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* CYLINDRICAL STOCHASTIC INTEGRAL

by

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Summary

In the first part of this study, we construct the stochastic integral $\int Y\, d\tilde{M}$ where Y is a "weakly" predictable process and $\tilde{M}$ is a "cylindrical" square integrable martingale. This last notion generalises the case where $\tilde{M}$ is a "white noise in time and space".

In the second part, this construction is extended, when M is not square integrable, by a regionalization procedure (Cf. [18]') ; this procedure is a generalization of the classical procedure of localization.

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A FIRST PART

CONSTRUCTION OF THE STOCHASTIC INTEGRAL WITH RESPECT TO PROCESSES OF LINEAR FUNCTIONNALS

INTRODUCTION

For the purpose of studying stochastic partial differential equations it is worth considering perturbations which are "white noise in time and in space". The mathematical expression of such and object is a cylindrical measure, or a linear random functional as studied for example in [1] or [8] . Considering the special case of "cylindrical brownian motion", several authors defined a stochastic integral with respect to such a stochastic process (cf. for example [7] and [11]). In [7] the operator valued processes, which are integrated with respect to the cylindrical brownian motion, are such that the integral process is a (Hilbert valued) Martingale.

The purpose of this part is to show that, in a very general context, it is possible to developp a theory of stochastic integration with respect to "cylindrical martingales", which extends in a natural way the classical L^2-stochastic integral with respect to square integrable martingales (real or Hilbert valued) as studied in [10], [14], [18] for example. This part generalizes and completes [14]'.

Hypotheses and notations are given in the first paragraph. In the second one, the notion of cylindrical martingale is defined and the particular case of "white noise in time and space" is specially studied. In the third paragraph, we define and study a process Q : the role of this process is analognous to the role of the "quadratic variation" for a real square integrable martingale. The stochastic integral is constructed in the fourth paragraph. The case where the processes considered are Hilbert-space valued is more specially studied in the fifth paragraph.

I - NOTATIONS

I-1 In all this paper we will assume that T is the closed intervall $[0,1]$ in $\mathbb{R}^+$, a basic probability space $(\Omega, \mathfrak{F}, P)$ and an increasing family $(\mathfrak{F}_t)_{t \in T \times \mathbb{R}^+}$ of sub-σ-algebras of $\mathfrak{F}$ with the usual following completion assumption : $\mathfrak{F}$ is P-complete and all the P-null sets in $\mathfrak{F}$ are in $\mathfrak{F}_t$ for every t.

$\mathcal{R}$ will mean the set of "predictable rectangles" : $(F \times]s,t]) \subset \Omega \times T$ where $s \leq t$, $s,t \in T$ and $F \in \mathfrak{F}_s$.

$\mathcal{A}$ will be the algebra of subsets of $\Omega \times T$ generated by $\mathcal{R}$.

$\mathcal{P}$ is the σ-algebra generated by $\mathcal{A}$, i.e. : the σ-algebra of predictable subsets of $T \times \Omega$.

I-2 $\mathcal{M}$ is the space of real martingales M which satisfy the following properties :

(i) $(M_t)_{t \in T}$ is a right continuous and with left hand limits process

(ii) $(M_t)_{t \in T}$ is a square-integrable process

(iii) $(M_t)_{t \in T}$ is defined up an indistinguability that is to say, if, $P\left[\sup_{t \in T} |M_t - N_t|\right] = 0$, then M and N correspond to the same element of $\mathcal{M}$.

(iv) $M_o = 0$

If M and N are two elements of $\mathcal{M}$, we consider $\langle M,N \rangle_{\mathcal{M}} = E\left[M_1 . N_1\right]$; it is well-known that $\mathcal{M}$ is an Hilbert space for this scalar product (cf. [17]).

I-3 $\mathbb{H}$ and $\mathbb{G}$ will denote real Banach spaces. The norm will be written : $||\cdot||_{\mathbb{H}}$, $||\cdot||_{\mathbb{G}}$, etc.... If $\mathbb{B}$ is a Banach space, then $\mathbb{B}'$ will denote the topological dual of B (set of continuous linear form) endowed, if not otherwise specified, with the dual Banach norm.

If $\mathbb{H}$ and $\mathbb{G}$ are Hilbert spaces, the scalar product in those spaces will be denoted by $\langle .,. \rangle_{\mathbb{H}}$, $\langle .,. \rangle_{\mathbb{G}}$, or simply, $\langle .,. \rangle$ if there is no possible confusion.

We recall that the algebraic tensor product $\mathbb{H} \otimes \mathbb{G}$ can be endowed with several norms, giving rise to several completions of $\mathbb{H} \otimes \mathbb{G}$:

- $\mathbb{H} \hat{\otimes}_1 \mathbb{G}$ is the completion for a norm such that every continuous bilinear mapping $b : (\mathbb{H} \otimes \mathbb{G}) \to \mathbb{K}$ can be factorized in a unique way as $b = u_b \circ \Pi$ where Π is the canonical inbedding $\Pi(x,y) = x \otimes y$ and u_b is a continuous linear mapping from $\mathbb{H} \hat{\otimes}_1 \mathbb{G}$ into $\mathbb{K}$, with same norm as b. The norm $\mathbb{H} \hat{\otimes}_1 \mathbb{G}$ is often called the trace-norm and denoted $||.||_{Tr}$. Recall that if $\mathbb{G} = \mathbb{H}$ is an Hilbert space and $b(x,y) = \langle x,y\rangle_{\mathbb{H}}$ the corresponding linear form u_b on $\mathbb{H} \hat{\otimes}_1 \mathbb{H}$ is called the trace-form and denoted Tr.

- If $\mathbb{H}$ and $\mathbb{G}$ are Hilbert spaces, $\mathbb{H} \hat{\otimes}_2 \mathbb{G}$ is a Hilbert space with scalar product the extension of $\langle x \otimes y,\ x' \otimes y'\rangle = \langle x,x'\rangle_{\mathbb{H}} \cdot \langle y,y'\rangle_{\mathbb{G}}$.

- $\mathbb{H} \hat{\otimes}_\varepsilon \mathbb{G}$ is a Banach space, the norm of which will be more easily described later.

The three topologies induced by the three considered topological tensor product on $\mathbb{H} \otimes \mathbb{G}$ are comparable and we have the canonical continuous injection.

$$\mathbb{H} \hat{\otimes}_1 \mathbb{G} \hookrightarrow \mathbb{H} \hat{\otimes}_2 \mathbb{G} \hookrightarrow \mathbb{H} \hat{\otimes}_\varepsilon \mathbb{G} .$$

I-4 There is a unique injective linear mapping of $\mathbb{H} \otimes \mathbb{G}$ into the vector space of linear operators with finite range from $\mathbb{H}$ into $\mathbb{G}$, associating to $x \otimes y$ the operator $h \rightsquigarrow \langle x,h\rangle_{\mathbb{H}}\, y$. This linear mapping has extensions which are :

1°) isometry from $\mathbb{H} \hat{\otimes}_1 \mathbb{G}$ onto $\mathcal{L}_1(\mathbb{H} ; \mathbb{G})$, the Banach space of nuclear operators from $\mathbb{H}$ into $\mathbb{G}$ with the trace norm ;

2°) isometry from $\mathbb{H} \hat{\otimes}_2 \mathbb{G}$ onto $\mathcal{L}_2(\mathbb{H} ; \mathbb{G})$ the Hilbert space of Hilbert-Schmidt operators from $\mathbb{H}$ into $\mathbb{G}$ with the Hilbert-Schmidt scalar product ;

3°) isometry from $\mathbb{H} \hat{\otimes}_\varepsilon \mathbb{G}$ onto $\mathcal{L}_c(\mathbb{H} ; \mathbb{G})$, the Banach space of compact operators with the usual norm of bounded operators.

