GUY MÉTIVIER

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SPECTRAL ASYMPTOTICS FOR THE $\bar{\partial}$-NEUMANN PROBLEM.

Guy METIVIER
Institut de Recherche
Mathématique de Rennes
Laboratoire Associé n° 305
Campus de Beaulieu
35 042 - RENNES CEDEX

0- INTRODUCTION.

With the rather extensive study of the $\bar{\partial}$-Neumann problem (see for instance Hörmander [5], Folland-Kohn [3], and the references there) it may be of some interest to give an asymptotic formula for the eigenvalues of this self adjoint non elliptic boundary value problem.

When considering the problem on $\Omega \subset \mathbb{C}^n$ we have (in a sense which is made precise in [9]) :

(0.1) $$N(\lambda) \sim c_i \lambda^n + B(\lambda)$$

where $N(\lambda)$ denotes the number of eigenvalues less or equal to λ, c_i is the "usual interior constant" for elliptic problems, and $B(\lambda)$ measures the contribution of the boundary .For elliptic boundary value problems it is well known that $B(\lambda)$ is negligible in front of the interior term, while for some degenerate problems the opposite phenomenom is occuring (see for instance [10] , [12] ...). Here, using min-max arguments, one can show, when $\partial\Omega$ is a C^∞ manifold, that $B(\lambda)$ is equivalent to the counting function $N_b(\lambda)$ of the eigenvalues of a pseudodifferential operator on the boundary $\partial\Omega$. When this operator is subelliptic with loss of one derivative (and in our case, that means that condition $Z(q)$ is satisfied), one could make use of the results of Menikoff-Sjöstrand [8] (suitably extended to systems) and obtain that

(0.2) $$N_b(\lambda) \sim c_b \lambda^n$$

So it appears that for the $\bar{\partial}$-Neumann problem the boundary term $B(\lambda)$ has the same order of growth as the interior term $c_i\ \lambda^n$, and one can expect a formula of the kind

$$(0.3) \qquad N(\lambda) \sim (c_i + c_b)\lambda^n$$

However, using the ideas of [11], we will give in this paper a self contained and indepndant proof of (0.3), the main interest of which being that it does not require $\partial\Omega$ to be very smooth (C^2 will be sufficient). Also we will localize (0.3) and show that the spectral function satisfies (see section 2 for a precise statement):

$$(0.4) \qquad \mathrm{tr}\ e(\lambda;z,z) \sim c_o\ \lambda^n + \lambda^{n+1} \int_o^\infty e^{-2\lambda\tau d(z)}\ 2\tau\ c(z,\tau)d\tau$$

where $d(z)$ denotes the distance of $z \in \Omega$ to the boundary $\partial\Omega$; the constants c_i, c_b, c_o and $c(z,\tau)$ occuring in (0.3) and (0.4) will be explicited in sections 1 and 2.

1.- STATEMENT OF THE RESULT.

For a detailed presentation of the $\bar{\partial}$-Neumann problem we refer the reader to [3], [5] (see also the references given there) We just recall now what is necessary for our purpose.

Let Ω be an open set in $\mathbb{C}^n$ whose boundary $\partial\Omega$ is of class C^2 ; we note $z = x+iy$ the points in $\mathbb{C}^n$, and dxdy the Lebesgue measure ; $\mathbb{C}^n \simeq \mathbb{R}^{2n}$ is equipped with the Euclidian metric and dS in the induced measure on $\partial\Omega$. Furthermore we select a C^2 real valued function ϕ, such that $\phi < 0$ in Ω, $\phi = 0$ and $d\phi \neq 0$ on $\partial\Omega$.

We shall note $L^2(\Omega;q)$, $H^k(\Omega;q)$, $C^\infty(\bar{\Omega};q)$... etc the spaces of $(0,q)$ forms on Ω

$$(1.1) \qquad u = \sum_{|J|=q} u_J\ d\bar{z}_J$$

with $L^2(\Omega)$, $H^k(\Omega)$, $C^\infty(\bar{\Omega})$...etc coefficients. Also we set :

$$||u||^2_{L^2(\Omega;q)} = 2^q \sum_{|J|=q} ||u_J||^2_{L^2(\Omega)}$$

Here and below J is an ordered sequence $(j_1,\dots,j_q)$ with $1 \leqslant j_1 < j_2 < \dots < j_q \leqslant n$, of length $q = |J|$; in the sequel we shall note $\{J\}$ the subset $\{j_1,\dots,j_q\}$ of $\{1,\dots,n\}$.

The operator $\bar\partial$ acts from $C^\infty(\bar\Omega;q)$ into $C^\infty(\bar\Omega;q+1)$ and its formal adjoint, Θ from $C^\infty(\bar\Omega;q)$ into $C^\infty(\bar\Omega;q-1)$.

For u and v in $C^\infty(\bar\Omega,q)$ we set :

$$a(u,v) = (\bar\partial u,\bar\partial v)_{L^2(\Omega;q+1)} + (\Theta u,\Theta v)_{L^2(\Omega;q-1)}$$

and

$$Q(u,v) = a(u,v) + (u,v)_{L^2(\Omega;q)}$$

Let $\sigma(\Theta)(z,\zeta)$ be the symbol of Θ and let $D(\Omega;q)$ be the space of the $u \in C^\infty(\bar\Omega;q)$ satisfying the boundary condition :

(1.2) $$\sigma(\Theta)(z,d\phi(z))u(z) = 0 \quad \forall z \in \partial\Omega.$$

At last let $V(\Omega;q) \hookrightarrow L^2(\Omega;q)$ be the completion of $D(\Omega;q)$ for the norm $(Q(.,.))^{1/2}$. Then the operator $\square$ on (o,q)-forms is the operator associated via the variationnal method to the space $V(\Omega;q)$ and the sesquilinear form a :

$$a(u,v) = (\square u,v)_{L^2(\Omega,q)}$$

for u in the domain of $\square$ and v in $V(\Omega;q)$.

A classical manipulation shows that for u and v in $D(\Omega;q)$ one can write :

(1.3) $$a(u,v) = 2^{q+1}\left\{\sum_{\substack{1\leqslant j\leqslant n\\ |J|=q}} \left(\frac{\partial u_J}{\partial \bar z_j}, \frac{\partial v_J}{\partial \bar z_j}\right)_{L^2(\Omega)} + \frac{1}{2}\sum_{|J|=|K|=q}\int_{\partial\Omega} c_{J,K}u_J\bar v_K dS\right\}$$

where the $c_{J,K}$'s are continuous functions on $\partial\Omega$ and satisfy $\overline{c_{J,K}} = c_{K,J}$. For each $z \in \partial\Omega$ let T_z be the (n-1) dimensional space of the holomorphic tangent vectors and let $L_z^{(q)}$ be the restriction to $\Lambda^q T_z$ of the hermitian form defined by the matrix $(c_{J,K})_{|J|=|K|=q}$; note that $L_z^{(q)}$ can be intrinsecally defined by a formula :

$$L_z^{(q)}(A,B) = \langle \mathcal{L}_z^{(q)}, A \wedge \bar{B} \rangle \quad , \quad A,B \in \Lambda^q T_z$$

with $\mathcal{L}_z^{(1)} = 2\,\partial(\frac{1}{|d\phi|}\bar{\partial}\phi)$, and $\mathcal{L}_z^{(q)} = \pm \frac{1}{q!}\ \omega\wedge\cdots\wedge\omega\wedge\mathcal{L}_z^{(1)}$ where the sign depends only on q, and where there are q-1 products of the metric form $\omega = \sum_{j=1}^{n} dz_j \wedge d\bar{z}_j$.

Recall that $L_z = L_z^{(1)}$ is called the Levi-form at $z \in \partial\Omega$.

Now our basic assumption will be :

<u>CONDITION Z(q)</u> : at each point of $\partial\Omega$ the Levi-form has at least n-q positive eigenvalues or at least q+1 negative ones.

It will be shown that if Ω is bounded and satisfies condition Z(q), then the embedding $V(\Omega;q) \hookrightarrow L^2(\Omega;q)$ is compact, and then the spectrum of $\Box$ (which is clearly self adjoint and non negative) is discrete. In that situation we note $N_q(\lambda)$ the number of eigenvalues of $\Box$ (on (0,q)-forms) less or equal to λ.

<u>THEOREM 1</u> : Let Ω be a bounded open set in $\mathbb{C}^n$ whose boundary is of class C^2 and satisfies condition Z(q) for some $q \leq n$. Then, as $\lambda \to +\infty$ we have :

$$(1.4) \qquad N_q(\lambda) = \lambda^n \left\{ \binom{n}{q} \frac{\operatorname{meas} \Omega}{(2\Pi)^n\, n!} + \int_{\partial\Omega} c(z)\,dS + o(1) \right\}$$

with $c(z) = 0$ if $q = n$ and $c(z)$ given when $q < n$ by :

$$(1.5) \qquad c(z) = \frac{1}{2.(2\pi)^n\, n!} \int_0^\infty tr(e^{-\tau L_z^{(q)}})\, e^{-\tau tr^- L_z} \det\{\tau|L_z|(1-e^{-\tau|L_z|})^{-1}\}\,d\tau.$$

In formula (1.5) $tr^- L_z$ is the sum of the negative parts of the eigenvalues of L_z, $|L_z|$ is the non negative square root of $L_z^* L_z$, and $f(s) = \frac{s}{1-e^{-s}}$ is seen as an holomorphic function near the real axis, so that $f(\tau|L_z|)$ is well defined.

REMARK 1 : When $q = n$, condition $Z(n)$ is always satisfied, and the operator $\square$ is known to be elliptic with elliptic (Dirichlet) boundary condition ; in that case formula (1.4) is well known so in the sequel we shall always assume that $q < n$.

REMARK 2 : Let $\mathcal{J}_q$ be the set of the sequences J of length q such that $n \notin \{J\}$. If $\mu_1,\dots,\mu_{n-1}$ are the eigenvalues of the Levi-form, then the eigenvalues of $L_z^{(q)}$ are :

$$(1.6) \qquad \mu_J = \sum_{j\in J} \mu_j \quad , \quad J \in \mathcal{J}_q$$

and setting $\widetilde{\mu}_J = \mu_J + \mathrm{tr}^- L_z = \mu_J + \sum_{j=o}^{n-1} \mathrm{Max}(0,-\mu_j)$, we see that (1.5) can be written :

$$(1.7) \qquad c(z) = \frac{1}{2(2\pi)^n\, n!} \sum_{J\in \mathcal{J}_q} \int_o^\infty e^{-\tau\widetilde{\mu}_J} \prod_{j=1}^{n-1} \left(\frac{\tau|\mu_j|}{1-e^{-\tau|\mu_j|}}\right) d\tau$$

Note that condition $Z(q)$ is precisely equivalent to the fact that $\widetilde{\mu}_J > 0$ for all $J \in \mathcal{J}_q$, so that it is now clear that the integral (1.5) defines a continous function on $\partial\Omega$.

Before beginning the proof we recall the very important following result:

THEOREM *(Hörmander [5]) : if Ω satisfies condition $Z(q)$ then for any $z^o \in \partial\Omega$ there are a neighborhood $\mathcal{O}$ of z^o and a constant C such that :*

$$(1.7) \qquad \sum_{j,J} \left\|\frac{\partial u_J}{\partial \bar{z}_j}\right\|^2_{L^2(\Omega)} + \sum_J \int_{\partial\Omega} |u_J|^2 \leq C\, Q(u,u)$$

for all (0,q) forms $u \in C^\infty(\bar{\Omega};q)$ supported in $\bar{\Omega} \cap \mathcal{O}$

In fact, in [5] this theorem is proved either when Ω is strongly pseudo-convex or when $\partial\Omega$ is of class C^3, but it can be easily extended to the case where $\partial\Omega$ is of class C^2 ; for the case where that would not be already written in the litterature we shall briefly discuss that point in the appendix.

At last let us point out the following simple consequence of this theorem : if $u \in V(\Omega;q)$, then u has a trace on $\partial\Omega$ which belongs to $L^2(\partial\Omega)$, and (1.2) holds a.e.

2.- REDUCTION TO THE BOUNDARY.

In the remainder of the paper Ω is a given open set in $\mathbb{C}^n$ with C^2 boundary satisfying condition $Z(q)$ for some $q < n$. As a differential operator in Ω, $\square$ is given by (cf. (1.3)) :

$$\square\Big(\sum_{|J|=q} u_J\, d\bar{z}_J\Big) = -\sum_{|J|=q} \frac{1}{2}\,(\Delta u_J)\, d\bar{z}_J \tag{2.1}$$

and $\square$ being elliptic and smooth it follows that the spectral function $e(\lambda;z,w)$ (i.e the distribution kernel of the spectral resolution $E(\lambda)$ of $\square$) is C^∞ on $\Omega \times \Omega$. To be clear, we recall that we are dealing with a system and that $e(\lambda;z,w)$ is then a C^∞ function on $\Omega \times \Omega$, valued in the space of linear operators in the $\binom{n}{q}$ dimensional space of the $(0,q)$-co-vectors.

