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LEAST SQUARES APPROXIMATIONS TO FIRST ORDER ELLIPTIC SYSTEMS

by

J. A. Nitsche

Summary: Linear boundary value problems for elliptic systems in the sense of Petrowski are considered. Using linear finite elements a least squares method is discussed. The concept of nearly zero boundary conditions - i.e. the boundary condition is imposed in the nodes on the boundary exactly - gives quasi-optimal error estimates in the L_2- and W_2^1-norms.

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1. Notations, the Analytic Problem

In the following $u, v, \dots$ will denote pairs (u^1, u^2), $(v^1, v^2), \dots$ of functions defined in a bounded domain $\Omega \subseteq R^2$ with boundary $\partial\Omega$ sufficiently smooth. If both components are in $L_2(\Omega)$ resp. the Sobolev-spaces $W_2^k(\Omega)$ we will write $u \in H_o$ resp. $u \in H_k$. We will also use the notation

$$(1)\qquad \begin{aligned} (u,v) &= (u^1,v^1)_{L_2(\Omega)} + (u^2,v^2)_{L_2(\Omega)} , \\ (u,v)_k &= (u^1,v^1)_{W_2^k(\Omega)} + (u^2,v^2)_{W_2^k(\Omega)} \end{aligned}$$

and

$$(2)\qquad \|u\| = (u,u)^{1/2} , \quad \|u\|_k = (u,u)_k^{1/2} .$$

As a model problem we will consider the elliptic system

$$(3)\qquad L\,u = f \qquad \text{in } \Omega ,$$

i.e.

$$(3)\qquad \left.\begin{aligned} L^1 u &= u^1_x - u^2_y + a^{11}u^1 + a^{12}u^2_y = f^1 \\ L^2 u &= u^1_y + u^2_x + a^{21}u^1 + a^{22}u^2 = f^2 \end{aligned}\right\} \text{ in } \Omega .$$

We remark that any elliptic system in the sense of Petrovski is - up to a coordinate-transformation-equivalent to (3), see

HAACK-HELLWIG [1] . In addition to (3) we impose the boundary condition

$$(4) \qquad l^1(u) = u^1 \cos \sigma + u^2 \sin \sigma = 0 \qquad \text{on} \quad \partial\Omega \ .$$

Essential for the solvability of the boundary value problem is the index of $\sigma = \sigma(s)$ - s arc-length on $\partial\Omega$ - defined by

$$(5) \qquad \operatorname{ind}(\sigma) = \frac{1}{2\pi} \oint_{\partial\Omega} \partial\sigma \ .$$

In case of $n = \operatorname{ind}(\sigma) \geq 0$ then (3), (4) is always solvable, the number of linear independent solutions of the homogeneous problem $(f = 0)$ is $2n + 1$. In case of $n < 0$ (3), (4) possesses a solution only if f fulfills $2|n|-1$ linear independent integral relations.

In order to characterize in case of $n \geq 0$ a special solution to following way is possible: Let $\mathfrak{P}_n(\partial\Omega)$ be a subspace of $L_2(\partial\Omega)$ of dimension $2n + 1$ similar to the space of trigonometric functions of order n . This means there is up to a factor exactly one element in $\mathfrak{P}_n(\partial\Omega)$ with zeros in $2n$ prescribed points on $\partial\Omega$, not necessarily distinct. Then there is one and only one solution of (3), (4) such that

$$(6) \qquad \oint_{\partial\Omega} p_i \, l^2(u) \, ds = r_i \qquad (i = 1,\dots,2n+1) \ .$$

Here

$$(7) \qquad l^2(u) = -u^1 \sin \sigma + u^2 \cos \sigma$$

denotes the orthogonal complement of the boundary condition (4), the set $\{p_i\}$ forms a basis of $\mathfrak{P}_n(\partial\Omega)$ and $\{r_i\}$ are fixed real numbers.

In case of a non-negative index n there are especially $2n + 1$ solutions u_j of (3), (4) with $f = 0$ according to

(8) $$\oint_{\partial\Omega} p_i \, l^2(u_j) ds = \delta_{ij} \quad .$$

For negative indices $n = \text{ind } (\sigma)$ on the other hand there exists one and only one solution of (3) such that

(9) $$l^1(u) \in \mathfrak{P}_{2|n|-i} \quad .$$

In the following we will impose the conditions (6) for non-negative indices of σ and use the weakened form (9) for negative indices.