In as much $x \otimes y$ can be identified with a bilinear continuous form on $(\mathbb{H} \times \mathbb{G})$ or a continuous linear form on $\mathbb{H} \otimes \mathbb{G}$, through the formula

$$\langle x \otimes y,\ x' \otimes y'\rangle = \langle x,x'\rangle_{\mathbb{H}} \cdot \langle y,y'\rangle_{\mathbb{G}}$$

there is also a continuous linear extension of the preceding linear mapping, into an isometry from $(\mathbb{H} \hat{\otimes}_1 \mathbb{G})'$ onto $\mathcal{L}(\mathbb{H} ; \mathbb{G})$, the Banach space of linear bounded operators from $\mathbb{H}$ into $\mathbb{G}$ with the usual norm. (This isometry is in fact the one which associates to a bounded bilinear b on $(\mathbb{H} \times \mathbb{G})$ the bounded linear operator $\tilde{b}$ in $\mathcal{L}(\mathbb{H} ; \mathbb{G})$ such that $\langle \tilde{b}(x),\ y\rangle = b(x,y)$.).

I-5 We shall note $\mathcal{L}(\mathbb{H}, \mathbb{G}_\sigma)$ the vector space of the linear continuous operators from $\mathbb{H}$ with its strong topology into $\mathbb{G}$ with its weak topology $\sigma(\mathbb{G}, \mathbb{G}')$. If u is an element of $\mathcal{L}(\mathbb{H}, \mathbb{G}_\sigma)$, the adjoint u^* of u is defined as a linear continuous operator from G' with its weak topology $\sigma(\mathbb{G}', \mathbb{G})$ into H' with its weak topology $\sigma(\mathbb{H}', \mathbb{H})$.

I-6 Random variables with values in $\mathbb{H}$ will be strongly $\mathcal{F}$-measurable mapping from Ω into $\mathbb{H}$. If such a random variable X has the property $E(||X||^2_{\mathbb{H}}) < \infty$, then $\omega \rightsquigarrow X(\omega) \otimes X(\omega)$ is a strongly measurable random variable with values in $\mathbb{H} \otimes \mathbb{H}$, and as $||x \otimes y||_{Tr} = ||x||_{\mathbb{H}}\, ||y||_{\mathbb{H}}$, $X \otimes X$ is an integrable mapping from Ω into $\mathbb{H} \otimes_1 \mathbb{H}$. As a consequence $E(X \otimes X) \in \mathbb{H} \hat{\otimes}_1 \mathbb{H}$ and is called the covariance of the variable X.

If to X is moreover associated the continuous mapping $\tilde{X}$: $h \rightsquigarrow \langle X,h\rangle$ from $\mathbb{H}$ into $L^2_{\mathbb{R}}(\Omega, \mathcal{F}, P)$, this mapping appears to be Hilbert-Schmidt. And it can be shown that conversely to every linear mapping $\tilde{X}$ from $\mathbb{H}$ into $L^2_{\mathbb{H}}(\Omega, \mathcal{F}, P)$ there can be associated a random variable X with values in $\mathbb{H}$, such that $\langle X,h\rangle = \tilde{X}(h)$ a.s, if and only if $\tilde{X}$ is Hilbert-Schmidt. The Hilbert Schmidt norm $||\tilde{X}||_2$ of $\tilde{X}$ is then equal to $\sqrt{E(||X||^2_{\mathbb{H}})}$.

I-7 To abbreviate the writting we will writte

$$\mathcal{R}^p = L^p_{\mathbb{R}}(\Omega, \mathcal{F}, P) \qquad p \geq 0$$

$$\mathcal{R}^p_t = L^p_{\mathbb{R}}(\Omega, \mathcal{F}_t, P) \qquad p \geq 0$$

I-8 The norm in $\mathcal{L}_1(\mathbb{H} ; \mathbb{G})$ will be written $||\cdot||_{Tr}$, the norm in $\mathcal{L}_2(\mathbb{H} ; \mathbb{G})$: $||\cdot||_{H.S}$, the norm in $\mathcal{L}(\mathbb{H} ; \mathbb{G})$: $||\cdot||_b$.

II - CYLINDRICAL MARTINGALE

II-1 Definition

If H is a Banach space, we shall say that $\tilde{M}$ is a 2-cylindrical martingale on H' if $\tilde{M}$ is an element of $\mathcal{L}(H', \mathcal{M})$, that is to say that $\tilde{M}$ is a linear continuous mapping from H' into $\mathcal{M}$ (for the strong topology of H' and the Hilbert space topology of $\mathcal{M}$).

II-2 Quadratic Doléans's measure

Let $\tilde{M}$ be a 2-cylindrical martingale on the Banach space H. We consider the function m defined on $[\mathcal{R} \times (H' \otimes H')]$ by :

$$\forall A = (F\times]s,t]) \in \mathcal{R}\ , \quad \forall (h,g) \in (H' \times H')\ ,$$

$$[m(A)](h \otimes g) = E\{1_F\ [\tilde{M}_t(h).\tilde{M}_t(g) - \tilde{M}_s(h).\tilde{M}_s(g)]\}$$

It is well-known that, for each element (h,g) of $(H' \times H')$, there is a unique real measure defined on $\mathcal{P}$ which is an extension of $m(.)(h \otimes g)$.

Then, this extension m defines a mapping from $\mathcal{P}$ into the algebraic dual of $(H' \otimes H')$.

We shall say that this extension m is the quadratic Doléans' measure of $\tilde{M}$. In fact, we are essentially interested in the case where the total variation of m is finite, m being considered as an application of $\mathcal{P}$ into the Banach space $(H' \hat{\otimes}_1 H')'$, dual of the tensor product $H' \otimes_1 H'$.

That is the case (Cf II-5 below) in particular when $\tilde{M}$ is a "white noise in time and in space" : in our context, the mathematical definition of such a process is the following :

II-3 Cylindrical brownian process (definition)

Let H be a Hilbert space and $\tilde{w}$ a cylindrical martingale on H. We shall say that $\tilde{w}$ is a cylindrical brownian process if, for each finite family $(h_k)_{1\leq k\leq n}$ constituted of elements of H, $(\tilde{w}_t(h_k))_{1\leq k\leq n}$ is an n-dimensionnal brownian motion such that $E\ [w_t(h_i).\ w_t(h_j)] = t<h_i\ ,\ h_j>_H$ for all pairs $(h_i\ ,\ h_j)$ of elements of H.

II-4 Proposition (elementary properties of the cylindrical brownian motion)

Let H be a Hilbert space. Let $\tilde{w}$ be a cylindrical brownian motion on H. Then, we have :

1°) $\tilde{w}$ is an isometry from H into $\mathcal{M}$

2°) For all elements h of H and for all elements $A = (F\times]s,t])$ of $\mathcal{R}$, we have :

$$E\{1_F\ [\tilde{w}_t(h) - \tilde{w}_s(h)]^2\} = [(P \otimes \mu)(A)]\ .\ ||h||^2_H$$

3°) Let m be the quadratic Doléans's measure of $\tilde{w}$. Then, the total variation of m is $v = P \otimes \mu$ where μ is the Lebesgue measure

4°) Let $Q = \frac{dm}{dv}$ be the Radon-Nikodym derivative of m with respect to v. Then, for all elements (t,ω) of $(T \times \Omega)$ and for all elements of $(H \times H)$,

$$Q\ [h \otimes g]\ (t,\omega) = <h,g>_H$$

Proof

1°) For each element (h,g) of $(H \times H)$, we have :

$$<\tilde{w}(h), \tilde{w}(g)>_{\mathcal{M}} = E\ [\tilde{w}_1(h).\tilde{w}_1(g) - \tilde{w}_o(h).\tilde{w}_o(g)] = <h,g>_H$$

then $\tilde{w}$, considered as an element of $\mathcal{L}(H, \mathcal{M})$, is an isometry.

