PUBLICATIONS MATHÉMATIQUES ET INFORMATIQUES DE RENNES

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Publications des séminaires de mathématiques et informatique de Rennes, 1976, fascicule S5
« Journées « éléments finis » », , p. 1-12
<http://www.numdam.org/item?id=PSMIR_1976___S5_A6_0>

NUMERICAL SOLUTION OF THE TRANSONIC EQUATION BY THE FINITE ELEMENT METHOD VIA OPTIMAL CONTROL.

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ABSTRACT

It is shown that the transonic equation for compressible potential flow is equivalent to an optimal control problem of a linear distributed parameter system. This problem can be discretized by the finite element method and solved by a conjugate gradient algorithm. Thus a new class of method for solving the transonic equation is obtained. It is particularly well adapted to problems with complicate two or three dimensional geometries and shocks.

1. PLAN

2. INTRODUCTION

The transonic equation is a non linear partial differential equation which has an elliptic behavior in the subsonic regions of the flow and a hyperbolic behavior in the supersonic regions. At the interface the normal component of the speed of the flow can be discontinuous (shocks). Some finite difference methods have been successfully developped even for flows around simple 3-D objects (Jameson (1974), Garabedian-Korn (1971)).

However the method of finite difference is not well suited to complicate geometries. An alternative approach using finite elements was studied by Gelder (1971), Norries & de Vries (1973), Periaux (1975) but their methods explode at supersonic speeds. Following Gelder's approach we shall replace the transonic equation by the minimization of a functional in an abstract space, a problem which can be solved by the methods of the theory of calculus of variations and optimal control theory.

3. STATEMENT OF THE PROBLEM

Stationary adiabatic monophasic compressible flows, in which the effects of viscosity are neglected, are well described by the set of equations

(3.1) $$\nabla\cdot(\rho u) = 0 \quad \left(\frac{\partial \rho u_1}{\partial_{x_1}} + \frac{\partial \rho u_2}{\partial_{x_2}} + \frac{\partial \rho u_3}{\partial_{x_3}}\right) = 0$$

(3.2.) $$\rho = \rho_o\left(1 - \frac{\gamma-1}{\gamma+1}\,\frac{|u|^2}{C_*^2}\right)^{\frac{1}{\gamma-1}}$$

(3.3) $$u = \nabla\phi \quad \left(u_i = \frac{\partial\phi}{\partial x_i},\ i = 1,2,3\right)$$

where ρ is the density, u is the speed of the fluid and where ρ_o, C_* and γ are constants (γ=1.4 for di-atomic gas, see Landau-Lifchitz (1971). We shall denote $k = \frac{\gamma-1}{\gamma+1}\,\frac{1}{C_*^2}$, $\alpha=1/\gamma-1$. Therefore, if Ω is the region occupied by the fluid, one must solve the nonlinear partial differential equation :

(3.4) $$\nabla\cdot\ (1-k|\nabla\phi|^2)^{\alpha}\nabla\phi = 0 \text{ in } \Omega$$

with the boundary conditions

(3.5) $$\phi|\Gamma_1 = \phi_1$$

(3.6) $$\frac{\partial\phi}{\partial n}\,|\Gamma_2 = g_2$$

Where Γ_1 and Γ_2 are parts of the boundary $\partial\Omega$ of Ω. We shall assume that $\Gamma_1 \cup \Gamma_2 = \partial\Omega$ and $\Gamma_1 \cap \Gamma_2 = 0$. In addition, if there are shocks (i.e. lines or surfaces where the tangential speed of the flow is continuous but the speed normal to these lines or surfaces is discontinuous) then, across the shock :

$$(3.7) \qquad (\rho u)^+ = (\rho u)^- \qquad \text{(Rankine-Hugoniot condition)}$$

$$(3.8) \qquad u_n^+ \leqslant u_n^- \qquad \text{(entropy condition)}$$

where it is understood that the particules of the fluid move from - to +.

