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On Doleans-Föllmer's measure for quasi-martingales and a Pellaumail's extension theorem

by

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Let $(\Omega, (\mathfrak{F}_t)_{t \in \mathbb{R}^+}, P)$ be the usual setting for studying stochastic processes. The idea of associating with every adapted process $(X_t)_{t \in \mathbb{R}^+}$ a set function μ_X, defined on the algebra of subsets of $\mathbb{R}^+ \times \Omega$ generated by the family $\{]s,t] \times F ;\ 0 < s \leq t,\ \ F \in \mathfrak{F}_s\}$, through the formula

$$\mu_X(]s,t] \times F) = E[1_F \cdot (X_t - X_s)]$$

seems to have been used by C. Doleans in [2] for the first time. She proved that, if X is a supermartingale of local class D, then μ_X is σ-additive.

Recently Föllmer [5] proved, under particular conditions on $(\mathfrak{F}_t)$ (which forbid the usual assumption of completeness on the $\mathfrak{F}_t$'s and are of topological character), that μ_X is always σ-additive as soon as X is a L^1- bounded quasi-martingale, and that the property for X to be of class D is equivalent to: every evanescent predictable subset of $\mathbb{R}^+ \times \Omega$ is of μ_X measure zero. Moreover, Föllmer notes that the previous decomposition theorem of quasi-martingales (F-processes in the work of Orey [10]) as got by Orey, Fisk and Rao can be received as mere immediate consequences of known decomposition theorems for measures.

(*) This seminar was written during the author's stay at University of Minnesota - Minneapolis during the fall 1973.

In this lecture we intend to take over Föllmer's treatment without assuming topological properties for the σ-algebras $\mathfrak{F}_t$'s , and with the usual assumptions of completeness. The results are slightly different: the measure μ_X is only simply-additive, and the property of σ-additivity is in this case equivalent to the property of being of class D for X .

The first paragraphs (1 to 8) study the one to one correspondence $X \to \mu_X$ between quasi-martingales and a class of finitely additive-measures with bounded variation, which is an isomorphism of the order structures defined by the positive cone of negative sub-martingales and the positive cone of positive measures respectively.

The §4 and §5 study the σ-additivity or pure finitely additivity of μ in terms of the process X and states the corresponding decomposition theorem.

In §6 we have exposed the recent proof of the Doob. Meyers decomposition theorem for quasi-martingales, due to J. Pellaumail. It is simple and based upon the σ-additivity of the Dolean's measure, and has moreover the advantage of being immediately applicable to vector valued quasi-martingales.

1. Notations and definitions.

$(\mathfrak{F}_t)_{t \in \mathbb{R}^+}$ is an increasing family of sub-σ-algebras of a σ-algebra $\mathfrak{F}$ of subsets of Ω .

$(\Omega,\mathfrak{F},P)$ is a complete probability space. We set $\mathfrak{F}_\infty = \bigvee_{t \in \mathbb{R}^+} \mathfrak{F}_t$ (σ-algebra generated by $\bigcup_{t \in \mathbb{R}^+} \mathfrak{F}_t$) and

$$\mathfrak{n} = \{F : F \in \mathfrak{F}_\infty,\ P(F) = 0\} .$$

We make the following:

<u>Assumption</u>: $\mathcal{F}_t \supset \mathcal{N}$ for any t, and $(\mathcal{F}_t)_{t \in \mathbb{R}^+}$ is right-continuous.

We define the following systems of subsets of $\overline{\mathbb{R}}^+ \times \Omega$, (where $\overline{\mathbb{R}}^+ = [0,\infty])$.

A <u>predictable</u> <u>rectangle</u> is a subset $]s,t] \times F$ of $\overline{\mathbb{R}}^+ \times \Omega$ such that $s<t$ and $F \in \mathcal{F}_s$.

Let $\alpha \in [0,+\infty]$. We call $\mathcal{R}_\alpha$ the set of predictable rectangles in $]0,\alpha] \times \Omega$.

$\mathcal{U}_\alpha$ is the algebra of subsets of $[0,\alpha[\times \Omega$ which are finite union of predictable rectangles.

$\overline{\mathcal{U}}_\alpha$ is the algebra of subsets of $[0,\alpha] \times \Omega$ which are finite union of predictable rectangles.

$\mathcal{P}_\alpha$: is the σ-ring generated by $\mathcal{U}_\alpha$.

$\overline{\mathcal{P}}_\alpha$: is the σ-ring generated by $\overline{\mathcal{U}}_\alpha$.

The elements of $\mathcal{P}_\alpha$ (resp $\overline{\mathcal{P}}_\alpha$) are called the <u>predictable</u> <u>subsets</u> of $[0,\alpha[\times \Omega$ (resp $[0,\alpha] \times \Omega$).

The subsets of $\overline{\mathbb{R}}^+ \times \Omega$ included in some $[0,\alpha] \times \Omega$ with $\alpha < \infty$, will be said <u>bounded</u>.

For all the processes $X = (X_t)_{t \in \mathbb{R}^+}$ which will be considered we will define $X_\infty = 0$ (X_∞ is to be distinguished from $X_\infty^- = \lim_{t \to \infty} X_t$ p.s. if such a limit exists).