2°) We consider $h \in H$ and $A = (F\times]s,t]) \in \mathcal{R}$. The random variable $\tilde{w}_t(h) - \tilde{w}_s(h)$ being gaussian and orthogonal to $\mathcal{F}_s$, we have :

$$E\{1_F.\ [\tilde{w}_t(h) - \tilde{w}_s(h)]^2\} = P(F).E\{[\tilde{w}_t(h) - \tilde{w}_s(h)]^2\}$$

$$= P(F).(t-s).||h||^2_H = (P \otimes \mu)(A).||h||^2_H$$

3°) Let u be an element of $H \otimes H$ with $u = \sum_{(i,j) \in I\times I} \lambda_{i,j}\ .\ a_i \otimes a_j$ where $(a_i)_{i\in I}$ is an orthonormal family in H. Let m be the quadratic Doléans's measure of w. If $A = F\times]s,t]$ is an element of $\mathcal{R}$, we have :

$$<m(A),u> = \sum_{(i,j)\in I\times I} \lambda_{i,j} \cdot E\{1_F.[\tilde{W}_t(a_i)\,\tilde{W}_t(a_j) - \tilde{W}_s(a_i)\,\tilde{W}(a_j)]\}$$

$$= \sum_{i\in I} \lambda_{i,i} \cdot (P\otimes\mu)(A) \leq (P\otimes\mu)(A).||u||_1$$

then, $P \otimes \mu = v$ is the total variation of m considered as a measure with values in $(H \hat{\otimes}_1 H)'$.

4°) Moreover, for each pair (h,g) of elements of H,
$m(F\times]s,t])(h\otimes g) = (P\otimes\mu)(F\times]s,t]).<h,g>_H$ and this proves the 4°).

III - THE PROCESS Q

III-1 Hypotheses

For all the following parts, we consider a Banach space H and a 2-cylindrical martingale $\tilde{M}$ on H. We note m the quadratic Doléans's measure of $\tilde{M}$. We suppose that the total variation v of m is finite.

We writte Q the weakly predictable process, $(H \hat{\otimes}_1 H)'$-valued, Radon-Nikodym derivative of m with respect to v.

III-2 Proposition

Let v be a positive measure defined on the tribe of predictable sets. Let V be the increasing "natural" process (cf. [5]) associated to v. Let r be a real measure defined on the tribe of predictable sets : we suppose then $|r| \leq v$ if $|r|$ is the total variation of r. Let Q be the predictable Radon-Nikodym derivative of r with respect to v. For each element ω of Ω, the real function Q(.,ω) is a borelian function. Then, we can define the process $(R_t)_{t\in T}$ by $R_t(\omega) = \int_{]o,t]} Q(s,\omega).dV_s(\omega)$ (this integral being calculated by trajectories). Then, the process $(R_t)_{t\in T}$ is the "natural" process associated to Ω.

Proof

Let $\mathcal{C}$ be the class of the real predictable processes such that, for each trajectory ω, the real function Q(.,ω) is measurable with respect to the Borel tribe ; $\mathcal{C}$ is a vector space such that, if $(Q^n)_{n>o}$ is an increasing sequence of elements of $\mathcal{C}$ which converges to Q, then Q is also an element of $\mathcal{C}$; moreover, all the processes 1_A with $A\in\mathcal{R}$ are elements of $\mathcal{C}$. Then, $\mathcal{C}$ is the set of all predictable sets. Then, the process (R_t) is well-defined. If $Q = 1_A$ with $A\in\mathcal{R}$, it is evident that R is a predictable process. Then, the same property is satisfied for all bounded predictable processes by linearity and dominated convergence (cf, for example, the theorem of [14]).

To prove that R is the "natural" process associated to r, it is sufficient to prove that the Doléans's measure associated to R is r. Then, it is sufficient to prove that, if $A = F\times]s,t]$ is an element of $\mathcal{R}$, we have :

$$r(A) = E\,[1_F.(R_t - R_s)]$$

This property is evident if $Q = 1_B$ with $B\in\mathcal{R}$; it is also satisfied for each bounded predictable process Q by linearity and dominated convergence.

III-3 Theorem (properties of the process Q)

Let B be a separable Banach space. Let m be a function defined on the tribe of predictable sets with values in the dual B' of B ; we suppose that m is σ-additive for the topology σ(B',B) and the total variation v of m is finite. Then, this total variation is σ-additive. Let Q be the B'-valued and weakly predictable process, Radon-Nikodym derivative of m with respect to v. Let V be the increasing "natural" process associated to v. Let S be the B'-valued process defined by

$$S_t(\omega) = \int_{]o,t]} Q(s,\omega).dV_s(\omega)$$

this integral being a "weak integral by trajectories".

Then, for each element x of B', the process <S,x> is, up to an indistinguability , the "natural" process associated to the real measure <m,x> .

We call S the natural Process of m.

Proof

The σ-additivity of v and m is a well-known property for vector measures. Then Q is well-defined by a "weak" Radon-Nikodym theorem (cf. [12]). The end of the theorem is a corollary of the proposition III-2 above.

IV - CONSTRUCTION OF THE STOCHASTIC INTEGRAL

IV-1 Introduction

The purpose of this part is to define the stochastic integral $\int Y.d\tilde{M}$ where Y is a "weakly predictable" process with values in $\mathcal{L}(H, G_\sigma)$ (cf. I-5 above), and where $\tilde{M}$ is a cylindrical martingale on H such that the total variation of its quadratic Doléans's measure is finite.

For all this part, H and G are Banach spaces, H being reflexive, and $\tilde{M}$ is an element of $\mathcal{L}(H', \mathcal{M})$. We suppose than H' is a separable space. We note m the quadratic Doléans's measure of $\tilde{M}$ and v the total variation of m where m is considered as a $(H' \otimes_1 H')'$-valued measure. We suppose than $v(\Omega \times T) < +\infty$. We note Q the predictable process, Radon-Nikodym derivative of m with respect to v.

We shall say that a $\mathcal{L}(H, G_\sigma)$-valued process Y is "weakly predictable" if, for each element (h,g) of $H \times G'$, $\langle Y(h), g\rangle$ is a real predictable process.

IV-2 $\mathcal{L}$-step process and stochastic integral associated

We note $\mathcal{E}$ the set of the processes Y such that $Y = \sum_{i \in I} u_i . 1_{A(i)}$ where $(u_i)_{i \in I}$ is a finite family of elements of $\mathcal{L}(H, G_\sigma)$ and $(A(i))_{i \in I}$ is an associated family of elements of $\mathcal{A}$.

We remark that, in this situation, we can suppose that the sets $A(i)$ belong to $\mathcal{P}$ and are pairwise disjoint (cf. [18]).

Let Y be an element of $\mathcal{E}$ with $Y = \sum_{i \in I} u_i . 1_{A(i)}$ where, for each element i of I, $A(i) = (F(i) \times]s(i), t(i)])$. For each element g of G', let $\tilde{Z}(g)$ be the real martingale defined by :

$$[\tilde{Z}(g)]_t = \sum_{i \in I} \{ 1_{F(i)} \cdot \int_{]s(i)\wedge t, t(i)\wedge t]} d\,\tilde{M}\,[u_i^*(g)] \}$$

where u_i^* is the adjoint operator of u_i.

This defined a cylindrical martingale on G that we shall note $\int Y.d\tilde{M}$ and call the stochastic integral of Y with respect to $\tilde{M}$.

The problem is to extend this construction to "weakly predictable" processes Y as done, for example, in [14] in the case of processes in the strict sense.

