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Associated Graded Rings Derived from Integrally Closed Ideals and the Local Homological Conjectures

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Associated Graded Rings Derived from Integrally Closed Ideals and the Local Homological Conjectures [1]

by Melvin Hochster [2]

1. Introduction

The second and third sections of this paper can be read independently. The second section explores the properties of certain "associated graded rings", graded by the nonnegative rational numbers, and constructed using filtrations of integrally closed ideals. The properties of these rings are then exploited to show that if $x_1, x_2, \ldots, x_d$ is a system of paramters of a local ring R of dimension d, $d \geq 3$, and this system satisfies a certain mild condition (to wit, that R can be mapped to a local ring S, perhaps the completion of R, which is module-

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1 The material discussed here is related to but differs substantially from that presented in the author's talk at the Rennes meeting, which focused on the fact that most of the consequences, such as the new intersection conjecture [PS_2], [R], of the existence of big Cohen-Macaulay modules can be deduced from the apparently much weaker direct summand conjecture using the acyclicity, for a larger class of local rings of positive residual characteristic p, of a modified version of the Koszul complex (details will appear in [H_5]), and which also discussed how the direct summand conjecture might be proved from a K-theoretic study of intersection theory in a certain family of ambient local hypersurfaces over discrete valuation rings (details will appear in [H_4]).

2 The author was supported, in part, by a grant from the National Science Foundation, U.S.A.

finite over a regular local ring A such that the images of $x_1,\ldots,x_d$ are a system of parameters for A ; this is automatic in the equicharacteristic case, while in mixed characteristic it suffices that one of the x_i be equal to the characteristic), then the equation

$$x_1^a \ldots x_d^a = y_1x_1^b + \ldots + y_dx_d^b$$

cannot hold in R for positive integers a,b unless $a/b > 2/d$. See Theorem (2.10).

The reason for our interest in this statement is that the direct summand conjecture (that a regular Noetherian ring R, is a direct summand, as an R-module, of every module-finite extension algebra of R) is equivalent to the statement that with R, $x_1,x_2,\ldots,x_d$ as above, the equation

$$x_1^a \ldots x_d^a = y_1x_1^b + \ldots + y_dx_d^b$$

cannot hold in R unless $a \geq b$. This is still an open question when $d = 3$. For a number of years, the author was unable to settle even whether

$$x_1^2x_2^2x_3^2 = y_1x_1^3 + y_2x_2^3 + y_3x_3^3$$

was possible: the result proved in Section 2 shows that it is not possible. The author hopes that the techniques introduced here can be pushed to yield more information.

Interest in the direct summand conjecture has been magnified by the fact, established in [H_5], that it already implies most of the consequences of the existence of big Cohen-Macaulay modules, such as the old and new intersection theorems, and, hence, Bass' question and M. Auslander's zero divisor conjecture. See [A_1], [A_2], [B], [H_1], [H_2], [H_3], [H_5], [R], [PS_1], and [PS_2].

The third section deals with the construction of big Cohen-Macaulay modules for a certain class of complete local domains of mixed characteristic p (the equicharacteristic case is done in [H_2]). These are the domains which have an integral extension domain T with an element $q \in T$ such that q divides p in T and the Frobenius is an automorphism of T/qT. This result is related to the result in [H_3] for rings embeddable in a good way in rings of generalized Witt vectors, but is incomparable to it, and has several advantages, one of which is a substantially simpler proof. We note that the argument given is closely related to the ideas of Section 2, but has been developed so as to avoid any direct reference to that section.

The construction in § 2 is similar to one used by [Lejeune-Teissier, Transversalité, Polygones de Newton et installations (notes prises par M. Galbiati), Astérisque, 6-10 (1973), 75-119].

2. Integrally closed ideals and an associated graded ring.

Let $R \subset S$ be domains with S integral over R (perhaps, $S = R$) and let I be an ideal of R. Let t be an indeterminate over S. The integral closure S_1 of the Rees ring $R[It] = R[it : i \in I]$ in $S[t]$ has the form $S + J_1t + J_2t^2 + \ldots + J_kt^k + \ldots$ where the J_k are ideals of S. Evidently, S_1 is also the integral closure of $S[It] = S[ISt]$. The ideal J_1 is, by definition, the <u>integral closure</u> of I in S: evidently, it is the same as the integral closure of IS in S, and we denote it $\overline{IS}$ (or $\overline{I}$ if $R = S$). Using the t-grading, it is easy to see that $s \in \overline{IS}$ if and only if there exists an integer $m \geq 1$ and elements $i_1, \ldots, i_m$ with $i_k \in I^k$, $1 \leq k \leq m$, such that $s^m + i_1s^{m-1} + \ldots + i_m = 0$. We then refer to $z^m + \sum_{k=1}^m i_kz^{m-k}$ as an <u>equation of integral dependence</u> for s on I.

We set down some basic facts for future reference:

(2.1) <u>Lemma</u>. a) <u>If</u> S <u>is normal</u>, $\overline{uS} = uS$.

b) $\overline{IS} \cap R = \overline{I}$ (<u>in</u> R).

c) $\overline{\overline{IS}} = \overline{IS}$.

d) $\overline{IS}^n \subset \overline{I^nS}$

e) $\overline{I_1S}\ \overline{I_2S} \subset \overline{I_1I_2S}$.

