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ON WEAK CONVERGENCE OF PARAMETER ESTIMATORS OF GENERAL STATISTICAL PARAMETRIC MODELS

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SUMMARY

We consider a sequence of general statistical parametric models equipped with filtrations and the corresponding likelihood ratio processes. Using theorem about weak convergence of the likelihood ratio processes to the likelihood ratio process of the Gaussian model we obtain conditions for weak convergence of maximum likelihood estimators and Bayesian estimators.

KEYWORDS: general statistical parametric model, likelihood ratio process, Gaussian statistical parametric model, maximum likelihood estimator, Bayesian estimator, Hellinger process, jump measure, compensator of jump measure, weak convergence.

1. INTRODUCTION

We consider a sequence of general statistical parametric models $\mathcal{E}^n = \{(\Omega^n, F^n, \mathbb{F}^n), P^n_\theta, \theta \in \Theta^n\}$ with filtrations $\mathbb{F}^n$, $n \geq 1$. Here (Ω^n, F^n) are measurable spaces, the filtrations $\mathbb{F}^n = (F^n_t)_{t\geq 0}$ are such that $\bigvee_{t\geq 0} F^n_t = F^n$, $F^n_o = \{\emptyset, \Omega^n\}$, and $(P^n_\theta)_{\theta \in \Theta^n}$ are probability measures depending on the parameter θ which belongs to a closed convex set $\Theta^n \subseteq R^m$, $\Theta^n \subseteq \Theta^{n+1}$, $m \geq 1$.

Suppose that $\{0\} \in \Theta^n$ and for every $\theta \in \Theta^n$ the measure P^n_θ is absolutely continuous with respect to a σ-finite measure μ^n. Then for the general statistical parametric model $\mathcal{E}^n$ we can construct the likelihood ratio process $Z^n = (Z^n_t(\theta))_{t\geq 0, \theta \in \Theta^n}$ of the measure P^n_θ with respect to P^n_o with

$$Z^n_\tau(\theta) = \frac{dP^n_{\theta,\tau}}{d\mu^n_\tau} \Big/ \frac{dP^n_{o,\tau}}{d\mu^n_\tau} \qquad \text{(here } 2/0 = \infty\text{)}$$

for every $\mathbb{F}^n$-stopping time τ where $P^n_{\theta,\tau}$, $P^n_{o,\tau}$ and μ^n_τ are the restrictions of P^n_θ, P^n_o and μ^n to the σ-algebra F^n_τ.

As the "limit model" to $\mathcal{E}^n$ we consider the Gaussian one with a likelihood ratio process $Z = (Z_t(\theta))_{t\geq 0, \theta \in R^m}$ of the form

$$Z_t(\theta) = \exp\{N_t(\theta) - \frac{1}{2}\langle N(\theta)\rangle_t\} \qquad (1)$$

where $N = (N_t(\theta))_{t\geq 0, \theta\in R^m}$ is an a.s. continuous Gaussian field such that $N(\theta) = (N_t(\theta))_{t\geq 0}$ is a martingale for every $\theta \in R^m$ with $N_0(\theta) = 0$, $\langle N(\theta)\rangle_t = DN_t(\theta)$ and $N(0)\equiv 0$.

In this paper we formulate conditions for the weak convergence of the maximum likelihood and the Bayesian estimators $\hat{\theta}^n = (\hat{\theta}^n_t)_{t>0}$ of the parameter θ based on Z^n, to the corresponding estimators $\hat{\theta} = (\hat{\theta}_t)_{t>0}$ of the parameter θ of the Gaussian model. We also give conditions for the weak convergence of the likelihood ratio process $\hat{Z}^n = (Z_t(\hat{\theta}^n_t))_{t>0}$ with θ substituted by the estimator to the likelihood ratio process $\hat{Z} = (Z_t(\hat{\theta}_t))_{t>0}$ of the Gaussian model.

In particular, considering the Gaussian model having the special form:

$$Z_t(\theta) = \exp\{\theta W_t - \frac{1}{2}\theta^2 t\} \qquad (2)$$

where $W = (W_t)_{t\geq 0}$ is the standard Wiener process, we obtain conditions for the weak convergence of the maximum likelihood estimators and the Bayesian estimators to the process $(W_t/t)_{t>0}$ and also conditions for the weak convergence of the likelihood ratio process $\hat{Z}^n$ with θ substituted by the estimator to the process $(W_t^2/t)_{t>0}$. It should be noticed that these results about weak convergence together with the results of §9 of chapter 1 in [2] immediately give conditions for the asymptotic efficiency of the estimators of θ.

2. ON WEAK CONVERGENCE OF THE LIKELIHOOD RATIO PROCESSES

To obtain the theorems about weak convergence of estimators first we formulate the theorem about weak convergence of the corresponding likelihood ratio processes.