For this extension, the following remark is fundamental :

IV-3 Remark

Let Y be an element of $\mathcal{E}$ with $Y = \sum_{i \in I} u_i . 1_{A(i)}$ where the sets $(A(i))_{i \in I}$ are pairwise disjoint and, for each $i \in I$, $A(i) = F(i) \times]s(i), t(i)]$; then, we have :

$$||\tilde{Z}(g)||^2_{\mathcal{M}} = \sum_{i \in I} E \left\{ 1_{F(i)} ([\tilde{M}_{t(i)} - \tilde{M}_{s(i)}] \cdot [u_i^*(g)])^2 \right\}$$

the random variables $1_{F(i)} \cdot [\tilde{M}_{t(i)} - \tilde{M}_{s(i)}]\ [u_i^*(g)]$ being pairwise orthogonal in $L_2(\Omega, \mathcal{F}, P)$, we have also :

$$||Z(g)||^2_{\mathcal{M}} = \int_{\Omega \times T} [Y^*(g) \otimes Y^*(g)]\ d\,m$$

where Y^* is the adjoint of Y,

finally, we obtain :

$$||Z(g)||^2_{\mathcal{M}} = \int_{\Omega \times T} Q\ [Y^*(g) \otimes Y^*(g)]\ d\,v$$

The fundamental idea is to consider a norm (cf. IV-6 below) associated to this formula.

IV-4 Lemma (topological)

We consider $u \in (H' \hat{\otimes}_1 H')'$, $v \in \mathcal{L}(H, G_\sigma)$ and $g \in G'$. Let v^*

be the adjoint of v. We suppose that $\mathbb{H}'$ is separable. Let $\{x_n\}_{n>o}$ be a sequence of elements of $\mathbb{H}'$, dense in $\mathbb{H}'$. Let $\mathbb{H}'_n$ be the vector space generated by $\{x_k\}_{1\leq k\leq n}$. Let π'_n a projector of $\mathbb{H}'$ onto $\mathbb{H}'_n$ which is a contraction. Then, we have :

$$u\left[(v^* \circ g)\otimes(v^* \circ g)\right] = \lim_{n\to\infty} u\left[(\pi'_n \circ v^* \circ g)\otimes(\pi'_n \circ v^* \circ g)\right]$$

and $$\lim_{n\to\infty} u\left\{\left[(v^* \circ g) - (\pi'_n \circ v^* \circ g)\right]^{\otimes 2}\right\} = 0$$

Proof

If we consider v, g and $\varepsilon > o$, there exists $k > o$ and $w \in \mathbb{H}'_k$ such that $\|v^* \circ g - w\|_{\mathbb{H}'} \leq \varepsilon$; this implies , $\forall\, n \geq k$:

$$\|\pi'_n \circ v^* \circ g - v^* \circ g\|_{\mathbb{H}'} \leq \|\pi'_n \circ v^* \circ g - w\|_{\mathbb{H}'} + \|v^* \circ g - w\|_{\mathbb{H}'}$$

$$\leq \|\pi'_n \circ (v^* \circ g - w)\|_{\mathbb{H}'} + \varepsilon$$

$$\leq 2\varepsilon$$

then $\pi'_n \circ v^* \circ g$ converges strongly to $v^* \circ g$ in $\mathbb{H}'$. The lemma follows from the continuity of u for the strong topology of $\mathbb{H}'$ and from the continuity of the mapping $(x,y) \rightsquigarrow (x\otimes y)$ for the " trace norm " on $(\mathbb{H}' \hat{\otimes}_1 \mathbb{H}')$.

IV-5 **Preliminary proposition**

1°) We suppose that $\mathbb{H}'$ is a separable Banach space. Let Q be a $(\mathbb{H}' \hat{\otimes}_1 \mathbb{H}')'$-valued and weakly predictable process. Let Y be a "weakly predictable" $\mathcal{L}(\mathbb{H}, \mathbb{G}_\sigma)$-valued process (cf. the end of IV-1 above). Let Y^* be the adjoint of Y (with values in $\mathcal{L}(\mathbb{G}'_\sigma, \mathbb{H}'_\sigma)$). Then, for each element g of $\mathbb{G}'$, the process $Q\left[Y^*(g) \otimes Y^*(g)\right]$ is a real (positive) predictable process.

2°) We have an analoguous result if we put, in the 1°) above, the condition "for each element ω of Ω , the mapping $t \rightsquigarrow Q(t,\omega)$ is borelian" in the place of the condition "predictable".

Proof

We prove the 1°) : the proof of the 2°) is absolutely analoguous. We consider Q, Y and g. We consider also a sequence $(\pi'_n)_{n>o}$ of projectors as in the lemma IV-4 above. This lemma implies :

$$\lim_{n\to\infty} Q\left[(\pi'_n \circ Y^* \circ g)\otimes(\pi'_n \circ Y^* \circ g)\right] = Q\left[(Y^* \circ g)\otimes(Y^* \circ g)\right]$$

But, for each n, $Q\left[(\pi'_n \circ Y^* \circ g)\otimes(\pi'_n \circ Y^* \circ g)\right]$ is a real predictable process. Then, the same is true for $Q\left[(Y^* \circ g)\otimes(Y^* \circ g)\right]$.

IV-6 **Définitions of $\mathcal{Q}_g$**

Let Q be as in IV-5 and positive. For each element g of $\mathbb{G}'$, we shall denote by $\mathcal{Q}_g$ the vector space of the processes Y such that

(i) $\forall (t,\omega) \in\,]0,1] \otimes \Omega$, $Y(t,\omega)$ is a linear operator with domain $\mathfrak{D}\left[Y(t,\omega)\right]$ in H and range in $\mathbb{G}$ such that $Q\left[Y^*(g) \otimes Y^*(g)\right](t,\omega)$ is "well-defined" (cf. below)

(ii) $Q\left[Y^*(g) \otimes Y^*(g)\right]$ is a (real positive) predictable process (finite or infinite)

We give now two different examples where $Q\left[Y^*(g)\otimes Y^*(g)\right](t,\omega)$ is "well-defined".

1°/ Let (t,ω) be an element of $]0,1]\otimes\Omega$; $Q\left[X^*(g) \otimes Y^*(g)\right]$ is "well-defined" in the following case : we suppose that $\mathbb{H}$ and $\mathbb{G}$ are Hilbert spaces and let $Q^{1/2}(t,\omega)$ be the self-adjoint operator such that $Q^{1/2} \circ Q^{1/2} = Q$; if $\mathfrak{D}\left[X(t,\omega)\right] \supset \text{Range } (Q^{1/2})$ and if the linear operator $Y(t,\omega) \circ Q^{1/2}(t,\omega)$ is extendable into a bounded linear operator from H into G, then

$Q\,[Y^*(g) \otimes Y^*(g)]\ (t,\omega)$ is well-defined by :

$$Q\,[Y^*(g) \otimes Y^*(g)]\ (t,\omega) = [Y(t,\omega) \circ Q^{1/2}(t,\omega)]^*\ (g)$$

2°/ If Y is a process with values in $\mathcal{L}(H, G_\sigma)$ "weakly predictable" (cf. the end of IV-1 above) then the conditions (i) and (ii) above are satisfied (cf. the proposition IV-5 above).

IV-7 Definitions of N_g and $\bar{\mathcal{E}}_g$

For each element g of G' and for each process Y belonging to Q_g, we note : $N_g(Y) = \{ \int_{\Omega\times T} Q\,[Y^*(g) \otimes Y^*(g)].\,d\nu \}^{1/2}$

This quantity, finite or infinite, is well-defined (cf. IV-6 above). If $\tilde{M}$ is a cylindrical brownian motion (cf. II-3 above), we remark that

$$N_g(Y) = \{ \int_{\Omega\times T} ||Y^*(g)||^2.\,d\nu \}^{1/2}$$

We note $\bar{\mathcal{E}}_g$ the adherence of $\mathcal{E}$, for the semi-norm N_g above, in the set of all "weakly predictable" processes Y.