In fact, a), b), c) and d) are quite trivial from the point of view we have taken. (In d), note that $J_n = \overline{I^n S}$.) e) also is easy: If $s_i t_i$ is integral over $R[I_i t_i]$, $i = 1,2$, then $s_1 t_1 s_2 t_2$ is integral over $S_2 = R[I_1 t_1, I_2 t_2]$ and, by a bidegree argument, is in fact integral over the subring which is the sum of the pieces of bidegree (k,k), $k \geq 0$, i.e. over $R[I_1 I_2(t_1 t_2)]$.

A different viewpoint is that $\overline{IS} = \bigcap_V IV$, where V runs through all valuation rings of S. We refer the reader to [ZS], Appendix 4 for further details. (The term "complete ideal" is used instead of "integrally closed ideal" in [ZS].)

Now let S be a domain, J an ideal of S, and let $\mathbb{Q}^+$ be the nonnegative rational numbers. If $q \in \mathbb{Q}^+$ and we write $q = a/b$, it is understood that a,b are integers and $b \geq 1$. We define a filtration of S, indexed by $\mathbb{Q}^+$, as follows: if $q = a/b$, ${}_J S_q = \{s \in S : s^b \in \overline{J^a}\}$. We must first check that S_q (omitting the subscript "J") is well-defined. Suppose t is a positive integer and $S'_q = \{s \in S : s^{bt} \in \overline{J^{at}}\}$. Clearly, $S_q \subset S'_q$ since $s^b \in \overline{J^a}$ implies $(s^b)^t \in \overline{(J^a)^t}$ (by 2.1)d)), while $S'_q \subset S_q$ because an equation of integral dependence for s^{bt} on J^{at} may be viewed as one for s^b on J^a.

Let ${}_J S_q^+ = \bigcup_{q' > q} {}_J S_q$. We frequently omit the subscript "J" in the sequel.

(2.2) *Proposition*. a) S_q (*respectively*, S_q^+) *is an integrally closed ideal of* S.

b) $S_0 = S$ (*respectively*, $S_0^+ = \mathrm{Rad}\, J$) , $S_r \subset S_q$ (*respectively*, $S_r^+ \subset S_q^+$) *if* $r \geq q$, *and* $S_q \cdot S_r \subset S_{q+r}$ (*respectively*, $S_q^+ \cdot S_r \subset S_{q+r}^+$). $S_q^+ \subset S_q$.

c) *If* T *is a domain integral over* S , ${}_{JT}T_q \cap S = {}_J S_q$ (*respectively*, ${}_{JT}T_q^+ \cap S = {}_J S_q^+$.

d) *If* $J = (j_\lambda : \lambda \in \Lambda)S$, T *is a domain integral over* S *which contains, for each* λ, *a* b^{th} *root* h_λ *of* j_λ , *and* $J_1 = (h_\lambda : \lambda \in \Lambda)T$, *then* ${}_J S_{a/b} = \overline{J_1^a} \cap S$.

e) *If* $q \in \mathbb{Q}^+$ *and* $r = c/d \in \mathbb{Q}^+ - \{0\}$, *then if* s^r *denotes a* d^{th} *root of* s^c *which happens to lie in* S , $s^r \in S_q$ (*respectively*, S_q^+) *if and only if* $s \in S_{q/r}$ (*respectively*, $S_{q/r}^+$) .

Proof. The parenthetical statements about S_q^+ follow easily once the versions involving S_q are established. c) is clear, for if $s \in S$, s^b is integral over $J^a \iff$ it is integral over J^aT .

d) holds because $JT \subset J_1^b \subset \overline{JT} \implies J_1^b = JT \implies \overline{J_1^a T} = \overline{J_1^{ab}}$ and $t \in {}_J T_{a/b} \iff t^b \in \overline{J^a T} = \overline{J_1^{ab}} \iff t \in {}_{J_1}T_{ab/b} = {}_{J_1}T_a \iff t \in \overline{J_1^a}$. a) follows from d). b) is immediate from the definition. e) holds because, if $q = a/b$, $s^r \in S_{a/b} \iff s^{rbd} \in S_{ad} \iff s^{bc} \in S_{ad} \iff s \in S_{ad/bc}$. Q.E.D.

If $I \subset R \subset S$ with S integral over R we may write ${}_IS_q$ etc. for ${}_{IS}S_q$ etc.

Now define $[G_J(s)]_q$ or, briefly, $[G(S)]_q$ or, even more briefly G_q by

$$G_q = S_q/S_q^+ .$$

The map $S_q \times S_r \to S_{q+r}$ given by multiplication induces a map $G_q \times G_r \to G_{q+r}$ and hence a graded (by $\mathbb{Q}^+$) commutative ring structure on

$$G_J(S) = \oplus_{q \in \mathbb{Q}^+} [G_J(S)]_q .$$

Since $J^n \subset S_n$ when $n \geq 0$ is an integer, we have a homomorphism $\theta : gr_J(S) \to G_J(S)$, where $gr_J(S) = \oplus_{n \geq 0} J^n/J^{n+1}$ is the usual associated graded ring. If $I \subset R \subset S$, S integral over I, we have a commutative diagram:

$$\begin{array}{ccc} gr_I R & \longrightarrow & gr_{IS} S \\ \downarrow & & \downarrow \\ G_I(R) & \longrightarrow & G_I(S) \end{array}$$

In particular, we have a map $gr_I(R) \to G_I(S)$.

(2.3) Proposition. $G_J(S)$ is integral over Im $gr_I(R)$ and, hence, over Im $G_I(R)$. Moreover, $G_I(R) \to G_I(S)$ is injective.