From (2.1) we deduce immediatly, using a very classical result [4], that for $z \in \Omega$:

$$\lim_{\lambda \to +\infty} \lambda^{-n}\operatorname{tr} e(\lambda;z,z) = \binom{n}{q}\,(2\Pi)^{-2n} \int_{\{\xi\in\mathbb{R}^{2n}/|\xi|^2\leqslant 2\}} d\xi = \binom{n}{q}\,\frac{1}{(2\Pi)^n n!} \tag{2.2}$$

Furthermore the convergence is uniform in z if z remains in a compact set contained in Ω.

Moreover, if we denote by $d(z)$ the distance of $z \in \Omega$ to $\partial\Omega$, for each $\delta > 0$ there is C_δ such that :

$$\forall\lambda > 0,\ \forall z \in \Omega,\ d(z) \geqslant \delta/\sqrt{\lambda}\ :\ 0 \leqslant \operatorname{tr} e(\lambda;z,z) \leqslant C_\delta \lambda^n \tag{2.3}$$

Because the boundary value problem is non elliptic, this estimate is possibly not quite classical, and for the sake of completeness a very short proof will be given in the appendix.

The main problem is of course to study the spectral function near the boundary, and we will prove :

PROPOSITION 2.1. : For $z^\circ \in \partial\Omega$ and for $0 < \delta \leqslant 1$ we have :

$$(2.4) \qquad \lim_{\lambda \to +\infty} \{\lambda^{-1/2} \int_{|z-z^\circ| \leq \delta/\sqrt{\lambda}} |tr(e(\lambda;z,z) - g(\lambda;z,z^\circ)| dxdy\} = 0$$

with $$g(\lambda;z,z^\circ) = \lambda^{n+1} \int_0^\infty e^{-2\lambda\tau d(z)}\, 2\tau\, c(\tau,z^\circ)\, d\tau$$

$d(z)$ denoting the distance of $z \in \Omega$ to $\partial\Omega$, and $c(\tau,z^\circ)$ being a positive continous function such that $\int_0^\infty c(\tau,z^\circ)d\tau = c(z^\circ)$ where $c(z^\circ)$ is defined in (1.5).

A consequence of (2.4) is that ; for some $C > 0$:

$$(2.5) \qquad \frac{1}{\sqrt{\lambda}} \int_{|z-z^\circ| \leq \delta/\sqrt{\lambda}} tr(e(\lambda;z,z))\, dxdy \leq C$$

but we will also prove :

PROPOSITION 2.2. : for each compact set $\sum \subset \partial\Omega$, there are C, λ_0, δ_0 such that (2.5) holds for $\lambda > \lambda_0$, $\delta < \delta_0$, $z^\circ \in \sum$.

Indeed it could be proved that the convergence (2.4) is uniform in $z^\circ \in \sum$, but the weaker and easier result of proposition 2.2 is sufficient for our purpose.

PROOF OF THEOREM 1 : Ω is assumed to be compact and near $\partial\Omega$ there are local coordinates $z = (\sigma,t)$ with $\sigma \in \partial\Omega$ and $t = d(z)$. Then $dxdy = \rho(\sigma,t)\, dS\, (\sigma)\, dt$ with $\rho(\sigma,o)=1$. We choose δ small enough such that $|\sigma - z^\circ| \leq \delta/\sqrt{\lambda}$ and $o < t < \delta/\sqrt{\lambda}$ implies $|z - z^\circ| \leq 1/\sqrt{\lambda}$.

Note that $g(\lambda;z,z^\circ)$ only depends on t and

$$(2.6) \qquad \int_0^{\delta/\sqrt{\lambda}} g(\lambda;z,z^\circ)dt = \lambda^n \int_0^\infty c(\tau,z^\circ)\, (1 - e^{-2\delta\sqrt{\lambda}\tau})d\tau$$

The dimension of Ω is $2n-1$ and there are constants $C_1 < C_2$ such that :

$$(2.7) \qquad C_1\, \lambda^{-n} \leq \lambda^{-\frac{1}{2}} \int_{|\sigma-z^\circ| \leq \delta/\sqrt{\lambda}} dS(\sigma) \leq C_2\, \lambda^{-n}$$

and with (2.6) it follows that :

$$\lim_{\lambda\to+\infty} \{\lambda^{-\frac{1}{2}} \int_{|\sigma-z^{o}|\leqslant\delta/\sqrt{\lambda}} dS(\sigma)\left|\int_{o}^{\delta/\sqrt{\lambda}} g(\lambda;z,z^{o})dt - \lambda^{n}\ c(z^{o})\right|\} = 0 \tag{2.8}$$

Because c is a continous function we can replace in (2.8) $c(z^{o})$ by $c(\sigma)$; then making use of (2.8), propositions 2.1 and 2.2, and integrating in $z^{o} \in \partial\Omega$ we get :

$$\lambda^{-\frac{1}{2}} \int_{|\sigma-z^{o}|\leqslant\delta/\sqrt{\lambda}} dS(z^{o})\ dS(\sigma)\left|\int_{o}^{\delta/\sqrt{\lambda}} tre(\lambda;z,z)dt - \lambda^{n}\ c(\sigma)\right| \to 0$$

Therefore with (2.7) and because $\rho(\sigma,o) \equiv 1$, it follows that :

$$\lambda^{-n} \int_{d(z)\leqslant\delta/\sqrt{\lambda}} tre(\lambda,z,z)dxdy \longrightarrow \int_{\partial\Omega} c(\sigma)dS(\sigma) \tag{2.9}$$

Now theorem 1 follows immediatly from (2.2) (2.3) and (2.9) (included the compactness of the embedding $V(\Omega;q) \hookrightarrow L^{2}(\Omega;q)$).

3.- REDUCTION TO A PERTURBATION PROBLEM.

The main idea of this paper is to make use of dilations near the boundary in order to transform the asymptotic study into a perturbation problem.

From now on z° is a given point of $\partial\Omega$ and after a unitary change of coordinates we can assume that $z^\circ = 0$ and that, near z°, Ω is the set of points $z = (z', z_n) \in \mathbb{C}^{n-1} \times \mathbb{C}$ such that :

$$\text{(3.1)} \qquad \operatorname{Im} z_n > \varphi(z', \operatorname{Re} z_n)$$

where $\varphi(z',r)$ is a C^2 real valued function such that $\varphi(0) = 0$ and $d\varphi(0) = 0$. We can also assume that the Levi-form at z° is diagonal that is :

$$\text{(3.2)} \qquad \frac{\partial^2}{\partial z_j \partial \bar{z}_k}(0) = \frac{1}{2}\mu_j \delta_{j,k} \qquad \text{for } 1 \leqslant j,\ k \leqslant n-1$$

($\delta_{j,k}$ is the Kronecker's symbol).

Near z° we consider the following set of antiholomorphic vector fields :

$$\text{(3.3)} \qquad \begin{cases} L'_n = \dfrac{\partial}{\partial \bar{z}_n} \\[2ex] L'_j = \dfrac{\partial}{\partial \bar{z}_j} - \alpha_j(z', \operatorname{Re} z_n)\dfrac{\partial}{\partial \bar{z}_n} \quad \text{for } j = 1,\dots,n-1 \end{cases}$$

with $\alpha_j = 2i\ (1+i\ \frac{\partial\varphi}{\partial r})^{-1}\ \frac{\partial\varphi}{\partial \bar{z}_j}$.

We also orthonormalize the L'_j 's and get an orthonormal set $L_1,\dots,L_n$ of antiholomorphic vector fields with $L_1,\dots,L_{n-1}$ tangent to $\partial\Omega$:

$$\text{(3.4)} \qquad \begin{cases} L_j = \displaystyle\sum_{k=1}^{n-1} a_{j,k}(z)\ L'_k \qquad \text{for } j \leqslant n-1 \\[3ex] L_n = \displaystyle\sum_{k=1}^{n} a_{n,k}(z)\ L'_k \end{cases}$$

The coefficients $a_{j,k}$ are C^1 functions (near z°) and satisfy :

$$(3.5)\quad \begin{cases} a_{j,k}(z) = \sqrt{2}\,\delta_{j,k} + O(|z-z^\circ|) & \text{if } j \leqslant n-1,\ k \leqslant n-1 \\ a_{n,k}(z^\circ) = \sqrt{2}\,\delta_{j,k} & \text{for } k \leqslant n. \end{cases}$$

Now we flatten the boundary by considering the following change of variables :

$$(3.6)\quad \begin{cases} z = (z',z_n) \longrightarrow \kappa(z) = (z',\tilde{z}_n) \\ \tilde{z}_n = z_n + \psi(z',\mathrm{Im} z_n) - i\varphi(z',\mathrm{Re} z_n) \end{cases}$$

where $\psi(z',r)$ is the following real valued function :

$$(3.7)\quad \psi(z',r) = \mathrm{Re}\,\{\sum_{j<n} \frac{\partial^2\varphi}{\partial z_j \partial r}(0)\ z_j r\} - \mathrm{Im}\,\{\sum_{\substack{j<n\\k<n}} \frac{\partial^2\varphi}{\partial z_j \partial z_k}(0)\ z_j z_k\}.$$

Let us call $\mathcal{O}$ a neighborhood of z°, such that κ is a C^2-diffeomorphism from $\mathcal{O}$ onto $\mathcal{O}_1 = \kappa(\mathcal{O})$; . shrinking $\mathcal{O}$ if necessary we can assume the L'_j's are defined on $\overline{\mathcal{O}}$. In the sequel we shall note $\tilde{z}_n = t + is$.

Let M'_j be the image of L'_j under κ. With (3.2) and (3.7) we see that :

$$(3.8)\quad \begin{cases} M'_j = \dfrac{\partial}{\partial z_j} + \dfrac{1}{2}\beta_j \dfrac{\partial}{\partial t} & \text{for } j < n \\ M'_n = \dfrac{\partial}{\partial \bar{z}_n} + \dfrac{1}{2}\beta_n \dfrac{\partial}{\partial t} + \dfrac{1}{2}\gamma_n \dfrac{\partial}{\partial s} \end{cases}$$

With :

$$(3.9)\quad \begin{cases} \beta_j(z',t,s) = 2\dfrac{\partial\psi}{\partial z_j}(z',s) - \alpha_j(z',t)\,(1+i\dfrac{\partial\psi}{\partial r}) = \mu_j z_j + O(|z'|) + O(|\tilde{z}_n|) \\ \beta_n(z',t,s) = i\dfrac{\partial\psi}{\partial r}(z',s) \\ \gamma_n(z',t,s) = i\dfrac{\partial\varphi}{\partial r}(z',t) = \beta_n(z',t,s) + O(|z'|) + O(|\tilde{z}_n|). \end{cases}$$

From now on we drop the tildas, no confusion being possible between functions or operators in the new or old coordinates, and in the new ones we introduce the dilations :

$$(3.10) \qquad h_\rho(z',z_n) = (\rho z', \rho^2 z_n).$$

If we set $M'_{j,\rho} = \frac{1}{\rho}(h_\rho)_* M_j$ we get from (3.7) :

$$(3.11) \qquad \begin{cases} M'_{j,\rho} = \frac{\partial}{\partial z_j} + \frac{1}{2}\beta_{j,\rho}\frac{\partial}{\partial t} & \text{for } j < n \\ M'_{n,\rho} = \rho\frac{\partial}{\partial z_n} + \frac{1}{2}\beta_{n,\rho}\frac{\partial}{\partial t} + \frac{1}{2}\gamma_{n,\rho}\frac{\partial}{\partial s} \end{cases}$$

where $\beta_{j,\rho} = \rho\ \beta_j o h_\rho^{-1}$ and $\gamma_{n,\rho} = \rho\ \gamma_n o h_\rho^{-1}$ are defined on $\mathcal{O}_\rho = h_\rho(\mathcal{O}_1)$. From (3.9) one immediatly gets :

<u>LEMMA 3.1.</u> : for any $\delta > 0$, $\mathcal{O}_\rho$ contains the set $\{z \in \mathbb{C}^n \ / \ |z'| \leq \delta,\ |z_n| \leq \rho\delta\}$ if ρ is large enough. On this set the functions $\gamma_{n,\rho}$ and $\beta_{j,\rho}$ for $j = 1,\dots,n$ are bounded as well as their first order derivatives with a bound independant of ρ. Furthermore on compact sets we have the following convergences as $\rho \longrightarrow +\infty$:

$$M'_{j,\rho} \longrightarrow Z_j \overset{\text{def}}{=} \frac{\partial}{\partial z_j} = \frac{1}{2}\mu_j z_j \frac{\partial}{\partial t}$$

$$M'_{n,\rho} - (\rho + \beta_{n,\rho})\frac{\partial}{\partial \bar{z}_n} \longrightarrow 0.$$

Similarily we define $M_{j,\rho} = \frac{1}{\rho}(h_\rho o \kappa)_* L_j$ (for $j = 1,\dots,n$) and from (3.4) (3.5) and lemma 3.1 we deduce that $M_{j,\rho} \longrightarrow \sqrt{2}\, Z_j$ for $j < n$ and $M_{n,\rho} \sim \sqrt{2}\rho \frac{\partial}{\partial \bar{z}_n}$ (this will be made more precise in lemma 6.4).

