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L^{∞} - CONVERGENCE OF FINITE ELEMENT GALERKIN APPROXIMATIONS ON GALERKIN PROBLEMS

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0. Introduction

Let the model problem

$$\begin{aligned} \dot{u} - \Delta u &= f \qquad \text{in} \quad \Omega \times (0,T] \quad , \\ u &= 0 \qquad \text{on} \quad \partial\Omega \times (0,T] \quad , \\ u_{t=0} &= u_o \qquad \text{in} \quad \Omega \end{aligned}$$

be given. Here $\Omega \subseteq R^N$ is a bounded domain with $\partial\Omega$ sufficiently smooth. With the help of a finite-element-approximation-space $S_h \subseteq \overset{o}{H}_1$ the standard Galerkin approximation $u_h = u_h(t) \in S_h$ is defined by

$$\begin{aligned} (\dot{u}_h, \chi) + D(u_h, \chi) &= (f, \chi) \qquad \text{for} \quad \chi \in S_h \quad , \\ u_h(0) &= Q_h \, u_o \quad . \end{aligned}$$

Here $(.,.)$ is the $L_2(\Omega)$-scalar product, $D(.,.)$ the Dirichlet integral and Q_h an appropriate projection.

The aim of this paper is to derive L_∞-estimates for the error $e = e_h = u - u_h$. For the corresponding elliptic case this problem was solved by SCOTT, NITSCHE about three years ago. There is a certain feeling that the proofs for the parabolic case would (resp. should) be more or less a direct consequence.

Besides the case of one space-dimension seemingly only BRAMBLE-SCHATZ-THOMEE-WAHLBIN have attacked this problem. Their approach is to rewrite the Galerkin-equations in the form

$$e + T_h \, \dot{e} = (I - R_h) \, u \quad .$$

Here to any f the element $U_h = T_h f = R_h(-\Delta^{-1} f)$ is the Ritz-approximation on $-\Delta^{-1} f$ defined by $U_h \in S_h$ and

$$D(U_h, \chi) = (f, \chi) \qquad \text{for} \quad \chi \in S_h \quad .$$

We may also write

$$e = -\Delta^{-1}\dot{e} + (I-R_h)\,\Delta^{-1}\dot{e} + (I-R_h)u \quad .$$

In this way L_∞-estimates for e are reduced to L_∞-estimates of the Ritz-method applied to u on the one hand and to $\Delta^{-1}\dot{e}$ on the other and L_∞-estimates of $\Delta^{-1}\dot{e}$. The last term is bounded in L_∞ if $\dot{e}$ belongs to L_p with $p > n/2$. In this way

$$\|e\|_{L_\infty} \leq c\|\dot{e}\|_{L_p} + \ldots$$

is shown. Repeating this argument an estimate of the type

$$\|e\|_{L_\infty} \leq c\|\partial_t^{\nu} e\|_{L_2} + \ldots$$

can be derived. This finally leads to an optimal order of convergence - with respect to the subspaces - , but depending on the dimension the norms of the solution u entering the right hand side are stringent.

Independent of the space-dimension the validity of the estimates can be shown:

$$\|e\|_{L_\infty(L_\infty)} \leq c\, h^m \left\{ \|u\|_{L_\infty(W_\infty^m)} + \|\dot{u}\|_{L_\infty(W_\infty^m)} + \|\ddot{u}\|_{L_2(W_\infty^m)} \right\} \quad .$$

The proofs are given in an article to appear in R.A.I.R.O., anal. numer. Here we discuss the case of $N = 3$ space dimensions in some detail. The general case is only sketched.

1. Notations, L_2-Projection

In the following we will use the standard notations of the theory of partial differential equations. In addition we will work with weighted norms resp. semi-norms introduced three years ago here in Rennes.

For $x_o \in \Omega$ and $\rho > 0$ let

$$\mu = |x-x_o|^2 + \rho^2 \quad .$$

The weighted semi-norms are defined by

$$\|\nabla^k u\|_\alpha^2 = \sum_{|\xi|=k} \iint_\Omega \mu^{-\alpha} \, |D^\xi u|^2 \, dx \quad .$$

The first result of the mentioned paper was related to the L_2-projection:

Theorem 1: The L_2-projection onto a finite-element-space is bounded in the α-norm for any $\alpha \in R$:

$$\|P_h u\|_\alpha \leq c_\alpha \|u\|_\alpha \quad .$$

Generalizing this result in the same way it can be shown.

Theorem 2: Let S_h be a finite-element-space and P_h be the L_2-projection onto S_h . Then

$$\|u-P_h u\|_\alpha + h\|\nabla(u-P_h u)\|_\alpha \leq \inf_{\chi\in S_h} \left\{\|u-\chi\|_\alpha + h\|\nabla(u-\chi)\|_\alpha\right\} .$$

This theorem guarantees the simultaneous approximation property of the L_2-projection with respect to the L_2- and H_1-norm.

