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In [16] Williams defined a filtration which did not arise in the usual way in that it was not constructed from a Markov process. The natural question, posed implicitly by Walsh [14], is to study the properties which this filtration has in common with the others. We have already carried out this programme in the recurrrent case [11] - [13]. But the transient case, which is in some ways typified by BES(3), is more interesting because then the filtration has a discontinuity at the minimum. One can thus hope for results which go beyond those of Williams [19]. And this note is intended as the beginning of such an investigation. It is a preliminary version of a more definitive study.

The content can be summarised as follows. In the first section we define the excursion σ-field $\mathcal{E}^x$ of $BES(3)$ below the level x, and derive what we call an internal characterisation of the process which generates it. This is used to derive a projection result, which can be used to prove the conditional excursion theorem. The main problem to be faced next is proving that the excursion filtration is right continuous. The proof occupies the rest of the article and is carried out as follows. First we calculate, as explicitly as possible, the so-called CMO formulae of Williams [18]. This appears to complete the key step in [18], and to carry it out we need conditional excursion theory in its most sophisticated form [10]. The method differs from that of [9], though not fundamentally. Next we prove that the excursion compensator process $\tilde{L}(x,t)$ has a bicontinuous version. This is important for our subsequent work. The proof is quite hard, needing several messy estimates.

In the final section we put our results together and deduce the right continuity of the excursion filtration. The result seems quite deep. One reason for such an opinion is the observation by Walsh [14], that it has the Williams decomposition [19] as an easy corollary. We give some detail on this and other consequences.

§1 Excursions from a fixed level

We recall here the basic notions of the conditional excursion theory for the process $BES(3)$. This is the process which is the unique strong non-negative solution of the stochastic differential equation

$$X_t = x_0 + \beta_t + \int_0^t \frac{ds}{X_s}$$

driven by the Brownian motion β_t. We assume that $x_0 > 0$, this being the interesting case. It will be convenient to introduce the notation $BES^a(3)$ for a process which is equivalent in law to $X_t + a$.

The local time of the process X_t is defined for us by the occupation density formula

$$\int_0^t f(X_s)ds = \int_R f(a)L(a,t)da$$

This does not lead to any normalisation difficulties since the singular point is polar and so its local time is zero anyway. We now define the excursion field of X_t below the level x by using the recipe of Williams [16]. We write $A(x,t) = \int_0^t 1_{(X_s < 0)}ds$ and we denote by $\tau(x,.)$ the right continuous inverse of A(x,.). Then the excursion σ-field of the process X_t below the level x is $\mathcal{E}^x = \sigma\{X_{\tau(x,t)} : t \geq 0\}$.

As it happens this is not the best description of the excursion field for the purposes of calculation, since it tells nothing about the way the martingales contained in the process below x are related to the martingales of the original process. As in [8], and using the method of filtration enlargement expounded in [5], we give an **internal** description of the excursion field which involves only quantities determined directly from the process $\tilde{X}(x,t) = X_{\tau(x,t)}$. In fact the filtration is that of a Brownian motion stopped at an independent exponentially distributed local time.

To make this more precise we write T_x to be the hitting time of the semi-infinite interval $(-\infty, x]$ by the process X_t , and we let ρ_x denote the corresponding last leaving time. Then ρ_x is not a stopping time for the filtration $\mathcal{X}_t$ of X_t but $\eta_x = A(x, \rho_x)$ is clearly a stopping time for the process below x, since $\tilde{X}(x,t)$ dies then. The obvious thing to do is to add in the ρ_x information at the beginning. The relevant result (which we state below as Theorem 1.1) can be found, by the careful reader, in [5]. In order to formulate it we let $\mathcal{Y}_t$ be the new filtration formed by taking the right continuous completion of the smallest filtration containing $\mathcal{X}_t$ and the random variable $L(x, \rho_x)$. Let us also define V_t^x to be the $\mathcal{X}_t$ supermartingale which is the $\mathcal{X}_t$ optional (= predictable) projection of $1_{[0,\rho_x[}$ so that $V_t^x = M_t^x - D_t^x$ is its Doob-Meyer decomposition. Here M_t^x is the martingale part while D_t^x is the increasing process. They are each continuous.

Theorem 1.1 The following process is a Brownian motion in the new filtration $\mathcal{Y}_t$.

$$\beta_t^x = \beta_t - \int_0^{t\wedge\rho_x} \frac{1}{V_s^x} d < \beta, M^x >_s$$

$$+ \int_{t\wedge\rho_x}^{t} \frac{1}{1 - V_s^x} d < \beta, M^x >_s$$

The next thing is to calculate the required optional projection. In this particular case it is not too hard.

Lemma 1.2 The $\mathcal{X}_t$ optional projection is given explicitly by $V_t^x = \inf(1, x/X_t)$.

Proof: Let T be any stopping time and look at

$$\mathbf{P}[1_{[0,\rho_x[}(T)] = \mathbf{P}[T < \rho_x] = \mathbf{P}[\mathbf{P}_{X_T}[T_x < +\infty]; T < +\infty]$$

However since the scale function for the $BES(3)$ process is $-1/x$ the inner expectation computes as $\inf(1, x/X_T)$. From which the result follows by definition.

Consequently when we substitute in the theorem above we find that in the filtration $\mathcal{Y}_t$ the process X_t satisfies the SDE

$$X_t = x_0 + \beta_t^x + \int_0^{t\wedge\rho_x} 1_{(X_s < x)} \frac{ds}{X_s} + \int_{t\wedge\rho_x}^{t} \frac{ds}{X_s - x}$$

It may help to state this in a more informal way. The standard convention is that the sets $\{\rho_x = 0\}$ and $\{T_x = +\infty\}$ are equal.

Corollary 1.3 (a) On the set $\{T_x < +\infty\}$ we have the following behaviour.

(i) $\{X_t : 0 \leq t \leq T_x\}$ behaves like a Brownian motion stopped at the time T_x.
(ii) $\{X_t : T_x \leq t \leq \rho_x\}$ starts at x, behaves like a Brownian motion when above x, and like a BES(3) process when below x. It stops at the indicated stopping time ρ_x.
(iii) $\{X_t : t \geq \rho_x\}$ is a $BES^x(3)$ process which is independent of the entire σ-field $\mathcal{X}_{\rho_x}$.

(b) On the set $\{T_x = +\infty\}$ X_t is a $BES^x(3)$ process.

However we have yet to make good our promise of giving an internal description of $\mathcal{E}^x$. And we must also explain the death of the process in (ii) above in a more satisfactory way. This is derived as follows. See [8] or [6] for the description in the general case. We write down Tanaka's formula for $X_{t} \wedge x$ in the filtration $\mathcal{Y}_t$

$$X_t \wedge x = x_0 \wedge x + \int_0^t 1_{(X_s < x)} d\beta_s^x + \int_0^t 1_{(X_s < x)} \frac{ds}{X_s} - \tfrac{1}{2} L(x,t)$$

Next do the time change $t \mapsto \tau(x,t)$ which gives us the reflecting stochastic differential equation

$$\tilde{X}(x,t) = x_0 \wedge x + \tilde{\beta}(x,t) + \int_0^t \frac{ds}{\tilde{X}(x,s)} - \tilde{L}(x,t) \qquad (*)$$

which is valid for $0 \leq t \leq \eta_x$. Here $\tilde{\beta}(x,t) = \int_0^{\tau(x,t)} 1_{(X_s < x)} d\beta_s^x$ is a $\mathcal{Y}_{\tau(x,t)}$ Brownian motion by the theorem of Paul Lévy, and $\tilde{L}(x,t) = \frac{1}{2} L(x, \tau(x,t))$. For more information on reflecting stochastic differential equations see [2] or [6]. Notice also that the time η_x can be defined by the equation $\eta_x = inf\{t : \tilde{L}(x,t) = \frac{1}{2} L(x, \rho_x)\}$, the **point** being that *since* $L(x, \rho_x)$ *is known before the process starts* this defines a stopping time. So the internal description reads as follows.
First of all we consider the above equation (*) as an SDE driven by the stopped Brownian motion $\{\tilde{\beta}(x,t) : 0 \leq t \leq \eta_x\}$ and subject to the boundary conditions $\tilde{X}(x,t) \leq x$ and $\int_0^t (\tilde{X}(x,s) - x) d_s \tilde{L}(x,s) = 0$. Such a set-up is called a reflection problem and to our knowledge was first investigated in the article [2]. Now we have the precise statement.