Proof. If $q \in \mathbb{Q}^+$, $s \in {}_JS_q$, let $[s]_q$ be the image of s in $G_q = {}_JS_q/{}_JS_q^+$. Suppose $[s]_q \in G_q$, $q = a/b$. If $z^m + \sum_{k=1}^m i_k z^{m-k}$ is any equation of integral dependence for s^b on I^a (thus, $i_k \in I^{ak}$, not merely $I^{ak}S$), then $z^{mb} + \sum_{k=1}^m [i_k]_{kb}\, z^{mb-kb}$ is a (homogeneous) equation of integral dependence for $[s]_q$ on $I_m(\mathfrak{g}r_I(R) \to G_J(S))$ in $G_J(S)$. The injectivity is immediate from (2.2) d). Q.E.D.

In general, $\mathfrak{g}r_I(R) \to G_I(R)$ is not injective.

(2.4) Proposition. $G_J(S)$ is reduced.

Proof. This is immediate, from (2.2) e). Q.E.D.

We call (R,I) admissible if

i) R is normal

ii) $\overline{I^n} = I^n$, all n

iii) $\bigcap_n I^n = 0$

iv) if $r,r' \in R - \{0\}$ then $\mathrm{ord}_I(rr') = \mathrm{ord}_I r + \mathrm{ord}_I r'$, where $\mathrm{ord}_I r = \sup \{n : r \in I^n\}$.

(2.5) Proposition. If (R,I) is admissible and S is a domain integral over R, then $gr_I R \to G_I(S)$ is injective.

Proof. Let n be a nonnegative integer. The result reduces to the assertion that $S_n^+ \cap R \subset I^{n+1}$. Suppose $r \in S_n^+ \cap R$, say $r \in S_{n+a/b}$, $a,b > 0$. Then $r^b \in S_{nb+a} \cap R =$

$\overline{I^{nb+a}S} \cap R = \overline{I^{nb+a}} = I^{nb+a} \Longrightarrow \text{ord}_I\, r^b \geq nb + a \Longrightarrow b\, \text{ord}_I r$
$\geq nb + a \Longrightarrow \text{ord}_I r > n \Longrightarrow r \in I^{n+1}$. Q.E.D.

Note: We have not used the full strenth of iv): only that $\text{ord}_I r^n = n\, \text{ord}_I r$, $r \neq 0$.

(2.6) Corollary. If R is a regular Noetherian local ring with maximal ideal m and S is a domain integral over R, then $G_m(S)$ is an integral extension of the polynomial ring $\text{gr}_m(R)$.

(2.7) Proposition. Let R be a Noetherian domain, $I \subset R$ a proper ideal, and S a domain integral over R . Then $\bigcap_n \overline{I^S}_n = (0)$, where n runs through the positive integers.

Proof. If $s \in \bigcap_n S_n$, $s \neq 0$, then s has a multiple in R - {0}. Thus it suffices to show that $(0) = (\bigcap_n S_n) \cap R = \bigcap_n (S_n \cap R) = \bigcap_n \overline{I^n}$. It suffices to show this for a larger Noetherian domain $R_1 \supset R$ with I replaced by IR_1 . Choose $m \supset I$ maximal, and replace R by the normalization of the quotient of the completion of R by a prime disjoint from $R_m - 0$: the new ring is a complete normal local domain A with J = IA inside the maximal ideal. Since A is complete, it is pseudo-geometric, and the normalization $\sum_n \overline{J^n}\, t^n$ is module-finite over A[Jt] and hence finitely generated as a A-algebra. It follows that for some k, $\sum_{nk} \overline{J^{nk}}\, t^{nk}$ is generated over A by $\overline{J^k}\, t^k$, so that $\overline{J^{nk}} = (\overline{J^k})^n$, all n. Thus $\bigcap_n \overline{J^n} \subset \bigcap_n \overline{J^{nk}} = \bigcap_n (\overline{J^k})^n = (0)$.
Q.E.D.

(2.8) *Theorem. Let* (R,I) *be admissible and* S *a domain integral over* R.

a) *Let* $s \in S - \{0\}$ *satisfy* $g(z) = z^d + \sum_{i=1}^d s_i z^{d-i}$, $s_i \in S$, *and suppose moreover that* $s_i \in S_{q_i}$. *Let* $q = \min_i\{q_i/i\}$. *Then* $s \in S_q$.

b) *Let* $s \in S - \{0\}$ *have minimal polynomial* $f(z) = z^m + \sum_{i=1}^m r_i z^{m+i}$ *over the fraction field* F *of* R *(since* R *is normal, every* $r_i \in R$*). Then for every integer* $n \geq 0$, $s \in I^n S = S_n \iff \min_i\{\mathrm{ord}_I r_i/i\} \geq n$.

c) *For every* $s \in S - \{0\}$ *there is a unique* $q \in \mathbb{Q}^+$ *such that* $s \in S_q - S_q^+$. *If* $d = \mathrm{LCM}\{1,2,\ldots,m\}$, *where* m *is the degree of* s *over* F, *then* d_q *is an integer.*

We write $\mathrm{rat}_I s = q$, *and call* q *the* ***rational order*** *of* s *with respect to* I.

d) *If* $s \in S - \{0\}$, $\mathrm{rat}_I s^n = n(\mathrm{rat}_I s)$ *for every integer* n.

e) *If* $r \in R - 0$, $\mathrm{rat}_I r = \mathrm{ord}_I r$, *and for every* $s \in S - \{0\}$, $\mathrm{rat}_I(rs) = \mathrm{ord}_I(rs) = \mathrm{ord}_I(r) + \mathrm{rat}_I(s)$.

f) $\mathrm{rat}_I(s_1+s_2) \geq \min\{\mathrm{rat}_I(s_1), \{\mathrm{rat}_I(s_2)\}$, *with equality unless* $\mathrm{rat}_I(s_1) = \mathrm{rat}_I(s_2)$ *(view* $\mathrm{rat}_I(0)$ *as* $+\infty$*)*.

