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Some Open Problems in Ergodic Theory

SOME OPEN PROBLEMS IN ERGODIC THEORY
by
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This monograph is meant to be a sequel to my book ERGODIC THEORY, RANDOMNESS AND MECHANICAL SYSTEMS, and I will mainly consider open problems and general directions in which I think further research would be fruitful.

I will be fairly non-technical and the only background I will assume will be the introduction to my book or a survey article like [4] or [5].

In order to put the open problems in the right perspective I will start with a brief summary of the main results in my book.

Summary of Old Results

(1) Two Bernoulli shifts of the same entropy are isomorphic (Let Y

be a space with k points having measures $p_1 \cdots p_k$, $p_i > 0$, $\Sigma p_i = 1$.

Let Y_i be copies of Y . Let $X = \prod_{-\infty}^{\infty} Y_i$ be the product space with

product measure. $T_{p_1 \cdots p_k}$ shifts each sequence in X and is called

a Bernoulli shift. $T_{p_1 \cdots p_k}$ is isomorphic to $T_{q_1 \cdots q_e}$ if and

only if $\Sigma p_i \log p_i = \Sigma q_i \log q_i$.)

(1a) A more general result is: Two transformations of the same entropy
with "finitely determined" generators are isomorphic.

(1b) Two Bernoulli shifts of infinite entropy are isomorphic.

(2) If T is Bernoulli and P any partition, then (a) P,T is

"finitely determined" (b) P,T is "Very weak Bernoulli" (c) P,T

is the $\bar{d}$ limit of multi-step mixing Markov processes (see[1] for the

above definitions). Furthermore, if P,T satisfies (a), (b), or (c) .

then T acting on $\bigvee_{-\infty}^{\infty} T^i P$ is Bernoulli.

The above come from an analysis of the proof of (1). Two immediate

applications are that factors and roots of Bernoulli shifts are Bernoulli.

(3) The existence and uniqueness of Bernoulli flows

(We call S_t Bernoulli if S_{t_0} is a Bernoulli shift for some

t_0 . There is a flow, $\bar{S}_t$, which is simple to describe, and is

Bernoulli for all t . Furthermore, any Bernoulli flow is isomorphic

to $\bar{S}_{ct}$ for some c . [we get $\bar{S}_{ct}$ from $\bar{S}_t$ by flowing at velocity

c instead of velocity 1]. There also exists a unique Bernoulli flow

of infinite entropy). The criteria of (2) play a crucial role in (3).

(4) The following specific transformations or flows are Bernoulli

    (a) ergodic automorphisms of the n-torus

    (b) geodesic flow on a surface of negative curvature

    (c) multi-step mixing Markov shifts

The above are proved by an analysis of the specific systems which give criteria (2.b.). The results are due to Katznelson, Weiss, Friedman and myself.

There is also a group of negative results about K-automorphisms that involve a different circle of ideas. The central result is

(5) There is a K-automorphism that is not Bernoulli.

I will now try to outline some of the areas where I think further research would be fruitful.

K-automorphisms

I will start by listing some results that are not in my book.

(1) There is a transformation T that is not the direct product of a K-automorphism and a transformation of 0 entropy (T can even be taken to be mixing) [6], [7].

(2) Shields and I constructed an uncountable family of non-isomorphic K-automorphisms of the same entropy [8].

(3) The transformations in (2) are not isomorphic to their inverses [8].

(4) There is a K-automorphism with no square root (Jack Clark constructed a K-automorphism with no roots at all) [9].

(5)   Smorodinsky constructed a K-flow that is not Bernoulli [10].

(6)   Dan Rudolf constructed two non-isomorphic K-automorphisms with isomorphic squares (he can even get all powers > 1 to be isomorphic).

We should note that all but (1) are examples of properties that hold for Bernoulli shifts but not for K-automorphisms. There is a large class of questions of this sort. The following are still open.

(1)   Is every K-automorphism a direct product?   Could one of the factors be taken to be Bernoulli? Is the above also true for transformation of positive entropy?

(2) If two K-flows are isomorphic at all times, are they isomorphic? (the answer is not known even dropping the K restriction) Dan Rudolf constructed 2 K-flows that are isomorphic at all rational times but not at all times.

(3) Are weakly isomorphic K-automorphisms isomorphic ? (weak isomorphism means that each transformation is a factor of the other) Steve Polit showed that there are two non-isomorphic, mixing transformations that are weakly isomorphic.

