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IRRATIONAL ROTATIONS AND QUASI-ERGODIC MEASURES

par

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INTRODUCTION

Let ψ be an irrational rotation of the space X of reals modulo one. A probability measure on X is non singular if ψ takes null sets to null sets and quasiergodic if each ψ - invariant set has measure zero or one Examples of such measures are discrete measures carried on single ψ - orbits as well as Lebesgue measure. In the following, we show the existence of many other non singular quasiergodic measures by constructing for each $0 \leq p < 1$ a continuous probability measure μ_p which is non singular and quasiergodic, such that μ_p and μ_q are orthogonal if $p \neq q$. The method of construction :

To a given irrational α we associate in §1 a modified continued fraction $\{n_1, n_2, \ldots\}$. In § 2, we use the fraction to construct a space Ω_α of one - sided sequences $\omega = \{\omega_k\}$ of integers with $0 \leq \omega_k \leq n_k - 1$. It is helpful to think of $\{\omega_k\}$ as the entries in an infinite register ; we define an operation φ_α consisting of adding one to the initial place in the register with a " lopsided " right carry. For each point in $[0,1]$ we can then define in § 3 an " α - expansion " consisting of a sequence of Ω_α . Like the n - ary expansions, the α - expansion is unique except at a countable number of points. However, the α - expansion has the additional property that rotation by α modulo 1 on X is reflected by the operation φ_α on the space Ω_α . Using $(\Omega_\alpha , \varphi_\alpha)$ as a representation of rotation by α on X, it is not difficult to construct in § 4 the desired measures μ_p, which arise from product measures on a subset of Ω_α.

§ 1 - The modified continued fraction expansion. -

Let $\alpha \in (0,1)$ and define

$$S(\alpha) := 1 - \left\{\frac{1}{\alpha}\right\}$$

$$N(\alpha) := \left[\frac{1}{\alpha}\right] + 1 ,$$

where [ ] and { } denote integral and fractional parts respectively. Then

$$S :]0,1] \longrightarrow]0,1]$$

$$N :]0,1] \longrightarrow N_2 = \{2,3,4,\dots\}$$

and for each $\alpha \in]0,1]$ we can define a sequence $\{\alpha_k\}$ in $]0,1]$ and a sequence $\{n_k\}$ in N_2 recursively by

$$\alpha_1 := \alpha$$

$$\left.\begin{array}{l}\alpha_{k+1} := S(\alpha_k) \\ n_k := N(\alpha_k)\end{array}\right\} \quad (k \geq 1)$$

We write

$$\alpha = \{n_1, n_2, \dots\}$$

and call $\{n_k\}$ the modified continued fraction of α.

Proposition 1. -

a) If $\alpha = \{n_1, n_2, \dots\}$, then $\alpha_k = \{n_k, n_{k+1}, \dots\}$

b) The map $]0,1] \longrightarrow \prod_1^\infty N_2$ given by $\alpha \longrightarrow \{n_k\}$ is one - to - one and on to

c) Each $\alpha \in]0,1]$ is the limit of the fractions

$$\cfrac{1}{n_1 - \cfrac{1}{n_2 - \ddots - \cfrac{1}{n_k}}}$$

d) $\alpha \in]0,1]$ is rational iff almost all $n_k = 2$.

Proof :

a) is obvious from the definitions

b) thinking of the sequence $\{n_k\}$ corresponding to α as an ∞ - ary fractional expansion of α, we note that the sets

$$A_n := \{ \alpha \mid N(\alpha) = n \} = \left]\frac{1}{n}, \frac{1}{n-1}\right]$$

are disjoint and cover $]0,1]$ for $n \in \mathbb{N}_2$. MOreover, S maps each A_n to $(0,1)$ by

$$S(\alpha) = n - \frac{1}{\alpha} = \frac{1}{\alpha(n-1)} \cdot (n-1)(n\alpha - 1),$$

which is a linear map from $]\frac{1}{n}, \frac{1}{n-1}]$ to $]0,1]$ followed by the multiplication $\frac{1}{\alpha(n-1)}$ depending on n and varying from $\frac{1}{1-\frac{1}{n}}$ at 0 to 1 at 1 monotonically. To prove b) it is obviously sufficient to show that

$$\rho_k := \sup_{n_j \geq 2} \left| \{ \alpha \mid N(\alpha_j) = n_j \text{ for } 1 \leq j \leq k \} \right|$$

tends to zero as k goes to infinity, where $|I|$ denotes the length of the interval I. Now $|A_n|$ attains its maximum value for n = 2 and the multiplicative factor is always greater than or equal to 1 and decreases in n. This implies that

$$\rho_k = \left| \{ \alpha \mid N(\alpha_j) = 2 \text{ for } 1 \leq j \leq k\} \right| = 1 - c_k,$$

c_k being the left endpoint of the interval and satisfying

$$c_{k+1} = 2 - \frac{1}{c_k} \qquad (k \geq 1)$$

$$\frac{1}{2} \leq c_k < c_{k+1} < 1$$

Setting $c = \lim c_k$, we have

$$c = 2 - \frac{1}{c}$$

Or c = 1. Thus $\rho_k \longrightarrow 0$.

c) Denote by α'_k the fraction in c). The m. c. f. of α'_k is

$$\{n_1, n_2, \ldots, n_{k-1}, n_{k+1}, 2, 2, \ldots \}$$

Therefore, $|\alpha - \alpha'_k| \leq \rho_{k-1} \longrightarrow 0$ as $k \longrightarrow \infty$

d) If $\alpha = \{2,2,2,\ldots\}$, then $\alpha = \frac{1}{2-\alpha}$ implies $\alpha = 1$.