$\overline{\mathcal{U}}$ will be the algebra generated by $\overline{\mathcal{U}}_\infty$ and the sets $\{\{\infty\} \times F;\ F \in \bigcup_{t \in \mathbb{R}^+} \mathcal{F}_t\}$.

We recall that $\mathcal{U}_\alpha$ consists of those so-called "stochastic intervals" $]\sigma,\tau] = \{(u,w) : \sigma(w) < u \leq \tau(w)\}$ where σ and τ are two finitely valued stopping times.

A function f on $\overline{\mathbb{R}}^+ \times \Omega$ is said to be <u>evanescent</u> if $P(\{w : f(t,w) = 0 \text{ for all } t \in \overline{\mathbb{R}}^+\}) = 1$. A subset G of $\overline{\mathbb{R}}^+ \times \Omega$ is called evanescent if its indicator function 1_A is evanescent.

Two processes X and Y are said indistinguishable if $X - Y$ is evanescent.

2. Simply additive measures associated with quasi-martingales.

2.1 Definition

An adapted process X is said to be an F-process (Orey's definition) or a quasi-martingale on a compact interval $[0,\alpha]$ if

$$K_\alpha = \sup_{0 \leq t_1 < \ldots < t_k \leq \alpha} \sum_{i=0}^{k-2} E\,|X_{t_i} - E(X_{t_{i+1}} | \mathcal{F}_{t_i})| < +\infty$$

where the sup is to be taken on all the increasing finite sequences $t_1 < \ldots < t_k$ in $[0,\alpha]$.

Remark.

Such a process is clearly bounded in L^1 on $[0,\alpha]$.

2.2. Measures associated with a general adapted process.

We define the following functions m_X^α and μ_X^α (resp. $\bar{m}_X^\alpha$ and $\bar{\mu}_X^\alpha$) on $\mathcal{R}_\alpha$ (resp. $\bar{\mathcal{R}}_\alpha$), for every adapted real process X such that $\forall t\ X_t \in L^1(\Omega, \mathcal{F}_t, P)$

(2.2.1) $$m_X^\alpha(]s,t] \times F) = 1_F \cdot (X_t - X_s) \in L^1 \quad (\text{resp. } \bar{m}_X^\alpha \ldots)$$

(2.2.2) $$\mu_X^\alpha(]s,t] \times F) = E\, 1_F \cdot (X_t - X_s) \in \mathbb{R} \quad (\text{resp. } \bar{\mu}_X^\alpha \ldots) .$$

It is quite immediate that this function can be extended into simply additive measures on the algebra $\mathcal{R}_\alpha$ (resp. $\bar{\mathcal{R}}_\alpha$). It is clear that, if X

is a Banach valued process (in Banach space E), we can still define m_X^α and μ_X^α through formula (2.2.1) and (2.2.2). In this case m_X^α takes its values in $L_E^1(\Omega,\mathcal{F}_\alpha,P)$ and μ_X^α takes its values in $\mathbb{E}$.

The following proposition follows immediately from the definition

Proposition 1.

$\bar{\mu}_X^\alpha$ is positive (resp. negative, resp. zero) if and only if X is a submartingale (resp. a supermartingale, resp. a martingale) , on $[0,\alpha]$. Same statement for μ_X^α and $[0,\alpha[$.

Remark.

From the convention $X_\infty = 0$, it follows that $\bar{\mu}_X^\infty$ is positive. (resp. negative, resp. zero) if and only if X is a negative submartingale on $[0,\infty]$, (resp. a positive supermartingale, resp. a null-process).

Proposition 2.

For two finitely valued stopping time σ and τ , $\sigma \leqslant \tau$

$$\mu_X\,]\sigma,\tau] = E(X_\tau - X_\sigma) \qquad m_X\,]\sigma,\tau] = x_\tau \angle x_\sigma$$

Proof.

If $\{o = t_o<\ldots<t_n\}$ is a set including the values of σ and τ , σ and τ can be written.

$$\sigma = \sum_{i=o}^{n-1} (t_{i+1} - t_i)\, 1_{F_i} \qquad F_i \in \mathcal{F}_{t_i}$$

$$\tau = \mathop{}_{i=o}^{n-1} (t_{i+1} - t_i)\, 1_{G_i} \qquad G_i \in \mathcal{F}_{t_i} \text{ and } G_i \supset F_i$$

Then

$$]\sigma,\tau] = \bigcup_{i=1}^{n-1}]t_i , t_{i+1}] \times (G_i - F_i)$$

and the formulas of the proposition follow immediately from the definition of m_X, μ_X and the fact that

$$X_\tau = \sum_{i=0}^{n-1} (X_{t_{i+1}} - X_{t_i}) \, 1_{G_i} \, .$$

2.3. <u>More on the correspondence</u> $X \to \bar{\mu}_X^\infty$.

From the assumption $X_\infty = 0$, and the relation

$$\bar{\mu}_X^\infty(]t,s] \times F) = -E(1_F \cdot X_t)$$

it is clear that $\bar{\mu}_X^\infty \longleftrightarrow X$ is a one-to-one correspondence between finitely additive measures μ on $\bar{\mathfrak{A}}_\infty$ such that for every t , $F \to \mu(]t,\infty] \times F)$ is an absolutely continuous bounded measure on $\mathcal{F}_t$, and processes X such that $X_t \in L^1$ for all t (defined up to a modification).