We still do not know any "physical" or "natural" examples of non-Bernoul. K-automorphisms.

Factors

By a factor we mean the action of T on an invariant sub-sigma algebra.

Twenty years ago it was not even clear that every Bernoulli shift had a proper factor. The entropy theory, of course, shows that any transformation of positive entropy has lots of factors. In [6] there is an example of a mixing transformation with no factors.

What transformations can arise as factors of a given transformation? (the K-automorphisms are exactly those transformations having no factors of 0 entropy and transformations with continuous spectrum are those having no factors isomorphic to a rotation of the circle)

In the case where T is Bernoulli we have a complete answer – every factor is Bernoulli.

In the case of a K-automorphism one can try to determine the lattice structure of the Bernoulli factors.

Is there a unique maximal Bernoulli factor? or equivalently is the span of two Bernoulli factors Bernoulli?

We know the following: if T is a non-Bernoulli K-automorphism then for some k, T^k has 2 Bernoulli factors whose span is not Bernoulli (since the proof gives a simple illustration of how one applies the isomorphism theory we repeat it here).

Let P generate under T. Since T is K, $(T^k)^i P$ is ϵ independent for some k. The isomorphism theory implies that there is a $\tilde{P}$ close to P such that $(T^k)^i \tilde{P}$ is independent. Thus, $\tilde{P}$ generates a B-factor under T^k. If the span of B were B, $\bigvee_0^{k-1} T^i \tilde{P}$ would generate a B factor under T^k. This T^k factor is now invariant under T and since roots of B are B we get that it is B under T. Hence, $\tilde{P}, T$ if F.D. Thus P, T is the $\bar{d}$ closure of F.D. processes and hence is F.D. (this is a contradiction since T is not Bernoulli)[2].

A related question is: is every K-automorphism spanned by its Bernoulli factors?

The next kind of question and the one that we will be mainly concerned with here is: how is a factor imbedded in a transformation. We will say

that two factors of T are imbedded in the same way if there is an
automorphism of T taking one factor onto the other. We will be
especially interested in the case when T is Bernoulli.

There are several qualitative ways in which a factor $\mathscr{A}$ can be
imbedded.

(1) $\mathscr{A}$ can split off (that is, T is the direct product of $\mathscr{A}$ and
an orthogonal factor $\bar{\mathscr{A}}$) . If T is Bernoulli $\bar{\mathscr{A}}$ is automatically
Bernoulli. Otherwise we can study the case where $\bar{\mathscr{A}}$ is Bernoulli.

(2) Any factor properly containing $\mathscr{A}$ has strictly larger entropy
(we will say in this case that $\mathscr{A}$ is maximal given its entropy or
simply maximal).

(3) $\mathscr{A}$ has the same entropy as T .

We will now discuss these cases in more detail.

(1) $\mathscr{A}$ splits off with a Bernoulli complement. A very interesting
and deep recent development is Thouvenot's relativised isomorphism
theory, which deals with this case [11],[12],[13]. If P and H are partitio
Thouvenot introduces the idea of P being "finitely determined relative
to $\mathscr{A}$ " or "F.D. rel H" . (P is F.D. rel H if given ε there
is a δ and an n such that if $\bar{T},\bar{P},\bar{H}$ satisfies

(1) $\bar{T},\bar{H} \sim T,H,$ (2) $|H(P \vee H,T) - H(\bar{P} \vee \bar{H},\bar{T})| < \delta$

(3) $|\text{dist} \overset{n}{\underset{0}{\vee}} (T^i P \vee H) - \text{dist} \overset{n}{\underset{0}{\vee}} \bar{T}^i (\bar{P} \vee \bar{H})| < \delta$

then $\bar{d}_{H,\bar{H}}$ (P $\vee$ H,T , $\bar{P}$ $\vee$ $\bar{H},\bar{T}$) $< \varepsilon$ where $\bar{d}_{H,\bar{H}}$ means that when
superimposing the $T^i(P \vee H)$ on the $\bar{T}^i(\bar{P} \vee \bar{H})$ the $T^i H$ must fit exactly.)

If the factor generated by P is Bernoulli and orthogonal to the factor generated by H then P is F.D. rel H .