Theorem 1.4 (i) $\tilde{X}(x,t)$ *is the unique solution of the equation* (*).

(ii) Both $\tilde{L}(x,t)$ and $\tilde{X}(x,t)$ are continuous processes adapted to the filtration of $\tilde{\beta}(x,t)$.

(iii) The σ-field $\mathcal{E}^x$ is the σ-field generated by the process $\{\tilde{\beta}(x,t) : 0 \le t \le \eta_x\}$.

(iv) The σ-fields $\mathcal{E}^x$ and $\mathcal{Y}_{r(x,t)}$ are conditionally independent given $\tilde{\mathcal{X}}(x,t)$.

[$\tilde{\mathcal{X}}(x,t)$ is defined as the completed right continuous filtration generated by $\tilde{X}(x,t)$ and the random variable η_x.]

Proof: (i) This is reduced to the result of [2] by expanding the probability space so that the driving Brownian motion runs for all time. Then we stop when the local time $\tilde{L}(x,t)$ hits the pre-determined level $\frac{1}{2}L(x,\rho_x)$. But (ii) follows in the same way, as does (iii).

(iv) It suffices to prove that if F is any bounded $\mathcal{E}^x$ measurable random variable then the projection of F onto $\mathcal{Y}_{r(x,t)}$ and $\tilde{\mathcal{X}}(x,t)$ is the same. But for this one can use the martingale calculus. We define $F_t = \mathbf{E}[F|\tilde{\mathcal{X}}(x,t)]$. Then this is a bounded $\tilde{\mathcal{X}}(x,t)$ martingale, so by (iii) we can use the representation theorem of Ito to write this as a stochastic integral

$$F_t = F_0 + \int_0^{t\wedge\eta_x} u_s d_s\tilde{\beta}(x,s)$$

where u_t is $\tilde{\mathcal{X}}(x,t)$ predictable. Which makes it clear that F_t is a $\mathcal{Y}_{r(x,t)}$ martingale, so that the proof is complete.

The importance of Theorem 1.4 for our purposes is that it leads to a very useful projection result [8] Lemma 4.2. Rather than repeat that proof here we reformulate it following the suggestion of T. Jeulin [6]. Our way is slightly different. He always stays in the original time scale while I work with the time-change. The problem in question concerns the connection between the filtration $\mathcal{Y}_{r(x,t)}$ and its sub-filtration $\tilde{\mathcal{X}}(x,t)$ which we have already looked at a little in Theorem 1.4.

Lemma 1.5 If N_t is any bounded totally discontinuous $\mathcal{Y}_{r(x,t)}$ martingale then N_t is an $\{\mathcal{Y}_{r(x,t)} \vee \mathcal{E}^x\}$ martingale.

Proof: If F is bounded and $\mathcal{E}^x$ measurable then the $\tilde{\mathcal{X}}(\tau(x,t))$ martingale $F_t = \mathbf{E}[F|\tilde{\mathcal{X}}(\tau(x,t))]$ is continuous and a $\mathcal{Y}_{\tau(x,t)}$ martingale by Theorem 1.4 (iii) and (iv) respectively. Hence N_tF_t is a $\mathcal{Y}_{\tau(x,t)}$ martingale. But we are required to prove that $\mathbf{E}[(N_t - N_s)FA] = 0$ for every bounded $\mathcal{Y}_{\tau(x,s)}$ measurable A. However this can be written as $\mathbf{E}[(N_tF_t - N_sF_s)A]$. Which is zero.

The crucial result in this direction is then stated as follows. To prepare we make a few remarks. Note that if $\tilde{T}$ is a $\mathcal{Y}_{\tau(x,t)}$ stopping time then $\tau(x,\tilde{T})$ is a $\mathcal{Y}_t$ stopping time. And it is clear that the corresponding σ-algebras are the same. But the converse is a little more tricky. Suppose that S is any $\mathcal{X}_t$ stopping time. Then the definition shows that $A(x,S)$ is an $\mathcal{X}_{\tau(x,t)}$ stopping time. But in this case the appropriate σ-field is $\mathcal{X}_{U(x)}$ where $U(x) = \inf\{t : t > S, X_t{=}x\}$.

Corollary 1.6 (Projection Result) If N_t is as above then for any stopping time $\tilde{T}$ of the filtration $\mathcal{Y}_{\tau(x,t)} \vee \mathcal{E}^x$ we have

$$\mathbf{E}[N_{\tilde{T}} - N_0 | \mathcal{E}^x] = 0.$$

Proof: This is just the Doob optional stopping theorem in the context of the lemma.

We have set this up separately for emphasis. It is important to note that $\tilde{T}$ can be any $\mathcal{E}^x$ measurable time. This is the crucial remark needed to prove the results of Walsh ([8] Theorem 4.7 for example).

So far we have not had occasion to use any excursion theory at all. We now introduce the setting. At this stage anyone unfamiliar with the theory is recommended not to notice that the excursion measure has a probabilistic interpretation at all. It is best to think of things simply from the point process angle. The excursion space $\mathcal{W}^x$ is the collection of all continuous paths γ, taking values in the semi-infinite interval $[x, +\infty)$, which satisfy $\gamma(0) = x$ and which are absorbed when next they return to the level x. The symbol Δ shall denote the (ubiquitous) null excursion and on the set $\mathcal{W}^x$ we let $\mathcal{Q}^x$ be the associated *Brownian* excursion measure. Our justification for this choice is of course the Corollary 1.3 (a) (iii), and a full detailed description of the σ-finite measure $\mathcal{Q}^x$ can be found in [10]. Then we define the excursion space of X_t to be $\mathcal{W} = \bigcup_x \mathcal{W}^x$. This is a Fréchet space in the compact open topology.

The next thing is to define the excursion process from the region [0,x] to be

$$\mathcal{E}^x(\omega, s) = \{X_t : \tau(x, s)- \leq t \leq \tau(x, s)\}$$

This is valid when s is a jump time of $\tau(x, s)$, otherwise $\mathcal{E}^x(\omega, s)$ is defined as Δ. The excursion process takes its values in the space $\mathcal{W}^x$, with the measure $\mathcal{Q}^x$. Note that the so-called excursion measure itself, which we can denote by $\mathcal{Q}$, is the direct sum of the various $\mathcal{Q}^x$'s. It will be clear that an excursion functional is a measurable function $\mathcal{A}^x : \mathcal{W}^x \mapsto R$. Then the following can be regarded as a weak version of the conditional excursion theorem. The full treatment of the theory can be found in Maisonneuve [7] where he deals with the general situation for Markov processes.

Theorem 1.7 Let $\mathcal{A}^x$ be any function defined on the excursion space $\mathcal{W}^x$ such that $\mathcal{Q}^x[\mathcal{A}^x] < +\infty$ Then

$$\sum_{0<s\leq t} \mathcal{A}^x \circ \mathcal{E}^x(\omega, s) - \mathcal{Q}^x[\mathcal{A}^x]\tilde{L}(x, t)$$

is an $\mathcal{Y}_{\tau(x,t)} \vee \mathcal{E}^x$ martingale up to time η_x.

Proof: By Corollary 1.3 (ii) we can apply the argument of [10] where we used the process $e^{-\lambda t} R_\lambda(B_t)$, $R_\lambda^x f(x)$ being the resolvent of Brownian motion killed when it first enters $(-\infty, x]$ (actually it is blatantly clear *afterwards* that this is the correct process to look at).