Note that (3.4) multiplied by $w \in C^1(\Omega)$ and integrated by parts, leads to

$$(3.9) \qquad \int_\Omega (1-k|\nabla\phi|^2)^\alpha \nabla\phi \, \nabla w dx = \int_\Gamma (1-k|\nabla\phi|^2) \; g_2 w d\Gamma_2$$

$$\forall w \in C^1(\Omega) \text{ s.t. } w|\Gamma_1 = 0 \; ; \; \phi|\Gamma_1 = 0$$

If the notion of derivative is extended and the space $C^1(\Omega)$ is replaced by $H^1(\Omega) = \{w \in L^2(\Omega) \; |\nabla w \in (L^2(\Omega))^3\}$ then (3.9) is called a weak formulation of (3.4)-(3.6). Note that it contains (3.7).

4. GELDER'S ALGORITHM FOR SUBSONIC FLOW

For notational convenience we suppose $g_2|\Gamma_2 = 0$. Consider the functional

$$(4.1) \qquad E_o(\phi) = - \int_\Omega (1-k|\nabla\phi|^2)^{\alpha+1} dx$$

we shall say that ϕ is a stationary point of E_o on

$$(4.2) \qquad H^1_{o1}(\Omega) = \{\phi \in H^1(\Omega) | \phi| \; \Gamma_1 = 0\}$$

if

$$\delta E_o = E_o(\phi+\delta\phi) - E_o(\phi) = o(\delta\phi) \quad \forall \delta\phi \in H^1_{o1}(\Omega)$$

Since, from (4.1)

(4.3) $$\delta E_o = \int_\Omega 2k(\alpha+1)(1-k|\nabla\phi|^2)^\alpha \nabla\phi\nabla\delta\phi dx + o(\delta\phi)$$

any stationary point of E_o on $H^1_{o1}(\Omega)$ satisfies

$$\int_\Omega (1-k|\nabla\phi|^2)^\alpha \nabla\phi\nabla w dx = 0 \quad \forall w \in H^1_{o1}(\Omega)$$

Thus all stationary points of E_o on $H^1_{o1}(\Omega)$ such that $\phi|\Gamma_1 = \phi_o|\Gamma_1$ and which satisfy (3.8) are solutions of our problem.

Let us look at

$$\frac{d^2}{d\lambda^2}(E(\phi+\lambda\delta\phi))|_{\lambda=0} = 2k(\alpha+1)\int_\Omega (1-k|\nabla\phi|^2)^\alpha [\nabla\delta\phi\nabla\delta\phi - \frac{2k\alpha(\nabla\phi\cdot\nabla\delta\phi)^2}{(1-k|\nabla\phi|^2)}]\, dx$$

with our notation the mach number is such that

$$M^2 = 2k\alpha(1-k|\nabla\phi|^2)^{-1}\ |\nabla\phi|^2$$

therefore, if θ is the angle between $\nabla\phi$ and $\nabla\delta\phi$;

$$\frac{d^2E}{d\lambda^2} = -2k(\alpha+1)\int_\Omega \rho(1-M^2\cos^2\theta)|\nabla\delta\phi|^2\, dx$$

This shows that if in some part of the fluid $M>1$, E is not convex and the solution of (3.4)-(3.8) is only a saddle point of E. On the other hand, if $M<1$ in Ω then E is convex and the solution of (3.4)-(3.8) is a minimum of E. This fact was utilized by Gelder (1971) and Periaux (1975) for constructing a solution of (3.4)-(3.8). The functional E is minimized by a gradient method with respect to the $H^1(\Omega)$-norm ; i.e. $\{\phi_n\}_{n\geqslant 2}$ is constructed by solving for $\phi_{n+1} \in H^1(\Omega)$:

$$\int_\Omega \rho_n \nabla\phi_{n+1}\nabla w d\Omega = 0 \quad \forall w \in H^1_{o1}(\Omega)\ ,\ (\phi_{n+1}-\phi_1)|\Gamma_1 = 0$$

This method works very well (less than 15 iterations in most cases) and it is desirable to construct a method as near to it as possible, for supersonic flows.