The relativised isomorphism theorem says that that if P is F.D. rel H , $\overline{P}$ is F.D. rel $\overline{H}$, $H(P \vee H, T) = H(\overline{P} \vee \overline{H}, \overline{T})$, and $H, T \sim \overline{H}, \overline{T}$, then there is an isomorphism between T acting on $\overset{\infty}{\underset{-\infty}{\vee}} T^i(P \vee H)$ and and $\overline{T}$ acting on $\overset{\infty}{\underset{-\infty}{\vee}} \overline{T}^i(\overline{P} \vee \overline{H})$ taking H onto $\overline{H}$.

Thus, if $P \vee H$ generate and P is F.D. rel H , then $\overset{\infty}{\underset{-\infty}{\vee}} T^i P$ splits off with a Bernoulli complement.

Relativising the characterizations of partitions of Bernoulli shifts, Thouvenot gets that if $\overset{\infty}{\underset{-\infty}{\vee}} T^i H$ splits off with a Bernoulli complement then any P is F.D. rel H .

The idea of V.W.B. can also be relativised (Thouvenot-Rahe).

Here are some applications

(a) (Thouvenot)[12]Any factor of the direct product of a transformation of 0 entropy and a Bernoulli shift also has this form. (Is the same true for 0 entropy x K automorphism?)

(b) (Thouvenot and Shields)[13] The processes arising from transformation of the above form are $\overline{d}$ closed.

In the case where T is a Bernoulli shift we have

(c) (Thouvenot) If $T^i(P \vee Q)$ are independent, then $\overset{\infty}{\underset{-\infty}{\vee}} T^i P$ splits off.

Rahe showed that if we lump together some states of a Markov process with no transitions of 0 probability then the resulting factor splits off.

(d) (Rahe) If $\mathscr{A}$ splits off under T^2 , then it splits off under T .

(e) The special examples of K-automorphism that are not Bernoulli, constructed by Shields and myself all have a Bernoulli factor that splits off.

Here are some open questions that the relativised theory has a good chance of solving

(a) What is the lattice structure of the factors of a Bernoulli shift that split off? Are they closed under intersections and spans?

(b) Does every transformation of positive entropy (or K-automorphism) have a Bernoulli factor that splits off?

(c) If every K-factor of T is Bernoulli is T the direct product of a Bernoulli shift and a transformation of 0 entropy? (the converse follows from Thouvenot's theorem (a))

The above gives one the feeling that if $\mathscr{A}$ splits off with a Bernoulli complement, then the way in which $\mathscr{A}$ is imbedded, is in some sense "Bernoulli", or that in some sense the "transformation relative to $\mathscr{A}$" is Bernoulli.

(2) $\underline{\mathscr{A} \text{ is maximal given its entropy.}}$ We will now only consider the case where T is Bernoulli. It is not hard to see that if $\mathscr{A}$ splits off, then $\mathscr{A}$ is maximal given its entropy, and it is natural to ask whether the converse is true. In [15] we show

(a) There is a factor of a Bernoulli shift that is maximal given its entropy but does not split off.

The case where $\mathscr{A}$ is maximal corresponds, in some sense, to the theory of K-automorphisms. The reasons for believing this are the following: Since $\mathscr{A}$ is not contained in a proper factor of the same entropy, the action of T relative to $\mathscr{A}$ is somehow analogous to the action of a transformation with no factors of 0 entropy. Furthermore, the factor in (a) is obtained by taking a skew product with a K-automorphism that is not Bernoulli. Thus, (a) can in some sense be regarded as a "relativised" version of the existence of a K-automorphism that is not

Bernoulli. ((a) intertwines the positive theory of Bernoulli shifts and the negative theory of K-automorphisms and in fact we use the positive theory to prove the skew product to be Bernoulli.)

(The example of a transformation whose 0-entropy factor does not split off can also be thought of as a "relativised" version of a K-automorphism that is not Bernoulli.)

The above analogies lead to the following conjectures (in the case when T is Bernoulli).

(b) There are uncountably many different ways that a maximal factor of given entropy can be imbedded.

(c) There are maximal factors that are imbedded the same way under T^2 but not T .

(d) There is a maximal factor that is imbedded differently under T and T^{-1} .

(e) There is a maximal factor that is not invariant under any square root of T .

(3) $\mathscr{A}$ has the same entropy as T. In this case the action of T "relative to $\mathscr{A}$ " is analogous to a 0-entropy transformation. We have no results about this case but one would expect notions of mixing, weak mixing, etc. to relativise in a reasonable way, and we could ask how much of the 0-entropy theory relativises. The simplest question along these lines is the following:

(a) If two factors of a Bernoulli shift, both have 2 point fibers, are they imbedded in the same way?