For functions $p, q \in L_2(\partial\Omega)$ we will use

(10) $$\begin{aligned} < p,q > &= \oint p \, q \, ds \quad , \\ |p| &= < p,p >^{1/2} \quad . \end{aligned}$$

The conditions (6) may be rewritten:

(6') $$< p_i, l^2(u) > = r_i \qquad (i = 1,\ldots,2n+1) \quad .$$

For $n = \text{ind } (\sigma) \geq 0$ - only this case will be considered here - let H_1^σ be the set of pairs $u = (u^1,u^2)$ with $u \in H_1$ and fulfilling (4) and $H_1^{\sigma.r}$ the subspace fulfilling (6).

For the sake of simplicity the coefficients a^{ij} in (3) as well as σ in (4) are assumed sufficiently smooth. Without loss of generality the elements of $\mathfrak{P}_n(\partial\Omega)$ can be chosen sufficiently smooth and therefore $\mathfrak{P}_n(\partial\Omega)$ can be extended to a space $\mathfrak{P}_n(\overline{\Omega})$ of functions defined in $\overline{\Omega}$.

For the modified boundary vlaue problems (3), (4), (6) in case of a non-negative index resp. (3), (9) in case of a negative index shift-theorems of the type

$$\|u\|_{k+1} \leq c \left\{\|f\|_k + \Sigma\, |r_i|\right\} \tag{11}$$

are valid. c depends besides of σ , a^{ik} on $\partial\Omega$ and k .

The statements of these sections are consequences of the theory on elliptic systems developed by VEKUA [3] .

2. Finite Element Spaces

Let Γ_h be a subdivision of Ω into generalized triangles Δ, i.e. Δ is a triangle if $\bar{\Delta}$ and $\partial\Omega$ have in common at most one point and otherwise one of the sides of Δ may be curved. We will only consider regular subdivisions: For $\varkappa > 1$ fixed there are to any $\Delta \in \Gamma_h$ two circles with radii $\varkappa^{-1}h$ and $\varkappa h$ contained in Δ resp. containing Δ.

The finite element spaces $S_h = S_h(\Gamma_h)$ we will work with consist of pairs $\chi = (\chi^1, \chi^2)$ of functions having the properties

i) $\chi \in H_1$, i.e. $\chi^i \in W_2^1(\Omega)$,

ii) χ^i restricted to any $\Delta \in \Gamma_h$ is linear.

Let $\{P_\nu\}$ be the set of nodes of Γ_h on $\partial\Omega$. The subspace $S_h^\sigma \subseteq S_h$ consists of those elements with

$$(12) \qquad l^1(\chi)_{|P_\nu} = 0 \quad .$$

Finally $S_h^{\sigma.r} \subseteq S_h^\sigma$ consists of elements with

$$(13) \qquad < p_i \, , \, l^2(\chi) > = r_i \qquad (i = 1, \ldots, 2n+1) \, .$$

Then the following approximation property holds:

<u>Lemma:</u> Let $n = \text{ind}\,(\sigma) \geq 0$. There is a linear projection-operator $Q_h = Q_h^{\sigma.r} : H_1^{\sigma.r} \to S_h^{\sigma.r}$

with

$$(14) \qquad \|u-Q_h u\|_1 \leq c\ h\ \|u\|_2$$

for $\qquad u \in H_2^{\sigma \cdot r} = H_1^{\sigma \cdot r} \cap H_2 \quad .$

First let I_h denote the linear interpolation. For $u \in H_2^{\sigma}$ we have obviously $I_h u \in S_h^{\sigma}$ and

$$(15) \qquad \|u-I_h u\|_l \leq c\ h^{2-l}\|u\|_2 \qquad (l = 0,1) \quad .$$

Further we get

$$(16) \qquad |r_i - \langle p_i\ ,\ l^2(I_h u) \rangle| \leq |p_i|\ |l^2(u-I_h u)|$$

and because of

$$(17) \qquad |z|^2 \leq \|z\|_{L_2(\Omega)}\ \|z\|_{W_2^1(\Omega)}$$

for any $z \in W_2^1(\Omega)$ therefore

$$(18) \qquad |r_i - \langle p_i\ ,\ l^2(I_h u) \rangle| \leq c\ h^{3/2}\ \|u\|_2 \quad .$$

In order to get Q_h we add a proper combination of the interpolated homogeneous solutions of (8)

$$(19) \qquad Q_h u = I_h u + \sum_{i=1}^{2n+1} \alpha_i\ I_h u_i \quad .$$

The conditions on $\{\alpha_i\}$ are

(20) $$\sum_{i=1}^{2n+1} < p_j \, , \, l^2(I_h u_i) > \alpha_i = < p_j, \, l^2(u-I_h u) >$$

$$(j = 1,\ldots,2n+1) \quad .$$

Because of

(21) $$|l^2(u_i - I_h u_i)| \leq c \; h^{3/2}$$

and (8) the inverse of the matrix of the linear equations (20) is bounded away from zero for h small enough. Using the bound (18) for the right hand side of (20) we get