Thus by c) any α whose m. c. f. ends in two is rational.

Conversely, if $\alpha = \frac{p}{q}$ is rational with p < q, then $\alpha_2 = \frac{q'}{p}$ has a denomination smaller than $\alpha_1 = \alpha$. By induction $\alpha_q = 1$ and $n_q = n_{q+1} = \ldots = 2$;

§ 2 - The dynamical systems $(\Omega_\alpha , \varphi_\alpha)$. -

In this section α denotes a fixed irrational in $]0,1)$ with m.c.f. $\{n_k\}$. We set

$$\Omega = \sum_{k=1}^{\infty} \{0,1,\dots,n_k - 1\}.$$

Definition :

1) A block $\omega_{i+1} \; \omega_{i+2} \dots \omega_{i+k}$ with $i \geq 0$ and $k \geq 1$ will be called k - critical if

$$\omega_{i+j} = n_{i+j} - 2 \qquad (1 \leq j \leq k - 1)$$

$$\omega_{i+k} = n_{i+k} - 1$$

2) A block $\omega_i \; \omega_{i+1} \dots \omega_{i+k}$ with $i \geq 1$ and $k \geq 1$ is non - admissible if

$$\omega_i = n_i - 1$$

$$\omega_{i+1} \; \omega_{i+2} \dots \omega_{i+k} \qquad k \text{ - critical}$$

3) $\omega \in \Omega$ is called k - critical if $\omega_1 \; \omega_2 \; . \; . \; . \; \omega_k$ is k - critical and non - critical if it is not k - critical for any $k \geq 1$.

4) $\omega \in \Omega$ is admissible if it contains non non - admissible blocks

Let Ω_α be the set of admissible points of Ω .

For $\omega \in \Omega_\alpha$ define $\varphi_\alpha(\omega) = \omega'$ as

$$\omega'_1 : = \omega_1 + 1$$

$$\omega'_j : = \omega_j \qquad (j \geq 2)$$

if ω is non - critical and as

$$\omega'_1 = \omega'_2 = . \; . \; . = \omega'_k = 0$$

$$\omega'_{k+1} : = \omega_{k+1} + 1$$

$$\omega'_j : = \omega_j \qquad (j \geq k + 2)$$

if ω is k - critical with $k \geq 1$.

For ease of expression we set

$$\tilde{\omega} : = \tilde{\omega}_1 \; \tilde{\omega}_2 \dots \text{ with } \quad \tilde{\omega}_i = 0 \quad (i \geq 1)$$

$$\bar{\omega} : = \bar{\omega}_1 \; \bar{\omega}_2 \dots \text{ with } \quad \bar{\omega}_i = n_i - 2 \quad (i \geq 1)$$

$$\hat{\omega} : = \hat{\omega}_1 \; \hat{\omega}_2 \dots \text{ with } \quad \hat{\omega}_1 = n_1 - 1$$

$$\hat{\omega}_i = n_i - 2 \quad (i \geq 2)$$

Proposition 2 :

a) Ω_α is a compact subset of Ω.

b) φ_α is one - to - one and $\varphi_\alpha(\Omega_\alpha) = \Omega_\alpha - \{\tilde{\omega}\}$

c) φ_α is continuous except at $\bar{\omega}$

Proof :

a) the set of ω for which $\omega_i\ \omega_{i+1} \ldots \omega_{i+k}$ is not non - admissible is a finite union of cylinders, and Ω_α is the intersection of all such sets.

b) Note first that $\omega_1 + 1, \omega_2\ \omega_3 \ldots \omega_k$ is non - admissible if $k \geq 2$ and $\omega_1\ \omega_2 \ldots \omega_k$ is k - critical. Therefore, if $\omega \in \Omega_\alpha$ is non - critical, $\varphi_\alpha(\omega) \in \Omega_\alpha$. Next note that if $\omega_1 \ldots \omega_k$ is k - critical and $\omega \in \Omega_\alpha$ then $\omega_{k+1}\ \omega_{k+2} \ldots \omega_{k+j}$ is not j - critical for any j, because otherwise $\omega_k \ldots \omega_{k+j}$ would be non - admissible. Therefore, $\varphi_\alpha(\omega) \in \Omega_\alpha$ if ω is k - critical. If $\omega \in \Omega_\alpha$ does not start with 0, there is obviously exactly one (non - critical) point of Ω_α whose φ_α - image is ω. If $\omega_1 = \omega_2 = \ldots = \omega_k = 0$ and $\omega_{k+1} > 0$, then the unique k - critical point ω'' with $\varphi_\alpha(\omega'') = \omega$ is given by

$$\omega''_i = n_i - 2 \qquad (1 \leq i \leq k - 1)$$

$$\omega''_k = n_i - 1$$

$$\omega''_{k+1} = \omega_{k+1} - 1$$

$$\omega''_j = \omega_j \qquad (j \geq k + 2)$$

Thus only $\tilde{\omega}$ remains without a pre - image.

c) If $\omega \neq \bar{\omega}$, then the property of ω of being non - critical or k - critical extends to a neighborhood of ω and φ_α is continuous because it changes at most the first k + 1 coordinates.

