma A e_{-\gamma} = (a(x,\xi)+i\mu\{a,\gamma\}+O(\mu^2))^{w_\lambda}$. We write our operator as $P = (p(x,\xi) + \lambda^{-1}p_1(x,\xi,\lambda))^{w_\lambda}$. Then

$$P_\gamma = (p(x,\xi) + i\mu\{p,\gamma\}(x,\xi) + \lambda^{-1}p_1(x,\xi,\lambda) + O(\mu^2))^{w_\lambda}. \tag{66}$$

We use also a multiplier $M = (m' + i\mu m'')$, with two real functions $m' \in S(1,g)$ and $m'' \in S(1,g)$. We get an energy estimate from the computation of $Im(P_\gamma u, M^* u)$. We have

$$Im(MP_\gamma) = (m' Imp + \mu(m'' Rep + m'\{Rep,\gamma\}) + \lambda^{-1}(m' Imp_1 + \{Rep, m'\}) + O(\mu\lambda^{-1}))^{w_\lambda}. \tag{67}$$

We must make the symbol in 67 positive. The first term $m' Imp$ is nonegative if $m' \geq 0$. We now concentrate on the second term $m'' Rep + m'\{Rep,\gamma\}$. Let $m''/m'(\rho) = c(\rho)$ be a C^∞ function. $cRep + \{Rep,\gamma\}$ is null on N' so the best possible choice of γ is to make it transversally elliptic on N'.
$\gamma = \gamma_0 + \gamma_1$, $c = c_0 + c_1$, γ_0 and c_0 are the functions which appears in the statement of Theorem 2, γ_1 and c_1 are constructed below. If γ_1 is null at the second order on N', the hessian of $c_1 Rep + \{Rep, \gamma_1\}$ at $\rho \in N'$ is given by the fundamental matrix $c_1(\rho)F_{Rep}(\rho) + [F_{Rep}, F_{\gamma_1}](\rho)$.

We shall localize at points of N', let $\rho \in N'$ we note

$$G(\rho) = c_1(\rho)F_{Rep}(\rho) + [F_{Rep}, F_{\gamma_1}](\rho). \tag{68}$$

The assumption (H_4) implies that at each point in N' there is a symplectic basis such that the hessian of F_{Rep} is a sum of terms

(i) $Q(x,\xi) = ax.\xi$, with $Spec(a) \subseteq \{z \in \mathbb{C};\ Rez > 0\}$

(ii) $Q(x,\xi) = \alpha(x^2+\xi^2)$, with $\alpha > 0$.

(iii) $Q(x,\xi) = 0$.

We shall find appropriate quadratic form γ_1 and constant c_1 at ρ so that G in 68 is positive and piece them together. If we are in case (i), we chose $\gamma_1(x,\xi) = (\alpha x, x) - (\beta\xi,\xi)$ α and β are two positive matrices so that $\sigma((x,\xi),[F_{Rep},F_{\gamma_1}](x,\xi)) \geq 1/C(x^2+\xi^2)$; any $c_1 > 0$ will fit.
If we are in case (ii), we take $\gamma_1(x,\xi) = -k(x^2+\xi^2)$ with $k > 0$ and small with respect to c_1.
In case (iii) $\gamma_1 = 0$. Therefore we can construct functions $\gamma(\rho)$ and $c(\rho)$, such that

$$(cRep + \{Rep,\gamma\})(\rho) \geq C^{-1}d(\rho, N')^2. \tag{69}$$

Moreover if γ_1 is small with respect to γ_0 we shall have $\{\gamma > 0\} \cap \partial\omega \cap OF(u) \subseteq \{\gamma_0 > -\varepsilon\} \cap \partial\omega \cap OF(u) = \emptyset$.

We choose $m'(\rho) = \varphi(\rho)^2$, where φ is a C^∞ function supported by ω. $Imp_1 > 0$ is positive, while $\{Rep, m'\}$ is supported near $\partial\omega$. We derive the estimate

$$Im(P_\gamma u, M^* u) \geq c\mu(\sum_j |v_j(\varphi u)|^2) + c\lambda^{-1} |\varphi u|^2 + O(\lambda^{-1}) |\psi u|^2 + O(\lambda^{-1}\mu) |u|^2 \tag{70}$$

the v_j form a set of equations of N', ψ is supported near $\partial\omega$. We replace u by $e'_\gamma u$ and we note $M_\gamma = e'_\gamma M e'_\gamma$. In the following the third term in 70 could be neglected since $OF(u) \cap \partial\omega \cap \{\gamma > 0\} = \emptyset$.

