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B. MERCIER

J. RAPPAZ

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EIGENVALUE APPROXIMATION

VIA NON-CONFORMING AND HYBRID FINITE ELEMENT METHODS

B. MERCIER

J. RAPPAZ *

(mars 1978)

We give some error estimates for the approximation of eigenvalue problems via abstract mixed-hybrid finite element method.

As a main application, we show how to apply these results to the case of non conforming approximation of a standard 2nd order elliptic problem.

Centre de Mathématiques Appliquées - Ecole Polytechnique (ERA 747 - CNRS)
91128 PALAISEAU CEDEX - FRANCE

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INTRODUCTION.

Let X, W and H be three Hilbert spaces satisfying X included and dense in H, the injection from X into H being continuous. Let a and b be two continuous bilinear forms, defined on $X \times X$ and $X \times W$ respectively.

We consider the following eigenvalue problem :

Find $\lambda \in \mathbb{C}$, $\{u,p\} \in X \times W$

(0.1.a) $$a(u,v) + b(v,p) = \lambda(u,v), \quad \forall\, v \in X ;$$

(0.1.b) $$b(u,q) = 0 , \qquad \forall\, q \in W ;$$

where $(.,.)$ *denotes the scalar product of* H .

The purpose of this work is to study the effect of the approximation when X and W are replaced by finite dimensional spaces X_h and W_h . Let

$$V = \left\{ v \in X \ : \ b(v, q) = 0 \ , \ \forall\, q \in W \right\} ;$$

and

$$V_h = \left\{ v_h \in X_h : b(v_h, q_h) = 0 \ , \ \forall\, q_h \in W_h \right\} ;$$

We notice that Problem (0.1) can be written in a more standard way :

Find $\lambda \in \mathbb{C}$ *and* $u \in V$ *such that*

(0.2) $$a(u,v) = \lambda(u,v), \qquad \forall\, v \in V .$$

The conforming case, where $V_h \subset V$, has been widely studied. We mention STRANG-FIX [16], for the compact and self-adjoint case, BABUSKA-AZIZ [1], BRAMBLE-OSBORN [3], FIX [9], OSBORN [13], for the compact case. For noncompact cases, we refer the reader to RAPPAZ [14], DESCLOUX-NASSIF-RAPPAZ [8]. The case where the right hand side of (0.1) is 0 and $\lambda(p,q)_W$ instead of $\lambda(u,v)$ and 0, has been studied by CANUTO [6], with a somewhat different technique than the one we use here.

We heard that the case where the same right hand side is $\lambda(u,v)$ and $\lambda(p,q)$, instead of $\lambda(u,v)$ and 0, would have been considered by KOLATA [12].

Note that only the present formulation (0.1) can take into account the case of non conforming elements (see RAVIART-THOMAS [15]), or hybrid elements (see BABUSKA [2], BREZZI [4], THOMAS [17]).

In Section 1, we review the abstract error estimates of OSBORN [13], simplified by DESCLOUX-NASSIF-RAPPAZ [8].

In Section 2, we apply the error estimates to the approximation of the abstract problem (0.1).

In Section 3, we specialize to the case of non conforming elements for the Dirichlet problem.

Finally, in Section 4, we add some remarks and comments.

1 - ERROR ESTIMATES FOR EIGENVALUE PROBLEMS.

We shall make a brief review of the general results of OSBORN [13] and DESCLOUX-NASSIF-RAPPAZ [8]. However, we shall derive the estimates for eigenvalue problems in a somewhat restricted framework.

Let H be a real Hilbert space, with the scalar product $(.,.)$ and the norm $|.|$. We denote by $\langle H \rangle$ the Banach space of compact linear operators $A : H \to H$, endowed with the usual operator norm $\|A\| \equiv \sup_{\substack{u \in H \\ |u|=1}} |Au|$.

Let $A \in \langle H \rangle$ and $\|A_h\| \subset \langle H \rangle$ be a family of operators indexed by a positive parameter h , such that

(1.1) A and A_h are self-adjoint (h arbitrary) ;

(1.2) $\lim_{h \to 0} \|A - A_h\| = 0$.

The compactness and the self-adjointness of A and A_h show that their spectrum has no other accumulation point than the origin, and is located on the real line.