We consider the Skorohod space $\mathbb{D}(R_+, \mathbb{C}_{loc}(R^m))$ of the functions $Z: t \rightsquigarrow (Z_t(\theta))_{\theta \in R^m}$ where $t \in R_+$, with values in the space $\mathbb{C}_{loc}(R^m)$ of real continuous functions with local uniform metric. The weak convergence of the likelihood ratio processes will be considered in the subspace

$$(\mathbb{D}\otimes\mathbb{C})_\infty = \{z \in \mathbb{D}(R_+, \mathbb{C}_{loc}(R^m)): \lim_{L\to\infty} \sup_{1/N \le t \le N} \sup_{|\theta|>L} Z_t(\theta)=0, \forall N>0\}$$

of the Skorohod space $\mathbb{D}(R_+, \mathbb{C}_{loc}(R^m))$.

To formulate the theorem we denote by $h^n(\theta,\theta') = (h_t^n(\theta,\theta'))_{t\ge 0}$ the Hellinger process corresponding to P_θ^n, $P_{\theta'}^n$, and $\mathbb{F}^n$ (see [5], [7], [4]). We set $Q^n = (P_\theta^n + P_o^n)/2$ and define

$$\zeta_t^n(\theta) = dP_{\theta,t}^n / dQ_t^n \text{ and } f_t^n(x,\theta) = |\sqrt{1-x/\zeta_{t-}^n(\theta)} - \sqrt{1+x/\zeta_{t-}^n(0)}|$$

with $x/0=\infty$ for every $\theta, \theta' \in \Theta^n$, $t>0$ and $x \in R$. We will denote by $\nu^n(\theta)$ the compensator of the jump measure [3],

[8] of the process $\zeta^n(0) = (\zeta_t^n(0))_{t\geq 0}$ with respect to $(Q^n, \mathbb{F}^n)$ and by the sign "*" the integral with respect to a compensator. The symbols $\mathcal{E}(\cdot)$ and $I(\cdot)$ will be used for the Doleans-Dade exponential function [3] and the indicator function respectively.

We consider functions $\varphi_t^n(y)$ where $y>0$, $t>0$, $n\geq 1$ and $\varphi_t^\infty(y) = \overline{\lim_{n\to\infty}}\, \varphi_t^n(y)$, which satisfy the conditions:

the functions $y^{m+r-1}(\varphi_t^n(y))^{1-m/\beta}$ *monotonically decrease with respect to* y *and are integrable at infinity for* $t>0$ *and* $1\leq n\leq\infty$ *where* $r\geq 0$ *and* $\beta>0$ *are constants.* (3)

THEOREM 2.1 (see [11]). *Suppose the following conditions are satisfied:*

1) *for every* $\varepsilon>0$, $t>0$ *and* $\theta \in \Theta^{n_o}$ *with* $n_o\geq 1$

$$((f_t^n(x,\theta))^2\, I(f_t^n(x,\theta)\geq\varepsilon) * \nu^n(\theta))_t \xrightarrow{P_o^n} 0,$$

2) *for every* $t>0$ *and* $\theta,\theta' \in \Theta^{n_o}$ *with* $n_o\geq 1$

$$h_t^n(\theta,\theta') \xrightarrow{P_o^n} \frac{1}{8} \langle N(\theta) - N(\theta')\rangle_t$$

3) *there exist constants* $\beta>m$, $r\geq 0$, $d\geq 0$ *and* $D\geq 0$ *such that for every* $t>0$ *and sufficiently large* $L>0$

$$\sup_n \sup_{|\theta|\leq L, |\theta'|\leq L} \frac{P_\theta^n(h_t^n(\theta,\theta')\geq DL^r|\theta-\theta'|^\beta)}{|\theta-\theta'|^{2\beta}} \leq dL^{2r} < \infty,$$

4) *there exist constants* $c \geq 0$, $C \geq 0$ *and functions* $\varphi_t^n(y)$ *satisfying the conditions* (3), *such that for every* $t>0$ *and sufficiently large* $L>0$

$$\sup_n \sup_{|\theta| \geq L} \frac{P_o^n(\mathcal{E}(-h^n(\theta,0))_t \geq C\varphi_t^n(|\theta|))}{(\varphi_t^n(|\theta|))^2} \leq c < \infty .$$

Then there exist likelihood ratio processes Z^n *having* (P_o^n-a.s.) *paths in* $(\mathbb{D} \otimes \mathbb{C})_\infty$ *and*

$$Z^n \xrightarrow{w(P_o^n)} Z \qquad (4)$$

where Z *is defined by* (1).

