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A Non-Central Functional Limit Theorem for Quadratic Forms in Martingale Difference Sequences

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Abstract

For quadratic forms with nulls on the diagonal the partial-sum process is both a martingale and a stochastic integral. Using corresponding tools we derive a result on the convergence to the double Wiener-Ito integral.

Let $\{A^n = (a^n_{i,j})\}_{n\in\mathbb{N}}$ be a sequence of infinite matrices with nulls on the diagonals:

$$a^n_{i,i} = 0, \qquad i = 1, 2, \ldots, \; n \in \mathbb{N}. \tag{1}$$

Let $\{X_j\}_{j\in\mathbb{N}}$ be a martingale difference sequence with respect to some filtration $\{\mathcal{F}_j\}_{j\in\mathbb{N}\cup\{0\}}$ such that

$$E(X_j^2|\mathcal{F}_{j-1}) = 1, \qquad j = 1, 2, \ldots. \tag{2}$$

Then for each $n \in \mathbb{N}$ the process

$$S_{n,k} = \sum_{1\le i,j\le k} a^n_{j,i} X_i X_j = \sum_{j\le k} \left(\sum_{i<j} (a^n_{j,i} + a^n_{i,j}) X_i \right) X_j \qquad k = 1, 2, \ldots, \tag{3}$$

is both a *square integrable* martingale and a stochastic integral with respect to $\{\mathcal{F}_j\}$.

Using the martingale structure, a Functional Central Limit Theorem for suitably scaled $\{S_{n,k}\}_{k\in\mathbb{N}}$ was proved in [JaMé90]. In the present paper we state a non-central functional limit theorem based on limit theorems for stochastic integrals given in [JMP89]. The limit is identified with a double Wiener-Ito stochastic integral (see e.g. [Ito51] or [Maj81]). In some sense it is not surprising: such limits arise, for example, in limit theory for quadratic forms in stationary gaussian sequences exhibiting long-range dependence (see [Ros79], also [SuHo86]).

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For each $n \in \mathbb{N}$, let $\overline{\mathbb{A}^n}$ be the symmetrization of $\mathbb{A}^n$, i.e. the matrix with entries $(a^n_{i,j} + a^n_{j,i})/2$. For $\mathbb{A}^n$, define its representation in $L^2([0,T]^2)$ by the formula

$$\widetilde{\mathbb{A}^n}(u,v) = a^n_{[nu],[nv]} \quad \text{for} \quad T \geq u, v \geq 0. \tag{4}$$

Finally, let

$$X^n(t) = n^{-1/2} \sum_{1 \leq j \leq [nt]} X_j \tag{5}$$

$$Y^n(t) = n^{-1} \sum_{1 \leq i,j \leq [nt]} a^n_{i,j} X_i X_j. \tag{6}$$

Theorem *Suppose $\mathbb{A}^n$'s satisfy (1) and $\{X_j\}$ is a martingale difference sequence such that (2) holds.*

If $\widetilde{\overline{\mathbb{A}^n}}$ converges in $L^2([0,T]^2)$ to some function A, then

$$Y^n \underset{\mathcal{D}}{\longrightarrow} Y, \tag{7}$$

where

$$Y(t) = \int\int A(u,v) I_{[0,t]^2}(u,v)\, dW_u dW_v \tag{8}$$

is the classical Wiener-Ito integral.

PROOF. For each $\varepsilon > 0$ there are continuous functions ϕ_l, ψ_l, $l = 1, 2, \ldots, m$ and numbers $\alpha_1, \ldots, \alpha_m$ such that

$$\Phi(s,t) = \sum_{l=1}^{m} \alpha_l \cdot \frac{\phi_l(s)\psi(t) + \psi_l(s)\phi_l(t)}{2}$$

satisfies

$$\|A - \Phi\|_2 < \varepsilon. \tag{9}$$

Hence for $n \geq n_1$, $\|\widetilde{\overline{\mathbb{A}^n}} - \Phi\|_2 < 2\varepsilon$ and by continuity of Φ $\|\widetilde{\overline{\mathbb{A}^n}} - \tilde{\Phi}_n\|_2 < 3\varepsilon$ for $n \geq n_2$, where $\tilde{\Phi}_n(s,t) = \Phi([ns]/n, [nt]/n)$.

If

$$Z^n(t) = n^{-1} \sum_{1 \leq i \neq j \leq [nt]} \Phi(i/n, j/n) X_i X_j, \tag{10}$$

then for $n \geq n_2$

$$E \sup_{1 \leq t \leq T} |Y^n(t) - Z^n(t)|^2 \leq 4E|Y^n(T) - Z^n(T)|^2 \leq 4\|\widetilde{\overline{\mathbb{A}^n}} - \tilde{\Phi}_n\|_2^2 \leq 4 \cdot 9\varepsilon^2. \tag{11}$$

Let

$$Z(t) = \int\int \Phi(u,v) I_{[0,t]^2}(u,v)\, dW_u dW_v. \tag{12}$$

By the isometry property for Wiener-Ito integrals

$$E \sup_{1 \leq t \leq T} |Y(t) - Z(t)|^2 \leq 4E|Y(T) - Z(T)|^2 = 4\|A - \Phi\|_2^2 \leq 4\varepsilon^2. \tag{13}$$