Two related questions are

(b) If T is the skew product of a Bernoulli shift and a transformation acting on two points, is it true that either the skew product is a direct produc or T is Bernoulli?

(c) Rudolph has shown that one can start with a fixed K-automorphism that is not Bernoulli and form uncountably many non-isomorphic K-automorphisms by taking skew products with transformations on 2 points.

We could study the ways of imbedding partitions as well as factors. (we say that two partitions, P and $\overline{P}$ are imbedded in the same way if there is an automorphism of T taking P onto $\overline{P}$). The question then arises as to whether this is the same problem as classifiying factors, that is: If P and $\overline{P}$ are imbedded differently are the factors they generate imbedded differently? Equivalently: Does every automorphism of a factor extend to an automorphism of T ? The same example as (a) above gives us:

There is a maximal factor $\mathscr{A}$ of a Bernoulli shift and an automorphism of $\mathscr{A}$ that does not extend to T .

Does every maximal factor that does not split off have a non-extendable automorphism?

Another kind of question about factors is the extent to which they determine the transformation. Thouvenot [14] – extending a result of Shields – showed that if T is Bernoulli and $\overline{T}$ any transformation on the same space having the same factors then $\overline{T}$ is T or T^{-1} . This result is not completely general since there are non-isomorphic (0-entropy-mixing) transformations with no factors. What happens for K-automorphisms?

The study of factors is closely related to the study of automorphisms. What automorphism can commute with a Bernoulli shift, T ? (the answer is easier for infinite entropy because we can represent T as a shift on the product of intervals.) Is T equal to its double commutator?

Smooth partitions

There are many examples of diffeomorphisms of compact manifolds with a smooth invariant measure that are known to be Bernoulli (we already mentioned the ergodic automorphisms of the n-torus and geodesic flow on surfaces of negative curvature). In these cases, the differential structure singles out a special class of partitions - the smooth partitions. (we say that a partition is smooth if the boundary of each set in the partition is a compact piecewise smooth differentiable submanifold).

For the moment let us restrict our attention to an ergodic automorphism of the 2 torus.

Bowen [16] proved the following:

(1) Any smooth partition is "weak Bernoulli" - a stronger property than "very weak Bernoulli".

The following is an example of the kind of pathology that "weak Bernoulli" rule out: It is shown in [48] that a Bernoulli shift, T , has a partition, P , such that $\bigvee\limits_{i > |n|} T^i P$ is the whole σ-algebra for all n . Such a P,T cannot be "Weak Bernoulli" and thus smooth partitionscannot exhibit the above pathology.

(2) There is no smooth independent generating partition.

((2) is striking in the light of Berg, Adler and Weiss' explicit construction of a smooth - actually piecewise linear-generating Markov partition.)[17], [18]

It is not known whether or not there are smooth independent partitions (that do not generate).

The Thouvenot theory can be applied to this case to show:

(3)   If the factor $\mathscr{A}$ has a smooth generator and is maximal given its entropy then it splits off.   (We thus have maximal factors with no smooth generators.)

It may be possible, in the light of (3) to classify the factors with a smooth generator.

The above is not restricted to automorphisms of the 2-torus. Bowen proved that (1) holds for any Anosov diffeomorphisms and (2) holds for any Anosov diffeomorphisms of the n-torus, the question being left open for general Anosov diffeomorphisms.

The next system to study would be the geodesic flow on surfaces of negative curvature.

It would also be interesting to study partitions into sets whose boundaries have measure 0 . (In the case of mechanical systems such a partition would represent a measurement that, with probability one, would not change under small perturbations.) Can we find such a partition that is independent, or an independent generator?

The action of more general groups

The study of a single transformation and its iterates can be thought of as the study of the action of the integers Z , while the study of flows is the study of the action of the reals R . In the case of mechanical systems the action of Z and R usually represents the passage of time. Mechanical systems can, however, have other automorphisms, and this provides some motivation for the study of the action of general groups.

Weiss pointed out that for any countable group   G and probabilities $p_1 \cdots p_k$ $(\Sigma p_i = 1)$ we have a Bernoulli $G_{p_1 \cdots p_k}$ action.   (This is defined as follows: Let Y have k points having measure $p_1 \cdots p_k$. Let $X = \prod_{i=G} Y_i$ where Y_i is a copy of Y , and the measure is product

measure. Thus, each $x \in X$ is a sequence $\{a_i\}_{i \in G}$, $a_i \in Y$, and if $g \in G$ then $g\{a_i\} = \{b_i\}, b_i = a_{i \cdot g}$. This is the usual definition when $G = Z$.)