Any value $\lambda \neq 0$ of the spectrum of A is an eigenvalue of A with a finite multiplicity* m , and can be isolated from the spectrum : there exists $\varepsilon > 0$ such that the interval $[\lambda-\varepsilon,\lambda+\varepsilon]$ does not contain any other eigenvalue of A than λ .

We shall denote by E the eigenspace associated to λ (and A), and by $P : H \to E$, the orthogonal projection operator onto E .

Assumption (1.2) shows that, for $h < h_0$ sufficiently small, there exists exactly m eigenvalues of A_h lying in the interval $[\lambda-\varepsilon,\lambda+\varepsilon]$. We call those $\lambda_{1h},\ldots,\lambda_{mh}$. Note that some of them may coincide since we count them according to their multiplicity. We call E_h the sum of all the eigenspaces corresponding to the λ_{ih} , and $P_h : H \to E_h$, the orthogonal projection operator onto E_h . Note that $\dim E = \dim E_h = m$, and that

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* *Note that algebraic and geometric multiplicity coincide for self-adjoint operators.*

$$\lim_{h \to 0} \| P - P_h \| = 0 \quad \text{(see [10,p.438])}.$$

Finally, we introduce

$$\| (A - A_h)\big|_E \| = \sup_{\substack{u \in E \\ |u|=1}} |(A - A_h)\, u| ,$$

and recall the following result due to OSBORN [13] :

<u>*THEOREM 1*</u> *: Assuming that (1.1) and (1.2) hold, there exist* C *and* $h_0 > 0$ *such that, for* $h \leqslant h_0$

(1.3) $$\| \lambda - \lambda_{ih} \| \leqslant C \left\{ \| (A - A_h)\big|_E \|^2 + \sup_{\substack{u \in E \\ |u|=1}} \ \sup_{\substack{v \in E \\ |v|=1}} |(Au - A_h u, v)| \right\} ;$$

(1.4.a) $$\sup_{\substack{u \in E \\ |u|=1}} |u - P_h u| \leqslant C \ \| (A - A_h)\big|_E \| ;$$

(1.4.b) $$\sup_{\substack{u_h \in E_h \\ |u_h|=1}} |u_h - P u_h| \leqslant C \ \| (A - A_h)\big|_E \| . \quad \blacksquare$$

For the convenience of the reader, we shall give a simplified proof of this result.

<u>Proof</u> : Let $\Lambda_h : E \to E_h$ be the restriction of P_h to E, i.e. $\Lambda_h = P_h\big|_E$. For $u \in E$, we have

$$|\Lambda_h u| \geqslant |u| - |(P - P_h)u| \geqslant (1 - \| P - P_h \|)\, |u| \quad ;$$

therefore, for h sufficiently small, Λ_h is one to one from E onto E_h, and has a uniformly bounded inverse Λ_h^{-1}; there exists C_1 independent on h and $h_0 > 0$, such that

(1.5) $$|\Lambda_h^{-1} u_h| \leqslant C_1 \ |u_h| \ , \text{ for } u_h \in E_h \text{ and } h \leqslant h_0 \ .$$

We define the operators $\hat{A}$ and $\hat{A}_h : E \to E$ by

$$\hat{A} = A\big|_E \quad \text{(restriction of } A \text{ to } E),$$
$$\hat{A}_h = \Lambda_h^{-1} A_h\big|_{E_h} \Lambda_h .$$

We check that λ is the unique eigenvalue of $\hat{A}$ and $(\lambda_{ih})_{1\leq i\leq m}$ are the eigenvalues of $\hat{A}_h$ (however, the eigenvectors of $\hat{A}_h$ have changed and are not orthogonal any more).

From WILKINSON [18,p.80], we have

$$|\lambda - \lambda_{ih}| \leq c_2 \, \|\hat{A} - \hat{A}_h\|_E \, ,$$

where the norm is the operator norm on E .