Now we go back to the $\bar{\partial}$-Neumann problem ; first we introduce $\omega_1,\dots,\omega_n$ the orthonormal dual basis of $L_1,\dots,L_n$ in the space of $(0,1)$-forms ; as usual, if J is the ordered sequence $(j_1,\dots,j_q)$ we set $\omega_J = \omega_{j_1}\wedge\dots\wedge\omega_{j_q}$. If u is a $(0,q)$-form with support in $\mathcal{O}\cap\bar{\Omega}$ we can write it in the ω_J's basis :

$$(3.12) \qquad u = \sum_{|J|=q} u_J\, \omega_J$$

and then :

$$(3.13) \qquad ||u||^2_{L^2(\Omega;q)} = \sum_{|J|=q} ||u_J||^2_{L^2(\Omega)}$$

In the decomposition (3.12) the boundary condition (1.2) becomes :

(3.14) $\quad u_{J|\partial\Omega} = 0$ whenever $n \in \{J\}$

We note $\Omega_\rho = h_\rho \circ \kappa(\Omega \cap \mathcal{O}) = \{z \in \mathcal{O}_\rho \ , \ \mathrm{Im}\, z_n > 0\}$ and we introduce the following unitary operator from $L^2(\Omega \cap \mathcal{O})$ on $L^2(\Omega_\rho)$:

(3.15) $$\mathcal{K}_\rho f = \rho^{-(n+1)} \ (|\det d\kappa|^{-1/2} f) \ \circ (h_\rho \circ \kappa)^{-1}$$

With the decomposition (3.10) we extend $\mathcal{K}_\rho$ acting component by component from $L^2(\Omega \cap \mathcal{O} \ ; \ q)$ on the space $(L^2(\Omega_\rho))^{\binom{n}{q}}$ which will be noted $\mathcal{L}^2(\Omega_\rho \ ; \ q)$. We also introduce $\mathcal{V}(\Omega_\rho \ ; \ q)$ the image under $\mathcal{K}_\rho$ of the space of the $u \in V(\Omega, q)$ with support contained in $\bar{\Omega} \cap \mathcal{O}$.

For u and v in $\mathcal{V}(\Omega_\rho, q)$ we define :

(3.16) $$a_\rho(u,v) = \frac{1}{\rho^2} \, a \ (\mathcal{K}_\rho^{-1} u, \mathcal{K}_\rho^{-1} v)$$

and from (1.3) it follows immediatly that a_ρ has the form :

(3.17) $$\begin{cases} a_\rho(u,v) = \sum\limits_{j,J} (M_{j,\rho} \, u_J, \, M_{j,\rho} \, v_J)_{L^2(\Omega_\rho)} + \\ \frac{1}{\rho} \sum\limits_{j,J,K} (M_{j,\rho} \, u_J, \, \alpha_{j,J,K,\rho} \, v_K)_{L^2(\Omega_\rho)} + (\alpha_{j,J,K,\rho} \, u_K, \, M_{j,\rho} \, v_J)_{L^2(\Omega_\rho)} + \\ \frac{1}{\rho^2} \sum\limits_{J,K} (\beta_{J,K,\rho} \, u_J, \, v_K)_{L^2(\Omega_\rho)} + \sum\limits_{J,K} \int (\gamma_{J,K,\rho} \, u_J \, v_K)_{|s=0} \, dx'dy'dt. \end{cases}$$

where the $\alpha_{j,J,K,\rho}$, $\beta_{J,K,\rho}$ and $\gamma_{J,K,\rho}$ are continuous functions on $\mathcal{O}_\rho$, bounded uniformly for $\rho \geqslant 1$; furthermore, due to condition (3.14) the integral over $s = 0$ only involves the sequences J and K which do not contain n. At last because the Levi-form at z° is diagonal, (see (3.2)) the matrix $c_{J,K}(z^\circ)$ (for J and K which do not contain n) which appears in (1.3) is also diagonal (see the intrinsic meaning of this matrix recalled after (1.3)). Therefore it follows that, as ρ goes to infinity :

$$(3.18) \qquad \begin{cases} \gamma_{J,K,\rho} \text{ converges on compact sets, to } 0 \text{ if } J \neq K \text{ and to } \mu_J = \sum_{j\in J} \mu_j \text{ if} \\ J = K. \end{cases}$$

From the main estimate (1.7) we deduce that if σ is small enough. There is a constant C such that for all $\rho \geqslant 1$ and $u \in \mathcal{V}(\Omega_\rho \ ; q)$:

$$(3.19) \qquad \sum_{j,J} ||M'_{j,\rho}\, u_J||^2_{L^2(\Omega_\rho)} + \sum_J \int |u_J|^2_{|\delta=0}\, dx'dy'dt \leqslant C\, \{a_\rho(u,u) + \frac{1}{\rho^2}||u||^2_{\mathcal{L}^2(\Omega_\rho)}\}$$

Now we go to the limit as $\rho \to +\infty$. From (3.19) and the equivalence $M'_{n,\rho} \sim \sqrt{2}\,\rho \frac{\partial}{\partial \bar{Z}_n}$, it is clear that on the "limit space" we must have $\frac{\partial u}{\partial \bar{z}_n} = 0$. More precisely, noting $\mathbb{C}^n_+$ the half space $\operatorname{Im} z_n > 0$, we define $\mathcal{D}(\mathbb{C}^n_+, q)$ to be the space of the $(u_J)_{|J|=q} \in (H^2(\mathbb{C}^n_+))^{\binom{n}{q}}$ with compact support in the Z'-directions, satisfying :

$$(3.20) \qquad \frac{\partial}{\partial \bar{z}_n} u_J = 0 \text{ for all } J$$

and also satisfying the boundary condition, analogous to (3.14) :

$$(3.21) \qquad u_{J|\operatorname{Im} z_n = 0} = 0 \text{ whenever } n \in \{J\}.$$

In fact, with (3.20), (3.21) is equivalent to :

$$(3.21)' \qquad u_J = 0 \quad \text{when} \quad n \in \{J\}.$$

On $\mathcal{D}(\mathbb{C}^n_+, q)$ we define the following sesquilinear form :

$$(3.22) \qquad a_\infty(u,v) = \sum_{n \notin \{J\}} \{2 \sum_{j=1}^{n-1} (Z_j u_J, Z_j v_J)_{L^2(\mathbb{C}^n_+)} + \mu_J \int (u_J v_J)_{|\sigma=0}\, dx'dy'dt\}.$$

We shall show in section 5 that a_∞ is positive.

Let $\mathcal{H}$ be the closure of $\mathcal{D}(\mathbb{C}^n_+ \ ; q)$ in $\mathcal{L}^2(\mathbb{C}^n_+ \ ; q)$ and let $\mathcal{P}$ be the orthogonal projector on $\mathcal{H}$; let $\square_\infty$ be the self adjoint operator in $\mathcal{H}$ associated to a_∞ ; let $E_\infty(\lambda)$ be the spectral resolution of $\square_\infty$ and let $\tilde{E}_\infty(\lambda)$ be the operator $E_\infty(\lambda)\mathcal{P}$. The following lemma will be proved in section 5 :

LEMMA 3.2 : the distribution kernel of $\tilde{E}_\infty(\lambda)$ is a C^∞ (matrix valued) function $e(\lambda,.,.)$ on $\mathbb{C}^n_+ \times \mathbb{C}^n_+$ and satisfies :

$$\operatorname{tr} e_\infty(1;z,z) = \int_0^\infty e^{-2\,\operatorname{Im}_n 2\tau} c(\tau,z^\circ)d\tau$$

where $c(\tau, z^\circ)$ is as in proposition 2.1.

Now the main idea of this paper is to deduce proposition 2.1 from the convergence "$a_\rho \to a_\infty$". In order to make this idea precise we introduce a cut-off function $\Theta \in C_0^\infty(\Theta)$, which is equal to 1 in a neighborhood of z°, and we consider the operator :

$$E_\rho(\lambda) = \mathcal{K}_\rho \theta\, E(\rho^2\lambda)\, \theta\, \mathcal{K}_\rho^{-1} \tag{3.23}$$

(If we could take $\theta = 1$, $E_\rho(\lambda)$ would be the spectral resolution of an operator associated to a_ρ). In terms of kernels, (3.23) means that :

$$\theta(Z)\,\theta(w)\,e(\rho^2\lambda;z,w) = \rho^{2n+2}|\det d\kappa(z)|^{\frac{1}{2}}\,|\det d\kappa(w)|^{\frac{1}{2}}\,e_\rho(\lambda;\, h_\rho o\kappa(z),\, h_\rho o\kappa(w)). \tag{3.24}$$

From the convergence "$a_\rho \to a_\infty$" we will deduce :

PROPOSITION 3.3 : for any $\delta > 0$ and $\lambda > 0$, we have :

$$\lim_{\rho\to+\infty} \frac{1}{\rho} \int_{\substack{|z'|\leq\delta \\ |z_n|\leq\rho\delta}} |\operatorname{tr} e_\rho(\lambda;z,z) - \operatorname{tr} e_\infty(\lambda;z,z)|\,dxdy = 0$$

PROOF THAT PROPOSITION 3.3 IMPLIES PROPOSITION 2.1.

For each $\delta > 0$ there is $\delta' > 0$ such that if $|z-z^\circ| < \delta/\rho$ then $w = h_\rho o\kappa(z)$ satisfies $|w'| \leq \delta'$ and $|w_n| \leq \rho\delta'$. Therefore with lemma 3.2 and (3.24), proposition 3.3 implies that :

$$\lim_{\rho\to+\infty} \frac{1}{\rho} \int_{\rho|z-z^\circ|<\delta} |\operatorname{tr} e(\rho^2;z,z) - \tilde{g}(\rho^2;z,z^\circ)|\,dxdy = 0$$

with :

$$\tilde{g}(\rho^2;z,z^\circ) = \rho^{2n+2}|\det d\kappa(z)|\int_0^\infty e^{-2\rho^2\tau(\operatorname{Im}_n - \varphi(z',\operatorname{Re} z_n))} 2\tau\, c(\tau,z^\circ)d\tau.$$

Because $\operatorname{Im} z_n - \Psi(z', \operatorname{Re} z_n) = \chi(z)\, d(z)$ where $d(z)$ denotes the distance of z to $\partial\Omega$ and χ is continuous and satisfies $\chi(z^\circ) = 1$, and because $|\det d\kappa(z^\circ)| = 1$, it is easy to see that :

$$\frac{1}{\rho} \int_{\rho|z-z^\circ|\leqslant\delta} |\tilde{g}(\rho^2,z,z^\circ) - g(\rho^2,z,z^\circ)|\, dxdy \to 0$$

where $g(\rho^2,z,z^\circ)$ is given in proposition 2.1.

Similarily proposition 2.2 is an immediate consequence of (3.22) and of the following result :

PROPOSITION 3.4 : there are δ, c and ρ_o such that for $\rho \geqslant \rho_o$:

$$\int_{\substack{|z'|\leqslant\delta \\ |z_n|\leqslant\rho\delta}} \operatorname{tr} e_\rho(1;z,z)\, dxdy \leqslant \rho\, C$$

Furthermore δ, C and ρ_o can be chosen independantly of z° if z° remains in a compact subset of $\partial\Omega$.

Now the remainder of the paper is devoted to the proof of propositions 3.3 and 3.4.

4.- A FEW LEMMAS.

Before going on we must prove a few preliminary results. Following [3] we introduce the norm :

$$(4.1) \qquad |||u|||_\sigma = \left(\int (1+|\zeta'|^2+\tau^2)^\sigma\, |\tilde{u}(\zeta',\tau,s)|^2 d\xi' d\tau ds\right)^{\frac{1}{2}}$$

where $\tilde{u}$ denotes the partial Fourier transform relative to (z',t), and where the integral is taken for $(\zeta',\tau,s) \in \mathbb{R}^{2n-2} \times \mathbb{R} \times \mathbb{R}_+$.

For further use we remark that if α is a C^1 function with compact support, then for every $|\sigma| \leqslant 1$ we have :

$$(4.2) \qquad |||\alpha u|||_{\sigma} \leqslant C\ |||u|||_{\sigma}$$

where C depends only on the dimension n and the norm of α in C^1.

For $\rho \geqslant 1$ we introduce the space W_ρ of the $u \in L^2(\mathbb{C}^n_+)$ such that :

$$(4.3) \qquad ||u||^2_{W_\rho} = |||u|||^2_{\frac{1}{2}} + \rho^2\ |||\frac{\partial u}{\partial \bar{z}_n}|||^2_{-\frac{1}{2}} < +\infty.$$

Of course only the norm depends on ρ, and we also introduce the space W_∞ of the $u \in W_1$ such that $\frac{\partial u}{\partial \bar{z}_n} = 0$; W_∞ is equipped with the norm $|||.|||_{1/2}$. As usual now we note $\mathbf{W}$ the space of the $u = (u_J)_{|J|=q}$ such that $u_J \in W_\rho$ for all $|J| = q$.