The mapping $Y \rightsquigarrow \int Y.d\tilde{M}(g) = \tilde{Z}(g)$, defined for $Y \in \mathcal{E}$, admits an unique extension to $\bar{\mathcal{E}}_g$ which is a linear continuous mapping from $\bar{\mathcal{E}}_g$ (with the topology associated to N_g) into $\mathcal{M}$ (cf. the remark IV-3 above).

IV-8 Definition of $\bar{\mathcal{E}}^b$

We shall say that a process Y belong to $\bar{\mathcal{E}}^b$ if the two following properties are satisfied :

(i) for each element g of G', $Y \in \bar{\mathcal{E}}_g$ (cf. IV-7 above)

(ii) $\left\{ \sup_{g\in G',\ ||g||\leq 1} N_g(Y) \right\} = N^b(Y) < +\infty$

The mapping $Y \rightsquigarrow \int Y.d\tilde{M}$ is extendable in a continuous linear mapping from $\bar{\mathcal{E}}^b$ (with the topology associated to $N^b(.)$) into $\mathcal{C}_b(G', \mathcal{M})$. For each element Y of $\bar{\mathcal{E}}^b$, we shall note $\int Y.d\tilde{M}$ the cylindrical martingale associated to Y and we shall call it the stochastic integral of Y with respect to $\tilde{M}$.

IV-9 Theorem

We consider the hypotheses given in IV-1. In this case, for each element g of G', $\bar{\mathcal{E}}_g$ (cf. IV-7 above) contains the vector space of the processes Y with values in $\mathcal{L}(H, G_\sigma)$ (cf. I-5 above), "weakly predictable" (cf. the end of IV-1) and such that $N_g(Y) < +\infty$.

Moreover, $\bar{\mathcal{E}}^b$ (cf. IV-7 above) contains the vector space of the processes with values in $\mathcal{L}(H, G_\sigma)$, "weakly predictable" and such that $N^b(Y) < +\infty$ (where $N^b(Y) = \sup_{g\in G',\ ||g||\leq 1} \{ \int_{\Omega\times T} Q\,[Y^*(g) \otimes Y^*(g)].d\nu \}^{1/2}$).

Proof

The second part of this theorem is an easy consequence of the first part.

To prove the first part, we consider a "weakly predictable" process Y such that $N_g(Y) < +\infty$. Let g be an element of G'. Let $(\pi'_n)_{n>0}$ a sequence of projectors as in the lemma IV-4 above. We consider :

$$A_n = \{(\omega,t) : Q\,[(\pi'_n \circ Y^* \circ g)^{\otimes 2}] \leq 2\,Q\,[(Y^* \circ g)^{\otimes 2}]\ \}$$

$$Y_n = 1_{A(n)}\,.\,[Y \circ (\pi'_n)^*]$$

(Y^* is well-defined because H is reflexive)

The sequence of processes $Q\,[(\pi'_n \circ Y^* \circ g - Y^* \circ g)^{\otimes 2}]$

converges to zero (cf. the lemma IV-4 above) ; but, for each n, if $\omega \in A(n)$,

$$Q\,[(\pi'_n \circ Y^* \circ g - Y^* \circ g)^{\otimes 2}] \leq 3\,Q\,[(Y^* \circ g)^{\otimes 2}]$$

Then, we have $\lim_{n\to\infty} N_g(Y-Y_n) = 0$ by the Lebesgue dominated convergence theorem. Moreover it is easily seen than Y_n belongs to $\bar{\mathcal{E}}_g$ since $\langle Y_n, g\rangle$ is strongly predictable and $N_g(Y_n) < +\infty$. Then Y belongs also to $\bar{\mathcal{E}}_g$.

IV-10 Theorem

We consider the hypotheses given in IV-1. In this case, the mapping $Y \rightsquigarrow \int Y.d\tilde{M}$ *is a linear isometry from* $\bar{\mathcal{E}}^b$ *(with the semi-norm* $N_b(.)$*) into* $\mathcal{L}(G', \mathcal{M})$ *(cf. the end of I-4).*

Moreover if Y is an element of $\bar{\mathcal{E}}^b$ *, the quadratic Doléans's measure z of* $\vec{Z} = \int Y.d\tilde{M}$ *is the* $(G' \otimes_1 G')'$*-valued measure defined by*

$$z(g_1 \otimes g_2) = \int \{ Q\,[Y^*(g_1) \otimes Y^*(g_2)] \}\, dv$$

for each element $(g_1 \otimes g_2)$ *of* $G' \otimes G'$

If $\mathbb{H}$ *and* $\mathbb{G}$ *are reflexive Banach spaces, the process* $Q\,[Y^*(.) \otimes Y^*(.)]$ *is a* $(G' \hat{\otimes}_1 G')'$*-valued process.*

In this case, the total variation r of z (considered as a $(G' \hat{\otimes}_1 G')'$*-valued measure) is such that* $dr = ||Q\,[Y^*(.) \otimes Y^*(.)]\,||_{(G' \hat{\otimes}_1 G')'}\, dv$

Then, this total variation is σ-finite.

Proof

1°) The norm of $\int Y.d\tilde{M}$ considered as an element of $\mathcal{L}(\mathbb{G}', \mathcal{M})$ is equal to

$$\left|\left| \int Y.dM \right|\right|_{\mathcal{L}(\mathbb{G}', \mathcal{M})} = \sup_{g \in \mathbb{G}',\, ||g|| \leq 1} N_g(Y)$$

$$= ||Y||_b$$ then the mapping $Y \rightsquigarrow \int Y.d\tilde{M}$ is an isometry.

2°) Let Y be an element of $\mathcal{E}$ with $Y = \sum_{i \in I} u_i . 1_{A(i)}$ where the sets $(A(i))_{i \in I}$ are pairwise disjoint. Let i be an element of I and let B be an element of $\mathcal{R}$ contained in A(i) with $B = F \times]s,t]$. For each element (g_1, g_2) of $(\mathbb{G}' \times \mathbb{G}')$, we have :

$$(z(B))(g_1 \otimes g_2) = E\,\{1_F . [(\tilde{M}_t \circ u_i^* \circ g_1)(\tilde{M}_t \circ u_i^* \circ g_2) - (\tilde{M}_s \circ u_i^* \circ g_1)(\tilde{M}_s \circ u_i^* \circ g_2)]\}$$

$$= \int_B Q\,[Y^*(g_1) \otimes Y^*(g_2)]\, dv$$

Then the same equality is true for each predictable set B and for each element Y of $\bar{\mathcal{E}}^b$ by linearity and density.

3°) For each $(t,\omega) \in (T \times \Omega)$, $Y^*(t,\omega)$ is a continuous linear mapping from $\mathbb{G}'$ with the topology $\sigma(\mathbb{G}', \mathbb{G})$ into $\mathbb{H}'$ with the topology $\sigma(\mathbb{H}', \mathbb{H})$.

If $\mathbb{H}$ and $\mathbb{G}$ are reflexive Banach spaces, this implies than $Y^*(t,\omega)$ is a continuous linear mapping from $\mathbb{G}'$ with its strong topology into $\mathbb{H}'$ with its strong topology (cf. [4]).

Then, $Q\,[Y^*(.) \otimes Y^*(.)]$ induces a process with values in $(\mathbb{G}' \hat{\otimes}_1 \mathbb{G}')'$, weakly predictable. Then, the end of the theorem is evident.

V - MORE WHEN $\mathbb{H}$ AND $\mathbb{G}$ ARE HILBERT SPACES

V-1 Introduction

If $\mathbb{H}$ is a separable Hilbert space, the previous proofs can be a little simplified ; actually $\mathbb{H}$ and $\mathbb{H}'$ can be identified and, in the lemma IV-4, it is convenient to consider an orthonormal basis $\{x_n\}_{n>0}$ and the orthogonal projector on the space $\mathbb{H}'_n$ generated by $\{x_k\}_{1 \leq k \leq n}$.