REMARK 2.1. The conditions 1) and 2) of theorem 2.1 provide the convergence of the finite dimensional distributions of Z^n to the ones of Z. The condition 1) is the conditional variant of the Lindeberg condition, and the condition 2) is the one about convergence of the triangle brackets of the martingale parts of $\ln Z^n(\theta)$. The conditions 3) and 4) provide the tightness of the distributions of Z^n in $(\mathbb{D} \otimes \mathbb{C})_\infty$. The meaning of these conditions is the following: the condition 3) guarantees sufficiently quick convergence in variation of the difference $P_\theta^n - P_{\theta'}^n$, when $\theta' \to \theta$, and the condition 4) provides sufficiently good separability of the measures

P_θ^n and $P_{\theta'}^n$, when the value of $|\theta-\theta'|$ is large. Notice also that the questions concerned with the calculation of the Hellinger process which appears in the conditions of theorem 2.1. were considered in [7] and [6].

REMARK 2.2. From the analysis of the proof of theorem 2.1 given in [11] we get that the additional conditions as $P_\theta^n \ll P_o^n$, $\forall \theta \in \Theta^n$, and the completeness of the filtration $\mathbb{F}^n$ with respect to the measure P_o^n used there, can be omitted.

REMARK 2.3. From the proof of theorem 2.1 we have the following: if we make the reparametrization $\theta \rightsquigarrow \theta+\theta_o$ with some $\theta_o \in \Theta^{n_o}$ and suppose that the conditions of theorem 2.1 are satisfied uniformly over $\Theta_o \in \mathcal{K} \subseteq \Theta^{n_o}$ where $\mathcal{K}$ is a compact set, then the weak convergence of Z^n to Z will be uniform over $\Theta_o \in \mathcal{K}$.

REMARK 2.4. Suppose that the conditions 1) and 2) of theorem 2.1 are satisfied for $0<t\leq T$ and the conditions 3) and 4) are fulfilled for $t=T$. Then using the methods of the proof of theorem 2.1 and theorem 5 in [10] we get the weak convergence in the space $\mathbb{C}(R^m)$:

$$Z_T^n \xrightarrow{w(P_o^n)} Z_T$$

where $Z_T^n = (Z_T^n(\theta))_{\theta \in R^m}$ and $Z_T = (Z_T(\theta))_{\theta \in R^m}$.

3. WEAK CONVERGENCE OF MAXIMUM LIKELIHOOD ESTIMATORS

We consider the maximum likelihood estimators of θ based on Z^n and Z:

$$\hat{\theta}_t^n = \arg \sup_{\theta \in R^m} Z_t^n(\theta), \quad \hat{\theta}_t = \arg \sup_{\theta \in R^m} Z_t(\theta), \quad t>0,$$

where they are equal to any point of the maximum with respect to θ if there are several such points and $\hat{\theta}_t^n = \infty$ and $\hat{\theta}_t = \infty$ if the supremum with respect to θ is not achieved. In the same way we define $\hat{\theta}_{t-}^n$ and $\hat{\theta}_{t-}$ substituting $Z_t^n(\theta)$ and $Z_t(\theta)$ by $Z_{t-}^n(\theta)$ and $Z_{t-}(\theta)$ respectively.

THEOREM 3.1. *Suppose that for every* $t>0$ *the maximum likelihood estimators* $\hat{\theta}_t^n$, $\hat{\theta}_{t-}^n$, $\hat{\theta}_t$ and $\hat{\theta}_{t-}$ *are unique. Then under the conditions of theorem* 2.1 *for* $\hat{\theta}^n = (\hat{\theta}_t^n)_{t>0}$ *and* $\hat{\theta} = (\hat{\theta}_t)_{t>0}$ *we have*

$$(\hat{\theta}^n, Z^n) \xrightarrow{w(P_o^n)} (\hat{\theta}, Z) \tag{5}$$

in the Skorohod space $\mathbb{D}((0,\infty), R^m \otimes \mathbb{C}(R^m))$.

PROOF. Since Z^n and Z belong to $(\mathbb{D}\otimes\mathbb{C})_\infty$ we have that $|\hat{\theta}_t^n|<\infty$, $|\hat{\theta}_{t-}^n|<\infty$ and $|\hat{\theta}_t|<\infty$, $|\hat{\theta}_{t-}|<\infty$ for every $t>0$ and also that the processes $\hat{\theta}^n$ and $\hat{\theta}$ have paths in the space $\mathbb{D}((0,\infty), R^m)$.