To set (1.3), we shall estimate the quantity $\|\hat{A} - \hat{A}_h\|_E$. For $u,v \in E$, we have

$$\begin{aligned}(\hat{A}u - \hat{A}_h u,v) &= (Au - \Lambda_h^{-1}A_h P_h u ,v) = (Au - \Lambda_h^{-1}P_h A_h u ,v) \\ &= (\Lambda_h^{-1}P_h (A - A_h)u,v) \\ &= ((A - A_h)u,v) + ((\Lambda_h^{-1}P_h - I)(A - A_h)u,v).\end{aligned}$$

Moreover, as

$$((\Lambda_h^{-1}P_h - I)(A - A_h)u,P_h v) = (P_h(\Lambda_h^{-1}P_h - I)(A - A_h)u,v) = 0 ,$$

we get

$$(\hat{A}u - \hat{A}_h u,v) = (Au - A_h u,v) + ((\Lambda_h^{-1}P_h - I)(A - A_h)u,v - P_h v) .$$

Using (1.5), we get

$$\|\hat{A} - \hat{A}_h\|_E \leq c \left\{ \sup_{\substack{u \in E \\ |u|=1}} \sup_{\substack{v \in E \\ |v|=1}} |(Au - A_h u,v)| + \|(A - A_h)\big|_E\| \, . \sup_{\substack{v \in E \\ |v|=1}} |v - P_h v| \right\} .$$

In order to complete the proof of (1.3), we need (1.4.a) which we prove now. In this purpose, the easiest way is to extend the Hilbert space H from the real to the complex case. Let Γ denote the circle of the complex plane C centered at λ and with radius ε . The projection operators P and P_h can be written as Dunford integrals :

$$P = \frac{1}{2\pi i} \int_\Gamma (z - A)^{-1} \, dz ,$$

$$P_h = \frac{1}{2\pi i} \int_\Gamma (z - A_h)^{-1} \, dz .$$

For $u \in E$, we have

$$|u - P_h u| = |(P - P_h)u| = \frac{1}{2\pi} \left| \int_\Gamma \left[(z - A)^{-1} - (z - A_h)^{-1}\right] u \, dz \right|$$

$$= \frac{1}{2\pi} \left| \int_\Gamma (z - A_h)^{-1} (A - A_h)(z - A)^{-1} u \, dz \right| .$$

We check easily that assumption (1.2) implies the existence of C_3 and $h_1 \geq 0$ such that, for $h \leq h_1$ and $z \in \Gamma$:

$$\| (z - A_h)^{-1} \| \leq C_3 ,$$

$$\| (z - A)^{-1} \| \leq C_3 ;$$

as $(z - A)^{-1} u \in E$, we have

$$|u - P_h u| \leq C_3^2 \, \| (A - A_h)|_E \| \, . \, |u| ,$$

which sets (1.4.a) ; the derivation of (1.4.b) uses the same techniques. ■

2 - ESTIMATES FOR ABSTRACT NON CONFORMING PROBLEMS.

We shall apply the previous estimates to the abstract problem considered in Introduction.

We make the following assumptions :

(2.1) $a(.,.)$ *is symmetrical and coercive on* V: *there exists* $\alpha > 0$ *such that* $a(v,v) \geqslant \alpha \|v\|_V^2$, $\forall v \in V$;

(2.2) *the imbedding* $V \to H$ *is compact* ;

(2.3) *there exists* $\beta > 0$ *such that*

$$\inf_{\substack{q \in W \\ |q|_W = 1}} \sup_{\substack{v \in X \\ |v|_X = 1}} b(v,q) \geqslant \beta .$$

These assumptions imply the continuity of the operators

$$A : H \to V \quad ; \quad B : H \to W ,$$

where, for $f \in H$, $Af \in V$ and $Bf \in W$ are uniquely defined by the relation

(2.4) $$a(Af,z) + b(z,Bf) = (f,z), \ \forall z \in W.$$

Note that A is self-adjoint since a is symmetrical ; moreover, A is compact in view of (2.2).

The eigenvalue problem (0.2) (and then (0.1)) is equivalent to the eigenvalue problem $Au = \lambda u$ which has been considered in the previous section, with the operator A defined in (2.4). Note that λ is indeed the inverse of the λ appearing in (0.1.a).

The approximate operator $A_h : H \to V_h$ is defined together with $B_h : H \to W_h$ in the following way : for $f \in H$, $A_h f \in V_h$ and $B_h f \in W_h$ are solution of

(2.5) $$a(A_h f, z_h) + b(z_h, B_h f) = (f, z_h).$$

We assume that A_h converges to A in $\langle H \rangle$, which requires some compatibility between X_h and W_h (see [4], [5], [17]).