5. FORMULATION VIA OPTIMAL CONTROL

Along the line of §5 we shall look for functionals which have the solution of (3.4)-(3.8) for minimum. Several functional where studied in Glowinski-Pironneau (1975) and Glowinski-Periaux-Pironneau (1976). In this presentation we shall study the following functional :

$$(5.1) \qquad E(\xi) = \int_\Omega \rho(|\nabla\xi|^2)|\nabla(\phi-\xi)|^2\, dx\,, \quad \rho(|\nabla\xi|^2) = (1-k|\nabla\xi|^2)^\alpha$$

where $\phi = \phi(\xi)$ is the solution in $H^1(\Omega)$ of

$$(5.2) \qquad \int_\Omega \rho(|\nabla\xi|^2)\, \nabla\phi\nabla w dx = 0 \quad \forall w \in H^1_{o1}(\Omega)\ , \ \phi|\Gamma_1 = \phi_1$$

Proposition 1

Given $\varepsilon > 0$ small the problem

$$(5.3) \qquad \min\ \{E(\xi)\,|\,\xi \in \Xi\}$$

where $\Xi = \{\xi \in H^1(\Omega)|\ \xi|\Gamma_1 = \phi_1 \quad |\nabla\xi(x)| \leqslant k^{-1/2}(1-\varepsilon)\ \text{ a.e. } x \in \Omega\}$ has at least one solution and if $\Delta\xi(x) < +\infty\ \forall x \in \Omega$, it is a solution of (3.4)-(3.8) if it exists.

Proof

Let $\{\xi_n\}$ be a minimizing sequence of E then $\xi_n \in \Xi$ implies that $\|\nabla\xi_n\|^2 < k^{-1}(1-\varepsilon)^2\int_\Omega dx$, therefore a subsequence (denoted $\{\xi_n\}$ also) converging towards a $\bar{\xi} \in \Xi$ can be extracted.

From the definition of ρ, and $\bar{\phi}$,

$$(5.4) \qquad \varepsilon^\alpha\|\nabla(\phi_n-\bar{\phi})\|^2 \leqslant \int_\Omega \rho_n\ \nabla(\phi_n-\bar{\phi})\ \nabla(\phi_n-\bar{\phi})\ dx = \int_\Omega(\rho_n-\bar{\rho})\nabla\bar{\phi}\nabla(\phi_n-\bar{\phi})dx$$
$$\leqslant \|\nabla(\phi_n-\bar{\phi})\|\,\Big|\int_\Omega (\rho_n-\bar{\rho})\nabla\bar{\phi}dx\Big|$$

But $\rho_n \to \bar{\rho}$ weakly therefore $\phi_n \to \bar{\phi}$ strongly in $H^1(\Omega)$. The functional E is convex and continuous in ξ-ϕ therefore it is weakly l.s.c. so that

(5.5) $$E(\bar{\xi}) \leqslant \lim_{n\to\infty} E(\xi_n)$$

Therefore (5.3) has at least one solution. Since any solution of (3.4)-(3.8) is a solution of 5.3 with $\phi=\xi$ and $E(\xi) = 0$, then $E(\bar{\xi}) = 0$ and $\bar{\xi} = \bar{\phi}$; therefore $\bar{\xi}$ (and $\bar{\phi}$) is a solution of (3.4)-(3.7) condition (3.8) can be rewritten : $\nabla \cdot u < +\infty$, hence $\Delta\xi < +\infty$.