However by using stochastic integration we can considerably strengthen the previous result. Classically Theorem 1.5 states the bald fact that the excursion process is a Poisson point process when one conditions by a suitable rate function. The statement given below is analogous to marking the point process by putting extra information in at each jump. And it is this which is most useful for doing calculations. Incidentally be careful not to confuse the two uses of the letter $\mathcal{E}$. It is used both for the excursion σ-field and for the excursion process.

Corollary 1.8 Let $\mathcal{A} : \mathcal{W} \times \Omega \times R^+ \mapsto R$ be a Borel measurable function such that for each fixed $\gamma \in \mathcal{W}^x$ the process $\mathcal{A}(\gamma, \omega, t)$ is $\mathcal{X}_{r(x,t)}$ optional. Then

$$\mathbf{E}[\sum_{0<s\leq t\wedge\eta_x} \mathcal{A}(\mathcal{E}^x(\omega,s),\omega,s) - \int_0^{t\wedge\eta_x} \mathcal{Q}^x[\mathcal{A}(.,\omega,s)]d_s\tilde{L}(x,s)|\mathcal{E}^x] = 0$$

provided that $\mathbf{E}[\int_0^t \mathcal{Q}^x[\mathcal{A}(.,\omega,s)]d_s\tilde{L}(x,s)] < +\infty$. Furthermore under $\mathcal{Q}^x$ the canonical process on $\mathcal{W}^x$ is strongly Markovian.

Proof: This essentially involves nothing more than noting that if we integrate the martingale against any predictable process then we obtain another martingale. So the result for $\mathcal{A}$ predictable is immediate while the generalisation to $\mathcal{A}$ optional follows since $\tilde{L}(x,t)$ is continuous and therefore does not charge the countable union of stopping time graphs where an optional and predictable process differ. For more details we refer to [10].

Notice how we have required that $\mathcal{A}$ be defined on the entire excursion space. This is because later we wish to consider excursions of the process from random levels. In general the conditional excursion theorem really describes how to project functionals of the process X_t onto the excursion filtration. Corollary 1.8 corresponds only to the most difficult part. An arbitrary functional can be decomposed in three parts, supported respectively on the time intervals $[0, T_x], [T_x, \rho_x]$ and $[\rho_x, +\infty[$. These may be termed the initial excursion, the final excursion and the interim process. In fact many of our calculations below are carried out by first doing this reduction. And our convention is to say that there is only a final excursion on the set where T_x is infinite.

We finish off the section by stating the results of Ray-Knight in the appropriate form. Calculations and proofs are to be found in [8]. Recall first that a $BES^2(k)$ process is the unique strong solution of the SDE

$$Z_a = Z_0 + \int_0^a \sqrt{Z_b}d\beta_b + ka$$

Thus the $BES^2(0)$ process is a martingale which is absorbed as soon as it hits zero. Walsh has given an interesting explanantion as to 'why' these appear in connection with the local time laws.

Theorem 1.9 (Ray-Knight) The local time of a $BES(3)$ process started at x_0 can be described as follows.

(a) On the set $\{T_a < +\infty\}$
(i) $\{L(x, \tau(a,t) \wedge \rho_a), x \geq a\}$ is a $BES^2(2)$ process, started at $L(a, , \tau(a,t) \wedge \rho_a)$, until level x_0 and is a $BES^2(0)$ process thereafter.
(ii) $\{L(x, \infty) - L(x, \rho_a), x \geq a\}$ is a $BES^2(2)$ process independent of (i).

(b) On the set $\{T_a = +\infty\}$
(i) $\{L(x, \infty), x \geq x_0\}$ is a $BES^2(2)$ process.

In [8] there is a more detailed investigation of the Ray-Knight properties using essentially the exponential martingale technique. One can also profit from consulting the article of Jeulin [6].

§2. CMO formulae

To begin we introduce some notation. In fact we shall use $\mathbf{E}^x$ to denote the expectation operator for the $BES^x(3)$ process. And we shall use the symbol $\tilde{\mathbf{E}}^x$ to represent the expectation associated to Brownian motion killed at time T_x. $\tilde{\tilde{\mathbf{E}}}$ shall represent the expectation associated to Brownian motion. The corresponding resolvent operators, which are easier to calculate explicitly than the transition densities, will be noted respectively as $R_\lambda^x f$, $\tilde{R}_\lambda^x f$ and $\tilde{\tilde{R}}_\lambda f$. It is also convenient to write $\bar{R}_\lambda^x f$ for the resolvent of the $BES(3)$ process killed at time T_x.

The basic raw material for this section is supplied by the calculation of the resolvent operators. These can be found in [4], and in many other books and papers. So at the risk of boring the reader the explicit formulae are given by following. Remark that in our notation $\tilde{R}_\lambda^a f'(a+)$ denotes the derivative of $\tilde{R}_\lambda^a f(y)$ in the variable y, evaluated at $y=a+$.

$$\tilde{R}_\lambda^a f(x) = \frac{1}{\sqrt{2\lambda}} \int_a^\infty \left(e^{-\sqrt{2\lambda}|x-y|} - e^{-\sqrt{2\lambda}|2a-x-y|} \right) f(y)dy$$

$$\tilde{R}_\lambda^a f'(a+) = 2 \int_a^\infty e^{-\sqrt{2\lambda}|a-y|} f(y)dy$$

$$R_\lambda^a f(x) = 2 \int_a^x x^{-\frac{1}{2}} y^{\frac{3}{2}} I_{\frac{1}{2}}(\sqrt{2\lambda}(y-a)) K_{\frac{1}{2}}(\sqrt{2\lambda}(x-a)) f(y)dy$$

$$+2 \int_x^\infty x^{-\frac{1}{2}} y^{\frac{3}{2}} I_{\frac{1}{2}}(\sqrt{2\lambda}(x-a)) K_{\frac{1}{2}}(\sqrt{2\lambda}(y-a)) f(y)dy$$

$$\bar{R}_\lambda^a f(x) =$$

$$2 \int_a^x x^{-\frac{1}{2}} y^{\frac{3}{2}} \left[I_{\frac{1}{2}}(\sqrt{2\lambda} y) - \frac{I_{\frac{1}{2}}(\sqrt{2\lambda} a)}{K_{\frac{1}{2}}(\sqrt{2\lambda} a)} K_{\frac{1}{2}}(\sqrt{2\lambda} y) \right] K_{\frac{1}{2}}(\sqrt{2\lambda} x) f(y)dy$$

$$+2 \int_x^\infty x^{-\frac{1}{2}} y^{\frac{3}{2}} \left[I_{\frac{1}{2}}(\sqrt{2\lambda} x) - \frac{I_{\frac{1}{2}}(\sqrt{2\lambda} a)}{K_{\frac{1}{2}}(\sqrt{2\lambda} a)} K_{\frac{1}{2}}(\sqrt{2\lambda} x) \right] K_{\frac{1}{2}}(\sqrt{2\lambda} y) f(y)dy$$

Here $I_{\frac{1}{2}}$ and $K_{\frac{1}{2}}$ are the usual modified Bessel functions.

The really difficult part is now to calculate, as explicitly as possible, the projection onto (say) $\mathcal{E}^a$ of a dense set of functionals of the path. The most 'convenient choice' for these are the so-called CMO formulae introduced by Williams in his sketch [18]. But even in this case we have considerable complication. Our method here is slightly different from that of [9], since in that article there is no mention of excursion theory. But if we prepare carefully then it is possible to do the proofs more cleanly. So first we write

$$K_t(n,\lambda,\mathbf{f}) = K_t(\lambda_1,\lambda_2,\ldots,\lambda_n; f_1, f_2, \ldots f_n)$$

$$= \int_0^t dt_n e^{-\lambda_n t_n} f_n(X_{t_n}) \int_0^{t_n} \cdots \int_0^{t_2} dt_1 e^{-\lambda_1 t_1} f_1(X_{t_1})$$

where the functions $\{f_n\}$ are always assumed to be continuous and to have compact support. Also it will be convenient to let $K_t(0,\lambda,\mathbf{f}) \equiv 1$. Given our formula [11] Lemma 3.2 for calculating the excursion measure of a functional first step is now as follows.