At last for $\delta > 0$ and $\rho \geqslant 1$ we note $\omega_{\rho,\delta}$ the set $\{z \in \bar{\mathbb{C}}^n_+ \ / \ |z'| \leqslant \delta$, $|z_n| \leqslant \rho\delta\}$.

<u>LEMMA 4.1</u> : for any $\delta > 0$ there is C such that for every $\rho \geqslant 1$, every $u \in \mathcal{V}(\Omega_\rho\ ;\ q)$ with support in $\omega_{\rho,\delta}$:

$$(4.4) \qquad ||u||^2_{W_\rho} \leqslant C\ \{a_\rho(u,u) + ||u||^2_{\mathcal{L}^2(\Omega_\rho\ ;\ q)}\}.$$

<u>PROOF</u> : from theorem 2.4.5 of Folland-Kohn [3] we know that (4.4) holds for $\rho = 1$ if $\mathcal{O}$ is small enough (although they assume Ω to be smooth their proof is valid without changing a word if $\partial\Omega$ is only of class C^2). Using the dilations h_ρ we immediatly get the estimates :

$$(4.5) \qquad |||\frac{\partial u}{\partial t}|||^2_{-1/2} + |||\frac{\partial u}{\partial s}|||^2_{-1/2} \leqslant C\ \{a_\rho(u,u) + \frac{1}{\rho^2}\ ||u||^2_{\mathcal{L}^2(\Omega_\rho\ ;\ q)}\}$$

for all $u \in \mathcal{V}(\Omega_\rho\ ;\ q)$ (without any condition on the support of u).

$\delta > 0$ being given, let $\chi \in C^\infty_0(\mathbb{C}^n)$ such that $\chi(z) = 1$ for $z \in \omega_{1,\delta}$ and $\chi(z) = 0$ for $|z| \geqslant 4\delta$; for $\rho \geqslant 1$ we set $\chi_\rho(z',z_n) = \chi(z',z_n/\rho)$ so that for $u \in \mathcal{V}(\Omega_\rho\ ;\ q)$ supported in $\omega_{\rho,\delta}$ we have $u = \chi_\rho\ u$; then from lemma 3.1 and (4.2) we deduce that :

$$|||\beta_{j,\rho} \frac{\partial u}{\partial t}|||_{-1/2} \leq C \, ||| \frac{\partial u}{\partial t} |||_{-1/2}$$

$$|||\gamma_{n,\rho} \frac{\partial u}{\partial s}|||_{-1/2} \leq C \, ||| \frac{\partial u}{\partial s} |||_{-1/2}$$

Therefore with (4.5), (3.11) and the main estimate (3.19) we see that :

$$\sum_{j=1}^{n-1} ||| \frac{\partial u}{\partial \bar{z}_j} |||^2_{-1/2} + \rho^2 \, ||| \frac{\partial u}{\partial \bar{z}_n} |||^2_{-1/2} \leq C \, \{a_\rho(u,u) + \frac{1}{\rho^2} \, ||u||^2_{2(\Omega_\rho;q)}$$

and with (4.5) the estimate (4.4) follows.

In order to study with more details the "convergence $W_\rho \to W_\infty$" we introduce the space H of the $u \in L^2(\mathbb{C}^n_+)$ such that $\frac{\partial u}{\partial \bar{z}_n} = 0$. It is easy to see that $u \in H$ has a trace, noted γu, on the hyperplane $s = 0$ and more precisely we have :

LEMMA 4.2. : the operator

(4.6) $$(R\,\phi)\ (z',z_n) = \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{iZ_n\tau}\, \Phi(z',\tau)\ \sqrt{2\tau}\ d\tau$$

is unitary from $L^2(\mathbb{C}^{n-1} \times]0,\infty[)$ onto H and

$$\Phi(z',\tau) = \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{2\tau}} \ \mathcal{F}_t(\gamma R\phi)\ (z',\tau)$$

where $\mathcal{F}_t$ denotes the partial Fourier transform in the t-variable.

The proof is immediate and left to the reader.

The adjoint operator R^* acting from $L^2(\mathbb{C}^n_+)$ onto $L^2(\mathbb{C}^{n-1} \times]0,\infty[)$ is given by :

(4.7) $$(R^* u)\ (z',\tau) = \frac{\sqrt{2\tau}}{\sqrt{2\pi}} \int_0^\infty (\mathcal{F}_t u)\ (z',\tau,s)\, e^{-s\tau} ds.$$

and the ortogonal projector P form $L^2(\mathbb{C}^n_+)$ on H is $P = R\,R^*$.

Now we can state :

LEMMA 4.3. There is a constant C such that for all $\rho \geqslant 1$ and $u \in W_\rho$:

i) $\quad ||(1-P)u||W_\rho \leqslant C_o \, ||u||W_\rho$

ii) $\quad |||\frac{\partial}{\partial s}(1-P)u|||_{-1/2} + |||\frac{\partial}{\partial t}(1-P)u|||_{-1/2} \leqslant C_o \, |||\frac{\partial u}{\partial \bar{z}_n}|||_{-1/2}$

iii) $\quad \int \frac{|\tau|}{(1+|\zeta'|^2+\tau^2)^{1/2}} \, |\gamma(1-P)u|^2 \, d\zeta' \, d\tau \leqslant C_o \, |||\frac{\partial u}{\partial \bar{z}_n}|||_{-1/2}$

REMARK. From definitions 4.1 and 4.3 we see that for $u \in W_\rho$:

$$|||u|||_{1/2} + |||\frac{\partial u}{\partial s}|||_{-1/2} \leqslant 2 \, |||u|||W_\rho$$

and it follows that u has a trace $\gamma u \in L^2(\mathbb{C}^{n-1} \times \mathbb{R})$ on $\rho = 0$, so $\gamma(1-P)u$ is also well defined.

PROOF : for simplicity we set $f = 2i \widetilde{\frac{\partial u}{\partial \bar{z}_n}}$ and $v = \widetilde{(1-P)u}$. We have $\frac{\partial v}{\partial s} + \tau v = -f$ and it is easy to check that :

$$(4.8) \qquad v(\zeta',\tau,s) = \int_s^\infty f(\zeta',\tau,\sigma)e^{-(s-\sigma)\tau} \, d\sigma = \tilde{u}(\zeta',\tau,s) \quad \text{where } \tau < 0 \text{ and}$$

$$(4.9) \qquad v(\zeta',\tau,s) = e^{-s\tau}\int_o^\infty f(\zeta',\tau,\sigma)e^{-\sigma\tau} \, d\sigma - \int_o^s f(\zeta',\tau,\sigma)e^{-(s-\sigma)\tau} d\sigma$$

$$= \tilde{u}(\zeta',\tau,s) - 2\tau \, e^{-s\tau} \int_o^\infty \tilde{u}(\zeta',\tau,\sigma)e^{-\sigma\tau} \, d\sigma$$

when $\tau > 0$. From (4.8) and (4.9) we deduce the following estimates :

$$(4.10) \qquad \begin{cases} \int_o^\infty |v(\zeta',\tau,s)|^2 ds \leqslant \int_o^\infty |u(\zeta',\tau,s)|^2 ds \\ \tau^2 \int_o^\infty |v(\zeta',\tau,s)|^2 ds \leqslant 4 \int_o^\infty |f(\zeta',\tau,s)|^2 ds \\ |\tau| \, |v(\zeta',\tau,o)|^2 \leqslant \int_o^\infty |f(\zeta',\tau,s)|^2 ds \end{cases}$$

from which lemma 4.3 follows immediatly.

LEMMA 4.4 : for $u \in W_\rho$ and $v \in W_\infty$ we have :

$$|(\gamma\, v, \gamma\, (1-P\ )u)_{L^2(\mathbb{C}^{n-1}\times\mathbb{R})}| \leq \frac{C_o}{\rho}\, ||u||_{W_\rho}\ ||v||_{W_\infty}$$

Proof : with the point iii) of the preceding lemma, it is sufficient to notice, making use of lemma 4.2, that for $v \in W_\infty$:

$$\int \frac{(1+|\zeta'|^2+|\tau|^2)^{1/2}}{2\tau}\, |\widetilde{\gamma\ v}|^2 d\zeta' d\tau = |||v|||^2_{1/2} = ||v||^2_{W_\infty}$$

LEMMA 4.5 : for each $f \in L^2(\mathbb{C}^n_+)$ we have :

$$\lim_{\rho\to+\infty} \left\{ \operatorname*{Sup}_{||u||_{W_\rho}\leq 1} |(f,(1-P\ )u)_{L^2(\mathbb{C}^n_+)}| \right\} = 0.$$

Proof : it is sufficient to prove the convergence when f is in a dense subspace of $L^2(\mathbb{C}^n_+)$, and it follows from lemma 4.3, ii) if :

$$\int \frac{(1+|\zeta'|^2+|\tau|^2)^{1/2}}{|\tau|^2}\, |\hat{f}(\zeta',\tau,s)|^2 d\zeta' d\tau ds \quad < +\infty .$$

In section 7 we shall need another kind of information about the W_ρ's, namely estimates of the n-diameters of their unit ball. Let us recall [7] that if B is a bounded subset of an Hilbert space H, then the ν-diameter of B in H is :

$$d_\nu(B,H) = \operatorname*{Inf}_{\dim G=\nu}\ \operatorname*{Max}_{u\in B}\ \operatorname*{Inf}_{v\in G} ||u-v||_H$$

These estimates will be usefull in order to give bounds to the Kernels $e_\rho(\lambda)$ (see Proposition 3.4) : indeed let us recall that an operator T acting from an Hilbert space H_1 into another one H, is said to be of class $\mathcal{C}_p(H_1,H)$ (see [1]) if :

$$(4.11) \qquad ||T||^p_{\mathcal{C}_p(H_1,H)} = \sum_{\nu=0}^{\infty} \{d_\nu(T\underline{H}_1,H)\}^{\,p} \quad < +\infty.$$

Here $\underline{H}_1$ denotes the unit ball of H_1, and this notation will be systematically used in the sequel. We also recall that if $H_1 = H = L^2(\mathbb{C}^n_+)$, any $T \in \mathcal{C}_1(H,H)$ is associated to a Kernel $T(z,w)$, defined for almost every z and almost every w, and :

$$(4.12) \qquad \int_{\mathbb{C}^n_+} |T(z,z)|\, dxdy \leqslant \|T\|_{\mathcal{C}_1(H,H)}$$

Before beginning we also remark that $d_\nu(\underline{H}_1,H) < \delta$ is equivalent to the existence of a $\delta' < \delta$ and of a subspace $G \subset H_1$ of codimension less or equal to ν such that $\|u\|_H \leqslant \delta'\|u\|_{H_1}$ for all $u \in G$.

Now we state :

LEMMA 4.6. let $\chi \in C_0^\infty(\mathbb{C}^n_+)$ and let $\chi_\rho(z',z_n) = \chi(z',\frac{z_n}{\rho})$. There is a constant C_1 such that for all $\rho \geqslant 1$ and all $\nu \in N$:

$$d_\nu(\chi_\rho \underline{W}_\rho \ , \ L^2(\mathbb{C}^n_+)) \leqslant C_1(1 + \frac{\nu}{\rho})^{-1/4n}$$

Proof : for $u \in W_\rho$ we extend $v = (1-P_0)u$ for $s < 0$ by setting $v(z',t,s) = v(z',t,-s)$. From lemma 4.3 we deduce that :

$$(4.13) \qquad \int (1+|\zeta'|^2+\rho^2\tau^2+\rho^2\sigma^2)^{1/2}|\hat{v}(\zeta',\tau,\sigma)|^2 d\zeta' d\tau d\sigma \leqslant C_0 \ \|u\|^2_{W_\rho}$$

where $\hat{v}$ denotes the Fourier transform of v in all the variable. Let us call X_ρ the space of the $u \in L^2(\mathbb{C}^n)$ such that the integral in the left hand side of (4.13) is finite. Using the change of variables $(z',z_n) \to (z',z_n/\rho)$ we see that :

$$d_\nu(\chi_\rho \underline{X}_\rho, L^2(\mathbb{C}^n)) = d_\nu(\chi_1 \ \underline{X}_1, L^2(\mathbb{C}^n) \leqslant C(1 + \nu)^{-1/4n}$$

the last inequality being a consequence of the results of [2] and of the fact that $X_1 = H^{1/2}(\mathbb{C}^n)$. It follows that :

$$(4.14) \qquad d_\nu(\chi_\rho(1-P\ )\underline{W}_\rho, L^2(\mathbb{C}^n_+)) \leqslant C_0C(1 + \nu)^{-1/4n}$$