Proposition 2

If $\xi|\Gamma_1 = \phi_1 \quad \delta\xi|\Gamma_1 = 0$, then

(5.6) $$E(\xi+\delta\xi) - E(\xi) = 2\int_\Omega \rho(|\nabla\xi|^2)(1+\tfrac{1}{2}M^2(1-|\nabla\xi|^2\cdot|\nabla\phi|^{-2}))\nabla\xi\cdot\nabla\delta\xi dx + o(\delta\xi)$$

$(M^2 = -2\rho'\rho^{-1}|\nabla\phi|^2 = -2k\alpha(1-k|\nabla\phi|^2)^{-1}|\nabla\phi|^2)$

Proof

From (5.1) and (5.2)

(5.7) $$E(\xi+\delta\xi)-E(\xi) = 2\int_\Omega [2\rho'\nabla\xi\cdot\nabla\delta\xi|\nabla(\phi-\xi)|^2-\rho\nabla(\phi-\xi)\nabla\delta\xi+\rho\nabla(\phi-\xi)\cdot\nabla\delta\phi]dx + o(\delta\xi) + o(\delta\phi)$$

where

$$\rho' = -k\alpha(1-k|\nabla\xi|^2)^{\alpha-1}$$

From (5.3)

(5.8) $$\int_\Omega \rho\nabla\delta\phi\nabla w dx = -\int_\Omega 2\rho'\nabla\xi\cdot\nabla\delta\xi\nabla\phi\cdot\nabla w dx+o(\delta\xi) \qquad \forall w \in H^1_{o1}(\Omega)$$

and since $\rho(|\nabla(\xi+\delta\xi)|^2)$ is bounded from below by a positive number, there exists K such that $||\nabla\delta\phi|| \leqslant K||\nabla\delta\xi||$

Therefore, by letting $w = \phi-\xi$ in (5.8), (5.7) becomes

$$\delta E = -2\int_\Omega [\rho\nabla(\phi-\xi)\cdot\nabla\delta\xi + \rho'(|\nabla\phi|^2-|\nabla\xi|^2)\nabla\xi\cdot\nabla\delta\xi]\, dx$$

and from (5.2) the term $\rho\nabla\phi\nabla\delta\xi$ disappears.

Corollary 1

If $\bar{\xi}$, $\bar{\phi}$ is a stationary point of E, it satisfies :

(5.9) $$\nabla\cdot[\bar{\rho}(1+\frac{\bar{M}^2}{2}(1-|\nabla\bar{\xi}|^2|\nabla\bar{\phi}|^{-2})\nabla\bar{\xi}]=0 \text{ in } \Omega$$

(5.10) $$\bar{\rho}(1+\frac{\bar{M}^2}{2}(1-|\nabla\bar{\xi}|^2|\nabla\bar{\phi}|^{-2})\frac{\partial\bar{\xi}}{\partial n}|_{\Gamma_2}=0\ ;\ \bar{\xi}|\Gamma_1=\phi_1$$

Remark : It should be noted that in most cases (5.3) has no other stationary point than the solutions of (3.4)-(3.7). Indeed let (x_ξ,y_ξ,z_ξ) be a curvilinear system of coordonate such that

$$\nabla\xi=(\frac{\partial\xi}{\partial x_\xi},o,o)$$

Then, from (5.9),(5.10)

(5.11) $$\frac{\partial}{\partial x_\xi}[\bar{\rho}(1+\frac{\bar{M}^2}{2}(1-|\nabla\bar{\xi}|^2|\nabla\bar{\phi}|^{-2})\frac{\partial\xi}{\partial x_\xi}]=0,\ \frac{\partial\bar{\xi}}{\partial n}|_{\Gamma_2}=0$$

$$\text{or } \bar{M}^2(1-|\nabla\bar{\xi}|^2|\nabla\bar{\phi}|^{-2})|_{\Gamma_2}=-2,\ \bar{\xi}|\Gamma_1=\phi_1$$

This system looks like the one dimensional transonic equation for a compressible fluid with density

$$\bar{\rho}(1+\frac{\bar{M}^2}{2}(1-|\nabla\bar{\xi}|^2|\nabla\bar{\phi}|^{-2}))$$

Therefore, if the ξ-stream lines meet two boundaries and $\Delta\xi<+\infty$ at the shocks and

$$1+\frac{\bar{M}^2}{2}(1-|\nabla\bar{\xi}|^2|\nabla\bar{\phi}|^{-2})>0$$

then $\bar{\phi}=\bar{\xi}$.