Moreover, we have some other results.

V-2 We consider the hypotheses given in IV-1. Moreover, we suppose that $\mathbb{H}$ is a separable Hilbert space. The process Q takes its values in the cône of positive elements of $(\mathbb{H} \otimes_1 \mathbb{H})'$. Then, there exists a process $Q^{1/2}$, with values in the set of self-adjoint linear mappings from $\mathbb{H}$ into $\mathbb{H}$, such that, for each $(t,\omega,h) \in (T \times \Omega \times \mathbb{H})$,

$$||Q_t^{1/2} \circ h||^2_{\mathbb{H}}(\omega) = Q_t\,[h \otimes h]\,(\omega)$$

Then we have :

$$N_g(Y) = \int ||Q^{1/2} \circ Y^*(g)||^2_{\mathbb{H}} . dv$$

If $\vec{M}$ is a cylindrical brownian motion (cf. II-3 above), for each element (t,ω,h) of $(T \times \Omega \times \mathbb{H})$, $[Q_t^{1/2}(\omega)]\,(h) = h$.

V-3 Condition to obtain a genuine process

In the preceding parts, we have supposed that $\vec{M}$ is an element of $\mathcal{L}(H', \mathcal{M})$.

If H is a Hilbert space and if $\vec{M}$ is an element of $\mathcal{L}_2(H', \mathcal{M})$ (cf. I-4 above), there exists a genuine process M, with values in H, such that, for each element h of H, $\vec{M}(h) = \langle M,h\rangle$. In this case, the quadratic Doléans's measure of M takes its values in $(H \otimes_\varepsilon H)' = H \hat{\otimes} H$ which identifies itself as such, as a subspace of $(H \hat{\otimes} H)'$. Moreover, (cf. [16]), Q takes its values in $H \hat{\otimes} H$ and is strongly predictable.

If $\vec{M}$ is an element of $\mathcal{L}(H', \mathcal{M})$ and if H and G are Hilbert spaces, it is interesting to obtain a sufficient condition on Y such that $\vec{Z} = \int Y.d\vec{M}$ is an element of $\mathcal{L}_2(H', \mathcal{M})$: in this case, there exists a genuine process Z associated to $\vec{Z}$ as above.

The following theorem gives such a sufficient condition.

V-4 Theorem

We consider the hypotheses given in IV-1 and we suppose that H and G are separable Hilbert spaces.

Let Y be a $\mathcal{L}(H, G_\sigma)$-valued (cf. I-5) process weakly predictable (cf. the end of IV-1).

a) The process $||Y \circ Q^{1/2}||_{H.S.}$ is a real predictable process.

b) Then, we can define :

$$N_2(Y) = \{ \int_{T\times\Omega} ||Y \circ Q^{1/2}||^2_{H.S} \,.\, d\nu \}^{1/2} \leq +\infty$$

$$\bar{\mathcal{E}}^2 = \{ Y : Y \in \bar{\mathcal{E}}^b , N_2(Y) < +\infty \}$$

c) The mapping $Y \rightsquigarrow N_2(Y)$ is an hilbertian semi-norm on $\bar{\mathcal{E}}^2$ associated with the positive bilinear form defined by, if $(g_k)_{k\geq o}$ is an orthonormal basis of G :

$$(Y_1 , Y_2) \rightsquigarrow \int_{\Omega\times T} \{ \sum_{k=o}^{\infty} Q [Y_1^*(g_k) \otimes Y_2^*(g_k)] \}.d\nu$$

d) The mapping $Y \rightsquigarrow \int Y.d\vec{M}$ induces a linear isometry from $\bar{\mathcal{E}}^2$ into $\mathcal{L}_2(G, \mathcal{M})$; then, if Y is an element of $\bar{\mathcal{E}}^2$, there exists a genuine process Z, with values in G, such that, for each element g of G :

$$\langle Z,g\rangle = \vec{Z}(g) = (\int Y.d\vec{M})(g)$$

e) Moreover, if Y is an element of $\bar{\mathcal{E}}^2$, the quadratic Doléans's measure z of $\vec{Z} = \int Y.d\vec{M}$ is the $(G \hat{\otimes} G)$-valued measure defined by

$$z(g_1 \otimes g_2) = \int \{ Q \circ [Y^*(g_1) \otimes Y^*(g_2)] \} .d\nu$$

The $(G \hat{\otimes} G)$-valued "natural" process $\langle z\rangle$ associated to z is related to the $(H \hat{\otimes} H)'$-valued "natural" process $\langle m\rangle$ associated to m by (see III-3)

$$\langle z\rangle_t = \int_o^t d \langle m\rangle \circ (Y_s^* \otimes Y_s^*)$$

Proof

a) Let $(g_k)_{k\geq o}$ an orthonormal basis of G. We have :

$$(||Y \circ Q^{1/2}||_{H.S})^2 = \lim_{n\to\infty} \{ \sum_{k=o}^{n} ||Q^{1/2} \circ Y^*(g_k)||^2_H \}$$

$$= \lim_{n\to\infty} \{ \sum_{k=o}^{n} Q [Y^*(g_k) \otimes Y^*(g_k)] \}$$

then $||Y \circ Q^{1/2}||_{H.S}$ is a real predictable (cf. IV-5 1°) above) process (finite or infinite).

b) For each element g of G, we have $N_g(Y) \leq N_2(Y)$. Then, $\bar{\mathcal{E}}^2$ is the vector space of $\mathcal{L}(H, G_\sigma)$-valued and "weakly predictable" (cf. the end of IV-1) processes Y such that $N_2(Y) < +\infty$.

c) is evident. If $\tilde{Q}$ is canonically associated to Q, we remark that the considered positive linear form can also be written :

$$(Y_1, Y_2) \rightsquigarrow \int_{\Omega\times T} \mathrm{Tr}\ (Y_1 \tilde{Q} Y_2^*).d\nu$$

d) Let Y be an element of $\mathcal{E}$ with $Y = \sum_{i \in I} u_i . 1_{A(i)}$ and, $\forall\, i \in I$, $A(i) = F(i) \times]s(i), t(i)]$ (the sets $(A(i))_{i \in I}$ being pairwise disjoint) ; let $(g_n)_{n \geq o}$ an orthonormal basis in $\mathbb{G}$. The square of the norm of $\int Y.dM$ in $\mathcal{L}_2(G, \mathcal{H})$ is equal to :

$$(\| \int Y.d\tilde{M} \|_{\mathcal{L}_2(\mathbb{G}, \mathcal{H})})^2 = (\| \int Y.d\tilde{M} \|_{H.S.})^2$$

$$= \sum_{n=o}^{\infty} \| (\int Y.d\tilde{M}) (g_n) \|^2_{\mathcal{H}}$$

$$= \sum_{n=o}^{\infty} \{ \sum_{i \in I} E \{ 1_{F(i)} [(\tilde{M} o u_i^* o g_n)_{t(i)} - (\tilde{M} o u_i^* o g_n)_{s(i)}]^2 \} \}$$

$$= \sum_{i \in I} \int_{F(i) \times]s(i), t(i)]} \{ \sum_{n \geq o} \| Q^{1/2} o\; u_i^*(g_n) \|^2_{\mathbb{H}} \} d\nu$$

$$= \int_{T \times \Omega} \| Q^{1/2} o\; Y^* \|^2_{H.S} . d\nu$$

then the mapping $Y \rightsquigarrow \int Y.d\tilde{M}$ is an isometry .

e) The proof of e) is the same as the proof of IV-10 2°) (cf. also, V-3).