By (4) and the Skorohod representation theorem we can find a probability space $(\bar{\Omega}, \bar{F}, \bar{P})$ and the processes

$\bar{Z}^n$ and $\bar{Z}$ with the same distributions as Z^n and Z respectively such that for every $N>0$ ($\bar{P}$-a.s.)

$$\sup_{1/N\le t\le N}\ \sup_{\theta\in R^m} |\bar{Z}_t^n(\theta) - \bar{Z}_t(\theta)| \to 0 \tag{6}$$

as $n\to\infty$. If we suppose that at the same time $\sup_{1/N\le t\le N} |\hat{\theta}_t^n-\hat{\theta}_t|\not\to 0$ as $n\to\infty$ then there exists the sequence $(t_n)_{n\ge 1}$, $1/N\le t_n\le N$, such that $|\hat{\theta}^n_{t_n} - \hat{\theta}_{t_n}| \not\to 0$ as $n\to\infty$. Since the sequence $(t_n)_{n\ge 1}$ belongs to the set $[1/N, N]$ we can suppose $t_n\to t_o$ and $t_n\ge t_o$ or $t_n<t_o$ for every $n\ge 1$. If $t_n\ge t_o$ then by right-continuity of the process $\hat{\theta}$ we have $|\hat{\theta}^n_{t_n} - \hat{\theta}_{t_o}| \not\to 0$ as $n\to\infty$; if $t_n<t_o$ then $|\hat{\theta}^n_{t_n} - \hat{\theta}_{t_o-}| \not\to 0$ as $n\to\infty$. Since this contradicts to (6), we have (5).■

COROLLARY 3.1. *Suppose that the limit process in* (4) *has the form* (2) *and the maximum likelihood estimator* $\hat{\theta}_t^n$ *and* $\hat{\theta}_{t-}^n$ *are unique. Then under the conditions of theorem* 2.1 *we get*

$$\hat{\theta}^n \xrightarrow{w(P_o^n)} (W_t/t)_{t>0} \tag{7}$$

where $W = (W_t)_{t\ge 0}$ *is the standard Wiener process.*

PROOF. In this case the maximum likelihood estimators $\hat{\theta}_t$ and $\hat{\theta}_{t-}$ for the limit process in (4) are unique, $\hat{\theta}_t = \hat{\theta}_{t-} = W_t/t$ and the result follows from theorem 3.1.■

Consider now the weak convergence of the likelihood ratio processes $\hat{Z}^n = (Z_t^n(\hat{\theta}_t^n))_{t>0}$ with the parameter θ substituted by the maximum likelihood estimators.

THEOREM 3.2. *Assume the conditions of theorem* 3.1. *Then we have the weak convergence*

$$\hat{Z}^n \xrightarrow{w(P_o^n)} \hat{Z}$$

in the Skorohod space $\mathbb{D}((0,\infty), R)$ *where* $\hat{Z} = (Z_t(\hat{\theta}_t))_{t>0}$.

PROOF. If $Z \in (\mathbb{D} \otimes \mathbb{C})_\infty$ and $\hat{\theta} \in \mathbb{D}((0,\infty), R)$ then the process $\hat{Z} = (Z_t(\hat{\theta}_t))_{t>0}$ has path in $\mathbb{D}((0,\infty), R)$.

Because of (5) we need to prove only the continuity of the map $(\hat{\theta}, Z) \rightsquigarrow (\hat{Z})$ for the Skorohod metric when the limit processes in (5) are continuous with respect to t.

Suppose that $(X^n, Y^n) \to (X, Y)$ as $n\to\infty$ where $Y^n, Y \in (\mathbb{D}\otimes\mathbb{C})_\infty$, $X^n, X \in \mathbb{D}((0,\infty), R)$ and the functions X and Y are continuous with respect to t. Then for every $h>0$ and $N>0$ such that $\sup_{1/N\le t\le N} |X_t^n - X_t| \le h$ we have

$$\sup_{1/N\le t\le N} |Y_t^n(X_t^n) - Y_t(X_t)| \le \sup_{1/N\le t\le N} \sup_{\theta\in R^m} |Y_t^n(\theta) - Y_t(\theta)| +$$

$$+ \sup_{1/N\le t\le N} \sup_{|\theta-\theta'|\le h} |Y_t(\theta) - Y_t(\theta')| \qquad (8)$$

and the right-hand side of this estimation tends to zero as $h \to 0$. Hence, $Y^n(X^n) \to Y(X)$ as $n \to \infty$ for the Skorohod metric. ■

COROLLARY 3.2. *Assume the conditions of corollary 3.1. Then*

$$\hat{Z}^n \xrightarrow{w(P_o^n)} (\exp(W_t^2/(2t)))_{t>0}.$$

REMARK 3.1. Note that under the conditions of remark 2.4 and the assumption about uniqueness of $\hat{\theta}_T$, $T>0$, we also have

$$\hat{\theta}_T^n \xrightarrow{w(P_o^n)} \hat{\theta}_T, \quad \hat{Z}_T^n \xrightarrow{w(P_o^n)} \hat{Z}_T,$$

and in the special case (2) we get

$$\hat{\theta}_T^n \xrightarrow{w(P_o^n)} W_T/T, \quad \hat{Z}_T^n \xrightarrow{w(P_o^n)} \exp(W_T^2/(2T)).$$