Proof. a) Let b be a common denominator for the q_i/i, let $q_i = a_i i/b$, and let J be the ideal generated by b^{th} roots of the generators of I in an integral extension domain T of S. Let $a = \min_i a_i$, so that $q = a/b$. The hypothesis implies that $s_1 \in \overline{J^{a_i i}} \subset \overline{J^{ai}}$, so that st is integral over $\sum_n \overline{J^{an}t^n}$ which implies $s \in \overline{J^a} \cap S = S_{a/b} = S_q$.

b) $s \in S$ is in $\overline{I^n S} \iff st^n$ is integral over $R[It]$. Since $I^n = \overline{I^n}$, all n and R is normal, $R[It]$ is normal, and so st^n is integral over $R[It] \iff$ the coefficients of its minimal polynomial over $F(t)$ are in $R[It]$. But this minimal polynomial is $(t^n)^m f(z/t^n) = z^m + \sum_{i=1}^m t^{ni} r_i z^{m-i}$, and so $s \in \overline{I^n S} \iff r_i \in I^{ni}$, $1 \leq i \leq m \iff \mathrm{ord}_I r_i \geq ni$, all $i \iff \min_i\{\mathrm{ord}_I\, r_i/i\} \geq n$, as required.

c) Let $s \in S - \{0\}$ be given with m, $f(z)$ as in b), and let $\mathcal{Q} = \{q \in \mathbb{Q}^+ : s \in S_q\}$. $\mathcal{Q}$ is bounded above by any integer h which exceeds $\min_i\{\mathrm{ord}_I(r_i)/i\}$. To complete the proof, it will suffices to show that given any $q \in \mathcal{Q}$, there exists $r \in \mathcal{Q}$ with $r \geq q$ and dr an integer, for then the largest element of $\mathcal{Q} \cap \{a/d$: c a nonnegative integer with $c \leq hd\}$, which will be a nonempty finite set, must also be the largest element of $\mathcal{Q}$. Suppose T is a domain integeral over S which contains all the roots $s = s_1,\ldots,s_m$ of $f(z)$. Suppose $q = b/a$, and let g be an equation of integral dependence for s^b on I^a , with coefficients in R . Then $s_1,\ldots,s_m$ all satisfy this equation, so that every $s_j \in S_q$. It follows that the k^{th} elementary symmetric function, $\pm r_j$, of the s_j is in $S_{kq} \cap R$. Choose an integer w_k such that

$w_k - 1 < k_q \le w_k$. Then $r_j \in S_{kq} \cap R \subset S^+_{w_k-1} \cap R = I^{w_k}$, i.e. $\mathrm{ord}_I r_j \ge w_k$, so that $r_j \in S_{w_k}$. By a), $s \in S_r$, where $r = \min_k \{w_k/k\}$, and, clearly, dr is an integer. This establishes a).

d) is immediate from Proposition (2.2c).

e) Clearly, $\mathrm{rat}_I r \ge \mathrm{ord}_I r$. But if $\mathrm{ord}_I r = n$ and $\mathrm{rat}_I r > n$, then $[r]_n \longmapsto 0$ under $\mathrm{gr}_I R \to G_I(S)$, contradicting Proposition (2.5). This establishes the first statement in e).

To prove the second statement, suppose $r \in R - \{0\}$, $s \in S - \{0\}$, $\mathrm{ord}_I r = n$, $\mathrm{rat}_I s = \frac{a}{b}$, and $\mathrm{rat}_I rs = n + \frac{a}{b} + \frac{c}{b}$, where c is a positive integer and b is a common denominator for $\mathrm{rat}_I s$ and $\mathrm{rat}_I rs$. (It is clear that $\mathrm{rat}_I rs \ge n + \frac{a}{b}$.) Then $\mathrm{ord}_I r^b = bn$, $\mathrm{rat}_I s^b = a$, and $\mathrm{rat}_I r^b s^b = bn + a + c$. Thus, if $f(z)$ is the minimal polynomial of s^b over F, say $f(z) = z^m + \sum_{i=1}^m r_i z^{m-i}$, then the minimal polynomial of $r^b s^b$ is $(r^b)^m f(z/r^b) = z^m + \sum_{i=1}^m r_i (r^b)^i z^{m-i}$. Since $r^b s^b \in S_{bn+a+c}$ we have that $\min_i \{\mathrm{ord}_I (r_i (r^b)^i)/i\} \ge bn + a + c$. Since (R,I) is admissible, $\mathrm{ord}_I (r_i r^b)^i/i = (\mathrm{ord}_I\, r_i)/i + b\, \mathrm{ord}_I r = (\mathrm{ord}_I r_i)/i + bn$. Thus, $\min_i \{(\mathrm{ord}_I r_i)/i\} + b_n \ge bn+a+c$ and $\min_i \{(\mathrm{ord}_I r_i)/i)\} \ge a + c$. By a) (or b)) we have $\mathrm{rat}_I s^b \ge a + c$, a contradiction.

f) is clear. Q.E.D.