B SECOND PART

LOCALIZATION AND REGIONALIZATION

I - DOOB-MEYER DECOMPOSITION THEOREM

I-1 Definition

Let us writte $\mathcal{L}^p_t = L^p(\Omega, \mathcal{F}_t, P)$ where $p \geq o$. Let $\mathbb{B}$ be a Banach space. Then a family $\tilde{X} = (\tilde{X}_t)_{t \in T \subset \mathbb{R}}$, where $\tilde{X}_t \in \mathcal{L}(\mathbb{B} ; \mathcal{H}^p_t)$ for every t, will be called a p-process of stochastic linear functionals (S.L.F.) on $\mathbb{B}$.

If for every $h \in \mathbb{B}$, the real process $(\tilde{X}_t(h))_{t \in T}$ is a martingale, the process $\tilde{X}$ will be called a p-cylindrical martingale. This is a generalization of the definition given in A - II-1.

I-2 Doleans' measure of a process of linear functionals

We extend here the concept of Doleans' measure as first defined in [5] for real sub-martingale and extended since then to vector valued quasi-martingales (see, for ex., [15]).

To every process $\tilde{X}$ of S.L.F. on the Banach space $\mathbb{B}$, we associate the additive functions $\tilde{\alpha}_X$ with values in B' defined on the set $\mathcal{R}$ of predictable rectangles by

(I-2-1) $$\tilde{\alpha}_X( ]s,t] \times F) = E\, [1_F . (\tilde{X}_t - \tilde{X}_s)] \in \mathbb{B}'$$

Such a function on $\mathcal{R}$ has clearly an additive extension to the algebra $\mathcal{A}$ generated by $\mathcal{R}$. We call it $\tilde{\alpha}_X$ again.

I-3 Definition

If the additive function $\tilde{\alpha}_X$ on $\mathcal{A}$ has a bounded variation (for the norm of $\mathbb{B}'$), the process $\tilde{X}$ of stochastic linear functionals, will be called a <u>cylindrical quasi-martingale on B.</u>

This clearly generalizes the classical definition (see [15]). We have then the

I-4 Proposition

For $\tilde{X}$ to be a generalized quasi-martingale, it is necessary and sufficient that the family of real additive measures (α_X^h) associated with the real processes $(\tilde{X}(h))_{||h||\leq 1}$ be of bounded variation, and that the set of those variations $|\alpha_X^h|$ has a supremum in the ordered set of bounded positive measures.

Proof

This comes from the fact that the total variation of $\tilde{\alpha}_X$ can be approximated by sums of the type

$$\sum_i E(1_{F_i} |X_{t_i}(h_i) - X_{s_i}(h_i)|) \qquad h_i \in \mathbb{B},\ ||h_i|| \leq 1$$

while the supremum of the variations can be approximated by sums of the type

$$\sum_i |\alpha_X^{h_i}\,]s_i, t_i] \times F_i| = \sum_i |E\{1_{F_i}[X_{t_i}(h_i) - X_{s_i}(h_i)]\}|$$

It is easily seen that both supremum coincide.

I-5 Doob-Meyer decomposition theorem

Let $\tilde{X}$ be a cylindrical quasi-martingale on B (cf. I-3 above). Let $\alpha_{\tilde{X}}$ be the Doleans's measure of $\tilde{X}$ and let v be the total variation of $\alpha_{\tilde{X}}$. Let $(V_t)_{t\in T}$ be the real "natural" process associated to v. Let $(Z_t)_{t\in T}$ be the B'-valued and weak predictable process, Radon-Nikodym derivative of $\alpha_{\tilde{X}}$ with respect to v. Let $(Y_t)_{t\in T}$ the process defined by :

$$Y_t = \int_0^t Z_s(\omega)\, .\, d\, V_s(\omega)$$

This integral being a weak integral calculated "by trajectories". Let $\tilde{Y}$ be the cylindrical process associated to Y by $\tilde{Y}(h) = \langle Y,h\rangle$.
Then, $\tilde{X} - \tilde{Y}$ is a cylindrical martingale. Moreover, for each element h of B, $\langle Y,h\rangle$ is the "natural" process associated to $\tilde{X}(h)$, up to an indistinguashibility.

Then to define the stochastic integral with respect to $\tilde{X}$, it is sufficient to define the stochastic integral with respect to $\tilde{Y}$ (weak integral "by trajectories") and with respect to the cylindrical martingale $\tilde{X}-\tilde{Y}$.

Proof

This theorem is a mere corollary of the theorem A - III-3 above. This theorem generalizes theorem in [18].

II - LOCALIZATION

II-1 Stopped cylindrical process

Let σ be a stopping time. Let $\tilde{X}$ be a cylindrical process on the Banach space $\mathbb{B}$. Let $\tilde{Z}$ the cylindrical process defined by :

$$\forall\, h \in \mathbb{B}',\quad [\tilde{Z}(h)]_t = [\tilde{X}(h)]_{t\wedge\sigma}$$

(where $[\tilde{X}(h)]_{t\wedge\sigma}$ is the real process $\tilde{X}(h)$ stopped at σ).

Then, we shall say that $\tilde{Z}$ is the cylindrical process $\tilde{X}$ stopped at σ and we shall note $\tilde{Z}_t = (\tilde{X}_t)_{t\wedge\sigma}$.

II-2 Local cylindrical process (definitions)

We shall say that $\tilde{X}$ is a local 2-cylindrical martingale (resp. a local cylindrical quasi-martingale), if there exists an increasing sequence $(\sigma_n)_{n>0}$ of stopping times such that.

(i) $\lim_{n\to\infty} P[\sigma_n < 1] = 0$

(ii) $\forall\, n$, *the cylindrical process $\tilde{X}$ stopped at σ_n is a 2-cylindrical martingale (resp. a cylindrical quasi-martingale).*

It is easily seen that the previous results (construction of the stochastic integral, Doob-Meyer decomposition theorem) can be extended to local cylindrical process as in the real case. The following proposition gives a sufficient condition to have a local 2-cylindrical martingale.

II-3 Proposition

We consider a Banach space such that its dual $\mathbb{B}'$ is separable and a cylindrical process $\tilde{X}$ on $\mathbb{B}$ such that :

(i) for each element h of $\mathbb{B}'$, $\tilde{X}(h)$ is a local square integrable real martingale,

(ii) for each stopping time σ, the mapping $(\tilde{X}_{t\wedge\sigma})$ from $\mathbb{B}'$, with its strong topology, into $L^2(\Omega,\mathcal{F},P)$, with its strong topology, is continuous.

Then, $\tilde{X}$ is a local 2-cylindrical martingale.

Proof

Let a be the positive measure defined on $\Omega\times T$ by $a = P\otimes\mu$ where μ is the Lebesgue measure on T. Let $(h_n)_{n>o}$ be a sequence of elements of $\mathbb{B}'$, dense in $\mathbb{B}'$. For each integer $n>o$, let $(\sigma'(n,k))_{k>o}$ be an increasing sequence of stopping times such that, for each integer $k>o$, $a(]\sigma'(n,k),1]) \leq 2^{-(n+k)}$ and the real process $\tilde{M}(h_n)$ stopped a $\sigma'(n,k)$ is a square integrable martingale. For each integer k, let be $\sigma(k)$ the stopping time defined by

$$\sigma(k) = \inf_{n>o} \sigma'(n,k) \text{ . We have :}$$

$$a(\,]\sigma(k),1]\,) \leq \sum_{n>o} a(\,]\sigma'(n,k),1]\,) \leq 2^{-k}$$

then the increasing sequence $(\sigma(k))_{k>o}$ satisfies the properties II-2-(i) and (ii).