6. DISCRETIZATION AND NUMERICAL SOLUTIONS

Let $\mathcal{T}_h$ be a set of triangles or tetraedra of Ω where h is the length of the greatest side. Suppose that

$$\bigcup_{T\in\mathcal{T}_h} T\subset\Omega\ ,\ T_1\cap T_2=\quad\text{or a vertex } \forall T_1,T_2\in\mathcal{T}_h,$$

Let $\Omega_h=\overline{\overset{o}{\bigcup_{T\in\mathcal{T}_h}T}}$ and Γ_{1h}, Γ_{2h} parts of $\partial\Omega_h$ which approximate Γ_1 and Γ_2.

Let $\mathcal{H}_h$ an approximation of $H^1(\Omega)$:

(6.1) $$\mathcal{H}_h = \{w_h \in C^o(\Omega_h) | w_h \text{ linear on } T \quad \forall T \in \mathcal{T}_h\}$$

Note that any element of $\mathcal{H}_h$ is completely determined by the values that it takes at the nodes of $\mathcal{T}_h$. Therefore if we assume that $\mathcal{T}_h$ has $N = n+p+m$ nodes P_i with $P_i \in \Gamma_{1h}$ if $i > n+p$, $P_i \in \Gamma_{2h}$ if $i \in]n,n+p]$, and if we define $w_i \in \mathcal{H}_h$ by

(6.2) $$w_i = 1 \text{ at node 1 and zero at all other nodes}$$

Then any function $w \in \mathcal{H}_h$ is written as

(6.3.) $$\phi = \Sigma\alpha_i w_i$$

Algorithms 1

Let $\xi_h = \sum_{i=1}^{N} \xi^i w_i$, then (5.2) becomes

(6.4) $$\int_\Omega (1-k|\nabla\xi_h|^2)^\alpha \nabla\phi_h \nabla w_i dx = 0 \qquad i=1,\dots,n+p$$
$$\phi_h = \sum_{i=1}^{n+p} \phi^i w_i + \sum_{n+p+1}^{N} \phi_1^i w_i$$

and (5.6) becomes

(6.5) $$\frac{1}{2}\delta E_h = \sum_{i=1}^{N} \delta\xi^i \delta E_h^i + o(\delta\xi^i)$$

(6.6) $$\delta E_h^i = \int_\Omega [\rho-\rho'(|\nabla\phi_h|^2-|\nabla\phi_h|^2)]\nabla\xi_h \cdot \nabla w_i \, dx$$

Consider the following algorithm

Step 0 Choose $\mathcal{T}_h$, ξ_{ho} set j=0

Step 1 Compute ϕ_{hj} by solving (6.4) with $\xi_h = \xi_{hj}$

Step 2 Compute $\{\delta E_{hj}^i, i = 1,\dots,N\}$ by (6.6)

Step 3 Compute $\delta\xi_h = \sum_{i=1}^{n+p} \delta\xi^i w_i$ by solving

(6.7) $$\int_{\Omega_h} \nabla\delta\xi_h \nabla w_i dx = \delta E^i_{hj}, \quad i=1,\dots,n+p$$

Step 4 Compute an approximation $\bar{\lambda}_j$ of the solution of

(6.8) $$\min_{\lambda\in[0,1]} \int_{\Omega_h} \rho(\lambda) |\nabla(\xi_h(\lambda) - \phi_h(\lambda))|^2 dx$$

where

$$\xi_h(\lambda) = \sum_{i=1}^{N} (\xi^i_{hj} - \lambda\delta\xi^i_h) w_i$$

Step 5 Set $\xi_{h_{j+1}} = \xi_h(\bar{\lambda}_j)$, $j=j+1$ and go to step 1.