Lemma 2.1 We have that

$$\mathbf{E}_y[K_{T_a}(n,\lambda,\mathbf{f})] = \bar{R}^a_{\mu_1}[f_1 \bar{R}^a_{\mu_2}[f_2 \ldots [\bar{R}^a_{\mu_n}[f_n]\ldots](x_0)$$

where $\mu_i = \lambda_i + \ldots + \lambda_n$. And there is an analogous result for $\tilde{\mathbf{E}}$.

Proof: These are done in exactly the same way the only difference being that the hitting time may be infinite in the $BES(3)$ case. And we only look at this one. By performing a change in the order of integration we get

$$\mathbf{E}[\int_0^{T_a} dt_n e^{-\lambda_n t_n} f_n(X_{t_n}) \int_0^{t_n} \cdots \int_0^{t_2} dt_1 e^{-\lambda_1 t_1} f_1(X_{t_1})] =$$

$$\mathbf{E}[\int_0^{T_a} dt_1 e^{-\lambda_1 t_1} f_1(X_{t_1}) \int_{t_1}^{T_a} \cdots \int_{t_{n-1}}^{T_a} dt_n e^{-\lambda_n t_n} f_n(X_{t_n})]$$

Now we can apply the Markov property of the $BES(3)$ process killed at time T_a at the times $t_{n-1}, \ldots t_1$ successively to get the required expression.

The above result needs to be slightly extended when we wish to do calculations. So we insert the trivial result which follows.

Lemma 2.2

$$\mathbf{E}_y[e^{-\mu T_a}K_{T_a}(n,\lambda,\mathrm{f})] =$$

$$\mathbf{E}_y[-\mu\int_0^{T_a} e^{-\mu t}K_t(n,\lambda,\mathrm{f})dt + \int_0^{T_a} e^{-(\lambda+\mu)t}f_n(X_t)K_t(n{-}1,\lambda,\mathrm{f})dt]$$

with an analogous result if we use the Brownian expectation.

Proof: We use integration by parts to write expand the l.h.s. and the result is immediate.

The point is of course that these can be evaluated using Lemma 2.1. But there is more. The following special case is extremely important, and is by no means obvious. We write Γ to be the excursion functional defined by the relation

$$\Gamma(f;\lambda,\nu)\circ\mathcal{E}^a(\omega,t) = e^{-\mu T_a}\int_0^{T_a} e^{-\lambda s}f(X_s)ds\circ\theta_{\tau(a,t)-}$$

It will be convenient later on to let $\Gamma(f;\lambda,0)=\Gamma(f;\lambda)$.

Corollary 2.3 The excursion measure of $\Gamma(f;\lambda,\mu)$ is given by

$$-\mu\tilde{R}^a_{\lambda+\mu}[f\tilde{\mathrm{R}}^a_\lambda 1]'(a+) + \tilde{\mathrm{R}}^a_{(\lambda+\mu)}f'(a+)$$

Proof: In the case $n=1$ we can evaluate

$$\tilde{\mathbf{E}}_x[e^{-\mu T_a}\int_0^{T_a} e^{-\lambda s}f(X_s)ds] = -\mu\tilde{R}^a_{\lambda+\mu}[f\tilde{\mathrm{R}}^a_\lambda 1](y) + \tilde{\mathrm{R}}^a_{(\lambda+\mu)}f(y)$$

The proof is completed by differentiating this in y at $y=a+$ (see [11] for the reason why).

Lemma 2.4 For $\lambda>0$, $\quad t<\rho_a$ we have

(a) $\tilde{\mathbf{E}}_x[e^{-\lambda T_a}] = \exp\{-\sqrt{2\lambda}(y-a)^+\}$

(b) $\mathbf{E}[\exp\{-\lambda[\tau(a,t)-\tau(a,0)]\}|\mathcal{E}^a] = \exp\{-\lambda t-\sqrt{2\lambda}\tilde{L}(a,t)\}$

Moreover (b) remains true if t is replaced by any $\mathcal{E}^x$ measurable time.

Proof: The proof of (a) is too well known for us to bother with the proof here. We therefore concentrate on (b). Note first of all that by the strong Markov property at time $\tau(a,0)$ we can assume that $x_0 \leq a$ (if $\tau(a,0)$ is infinite then there is nothing to prove). First we write $e^{-\lambda t} = 1 - \lambda \int_0^t e^{-\lambda s} ds$. Then if we do the time change $t \mapsto \tau(a,t)$ we can write this as

$$\exp\{-\lambda\tau(a,t)\} = 1 - \lambda \int_0^{\tau(a,t)} e^{-\lambda s}\, ds$$

$$= 1 - \lambda \sum_{0 < s \leq t} \int_{\tau(a,s)-}^{\tau(a,s)} e^{-\lambda u} du - \lambda \int_0^t \exp\{-\lambda\tau(a,s)\} ds$$

Now let us take the projection of this onto $\mathcal{E}^a$. The only term which gives difficulty is the jump term, but this can be written as $-\lambda \sum \Gamma(1;\lambda) \exp\{-\lambda\tau(a,s)\text{-}\}$ whose projection we can calculate to be $-\lambda \int_0^t \mathcal{Q}[\Gamma(1;\lambda)] \mathbf{E}[\exp\{-\lambda\tau(a,s)\text{-}\}|\mathcal{E}^a] d_s\tilde{L}(a,s)$. However from the previous result $\mathcal{Q}[\Gamma(1;\lambda)] = 2/\sqrt{2\lambda}$ so the projection gives us the equation

$$\mathbf{E}\exp\{-\lambda\tau(a,t)\}|\mathcal{E}^a] =$$

$$-\lambda \int_0^t \mathbf{E}[\exp\{-\lambda\tau(a,s)\text{-}\}|\mathcal{E}^a](dt + (2/\sqrt{2\lambda}) d_t\tilde{L}(a,t))$$

We obtain the result by solving this integral equation noting how continuity of the integrator allows replacement of $\tau(a,s)$- by $\tau(a,s)$. The final comment is justified since it is true for simple random variables, and one can prove it for the other times by using the dominated convergence theorem. For this we note that, by Theorem 1.4, $\tilde{L}(a,.)$ is continuous.

This is by no means the easiest proof, the use of the excursion theorem being much too extreme a weapon, though it is claimed that this is how the result was discovered. See [8] for an easier proof based on the ideas of [16].

We are going to do our calculation by induction. The key to the entire argument is to use the continuity of the local time to simplify the calculation of the jump terms. We formalise this now since the argument is much used later on.

Lemma 2.5 Let U_t be a $\mathcal{Y}_t$ optional bounded process. Then we have

$$\mathbf{E}[e^{-\mu t} \int_{T_a}^{\rho_a} e^{-\lambda t} f(X_t) U_t dt | \mathcal{E}^a] =$$

$$\int_0^{\eta_a} \mathbf{E}[\exp\{-(\lambda+\mu)\tau(a,t)\}U_{\tau(a,t)}|\mathcal{E}^a]\left[\mathcal{Q}[\Gamma(f;\lambda,\mu)]\,d_t\tilde{L}(a,t)+f(\tilde{X}(a,t))dt\right]$$

Proof: By the $\mathcal{E}^a$ measurability of $\tilde{X}(a,t)$ (see Theorem 1.4) it suffices to consider only the jump term. So we look at

$$\sum \exp\{-(\lambda+\mu)\tau(a,t)\}U_{\tau(a,t)}\Gamma(f;\lambda,\mu)\circ\mathcal{E}(\omega,t)$$

By the conditional excursion theorem in its general form, because $U_{\tau(a,t)}$ is $\mathcal{Y}_{\tau(a,t)}$ optional, we know that this projects to the required result. But since U_t is optional it follows that we can change it as required on the jumps and still obtain the same projection because $\tilde{L}(a,t)$ is continuous. Which proves the result.