2.3 <u>Theorem 1.</u>

$\bar{\mu}_X^\alpha$ of bounded variation on $\bar{\mathfrak{A}}_\alpha \Longleftrightarrow X$ is an F-process on $[0,\alpha]$. In this case $|\bar{\mu}_X^\alpha|(]0,\alpha] \times \Omega) = K_\alpha$.

<u>Proof.</u>

By definition, for every predictable $]s,t] \times F$,

(2.3.1) $$|\bar{\mu}_X^\alpha|(]s,t] \times F) = \sup \sum_i |E(1_{F_i} \cdot (X_{t_i} - X_{s_i}))|$$

where $(]s_i,t_i] \times F_i)$ is any family of disjoint rectangles included in $]s,t] \times F$.

By taking a finer partition if necessary, one may assume that the partition on the right-hand side of (2.3.1) is of the following form

(2.3.2) $$\{]t_k,t_{k+1}] \times F_{k,\ell} \, , \quad s \le t_0 < \ldots < t_n \le t \, , \quad \ell = 1 \ldots m_k \} \, .$$

Let us then denote

$$\Lambda_k = [E(X_{t_{k+1}} \mid \mathcal{F}_{t_k}) - X_{t_k} > 0] \ .$$

It is clear, from $F_{k,\ell} \in \mathcal{F}_{t_k}$, that

$$(2.3.3) \qquad \sum_{k,\ell} |E(X_{t_{k+1}} - X_{t_k}) \cdot |F_{t_{k,\ell}}| = \sum_k \sum_\ell E|1_{F_{k,\ell}} E(X_{t_{k+1}} - X_{t_k} \mid \mathcal{F}_{t_k})|$$

$$\leq \sum_k E|E(X_{t_{k+1}} - X_{t_k} \mid \mathcal{F}_{t_k})| \ .$$

On the other hand it is trivial that for each division $s = t_0 < \dots < t_k = t$

$$\bar{\mu}_X^a(]s,t) \times F) \geq \sum_k E|1_{\Lambda_k} \cdot (X_{t_{k+1}} - X_{t_k})| + \sum_k E|1_{\Lambda_k} \cdot (X_{t_{k+1}} - X_{t_k})| \ .$$

The two last inequalities imply the theorem.

<u>Theorem 1'.</u>

If X is Banach valued, the same conclusions as in Theorem 1 hold for the Banach valued finite additive-measure $\bar{\mu}_X^a$.

<u>Proof.</u>

With the same notations as in the proof of Theorem 1

$$|\bar{\mu}_X^a|(]s,t] \times F) = \sup \sum_{k,\ell} E(1_{F_{k,\ell}} \cdot E(X_{t_{k+1}} - X_{t_k} \mid \mathcal{F}_{t_k}))$$

where the sup is taken over all the partitions of the form (2.3.2). Inequality (2.3.3) is proven exactly the same way.

For every ϵ , there exists a step function

$$\varphi_k = \sum_\ell 1_{F_{k,\ell}} \cdot x'_{k,\ell} \ , \quad x'_{k,\ell} \in E' \quad \|\varphi_k\|_\infty \leq 1$$

such that

$$|E(X_{t_{k+1}} - X_{t_k} | \mathcal{F}_{t_k})| \geq \langle \varphi_k, \ E(X_{t_{k+1}} - X_{t_k}) | \mathcal{F}_{t_k}) \rangle$$

$$\geq |E(X_{t_{k+1}} - X_{t_k} | \mathcal{F}_{t_k})| - \frac{\epsilon}{n} \ .$$

From here it is easily seen that for every ϵ

$$\bar{\mu}_X^{\alpha}(]s,t] \times F) \geq \sum_{k=1}^{n} |E(X_{t_{k+1}} - X_{t_k} | \mathcal{F}_{t_k})| - \epsilon \ .$$

And then

$$\bar{\mu}_X^{\alpha}(]s,t] \times F) \leq K_{\alpha}$$

3. <u>Bounded variation of</u> μ_X <u>and regularity of trajectories of</u> X .

<u>Theorem 2</u>. (Orey).

Let X be a separable real quasi-martingale on $[0,\alpha]$. Almost surely the trajectories have left and right limits.

<u>Proof</u>.