Hence it is enough to prove that

$$Z^n \underset{\mathcal{D}}{\longrightarrow} Z \tag{14}$$

on the space $\mathbb{D}([0,T])$. But

$$\begin{aligned}
Z_n(t) &= 2n^{-1} \sum_{1\le i<j\le[nt]} \Phi(i/n, j/n) X_i X_j \\
&= \sum_{l=1}^{m} \alpha_l \left(\sum_{1\le i<j\le[nt]} \phi_l(i/n)\psi_l(j/n) \frac{X_i}{\sqrt{n}}\frac{X_j}{\sqrt{n}} + \right. \\
&\qquad \left. + \sum_{1\le i<j\le[nt]} \psi_l(i/n)\phi_l(j/n) \frac{X_i}{\sqrt{n}}\frac{X_j}{\sqrt{n}} \right) \\
&= \sum_{l=1}^{m} \alpha_l \left(\int_o^t \psi_l(v) \left(\int_0^{v-} \phi_l(u)\, dX_u^n \right) dX_v^n + \right. \\
&\qquad \left. + \int_0^t \phi_l(v) \left(\int_0^{v-} \psi_l(u)\, dX_u^n \right) dX_v^n \right) \\
&\underset{\mathcal{D}}{\longrightarrow} \sum_{l=1}^{m} \alpha_l \left(\int_o^t \psi_l(v) \left(\int_0^{v-} \phi_l(u)\, dW_u \right) dW_v + \right. \\
&\qquad \left. + \int_0^t \phi_l(v) \left(\int_0^{v-} \psi_l(u)\, dW_u \right) dW_v \right) \\
&= Z(t),
\end{aligned}$$

where the convergence in distribution holds by [JMP89, Theorem 2.6] □

Example 1. Let $f : [0,1]^2 \to \mathbf{R}^1$. Define quadratic forms

$$a_{i,j}^n = f(i/n, j/n).$$

If $g_n(u,v) = f([nu],[nv]) \longrightarrow_{L^2} f(u,v)$, then

$$n^{-1} \sum_{1\le i,j\le[nt]} a_{i,j}^n X_i X_j \underset{\mathcal{D}}{\longrightarrow} \int_0^1 \int_0^t \overline{f}(u,v)\, dW_u\, dW_v,$$

where $\overline{f}$ is the symmetrization of f.

Example 2. Let

$$c_0 = 0, c_1 = b_1, c_2 = b_2, \ldots, c_d = b_d, c_{d+1} = b_1, c_{d+2} = b_2, \ldots$$

Define a matrix A by

$$a_{i,j} = c_{|i-j|}.$$

In this case $\widetilde{\mathsf{A}^n}$ does not converge, so we cannot apply directly our theorem.

Take $n = N \cdot d$ and define a rearranging of coordinates:

$$\begin{array}{rcl} e_1 & \mapsto & e_1 \\ e_2 & \mapsto & e_{N+1} \\ e_3 & \mapsto & e_{2N+1} \\ & \vdots & \\ e_d & \mapsto & e_{(d-1)N+1} \\ e_{d+1} & \mapsto & e_2 \\ e_{d+2} & \mapsto & e_{N+2} \\ & \vdots & \end{array}$$

(The general formula is of the form $e_{k \cdot d+i} \mapsto e_{(i-1)N+k+1}$ if $0 \leq k \leq N-1$, $1 \leq i \leq d$).

Under this rearranging, the distribution of X^n remains unchanged, while A^n transforms to the form $\mathsf{D}^n = (d_{p,q}^{\,n})$, where for $p = (i-1)N + k + 1$ and $q = (j-1)N + l + 1$

$$d_{p,q}^{\,n} = \begin{cases} 0 & \text{if } \ j = i, \ l = k \\ b_d & \text{if } \ j = i, \ l \neq k \\ b_{|j-i|} & \text{if } \ j > i, \ l \geq k \text{ or } j < i, \ l \leq k \\ b_{d-|j-i|} & \text{if } \ j > i, \ l < k \text{ or } j < i, \ l > k. \end{cases}$$

It is easy to see that now $\widetilde{\mathsf{D}^n} \to D$ in $L^2([0,1]^2)$, where

$$D(u,v) = \begin{cases} 0 & \text{if } \ u = v \\ b_d & \text{if } \ [d \cdot u] = [d \cdot v], \ u \neq v \\ b_{|[d\cdot u]-[d\cdot v]|} & \text{if } \ [d \cdot u] > [d \cdot v] \text{ and } \{d \cdot u\} \geq \{d \cdot v\} \\ & \qquad \text{or} \\ & \quad\ [d \cdot u] < [d \cdot v] \text{ and } \{d \cdot u\} \leq \{d \cdot v\} \\ b_{d-|[d\cdot u]-[d\cdot v]|} & \text{if } \ [d \cdot u] > [d \cdot v] \text{ and } \{d \cdot u\} < \{d \cdot v\} \\ & \qquad \text{or} \\ & \quad\ [d \cdot u] < [d \cdot v] \text{ and } \{d \cdot u\} > \{d \cdot v\} \end{cases}$$

Here $[x]$ is the integer part of x and $\{x\} = x - [x]$.

The convergence is easily extendible from subsequence $N \cdot d$ to the whole $\mathbb{N}$. But we **do not have the functional convergence**, since

$$n^{-1} \sum_{1 \leq i,j \leq [nt]} a_{i,j}^n X_i X_j$$

and

$$n^{-1} \sum_{1 \leq i,j \leq [nt]} d_{i,j}^n X_i X_j$$

have nothing common except for $t = 1$, where they coincide.

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