The next theorem is important because it enables us to reduce nth order formulae to lower order cases, and hence we can carry out inductive arguments. The main application of this is given in section 4.

Theorem 2.6 For each integer n

$$\mathbf{E}[\exp\{-\mu\tau(a,t\wedge\eta_a)\}\int_0^{\tau(a,t)\wedge\rho_a} e^{-\lambda s}f(X_s)K_s(n,\lambda,\mathrm{f})ds|\mathcal{E}^a]=$$

$$\mathbf{E}[\exp\{-\mu T_a\}\int_0^{T_a} e^{-\lambda s}f(X_s)K_s(n,\lambda,\mathrm{f})ds]\exp\{-\mu t\wedge\eta_a-\sqrt{2\mu}\tilde{L}(a,t)\}+$$

$$\int_0^{\eta_a\wedge t} f(\tilde{X}(a,s))\exp\{-\sqrt{2\mu}[\tilde{L}(a,t)-\tilde{L}(a,s)]-\mu(t\wedge\eta_a-s)\}$$

$$\mathbf{E}[\exp\{-(\mu+\lambda)\tau(a,s)\}K_{\tau(a,s)}(n,\lambda,\mathrm{f})|\mathcal{E}^a]ds \quad +$$

$$\{-\mu\tilde{R}^a_{\lambda+\mu}[f\tilde{\mathrm{R}}^a_\lambda 1]'(a+)+\tilde{\mathrm{R}}^a_{(\lambda+\mu)}f'(a+)\}\int_0^{t\wedge\eta_a}\exp\{-\sqrt{2\mu}[\tilde{L}(a,t)-\tilde{L}(a,s)]-\mu(t\wedge\eta_a-s)\}$$

$$\mathbf{E}[\exp\{-(\mu+\lambda)\tau(a,s)\}K_{\tau(a,s)}(n,\lambda,\mathrm{f})|\mathcal{E}^a]d_s\tilde{L}(a,s).$$

Proof: We break down the expectation in the usual way. By the strong Markov property the first part is given by

$$\mathbf{E}[\exp\{-\mu\tau(a,t\wedge\eta_a)\}\int_0^{T_a} e^{-\lambda t}f(X_t)K_t(n,\lambda,\mathrm{f})dt|\mathcal{E}^a]=$$

$$\mathbf{E}[\exp\{-\mu T_a\}\int_0^{T_a} e^{-\lambda t} f(X_t)K_t(n,\lambda,\mathbf{f})dt]\mathbf{E}[\exp\{-[\tau(a,t)-\tau(a,0)]\}|\mathcal{E}^a].$$

From which we get the answer by Lemma 2.4. The next part we look at is

$$\exp\{-\mu\tau(a,t)\}\int_{T_a}^{\tau(a,t)\wedge\rho_a} e^{-\lambda s} f(X_s)K_s(n,\lambda,\mathbf{f})ds$$

which by the previous lemma projects in two parts. The first part is

$$\mathbf{E}[\exp\{-\mu\tau(a,t)\}\int_0^{t\wedge\eta_a}\exp\{-\lambda\tau(a,s)\}f(\tilde{X}(a,s))K_{\tau(a,s)}(n,\lambda,\mathbf{f})ds|\mathcal{E}^a]$$

Using Fubini to bring the conditional expectation inside we see that we must find $\mathbf{E}[\exp\{-\mu\tau(a,t\wedge\eta_a)+\lambda\tau(a,s)\}f(\tilde{X}(a,s))K_{\tau(a,s)}(n,\lambda,\mathbf{f})|\mathcal{E}^a]$. But for this we use Theorem 1.4 (iv) to write it as

$$\mathbf{E}[\exp\{-(\mu+\lambda)\tau(a,s)\}f(\tilde{X}(a,s))K_{\tau(a,s)}(n,\lambda,\mathbf{f})|\mathcal{E}^a]\mathbf{E}_{\tilde{X}(a,s)}[\exp\{-\mu\tau(a,t\wedge\eta_a-s)\}|\mathcal{E}^a]$$

$$= f(\tilde{X}(a,s))\mathbf{E}[\exp\{-(\mu+\lambda)\tau(a,s)\}K_{\tau(a,s)}(n,\lambda,\mathbf{f})|\mathcal{E}^a]$$
$$\exp\{-\sqrt{2\mu}[\tilde{L}(a,t)-\tilde{L}(a,s)]-\mu(t\wedge\eta_a-s)\}$$

where for the last part we can quote Lemma 2.4. Finally we look at the jump term which, from the previous lemma has the same projection as

$$\sum_{0<s\le t\wedge\eta_a}\exp\{-(\mu+\lambda)\tau(a,s)\}\Gamma(f;\lambda,\mu)\circ\mathcal{E}(\omega,s)K_{\tau(a,s)}(n,\lambda,\mathbf{f})$$

the projection itself being given by

$$\mathcal{Q}[\Gamma(f;\lambda,\mu)]\int_0^{t\wedge\eta_a}\mathbf{E}[\exp\{-(\mu+\lambda)\tau(a,s)\}K_{\tau(a,s)}(n,\lambda,\mathbf{f})|\mathcal{E}^a]d_s\tilde{L}(a,s)$$

And the result we want follows by Corollary 2.3.

Allowing the luxury of a comment as to why these formulae work so comparatively easily compared with any others we can isolate two reasons. The first is that the parts which drag along are exponentials, and consequently have the simplest possible multiplicative properties. Another is the trick of pushing the time off to infinity at every opportunity. This has the effect of making everything homogeneous Markovian, so avoiding the complications of exiting after constant times. In any case for the sake of completeness we now show how to compute the first order CMO formula.

Theorem 2.7 Let f be a bounded continuous function having compact suuport. Then

$$\mathbf{E}[\int_0^\infty e^{\lambda t} f(X_t)dt|\mathcal{E}^a] =$$

$$\bar{R}^a_\lambda f(x_0) + \int_0^{\eta_a} \exp\{-\sqrt{2\lambda}(x_0-a)^+ - \lambda t - \sqrt{2\lambda}\tilde{L}(a,t)\} f(\tilde{X}(a,t))dt$$

$$\exp(-\sqrt{2\lambda}(x_0-a)^+)\tilde{\bar{R}}^a_\lambda f'(a+) \int_0^{\eta_a} \exp\{-\lambda t - \sqrt{2\lambda}\tilde{L}(a,t)\} d_t\tilde{L}(a,t)$$

$$+ \inf(1, \frac{a}{x_0}) \exp\{-\sqrt{2\lambda}(x_0-a)^+ - \lambda\eta_a - \sqrt{2\lambda}\tilde{L}(a,\eta_a)\} R^a_\lambda f(a)$$

Proof: Using the strong Markov property of $BES(3)$ at time T_a we find that

$$\mathbf{E}[\int_0^{T_a} e^{-\lambda t} f(X_t)dt|\mathcal{E}^a] = \mathbf{E}[\int_0^{T_a} e^{-\lambda t} f(X_t)dt] = R^a_\lambda f(x_0)$$

Next we consider $\mathbf{E}[\int_{T_a}^{\rho_a} e^{-\lambda t} f(X_t)dt|\mathcal{E}^a]$. which by Lemma 2.5 is equal to

$$\int_0^{\eta_a} \mathbf{E}[\exp(-\lambda \tau(a,t))|\mathcal{E}^a] f(\tilde{X}(a,t))dt \qquad +$$

$$\int_0^{\eta_a} \mathcal{Q}[\Gamma(f;\lambda)]\mathbf{E}[\exp(-\lambda\tau(a,t)-)|\mathcal{E}^a] d_t\tilde{L}(a,t)$$

However the excursion measure is evaluated from Corollary 2.3 while we can compute the conditional expectation by Lemma 2.4. The final term is clearly

$$\mathbf{P}[T_a < +\infty]\mathbf{E}[\exp\{-\sqrt{2\lambda}(x_0-a)^+ - \lambda\rho_a\}|\mathcal{E}^a]\bar{\mathbf{E}}^a_a[\int_0^\infty e^{-\lambda t} f(X_t)dt]$$

and this can be written as required, again from Corollary 2.3 and Lemma 2.4.