It remains to study the ν-diameters of $\underline{W_\infty}$; denoting by $\mathcal{F}_z$, the Fourier transform with respect to the z' variables, we introduce for $\lambda > 0$ the operator G_λ in $L^2(\mathbb{C}^{n-1} \times]0,\infty[)$ given by :

$$(\mathcal{F}_{z'}G_\lambda\phi)\ (\xi',\tau) = (\mathcal{F}_{z'}\phi)(\xi',\tau) \quad \text{when } 1 + |\zeta'|^2 + \tau^2 \leqslant \lambda^2$$

$$= 0 \quad \text{when } 1 + |\zeta'|^2 + \tau^2 > \lambda^2$$

The operator $F_\lambda = R\, G_\lambda\, R^*$ is an orthogonal projector in $L^2(\mathbb{C}^n_+)$; its kernel is C^∞ and with (4.6) (4.7), we get that :

$$F_\lambda(z,z) = (2\pi)^{-2n+1} \int_{\substack{1+|\zeta'|^2+\tau^2 \leqslant \lambda^2 \\ \tau>0}} e^{-2\tau \operatorname{Re} z_n}\, 2\tau\, d\xi'\, d\tau$$

from which it follows that :

(4.15) $$\int |\chi_\rho(z)|^2\, F_\lambda(z,z)\, dxdy \leqslant C'\rho\ \lambda^{2n-1}$$

Now for $u \in W_\infty$ we have :

(4.16) $$||u - F_\lambda u||_{L^2(\mathbb{C}^n_+)} \leqslant \frac{1}{\sqrt{\lambda}}\, |||u|||_{W_\infty}$$

Because the integral in the left hand side of (4.15) is the Hilbert-Schmidt norm of $\chi_\rho F_\lambda$ we have :

(4.17) $$\nu d^2_\nu(\chi_\rho F_\lambda \underline{W_\infty}, L^2(\mathbb{C}^n_+)) \leqslant C'\rho\ \lambda^{2n-1}$$

and choosing $\lambda = (\frac{\nu}{C'\rho})^{1/2n}$ we get from (4.16) and (4.17) :

$$d_\nu(\chi_\rho \underline{W_\infty},\ L^2(\mathbb{C}^n_+)) \leqslant 2\ (\frac{\nu}{C'\rho})^{-1/4n}$$

Because $||.||_{L^2} \leqslant ||.||_{W_\infty}$ we always have $d_\nu(\chi_\rho \underline{W_\infty},\ L^2(\mathbb{C}^n_+)) \leqslant 1$ and because of lemma 4.3 we have $\chi_\rho\ P\ \underline{W_\rho} \subset (C_0+1)\underline{W_\infty}$; finally we have just proved the estimate :

(4.18) $$d_\nu(\chi_\rho P\ \underline{W_\rho},\ L^2(\mathbb{C}^n_+)) \leqslant C''(1 + \frac{\nu}{\rho})^{-1/4n}$$

Lemma 4.6 now follows from (4.14), (4.18) and the inequality :

$$d_{\nu'+\nu''}(\chi_\rho \underline{W_\rho}, L^2(\mathbb{C}^n_+)) \leq d_{\nu'}(\chi_\rho (1-P)\underline{W_\rho}, L^2(\mathbb{C}^n_+)) + d_{\nu''}(\chi_\rho\, P\, \underline{W_\rho}, L^2(\mathbb{C}^n_+))$$

<u>LEMMA 4.7</u> : let χ and χ_ρ as in lemma 5.6 and let $\zeta \in C^\infty(\mathbb{R})$ be such that $\zeta(s) = 0$ for $s \leq 1$ and $\zeta(s) = 1$ for $s \geq 2$. We set $\zeta_T(s) = \zeta(s_{/T})$. Then there is C_2 such that for all $\rho \geq 1$ and $\nu \in \mathbb{N}$:

$$d_\nu(\zeta_T \chi_\rho \underline{W}_\rho, L^2(\mathbb{C}^n_+)) \leq C_2 (1 + \frac{T\nu}{\rho})^{-1/4n}$$

<u>LEMMA 4.8</u> : let $\psi' \in C^\infty_0(\mathbb{C}^{n-1})$ and let $\psi(z', z_n) = \frac{\psi'(z')}{z_n + i}$. There is C_3 such that for all $\nu \in \mathbb{N}$:

$$d_\nu(\psi \underline{W}_1, L^2(\mathbb{C}^n_+)) \leq C_3 (1 + \nu)^{-1/12n}$$

The proofs are quite similar to the proof of lemma 5.6 and we don't give the details. Let us just mention that (4.15) is to be replaced by :

$$\int |\zeta_T \chi_\rho (z)|^2 \ F_\lambda(z,z) dxdy \leq C \frac{\rho}{T} \ \lambda^{2n-2}$$

and

$$\int |\psi(z)|^2 \ F_\lambda(z,z) dxdy \leq C \ \lambda^{2n-1}$$

Because $\zeta_T\ \chi_\rho = 0$ if $\rho \leq C_4 T$, it is an immediate consequence of (4.14) that :

$$d_\nu(\zeta_T \chi_\rho (1-P\ ) \underline{W}_\rho, L^2(\mathbb{C}^n_+)) \leq C_5 (1 + \frac{T\nu}{\rho})^{-1/4n}$$

and for lemma 4.8, (4.14) is to be replaced by :

$$d_\nu(\psi (1-P\ ) \underline{W}_1, L^2(\mathbb{C}^n_+)) \leq C (1 + \nu)^{-1/12n}$$

which is a consequence of :

$$d_\nu(\frac{\psi'}{|z_n| + 1} \underline{H^{1/2}(\mathbb{C}^n)}, L^2(C^n)) \leq C (1 + \nu)^{-1/12n}$$

We end this section with the following remark : it is clear that the estimates of lemmas 4.6, 4.7, and 4.8 are still valid if we replace W_ρ by $\mathcal{U}_\rho$ and $L^2(\mathbb{C}^n_+)$ by $\mathcal{L}^2(\mathbb{C}^n_+; q)$.

5.- THE LIMIT PROBLEM.

In this section we study with more details the operator $\square_\infty$ introduced in section 3. We shall make an essential use of lemma 4.2 in order to reduce the problem to the boundary.

We note $\mathcal{J}_q$ the set of the ordered sequences J of length $|J| = q$ such that $n \notin \{J\}$, and $\mathcal{H}_o$ the space of the $\phi = (\phi_J)_{J \in \mathcal{J}_q}$ with $\phi_J \in L^2(\mathbb{C}^{n-1} \times \mathbb{R}_+)$. For $\phi \in \mathcal{H}_o$ we define $u = \mathcal{R}\phi \in \mathcal{L}^2(\mathbb{C}^n_+, q)$ by setting $u_J = 0$ if $n \in \{J\}$ and $u_J = R\,\phi_J$ if $n \notin \{J\}$.

From (3.20) (3.21) and lemma 4.2 we see that $\mathcal{D}(\mathbb{C}^n_+ ; q)$ is the image under $\mathcal{R}$ of the space $\mathcal{D}_o$ of the $\phi \in \mathcal{H}_o$ with compact support in z' and satisfying $\tau^\beta \partial^\alpha_{z'} \phi_J \in L^2(\mathbb{C}^{n-1} \times \mathbb{R}_+)$ for $|\alpha|+|\beta| \leqslant 2$. Because $\mathcal{D}_o$ is dense in $\mathcal{H}_o$, it follows that the closure $\mathcal{H}$ of $\mathcal{D}(\mathbb{C}^n_+ ; q)$ in $\mathcal{L}^2(\mathbb{C}^n_+ ; q)$ is simply $\mathcal{R}\,\mathcal{H}_o$.

Making use once more of lemma 4.2 we get that for $u = \mathcal{R}\,\phi \in \mathcal{D}(\mathbb{C}^n_+ ; q)$ we have :

(5.1) $$a_\infty(u,u) = b(\phi,\phi)$$

with

$$b(\phi,\phi) = \sum_{J \in \mathcal{J}_q} \left\{ \sum_{j=1}^{n-1} 2\|B_j\,\phi_J\|^2_{L^2(\mathbb{C}^{n-1} \times \mathbb{R}_+)} + 2\,\mu_J\|\sqrt{\tau}\,\phi_J\|^2_{L^2(\mathbb{C}^{n-1} \times \mathbb{R}_+)} \right\}$$

and

$$B_j = \frac{\partial}{\partial \bar{z}_j} + \frac{1}{2}\,\mu_j\,\tau\,z_j \qquad \text{for } j = 1,\dots,n-1.$$

The form b is non negative on $\mathcal{D}_o$; let $\mathcal{B}$ be the operator in $\mathcal{H}_o$ associated to b, and let $F(\lambda)$ be its spectral resolution. We clearly have :

(5.2) $$\tilde{E}_\infty(\lambda) = \mathcal{R}\,F(\lambda)\,\mathcal{R}^*.$$

PROOF OF LEMMA 3.2.

On $\mathbb{C}^{n-1}$ we consider for $J \in \mathcal{J}_q$ the operator :

$$A_J = 2 \sum_{j=1}^{n-1} \left(-\frac{\partial}{\partial z_j} + \frac{1}{2}\,\mu_j\,z_j \right) \left(\frac{\partial}{\partial \bar{z}_j} + \frac{1}{2}\,\mu_j\,z_j \right) + 2\,\mu_J$$

and in $\left\{L^2(\mathbb{C}^{n-1})\right\}^{\binom{n-1}{q}}$ the diagonal operator $\mathcal{A} = (A_J)_{J\in \mathcal{J}_q}$. A_J is elliptic, self adjoint and bounded from below by $2\,\widetilde{\mu}_J = 2\,\mu_J + 2 \sum\limits_{j<n} \mathrm{Max}(0,-\mu_j)$; therefore $\mathcal{A}$ is bounded from below by $2\,\widetilde{\mu} = \mathrm{Inf}\ 2\,\widetilde{\mu}_J$, and due to condition $Z(q)$, $\widetilde{\mu} > 0$. It follows that the sepctral resolution of $\mathcal{A}$ has a C^∞ kernel $F_0(\lambda\ ;\ z',\ w')$ on $\mathbb{C}^{n-1} \times \mathbb{C}^{n-1}$; that $F_0(\lambda\ ;\ z',\ w') = 0$ for $\lambda < \widetilde{\mu}$; and that $F_0(\lambda\ ;\ z',\ w') = O(\lambda^{\,n-1})$ uniformly for z' and w' in a compact set.

With (5.2), formulas (4.6) (4.7), and the use of the homogeneity we see that $\widetilde{E}_\infty(\lambda)$ has a C^∞ kernel given by :

$$(5.3)\qquad \widetilde{E}_\infty(\lambda\ ;\ z,w) = \frac{1}{2\pi}\int_0^\infty e^{i(z_n-\overline{w}_n)\tau}\, 2\,\tau^n\, F_0(\lambda/\tau,\ \sqrt{\tau}z',\ \sqrt{\tau}w')\,d\tau$$

(Indeed the integral is over $]0,\lambda/\widetilde{\mu}[$, and absolutely convergent at $\tau = 0$).

Now we remark that, for each $w' \in \mathbb{C}^{n-1}$, $\mathcal{A}$ commutes with the unitary operator :

$$u(z') \longrightarrow v(z') = u(z'-w')\ e^{\mathrm{Im}\sum\limits_{j=1}^{n-1} \mu_j z_j \overline{w}_j}$$

and so does its spectral resolution ; therefore we have $F_0(\lambda;z',z') = F_0(\lambda;0,0)$ and with (5.3) we get the formula :

$$(5.4)\qquad \mathrm{tr}\ \widetilde{E}_\infty(\lambda;z,z) = \frac{1}{2\pi}\int_0^\infty e^{i2\tau \mathrm{Im} z_n}\, 2\tau^n \mathrm{tr}F_0(\lambda/\tau;0,0)\,d\tau$$

The proof of lemma 3.2 will be complete il we show that :

$$c = \frac{1}{2\pi}\int_0^\infty \tau^{n-1}\ \mathrm{tr}\ F_0(\lambda/\tau;0,0)\,d\tau$$

is exactly equal to the value at z° of the function defined in (1.5). First we note that :

$$\int_0^\infty \tau^{n-1}\ F_0(\lambda/\tau)\,d\tau = \frac{1}{n!}\int_0^\infty e^{-\tau\mathcal{A}}\,\tau^{n-1}\ d\tau$$

and setting $G_J(\tau) = e^{-\tau(A_J)}$ we get :

$$(5.5) \qquad c = \sum_{J\in\mathcal{J}_q} \frac{1}{2\pi}\frac{1}{n!}\int^{\infty} \tau^{n-1}\, G_J(\tau;0,0)\,d\tau .$$

The Kernels $G_J(\tau,0,0)$ have been worked out by many authors ([8], [13] ...) and it is easy to see that :

$$G_J(\tau;z',0) = (2\pi)^{-2(n-1)} \int e^{i(x'.\xi'+y'.\eta'+\Phi_J(\tau,z',\zeta'))}\, d\xi' d\eta'$$

with $z' = x'+iy'$, $\zeta' = \xi'+i\eta'$ and

$$\Phi_J(\tau;z',\zeta') = \frac{1}{2}\sum_{j=1}^{n-1} |\mu_j|\,|z_j|^2 - \sum_{j=1}^{n-1} C_j(\tau)\,|\zeta_j|^2 - 2\tilde{\mu}_J\tau$$

where $\tilde{\mu}_J = \mu_J + \sum_{j=o}^{n-1} \mathrm{Max}(0,-\mu_j)$ and $C_j(\tau) = \dfrac{1-C^{-2|\mu_j|\tau}}{4|\mu_j|}$.