(2.9) Corollary. *Let (R,I) be admissible and S a domain integral over R. Then every nonzero element of $\mathrm{gr}_I R$ is a nonzero divisor on $G_I(S)$.*

In particular, this holds when R *is regular local and* I *is the maximal ideal.*

Proof. It suffices to show that if $[r]_n$, $[s]_{a/b}$ are nonzero forms in $gr_I R$, $G_I(S)$ respectively, then $[rs]_{n+a/b} \neq 0$, which is precisely the content of Theorem (2.8e). Q.E.D.

The direct summand conjecture (see $[H_1]$, $[H_2]$, $[H_4]$) is equivalent to the assertion that if R is a regular local ring of dimension d with maximal ideal $(x_1,\ldots,x_d)$ (so that $x_1,\ldots,x_d$ is a *regular* system of parameters) and S is a domain integral over R, then $x_1^a,\ldots,x_d^a \notin (x_1^b,\ldots,x_d^b)S$ if $a < b$. Until recently, the author was unable to establish this even if $d = 3$, $a = 2$, $b = 3$. The next result, while still a long way from what is needed, at least handles this case.

(2.10) *Theorem. Let* R *be a regular local ring of dimension* d *with maximal ideal* $(x_1,\ldots,x_d)$ *and let* $S \supset R$ *be a ring integral over* R.

Suppose a,b *are positive integers such that* $(x_1,\ldots,x_d)^a \in (x_1^b,\ldots,x_d^b)S$. *Then* $a/b > 2/d$ *if* $d \geq 3$.

Proof. Suppose $(x_1 \ldots x_d)^a = \sum_{i=1}^d y_i x_i^b$. By killing a minimal prime of S disjoint from $R - \{0\}$, we may suppose that S is a domain. Assume that $a/b \leq 2/d$. The direct summand conjecture is known in the equicharacteristic case $[H_1]$. Hence, we may assume that R is of mixed characteristic and hence that the fraction field F of R has characteristic 0. We may replace S by $R[y_1,\ldots,y_d]$ and so assume that S is module-finite over R. Let L be a finite Galois field extension of F containing $y_1,\ldots,y_d$, let $G = \mathrm{Aut}_F L$ be the Galois group, and enlarge S further to contain all the $g(y_i)$, $g \in G$. Let G have order h. By applying each element g of G to the equation $(x_1,\ldots,x_d)^a = \sum_{i=1}^d y_i x_i^b$, we obtain a system of equations

$$(x_1,\ldots,x_d)^a = \sum_{i=1}^d g(y_i) x_i^b , \quad g \in G ,$$

which, for a fixed i we can rewrite as

$$g(y_i) x_i^b = (x_1,\ldots,x_d)^a - \sum_{j\neq i} g(y_i) x_j^b$$

If we take the sum of the products of these equations k at a time, $1 \leq k \leq h$, then the right hand side becomes a linear combination of the products of k-element subsets of $\{(x_1 \ldots x_d)^a, x_1^b,\ldots,\widehat{x_i^b},\ldots,x_n^b\}$, where $\widehat{\ }$ indicates omission, with coefficients invariant under G, i.e. in R, while the left hand side is $\sigma_k x_i^{kb}$, where σ_k is the kth elementary symmetric function of the $g(y_i)$, $g \in G$. If $\mathfrak{A}_i = ((x_1 \ldots x_d)^a, x_1^b,\ldots,\widehat{x_1^b},\ldots,x_n^b) \subset R$, we then have

$$\sigma_k x_i^{kb} \in \mathfrak{A}_i^k \Longrightarrow \sigma_k \in \mathfrak{A}_i^k : x_i^{kb} .$$

Since $x_1,\ldots,x_d$ is a regular sequence in R, one can compute $\mathfrak{a}^k : x_i^{kb}$ just as though $x_1,\ldots,x_b$ were indeterminates in a polynomial ring [EH]. One obtains:

$$\mathfrak{a}_i^k : x_i^{kb} = \sum(((x_1 \ldots x_d)^{k_0 a} \Pi_{1 \leq j \leq d, j\neq i}\, x_j^{k_j b}) : (x_i^{kb}))$$

where the summation is extended over all d-tuples $(k_0,k_1,\ldots,k_{i-1},k_{i+1},\ldots,k_d)$ of nonnegative integers such that $\sum_{j\neq i} k_j = k$. Since $b > a$ and $k_0 \leq k$, we obtain, writing $z_i = x_1\ldots x_{i-1}\cdot x_{i+1} \ldots x_d$, $\mathfrak{a}_i^{kb}: x_i^{kb} = \sum(z_i^{k_0 a})\Pi_{j\neq i}(x_j^{k_j b}) = \mathfrak{b}_i^k$ where $\mathfrak{b}_i = (z_i^a, x_1^b,\ldots,\widehat{x_i^b},\ldots,x_d^b)R$. Since y_i satisfies the equation

$$\Pi_{g \in G}(Y - g(y_i)) = 0$$

in which the coefficient of Y^k is $\pm \sigma_k$, and since $\sigma_k \in \mathfrak{a}_i^k : x_i^{kb} = \mathfrak{b}_i^k$, we find that y_i is integral over $\mathfrak{b}_i$, $1 \leq i \leq d$.

Now let T be the integral closure of S in an algebraic closure of its fraction field. T is integral over the regular local ring R, and if m denotes the maximal ideal of R, we may form the big associated graded ring $G_m(T)$.

Now $\text{ord}(x_1 \ldots x_d)^a = da$, $\text{ord}\, x_j^b = b$, $\text{ord}\, z_i^a = (d-1)a$, and so $\text{rat}\, y_j \geq \min\{(d-1)a, b\}$.