4. WEAK CONVERGENCE OF FUNCTIONALS OF INTEGRAL TYPE OF Z^n

Let $\ell(\theta)$, $\theta \in R^m$, be a nonnegative continuous function of polynomial growth with $\ell(0)=0$. Let $(Q^n)_{n\geq 1}$ be probability measures on R^m admitting the densities

$q^n(\theta)$, $\theta \in R^m$, with respect to the Lebesgue measure. Assume that $\sup_n \sup_{\theta \in R^m} q^n(\theta) < \infty$ and that $q^n(\theta) \to q(\theta)$, $\forall\, \theta \in R^m$. For every $t>0$ we set

$$\Psi_t^n(u) = \int_{R^m} \ell(u-\theta)\; Z_t^n(\theta)\; q^n(\theta)d\theta,$$

$$\Psi_t(u) = \int_{R^m} \ell(u-\theta)\; Z_t(\theta)\; q(\theta)\; d\theta. \tag{9}$$

We are interested in conditions for the weak convergence of $\Psi^n = (\Psi_t^n(u))_{t>0,\ u\in R^m}$ to $\Psi = (\Psi_t(u))_{t>0,\ u\in R^m}$ in the Skorohod space $\mathbb{D}((0,\infty),\ \mathbb{C}_{loc}(R^m))$.

As in the condition (3) we introduce functions $\varphi_t^n(y)$ with $y>0$, $t>0$, $n\geq 1$ and $\varphi_t^\infty(y) = \overline{\lim}_{n\to\infty} \varphi_t^n(y)$ satisfying the conditions:

the functions $y^{m+r-1}(\varphi_t^n(y))^{\beta/(\beta+m)}$ *monotonically decrease with respect to* y *and are integrable at infinity for* $t>0$ *and* $1\leq n\leq\infty$ *where* $r\geq 0$ *and* $\beta>0$ *are constants.* (10)

We set

$$\psi_{t,L}^n(u) = \int_{|\theta|>L} \ell(u-\theta)\; Z_t^n(\theta)q^n(\theta)d\theta,$$

$$\psi_{t,L}(u) = \int_{|\theta|>L} \ell(u-\theta)\; Z_t(\theta)q(\theta)d\theta \tag{11}$$

and prove the following lemma.

LEMMA 4.1. *Suppose the conditions* III) *and* IV) *of theorem* 2.1 *with* $\beta>1$ *and substitution of* (3) *by* (10). *Then for every* $N>0$, $L>0$, $u\in R^m$ *and* $\varepsilon>0$

$$P_o^n\Big(\sup_{1/N\le t\le N}\psi_{t,L}^n(u)\ge\varepsilon\Big)\le c(\varepsilon)\int_{L/2}^{\infty}(y+1)^{m+r-1}(\varphi_{1/N}^n(y))^{\beta/(\beta+m)}dy, \quad (12)$$

$$P\Big(\sup_{1/N\le t\le N}\psi_{t,L}(u)\ge\varepsilon\Big)\le c(\varepsilon)\int_{L/2}^{\infty}(y+1)^{m+r-1}(\varphi_{1/N}^{\infty}(y))^{\beta/(\beta+m)}dy \quad (13)$$

where r *is the degree of the polynomial growth of* $\ell(\cdot)$ *and* $c(\varepsilon)$ *is a positive constant.*

PROOF. Note that $\psi_L^n(u)=(\psi_{t,L}^n(u))_{t>0}$ is an optional process and for every $N>0$, $L>0$, $u\in R^m$ and $\varepsilon>0$ consider the stopping time

$$\tau=\inf\{1/N\le t\le N:\ \psi_{t,L}^n(u)\ge\varepsilon\} \quad (14)$$

with $\inf\{\emptyset\}=N$. Then

$$P_o^n\Big(\sup_{1/N\le t\le N}\psi_{t,L}^n(u)\ge\varepsilon\Big)\le P_o^n(\psi_{\tau,L}^n(u)\ge\varepsilon/2)\ . \quad (15)$$

Using the fact that $\ell(u-\theta)$ is smaller then a polynomial of degree r and also the estimation (37) of theorem 2 in [9] we obtain

$$P_o^n(\psi_{\tau,L}^n(u)\ge\varepsilon/2)\le c(\varepsilon)\int_{L/2}^{\infty}(y+1)^{m+r-1}(\varphi_{1/N}^n(y))^{\beta/(\beta+m)}dy, \quad (16)$$

for every stopping time $\tau\ge1/N$ and, hence, (12).

Using the weak convergence (4), the Skorohod representation theorem and Fatou's lemma, from the estimation (12) we obtain (13).

