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COTORSION PROPERTIES

par

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(1) From the functor $\mathrm{Hom}(.,\mathbb{Z})$

Given any (abelian) group G, we denote the group

$$\mathrm{Hom}(G,\mathbb{Z}) \text{ by } G^+ ;$$

and, for any homomorphism $\alpha : G \longrightarrow H$ of (abelian) groups, the notation

$$\alpha^+ : H^+ \longrightarrow G^+$$

will be used for the homomorphism induced by α.

There is the canonical homomorphism

$$L_G : G \longrightarrow G^{++}$$

where, for $g \in G$, $L_G(g)$ sends $\varphi \in G^+$ into $\varphi(g)$.

The equation

$$\alpha^{++} \circ L_G = L_H \circ \alpha$$

is proved straightforward.

Let K be a subgroup of G. Then

$$\overline{K} = \bigcap_{\substack{\varphi \in G^+ \\ \varphi(K)=0}} \ker \varphi$$

is a group between K and G.

For a differend description of $\overline{K}$ let $A = \mathrm{im}\ (G^+ \longrightarrow K^+)$.

Then

$$\overline{K} = L_G^{-1}(A^+) \tag{1}$$

For a proof of (1), we just observe that - for $g \in G$ - $L_G(g) \in A^+$ means, that the evaluation $\varphi \longmapsto \varphi(g)$ ($\varphi \in G^+$) is zēro whenever $\varphi \in H^+$ i.e. whenever $\varphi \in G^+$ and $\varphi(K) = 0$. Yet $\varphi(K) = 0$ and $\varphi(\overline{K}) = 0$ are equivalent.

The notation of the following exact sequences will be refered to several times

$$0 \longrightarrow K \longrightarrow G \longrightarrow H \longrightarrow 0 \tag{2}$$

$$0 \longrightarrow H^+ \longrightarrow G^+ \longrightarrow A \longrightarrow 0 \tag{3}$$

$$\begin{tikzcd} 0 \arrow[r] & A^{+} \arrow[r] & G^{++} \arrow[r, "\alpha^{++}"] & H^{++} & \\ 0 \arrow[r] & K \arrow[r] \arrow[u, "\varkappa"] & G \arrow[r, "\alpha"] \arrow[u, phantom, "L_G"] & H \arrow[r] \arrow[u, "L_H"'] & 0 \end{tikzcd} \tag{4}$$

(in (4), $\varkappa$ is the restriction of L_G).

<u>Lemma 1</u> : If (3) splits and L_G is surjective, then

$$(A^{+} \,/\, L_G(K))^{+} = 0.$$

Proof : Let $\varphi : A^{+} \longrightarrow \mathbb{Z}$ such that $\varphi(L_G(K)) = 0$ be given. There exists an extension $\bar{\varphi} : G^{++} \longrightarrow \mathbb{Z}$ of φ . Now $\bar{\varphi}(L_G(\bar{K})) = 0$; but, by equation (1),

$$L_G(\bar{K}) = A^{+}.$$

<u>Proposition</u> (R.J. Nunke) : Let, in the sequence (2), $G = \mathbb{Z}^{N}$.

Then

$$H^{++} \cong \mathbb{Z}^{J} \text{ where } J \text{ is countable,}$$

$(\ker L_H)^{+} = 0$, and $H \cong \ker L_H \oplus H^{++}$.

Proof : Refering to sequence (3) and diagramm (4), G^{+} is countable free by a theorem of E.C. Zeeman (see Fuchs II, Cor. 94.6), so A is free, and (3) splits. Thus $G/K \cong G^{++}/K \cong A^{+}/K \oplus H^{++}$. Also, $A^{+}/K \cong \ker L_H$, and $(A^{+}/K)^{+} = 0$ by lemma 1.

Actually, this is the elementary proof of Nunke's theorem, which - in addition - tells that $\ker L_H$ is cotorsion [R.J. Nunke : Slender groups Acta Sci. Math. Szeged <u>23</u>, 67-75 (1962) ; see Griffith, Thm. 153].

(2) <u>Algebraically compact groups</u>

<u>Définition</u> : The (abelian) group G is called.

(i) <u>Cotorsion</u> if, whenever $G \subset E$ is a subgroup such that E/G is torsionfree then G splits off,

and

(ii) <u>Pure injective</u> if, whenever $G \subset E$ is a pure subgroup then G splits off.