The proof goes as the traditional proof for martingales due to Doob. (cf. [11] p.). Let a and b be two real numbers $a < b$. Let $S = \{s_1 < s_2 < \ldots < s_{2n}\} \subset [0,\alpha]$. We define the times of up crossings and down crossings over $[a,b]$, as follows:

$$\sigma_1 = s_1 \qquad \sigma_{2k} = \begin{cases} \inf\{s : s \in S, \ s > \sigma_{2k-1}, \ X(s) \leq a\} \text{ if } \{ \ \} \neq \emptyset \\ \sigma_{2k-1} \text{ if } \{ \} = \emptyset \end{cases}$$

$$\sigma_{2k+1} = \begin{cases} \inf\{s : s \in S, \ s > \sigma_{2k}, \ X(s) \geq b\} \text{ if } \{ \ \} \neq \emptyset \\ \sigma_{2k} \text{ if } \{ \ \} = \emptyset \end{cases} .$$

The condition of bounded variation on μ_X^a implies

$$K_a = |\mu_X^a|\,]0,a]\times\Omega) \geq \sum_{k=1}^{n-1} |E(X_{\sigma_{2k+1}} - X_{\sigma_{2k}})| \quad .$$

Because of the positivity of $X_{\sigma_{2k+1}} - X_{\sigma_{2k}}$:

$$K_a \geq \sum_{k=1}^{n-1} E|X_{\sigma_{2k+1}} - X_{\sigma_{2k}}|$$

$$\geq \sum_{k=1}^{n-1} j\cdot(b-a)P(F_{S,j}^{(a,b)})$$

where

$$F_{S,j}^{(a,b)} = \{w : j \text{ among the } X_{\sigma_{2k+1}} - X_{\sigma_{2k}} \text{ on } > 0\} \; .$$

We may then consider a dense denumerable set S in $[0,a]$, and an increasing sequence (S_n) of finite subsets of S such that $S = \bigcup_n S_n$, and the corresponding sets $F_{S_n,j}^{(a,b)}$. From

$$P(F_{S_n,j}^{(a,b)}) \leq \frac{K_a}{j\cdot(b-a)}$$

we deduce that the set Ω_∞ of trajectories having infinitely many crossings over $[a,b]$, on the set S , has probability 0 .

The property of the theorem is deduced from there, by the usual argument.

4. Decomposition theorems.

We recall that an additive function μ on an algebra $\mathfrak{A}$ of sets is the difference of two positive additive functions μ^+ and μ^- if and only if μ is of bounded variation on any set A of $\mathfrak{A}$, i.e.: if
$\forall\; A\in\mathfrak{A}$ $|\mu|(A) = \sup\{\sum_i \mu(A_i) : (A_i)$, any finite partition of A , $A_i\in\mathfrak{A}\} < \infty$,
one has

$$|\mu|(A) = \mu^+(A) + \mu^-(A) \; .$$

One may view this as a Riesz decomposition in the ordered space (completely reticulated: see Bourbaki Integration I §1) of relatively bounded linear form on the space of step functions on $\mathcal{A}$. Every simply additive function μ, with bounded variation, is isomorphically (linearly and for the order) associated with a linear form $\tilde{\mu}$ by

$$\langle \tilde{\mu}, \sum_i A_i |_{s_i} \rangle = \sum_i \alpha_i \, \tilde{\mu}(s_i) \; .$$

We recall too, that the σ-additive-functions on $\mathcal{A}$ are easily seen to constitute a Riesz Band (cf. Bourbaki, Ref. above).

The band of the simply additive functions, which are orthogonal ("étrangères") to all σ-additive-functions is formed from all the so called "purely finitely additive functions," which may be characterized in the following way:

μ is purely finitely additive, if $(0 \leq \nu \leq |\mu|$ and ν σ-additive$) \Longrightarrow \nu = 0$.

Every finitely additive measures with bounded variation is the sum $\mu_\sigma + \mu_s$ of a σ-additive measure and a purely finitely additive one. The decomposition is unique.

These decomposition theorems give us immediately the following

4.1. Theorem 3.

Every quasi-martingale X on $[0,\alpha]$ is the difference of two positive L^1-bounded supermartingales X^+ and X^- : $X_t = X^- - X_t^+$. The decomposition is unique if we assume $X_\alpha = 0$ and impose $X^+(\alpha) = X^-(\alpha) = 0$ and: for every $\epsilon > 0$ there exists a sequence $\tau_1 < \ldots < \tau_n$ of finitely valued stopping times with values in $[0,\alpha]$ and two subsequences (τ_i'), (τ_j'') whose union is (τ_i) such that

(4.1.2) $$\sum_i E(X^+_{\tau_i'} - X^+_{\tau_{i+1}'}) + \sum_j E(X^-_{\tau_j''} - X^-_{\tau_{j+1}''}) \leq \epsilon \; .$$

Proof.

Decompose $\mu_X^a = \mu_X^{a+} - \mu_X^{a-}$, and take

$$X_t^+ = \left(\frac{d\nu_t^+}{dP}\right)_{\mathcal{F}_t} , \quad X_t^- = \left(\frac{d\nu_t^-}{dP}\right)_{\mathcal{F}_t} \quad : \text{ Radon-Nikodym derivatives of}$$

the measures ν_t^+ and ν_t^- defined on $\mathcal{F}_t$ by

$$\nu_t^+(F) = \mu_X^{a,+}(]t,a] \times F)$$

and

$$\nu_t^+(F) = \mu_X^{a,-}(]t,a] \times F) \ .$$

The decomposition $X_t = X_t^- - X_t^+$ follows from

$$\mu_X^a(]t,a] \times F) = -E(1_F \cdot X_t) = -E(1_F \cdot X_t^+) + E(1_F \cdot X_t^-)$$

as to the unicity condition of the theorem, it expresses only that $\inf(\mu_X^{a,+}, \mu_X^{a,-}) = 0$.