(22) $$|\alpha_i| \leq c \; h^{3/2} \; \|u\|_2 \quad .$$

Therefore

(23) $$\|u - Q_h u\|_1 \leq \|u - I_h u\|_1 + \text{Max} \left\{ \|I_h u_i\|_1 \right\} \sum_{1}^{2n+1} |\alpha_i|$$

which gives (14) .

3. Least Squares Method, Error Estimates

For $u \in H_1$ and $v \in H_1$ resp. $w \in H_0$ let us define the bilinear functionals

$$(24)\qquad \begin{aligned} a(u,v) &= (Lu,\ Lv) \\ &= (L^1u,\ L^1v) + (L^2u,\ L^2v) \end{aligned}$$

resp.

$$(25)\qquad \begin{aligned} b(u,w) &= (Lu,\ w) \\ &= (L^1u,\ w^1) + (L^2u,\ w^2) \quad . \end{aligned}$$

Obviously we have

$$(26)\qquad a(u,v) = b(u,\ Lv) \quad .$$

We get by partial integration - for $w \in H_1$ -

$$\begin{aligned} b(u,w) = (u,\ {}^*Lw) &+ \\ &+ \oint l^1(u)\left\{w^1 \sin\,(\sigma+\gamma) - w^2 \cos(\sigma+\gamma)\right\}ds \\ &+ \oint l^2(u)\left\{w^1 \cos\,(\sigma+\gamma) + w^2 \sin(\sigma+\gamma)\right\}ds \end{aligned}$$

with γ denoting the angle between the tangent at a point of $\partial\Omega$ and the x-axis, and

$$(28)\quad \begin{aligned} {}^{*}L^{1}w &= -w^{1}_{x} - w^{2}_{y} + a^{11}w^{1} + a^{21}w^{2}, \\ {}^{*}L^{2}w &= w^{1}_{y} - w^{2}_{x} + a^{12}w^{1} + a^{22}w^{2} \end{aligned}$$

being the adjoint of the differential operator L.

If

$$(29)\qquad n = \operatorname{ind}(\sigma) \geq 0$$

then the index of the boundary condition

$$(30)\qquad w^{1}\cos(\sigma+\gamma) + w^{2}\sin(\sigma+\gamma) = 0$$

with respect to the operator ${}^{*}L$ - w^{1} and w^{2} have to be interchanged in order to give the Cauchy-Riemann principle part - is

$$(31)\quad \begin{aligned} n^{*} &= \operatorname{ind}(\sigma^{*}) \\ &= \operatorname{ind}\left(\tfrac{\pi}{2}-\sigma-\gamma\right) = -n-1 . \end{aligned}$$

According to (9) then (30) has to be modified

$$(32)\qquad w^{1}\cos(\sigma+\gamma) + w^{2}\sin(\sigma+\gamma) \in P_{2n+1} \qquad \text{on } \partial\Omega$$

in order that

$$(33)\qquad {}^{*}Lw = g \qquad\qquad \text{in } \Omega$$

together with (32) has a unique solution.

For simplicity we will consider the boundary value problem (3), (4), (6) only with $r_i = 0$. Then we have the duality relation

$$(34) \qquad (u,g) = b(u,w) \quad .$$

Further let v be defined by

$$(35) \qquad \begin{aligned} L \; v &= w && \text{in } \Omega \;, \\ l^1(v) &= 0 && \text{on } \partial\Omega \;, \\ < p, \; l^2(v) > &= 0 && \text{for } p \in P_{2n+1} \;. \end{aligned}$$

Then we have

$$(36) \qquad (u,g) = a(u,v) \quad .$$

Using the shift theorem (11) we get $v \in H_{k+2}$ for $g \in H_k$.

In order to approximate the solution u of (3), (4), (6) - with $r_i = 0$ - we use the least squares method: The approximation $u_h \in S_h^{\sigma \cdot o}$ is defined by

$$(37) \qquad a(u_h, \chi) = (f, L\chi) \qquad \text{for } \chi \in S_h^{\sigma \cdot o} \quad .$$

Though $a(.,.)$ is positive definite in $H_1^{\sigma \cdot o}$ it might only be semi-definite in $S_h^{\sigma \cdot o}$. With $e = u - u_h$ and - using an appropriate approximation U_h on u in $S_h^{\sigma \cdot o}$ - $\varepsilon = u - U_h$ and therefore $e = \varepsilon + \Phi$ with $\Phi = U_h - u_h \in S_h^{\sigma \cdot o}$ we get

(38) $$a(e,\chi) = 0 \qquad \text{for} \quad \chi \in S_h^{\sigma,0}$$

resp.