We freely use the commonly known properties of cotorsion and pure injective groups as they may be found in the books of Fuchs or Griffith. Here are some of them.

a) Since E/G being torsion-free implies G to be a pure subgroup of E, any pure injective group is cotorsion.

b) If G is a torsion-free cotorsion group then G is pure injective (see Fuchs I, Cor. 54.5).

c) $\mathrm{Ext}(\mathbb{Q},G) = 0$ suffices for G to be cotorsion.

d) Any $\mathrm{Ext}(A,B)$ is cotorsion. A possible proof by homological algebra uses $\mathrm{Ext}(\mathbb{Q},\mathrm{Ext}(A,B)) \cong \mathrm{Ext}(\mathrm{Tor}(\mathbb{Q},A),B)$.

e) G cotorsion implies $G^{+} = 0$. For this, as direct summands of cotorsion groups are cotorsion, we observe that $\mathbb{Z}$ is not cotorsion. There are homomorphic images H of $\mathbb{Z}^{I}$, where $|I| = \aleph_1$, which have $H^{+} = 0$, but are not cotorsion (see proposition 2 below).

(3) <u>On cotorsion properties of homomorphic images of $\mathbb{Z}^{I}$</u>

<u>Examples</u> : If H is as homomorphic image of $\mathbb{Z}^{I}$, and if $|I| \leq \aleph_0$ then, by Nunke's theorem cited above, $\ker L_H$ is cotorsion as well as L_H is surjective. Neither property is preserved as soon as I is uncountable, as shown by the following examples.

(i) By proposition 2, $H = \mathbb{Z}^{\mathbb{R}}/\mathbb{Z}^{(\mathbb{R})}$ is not cotorsion, although, by Zeeman's theorem, $H^{+} = 0$ (thus $\ker L_H = H$).

(ii) Choose a free (abelian) group F and a subgroup $K \subset F$ such that $\overline{K}/K \cong \mathbb{Z}$ (here the indicates annihilation within F). Next, choose $G = \mathbb{Z}^{I} \supset F$ such that $G^{+} \longrightarrow F^{+}$ is surjective and $\ker L_{G/\underline{F}} = 0$. Then, annihilation of K within G yields K again, and $\ker L_{G/K} \cong \mathbb{Z}$. $L_{G/K}$ is not surjective in this example.

(iii) Let, keeping the notation of diagramm (4), $H = \mathbb{Z}^{\mathbb{N}}$ and G free. Then coker $L_A \cong \operatorname{Ext}(\mathbb{Z}^{\mathbb{N}},\mathbb{Z})$. L_A is injective in this example.

Problem : The following problem seems to be open. If H is a homomorphic image of $\mathbb{Z}^I$ (I uncountable), is then $(\ker L_H)^+ = 0$ provided L_H is surjective ? Actually, this problem is easily answered to the affirmative, once one assumes that white head groups are free. So, the real point is to state it within ZF (+GCH).

Proposition 1 : If $H = \mathbb{Z}^I/K$, and if $\mathbb{N} \subset I$ such that $C = \mathbb{Z}^{I\setminus\mathbb{N}}/K \cap \mathbb{Z}^{I\setminus\mathbb{N}}$ is cotorsion then $\ker L_H$ is cotorsion.

Proof : H is a homomorphic image of $\mathbb{Z}^{\mathbb{N}} \oplus C$. So, the following lemma will be sufficient.

Lemma 2 : In the notation of diagramm (4), let $G = \mathbb{Z}^{\mathbb{N}} \oplus C$, where C is cotorsion. Then $\ker L_H$ is cotorsion.

Proof: From diagramm (4) we the well-known ker - coker - sequence

$$C = \ker L_G \xrightarrow{\lambda} \ker L_H \xrightarrow{\mu} \operatorname{coker} \mathcal{K} \longrightarrow \operatorname{coker} L_G = 0$$

(see Mac Lane, Homology. Chap. II, lemma 5.2), and from it

$$0 \longrightarrow \ker \mu \longrightarrow \ker L_H \longrightarrow \operatorname{coker} \mathcal{K} \longrightarrow 0.$$

Now firstly, $\ker \mu = \sin \lambda$ is cotorsion as homomorphic image of C. Secondly, in ouder to prove that coker $\mathcal{K}$ is cotorsion, it will be sufficient to prove coker $\mathcal{K} = 0$, as A is countable free, and by Nunke's theorem. Yet coker $\mathcal{K} = 0$ follows from lemma 1, as L_G is surjective.

Proposition 2 : $\mathbb{Z}^{\mathbb{R}}/\mathbb{Z}^{(\mathbb{R})}$ fails to be algebraically compact.

For a proof see manuscripta math. 11, 103-109 (1974).

Problem : Let $B(I) = \{f \in \mathbb{Z}^I \ / \ f \text{ bounded}\}$. Then $\mathbb{Z}^I/B(I)$ is algebraically compact, in fact divisible as pointed out by P. Hill. So, one might ask for groups K between $\mathbb{Z}^{(I)}$ and $B(I)$ as small as possible such that $\mathbb{Z}^I/K$ is algebraically compact.

The question for subgroups K of Z^I, whether $\mathbb{Z}^{(I)} \subset K$ or not, such that $\mathbb{Z}^I/K$ is cotorsion would be a most general one, as any cotorsion group is a homomorphic image of some group $\mathbb{Z}^I$.