§3. Bicontinuity of $\tilde{L}(x,t)$

One of the most important facts that we need to establish is the bicontinuity of the process $\tilde{L}(x,t)$ as a two parameter process. This is hard enough in the Brownian case, and is even harder here. The complications are that, whereas the law of the Brownian local time is well-known and has a simple form, this is not true for the local time at constant level of $BES(3)$. Furthermore, the law of the last leaving time plays an essential role. So we first need to calculate both of these.

Instead of computing the law of ρ_a directly we look at that of η_a. Recall how η_a is the death time of the process $\tilde{X}(a,t)$, alias the time X_t spends in the interval $[0,a]$.

Lemma 3.1 The law of η_a is given by

$$\mathrm{E}[\exp\{-\frac{\mu^2}{2}\eta_a\}] = \frac{f_a(X_0)}{f_a(+\infty)}$$

where

$$\begin{aligned} f_a(x) &= I_{\frac{1}{2}}(\mu x) & (x \leq a) \\ &= I_{\frac{1}{2}}(\mu a) - a\mu I'_{\frac{1}{2}}(\mu a) + (a^2\mu/x) I_{\frac{1}{2}}(\mu a). & (x > a) \end{aligned}$$

Proof: This we do by a standard martingale technique.With f defined as above using Ito's formula we can check that $f_a(X_t)\exp\{-(\mu^2/2)\int_0^t 1_{(X_s<a)}ds\}$ is a bounded martingale. So by Doob's theorem at time $+\infty$ we evaluate

$$\mathrm{E}[\exp\{-\frac{\mu^2}{2}\eta_a\}] = \frac{f_a(X_0)}{f_a(+\infty)}$$

and the proof is finished.

Corollary 3.2 $\eta_{a+\epsilon}$ decreases to η_a almost surely.

Proof: It is enough to check convergence in law. Which follows from the continuity of the above expression in a.

We now seek an estimate for $\sup_{z<x}\tilde{L}(z,t)$. Unfortunately this is quite complicated. From this point on it is convenient to adopt the convention that C is always a constant, which need not have the same value from one line to the next. Hopefully it will be clear what we mean by a 'constant'.

Lemma 3.3 For x fixed

$$\mathbf{E}[\sup_{z<x} L^p(z,\infty)]$$

is finite for all values of $p \geq 0$.

Proof: By using the strong Markov property at the hitting time of x_0 we see that it suffices to compute with the case where X_t starts at zero. In this case the Ray-Knight theorem ([8] p.798 for example) implies that $\{L(z,\infty), z\geq 0\}$ is a $BES^2(2)$ process started at zero. However if B_t is a Brownian motion started at zero then by the inequality of Burkholder-Davis-Gundy [1] we see the existence of a constant C_p such that

$$\mathbf{E}[(B_t^*)^p] \leq C_p t^{p/2}$$

B_t^* being the maximal process. Since $L(z,\infty)$ is identical in law to the sum of the squares of two independent such processes the result follows.

This is one instance in which we are able to take advantage of the transience of the process. It will be seen later on that the estimates for times are also easier than in [11].

Lemma 3.4 If $x > a$ then

$$||\tilde{L}(x,t) - \tfrac{1}{2}L(a,\tau(x,t))||_n \leq C|x-a|^{\frac{1}{2}}$$

where x and a are fixed in a given compact set K bounded away from zero, with $t \leq t_0$.

Proof: We begin wth the defining equation for $BES(3)$ namely

$$X_t = x_0 + \beta_t + \int_0^t \frac{ds}{X_s}$$

We use the Tanaka formula on this to get

$$(X_t \wedge x) \vee a = (x_0 \wedge x) \vee a + \int_0^t 1_{(a<X_s<x)} d\beta_s + \int_0^t 1_{(a<X_s<x)} \frac{ds}{X_s} + \frac{1}{2}[L(a,t) - L(x,t)]$$

Now we rewrite this in the $\tau(x,t)$ time scale to obtain

$$\frac{1}{2}L(a,\tau(x,t)) - \tilde{L}(x,t) = (\tilde{X}_t \wedge x) \vee a - (x_0 \wedge x) \vee a - \int_0^t 1_{(a<\tilde{X}_s<x)} d\tilde{\beta}_s - \int_0^t 1_{(a<\tilde{X}_s<x)} \frac{ds}{\tilde{X}_s}$$

where $\tilde{\beta}_t$ is a new Brownian motion. We now apply the inequality of Burkholder-Davis-Gundy to estimate the r.h.s. By the triangle inequality

$$||\tfrac{1}{2}L(a,\tau(x,t)) - \tilde{L}(x,t)||_n \quad \leq \quad ||(\tilde{X}_t \wedge x) \vee a - (x_0 \wedge x) \vee a||_n \quad +$$

$$||\sup_{s<t} \int_0^s 1_{(a<\tilde{X}_u<x)} d\tilde{\beta}_u||_n \quad + \quad ||\sup_{s<t} \int_0^s 1_{(a<\tilde{X}_u<x)} \frac{du}{\tilde{X}_u}||_n$$

The first term is bounded by $(x-a)$. For the second we bound by the inequality of Burkholder-Davis-Gundy via the occupation density formula

$$\mathbf{E}[\left((\sup_{s<t} \int_0^s 1_{(a<\tilde{X}_u<x)} d\tilde{\beta}_u \right)^n] \quad \leq \quad C\mathbf{E}[\left(\int_0^t 1_{(a<\tilde{X}_s<x)} ds \right)^{\frac{n}{2}}]$$

$$\leq \quad C\mathbf{E}[\left(\int_a^x L(b,\tau(x,t)) db \right)^{\frac{n}{2}}] \, quad \leq \quad C|x-a|^{\frac{n}{2}} \mathbf{E}[\sup_{z\in K} L^{\frac{n}{2}}(z,\infty)].$$

Which gives the required bound by using Lemma 3.3. The estimate for the third term is similar but easier.

Lemma 3.5 We have the estimate

$$\mathbf{E}[(\tilde{L}(a,s) - \tilde{L}(a,t))^{2n}] \quad \leq \quad C|s-t|^n$$

for s,t in a fixed compact set.