If G is an uncountable group of measure preserving transformation, we call G Bernoulli if every countable subgroup is Bernoulli. In this case, as in the case of R, existence requires proof.

Katznelson-Weiss and Thouvenot [19], [20], and [24a] showed that for the n-dimensional integers Z^n, the isomorphism theory of Bernoulli shifts still holds:

(1) $Z^n_{p_1 \cdots p_k}$ is isomorphic to $Z^n_{q_1 \cdots q_e}$ if and only if

$\Sigma p_i \log p_i = \Sigma q_i \log q_i$.

The definitions of Finitely Determined and Very Weak Benoulli carry over and we get

(1a) Two Z^n actions are isomorphic if they have F.D. generators of the same entropy

(2) Any partition under a Z^n Bernoulli action is F.D. and V.W.B. . Furthermore, if a partition is F.D. or V.W.B. under Z^n, then Z^n acting on its span is Bernoulli (this implies that factors of Bernoulli actions are Bernoulli).

The above results were applied by Gallavotti, deLiberto and Russo to Ising spin systems [20], [21], [22], and [24b]. A 3-dimensional Ising spin system is a 3-dimensional lattice of particles, each of which has two possible spins. We can describe a configuration of the system by assigning a +1 or -1 to each integer point in 3-space. The Ising model consists of a probability distribution on these configurations.

It is usually assumed that this measure is invariant under spacial shifts. We thus have Z^3 acting on our probability space. In most cases this action can be shown to be Bernoulli (by establishing the V.W.B.

property).

The difficult part in extending the isomorphism theory to Z^n lies in the analogue of the Rochlin-Kakutani theorem. Lind [23][24] proved an analogue of this theorem for R^n . He then showed that R^n Bernoulli actions exist and the isomorphism theory extends to them.

Krieger [24c] and Kiefer [24d] extended the isomorphism theorem for Bernoulli shifts to a class of countable groups containing all infinite abelian groups

The kinds of questions that are open are the following

(1) For which groups does the isomorphism theory work? Are any two Bernoulli actions with the same entropy, of any countable groups isomorphic?

(2) How much of what we know about Z actions works for R actions or Z^n or R^n actions? For example, does the Thouvenot theory work for R ? How does one characterize the continuous time processes arising from Bernoulli flow, or the Z^n processes arising from Bernoulli actions (other than F.D. or V.W. B.)

(3) Which counterexamples extend to Z^n or R or R^n actions that are K but not Bernoulli?

(4) We can study certain types of behavior that can not arise in the case of Z or R . For example, we could study Z^n actions that are not Bernoulli but where each transformation is Bernoulli. Are there K-actions of Z^n where no transformation is Bernoulli?

Classical examples

Certain specific classical systems are known to be Bernoulli (we already mentioned the ergodic automorphism of the n-torus and geodesic flow on surfaces of negative curvature). I think that one of the most interesting and important features of the theory is that it can be applied to specific classical examples, and that there is an interplay between pure ergodic theory and other parts of mathematics. Because of the richness of this area I will be forced to be very sketchy, but I think that there is a lot to be done along these lines and that the future of the subject depends to a large degree in its application to other parts of mathematics.

(1)   Ergodic automorphisms of compact groups

Rokhlin and Yuzvinskii showed that all such automorphisms are K [25] and [26]. Bernoulliness was proved for certain special groups (n-torus by Katznelson [27], solenoid by Katznelson and Weiss [49], T^{∞} by Lind [28] and independently by Totoki [29] and a very wide class of groups by Lind). Thomas and Miles have recently shown that the ergodic automorphisms of any compact group are Bernoulli. Their method makes heavy use of number theory — to check the VWB property. Lind was able to make substantial simplifications using the Thouvenot theory.

The torus generalizes to locally compact Lie Groups modulo descrete subgroups. This case is still open.

(2)   Anosov systems and Axiom A .

Sinai and Anosov showed that any Anosov flow with a smooth invariant measure is K (except in the case of the suspension of an Anosov diffeomorphism under a constant function). [30],[31] Bunimovich and Ratner [32], [33] extended the Sinai-Anosov result to prove Bernoulliness (under the same conditions). These results were then extended to systems satisfying Smale's Axiom A and to Axiom A attractors by Bowen and Ruelle [34], [35], and [36]. The latter result sheds some light on turbulence.