We want to derive some error estimates for the approximation of an eigenvalue λ of A. For this respect, we apply the abstract estimates (1.3) and (1.4), where only the term $(Au - A_h u, v)$ needs to be evaluated more specifically.

THEOREM 2 : We have the estimate for $f, g \in H$:

$$|(Af - A_h f, g)| \leq C \left\{ \|(A - A_h)g\|_X \left(\|(A - A_h)f\|_X + \|(B - B_h)f\|_W \right) + \|(A - A_h)f\|_X \; \|(B - B_h)g\|_W \right\} .$$

Proof : We apply (2.4) to g with $z = (A - A_h)f$. We get

(2.6) $$(Af - A_h f, g) = a(Ag, (A - A_h)f + b((A - A_h f), Bg) ;$$

substracting (2.4) and (2.5), we get, for all $z_h \in X_h$,

(2.7) $$a((A - A_h)f, z_h) + b(z_h, (B - B_h)f) = 0 .$$

As $Ag \in V$, we also have

$$b(Ag, (B - B_h)f) = 0 ;$$

with (2.7), this implies

(2.8) $$a(Ag, (A - A_h)f) = a(Ag - z_h, (A - A_h)f) + b(Ag - z_h, (B - B_h)f).$$

Choosing $z_h = A_h g$ and using the continuity of a and b, this gives

$$|a(Ag, (A - A_h)f)| \leq C \; \|(A - A_h)g\|_X \left(\|(A - A_h)f\|_X + \|(B - B_h)f\|_W \right) .$$

On the other hand, as $Af \in V$ and $A_h f \in V_h$, we have

$$b((A - A_h)f, q_h) = 0 \;, \; \forall \, q_h \in W_h \;;$$

hence

$$b((A - A_h)f, Bg) = b((A - A_h)f, (B - B_h)g) \;,$$

where we have chosen $q_h = B_h g$. This gives

$$| \, b(A - A_h)f, Bg)| \leq C \, \|(A - A_h)f\|_X \; \|(B - B_h)g\|_W \;,$$

hence, (with (2.6), (2.8)), the desired result. ∎

As in Section 1, we shall consider an eigenvalue λ of A with multiplicity m, and the eigenvalues $(\lambda_{ih})_{1 \leq i \leq m}$ of A_h converging to λ. We denote again by E and E_h the corresponding eigenspaces.

We shall use the following norm for a linear operator $C : Y \to Z$:

$$\| C \|_{YZ} \equiv \sup_{\substack{u \to Y \\ \| u \|_Y = 1}} \| Cu \|_Z .$$

As a consequence of Theorems 1 and 2, we have

<u>*THEOREM 3*</u> *: Under assumption (1.2), one has the following estimate*

$$|\lambda - \lambda_{ih}| \leq C \, \| (A - A_h)|_E \|_{HX} \left\{ \| (A - A_h)|_E \|_{HX} + \| (B - B_h)|_E \|_{HW} \right\} ,$$

for $1 \leq i \leq m$. ■

<u>REMARK 2.1</u> : In the conforming case $(V_h \subset V)$, the quantity $\| (B - B_h)|_E \|_{HW}$ disappears in the previous result. ■

3 - APPLICATION TO NON CONFORMING ELEMENTS FOR THE DIRICHLET PROBLEM (SEE THOMAS [17]).

Let $\Omega \subset \mathbb{R}^d$ be a bounded open set of $\mathbb{R}^d$ (d=2,3). For the sake of simplicity, we assume that Ω is a polygone. We call $\mathcal{T}_h$ a family of triangulations of Ω. The elements of $\mathcal{T}_h$ are triangles in the case d=2, and tetraedrons in the case d=3.

For K an open set of $\mathbb{R}^d$, we denote by $H^m(K)$ the Sobolev space of functions, the derivative of which, up to the order m are in $L^2(K)$, and by $\|\cdot\|_{m,K}$, the associated norm (note that $\|\cdot\|_{0,K}$ is the norm of $L^2(K)$).