4.2. Extension of $\bar{\mu}_X^\infty$

Let us suppose that X is a F-process on $[0,\infty]$ (with the convention here made that $X_\infty = 0$). It follows immediately from the decomposition theorem 3 that $\forall \ F \in \bigcup_{t \in \mathbb{R}^+} \mathcal{F}_t$

$$\lim_{t \to \infty} \bar{\mu}_X^\infty(]t,\infty] \times F) = - \lim_{t \to \infty} E(1_F \cdot X_t) \quad \text{exists.}$$

It is then clear that if we set

$$\bar{\mu}_X(\{\infty\} \times F) = \lim_{t \to \infty} \bar{\mu}_X^\infty(]t,\infty] \times F)$$

and

$$\bar{\mu}_X(]s,t] \times F) = \bar{\mu}_X^\infty(]s,t] \times F) \ \text{ whenever } \ s < t \in [0, +\infty] \ ,$$

we define an additive extension $\bar{\mu}_X$ of $\bar{\mu}_X^\infty$ to the algebra called $\bar{\mathcal{A}}$ above.

It is evident that $\bar{\mu}_X$ is the difference of the extentions $\bar{\mu}_X^+$ and $\bar{\mu}_X^-$ of $\bar{\mu}_X^{\infty,+}$ and $\bar{\mu}_X^{\infty,-}$. As those extensions are such that $\inf(\bar{\mu}_X^+, \bar{\mu}_X^-) = 0$, they are respectively the positive part and negative part of $\bar{\mu}_X$.

From these definitions, follows immediately:

Proposition 3.

$(X_t)_{t \in \mathbb{R}^+}$ is a martingale if and only if $\bar{\mu}_X([0,\infty] \times \Omega) = 0$. $(X_t)_{t \in \mathbb{R}^+}$ is a potential (i.e. a positive supermartingale such that $\lim_{t \to \infty} E(X_t) = 0$) if and only if $\bar{\mu}_X \leq 0$ and $\bar{\mu}_X\{\infty\} \times \Omega = 0$.

Every quasi-martingale X can be written uniquely as

$$X = M + V^- - V^+$$

where V^- and V^+ are potentials verifying the condition (4.1.2) (X^+ and X^- being replaced by V^+ and V^- in the statement of this condition), and M is a martingale.

4.3. Theorem 4. (Orey)

Let $(\mathfrak{F}_n)$ be a decreasing sequence of σ algebras with $\mathfrak{F} = \bigcap_n \mathfrak{F}_n$. If the variables X_n verify

$$\sum_n E|E(X_n - X_{n+1}|\mathfrak{F}_{n+1})| < \infty ,$$

Then they are uniformly integrable.

Proof.

We refer to [10] for the proof, or the preceding theorem may be applied, and we can then use uniform integrability properties of supermartingales.

5. Characterization of σ-additive and purely finitely additive parts.

5.1. σ-additivity on $\mathcal{P}_\infty$.

We consider here the case when X being a quasi-martingale on every bounded interval $[0,\alpha]$, $\bar{\mu}_X^\infty$ is of bounded variation only on the algebra $\mathcal{P}_\sigma$ generated by bounded predictable rectangles. So we take only its restriction μ_X to $\mathcal{P}_\infty$ into consideration.

Definition.

We recall that a process X on $[0,\infty[$ is said to be of class D if the set $\{X_T : T$ any finite-stopping time$\}$ is uniformly integrable. It is said to be locally of class D if for every $\alpha < \infty$, the set $\{X_T : T$ any stopping time $\leq \alpha\}$ is uniformly integrable.

Proposition 4.

If $\bar{\mu}_X^\alpha$ is σ additive on $\bar{\mathcal{P}}_\alpha$, $\alpha \in \overline{\mathbb{R}}^+$, and if X is a.s. right continuous, then for every stochastic interval $]T,\alpha]$

$$\bar{\mu}_X^\alpha(]T,\alpha]) = E(X_\alpha) - E(X_T) .$$

Proof.

This proposition is trivially true for finitely valued stopping time T. Using for a stopping time T, an upper decreasing approximating sequence (T_n), of finitely valued stopping time T, we have then, from the σ-additivity,

$$\overline{\mu}_X^a(]T,a]) = \lim_n [E(X_a) - E(X_{T_n})] .$$

But as

$$\lim_n X_{T_n} = X_T \quad \text{a.s.}$$

applying Theorem 4 to the variable X_{T_n} and σ-algebras $\mathcal{F}_{T_n}$ we get the convergence of X_{T_n} towards X_T in L^1, and from there the proposition 4.

Theorem 5.

Let X be a real process, right continuous in L^1, which is a quasi-martingale on every bounded interval $[0,a]$.

Then μ_X^∞ is σ-additive if and only if X is locally of class D.

Proof.

Necessity.

Let $a < \infty$ and $\overline{\mu}_X^a$ the restriction of μ_X^∞ to $\overline{\mathcal{P}}_a = \mathcal{P}^\infty \cap [0,a] \times \Omega$. If $\overline{\mu}_X^a$ is σ-additive, its positive and negative parts are σ-additive too. Let us consider the positive part associated with the positive supermartingale X^-. From the σ-additivity of μ_X^+ $\lim_{t \downarrow s} E(X_t^- - X_s^-) = 0$. Then there exists a right-continuous version of X^-.