We introduce the additionnal notation : let m be an order function and g a metric a symbol $a(x,\xi) \in \tilde{S}(m,g)$ if it is the sum of a symbol in $S(m,g)$ supported by a neighborhod of the support of φ and a symbol of order $-\infty$. In the following m will have the form $\lambda^m(\ln\lambda)^p$ and $g = g_0$ or $m = \lambda^\gamma(\ln\lambda)^p$ and $g' = (\ln\lambda)^2 g_0$. Then we have

$$Im(Pu, M^*_\gamma u) \geq c(\mu \sum_j |v_j \varphi e'_\gamma u|^2) + \lambda^{-1}|\varphi e'_\gamma u|^2) + (R_{2\gamma-2,2}u, u). \tag{71}$$

with $R \in \tilde{S}(\lambda^{2\gamma-2}(\ln\lambda)^2, g')$. We shall use the notation $|u|_\gamma = |e'_\gamma u|$.

5.2 Concatenations.

We move the subprincipal symbol using multiplication by non elliptic operators, this is named concatenations.

$$Im(JU_N Pu, JM^*_\gamma U_N u) = Im([JU_N, P]\, u, JM^*_\gamma U_N u) + +Im(PJU_N, M^*_\gamma JU_N u) \tag{72}$$

where $U_N u = (U_\alpha u)_{|\alpha|=N}$ and $U_\alpha = (u_1^{w_\lambda})^{\alpha_1} \cdots (u_r^{w_\lambda})^{\alpha_r}$, J is a linear operator in the space $\mathbf{C}^{N'}$ of multi-indices of length N. We can apply inequality 71 to the second term of 72.

We compute the commutator $[P, U_N]$. Let P_0 be the principal part of P, i.e. $P_0 = (p)^{w_\lambda} = \sum_{1\leq j\leq r}(p_j u_j)^{w_\lambda}$ and $P = P_0 + i\lambda^{-1}(p_1)^{w_\lambda}$, p_1 is the sub principal symbol.

$$[U_\alpha, P_0] = \sum_{1\leq j\leq r,\, p+q=\alpha_j-1} (u_1^{w_\lambda})^{\alpha_1} \cdots (u_j^{w_\lambda})^p \left[u_j^{w_\lambda}, P_0\right] (u_j^{w_\lambda})^q \cdots (u_r^{w_\lambda})^{\alpha_r}. \tag{73}$$

But $\left[u_j^{w_\lambda}, P_0\right] = \sum_{1\le k\le r}\left[u_j^{w_\lambda}, (p_k u_k)^{w_\lambda}\right]$, $(p_k u_k)^{w_\lambda} = p_k^{w_\lambda} u_k^{w_\lambda} - \frac{1}{2i\lambda}\{p_k, u_k\}^{w_\lambda} + \mathcal{O}(\lambda^{-2})$, so $\left[u_j^{w_\lambda}, P_0\right] = \sum_{1\le k\le r}\left[u_j^{w_\lambda}, p_k^{w_\lambda}\right] u_k^{w_\lambda} + \mathcal{O}(\lambda^{-2})$.

We deduce then

$$
\begin{aligned}
&[U_\alpha, P_0] = \\
&\quad \sum_{1\le k\le r,\, 1\le j\le r} \frac{1}{i\lambda}\{u_j, p_k\}^{w_\lambda} \alpha_j (u_1^{w_\lambda})^{\alpha_1}\cdots(u_k^{w_\lambda})^{\alpha_{k+1}}\cdots(u_j^{w_\lambda})^{\alpha_j-1}\cdots(u_r^{w_\lambda})^{\alpha_r} \\
&\quad + \sum_{\beta<\alpha}(c_{\alpha,\beta})^{w_\lambda} U_\beta \qquad (74)
\end{aligned}
$$

where $c_{\alpha,\beta}$ are symbols of degree $-1-|\alpha|+|\beta|$.

We know that $Spec(\{p_k, u_j\}) \subset \{z \in \mathbf{C},\, Rez > c\}$.