We choose $H = L^2(\Omega)$ and $X = \prod_{K \in \mathcal{T}_h} H^1(K)$ with the norm $\|u\|_X \equiv \left(\sum_{K \in \mathcal{T}_h} \|u\|^2_{1,K}\right)^{\frac{1}{2}}$. To define the space W, we introduce

$$H(\mathrm{div},\Omega) = \left\{q \in (L^2(\Omega))^d : \mathrm{div}\, q \in L^2(\Omega)\right\},$$

where div is the divergence operator, and the following relation of equivalence

(R) $\quad p$ *is equivalent to* $q \iff p.n = q.n$ *on* ∂K *for all* $K \in \mathcal{T}_h$,

where n denotes the normal vector to the boundary ∂K of the element K, outwards directed. We choose then $W = H(\mathrm{div};\Omega)/\mathcal{R}$.

As a norm for $H(\mathrm{div};\Omega)$, we choose the following

$$\|q\|_{H(\mathrm{div};\Omega)} \equiv \sum_{K \in \mathcal{T}_h} \left(\|q\|^2_{0,K} + h_K \|\mathrm{div}\, q\|^2_{0,K}\right)^{\frac{1}{2}},$$

where h_K denotes the diameter of an element K.

We endow W with the corresponding quotient norm, and assume that $h = \max_{K \in \mathcal{T}_h} h_K$.

As for the bilinear forms, we choose

$$a(u,v) = \sum_K \int_{\partial K} \nabla u . \nabla v \quad dX \quad ,$$

$$b(u,v) = \sum_K \int_{\partial K} u(q.n) \; ds \quad ,$$

where the integral on ∂K is indeed the duality between $H^{\frac{1}{2}}(\partial K)$ and $H^{-\frac{1}{2}}(\partial K)$.

We notice that $V = H_0^1(\Omega) \equiv \left\{ v \in H^1(\Omega) \; : \; v = 0 \;\; \text{on} \;\; \partial\Omega \right\}$, and $u = Af$ is the solution of the Dirichlet problem

$$\begin{cases} -\Delta u = f & \text{on} \quad \Omega \; , \\ \quad u = 0 & \text{on} \quad \partial\Omega \; . \end{cases}$$

For the approximate problem, we choose $X_h = \prod_{K \in T_h} \mathbb{P}_k(K)$, where $\mathbb{P}_k(K)$ denotes the set of functions defined on K which are polynomials of degree less or equal to k.

We call $S_k(K)$ the subset of $(\mathbb{P}_k(K))^d$ made with vector functions q, *with a normal trace* $q.n$ *of degree less of equal to* $k-1$ *on each side of the element* K.

We define W_h as the subspace of W made with vector functions the restriction of which to each $K \in T_h$ is in $S_k(K)$. Note that the normal trace of such a function has no discontinuity across a side common to two elements since W is included in $H(\text{div};\Omega)$.

We notice that V_h *is not included in* $H_0^1(\Omega)$, and constitutes a so-called *non conforming finite element space.*

We recall the following result due to RAVIART-THOMAS [15,thm 4 & 5] :

<u>*THEOREM 4*</u> *: If the family of triangulations* T_h *is regular (that is, no angle tends to zero as* h *tends to zero), there exists a constant* C *independent of* h *such that*

$$\| (A - A_h) f \|_X + \| (B - B_h) f \|_W \leq C \, h^{\ell} \, \| f \|_{\ell-1,\Omega}$$

for all $\ell = 1,2,\ldots,k$ *, such that* $Af \in H^{\ell+1}(\ell)$. ■

We also have the following L -error estimates proved by THOMAS [17, chapter V, Thm 4.4] :

THEOREM 5 : Under the same assumptions as in Theorem 4, provided that Ω *is convex, we have*

$$\| (A - A_h) f \|_{0,\Omega} \leqslant C\, h^{\ell+1} \, \| f \|_{\ell-1,\Omega}$$

for all $\ell=1,2,\ldots,k$ *, such that* $Af \in H^{\ell+1}(\Omega)$.

For $f \in L^2(\Omega)$, Grisvard's regularity results for the Dirichlet problem, give $Af \in H^2(\Omega)$. Theorem 5 with m=1 shows the uniform convergence of A_h ot A and the validity of assumption (1.2) ; (2.2) results from the compactness of Sobolev's imbedding and implies the compactness of A .