Proof: Suppose that $s < t$. By the strong Markov property at the first time the process $\tilde{X}(a,t)$ hits zero after time s, it suffices to estimate $\mathbf{E}[\tilde{L}^{2n}(a,t)]$ for small values of t. For this we find the appropriate Green's function. Thus we try to compute

$$\mathbf{E}\left[\int_0^\infty \exp\{-\frac{\mu^2}{2}t - \gamma\tilde{L}(a,t)\}dt\right]$$

by following the recipe of [4]. Here the boundary conditions are that

(a) at $x = 0$ the solution is bounded

(b) at $x = a$ we have $f'(a-) = -\gamma f(a)$

while we must solve the differential equation

$$f'' + (2/x)f' = \mu^2 f.$$

The two linearly independent solutions of this are

$$x^{-\frac{1}{2}} I_{\frac{1}{2}}(\mu x) \quad \text{and} \quad x^{-\frac{1}{2}} K_{\frac{1}{2}}(\mu x)$$

Thus the solution which satisfies the condition (a) is

$$g_1(x) = x^{-\frac{1}{2}} I_{\frac{1}{2}}(\mu x).$$

The other solution we can write as

$$g_2(x) = x^{-\frac{1}{2}}[K_{\frac{1}{2}}(\mu x) + A I_{\frac{1}{2}}(\mu x)]$$

where we compute that

$$A = -\frac{2a[\mu K'_{\frac{1}{2}}(\mu a) + \gamma K_{\frac{1}{2}}(\mu a)] - K_{\frac{1}{2}}(\mu a)}{2a[\mu I'_{\frac{1}{2}}(\mu a) + \gamma I_{\frac{1}{2}}(\mu a)] - I_{\frac{1}{2}}(\mu a)}$$

The Wronskian of the two functions is $W = 1/x^2$ (which is just a multiple of the scale derivative) and so the desired quantity can be expressed as

$$2g_2(a)\int_0^a x^2 g_1(x)dx = 2g_2(a)\int_0^a x^{\frac{3}{2}} I_{\frac{1}{2}}(\mu x)dx$$

We remark at once that γ does not appear in the integral. Using the Wronskian relation we compute that

$$g_2(a) = -a^{-\frac{1}{2}}\left[I_{\frac{1}{2}}(\mu a) - 2a\gamma I_{\frac{1}{2}}(\mu a) - 2a\mu I'_{\frac{1}{2}}(\mu a)\right]^{-1}$$

We now evaluate $\mathbf{E}[\int_0^\infty \exp\{-\frac{\mu^2}{2}t\}\tilde{L}^n(a,t)dt]$ by taking the n^{th} derivative of this in γ. Using this it is easy to see that for large values of μ this behaves like a multiple of $1/\mu^{n+2}$. Since the bound is uniform in a the result follows.

Lemma 3.6 Under the above conditions, if $p \geq 2$, we have

$$||\tfrac{1}{2}L(a,\tau(x,t)) - \tilde{L}(a,t)||_p \leq C|x-a|^{\frac{1}{2}}$$

Proof: Since $\frac{1}{2}L(a,\tau(x,t)) = \tilde{L}(a, A(a,\tau(x,t)))$ we note the estimate

$$t - A(a,\tau(x,t)) = \int_a^x L(b,\tau(x,t))db \quad \leq \int_a^x L(b,\infty)db$$

which is conditionally independent of $\tilde{L}(a,t)$ given $\tilde{L}(a,\eta_a)$ by Theorem 1.4 and the Ray-Knight theorem. Now use the previous lemma to obtain

$$\mathbf{E}[(\tfrac{1}{2}L(a,\tau(x,t)) - \tilde{L}(a,t))^p|\tilde{L}(a,\eta_a)] \quad \leq \quad C\mathbf{E}[\left(\int_a^x L(b,\infty)db\right)^{\frac{p}{2}}|\tilde{L}(a,\eta_a)]$$

However Lemma 3.3 gives the bound $C|x-a|^{\frac{p}{2}}$. Which is as required.

The purpose of these calculations is to provide the input for an application of the Kolmogoroff Criterion. Recall the statement.

Kolmogoroff Criterion If X_t is a process indexed by R^d satisfying the moment estimate

$$\mathbf{E}[|X_t - X_s|^p \leq C|t-s|^{d+\gamma} \qquad (p>0\,,\gamma>0)$$

then X_t has a version which is almost surely Hölder continuous of order α for every $\alpha < \gamma/p$.

Note incidentally that in the Hölder condition

$$|X_t - X_s| \leq C|t-s|^\alpha \qquad (|t-s| < \delta)$$

it is the choice of the δ which is random. Compare for example the sharp estimate of Paul Lévy for Brownian motion.

Theorem 3.7 The process $\tilde{L}(x,t)$ has a version which is jointly Hölder continuous of order α for every value of $\alpha < \frac{1}{4}$.

Proof: To do this we put together the previous estimates. By the triangle inequality

$$||\tilde{L}(x,t)-\tilde{L}(a,s)||_n \leq ||\tilde{L}(x,t)-\tfrac{1}{2}L(a,\tau(x,t))||_n$$
$$+ \quad ||\tilde{L}(a,t)-\tfrac{1}{2}L(a,\tau(x,t))||_n \quad + \quad ||\tilde{L}(a,t)-\tilde{L}(a,s)||_n$$

Then this is bounded by $C[|x-a|^{\frac{1}{2}}+|t-s|^{\frac{1}{2}}]$. And so if we write n=$4m$ we obtain the estimate

$$\mathbf{E}[(\tilde{L}(x,t)-\tilde{L}(a,s))^{4m}] \leq C|(x,t)-(a,s)|^m$$

since we are working on an arbitrary fixed compact set K and thus $|x^{\frac{1}{2}} + t^{\frac{1}{2}}| \leq C|(x,t)|$. Which enables us to apply the Kolmogoroff criterion, and finish the proof.

§4. The excursion filtration.

As already indicated this section is concerned with the excursions of the process from a random level. This appears to have been considered first by Walsh [14], though the prototype is to be found in [17]. The first, and indeed very considerable problem, is to show that the excursion filtration is right continuous. The filtration as we have defined it is assumed to be complete and traditionally this suffices if the underlying Markov process is a 'good' one. However we have not as yet been able to exhibit the process. So we follow the standard pattern of trying to show that $\mathcal{E}^{x+}$ differs from $\mathcal{E}^{x}$ only by null sets of the measure $\mathbf{P}$. The main difficulty is in proving that we can apply the dominated convergence theorem to a suitably large class of projections. The other results in this section follow readily once we establish right continuity. In fact they are made to appear almost embarassingly naive. It is of interest to search for an easy proof of the right continuity. I have been unable to find one.

Theorem 4.1 The filtration $\{\mathcal{E}^{x}, x \geq 0\}$ satisfies the usual conditions.

Proof: Since each $\mathcal{E}^{x}$ is defined to be complete it suffices to prove right continuity. Namely that for F any bounded measurable function on Ω we have

$$\lim_{\epsilon \downarrow 0} \mathbf{E}[F|\mathcal{E}^{a+\epsilon}] \quad = \quad \mathbf{E}[F|\mathcal{E}^{a}]$$

almost surely. This proves that $\mathcal{E}^{x+}$ and $\mathcal{E}^{x}$ differ by no more than $\mathbf{P}$ null sets. And since it is enough to give the proof for a dense set of such F we can restrict ourselves to those functionals of the form $K_\infty(n, \lambda, \mathbf{f})$ (see section two for the definition of these) where the functions $\{f_n\}$ are all supported on a compact subset of $(x, +\infty)$. Thus the relevant projections compute as in section 2, though they are simpler in that the absolutely continuous term is missing. To begin with we have the evaluation from Theorem 2.7 of the first order CMO formula when f is supported above x.

$$\mathbf{E}[\int_0^\infty e^{-\lambda t} f(X_t) dt | \mathcal{E}^{a}] =$$

$$\bar{R}^a_\lambda f(x_0) + \exp(-\sqrt{2\lambda}(x_0-a)^+)\tilde{R}^a_\lambda f'(a+) \int_0^{\eta_a} \exp\{-\lambda t-\sqrt{2\lambda}\tilde{L}(a,t)\} d_t\tilde{L}(a,t)$$

$$+ \inf(1, \frac{a}{x_0}) \exp\{-\sqrt{2\lambda}(x_0-a)^+-\lambda\eta_a-\sqrt{2\lambda}\tilde{L}(a,\eta_a)\} R^a_\lambda f(a)$$