The operator $(z_\alpha) \to (\alpha_j z_{\alpha-(j)+(k)})$ is algebraically the operator $z_k\frac{\partial}{\partial z_j}$. With the notation $a_{j,k} = \{u_j, p_k\}$, and

$$
(\mathcal{A}_N U)_\alpha = \sum_{\alpha=\beta-(j)+(k),\, 1\le j,k\le r} a_{k,j}\beta_j U_\beta,
$$

we have

$$
[U_N, P_0] = i\lambda^{-1}(\mathcal{A}_N U_N) + (\sum_{\beta<\alpha}(c_{\alpha,\beta})^{w_\lambda} U_\beta)_\alpha. \qquad (75)
$$

The same result will hold for P since P_1 will contribute to the second term in 75.

We construct the linear operator J, such that $J\mathcal{A}_N(\rho_0)J^{-1} = ((\sum_j \alpha_j\lambda_j)\delta_{\alpha,\beta}) + o(N)$, the contribution $o(N)$ is due to that eventually $a(\rho_0)$ cannot be made diagonal. We see that the self-adjoint part of operator $M_\gamma J A_N J^{-1}$ is positive elliptic. Using the Gärding inequality for systems we have

$$
Im(JA_N U_N u, JM_\gamma^* U_N u) \ge cN|\varphi^{w_\lambda} e'_\gamma J U_N u|^2 - C_N(R_{2\gamma-1,2}U_N u, U_N u) \qquad (76)
$$

where $R_{2\gamma-1,2} \in \tilde{S}(\lambda^{2\gamma-1}(\ln\lambda)^2, g')$.

We estimate

$$
\begin{aligned}
&Im(J[U_N, P]u, JM_\gamma^* U_N u) \ge \\
&\quad cN\lambda^{-1}|\varphi e'_\gamma J U_N u|^2 + (R^{(N)}_{2\gamma-2,2}U_N u, U_N u) + \sum_{l<N} Re(JC_{N,l}U_l u, M_\gamma^* J_N u) \qquad (77)
\end{aligned}
$$

where $R^{(N)} \in \tilde{S}(\lambda^{2\gamma-2}(\ln\lambda)^2, g')$, the $C_{N,l}$ are operators of order $-1-N+l$. We estimate the third term in 77. The operator $M_\gamma = e'_\gamma(\varphi^{w_\lambda})^2 e'_\gamma + R_{2\gamma-1,2}$, then

$$
\begin{aligned}
&Re(M_\gamma J C_{N,l} U_l u, J U_N u) \leq \\
&\qquad \varepsilon\lambda^{-1}|\varphi^{w_\lambda} e'_\gamma J U_N u|^2 + C_{N,\varepsilon}\lambda^{-1-2N+2l}|U_l u|_\gamma^2 + C_N\lambda^{-3}(\ln\lambda)^4|U_N u|_\gamma^2
\end{aligned}
\tag{78}
$$

So we get

$$
\begin{aligned}
&Im(J\,[U_N, P]\,u, JM_\gamma^* u) \geq \\
&\qquad cN\lambda^{-1}|\varphi e'_\gamma J U_N u|^2 + (R^{(N)}_{2\gamma-2,2} U_N u, U_N u) + \sum_{l<N} R^{(N,l)}_{2\gamma-2N+2l} U_l u, U_l u)
\end{aligned}
\tag{79}
$$

If we chose N such that $Imp_1 + cN > 0$, using 71, 72 and 79 we obtain

$$
\begin{aligned}
&Im(JU_N Pu, JM_\gamma^* U_N u) \geq \\
&\quad cN\lambda^{-1}|\varphi^{w_\lambda} e'_\gamma J U_N u|^2 - C\lambda^{-2}(\ln\lambda)^4|\varphi_1^{w_\lambda} e'_\gamma U_N u|^2 \\
&\quad -C_N \sum_{0\leq k\leq N-1} \lambda^{-1-2N+2k}|\varphi_1^{w_\lambda} e'_\gamma U_k u|^2 + \mathcal{O}(\lambda^{-\infty})
\end{aligned}
\tag{80}
$$

where φ_1 is a function supported by a neighborhod of $supp\varphi$.

As in [5] the proof is based on a recurrence on the H^s regularity of the $U_k u$ in the domain $\{\gamma(x,\xi) > 0\} \cap \omega$.