It follows that :

$$G_J(\tau;0,0) = (2\pi)^{-(n-1)}\, e^{-2\tilde{\mu}_J\tau} \prod_{j=1}^{n-1} \left(\frac{2|\mu_j|}{1-e^{-2|\mu_j|\tau}}\right)$$

Substituting in (5.5) and changing 2τ into τ we get formula (1.7) and the proof of lemma 3.2 is complete.

We end this section with the following lemmas we shall need later on :

<u>LEMMA 5.1.</u> $\acute{\square}_\infty$ has no positive eigen values.

This is a consequence of (5.4) or, more directly, of the quasihomogeneity of $\square_\infty$.

LEMMA 5.2. $\mathcal{D}(\mathbb{C}^n_+ ;q)$ is contained and dense in the domain of $\square_\infty$; furthermore for $u \in \mathcal{D}(\mathbb{C}^n_+ ;q)$ and $v \in \mathcal{U}_1^\Gamma$ we have :

(5.6) $$\sum_{j,J} 2(Z_j^* Z_j u_J, v_J)_{L^2(\mathbb{C}^n_+)} + \sum_J \mu_J(\gamma u_J, \gamma v_J)_{L^2(\mathbb{C}^{n-1}\times\mathbb{R})}$$

$$= (\square_\infty u, \mathcal{P}v)_{L^2(\mathbb{C}^n_+)} + \sum_J \mu_J(\gamma u_J, \gamma(1-P)v_J)_{L^2(\mathbb{C}^{n-1}\times\mathbb{R})}$$

Proof : Indeed for u and v in $\mathcal{D}(\mathbb{C}^n_+ ; q)$ we have :

(5.7) $$a_\infty(u,v) = \sum_{j,J} 2(Z_j^* Z_j u_J, v_J) + \sum_J \mu_J(\gamma u_J, \gamma v_J) = (f,v)_{\mathcal{L}^2(\mathbb{C}^n_+;q)}$$

with $f_J = \sum_{j=1}^n 2Z_j^* Z_j u_J + \mu_J\ R(2\tau\ R^* u_J) \in L^2(\mathbb{C}^n_+)$.

It follows that $u \in D(\square_\infty)$ with $\square_\infty\ u = f$. Furthermore the second equality of (5.7) can be extended to $v \in \mathcal{H}$; splitting any $v \in \mathcal{U}_1^\Gamma$ into $\mathcal{P}v + (1-\mathcal{P})v$ and remarking that $Z_j^* Z_j u_J \in H$ if $u \in \mathcal{D}(\mathbb{C}^n_+ ; q)$ we immediatly get (5.6).

6. - STRONG CONVERGENCE.

The first step in the proof of proposition 3.3 is to get the strong convergence $E_\rho(\lambda) \longrightarrow \widetilde{E}_\infty(\lambda)$. Unfortunately, for technical reasons which will be clear in the next section, we also have to introduce another parameter and to study the translated operators of $E_\rho(\lambda)$ in the t-direction.

First, denoting by $R(\mu)$ the resolvant operator of $\square$ for $\mu \in \mathbb{C}\setminus[0,\infty]$, we set with the same function θ as in (3.23) :

(6.1) $$R_\rho(\mu) = \mathcal{K}_\rho\ \theta\ \rho^2 R(\rho^2\mu)\theta\mathcal{K}_\rho^{-1}$$

and we extend $R_\rho(\mu)$ as an operator in $\mathcal{L}^2(\mathbb{C}^n_+ ; q)$ by setting that $R_\rho(\mu)f$ is the extension by 0 oustide of Ω_ρ of $R_\rho(\mu)(f_{|\Omega_\rho})$

Then, as announced, we introduce for $\tau \in \mathbb{R}$ the translation operator :

$$(T_{\rho,\tau}f)\ (z',t,s) = f(z',t-\rho\tau,s)$$

and we set

(6.2) $$\tilde{R}_{\rho,\tau}(\mu) = T_{\rho,\tau}\ \tilde{R}_{\rho}(\mu)\ T^{-1}_{\rho,\tau}$$

On the other hand we note $R_\infty(\mu)$ the resolvant of $\Box_\infty$ in $\mathcal{H}$ and we extend it to $\mathcal{L}^2(\mathbb{C}^n_+;q)$ by setting $\tilde{R}_\infty(\mu) = R_\infty(\mu)\mathcal{P}$.

PROPOSITION 6.1.

Let Λ be a compact set in $\mathbb{C}\setminus[0,\infty]$, let δ be a positive number and let $f \in \mathcal{L}^2(\mathbb{C}^n_+;q)$. Then $\tilde{R}_{\rho,\tau}(\mu)f$ converges to $\tilde{R}_\infty(\mu)f$ as ρ goes to $+\infty$, uniformly for $\mu \in \Lambda$ and $|\tau| \leqslant \delta$.

Before giving the proof of this proposition we shall make a few remarks. Let us introduce the operators $M_{j,\rho,\tau}$ and the forms $a_{\rho,\tau}$ deduced by translation from the $M_{j,\rho}$'s and a_ρ's. They are defined on $\Omega_{\rho,\tau} = \{(z',t,s) \in \mathbb{C}^n_+/(z',t-\rho\tau,s) \in \Omega_\rho\}$. On the space $\mathcal{V}_{\rho,\tau} = T_{\rho,\tau}\ \mathcal{V}_\rho$ we introduce the norm :

$$||u||_{\rho,\tau} = \{\sum_{j=1}^{n} ||M_{j,\rho,\tau}u||^2_{\mathcal{L}^2} + ||u||^2_{\mathcal{L}^2}\}^{\frac{1}{2}}$$

First we note :

LEMMA 6.2. Let Λ be a compact set in $\mathbb{C}\setminus[0,\infty[$. Then there is $C_0 > 0$ such that for all $f \in \mathcal{L}^2(\mathbb{C}^n_+;q)$ all $\rho \geqslant 1$, all τ and all $\mu \in \Lambda$.

$$||\tilde{R}_{\rho,\tau}(\mu)f||_{\rho,\tau} \leqslant C_0||f||_{2}$$

$$||\tilde{R}_\infty(\mu)f||_{\mathcal{L}^2} \leqslant C_0||f||_{\mathcal{L}^2}$$

Proof : The second inequality is quite trival, while the first is a consequence of the following ones :

$$||\rho^2\ R(\rho^2\mu)g||_{L^2(\Omega;q)} \leqslant ||g||_{L^2(\Omega;q)}$$

$$||\rho\ R(\rho^2\mu)g||_{V(\Omega;q)} \leqslant ||g||_{L^2(\Omega,q)}.$$

Then we remark that $\widetilde{R}_{\rho,\tau}(\mu)$ looks like a resolvant for $a_{\rho,\tau}$ because we have :

<u>LEMMA 6.3</u>. Let Λ be a compact set in $\mathbb{C}\setminus[0,\infty[$ and let $\delta > 0$. There is C_1 such that for $f \in \mathcal{L}^2(\mathbb{C}^n_+;q)$, $w \in \mathcal{V}_{\rho,\tau}$ $|\tau| \leqslant \delta$ and $\rho \geqslant 1$ we have :

$$|(a_{\rho,\tau}-\mu)\ (\widetilde{R}_{\rho,\tau}(\mu)f,w) - (\theta_{\rho,\tau}f,\theta_{\rho,\tau}w)_{\mathcal{L}^2}| \leqslant C_1\ \rho^{-1}\ ||f||_{\mathcal{L}^2}\ ||w||_{\rho,\tau}$$

with $\theta_{\rho,\tau} = T_{\rho,\tau}\ (\theta o(h_\rho o\kappa)^{-1})$.

<u>Proof</u> : Let $\theta' \in C_0^\infty(\mathcal{O})$ be 1 on the support of θ. We set $g = \theta\cdot\kappa_\rho^{-1}\ T_{\rho,\tau}^{-1}f$, $u = \rho^2\ R(\rho^2\mu)g$ and $v = \theta'\ \kappa_\rho^{-1}\ T_{\rho,\tau}^{-1}w$. By definition we have :

$$(a_{\rho,\tau}-\mu)\ (\widetilde{R}_{\rho,\tau}(\mu)f,w) = \frac{1}{\rho^2}\ (a-\rho^2\mu)\ (\theta u,v)$$

and $\frac{1}{\rho^2}\ (a-\rho^2\mu)\ (u,\theta v) = (g,\theta v) = (\theta_{\rho,\tau}f,\ \theta_{\rho,\tau}w)$

Clearly we also have :

$$|a(\theta u,v) - a(u,\theta v)| \leqslant C\{||u||_{L^2}\sum_{j=1}^n ||\frac{\partial v}{\partial\bar{z}_j}||_{L^2} + ||v||_{L^2}\sum_{j=1}^n ||\frac{\partial u}{\partial\bar{z}_j}||_{L^2} + ||u||_{L^2}\ ||v||_{L^2}\}$$

and the lemma follows if we note that :

$$||u||_{L^2} + \frac{1}{\rho}\sum_{j=1}^n ||\frac{\partial u}{\partial\bar{z}_j}|| \leqslant C'\ ||f||_{\mathcal{L}^2}$$

$$||v||_{L^2} + \frac{1}{\rho}\sum_{j=1}^n ||\frac{\partial v}{\partial\bar{z}_j}|| \leqslant C'\ ||w||_{\rho,\tau}$$

It must be noted that if w has its support contained in the set $\{z/|z'| \leqslant \delta,\ |z_n| \leqslant \rho\delta\}$ then, with the notations of the proof above, we have $\theta = 1$ on the support of v provided that $|\tau| \leqslant \delta$ and ρ is large enough, so that we have exactly :

(6.3) $\quad (a_{\rho,\tau}-\mu)\ (\tilde{R}_{\rho,\tau}(\mu)f,w) = (f,w)_{\mathcal{L}^2}$

An important step in the proof of proposition 6.1 is the following lemma.

LEMMA 6.4. Let $\chi \in C_0^\infty(\mathbb{C}^n)$ be equal to 1 on a neighborhood of 0. For $\varepsilon > 0$ and $\rho > 0$ we set $\chi_{\rho,\varepsilon}(z',z_n) = \chi(\varepsilon z', \rho^{-1}\varepsilon z_n)$. Let $u \in \mathcal{D}(\mathbb{C}^n_+;q)$. Then for fixed δ and ε (ε small enough) we have uniformly in $|\tau| \leqslant \delta$:

$$\| M^*_{j,\rho,\tau}\, M_{j,\rho,\tau}(\chi_{\varepsilon,\rho}u) - 2\, Z_j^*\, Z_j u \|_{\mathcal{L}^2} \longrightarrow 0$$

$$\| M_{n,\rho,\tau}\ (\chi_{\varepsilon,\rho}u)\|_{\mathcal{L}^2} \longrightarrow 0$$

Proof : From (3.4) (3.5) we see that :

(6.4) $\quad M_{j,\rho,\tau} = \sum_k a_{j,k,\rho,\tau}\, M'_{k,\rho,\tau}$

with coefficients $a_{j,k,\rho,\tau}$ bounded as well as their derivatives and converging to $\sqrt{2}\,\delta_{j,k}$ on compact subsets.

From formula (3.11) we see that the derivatives $M'_{j,\rho,\tau}(\chi_{\varepsilon,\rho})$ are bounded, even for $j = n$, and converge to 0 on compact supsets of supp u if ε is small enough. (Recall that $u \in \mathcal{D}(\mathbb{C}^n_+;q)$ is compactly supported in z').

Now lemma 6.4 follows easily from (3.11), lemma 3.1 and (6.4).

Proof of proposition 6.1.

In view of lemma 6.2 it is sufficient to make the proof for f in a dense subspace of $\mathcal{L}^2(\mathbb{C}^n_+;q)$ and we assume from now on (see lemma 5.2) that $u = \tilde{R}_\infty(\mu)f$ belongs to $\mathcal{D}(\mathbb{C}^n_+;q)$.

As in lemma 6.4 we introduce $\chi \in C_0^\infty(\mathbb{C}^n)$ which is 1 near 0, and for $\varepsilon > 0$, $\rho > 0$ we set $\chi_{\rho,\varepsilon}(z',z_n) = \chi(\varepsilon z', \rho^{-1}\varepsilon z_n)$. We note $v_{\rho,\tau} = \tilde{R}_{\rho,\tau}(u)f$ and $v_{\rho,\tau,\varepsilon} = \chi_{\rho,\varepsilon}\, v_{\rho,\varepsilon}$.