THEOREM 4.1. *Suppose the conditions of theorem* 2.1 *with* $\beta>1$ *and the substitution of* (3) *by* (10). *Then*

$$\Psi^n \xrightarrow{w(P_o^n)} \Psi$$

as $n\to\infty$ *in the Skorohod space* $\mathbb{D}((0,\infty), \mathbb{C}_{loc}(R^m))$.

PROOF. To prove the convergence of the finite dimensional distributions we use the Kramer-Wold method and show that for every $c_i \in R$, $u_i \in R^m$, $t_i \in R_+$ with $1\le i\le k$, $k\ge 1$, we have

$$\sum_{i=1}^{k} c_i \Psi_{t_i}^n(u_i) \xrightarrow{w(P_o^n)} \sum_{i=1}^{k} c_i \Psi_{t_i}(u_i) . \tag{17}$$

For every $L>0$ we set

$$f_L^n = \sum_{i=1}^{k} c_i(\Psi_{t_i}^n(u_i)) - \psi_{t_i,L}^n(u_i)),\ f_L = \sum_{i=1}^{k} c_i(\Psi_{t_i}(u_i) - \Psi_{t_i,L}(u_i))$$

From (4), the Skorohod representation theorem and the convergence $q^n(\theta) \to q(\theta)$, $\forall\theta \in R^m$, and also from the uniform boundedness of the functions $\ell(\theta)$, $q^n(\theta)$, $Z_t^n(\theta)$ on compact sets of θ we get by the Lebesgue dominated convergence theorem that $f_L^n \to f_L$ as $n\to\infty$. Hence, for all continuity points y of the distribution of f_L we obtain

$$\lim_{n\to\infty} P_o^n\,(f_L^n \geq y) = P(f_L \geq y) \quad . \tag{18}$$

Note that for every $L>0$, $y>0$ and $\varepsilon>0$

$$0 \leq \overline{\lim_{n\to\infty}}\, P_o^n\Big(\sum_{i=1}^{k} c_i\, \Psi_{t_i}^n(u_i) \geq y\Big) - \underline{\lim_{n\to\infty}}\, P_o^n\Big(\sum_{i=1}^{k} c_i \Psi_{t_i}^n(u_i) \geq y\Big) \leq$$

$$\leq \overline{\lim_{n\to\infty}}\, P_o^n(f_L^n \geq y-\varepsilon) - \underline{\lim_{n\to\infty}}\, P_o^n(f_L^n \geq y+\varepsilon) + \tag{19}$$

$$+\, 2\overline{\lim_{n\to\infty}}\, P_o^n\Big(\sum_{i=1}^{k} |c_i|\, \psi_{t_i,L}^n(u_i) \geq \varepsilon\Big)$$

and by lemma 4.1

$$\lim_{L\to\infty} \overline{\lim_{n\to\infty}}\, P_o^n\Big(\sum_{i=1}^{k} |c_i| \cdot \psi_{t_i,L}^n(u_i) \geq \varepsilon\Big) = 0. \tag{20}$$

Then choosing $\varepsilon \downarrow 0$ such that $y-\varepsilon$ and $y+\varepsilon$ are points of continuity of the distribution of f_L, from (18), (19) and (20) after taking $\lim_{\varepsilon\to 0} \overline{\lim_{L\to\infty}}$ we obtain (17).

For the weak convergence of Ψ^n to Ψ in $\mathbb{D}((0,\infty), \mathbb{C}_{loc}(R^m))$ we have to prove the weak convergence in $\mathbb{D}([1/N,N], \mathbb{C}_{loc}(R^m))$ for every $N>0$. For that in turn according to [11] and the finite dimensional convergence proved before it is sufficient to show that

i) $\lim_{h\to 0} \sup_{n} P_o^n(w_{h,N}^u(\Psi^n)\geq\varepsilon) = 0, \forall u \in R^m, \forall N>0$,

ii) $\lim_{h\to 0} \sup_{n} P_o^n(\kappa_{h,N}^i(\Psi^n)\geq\varepsilon) = 0, \forall i\geq 1, \forall N>0$,

where $w_{h,N}^u(\Psi^n)$ is the modulus of continuity in $\mathbb{D}([1/N,N], R)$ for $\Psi^n(u)$ and

$$\kappa_{h,N}^i(\Psi^n) = \sup_{1/N\leq t\leq N} \sup_{|\theta-\theta'|\leq h, \theta,\theta'\in\mathcal{K}_i} |\Psi_t^n(\theta) - \Psi_t^n(\theta')|$$

where $(\mathcal{K}_i)_{i\geq 1}$ is an increasing sequence of compact sets, $\bigcup_{i=1}^{\infty} \mathcal{K}_i = R^m$.