THEOREM 6 : Assuming that the eigenspace E *, associated to an eigenvalue* λ *of* A *, is included in* $H^{\ell+1}(\Omega)$ *with* $1 \leqslant \ell \leqslant k$*, then there exists a constant* C *such that*

$$|\lambda - \lambda_{ih}| \leqslant C\, h^{2\ell} \quad \textit{for } i=1,\ldots,m\,.,$$

where m *denotes the multiplicity of* λ *and* λ_{ih} *the eigenvalues of* A_h *converging to* λ *. Moreover,* E_h *being the eigenspace associated to the* $(\lambda_{ih})_{1 \leqslant i \leqslant m}$ *, we have the following bound on the distance between* E *and* E_h

$$\sup_{\substack{u \in E \\ \|u\|=1}} \; \inf_{u_h \in E_h} \| u - u_h \|_0 \leqslant C\, h^{\ell+1} .$$

Proof : Note first that the norms $\| . \|_{0,\Omega}$ and $\| . \|_{\ell-1,\Omega}$ are equivalent on E which is finite dimensional.

From Theorems 4 and 5, we get

$$\| (A - A_h)|_E \|_{HX} \leqslant \sup_{\substack{f \in E \\ \|f\|_0=1}} \| (A - A_h) f \|_X \leqslant C\, h^{\ell} \;;$$

$$\| (A - A_h)\big|_E \|_{HH} \leqslant C\, h^{\ell+1} \quad ;$$

$$\| (B - B_h)\big|_E \|_{HW} \leqslant C\, h^{\ell} \quad .$$

The result follows then from Theorems 1 and 3. ■

4 - REMARKS AND COMMENTS.

1 - The estimates given in Theorem 6 are the same as in the conforming case, and give then the optimal order.

2 - To give an example of a polygonal domain, where the eigenvectors are in $H^{k+1}(\Omega)$ for k large enough, we mention the case where Ω is a square, where the eigenvectors are known explicitly.

3 - To get the same estimates in the case where Ω is not a polygone, it is necessary to use isoparametric elements, and we refer the reader to THOMAS [17] for a definition of the spaces X_h and W_h in this case.

4 - Estimate of the distance between E and E_h in the X-norm. Under the same assumptions as Theorems 6, we set the estimate

$$\sup_{\substack{u \in E \\ \|u\|_X=1}} \quad \inf_{u_h \in E_h} \|u-u_h\|_X \leq C\, h^{\ell} .$$

For simplicity, assume that $m=1$. For $u \in E$, we let $u_h = P_h u \in E_h$. We have $u = \frac{1}{\lambda} Au$; $u_h = \frac{1}{\lambda_h} A_h u_h$, hence

$$u-u_h = \frac{1}{\lambda}(A-A_h)u + \frac{\lambda_h-\lambda}{\lambda_h \lambda} A_h u + \frac{1}{\lambda_h} A_h(u-u_h).$$

We use then the uniform continuity of A_h ,

$$\|A_h f\|_X \leq C \|f\|_{0,\Omega} \qquad \text{for all } f \in H ,$$

which results from (2.5), and the discrete Poincaré inequality proved by THOMAS [17, chap.V].

5 - Other examples. The abstract framework considered in this paper contains the case of hybrid and mixed elements (see THOMAS [17], BREZZI-RAVIART [5], BREZZI [4]). A non conforming method for 4th order value problems has been considered by KIKUCHI [11], and the convergence of eigenvalues is proved by other methods. Eigenvalue problems for the Stokes equations are also a special case of our work when one considers the formulations of CROUZEIX-RAVIART [7].

6 - Unsymmetric case. It is also possible to extend the present method to the case of a nonsymmetric bilinear form a. However, the operator A is not self-adjoint any more. The method is not as simple as in the self-adjoint case. However, one gets the same estimate as in Theorem 6, but for the distance between λ and the average of λ_{ih}. We refer the interested reader to the work of OSBORN [13] to pick up the techniques for the non self-adjoint case. Note that BREZZI, RAVIART and THOMAS' results still hold when the bilinear form a is unsymmetric.

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