We want to reduce to the case where $(d-1)a \geq b$. We first choose c, a positive integer, such that

$$1/bc \leq (2/d) - 1/(d-1)$$

($d \geq 3$ here). We can then replace the regular ring R by $R^* = R[x_1^{1/c}, \ldots, x_d^{1/c}]$ (the new ring is still regular, since $x_1^{1/c}, \ldots, x_d^{1/c}$ generate the maximal ideal). If $\hat{x}_i = x_i^{1/c}$, we then have

$$(\hat{x}_1 \cdots \hat{x}_d)^{ac} = \sum y_i (\hat{x}_i)^{bc} .$$

Thus, we can replace a,b by ac, bc. We return to our old notation. We now can assume that $1/b < (2/d) - 1/(d-1)$. We next increase a by one succesively until we reach an integer a' such that $a'/b \geq 1/(d-1)$. Thus, we shall have $a' = a + t + 1$, where $(a + t)/b < 1/(d-1))$ while $(a + t + 1)/b \geq 1/(d-1)$. Then $a'/b = (a + t)/b + 1/b < 1/(d-1) + ((2/d) - 1/(d-1)) = 2/d$. Multiplying the equation

$$(x_1 \ldots x_d)^a = \sum y_i x_i^b$$

by $(x_1 \ldots x_d)^{t+1}$ yields

$$(x_1 \ldots x_d)^{a'} = \sum y_i' x_i^b$$

where $1/(d-1) \leq a'/b \leq 2/d$. Thus, without loss of generality we may assume that $1/(d-1) \leq a/b \leq 2/d$ (we again return to our previous notation).

Then

$$da = \mathrm{ord}(x_1 \ldots x_d)^a = \mathrm{rat}(\sum y_i x_i^b) \geq 2b.$$

This gives a contradiction at once unless $da = 2b$, since we have that $da \leq 2b$. Thus, there remains only to treat the subtler case where $da = 2b$. But we can now take "leading forms" to obtain the following equation in $G_m(T)$:

$$[(x_1 \cdots x_d)^a]_{2b} = \sum_i [y_i]_b [x_i^b]_b$$

or

$$(\#) \quad \Pi_i([x_i]_1)^a = \sum_i [y_i]_b ([x_i]_1)^b$$

But $G_m(T)$ is an integral extension domain of $K[\bar{x}_1, \ldots, \bar{x}_d]$, where $K = R/m$ and $\bar{x}_i = [x_i]_1$. Hence, equation (#) cannot hold for $b > a$ because the direct summand conjecture is known in the equicharacteristic case. Q.E.D.

3. A class of mixed characteristic rings possessing big Cohen-Macaulay modules.

Let (R,m) be a (Noetherian) complete local domain of mixed characteristic with residual characteristic $p > 0$. We shall say that an integral domain $T \supset R$ is a <u>wonderful</u> extension if

1) T is integral over R

2) there is an element $q \in T$ such that $p \in qT$ and the Frobenius is an automorphism of T/qT.

Condition 2) is equivalent to the assertion that qT is radical and every element of T has a p^{th} root modulo qT. Of course, T is not Noetherian. We do not know whether every complete local domain has a wonderful extension. The problem is to adjoin sufficiently many elements to serve as pth roots modulo (q) while not permitting ramification at q.

Our interest in this notion is motivated by:

(3.1) <u>Theorem</u>. <u>If</u> R <u>has a wonderful extension</u> T, <u>then</u> R <u>has a big Cohen-Macaulay module</u>.

<u>Proof</u>. R is module-finite over a complete regular local ring $A = V[[x_2,\ldots,x_n]]$, where V is a discrete valuation ring with maximal ideal pV. Let $x_1 = q$. We shall show that there is a T-module N such that $(x_1,\ldots,x_n)N \neq N$ and $x_1,\ldots,x_n$

a regular sequence on N. This implies that N is a big Cohen-Macaulay module for R[q], hence that there is a big Cohen-Macaulay module for R[q] which is A-free, and hence a big Cohen-Macaulay module for R for the system of parameters $p, x_2, \ldots, x_n$.

We recall from [H_2] that if no such T-module N exists, then there is a positive integer r and integers $k_0, \ldots, k_{r-1}$, $0 \leq k_i \leq n-1$ and a sequence

$$(M_0, m_0) = (T, 1) \xrightarrow{f_0} (M_1, n_1) \xrightarrow{f_1} \cdots \xrightarrow{f_2} (M_i, m_i) \xrightarrow{f_i} \cdots \xrightarrow{f_{r-1}} (M_r, m_r)$$

such that:

1) M_i is a T-module, $0 \leq i \leq r$

2) $m_i \in M_i$, $0 \leq i \leq r$.

3) $f_i : M_i \to M_{i+1}$ is a T-linear map such that $f_i(m_i) = m_{i+1}$, $0 \leq i \leq r-1$

4) f_i is a modification of M_i of type k_i , i.e. there is an element $u_i \in (x_1, \ldots, x_{k_i})M_i : x_{k_i+1}T$ such that f_i is the map

$$M_i \to (M_i \oplus T^{k_i})/T(u_i \oplus (x_1, \ldots, x_{k_i}))$$

induced by the inclusion $M_i \to M_i \oplus T^{k_i}$

5) $m_r \in (x_1, \ldots, x_n)M_r$.

The last condition makes the sequence of modifications "bad" in the terminology of $[H_2]$. We shall assume the existence of a bad sequence of modifications as above for fixed integers $r, k_0, \ldots, k_{r-1}$ and obtain a contradiction.

