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Minimal Sets Generated by a Substitution of Non Constant Length

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MINIMAL SETS GENERATED BY A SUBSTITUTION OF NON CONSTANT LENGTH

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Let P denote the alphabet $\{0,1\}$ and let

$$\Omega = P^{\mathbb{Z}} = \{\omega = \ldots\omega_{-1}\ \omega_0\ \omega_1\ \omega_2\ \ldots \mid \omega_i \in P\}$$

A substitution θ is a map from P to $\bigcup_{n\geq 2} P^n$. If card $\theta(0)$ = card $\theta(1$ the substitution is said of constant length. If not, it is said of non constant length.

For suitable i and j we can generate (1) a bisequence by the following:

$$\omega[-\text{card } \theta^{2n}(i),\ \text{card } \theta^{2n}(i)+\text{card } \theta^{2n}(j)] = \theta^{2n}(i)\ \theta^{2n}(j)\ ,\forall n \in \mathbb{N}$$

where $\omega(a,k)$ is the k-block of ω which begin at ω_a.

Under a condition of non-degeneracy, the dynamical system (O_ω,T) where O_ω is the orbit closure of ω, and T the shift, is minimal (1) and strictly ergodic (2) .

We can prove that the dynamical system associated with the substitution θ_1 defined by

$$\begin{cases} \theta_1(0) = 0^{n+1-p}\ 1\ 0^p \\ \theta_1(1) = 1\ \ 0^n \end{cases} \qquad \text{for } 0\leq p\leq n\ ,$$

has a purely discrete spectrum. (i^k, $i \in P$, $k \in \mathbb{N}$ means $\underbrace{i\ i\ldots i}_{n \text{ times}}$).

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(1) GOTTSCHALK. Substitution minimal sets. Trans. Amer. Math. Soc. 109 (1963)

(2) MICHEL. Stricte ergodicité d'ensembles minimaux de substitution. CRAS Paris, 278, (1974).

For this, we introduce the notion of "coïncidence-value of the substitution θ" by the following :

$$\text{if} \quad \theta^{\infty}(0) = \lim_{n\to\infty} \theta^{2n}(0) = \omega_1^0 \, \omega_2^0 \ldots \omega_n^0 \ldots$$

$$\theta^{\infty}(1) = \lim_{n\to\infty} \theta^{2n}(1) = \omega_1^1 \, \omega_2^1 \ldots \omega_n^1 \ldots ,$$

we say that n is a "coïncidence value of θ" if and only if

$$\omega_n^0 = \omega_n^1 .$$

If we denote the density of the coïncidence values of θ in $\mathbb{N}$ by $\delta_c(\theta)$, we can prove that $\delta_c(\theta_1)$ is equal to one.

Conversely if we consider the substitution θ_2 :

$$\begin{cases} \theta_2(0) = 0\ 1 \\ \theta_2(1) = 1\ 1\ 0\ 0 \end{cases}$$

we have $\delta_c(\theta_2) = \frac{1}{3}$ and we can prove that there is a continuous part in the spectrum of θ_2 .

Moreover, if we consider the two closely related substitutions θ_2' and θ_2'' defined by

$$\theta_2'(0) = \theta_2''(0) = 0\ 1$$

and

$$\begin{cases} \theta_2'(1) = 1\ 0\ 1\ 0 \\ \theta_2''(1) = 1\ 0\ 0\ 1 \end{cases} .$$

we can prove that θ_2' and θ_2'' have purely discrete spectra and that

$$\delta_c(\theta_2') = \delta_c(\theta_2'') = 1 .$$

More generally, we conjecture that the substitution θ_n :

$$\theta_n(0) = 0^n\ 1^n = \underbrace{0\ldots0}_{n \text{ times}}\ \underbrace{11\ldots1}_{n \text{ times}}$$

$$\theta_n(1) = 1^{2n}\ 0^{2n} = \underbrace{11\ldots1}_{2n \text{ times}}\ \underbrace{00\ldots0}_{2n \text{ times}} ,$$

has :

1) a purely discrete spectrum if n is even (then $\delta_c = 1$)

2) a mixed spectrum (partly continuous and partly discrete) if n is odd (then $\delta_c = \frac{1}{3}$).

At last, if we consider the constant-length case (3), we know that the so-called

- regular Toeplitz substitution generates a dynamical system with discrete spectrum. In this case $\delta_c = 1$.
- generalized Morse substitution generates a dynamical system with mixed spectrum and in this case $\delta_c = 0$.

Thus we can draw the following table, and we state the question

Is the following true :

$$\delta_c = 1 \Longrightarrow \text{discrete spectrum}$$

$$\delta_c < 1 \Longrightarrow \text{mixed spectrum ?}$$

(see the following page)

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(3) COVEN and KEANE. The structure of substitution minimal sets. (1971)

<table>
<tr><th colspan="3">SUBSTITUTION</th><th>δ_c</th><th>SPECTRUM</th></tr>
<tr><td rowspan="2">constant length</td><td colspan="2">Regular Toeplitz substitution
$(\theta(0) \neq \widetilde{\theta(1)})$ (1)</td><td>1</td><td>Discrete</td></tr>
<tr><td colspan="2">Generalized Morse substitution
$(\theta(0) = \widetilde{\theta(1)})$ (1)</td><td>0</td><td>Mixed</td></tr>
<tr><td rowspan="6">non constant length</td><td colspan="2">$\begin{cases} \theta(0) = 0^{n+1-p}\ 1\ 0^{p} \\ \theta(1) = 1\ 0^{n} \end{cases}$ $0 \leq p \leq n$</td><td>1</td><td>Discrete</td></tr>
<tr><td colspan="2">$\begin{cases} \theta(0) = 01 \\ \theta(1) = 1010 \end{cases}$ and $\begin{cases} \theta(0) = 01 \\ \theta(1) = 1001 \end{cases}$</td><td>1</td><td>Discrete</td></tr>
<tr><td colspan="2">$\begin{cases} \theta(0) = 0\ 1 \\ \theta(1) = 1\ 1\ 0\ 0 \end{cases}$</td><td>$\frac{1}{3}$</td><td>Mixed</td></tr>
<tr><td rowspan="2">$\begin{cases} \theta(0) = 0^{n}\ 1^{n} \\ \theta(1) = 1^{2n}\ 0^{2n} \end{cases}$</td><td>n even</td><td>1</td><td>Discrete</td></tr>
<tr><td>n odd</td><td>$\frac{1}{3}$</td><td>Mixed</td></tr>
<tr><td colspan="2">other cases</td><td>?</td><td>?</td></tr>
</table>

(1) with $\tilde{\omega}_k = 1 - \omega_k$