(3)   Billiards and the hard sphere gas

Sinai showed that the flows arising from the motion of a billiard ball on a square table with a convex obstance or the motion of a hard sphere gas in a box are K . Sinai's analysis can be used to get the V.W.B. criterion [37], [38] and [39] and thus the above flows are Bernoulli.

(4)   Nothing is known about a gas in a box where the particles can attract and repel each other. Not even ergodicity is proved.

I should remark here that it might be possible to get some information about the ergodic properties of a mechanical system even in cases where ergodicity cannot be proved or more accurately where the manifolds in the phase space on which it is ergodic cannot be identified. One could still hope to prove that on each such manifold the transformation is K or Bernoulli. We would thus know what kinds of measurements can arise from such systems.

(5)   Connections with the K.A.M. theory

It would be interesting to try to apply the Bernoulli theory to systems to which the Kolmogorov-Arnold-Moser theory applies. The K.A.M. theory allows one to prove stability or non-rendomness, the opposite in some sense of the very random Bernoulli behavior. Thus, one might be able to find systems for which the K.A. M. theory gives stability at low energies but is Bernoulli at high energies (or where Bernoulli components arise in the phase space after small perturbations). I. Kubo and Murata recently got some results in this direction [39a and [39b].

(6)   A statistical version of "structural stability". One example of statistical stability is the following: Consider geodesic flow on a surface of negative curvature and perturb the surface slightly. The perturbed flow is still a geodesic flow on a surface of negative curvature, hence, it is still a Bernoulli flow. We thus have that the perturbed flow is isomorphic

to the original flow after a rescaling of the time perimeter.

Results of Bowen and Ruelle [34], [36] show that small perturbations in the surface produce small changes in entropy then the above rescaling will be small. This implies that except for a set in the phase space of small measure the isomorphism moves points a small distance.

Which systems are statistically stable?

Theorems that say a certain class of processes are closed in the $\bar{d}$ sense also give a kind of stability result. For example, the processes P,T , T Bernoulli are $\bar{d}$ closed, and hence if Q,T is not Bernoulli it will remain not Bernoulli under small $\bar{d}$ perturbations (thus the K-automorphism that is not Bernoulli cannot become Bernoulli in the presence of a small amount of noise).

Are the processes that generate transformations satisfying the Pinsker conjecture $\bar{d}$ closed?

(7) Infinite particle systems

We already mentioned the Ising model (this is a static model and there is no time evolution defined for it). Liebowitz, Goldstein, Aizeman, Lanford, Presutti and Caldiera have studied infinite particle systems for which there is a time evolution [40], [41], [42], [42a], [42b], [42c]. Free particle systems are Bernoulli under space translations or time (this is easy to see) but not under space-time with respect to which they have 0-entropy. Liebowitz-Goldstein and Aizeman showed that the time evolution of freely moving hard rods (with velocities bounded away from 0) is Bernoulli, but not with respect to space-time. The implications of this is that the randomness of infinite particle systems should be reflected by their space-time ergodic theory but not by their space or time ergodic

theory separately. So far we have no reasonable examples of infinite particle systems that can be proved to be space-time Bernoulli.

(8) Number theoretic transformations

Several number theoretic transformation have been studied by Smorodinsky, Wilkinson, Rudolfer, Ito, Murata, Totoki, Adler and Weiss. The simplest examples are $x \to \beta x \mod 1$ for $\beta > 1$ and the continued fraction transformation [43], [44], [45], [46], [46a], [46b], [46c]. These transformations are not $1 - 1$, but there is a natural extension to a $1 - 1$ transformation and these have been shown to be Bernoulli. (Another way of saying this is the following: Take, for example, continued fractions. The continued fraction expansion of a number gives us a sequence of integers. We thus get a measure on sequences of integers. This measure could also be obtained from a process P,T where T is Bernoulli and we only look at the process from time 0 on. Thus, a.e. number has a continued fraction expansion, in which the frequencies of finite strings is that given by a process P,T, T Bernoulli.)

(9) Transformations described by simple formulas

There is a simple skew product which is easily seen to be K, but is probably not Bernoulli, although this has not been proven. Let T_1 and T_2 be 2-shifts (Bernoulli shifts on independent sequences of 0 and 1's). $T(x,y) = (T_1 x , T_2^{f(x)} y)$ where $f(x)$ is -1 or 1 depending on the oth coordinate of x.