Because the derivatives $M_{j,\rho,\tau}\,\chi_{\varepsilon,\rho}$ are bounded by $C\,\varepsilon$ it follows from formula (3.17) that for $\rho \geqslant 1$, we have :

$$|a_{\rho,\tau}(v_{\rho,\tau,\varepsilon},w) - a_{\rho,\tau}(v_{\rho,\tau},\chi_{\rho,\varepsilon}w)| \leqslant C_2\varepsilon||v_{\rho,\tau}||_{\rho,\tau}||w||_{\rho,\tau}$$

and with (6.3) and lemma 6.2 we see that there is C_3, such that for each $\varepsilon < 1$, and $\delta > 0$ one can find ρ_1 such that for $\rho \geqslant \rho_1$, $|\tau| \leqslant \delta$ and $w \in \mathcal{V}_{\rho,\tau}$ we have :

$$(6.5) \qquad |(a_{\rho,\tau}-\mu)\ (v_{\rho,\tau,\varepsilon},w) - (\chi_{\rho,\varepsilon}f,w)_{\mathcal{L}^2}| \leqslant C_3\ \varepsilon||w||_{\rho,\tau}$$

On the other hand, we set $u_{\rho,\varepsilon} = \chi_{\rho,\varepsilon}u$ and from formula (3.17) it follows that :

$$|(a_{\rho,\tau}-\mu)\ (u_{\rho,\varepsilon},w) - \sum_{j=1}^{n-1}(M^*_{j,\rho,\tau}\ M_{j,\rho,\tau}\ u_{\rho,\varepsilon},w)_{\mathcal{L}^2}$$

$$- \mu(u_{\rho,\varepsilon},w)_{\mathcal{L}^2} - \int(\gamma_{\rho,\tau}\ u_{\rho,\varepsilon},w)_{|s=o}\Big| \leqslant C_4\ \{||M_{n,\rho,\tau}u_{\rho,\varepsilon}||+\tfrac{1}{\rho}||f||_{\mathcal{L}^2}\}||w||_{\rho,\tau}$$

$$(6.6) \qquad (a_{\rho,\tau}-\mu)\,(u_{\rho,\varepsilon},w) = (f,\ w)_{\mathcal{L}^2} + \sum_J \mu_J\int(u_J,1-Pw)_{|s=o} + o(1)||w||_{\rho,\tau}$$

where $o(1)$ stands for a function (depending on $f,\varepsilon,\tau,\rho,\mu$) converging to o as $\rho \to +\infty$, uniformly for $\mu \in \Lambda$ and $|\tau| \leqslant \delta$.

Now we set $w_{\rho,\tau,\varepsilon} = v_{\rho,\tau,\varepsilon} - u_{\rho,\varepsilon}$ and we insert $w = w_{\rho,\tau,\varepsilon}$ in formulas (6.5) and (6.6). With lemmas 4.4 and 4.5 we see that :

$$|(a_{\rho,\tau}-\mu)\,(w_{\rho,\tau,\varepsilon},\ w_{\rho,\tau,\varepsilon})| \leqslant (o(1) + C_3\varepsilon)||w||_{\rho,\tau}$$

and with the main estimate (3.19) which is obviously still valid after the translations we get :

$$(6.7) \qquad ||v_{\rho,\tau,\varepsilon} - u_{\rho,\varepsilon}||_{\rho,\tau} \leqslant o(1) + C'_3\varepsilon.$$

From lemmas 6.3 and 6.5 we deduce that :

Making use of (3.18), lemmas 6.4 and 5.2 we get :

$$|(a_{\rho,\tau}-\mu)(v_{\rho,\tau}-v_{\rho,\tau,\varepsilon},w) - ((\theta_{\rho,\tau}-\chi_{\rho,\varepsilon})f,w)_{\mathcal{L}^2}|$$
$$\leq (C_1\,\rho^{-1}||f||_{\mathcal{L}^2} + C_3\varepsilon)\,||w||_{\rho,\tau}$$

and taking now $w = v_{\rho,\tau} - v_{\rho,\tau,\varepsilon}$ we get with the main estimate that :

$$(6.8)\qquad ||v_{\rho,\tau} - v_{\rho,\tau,\varepsilon}||_{\rho,\tau} \leq C_5\{(\rho^{-1} + \varepsilon + ||(\theta_{\rho,\tau}-\chi_{\rho,\varepsilon})f||_{\mathcal{L}^2}\}$$

For each $\delta > o$ and $\eta > o$ there is ε small enough such that for all $|\tau| < \delta$ and ρ large enough we have $||(\theta_{\rho,\tau}-\chi_{\rho,\varepsilon})f||_{\mathcal{L}^2} \leq \eta$. On the other and it is clear that $u_{\rho,\varepsilon} \to u$ as $\rho \to +\infty$ if ε is small enough. (u is assumed to be compactly supported in z'). So from (6.7) and (6.8) we conclude that $v_{\rho,\tau} \to u$ as $\rho \to +\infty$, uniformly for $|\tau| < \delta$, and proposition 6.1 is proved.

Similarly to (6.1) we set $E_{\rho,\tau}(\lambda) = T_{\rho,\tau}\, E_\rho(\lambda)\, T_{\rho,\tau}^{-1}$. Then we have :

PROPOSITION 6.5.

For $f \in \mathcal{L}^2(\mathbb{C}^n_+;q)$, $\lambda > 0$ and $\delta > 0$ we have :

$$\operatorname*{Max}_{|\tau|\leq\delta} ||(\tilde{E}_{\rho,\tau}(\lambda) - \tilde{E}_\infty(\lambda))f||_{\mathcal{L}^2(\mathbb{C}^n_+;q)} \to 0 \text{ as } \rho \to \infty .$$

Proof : This result is a consequence of proposition 6.1 and of theorem 1.15, Chap VIII of T. KATO [6]. (See also the proof of lemma 4.5 of [11]).

Because it is clear that $u_{\rho,\varepsilon}$ converges to u as $\rho \to +\infty$ proposition 6.1 is a consequence of (6.10) and 6.11).

Similarly to (6.3) we set $\widetilde{E}_{\rho,\tau}(\lambda) = T_{\rho,\tau}\, \widetilde{E}_\rho(\lambda)\, T^{-1}_{\rho,\tau}$ where $\widetilde{E}_\rho(\lambda)$ is the extension to $\mathcal{L}^2(\mathbb{C}^n_+,q)$ of $E_\rho(\lambda)$ defined in (3.21).

PROPOSITION 6.2. : for $f \in \mathcal{L}^2(\mathbb{C}^n_+,q)$, $\lambda > 0$ and $\delta > 0$ we have :

$$\max_{|\tau|\leq\delta} \|(\widetilde{E}_{\rho,\tau}(\lambda) - \widetilde{E}_\infty(\lambda))f\|_{\mathcal{L}^2(C^n_+,q)} \to 0 \quad \text{as } \rho \to +\infty$$

This proposition is a consequence of proposition 6.1, lemma 5.3 and a theorem of T. Kato [6]. (See also the proof of lemma 4.5 of [11]).

7.- ESTIMATES FOR THE SPECTRAL FUNCTION.

The second step in the proof of proposition 3.3 is to give a priori estimates for the spectral function. Here we follow closely the method of section 2.3 of [11] .

Let $f \in \mathcal{L}^2(\mathbb{C}^n_+,q)$ and let $u_{\rho,\tau} = \widehat{E}_{\rho,\tau}(\lambda)f$; if $v_{\rho,\tau} = E(\rho^2\lambda)\Theta \mathcal{K}^{-1}_\rho\, T^{-1}_{\rho,\tau} f$, v belongs to the domain of $\square^k$ for all $k \in \mathbb{N}$, and it follows that the functions :

$$u^{(k)}_{\rho,\tau} = T_{\rho,\tau}\mathcal{K}_\rho \Theta\; \rho^{-2k}\, \square^k\, v_{\rho,\tau} \tag{7.1}$$

are well defined and satisfy :

$$\|u^{(k)}_{\rho,\tau}\|_{\mathcal{L}^2(\mathbb{C}^n_+,q)} \leq \lambda^k\, \|f\|_{\mathcal{L}^2(\mathbb{C}^n_+,q)} \tag{7.2}$$

Note that $u^{(o)}_{\rho,\tau}$ is simply $u_{\rho,\tau}$. We have :

LEMMA 7.1 : let $\chi \in C^\infty_o(C^n)$ and let ρ be large enough so that $T_{\rho,\tau}\Theta_\rho = 1$ on the support of χ. Then :

$$a_{\rho,\tau}(\chi u_{\rho,\tau}^{(k)},\chi u_{\rho,\tau}^{(k)}) \leq \mathrm{Re}(\chi u_{\rho,\tau}^{(k+1)},\chi u_{\rho,\tau}^{(k)})_{\mathcal{L}^2} + C||M_{\rho,\tau}(\chi)u_{\rho,\tau}^{(k)}||_{\mathcal{L}^2}^2$$

where $M_{\rho,\tau}(\chi) = 1 + \underset{j}{\mathrm{Max}}|(M_{j,\rho,\tau}\chi)|^2$, and C is independant of f, k, χ, ρ and τ.

Proof : Commuting with χin formula (3.17) one gets :

$$a_{\rho,\tau}(\chi u_{\rho,\tau}^{(k)},\chi u_{\rho,\tau}^{(k)}) - \mathrm{Re}\ a_{\rho,\tau}(u_{\rho,\tau}^{(k)},\chi^2 u_{\rho,\tau}^{(k)}) \leq C||M_{\rho,\tau}(\chi)u_{\rho,\tau}^{(k)}||_{\mathcal{L}^2}^2$$

and because $T_{\rho,\tau}\ \Theta_\rho = 1$ on the support of χ, we have (cf. (6.5)) :

$$a_{\rho,\tau}(u_{\rho,\tau}^{(k)},\chi^2 u_{\rho,\tau}^{(k)}) = (u_{\rho,\tau}^{(k+1)},\chi^2 u_{\rho,\tau}^{(k)})_{\mathcal{L}^2}$$

and lemma 7.1 follows.

Now we set $N = 12n + 2$ and we consider a sequence of cut-off functions $\chi^{(j)} \in C_0^\infty(\mathbb{C}^n)$, $j = 0,\ldots,N$, such that $\chi^{(j+1)}\ \chi^{(j)} = \chi^{(j)}$. We introduce $\chi_\rho^{(j)}(z',z_n) = \chi^{(j)}(z',z_{n/\rho})$ and we remark that the $M_{\rho,\tau}(\chi^{(j)})$ are uniformy bounded for $\rho \geqslant 1$, $|\tau| \leqslant \delta$; furthermore $T_{\rho,\tau}\ \Theta_\rho$ is equal to 1 on the support of $\chi_\rho^{(j)}$ for $|\tau| \leqslant \delta$ and ρ large enough. Therefore, from proposition 4.1 and lemma 7.1 we get :

$$(7.3) \qquad ||\chi^{(j)}u_{\rho,\tau}^{(k)}||_{\mathcal{W}_\rho} \leq C_0\{||\chi_\rho^{(j)}u_{\rho,\tau}^{(k+1)}||_{\mathcal{L}^2} + ||\chi_\rho^{(j+1)}u_{\rho,\tau}^{(k)}||_{\mathcal{L}^2}\}$$

From lemma 4.6 we deduce that for $\mu \in \mathbb{R}$ there is a subspace Φ_μ of $\mathcal{W}_\rho$ of codimension less than $C_1\rho\ \mu^{4n}$ such that on Φ_μ we have :

$$(7.4) \qquad \mu||\chi_\rho^{(N)}v||_{\mathcal{L}^2} \leq ||v||_{\mathcal{W}_\rho}$$

If we assume that $\chi_\rho^{(j)}\ u_{\rho,\tau}^{(k)}$ belongs to Φ_μ for all j and k such that $1 \leqslant j+k \leqslant N$, then by induction on $N-(j+k)$, starting with 7.2, and using (7.3) and (7.4) we see that :

$$\mu^{N-j-k}||\chi_\rho^{(j)}u_{\rho,\tau}^{(k)}||_{\mathcal{L}^2} \leq C_2(2C_0)^{N-j-k}||f||_{\mathcal{L}^2}$$

and with (7.3) we get :

$$(7.5) \qquad \mu^{N-1}||\chi_\rho^{(0)}u_{\rho,\tau}||_{\mathcal{W}_\rho} \leq C_3||f||_{\mathcal{L}^2}$$

The integer ν being given we choose $\mu = \left(\frac{2\nu}{C_1\rho(N+1)(N+2)}\right)^{1/4n}$ and it follows from (7.5) that :

(7.6) $$d_\nu(\chi_\rho^{(o)}\ \tilde{E}_{\rho,\tau}(\lambda)\mathcal{L}^2,\mathcal{W}_\rho) \leqslant \frac{C_3}{\mu^{N-1}} = C_4\left(\frac{\rho}{\nu}\right)^{(N-1)/4n}$$

Because $\tilde{E}_{\rho,\tau}(\lambda)$ is a contraction the ν-diameter in the left hand side of (7.6) is also less than one, and summing up we have proved :

<u>LEMMA 7.2</u> : The operators $\chi_\rho^{(o)}\ \tilde{E}_{\rho,\tau}(\lambda)$ are in the trace class from $\mathcal{L}^2$ to $\mathcal{W}_\rho$ and, for each $\delta > 0$ there are C and ρ_o such that for $|\tau| \leqslant \delta$, $\rho \geqslant \rho_o$ we have :

$$||\chi_\rho^{(o)}\ \tilde{E}_{\rho,\tau}(\lambda)||_{\mathcal{C}_1(\mathcal{L}^2,\mathcal{W}_\rho)} \leqslant C\ \rho$$