We first claim that there exist linear maps $\phi_i : M_i \to T$, $0 \le i \le r$ such that $\phi_0 = \mathrm{id}\ T$, positive integers $e_0, \ldots, e_{r-1}$, elements $a_0, \ldots, a_{r-1} \in A - pA$ and elements $c_0, \ldots, c_{r-1} \in T$ such that for each i, $0 \le i \le r-1$, $c_i^{p^{e_i}} \equiv a_i$ modulo qT, the order d_i of $\bar{a}_i \in A/pA$ with respect to the maximal ideal of A/pA is $< \frac{1}{r} p^{e_i}$, and such that the diagram

$$\begin{array}{ccccccccccccc}
T & \xrightarrow{f_0} & M_1 & \xrightarrow{f_1} & M_2 & \xrightarrow{f_2} \cdots \xrightarrow{f_{i-1}} & M_i & \xrightarrow{f_i} & \cdots & \xrightarrow{f_{r-1}} & M_r \\
{\scriptstyle \mathrm{id}}\downarrow & & \downarrow{\scriptstyle \phi_1} & & \downarrow{\scriptstyle \phi_2} & & \downarrow{\scriptstyle \phi_i} & & & & \downarrow{\scriptstyle \phi_r} \\
T & \xrightarrow{c_0} & T & \xrightarrow{c_1} & T & \xrightarrow{c_2} \cdots \xrightarrow{c_{i-1}} & T & \xrightarrow{c_i} & \cdots & \xrightarrow{c_{r-1}} & T
\end{array}$$

commutes.

We construct the ϕ_i, e_{i-1}, a_{i-1}, c_{i-1} by recursion on i. If $i = 0$, we choose $\phi_0 = \mathrm{id}$. Now suppose ϕ_j, $j \le i$, and e_j, a_j, c_j, $j \le i-1$ have been constructed already, so that the conditions above are satisfied and the diagram

$$\begin{array}{ccccc}
T & \longrightarrow & \cdots & \longrightarrow & M_i \\
{\scriptstyle \mathrm{id}}\downarrow & & & & \downarrow{\scriptstyle \phi_i} \\
T & \longrightarrow & \cdots & \longrightarrow & T
\end{array}$$

commutes. It will suffice to construct $\phi_{i+1} : M_i \to T$, e_i, a_i and c_i such that $c_i^{p^{e_i}} \equiv a_i$ modulo qT, $d_i = \text{ord}\ \bar{a}_i < \frac{1}{r} p^{e_i}$ and

$$\begin{array}{ccc} M_i & \xrightarrow{f_i} & M_{i+1} \\ \phi \downarrow & & \downarrow \phi_{i+1} \\ T & \xrightarrow{c_i} & T \end{array}$$

commutes. We shall abbreviate by omitting the subscript i, and write ϕ, M, f, k and u for ϕ_i, M_i, f_i, k_i and u_i respectively. Now f is the map

$$M \to (M \oplus T^k)/T(u \oplus (x_1,\ldots,x_k))$$

where $u \in (x_1,\ldots,x_k)M : x_{k+1}T$, so that we may write $x_{k+1}u = \sum_{i=1}^{k} x_i m_i$, $m_1,\ldots,m_k \in M$. Applying ϕ we have $x_{k+1}(u) = \sum_{i=1}^{k} x_i \phi(m_i)$. We next prove a crucial lemma.

(3.2) *Lemma. Let* $t_1,\ldots,t_k$, $t \in T$ *be elements such that* $tx_{k+1} = t_1x_1 + \ldots + t_kx_k$. *Then we can choose* $a \in A - pA$ *such that for all positive integers* e *if* $c \in T$ *and* $c \equiv \bar{a}^{1/p^e}$ *module* qT, *then* $ct \in (x_1,\ldots x_k)T$.

Proof. If $k = 0$ we have $tx_1 = 0$ in T and so $t = 0$ and we may choose $b = 1$.

Note that $qt \cap A \supset pA$ and must have height one: since pA is prime, we have $qT \cap A = pA$. Let $\bar{A} = A/pA$, $\bar{T} = T/qT$.

Thus, $\bar{A} \hookrightarrow \bar{T}$. If K is the residue class field of V, $\bar{A} = K[[\bar{x}_2,\ldots,\bar{x}_n]]$. Choose $B \subset \bar{T}$ module-finite over $\bar{A}$ and containing $\bar{t}_1,\ldots,\bar{t}_k,\bar{t}$, where $^-$ denotes reduction modulo qT.

Let L be the fraction field of $\bar{A}$. Then $L \otimes_{\bar{A}} B$ is a finite-dimensional vector space over L, and we can choose $b_1,\ldots,b_m \in B$ such that $\{1 \otimes b_i : 1 \le i \le m\}$ is a vector space basis over L. It follows that $\bar{A}b_1 + \ldots + \bar{A}b_m \cong \bar{A}^m$ with $b_1,\ldots,b_m$ as free basis over $\bar{A}$ and that $\sum \bar{A}b_i \hookrightarrow B$ has a cokernel which is a torsion-module over $\bar{A}$. Choose $a \in A - pA$ such that $\bar{a}$ is a nonzero element of $\bar{A}$ which kills this cokernel: thus $\bar{a}B \subset \sum \bar{A}b_i$. The relation $tx_{k+1} = \sum_{j=1}^{k} t_j x_j$ yields, in B, $\bar{t}\,\bar{x}_{k+1} = \sum_{j=2}^{k} \bar{t}_j \bar{x}_j$. Applying the eth power of Frobenius and multiplying by $\bar{a}$ we obtain