2. A priori Estimates in Weighted Norms

We will use the identity

$$D(u,\mu^{-\alpha}u) = \|\nabla u\|_\alpha^2 + \iint u\nabla u\ \nabla\mu^{-\alpha} = \|\nabla u\|_\alpha^2 - \tfrac{1}{2}\iint u^2\Delta\mu^{-\alpha} .$$

Now by direct differentiations we have

$$\Delta\mu^{-\alpha} = - 2\alpha\mu^{-\alpha-2}(N\rho^2 + (N-2\alpha-2)\ r^2 .$$

For $0 < \alpha < N/2 - 1$ therefore $\Delta\mu^{-\alpha}$ is negative and

$$\mu^{\alpha+1}|\Delta\mu^{-\alpha}|$$

is bounded and bounded away from zero:

Lemma 1: Let $0 < \alpha < N/2 - 1$. Then for any $u \in \overset{o}{H}_1$

$$\|u\|_{\alpha+1}^2 + \|\nabla u\|_\alpha^2 \leq c\ D(u,\mu^{-\alpha}u) .$$

The case $\alpha = N/2 - 1$ is of special interest. Then

$$\Delta\mu^{-\alpha} = - N(N-2)\ \rho^2\ \mu^{-\alpha-2} .$$

Lemma 2: Let $\alpha = N/2 - 1$. Then for any $u \in \overset{o}{H}_1$

$$c\ \rho^2\|u\|_{\alpha+2}^2 + \|\nabla u\|_\alpha^2 \leq D(u,\mu^{-\alpha}u) .$$

For finite elements the L_∞-norm is bounded by the weighted norms if x_o and ρ are chosen properly. Especially we have for $\rho = \gamma h$ with γ fixed: Let $\varphi \in S_h$ and $\alpha = N/2 - 1$. Then

$$\|\varphi\|_{L_\infty} \leq \sup_{x_o \in \Omega} \rho \, \|\varphi\|_{\alpha+2} .$$

3. Error Estimates for the Galerkin Method in Case of N = 3 Space Dimensions.

The defining relation of the error $e = u - u_h$ is

$$(\dot{e}, \chi) + D(e, \chi) = 0 \qquad \text{for } \chi \in S_h .$$

Now we introduce the Ritz-approximation $U_h = R_h u$ and use the splitting

$$e = (u - U_h) - (u_h - U_h)$$

$$= \varepsilon - \Phi$$

with $\Phi \in S_h$. Further we assume the initial condition

$$u_h(0) = R_h u_o .$$

Therefore we have

$$\Phi(0) = 0 .$$

The defining relation for Φ is

$$(\dot{\Phi}, \chi) + D(\Phi, \chi) = (\dot{\varepsilon}, \chi) \qquad \text{for } \chi \in S_h .$$

Using the estimates of Section 2 we get for $\alpha = N/2 - 1 = 1/2$

$$c^{-1}\left\{\rho^2\|\Phi\|^2_{\alpha+2} + \|\nabla\Phi\|^2_\alpha\right\} \le D(\Phi,\mu^{-\alpha}\Phi)$$

and

$$D(\Phi,\mu^{-\alpha}\Phi) = D(\Phi,\mu^{-\alpha}\Phi-\chi) - (\dot{\varepsilon}-\dot{\Phi},\mu^{-\alpha}\Phi-\chi) + (\dot{\varepsilon}-\dot{\Phi},\mu^{-\alpha}\Phi) \quad .$$

If we choose $\chi = P_h(\mu^{-\alpha}\Phi)$ the L_2-projection then the middle term vanishes mostly. By pure approximation arguments we get

Lemma

$$\inf_{\chi\in S_h} D(\Phi,\mu^{-\alpha}\Phi-\chi) \le c\left(\frac{h}{\rho}\right)^2\left\{\rho^2\|\Phi\|^2_{\alpha+2} + \|\nabla\Phi\|^2_\alpha\right\} \quad .$$

For $\rho = \gamma\, h$ with a proper γ this gives

$$\rho^2\|\Phi\|^2_{\alpha+2} + \|\nabla\Phi\|^2_\alpha \le c(\dot{\varepsilon}-\dot{\Phi},\mu^{-\alpha}\Phi) \quad .$$

The left hand side is a bound of $\|\Phi\|^2_{L_\infty}$ if x_0 is chosen properly. The right hand side can be estimated by

$$(\dot{\varepsilon}-\dot{\Phi},\mu^{-\alpha}\Phi) \le \|\Phi\|_{L_\infty}\iint\mu^{-\alpha}|\dot{\varepsilon}-\dot{\Phi}| \le \|\Phi\|_{L_\infty}\|\dot{\varepsilon}-\dot{\Phi}\|\left\{\iint\mu^{-2\alpha}\right\}^{1/2}$$

In the case of $N = 3$ we have $2\alpha = 1$ and the last integral is bounded. This gives because of $e = \varepsilon - \Phi$