To obtain i) we note that for every L>0, N>0 and $u \in R^m$

$$w_{h,N}^u(\Psi^n) \leq w_{h,N}^u(\Psi^n - \psi_L^n) + 2 \sup_{1/N\leq t\leq N} \psi_{t,L}^n(u). \qquad (21)$$

By theorem 2.1 and the estimation

$$w_{h,N}^u(\Psi^n - \psi_L^n) \leq c(L)\ w_{h,N}(Z^n)$$

where c(L) is a positive constant and $w_{h,N}(\cdot)$ is the modulus of continuity in $\mathbb{D}([1/N, N], \mathbb{C}_{loc}(R^m))$, we get

$$\lim_{h\to 0} \sup_n P_o^n(w_{h,N}^u(\Psi^n - \psi_L^n) \geq \varepsilon) = 0, \; \forall u \in R^m. \tag{22}$$

In turn by lemma 4.1

$$\lim_{L\to\infty} \sup_n P_o^n\Big(\sup_{1/N\leq t\leq N} \psi_{t,L}^n(u) \geq \varepsilon\Big) = 0, \; \forall u \in R^m,$$

that together with (21) and (22) gives i).

For ii) we have for every L>0 and N>0

$$\kappa_{h,N}^i(\Psi^n) \leq \kappa_{h,N}^i(\Psi^n - \psi_L^n) + 2 \sup_{1/N\leq t\leq N} \sup_{u\in\mathcal{X}_i} \psi_{t,L}^n(u), \tag{23}$$

$$\kappa_{h,N}^i(\Psi^n - \psi_L^n) \leq \sup_{|\theta-\theta'|\leq h,\theta,\theta'\in\mathcal{X}_j} |\ell(\theta)-\ell(\theta')| \sup_{1/N\leq t\leq N} \sup_{|\theta|\leq L} Z_t^n(\theta)$$

where $\mathcal{X}_j$ is defined by $\{u-\theta : |\theta|\leq L, u\in\mathcal{X}_i\} \subseteq \mathcal{X}_j$. Since the function $\ell(\cdot)$ is continuous and the sequence $\sup_{1/N\leq t\leq N} \sup_{|\theta|\leq L} Z_t^n(\theta)$ is tight we obtain

$$\lim_{h\to 0} \sup_n P_o^n(\kappa_h^i(\Psi^n - \psi_L^n) \geq \varepsilon) = 0 .$$

For the second term on the right-hand side of (23) we use the estimator of lemma 4.1 and get ii).■

5. WEAK CONVERGENCE OF BAYESIAN ESTIMATORS

Suppose that for $\ell(\cdot)$, $q^n(\cdot)$ and $q(\cdot)$ the conditions of the previous section are satisfied. We denote by $\tilde{\theta}_t^n$ and $\tilde{\theta}_t$, $t>0$, the Bayesian estimators of the parameter θ corresponding to $\ell(\cdot)$, $q^n(\cdot)$, Z^n and $\ell(\cdot)$, $q(\cdot)$, Z respectively, i.e.

$$\tilde{\theta}_t^n = \arg \inf_{u\in R^m}(\Psi_t^n(u)), \quad \tilde{\theta}_t = \arg \inf_{u\in R^m}(\Psi_t(u)) \tag{24}$$

with $\tilde{\theta}_t^n = \infty$, $\tilde{\theta}_t = \infty$ if the infimum with respect to u is not achieved and they are equal to any point of minimum with respect to u if there are several points of minimum. In the same way we define $\tilde{\theta}_{t-}^n$ and $\tilde{\theta}_{t-}$ substituting $\Psi_t^n(u)$ and $\Psi_t(u)$ by $\Psi_{t-}^n(u)$ and $\Psi_{t-}(u)$ respectively.

THEOREM 5.1. *Suppose that the Bayesian estimators* $\tilde{\theta}_{t-}^n$, $\tilde{\theta}_t^n$, $\tilde{\theta}_t$ and $\tilde{\theta}_{t-}$ *are unique for every* $t>0$. *Assume also the conditions of theorem* 4.1 , *then for* $\tilde{\theta}^n = (\tilde{\theta}_t^n)_{t>0}$ *and* $\tilde{\theta} = (\tilde{\theta}_t)_{t>0}$ *we have the convergence*

$$(\tilde{\theta}^n, Z^n) \xrightarrow{w(P_o^n)} (\tilde{\theta}, Z) \tag{25}$$

in the Skorohod space $\mathbb{D}((0,\infty), R^m \otimes \mathbb{C}(R^m))$.