We define the stopping times

$$R_n = \inf\{t : X_t^- > n\} .$$

For $u < \alpha$

$$P(\bigcap_n [R_n \wedge u,\ u]) = 0 .$$

From the σ-additivity and proposition 4 we deduce

$$\lim_n E(X_u - X_{R_n \wedge u}) = 0 .$$

Using the same argument as in Meyer [9], p. 138, we will prove that this implies the uniform improbability of $\{X_T : T \leq u\}$.

Let us define

$$T'(w) = T(w) \quad \text{if} \quad X_{T(w)} \geq n$$

$$T'(w) = u \quad \text{if} \quad X_{T(w)} < n .$$

One has

$$R_n \wedge u \leq T'$$

and then

$$E(X^-_{R_n \wedge u}) \geq \int X^-_{T'}\, dP$$

$$\geq \int_{[X^-_T \geq n]} X^-_T\, dP + \int_{[X^-_T < n]} X^-_u\, dP .$$

Then

$$\int_{[u < R_n]} X^-_u\, dP + \int_{[u \geq R_n]} X^-_{R_n}\, dP - \int_{[X^-_T < n]} X^-_u\, dP \geq \int_{[X^-_T \geq n]} X^-_T\, dP$$

as $[u < R_n] \subset [X^-_T < n]$ then the positivity of X^- implies

$$\int_{[u \geq R_n]} X^-_{R_n}\, dP \geq \int_{[X^-_T \geq n]} X^-_T\, dP$$

which proves the uniform integrability property. We do the same reasoning for X^-_T .

Sufficiency.

We prove that for every decreasing sequence (H_n) of sets from $\mathfrak{F}$, such that $\cap_n H_n = \emptyset$

$$(5.1.1) \qquad \lim_{n\to\infty} \sup_{\substack{A \in \bar{\mathfrak{A}}_\alpha \\ A \subset [0,\alpha] \times H_n}} |\bar{\mu}_X^\alpha|(A) = 0 \quad .$$

We start with a finite partition (C_k) of $]0,\alpha] \times \Omega$ such that

$$\left| \Sigma |\bar{\mu}_X^\alpha(C_k)| - |\bar{\mu}_X^\alpha|[0,\alpha] \times \Omega \right| \le \epsilon \ ,$$

which implies

$$(5.1.2) \qquad \epsilon_k + |\bar{\mu}_X^\alpha(C_k)| \ge |\bar{\mu}_X^\alpha|(C_k) \quad \text{with} \quad \Sigma\, \epsilon_k \le \epsilon \ .$$

We will be finished if we can prove that for a suitable n

$$(5.1.3) \qquad \bar{\mathfrak{A}}_\alpha \ni A \subset C_k \cap]0,\alpha] \times H_n \Longrightarrow |\bar{\mu}_X^\alpha|(A) \le 2\epsilon_k \quad .$$

From (5.1.2) it follows easily that

$$(5.1.4) \qquad \forall \ A \in \bar{\mathfrak{A}}_\alpha \ , \quad A \subset C_k \ , \quad \epsilon_k + |\bar{\mu}_X^\alpha(A)| \ge |\bar{\mu}_X^\alpha|(A) \ .$$

We can find $]\sigma,\tau]$ where σ and τ are finitely valued stopping times such that $A \subset]\sigma,\tau] \subset C_k \cap]0,\alpha] \times H_n$. Then, from (5.1.4)

$$|\bar{\mu}_X^\alpha|(A) = |\bar{\mu}_X^\alpha| \ \sigma,\tau] \le \epsilon_k + |\bar{\mu}_X^\alpha(]\sigma,\tau])|$$

$$\le \epsilon_k + \left| \int_{H_n} (X_\tau - X_\sigma)\, dP \right| \ .$$

From the uniform integrability of the X_τ it is then possible to find n such that (5.1.3) holds. The theorem then follows from the lemma .

Lemma (Pellaumail).

Let λ be a finitely additive measure on $\overline{\mathfrak{A}}_\alpha$, with the following properties: it is of finite variation and

(i) $\forall F \in \mathfrak{F}_s$, $s < t$

$$\lim_{t \downarrow s} |\lambda(]s,t] \times F)| = 0$$

(ii) for every decreasing sequence (F_n) extracted from $\mathfrak{F}_\alpha$, such that $\bigcap_n F_n = \emptyset$

$$\lim_{n \to \infty} \sup_{\substack{A \in \overline{\mathfrak{A}}_\alpha \\ A \subset]0,\alpha] \times F_n}} |\lambda(A)| = 0 .$$

Then λ is σ-additive.

Proof.

We have to prove that for every decreasing sequence

(5.1.5) $\qquad (A_n)$, $A_n \in \overline{\mathfrak{A}}_\alpha$ and $\bigcap_n A_n = \emptyset$, $\lim_n \lambda(A_n) = 0$.