$$(\bar{a}\,\bar{t}^{p^e})\bar{x}_{k+1}^{p^e} = \sum_{j=2}^{k} (\bar{a}\,\bar{t}_j^{p^e})\bar{x}_j^{p^e}$$

and $\bar{a}\bar{t}^{p^e}$, $\bar{a}\bar{t}_j^{p^e} \in \sum \bar{A}b_i \cong \bar{A}^m \subset B$. The relation given is trivial in $\bar{A}^m$, so that we can choose $\theta_2,\ldots,\theta_j \in \bar{T}$ such that

$$\bar{a}\,\bar{t}^{p^e} = \sum_{j=2}^{k} \theta_j \bar{x}_j^{p^e}$$

Since the Frobenius is an automorphism of $\bar{T}$, we then have

$$\bar{a}^{1/p^e}\,\bar{t} = \sum_{j=2}^{k} \theta_j^{1/p^e}\,\bar{x}_j \in (\bar{x}_2,\ldots,\bar{x}_k)\bar{T}.$$

Now, if $c \equiv \bar{a}^{1/p^e}$ modulo qT we have, recalling that $q = x_1$, that $ct \in (x_1,\ldots,x_k)T$. Q.E.D.

We now return to the proof of the theorem. We apply the lemma with $t = \phi(u)$, $t_i = \phi(m_j)$. Thus, we can choose $a \in A - pA$ such that for every positive integer e, if $c \in T$ and $c \equiv a^{1/p^e}$ modulo qT, then $c\phi(u) \in (x_1,\ldots,x_k)T$. Now $\bar{a}$ will have some order d with respect to the maximal ideal $(\bar{x}_2,\ldots,\bar{x}_n)\bar{A}$ of $\bar{A} = K[[\bar{x}_2,\ldots,\bar{x}_n]]$. Choose $e_i = e$ so large that $d/p^e < 1/r$. Choose $c_i = c$ so that $c \equiv a^{1/p^e}$ modulo qT. All conditions will be satisfied if we can construct ϕ_{i+1} so that

$$\begin{array}{ccc} M & \xrightarrow{f} & (M \oplus T^k)/T(u \oplus (x_1,\ldots,x_k)) \\ \downarrow \phi & & \downarrow \phi_{i+1} \\ T & \xrightarrow{c} & T \end{array}$$

commutes. To give the map it suffices to give a map $\psi\colon M \oplus T^k \to T$ whose restriction to M is $c\phi$ and which assigns values $s_1,\ldots,s_k$ to the free generators of T^k in such a way that ψ kills $u \oplus (x_1,\ldots,x_k)$. The condition on $s_1,\ldots,s_k$ is then that

$$-c\phi(u) = \sum s_i x_i \quad .$$

Since $c\phi(u) \in (x_1,\ldots x_k)T$, it is possible to choose $s_1,\ldots,s_k$ with the required property.

Thus, it is possible to choose the ϕ_i, e_i, a_i, c_i as stated earlier. Now, since the diagram

$$\begin{array}{ccccc} (T,1) & \longrightarrow & \cdots & \longrightarrow & (M_r, m_r) \\ \downarrow \text{id} & & & & \downarrow \phi_r \\ (T,1) & \xrightarrow{c_0} & \cdots & \xrightarrow{c_{r-1}} & (T, c_0 \cdots c_{r-1}) \end{array}$$

commutes, the image of 1 in the lower right hand copy of T may be computed in two ways to show that

$$c_0 \cdots c_{r-1} = \phi_r(m_r) \in (x_1, \ldots, x_n)T$$

(since $m_r \in (x_1, \ldots, x_n)M_r$, by virtue of the assumption that we have a <u>bad</u> sequence of modifications). Modulo qT (or $x_1 T$) we have

$$\bar{c}_0 \cdots \bar{c}_{r-1} \in (\bar{x}_2, \ldots, \bar{x}_n)\bar{T}$$

whence

$$\bar{a}_0^{1/p^{e_0}} \cdots \bar{a}_{r-1}^{1/p^{e_{r-1}}} \in (\bar{x}_2, \ldots, \bar{x}_n)\bar{T}$$

for $e \geq \max\{e_0, \ldots, e_{r-1}\}$, we have

$$\bar{a}^{p^{e-e_0}} \cdots \bar{a}^{p^{e-e_{r-1}}} \in (\bar{x}_2^{p^e}, \ldots, \bar{x}_n^{p^e})\bar{T} \cap \bar{A} \subset (\bar{x}_2, \ldots, \bar{x}_n)^{p^e}\bar{A},$$

since $(\bar{x}_2, \ldots, \bar{x}_n)^{p^e}$ is <u>integrally closed</u>. But

$$\operatorname{ord}(\bar{a}_0^{p^{e-e_0}} \cdots \bar{a}_{r-1}^{p^{e-e_{r-1}}})$$

$$= \sum_{j=0}^{r-1} p^{e-e_j} \operatorname{ord} \bar{a}_j = p^e \sum_{j=0}^{r-1} \frac{\operatorname{ord} \bar{a}_j}{p^{e_j}}$$

$$< p^e \sum_{j=0}^{r-1} \frac{1}{r} = p^e, \text{ so that } \Pi_{j=0}^{r-1} \bar{a}_j^{p^{e-e_j}}$$

has order <u>strictly</u> less that p^e, a contradiction. Q.E.D.

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