Similarly one can prove :

<u>LEMMA 7.3</u> : we have the following estimates uniformy for $|\tau| \leqslant \delta$ and ρ large enough :

$$||\frac{\chi_\rho^{(o)}}{(z_n+i)^{N-1}}\ \tilde{E}_{\rho,\tau}(\lambda)||_{\mathcal{C}_1(\mathcal{L}^2,\mathcal{W}_\rho)} \leqslant C$$

<u>Proof</u> : the proof is quite similar to the preceding one : instead of lemma 4.6 we use lemma 4.8 ; instead of (7.3) we use the following consequence of lemma 7.1 and proposition 4.1 :

$$||\frac{\chi_\rho^{(j)}}{(z_n+i)^p}\ u_{\rho,\tau}^{(k)}||_{\mathcal{W}_\rho} \leqslant C\ \{||\frac{\chi_\rho^{(j)}}{(z_n+i)^p}\ u_{\rho,\tau}^{(k+1)}||_{\mathcal{L}^2} + ||\frac{\chi_\rho^{(j+1)}}{(z_n+i)^p}\ u_{\rho,\tau}^{(k)}||_{\mathcal{L}^2}\}$$

and (7.4) is now to be replaced by :

$$\mu\ ||\frac{\chi_\rho^{(N)}}{z_n+i}\ v||_{\mathcal{L}^2} \leqslant ||v||_{\mathcal{W}_\rho}$$

<u>LEMMA 7.4</u> : let ζ be as is lemma 4.7 ; then we have :

$$||\zeta_T\chi_\rho^{(o)}\tilde{E}_{\rho,\tau}(\lambda)||_{\mathcal{C}_1(\mathcal{L}^2,\mathcal{L}^2)} \leqslant C\rho\ T^{-1/N}$$

Proof : we already know that (7.5) holds if f in is a space of codimension less than $C_4\,\rho\,\mu^{4n}$ with lemma 4.7 we know that :

(7.7) $\quad \mu'\,\|\zeta_T\chi_\rho^{(N)}v\|_{\mathcal{L}^2} \leq \|v\|_{\mathcal{W}_\rho}$

if v is in a space of codimension less than $C\rho T^{-1}\mu'^{4n}$

ν being given we choose $\mu = (\frac{\nu}{2\rho C_4})^{1/4n}$ and $\mu' = (\frac{\nu T}{2\rho C_5})^{1/4n}$ and from (7.5) and (7.7) we get that :

$$d_\nu(\zeta_T\chi_\rho^{(o)}\tilde{E}_{\rho,\tau}(\lambda)\underline{\mathcal{L}^2},\mathcal{L}^2) \leq \mu^{-(N-1)}\mu'^{-1} = C_6(\frac{T^{1/N}\nu}{\rho})^{-N/4n}$$

and the lemma follows.

Proof of proposition 3.3

Let $\delta > 0$ and $\lambda > 0$ be given ; for $T > 0$ we choose $\chi \in C_o^\infty(\mathbb{C}^n)$ such that $\chi(z',z_n) = 1$ when $|z'| \leq \delta$ and $|z_n| \leq 2T$. Because the embedding of $\mathcal{W}_\rho$ in $\mathcal{W}_1$ is of norm less than 1 and because $\frac{1}{|z_n+i|}$ is bounded from below on support of χ we deduce from lemma 7.3 that there is $C > 0$ and $\rho_o > 0$ such that for $\rho \geq \rho_o$, $|\tau| \leq \delta$ we have :

(7.8) $\quad \|\chi\tilde{E}_{\rho,\tau}(\lambda)\|_{\mathcal{C}_1(\mathcal{L}^2,\mathcal{W}_1)} \leq C$

Now we use the following general fact : if E_ρ is a family of operators from an Hilbert space H into another one W, which is compactly embedded in H, if the trace norms $\|E_\rho\|_{\mathcal{C}_1(H,W)}$ are uniformy bounded and if E_ρ converges strongly to E_∞, then E_ρ converges to E_∞ in the trace norm $\mathcal{C}_1(H,H)$; furthermore if E_ρ depends on parameters, τ, if the strong convergence is uniform in τ, and if the given estimates for the trace norms in $\mathcal{C}_1(H,W)$ are uniform in τ, then the convergence in $\mathcal{C}_1(H,H)$ is also uniform with respect to τ.

From lemma 4.8, the embedding of $\{u \in W_1 \,/\, \text{supp}\, u \subset \text{supp}\, \chi\}$ into $\mathcal{L}^2$ is compact, and with (7.8) proposition 6.2 and (4.12), the result recalled above tells us that for given $T > 0$, $\delta > 0$, $\lambda > 0$:

$$(7.9)\qquad \max_{|\tau|<\delta}\left\{\int_{\substack{|z'|\leq\delta\\ |z_n|\leq 2T}} |\mathrm{tr}\,\tilde{e}_{\rho,\tau}(\lambda;z,z) - \mathrm{tr}\,\tilde{e}_\infty(\lambda,z,z)|\,dxdy \to 0 \quad \text{as } \rho\to\infty\right.$$

Because $\tilde{e}_{\rho,\tau}(\lambda;z,z) = \tilde{e}_\rho(\lambda;(z',t-\rho\tau+is)\,(z',t-\rho\tau+is))$ and because $\tilde{e}_\infty(\lambda;z,z)$ depends only on $\mathrm{Im}z_n$, (cf. lemma 3.3) we deduce from (7.9) that :

$$(7.10)\qquad \lim_{\rho\to+\infty}\left\{\frac{1}{\rho}\int_{\substack{|z'|\leq\delta\\ |t|\leq\delta\rho\\ 0<s<2T}} |\mathrm{tr}\,\tilde{e}_\rho(\lambda;z,z) - \mathrm{tr}\,\tilde{e}_\infty(\lambda,z,z)|\,dxdy\right\} = 0$$

On the other hand, from lemma (7.4) and (4.12) we see that :

$$(7.11)\qquad \int_{\substack{|z'|\leq\delta\\ |t|\leq\delta\rho\\ s\geq 2T}} |\mathrm{tr}\,\tilde{e}_\rho(\lambda,z,z)|\,dxdy \leq C\,\rho\,T^{-1/4N}$$

From lemma 3.3 we deduce that :

$$(7.12)\qquad \int_{\substack{|z'|\leq\delta\\ |t|\leq\delta\rho\\ s\geq 2T}} |\mathrm{tr}\,\tilde{e}_\infty(\lambda;z,z)|\,dxdy = \frac{\rho}{2\pi}\int_0^\infty e^{-2T\tau}\,c(\tau,z^0)\,d\tau$$

Letting T go to $+\infty$, proposition 3.3 is now a straightforward consequence of (7.10), (7.11), (7.12).

<u>Proof of proposition 3.4</u>.

An immediate consequence of lemma 7.2 and (4.12) it that :

$$\int_{\substack{|z'|\leq\delta\\ |z_n|\leq\rho\delta}} |\mathrm{tr}\,e_\rho(\lambda;z,z)|\,dxdy \leq C\,\rho$$

If we look closely at the proof of lemma 7.2 we see that the constant C depends only on the constants occuring in the main estimate (3.17) and in proposition 4.1, and on the bounds of the functions $M_{\rho\tau}(\chi_\rho^{(j)})$ occuring when lemma 7.1 is used. It is then almost obvious that C can be chosen independantly of z^0, if z^0 stays in a compact set included in $\partial\Omega$.

APPENDIX.

First we show that Hörmander's theorem stated in section 1 is still valid if $\partial\Omega$ is assumed to be only of class C^2. Let $z^\circ \in \partial\Omega$, we can assume that near z°, Ω is given by (3.1), and that the Levi form at z° is diagonal (i.e. (3.2)).

First we recall some convenient notations used in [3] : we set

$$E(u) = \left\{ \sum_{j=1}^{n} \left|\left|\frac{\partial u}{\partial \bar{z}_j}\right|\right|^2_{L^2(\Omega)} + ||u||^2_{L^2(\Omega)} + \int_{\partial\Omega} |u|^2 \right\}^{1/2}$$

and we denote by $R(u)$ any expression such that for each $\delta > 0$ there is a small neighborhood $\mathcal{O}$ of z° such that for $u \in C^1(\bar{\Omega})$ with support in $\mathcal{O} \cap \bar{\Omega}$ we have : $|R(u)| \leq \delta E(u)^2$.

Then we recall the following Green's formula :

$$\text{(A.1)} \qquad \left(\frac{\partial u}{\partial z_j}, \frac{\partial v}{\partial z_k}\right)_{L^2(\Omega)} = \left(\frac{\partial u}{\partial \bar{z}_k}, \frac{\partial v}{\partial \bar{z}_j}\right)_{L^2(\Omega)} + \int_{\partial\Omega} \left(\bar{\nu}_k \frac{\partial u}{\partial z_j} - \nu_j \frac{\partial u}{\partial \bar{z}_k}\right) \bar{v}$$

where (near z°) $\nu_j = \frac{1}{|d\phi|} \frac{\partial \phi}{\partial z_j}$ and $\phi(z) = \varphi(z', \text{Re } z_n) - \text{Im} z_n$

Let $L'_j = \frac{\partial}{\partial \bar{z}_j} - \alpha_j \frac{\partial}{\partial \bar{z}_n}$ $(j = 1,\dots,n-1)$ be the antiholomorphic tangent vector fields introduced in (3.3). We have $L'^{*}_j = -\frac{\partial}{\partial z_j} + \frac{\partial}{\partial z_n} \alpha_j$. Developping $||L'^{*}_j u||^2_{L^2(\Omega)}$ and making use of (A.1) we see that for $u \in C^1(\bar{\Omega})$:

$$||L'^{*}_j u||^2_{L^2(\Omega)} = ||L'_j u||^2_{L^2(\Omega)} + O(E(u)\,||u||_{L^2(\Omega)}) + \int_{\partial\Omega} \nu_n L'_j(a_j)\,|u|^2.$$

With the assumptions (3.1) and (3.2) and remarking that $||L'_j u||^2_{L^2(\Omega)} = \left|\left|\frac{\partial u}{\partial \bar{z}_j}\right|\right|^2_{L^2(\Omega)} + R(u)$, we see that :

$$\text{(A.3)} \qquad \left|\left|\frac{\partial u}{\partial \bar{z}_j}\right|\right|^2_{L^2(\Omega)} \geq -\frac{1}{2} \mu_j \int_{\partial\Omega} |u|^2 - O(E(u)\,||u||_{L^2(\Omega)}) + R(u).$$

Now if $u \in C^1(\Omega;q)$ has its support in a small neigborhood of z°, we can write it as in formula (3.10), with $C^1(\bar{\Omega})$ coefficients u_J ; if u satisfies the boundary condition (1.2) or equivalently (3.12) we deduce from formula (1.3) that :

$$(A.4) \qquad a(u,u) = 2 \sum_{j,J} \left|\left|\frac{\partial u_J}{\partial \bar{z}_j}\right|\right|^2 + \sum_J \mu_J \int_{\partial\Omega} |u_J|^2 + O(E(u)||u||_{L^2(\Omega)}) + R(u)$$

Note that the second term on the right makes sense because $u_J = 0$ on $\partial\Omega$ if $n \in \{J\}$, and if $n \notin \{J\}$ μ_J is, as before, equal to $\sum_{j\in J} \mu_j$.

Now it is well known that condition $Z(q)$, (A.3) and (A.4) imply the main estimate (1.7) (see for instance [3]).

At last we give a short proof of (2.3). For $z^\circ \in \Omega$ let h_{ρ,z° be the dilation $z \to z^\circ + \rho(z-z^\circ)$. If $e_{\rho,z^\circ}(\lambda)$ is the spectral function of $\Box$ in $L^2(\Omega_{\rho,z^\circ};q)$. With $\Omega_{\rho z^\circ} = h_{\rho,z^\circ}(\Omega)$, it is clear that :

$$(A.5) \qquad e(\rho^2\lambda;z,w) = \rho^{2n}\, e_{\rho,z^\circ}(\lambda;h^{-1}_{\rho,z^\circ}(z),\, h^{-1}_{\rho,z^\circ}(w)).$$

On the other hand if B_{δ,z° denotes the open ball of radius δ centered at z° we have for $k > n/2$.

$$(A.6) \qquad |u(z^\circ)| \leq C_\delta \{||\Delta^k u||_{L^2(B_{\delta,z^\circ})} + ||u||_{L^2(B_{\delta,z^\circ})}\}$$

and clearly C_δ depends only on δ.

If $d(z^\circ) \geq \delta/\rho$, Ω_{ρ,z° contains B_{δ,z° and it follows from (A.6) that :

$$(A.7) \qquad \mathrm{tr}\, e_{\rho,z^\circ}(\lambda;z^\circ,z^\circ) \leq C'_\delta\,(\lambda^k+1)$$

Now (2.3) is an immediate consequence of (A.5) and (A.7).

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