Suppose that for some class $\mathcal{C}$ of subsets of $[0,\alpha]$, $\mathcal{C}$ being stable with respect to finite unions and intersections, we have the property $\forall$ $A \in \overline{\mathfrak{A}}_\alpha$, $\forall \epsilon$, $\exists$ $C \in \mathcal{C}$ and $A' \in \overline{\mathfrak{A}}_\alpha$ such that

$$A' \subset C \subset A , \quad |\lambda(A - A')| \leq \epsilon .$$

Then if for every decreasing sequence (C_n) such that $C_n \in \mathcal{C}$ and $\bigcap_n C_n = \emptyset$, we have

(5.1.6) $$\lim_n \sup_{\substack{A \in \overline{\mathfrak{A}}_\alpha \\ A \subset C_n}} |\lambda(A)| = 0 .$$

We see immediately that (5.1.5) is true: take $A'_n \subset C_n \subset A_n$ with $|\mu(A_n - A'_n)| \leq \frac{\epsilon}{2^n}$. Then if we set

$$C'_n = \bigcap_{k\leq n} C_k , \qquad B_n = \bigcap_{k\leq n} A'_k \in \overline{\mathfrak{A}}_\alpha$$

we get a decreasing sequence (C'_n), extracted from $\mathcal{C}$ with void intersection and such that

$$\forall_n \ |\mu(A_n - B_n)| \leq \epsilon .$$

From (5.1.6) it is clear that $\lim\limits_n \lambda(B_n) = 0$ and then $\limsup\limits_n |\lambda(A_n)| \leq \epsilon$ for all ϵ.

We only have to prove (5.1.6) for a suitable class $\mathcal{C}$. We take for $\mathcal{C}$ the class of finite unions of rectangles of the type $]s,t] \times F$, $F \in \mathfrak{F}_s$. From property (i) it is clear that for every set $A = [s,t] \times F$ (and then for every finite union of such sets), it is possible to find $C = [s',t] \times F \subset A$ and $A' = [s',t] \times F$ such that $|\lambda(A - A')| = |\lambda(]s,s'] \times F)| \leq \epsilon$.

Let us take a decreasing family (C_n) of sets in $\mathcal{C}$ such that $\bigcap\limits_n C_n = \emptyset$. As, for every w, the set $C_n(w) = \{u : (u,w) \in C_n\}$ is compact in $\overline{\mathbb{R}}$:

$$\bigcup_k \{w : \bigcap_{n\leq k} C_n(w) = \emptyset\} = \Omega .$$

As $\{w : \bigcap\limits_{n\geq k} C_n(w) = \emptyset\} = F_n \in \mathfrak{F}_\alpha$

$$\bigcap_{n\leq k} C_n \subset [0,\alpha] \times F_n .$$

Property (ii) then implies (5.1.6).

5.2. σ-additivity on $\bar{\mathcal{P}}_\infty$.

The following theorems are mere corollaries of Theorem 5.

Theorem 5'.

Let X be a right continuous quasi-martingale on $[0,\infty]$. Then $\bar{\mu}_X^\infty$ is σ-additive if and only if X is of class D .

Theorem 5".

Let X be a right continuous process which is a quasi-martingale on every $[0,\alpha]$, $\alpha<\infty$. Let T be a stopping time such that $\{X_\sigma : \sigma \leq T$, σ stopping time$\}$ is uniformly integrable. Then $\bar{\mu}_X^\infty$ restricted to $[0,T] \cap \bar{\mathcal{P}}_\alpha$ is σ-additive.

Remark. The theorem 5" can be applied, in particular if $T=\inf\{t : X_t \geq n\}$.

5.3. Pure simple additivity of $\bar{\mu}_X$

Theorem 6.

Let X be a right continuous quasi-martingale on $[0,\infty]$. $\bar{\mu}_X^\infty$ is purely singly additive if and only if X is a local martingale, such that $\lim_{t\to\infty} X_t = 0$ a.s.

Proof.

Let $R_n = \inf\{t : |X_t| > n\}$, and $Y_t^n = X_{t\wedge R_n}$. As $(Y_t^n)_{t\in\mathbb{R}^+}$ is trivially a quasi-martingale of class D , and as

$$|\bar{\mu}_{Y^n}^\alpha| \leq |\bar{\mu}_X^\alpha|$$

$\bar{\mu}_{Y^n}^\alpha = 0$ if $\bar{\mu}_X^\infty$ is purely simply additive, which means in particular that Y^n is a martingale, and then X is a local martingale.

Let $X = M + V^+ - V^-$ be the decomposition of X as the sum of a martingale, and the difference of two potentials. It is known (and easy to check) that the σ-additive (and then in this case P-absolutely continuous) part of $\bar{\mu}_M$ is $\bar{\mu}_{M^\infty}$, where $(M_t^\infty)_{t \in \mathbb{R}^+}$ is the uniformly integrable martingale

$$M_t^\infty = E(\lim_{t \to \infty} M_t | \mathcal{F}_t) .$$

As

$$\lim_{t \to \infty} X_t = \lim_{t \to \infty} M_t \quad P. \text{ a.s.}$$

we see that if $\bar{\mu}_X$ is purely simply additive, then $\lim_{t \to \infty} X_t = 0$

Conversely, from what precedes, to prove that, for a local martingale X such that $\lim_{t \to \infty} X_t = 0$ a.s., $\bar{\mu}_X$ is purely simply additive, it suffices to prove that a potential V, which is a local martingale, is such that $\bar{\mu}_V$ is purely simply additive. But noticing that every process Y such that $0 \leq \bar{\mu}_Y \leq \bar{\mu}_V$ and which is σ-additive, has to be a potential which is a local martingale of class D, then a uniformly integrable martingale, is zero.