Adler and Shields [66] showed that if T_2 is an irrational rotation of a circle, then the above skew product is Bernoulli.

(10)   Differential structure

Ergodic theory came into being by abstracting the statistical

structure of mechanical systems and ignoring the differential or

topological structure.   I think that it is important to study the

interplay between these structures.   We already discussed smooth

partitions.   A general question is the following: Which abstract

transformations can arise from diffeomorphisms on a compact manifold

with a smooth invariant measure?   Because of the example of geodesic

flow we know that Bernoulli shifts can have a differential structure.

On the other hand, Kushnirenko showed [47] that only transformations

of finite entropy can have a differential structure.

Does every transformation of finite entropy have a differential

structure?   Are there non-Bernoulli K-automorphisms with a differential

structure?   What manifolds can support a Bernoulli flow?   (are there any

topological obstructions to randomness?)

(11)   Topological structure

In case T_1 and T_2 act on spaces X_1 and X_2 which have a

topological structure and are homeomorphisms we can ask if it is possible to

throw out invariant sets of measure   0 from X_1 and X_2 (obtaining

$\overline{X}_1$ and $\overline{X}_2$),and then find a   1 - 1   measure preserving homeomorphism

φ between $\overline{X}_1$ and $\overline{X}_2$ that commute with T_1 and T_2 .   In other

words, after ignoring sets of probability   0 , T_1 and T_2 have the

same statistical   and topological structure.   (In case T_1 and T_2 are

shifts with the product topology, the problem is just that of the

existence of finitary codes - discussed on page 23 .   From this point of

view, it is easy to see that a Bernoulli shift can have more than one

topological structure.)

(12)   Orbit structure

There is an old problem, which I believe is due to Kakutani. I will

try to explain it in our present context. Global analysis focusses on the

orbits of individual points in phase space while ergodic theory focusses on the

evolution of sets of points. One can ask how much of the statistical

or ergodic properties of a system are determined by the individual orbits.

One formulation of the problem is the following: We say that flows

S_t and $\overline{S}_t$ are equivalent under a variable time change if we can change

the speed along orbits so as to make them isomorphic. The speed change

may vary from point to point and will not in general be a simple rescaling

of the time parameter. Properties such as mixing are no longer invariants.

The properties of having finite, infinite or 0 entropy will however

be preserved. Are these the only ones? Is every K-flow equivalent

to the Bernoulli flow? (If we represent the flow as a flow built under a

function, a speed change amounts to a change in the function.)

Information theory

Ideas from information theory have had a great impact on ergodic theory

and it is reasonable to hope that recent developments in ergodic theory

will have some impact on information theory.

The connection between ergodic theory and coding is the following:

Start with two finite alphabets. We define a (sliding block) code of

length $2k + 1$ from sequences in the first alphabet to sequences in the

second by assigning, to each sequence of length $2k + 1$ of letters in the

first alphabet, one letter in the second alphabet. This gives us a

map on sequences in which the ith cordinate of the image sequence is determined by the coordinates between $i - k$ and $i + k$ of our original sequence.

We say that a sequence of codes of length k , converge (as $k \to \infty$) on a process P,T , if for a.e. P, T sequence and all j , the j coordinate of the image sequence is the same for all sufficiently long codes.

We define an infinite code as the limit of finite codes.

Now, suppose P generates under T and Q is any partition. Q can be approximated arbitrarily well by sets in $\overset{k}{\underset{-k}{\vee}} T^i P$. This means that there is an infinite code, c , which applied to P,T gives Q,T . If Q also generates there will also be an infinite code $\bar{c}$ from Q,T to P,T and for a.e. P,T sequence, if we first apply c and then $\bar{c}$ we get back the sequence we started with.

Because of the above, certain coding problems translate into problems of constructing partitions. This is a very different viewpoint and certain things which can be seen from this viewpoint (like the Rokhlin-Kakutani theorem) become extremely complicated from the coding viewpoint.

We will now give three applications of the isomorphism theorem to coding. These are about infinite codes and are very theoretical. Their main point is that they give clean mathematical theorems, and the messier $\varepsilon - \delta$ results about finite codes can be read off from them and viewed as approximations to an ideal reality.

The following results will be restricted to B processes, (processes P,T such that T is Bernoulli) but since these include all mixing processes of finite memory and their $\bar{d}$ limits, the restriction

does not seem serious.