We must modify the Proposition 1 of [5] to take care of the last terms in 80. We shall estimate $c_k = \sum_{0\leq l\leq k} \lambda^l |e'_\gamma \varphi^{w_\lambda} U_k u|$ by c_N, N is now fixed. To do that we use the equation

$$
Pu = f = \sum_{1\leq j\leq\leq r} p_j^{w_\lambda} u_j^{w_\lambda}(u) + (\frac{1}{2i\lambda}\{u_j, p_j\} + \lambda^{-1}p_1 + \mathcal{O}(\lambda^{-2}))^{w_\lambda}(u) \tag{81}
$$

We note

$$
p'_1 = \frac{1}{2i}\sum_{1\leq j\leq r}\{u_j, p_j\} + p_1 \tag{82}
$$

We use as above commutators with the operators U_k; from formula 75 we obtain

$$
\begin{aligned}
U_n f = U_n Pu &= i\lambda^{-1}A_n U_n(u) + \sum_{0\leq j\leq n-1} Q_{n,j} U_j(u) \\
&\quad + \sum_{1\leq j\leq\leq r} p_j^{w_\lambda} u_j U_n(u) + \lambda^{-1} p_1'^{w} U_n(u) + \mathcal{O}(\lambda^{-2})(u)
\end{aligned}
\tag{83}
$$

The $Q_{n,j}$ are operators of order $-1-n+j$.

It is a consequence of the assumptions that the matrix $\mathcal{A}'_n = \mathcal{A}_n + Id_{\mathbb{C}^{n'}} p'_1$ is invertible for any n in a neighborhod of ρ_0. Let $\mathcal{B}_n$ an operator of order γ such that $\mathcal{B}_n \mathcal{A}'_n = \varphi_0 e'_\gamma Id + \mathcal{O}(\lambda^{-\infty})$. We apply $\mathcal{B}_n$ on the members of equation 83

$$\mathcal{B}_n(f) = i\lambda \varphi_0^{w_\lambda} e'_\gamma U_n + \sum_{1 \le j \le r} \mathcal{B}_n P_j(U_n u) + \sum_{l<n} \mathcal{B}_n Q_{n,j} U_j u \qquad (84)$$

We multiply both members of 84 by φ^{w_λ}, using $\varphi^{w_\lambda}\varphi_0^{w_\lambda} = \varphi^{w_\lambda} + \mathcal{O}(\lambda^{-\infty})$, $[\mathcal{B}_n P_j, \varphi^{w_\lambda}] \in \tilde{S}(\lambda^{\gamma-1}(\ln \lambda), g')$ and $[\mathcal{B}_n Q_{n,j}, \varphi^{w_\lambda}] \in \tilde{S}(\lambda^{\gamma-2-n+j}(\ln \lambda), g')$ we obtain

$$\begin{aligned}
&|\varphi^{w_\lambda} e'_\gamma U_n u| \le \\
&\quad C(o(1) \sum_{1\le j\le r} |\varphi^{w_\lambda} u_j U_n u|_{\gamma+1} + \sum_{l<n} |\varphi^{w_\lambda} U_l u|_{\gamma-n+l} + \sum_{l\le n+1} |\varphi_1^{w_\lambda} U_l u|_{\gamma-n+l,1}) \\
&\quad + \mathcal{O}(\lambda^{-\infty}) \qquad (85)
\end{aligned}$$

This justify the notations $c_n = \sum_{j\le n} \lambda^j |e'_\gamma \varphi^{w_\lambda} u|$ and $d_n = (\ln \lambda) \sum_{j \le n} \lambda^j |e'_\gamma \varphi_1^{w_\lambda} u|$. We have proved

$$c_n \le o(1) c_{n+1} + k_0 \sum_{j<n} c_j + k_1 d_{n+1} + \mathcal{O}(\lambda^{-\infty}) \qquad (86)$$

k_0, k_1 are some constant.

The basic idea of Propsition 1 of [5] is to derive from 86 an upper bound of the c_j for $0 \le j \le N-1$ by c_N, where N is an integer choosen large enough with respect to the imaginary part of p_1.We shall need eventually to shrink ω accordingly. The d_j are controlled by using the steps of this recurrence. So we obtain by recurrence the smoothness of u in the domain $\{(x,\xi);\ \gamma(x,\xi) > 0\} \cap \omega$.

Now we can finish the proof as in [5].

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