We can now examine each term in turn. The first one $\bar{R}^a_\lambda f(x_0)$ is continuous in a as can be seen from the explicit formula given at the beginning of section two. The second term is continuous in a by Lemma 3.1, the bicontinuity of $\tilde{L}$, and the explicit form of the excursion measure which we have listed at the beginning of section two. And of course the last term is continuous in a, as we can also see from its explicit form. We now consider the higher order formulae

$$\mathbf{E}[\int_0^\infty e^{-\lambda t} f(X_t) K_t(n,\lambda,\mathbf{f}) dt | \mathcal{E}^a] =$$

$$\mathbf{E}[\int_0^{T_a} e^{-\lambda t} f(X_t) K_t(n,\lambda,\mathbf{f}) dt] + \mathbf{E}[\int_{T_a}^{\rho_a} e^{-\lambda t} f(X_t) K_t(n,\lambda,\mathbf{f}) dt | \mathcal{E}^a] +$$

$$\mathbf{E}[\int_{\rho_a}^\infty e^{-\lambda t} f(X_t) K_t(n,\lambda,\mathbf{f}) dt | \mathcal{E}^a]$$

Examining each of these terms in turn we see that the first calculates from Lemma 2.1. The second term is reduced, via Lemma 2.5, to the evaluation of

$$\mathbf{E}[\exp\{-\lambda\tau(a,t)\} K_{\tau(a,t)}(n,\lambda,\mathbf{f}) | \mathcal{E}^a]$$

for $t < \eta_a$. Which we can calculate from Lemma 2.2 and the reduction facilities offered via Theorem 2.6. Coming to the final term we see that it decomposes as

$$\mathbf{E}[\int_{\rho_a}^\infty e^{-\lambda t} f(X_t) dt \int_{\rho_a}^t e^{-\lambda_n s} f_n(X_s) K_s(n-1,\lambda,\mathbf{f}) ds | \mathcal{E}^a] \qquad +$$

$$\mathbf{E}[\int_{\rho_a}^\infty e^{-\lambda t} f(X_t) dt \int_0^{\rho_a} e^{-\lambda_n s} f_n(X_s) K_s(n-1,\lambda,\mathbf{f}) ds | \mathcal{E}^a] \qquad (**)$$

The second one of these can be factored by the strong Markov property at the time ρ_a to give simply

$$R^a_\lambda f(a) \mathbf{E}[e^{-\lambda\rho_a} \int_0^{\rho_a} e^{-\lambda_n s} f_n(X_s) K_s(n-1,\lambda,\mathbf{f}) ds | \mathcal{E}^a]$$

whereupon we operate with Theorem 2.6 as before. It remains only to remark that the first term (**) can be evaluated by the same sort of procedure, using Lemma 2.1 to calculate the expectations after time ρ_a. The fact that the limit gives what we want follows by the same argument as in the first order case applied inductively, since at each stage the projection is continuous in the variable a.

Since we are now in the realms of the general theory of processes we can consider stopping times Z in the excursion filtration. Recall the 'policy statement' in [1] to the effect that general theory is concerned with proving analogues of fixed time results for random times. In the present context this is **devastatingly** true in that we are able to trivialise the apparently deep result of Williams [19]. This was one of the more surprising results to emerge from [14] and here we simply repeat the proof given there. But first we prepare with some refinements, one of which is the following useful technical lemma. In this ξ denotes the minimum of the process X_t. By the strong Markov property it is attained at a unique time. The same argument shows that we have the equality of the set $\{\xi > x\}$ and the set $\{T_x < +\infty\}$.

Lemma 4.2 Let $Z \geq \xi$ be any $\mathcal{E}^x$ stopping time.

(a) If $\mathcal{F}(i, x)$ and $\mathcal{F}(f, x)$ are bounded measurable functionals defined on the initial and final excursions respectively then

$$\mathbf{E}[\mathcal{F}(i, x)\mathcal{F}(f, x)1_{(Z>x)}|\mathcal{E}^Z] = \tilde{\mathbf{E}}^x[\mathcal{F}(i, x)]\bar{\mathbf{E}}^x[\mathcal{F}(f, x)]1_{(Z>x)}$$

(b) If $\mathcal{A}$ is defined as at Corollary 1.6 then

$$\mathbf{E}[\sum_{0<s\leq t\wedge\eta_x} \mathcal{A}(\mathcal{E}^x(\omega, s), \omega, s)1_{(Z>x)}|\mathcal{E}^Z] =$$

$$1_{(Z>x)}\int_0^{t\wedge\eta_x} \mathbf{E}[\mathcal{Q}^x[\mathcal{A}(., \omega, s))]|\mathcal{E}^Z]d_s\tilde{L}(x, s)$$

Proof: (a) Let A be any element of $\mathcal{E}^Z$. Then by Corollary 1.3 we get

$$\mathbf{E}[\mathcal{F}(i, x)\mathcal{F}(f, x)1_{(Z>x)}1_A] = \tilde{\mathbf{E}}^x[\mathcal{F}(i, x)]\bar{\mathbf{E}}^x[\mathcal{F}(f, x)]\mathbf{E}[1_{(Z>x)}1_A]$$

since $1_{(Z>x)}1_A$ is $\mathcal{E}^x$ measurable. So, by definition, the proof is complete.

(b) This is carried out in the same way only now we use corollaries 1.6 and 1.8, noting how $1_{(Z>x)}\tilde{L}(x,.)$ is $\mathcal{E}^Z$ measurable.

Theorem 4.3 Let $Z \geq \xi$ be any $(\mathcal{E}^x, x \geq 0)$ stopping time.

(a) The initial excursion to Z, the final excursion from Z, and the process in between, are all mutually independent.

(b) The initial excursion has the law of a Brownian motion, independent of Z, stopped at the time T_Z.

(c) The final excursion has the law of a $BES^Z(3)$, driven by a Brownian motion independent of Z.

(d) If $\mathcal{A}$ is defined as at Corollary 1.6 and is continuous on the excursion space $\mathcal{W}$ then

$$\mathbf{E}[\sum_{0<s\leq t\wedge\eta_Z} \mathcal{A}(\mathcal{E}^Z(\omega,s),\omega,s)|\mathcal{E}^Z] = \int_0^{t\wedge\eta_Z} \mathbf{E}[\mathcal{Q}^Z[\mathcal{A}(.,\omega,s)]|\mathcal{E}^Z]d_s\tilde{L}(Z,s)$$

Proof:(a) In Lemma 4.2 (a) $\tilde{\mathbf{E}}^x[\mathcal{F}(i,x)]$ and $\bar{\mathbf{E}}^x[\mathcal{F}(f,x)]$ are continuous in x, as can be seen by looking at the explicit formulae at the beginning of section two. Now let $\{Z_n\}$ be a sequence of $\mathcal{E}^x$ measurable $\mathcal{E}^x$ stopping times such that $Z_n \downarrow Z$ a.s. But then we can replace x by Z_n in the statement of Lemma 4.2 (a) so that the result follows by taking the limit in n, using the dominated convergence theorem.

(b) This is the same sort of argument. One only needs to check that $\mathcal{Q}^x[\mathcal{A}]$ is continuous in x, which follows from the definition of the excursion measure via the semigroup density as in [10], and also the continuity of $\tilde{L}(x,t\wedge\eta_x)$ which we looked at in the previous section.

We now wish to apply this very powerful and surprising result. It is of course what Walsh calls the 'strong Markov property of the excursion process', though we would prefer to reserve this terminology for another use.

Lemma 4.4 ξ is a stopping time of the filtration $\mathcal{E}^x$.

Proof: This has already been noted since $\{T_x < +\infty\} = \{\xi < x\}$ and the filtration is right continuous.

Corollary 4.5 (Williams) A $BES(3)$ process X_t can be decomposed in the following manner.
(a) It is a Brownian motion B_t run until it hits the random variable ξ.
(b) Then it runs as a $BES^{\xi}(3)$ process Y_t.
(c) ξ is an independent uniform random variable on the interval $[0, x_o]$

Proof: Then only thing left to prove is that ξ has the uniform distribution on $[0, x_0]$. But this is immediate by definition of the scale function $s(x) = -(1/x)$ for $BES(3)$.

As already indicated this proof is much harder than it seems, due to the subtlety of proving the filtration right continuous. Nevertheless its air of inevitability is inescapable.

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