6. Pellaumail's proof of the Doob-Meyer's Decomposition theorem.

Theorem 7 (cf. [11]).

Let α be a positive finite measure on $\mathcal{P}_\infty$, such that $(A \in \mathcal{P}_\infty, A \text{ evanescent}) \Longrightarrow \alpha(A) = 0$.

Then, there exists an increasing process (c.t.r.), unique up to indistinguishability, such that $\forall\ s < t$, $\forall\ F \in \mathcal{F}_t$

(6.1.1) $$E[1_F \cdot (V_t - V_s)] = \int_{]s,t] \times \Omega} E(1_F | \mathcal{F}_{u^-})\, d\alpha$$

denoting by $E(1_F|F_{u^-})$ a left continuous (then predictable) version of the martingale $(E(1_F|F_u))_{u \leq t}$.

The process V thus defined is natural in the Meyer's sense (cf. [7] Chap. VIII).

<u>Proof</u>.

The unicity, up to indistinguishability, is quite trivial, V_t being necessarily such that

$$(6.1.2) \qquad \forall \quad F \in F_t \qquad E(1_F \cdot V_t) = \int_{(0,t] \times \Omega} E(1_F|F_{u^-})\, d\alpha \ .$$

We consider the following function on F_t

$$\alpha_t : F \to \int_{(0,t] \times \Omega} E(1_F|F_{u^-})\, d\alpha \ .$$

Using the martingale inequality and the Borel Cantelli lemma, we prove in a standard way, that from any decreasing sequence (g_n) of F_t - measurable functions, such that

$$\lim_n g_n = 0 \quad \text{p.s.}$$

we can extract a subsequence (g_{n_k}) such that, if

$$Y_{k(n)} = E(g_{n_k}|F_{u^-}) \ ,$$

we have

$$\text{a.s.} \qquad \lim_{k \to \infty} \sup_{0 \leq u \leq t} |Y_k(u,\omega)| = 0 \ .$$

The σ-additivity of α_t follows from this, and, denoting by $\bar{\Lambda}_t$ an expression of the Radon-Nikodym derivative $[\frac{d\alpha_t}{dP}]_{\mathcal{F}_t}$ one gets easily the following:

$$\forall \quad f \in \mathcal{L}^1(\Omega, \mathfrak{F}_t, P), \quad E(f \cdot \tilde{A}_t) \quad \int_{(0,t]} E(f|\mathfrak{F}_{u^-})\, d\alpha$$

$$\forall \quad f \in \mathcal{L}^1(\Omega, \mathfrak{F}_t, P) \ , \quad s < t$$

$$E(f \cdot (\tilde{A}_t - \tilde{A}_s)) = E(f \cdot \tilde{A}_t) - E(E(f|\mathfrak{F}_s) \cdot \tilde{A}_s)$$

$$= \int_{[s,t] \times \Omega} E(f|\mathfrak{F}_{u^-})\, d\alpha \ .$$

One then gets easily a modification V of $\tilde{A}$ having all the required properties.

Using the relation

$$E(Y_t \cdot V_t) = E \int_0^t Y_s \cdot dV_s$$

for a positive martingale Y and an increasing process V (cf. [7], Chap. VIII), one gets immediately

$$\int_{(0,t] \times \Omega} Y_{s^-} d\alpha = \int_{(0,t] \times \Omega} E(Y_t|\mathfrak{F}_{s^-})\, d\alpha$$

$$= E(Y_t \cdot V_t) = E \int_0^t Y_s \cdot dV_s$$

which proves the "naturality" of the process V in the sense of P.A. Meyer (cf. [7], Chap. VIII).

Corollary. Doob-Meyer's Decomposition Theorem.

If X is a L^1-bounded process which is a quasi-martingale on every finite interval $[0, a] \subset \mathbb{R}$, and is locally of class D, there is a unique (up to indistinguishability) decomposition

$$X = M + V$$

where M is a martingale, and V a process which is the difference of two increasing natural processes, vanishing at 0.

Proof.

We take the Doleans measure α associated with X and apply the preceding theorem to get V. As the Doleans measure associated with $X-V$ is zero, $X-V=M$ has to be a martingale.

7. Extension to the case of vector valued processes.

We have already noticed theorem 1' that some of the previous results could be restated without any change for Banach valued processes. But in this case, the notion of decomposing μ_X into a positive and negative part is meaningless.

The sufficient part of the proof of Theorem 5 can be applied without change to the vector situation. This is not the case for the necessity part of the proof.

As done in [6] and [11] the decomposition theorem of §6 extends without change to the case of a quasi-martingale X taking its values in the separable dual of a Banach space.

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