III - CYLINDRICAL REGIONAL 2-DISTRIBUTION PROCESS

III-1 Definition

Let $\tilde{X}$ be a cylindrical process on the Banach space $\mathbb{B}$. We shall say that $\tilde{X}$ is a cylindrical regional 2-distribution process if there exists a sequence $(F(n))_{n>o}$ of elements of $\mathcal{F}$ and a sequence $(a_n)_{n>o}$ of σ-finite positive measures on $\mathcal{P}$ such that

(i) for each element h of $\mathbb{B}'$ such that $||h|| \leq 1$, for each element A of $\mathcal{P}$ and for each real $\mathcal{R}$-step process with $\sup_{t,\omega} |Y_t(\omega)| \leq 1$

we have $$\left(||1_{F(n)}\cdot\int_A Y.d\,\tilde{X}(h)||_{L^2(\Omega,\mathcal{F},P)} \right)^2 \leq a_n(A)$$

(ii) for each integer n, $P\,[F(n)] \geq 1 - 2^{-n}$

As in the real case, to define the stochastic integral $\int Y\,d\,\tilde{X}$ of a $\mathcal{L}(H, G_\sigma)$-valued process Y with respect to $\tilde{X}$, it is sufficient to define this stochastic integral for each element ω of F(n) (for each integer n).(cf [18]'). Then, it is sufficient to define this stochastic integral with respect to a cylindrical 2-distribution process in the following sense :

III-2 Definition

Let $\tilde{X}$ be a cylindrical process on the Banach space $\mathbb{B}$. We shall say that $\tilde{X}$ is a cylindrical 2-distribution process if there exists a σ-finite real measure a on $\mathcal{P}$ such that, for each element h of $\mathbb{B}'$ with $||h|| \leq 1$, for each element A of $\mathcal{P}$ and for each real $\mathcal{R}$-step process with $\sup_{t,\omega} |Y_t(\omega)| \leq 1$ we have $\left(||\int_A Y.d\,\tilde{X}(h)||_{L^2(\Omega,\mathcal{F},P)} \right)^2 \leq a(A)$

We remark that, in the real case, this definition is a little different from the definition of a 2-distribution process given in [18]. It is convenient for our purpose which is not the same as in [18].

III-3 The process Q

Let $\mathbb{H}$ be a separable Hilbert space. Let $(h_n)_{n>o}$ be an orthonormal basis in $\mathbb{H}$. Let $\tilde{X}$ be a cylindrical 2-distribution process. Let a be a σ-finite positive measure on $\mathcal{P}$ such that, for each element $(h \times B)$ of $(\mathbb{H} \times \mathcal{R})$ and for each real $\mathcal{R}$-step process Y, we have :

$$\left(||\int_B Y.d\,\tilde{X}(h)||_{L^2(\Omega,\mathcal{F},P)} \right)^2 \leq ||h||^2_{\mathbb{H}} \,.\, a(B)$$

For each integer n, we consider the set J_n of all σ-finite positive measures m such that, for each real $\mathcal{R}$-step process Y and for each element B of $\mathcal{R}$ we have $\left(||\int_B Y.d\,\tilde{X}(h_n)||_{L^2(\Omega,\mathcal{F},P)} \right)^2 \leq m(B)$ (of course, a is an element of J_n). Let a_n be the lower bound of J_n and let S_n be the Radon-Nikodym derivative of a_n with respect to a (S_n is the lower bound in $L_1(\Omega\times T,\mathcal{P}, a)$ of the Radon-Nikodym derivatives $\frac{dm}{da}$ for $m \in J_n$).

For each element $(h \otimes h')$ of $\mathbb{H} \otimes \mathbb{H}$ with $h = \sum_{k>o} \alpha_k . h_k$ and $h' = \sum_{k>o} \alpha'_h . h_k$ and for each element ω of Ω we define :

$$Q(h \otimes h')(\omega) = \sum_{k>o} \alpha_k . \alpha'_k . S_k(\omega)$$

By hypothesis, for each integer k, $|S_k(\omega)| \leq 1$. Then, for each element ω of Ω, $Q(.)(\omega)$ is a real linear mapping defined on $(\mathbb{H} \otimes \mathbb{H})$ and this mapping is continuous for the trace-norm on $\mathbb{H} \otimes \mathbb{H}$: then, this mapping is extendable in a real linear continuous mapping defined on $(\mathbb{H} \hat{\otimes}_1 \mathbb{H})$ that we shall note also $Q(.)(\omega)$. The process Q is a $(\mathbb{H} \hat{\otimes}_1 \mathbb{H})'$-valued process.

III.4 The stochastic integral

Then, it is easily seen that the construction of the stochastic integral given in the paragraph A-IV above can be extended in the present case. Notably, if $\mathbb{G}$ is a Hilbert space, if Y a $\mathcal{A}$-step process with values in $\mathcal{L}(\mathbb{H}, \mathbb{G}_\sigma)$, for each element A of $\mathcal{A}$, for each element g of $\mathbb{G}$ with $||g|| \leq 1$, we have

$$\left(\left|\left| \int_A Y^*(g) . d\tilde{X} \right|\right|_{L^2(\Omega, \mathcal{F}, P)} \right)^2 \leq \int_A Q\left[Y^*(g) \otimes Y^*(g)\right] . da$$

(result analoguous to that of the remark A-IV - 3).

Let Y be a $\mathcal{L}(\mathbb{H}, \mathbb{G}_\sigma)$-valued weakly predictable process such that there exists an increasing sequence $(A(n))_{n>o}$ of element of $\mathcal{P}$ with :

$$\sup_{g \in \mathbb{G}, ||g|| \leq 1} \left\{ \int_{\Omega \times T} Q\left[Y^*(g) \otimes Y^*(g)\right] . da < +\infty \right\}$$

Then the stochastic integral $\tilde{Z} = \int Y . d\tilde{X}$ can be defined as in A-IV and $\tilde{Z}$ is a cylindrical 2-distribution process.

IV - EXAMPLE

This example shows that for a 2-cylindrical martingale $\tilde{M}$ on a Hilbert space $\mathbb{H}$, the total variation of the quadratic Doléans's measure m of $\tilde{M}$ is not necessarily finite (this total variation being calculated for m considered as a measure with values in $(\mathbb{H} \hat{\otimes}_1 \mathbb{H})'$).

Construction of the example :

Let (t_n) be a decreasing sequence to zero. Let $\mathbb{H}$ be an Hilbert space, (e_n) an orthonormal basis of $\mathbb{H}$ and let us define the cylindrical martingale :

$$\tilde{M}_t(h) = \sum_n 1_{[t_{n+1}, \infty[}(t) \; (h \mid e_n) \; . \; \frac{1}{\sqrt{t_n - t_{n+1}}} (\beta_{t_n \wedge t} - \beta_{t_{n+1}})$$

where (β_t) is a usual real standard Brownian motion. For every n, $(\tilde{M}_t(e_n))_{t \in \mathbb{R}^+}$ is a process with independant increments, zero on $]0, t_{n-1}]$, path-wise constant on $[t_n, \infty[$. For every $h \in \mathbb{H}$, the above series converge in $L^2(\Omega, \mathcal{F}_t, P)$ with

$$E\, |\tilde{M}_t(h)|^2 \leq ||h||^2_{\mathbb{H}}$$

defining a process with zero-mean independant increments. But considering the partition $( ]t_{n+1}, t_n] \times \Omega)_{n>o}$ of $]0, t_1] \times \Omega$, it is immediately seen that, $\tilde{X}$ beeing the process above defined :

$$\sum_n E \, || 1_\Omega (\tilde{X}_{t_n} - \tilde{X}_{t_{n+1}}) ||_{\mathcal{L}(\mathbb{H} \hat{\otimes} \mathbb{H})} \geq \sum_n E \frac{1}{\sqrt{t_n - t_{n+1}}} (\beta_{t_n} - \beta_{t_{n+1}})^2 = +\infty$$

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