Proposition 3

Let $\{\xi_{hj}\}_{j\geq 0}$ be a sequence generated by algorithm 1 such that $|\nabla\xi_{hj}(x)| \leq k^{-1/2}$ $\forall x, \forall j$. Every accumulation point of $\{\xi_{hj}\}_{j\geq 0}$ is a stationary point of the functional

(6.9) $$E_h(\xi_h) = \int_{\Omega_h} |\nabla(\phi_h - \xi_h)|^2 dx$$

Where $\phi_h = \phi_h(\xi_h)$ is the solution of (6.4), in

$$\Xi_h = \{\xi_h \in \mathcal{H}_h | \; |\nabla\xi_h(x)| \leq k^{-1/2} \;\; \forall x \in \Omega_h\}$$

Proof

Algorithm 1 is the method of steepest descent applied to minimize (6.9) in Ξ_h, with the norm

(6.10) $$\|\xi_h\|^2_h = \int_{\Omega_h} \nabla\xi_h \nabla\xi_h \, dx$$

Therefore $\{E_h(\xi_{hj})\}_j$ decreases until δE_{hj} reaches zero.

Remark 6.1 : (6.4) should be solved by a method of relaxation but (6.7) can be factorized once and for all by the method of Choleski.

Remark 6.2 : Problem (6.8) is usually solved by a Golden section search or a Newton method.

Remark 6.3 : Step 5 can be modified so as to obtain a conjugate gradient method.

Remark 6.4 : The restriction : $|u_{h_j}(x)| \leqslant k^{-1/2}$ in theorem 5.1 is not a problem if u is not too close to $k^{-1/2}$ otherwise one must treat this restriction as a constraint in the algorithm. Also, even though theorem (5.1) ensures the computation of stationary points only, it is a common experience that global minima can be obtained by this procedure if there is a finite number of local minima.

Remark 6.5 : The entropy condition $\Delta\xi_h < +\infty$ can be taken into account numerically. Let M(x) be a real valued function then $\Delta\xi_h \leqslant M(x)$ becomes, from (6.7)

(6.11) $$-\Sigma\, \bar{\lambda}_j\ \delta E^i_{hj} \leqslant M(x_i) \qquad i=1,\dots,n+p$$

Therefore, to satisfy (6.11) at iteration j+1, it suffices to take $\delta E^i_{hj} = 0$ in (6.7) for all i such that (6.11) at iteration j is an equality. This procedure amounts to control $\omega = \Delta\xi$ instead of ξ.

7. NUMERICAL RESULTS

The method was tested on a nozzle discretized as shown on figure 1, (300 triangular elements, 180 nodes). The Polak-Ribiere method of conjugate gradient was used with an initial control : $\Delta\xi = 0$ (incompressible flow). A mono-dimensional optimization subroutine based on a dychotomic search was given to us by Lemarechal. Several boundary conditions were tested

1°) subsonic mach number $M_\infty = 0.63$ at the entrance, zero potential on exit, the method had already converged in 10 iterations (to be compared with the Gelder-Periaux method) giving a criterium $E_{h10} = 2\ 10^{-13}$ ($E_{ho} = 10^{-4}$).

2°) Entrance and exit potential specified.

For a decrease of potential of $\phi_1 - \phi_2 = 0.7$ the method had converged in 20 iterations without including the entropy condition, giving a criterium of $E_{h20} = 5\ 10^{-7}$, the results are shown on figure 2.

3°) Supersonic mach number $M_\infty = 1.25$ at the entrance.

The method computes a solution that has a shock at the first section of discretization. An other boundary condition must be added. One iteration of the method takes 3" on an IBM 370/158 on this example.

A three dimensional nozzle is being tested : the result will be shown at the conference. 20 to 40 iterations are usually sufficient for the algorithm to converge. The results are in good agreement with the tabulated data. Simple and multi-bodies airfoils are also being tested. For them it is necessary to include the entropy condition ; 80 iterations are usually more than sufficient for the convergence.

8. CONCLUSIONS

Thus this method seems very promizing. It compares very well with the finite differences method available and it has the advantage of allowing complicate two and three dimensional geometries. This work illustrates the fact that optimal control theory is a powerful tool with unexpected applications sometimes.

ACKNOWLEDGEMENT

We wish to thank M. Periaux, Perrier and Poirier for allowing us to use their data files and computer and for their valuable comments.

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