(39) $$a(\Phi,\chi) = -a(\varepsilon,\chi) \qquad \text{for} \quad \chi \in S_h^{\sigma,0} \quad .$$

By

(40) $$\|\cdot\|' = a(.,.)^{1/2}$$

a semi-norm is defined. Obviously we have for $v \in H_1$

(41) $$\|v\|' \leq c \;\|v\|_1 \quad .$$

So we get from (39)

(42) $$\begin{aligned} \|\Phi\|' &\leq \|\varepsilon\|' \\ &\leq c \;\|\varepsilon\|_1 \leq c \; h \;\|u\|_2 \end{aligned}$$

and consequently

(43) $$\|e\|' \leq 2 \; c \; h \;\|u\|_2 \quad .$$

Next we identify $g = e$ in (34) and let w resp. $\mathbf{v}$ be the solutions of (32), (33) resp. (35). Then we get

$$\|e\|^2 = (e, {}^*Lw)$$

(44)

$$= (Le,w) - \oint \left\{l^1(e)\ {}^*l^2(w) - l^2(e)\ {}^*l^1(w)\right\} ds \quad .$$

Because of (13) and (32) the last term on the right hand side vanishes. Therefore we get

$$(45) \qquad \|e\|^2 = (Le,Lv) - \oint l^1(e)\ {}^*l^2(w)\ ds \quad .$$

We will estimate the two terms separately. Using the shift-theorem (11) we find $v \in H_2$ and $\|v\|_2 \leq c\|e\|$. With an appropriate approximation $\chi \in S_h^{\tau.o}$ on v we get with (38), (43)

$$(46) \qquad \begin{aligned} (Le,Lv) &= a(e,\ v-\chi) \\ &\leq \|e\|'\ c\ h\|v\|_2 \\ &\leq c\ h^2\|u\|_2\ \|e\| \quad . \end{aligned}$$

In order to find a bound for the second term in (45) we first notice - using (17) -

$$(47) \qquad |{}^*l^2(w)| \leq c\|e\| \quad .$$

Next we make use of conditions (12) which play the role of 'nearly zero boundary conditions' introduced in [2] . With arguments parallel to there we get

<u>Proposition 1:</u> To any $v \in H_1^{\tau.o} \cap H_2$ there is a $\chi \in S_h^{\tau.o}$ according to

$$(48) \qquad |l^1(v-\chi)| \leq c\ h^2\|v\|_2 \quad .$$

<u>Proposition 2:</u> For any $\chi \in S_h^{\sigma,0}$ nearly zero boundary conditions of the type

$$(49) \qquad |l^1(\chi)| \leq c\ h^{3/2}\ \|\chi\|_1$$

hold.

For κ-regular triangulations inverse properties

$$(50) \qquad \|\chi\|_1 \leq c\ h^{-1}\ \|\chi\|$$

hold for $\chi \in S_h$. Therefore (49) can be replaced by

$$(51) \qquad |l^1(\chi)| \leq c\ h^{1/2}\ \|\chi\| \quad .$$

Now let U_h be an approximation on u according to Proposition 1 and put $\phi = U_h - u_h \in S_h^{\sigma,0}$. Then we get

$$(52) \qquad \begin{aligned} |l^1(e)| &\leq |l^1(u-U_h)| + |l^1(\phi)| \\ &\leq c\ h^2\|u\|_2 + c\ h^{1/2}\ \|\phi\| \\ &\leq c\ h^2\|u\|_2 + c\ h^{1/2}\left\{\|u-U_h\| + \|u-u_h\|\right\} \\ &\leq c\ h^2\|u\|_2 + c\ h^{1/2}\|e\| \quad . \end{aligned}$$

The bounds (46) and (47), (52) give - see (45) -

(53) $\|e\|^2 \leq c\ h^2\|u\|_2\ \|e\| + c\ h^{1/2}\|e\|^2$

and so for h small enough

(54) $\|e\| \leq c\ h^2\|u\|_2$.

Since we now know e to be bounded we have as a consequence the unique solvability of the defining equations (37).

Using (50) we also get the error estimate

(55) $\|e\|_1 \leq c\ h\|u\|_2$.

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