PROOF. For every $L>0$ and $N>0$ we consider the stopping time

$$\tau = \inf\{1/N \le t \le N: |\tilde{\theta}^n_t| \ge L\}$$

with $\inf\{\emptyset\} = N$. Using the estimation of theorem 2 in [9] we get

$$P^n_o(\sup_{1/N\le t\le N} |\tilde{\theta}^n_t| = \infty) \le \lim_{L\to\infty} P^n_o(\sup_{1/N\le t\le N} |\tilde{\theta}^n_t| \ge L) \le$$

$$\le \lim_{L\to\infty} P^n_o(|\tilde{\theta}^n_\tau| \ge L/2) = 0.$$

Therefore, the sequence $(\tilde{\theta}^n_t)_{1/N\le t\le N}$ is uniformly bounded (P^n_o-a.s.) for every $N>0$ and since the functional $f(\cdot) = \arg\inf_{u\in R^m}(\cdot)$ is continuous when infinum is achieved in unique point, the process $\tilde{\theta}^n$ has the paths in the space $\mathbb{D}((0,\infty), R^m)$. In the same way we obtain that the process $\tilde{\theta}$ has the paths in the space $\mathbb{D}((0,\infty), R^m)$, too.

By (4) and the Skorohod representation theorem one can find a probability space $(\bar{\Omega}, \bar{F}, \bar{P})$ and the processes $\bar{Z}^n$ and $\bar{Z}$ for every $N>0$ ($\bar{P}$-a.s)

$$\sup_{1/N\le t\le N} \sup_{\theta\in R^m} |\bar{Z}^n_t(\theta) - \bar{Z}_t(\theta)| \to 0$$

as $n\to\infty$. We define $\bar{\Psi}^n_t(u)$, $\bar{\psi}^n_{t,L}(u)$ and $\bar{\Psi}_t(u)$, $\bar{\psi}_{t,L}(u)$ by (9) and (11) with replacing Z^n and Z by $\bar{Z}^n$ and $\bar{Z}$. Because of lemma 4.1 $\bar{P}(\lim_{L\to\infty} \sup_{1/N\le t\le N} \sup_{u\in\mathcal{X}_i} \bar{\psi}_{t,L}(u)>0)=0$ for every $i\ge 1$ and applying the Lebesgue dominated convergence theorem we obtain $\sup_{1/N\le t\le N} \sup_{u\in\mathcal{X}_i} |\bar{\Psi}^n_t(u) - \bar{\Psi}_t(u)|\to 0$ as $n\to\infty$. After that in the same way as in theorem 3.1 we prove that $\sup_{1/N \le t \le N} |\tilde{\theta}_t^n - \theta_t| \to 0$ as $n \to \infty$ and hence, (25) follows.∎

COROLLARY 5.1. *Suppose that* $Z = (Z_t(\theta))_{t \ge 0}$ *has the form* (2) *and* $\tilde{\theta}$ *are the Bayesian estimators for the continuous nonnegative loss function* $\ell(\cdot)$ *having at most polynomial grows and such that* $\ell(0)=0$, $\ell(\theta)=\ell(-\theta)$ *and the set* $\{\ell(\theta)<c\}$ *is convex and bounded for every* $c>0$, *and for the a priory density* $q(\theta)\equiv 1$, $\forall \theta \in R^m$. *Assume the conditions of theorem* 4.1 *and the uniqueness of the Bayesian estimators* $\tilde{\theta}_t^n$, $\tilde{\theta}_{t-}^n$ *for every* $t>0$ *then*

$$\tilde{\theta}^n \xrightarrow{w(P_o^n)} (W_t/t)_{t>0} \quad .$$

PROOF. Since in this case

$$\Psi_t(u) = \int_{R^m} \ell(u-\theta)\exp(\theta W_t - \frac{1}{2}\theta^2 t)d\theta = \exp(W_t^2/(2t))\cdot$$

$$\cdot \int_{R^m} \ell(u-\theta)\exp(-t(\theta - W_t/t)^2/2)d\theta$$

we obtain by setting $v=\theta-W_t/t$ and using the analog of the Anderson's lemma [2] that $\tilde{\theta}_t = W_t/t$ and the result follows from theorem 5.1.∎

THEOREM 5.2. *Suppose the conditions of theorem 5.1. Then for the processes* $\hat{Z}^n = (Z_t^n(\tilde{\theta}_t^n))_{t>0}$ *and* $\hat{Z} = (Z_t(\tilde{\theta}_t))_{t>0}$ *we have the weak convergence*

$$\hat{Z}^n \xrightarrow{w(P_o^n)} \hat{Z}$$

in the Skorohod space $\mathbb{D}((0,\infty), R)$.

PROOF: the same as in theorem 3.2. ∎

COROLLARY 5.2. *Assume the conditions of corollary* 5.1. *Then*

$$\hat{Z}^n \xrightarrow{w(P_o^n)} (\exp(W_t^2/(2t)))_{t>0} \quad .$$

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