(1)   Noiseless channels

Suppose we have a channel that send k letters. If P,T is a
B process (i.e. T is Bernoulli) and H(P,T) ≤ log k , then P,T
can be coded (by an infinite code) into an independent process on k
letters, sent over the channel and recovered (by another infinite code
applied to the independent process).

Since infinite codes are the limits of finite codes we get results
about finite codes. (These codes have the advantage over the usual
block codes that they are stationary and can be improved without much of a
change.)

(2)   Noisy channels

For simplicity let us assume that our channel sends only 0's and 1's
and that we make an error with probability p independently of the past
errors and what is being sent. The capacity, C , of this channel is then
$|\log \frac{1}{2} - (p \log p + (1-p) \log (1-p)|$, Gray and I showed the following:
given ε , there is a process Q,T , on 2 symbols, T Bernoulli and
H(Q,T) > C - ε such that Q,T can be sent across our channel and
recovered exactly, with probability one   (it is impossible to get
H(Q,T)= C). (call such a Q,T invulnerable).

The point of the above is that, in some theoretical way it reduces the
noisy channel problem to the noiseless one. We can take any B process
P,T , H(P,T) = H(Q,T) , code it into Q,T , send Q, T , recover Q,T
and then recover P,T .

The invulnerable processes Q,T shed some light on the question of
how badly behaved B processes can be. It was shown in [48] that there

is a B process, Q,T such that $\bigcap_{n=1}^{\infty} \bigvee_{i > |n|} T^i Q$ is the whole σ-algebra.

This means that if we make a finite number of changes in Q,T , then the

original sequence can be recovered with probability one. The invulnerable

processes give a more striking example of this kind of behavior. Furthermore,

the coding from an independent process to an invulnerable process must

be very delicate because if we change one symbol of our independent

process, then with probability one the encloded sequences will be changed

in a "very large" number of places.

(3)   Cryptography

Suppose that you know that you are receiving a process P,T or a certain

automorphism of P,T , but you do not know which. Then there is no way

of deciding without further information (since the statistics of P,T or its

automorphism are the same). Since B-processes seem to include all processes

one might reasonably want to send, the automorphisms of B-processes seem to

be relevant to cryptography. If P,T is a B-process, then by definition

T is Bernoulli and because of the isomorphism theorem we can find an

independent generator Q with two sets of equal measure and thus an

automorphism. (A minor variant would be to code P,T in a non-secrete

way onto Q,T , then either send Q,T as is or exchange the two equally

probable letters in a secrete way. Then send Q,T or its automorphism

and recover P,T.)

The above applications are of course highly theoretical and highly

impractical. However, we hope,and there is some reason to belive,that

the partition viewpoint will actually be useful in producing real codes.

[50] , [51], [52], and [53].

(4)   One sided and finitary codes

One can study the existence of infinite codes with certain special properties. An infinite code is said to be one sided if it depends only on the past of the process. Sinai's original weak isomorphism theorem was one sided, thus any ergodic process can be coded one-sidedly onto any independent process of the same or smaller entropy, see [5]], [53]. It is not known if the above result remains true if we replace the independent process by a B-process. It is easy to see, however, that for two different independent processes of the same entropy we cannot find an invertible code such that both the code and its inverse depend only on the past. (We cannot even find invertible codes, φ , between certain independent processes such that φ and φ^{-1} only look a finite number of steps into the future.) The avove type of question is discussed very thoroughly in [56], [57], [58], [6 and [61]. One of the first one sided codings is due to Rosenblatt [63], [64],

We say that a code is finitary if each coordinate, i , of the encloded sequence depends on only a finite number k(w,i) of coordinates of our input sequence, w . Almost nothing is known about finitary codes. The codes produce by Adler and Weiss [18], Mesalkin [62] and Bowen [16] are finitary and one can construct B-processes that are not finitary codings of any independent process. Finitary codes and partitions with boundary of measure 0 are closely related. For example, a process can be represented as a partition of the 2-torus with measure-0 boundary if and only if it can be coded finitarily from the Adler-Wei partition (or any partitions P such that the diameter of the atoms in

$\bigvee\limits_{-n}^{n} T^i P$ goes to 0). Because the Adler-Weiss codes (between automorphisms of 2-tori of the same entropy) are finitary, they are also-after ignoring sets of measure 0-homeomorphisms.

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