Theorem 3: Let $N = 3$. Then for any fixed time t

$$\|e\|_{L_\infty} \le c\left\{\|\varepsilon\|_{L_\infty} + \|\dot{\varepsilon}\|_{L_2} + \|\dot{e}\|_{L_2}\right\} \quad .$$

The L_2-error estimates

$$\|e\|_{L_\infty(L_2)} \le \|\varepsilon\|_{L_\infty(L_2)} + c\|\dot{\varepsilon}\|_{L_2(L_2)}$$

are well-known. We apply this with e replaced by $\dot{e}$ and ε replaced by $\dot{\varepsilon}$. In this way we come to

<u>Theorem 4:</u> Let $N = 3$. Then

$$\|e\|_{L_\infty(L_\infty)} \le c\left\{\|\varepsilon\|_{L_\infty(L_\infty)} + \|\dot{\varepsilon}\|_{L_\infty(L_2)} + \|\ddot{\varepsilon}\|_{L_2(L_2)}\right\} .$$

For u sufficiently regular this leads to optimal error estimates of the Galerkin approximation.

4. Two Types of Error Estimates in Arbitrary Dimensions.

Using the splitting $e = \varepsilon - \Phi$ of above we get for arbitrary N of space dimensions with $\alpha = N/2 - 1$ similar to above with $\chi = P_h(\mu^{-\alpha}\Phi)$

$$(\dot{\Phi}, \Phi)_\alpha + \|\nabla\Phi\|_\alpha^2 + k\rho^2\|\Phi\|_{\alpha+2}^2 =$$
$$= D(\Phi, \mu^{-\alpha}\Phi - \chi) - (\dot{\varepsilon}, \mu^{-\alpha}\Phi - \chi) + (\dot{\varepsilon}, \Phi)_\alpha .$$

By approximation arguments we come to

$$(\Phi, \dot{\Phi})_\alpha \le c\left\{\|\Phi\|_\alpha^2 + \|\dot{\varepsilon}\|_\alpha^2\right\}$$

if $\rho \ge \gamma h$ with γ properly chosen. This leads to the interesting a priori inequality

$$\|\Phi(t)\|_\alpha^2 \le \|\Phi(0)\|_\alpha^2 + c\int_0^t \|\dot{\varepsilon}(\tau)\|_\alpha^2\, d\tau .$$

A first - but non-optimal - L_∞-result is the consequence. If x_0 is chosen properly then $\|\Phi\|_\alpha$ is a bound of $h\|\Phi\|_{L_\infty}$. Therefore we get

$$\|\Phi\|^2_{L_\infty(L_\infty)} \leq c\ h^{-2}\|\dot{\varepsilon}\|^2_{L_2(\alpha)} = c\ h^{-2} \int_0^T \|\dot{\varepsilon}\|^2_\alpha\, d\tau\ .$$

The α-norm is bounded by the L_p-norm for $p > N$. Therefore we have

<u>Theorem 5:</u> Let $N > 3$ be arbitrary and $p > N$. Then

$$\|e\|_{L_\infty(L_\infty)} \leq \|\varepsilon\|_{L_\infty(L_\infty)} + c\ h^{-1}\|\dot{\varepsilon}\|_{L_2(L_p)}$$

Similarily we can get

$$\|\dot{\Phi}(t)\|^2_\alpha \leq \|\dot{\Phi}(0)\|^2_\alpha + c \int_0^t \|\ddot{\varepsilon}(\tau)\|^2_\alpha\, d\tau\ .$$

For $\Phi(0) = 0$ we find

$$\|\dot{\Phi}(0)\|_\alpha \leq c\|\dot{\varepsilon}(0)\|_\alpha$$

and in this way

$$\sup \|\dot{\Phi}(t)\|_\alpha \leq \sup \|\dot{\varepsilon}(t)\|_\alpha + c \left\{\int_0^T \|\ddot{\varepsilon}\|^2_\alpha\, d\tau\right\}^{1/2}\ .$$

The link to L_∞-estimates of Φ and therefore of e is

Theorem 6: Let $N > 3$ and $\alpha = N/2 - 1$. For $\rho \geq \gamma h$ with γ chosen properly

$$\|\Phi\|_{\alpha+2} + \|\nabla\Phi\|_{\alpha+1} \leq c \ \rho^{-1} \ \|\dot{\varepsilon} - \dot{\Phi}\|_{\alpha} \ .$$

The proof of this theorem is quite lengthy, the lines of it still being the same as in the paper three years ago.

Since now $\alpha + 2 = N/2 + 1$ for $x_o \in \Omega$ appropriate and $\rho = \gamma h$ we get

$$\|\Phi\|_{L_{\infty}} \leq c \ \rho \ \|\Phi\|_{\alpha+2} \ .$$

This finally gives the error estimate stated at the end of the introduction.

In the mentioned paper to appear in R.A.I.R.O. a detailed bibliography is given. We suppress this here.