The trouble at $\bar{\omega}$ is that the point whose image should be $\tilde{\omega}$ is missing. By inserting a backward orbit for $\tilde{\omega}$ and modifying the topology suitably, this problem can be rectified, and φ_α made into a homeomorphism. We shall have no need for this in the following.

§ 3 - α - expansions.

Let $X = \mathbb{R}/_{\mathbb{Z}}$ denote the reals modulo one and $\psi_\alpha(x) = x + \alpha \mod 1$ rotation by α. We fix an irrational $\alpha \in (0,1)$ with the corresponding sequences $\{\alpha_k\}$ and $\{n_k\}$ as in § 1. Define

$$\beta_k := \prod_{j=1}^{k} \alpha_j \qquad (k \geq 1)$$

and

$$\Pi(\omega) := \sum_{k=1}^{\infty} \omega_k \beta_k \qquad (\omega \in \Omega_\alpha)$$

Proposition 3. -

a) π maps Ω_α onto $[0,1]$ (and hence onto X)

b) π is one - to - one except at a countable number of points where it is two - to - one

c) π is continuous

d) $\pi \circ \psi_\alpha = \psi_\alpha \circ \pi$

Proof. -

Let "$\prec$" denote the lexicorgraphical ordering in Ω_α. With respect to this ordering, $\tilde{\omega}$ is the smallest element, $\hat{\omega}$ is the largest element, and $\omega \prec \eta$ with no point in between them if and only if there exists $k \geq 1$ such that

$$\omega_i = \eta_i \qquad (i < k)$$

$$\omega_k + 1 = \eta_k$$

$$\omega_{k+j} = \hat{\omega}_j \qquad (j \geq 1)$$

$$\eta_{k+j} = 0$$

We shall need some formulae :

1) $\lim_{k\to\infty} \beta_k = 0$:

Since infinitely many η_k are greater than 2, infinitely many α_k are less than or equal to $\frac{1}{2}$.

2) Let $\omega_1 \ \omega_2 \ldots \omega_k$ be the initial k - block of an admissible sequence. Then

$$1 - \sum_{j=1}^{k} \omega_j \beta_j \geq \beta_k \left[1 - \alpha_{k+1}\right]$$

with equality if $\omega_j = \hat{\omega}_j$ $(1 \le j \le k)$

Here there are two cases : if $\omega_1 \le n_1 - 2$, then

$$1 - \sum_{j=1}^{k} \omega_j \beta_j \ge 1 - (n_1 - 2)\alpha_1 - \alpha_1 \sum_{j=2}^{k} \omega_j \beta'_j$$

$$= 2\alpha_1 - \alpha_1\alpha_2 - \alpha_1 \sum_{j=2}^{k} \omega_j \beta'_j$$

$$> \alpha_1 \left(1 - \sum_{j=2}^{k} \omega_j \beta'_j\right)$$

where we have set $\beta'_j = \alpha_2 \alpha_3 \ldots \alpha_j$ $(j \ge 2)$ and if $\omega_1 = n_1 - 1$, then

$$1 - \sum_{j=1}^{k} \omega_j \beta_j = 1 - (n_1 - 1)\alpha_1 - \alpha_1 \sum_{j=2}^{k} \omega_j \beta'_j$$

$$= \alpha_1 - \alpha_1\alpha_2 - \alpha_1 \sum_{j=2}^{k} \omega_j \beta'_j$$

Setting $\omega'_2 = \omega_2 + 1$, $\omega'_j = \omega_j$ $(j \ge 3)$, we have then

$$1 - \sum_{j=1}^{k} \omega_j \beta_j = \alpha_1 \left(1 - \sum_{j=2}^{k} \omega'_j \beta'_j\right).$$

Now, if $\omega_1 \omega_2 \ldots \omega_k$ is admissible and $\omega_1 = n_1 - 1$, then also $\omega'_2 \ldots \omega'_k$ must be admissible. Therefore, we can use induction ; noting that

$$1 - (n_1 - 1)\alpha_k = \alpha_k (1 - \alpha_{k+1})$$

we arrive at the desired result.

3) $\pi(\check{\omega}) = 0$ and $\pi(\hat{\omega}) = 1$:

the first one is obvious, and we have by 2) and 1)

$$1 - \sum_{j=1}^{k} \hat{\omega}_j \beta_j = \beta_k \left[1 - \alpha_{k+1}\right] \xrightarrow[k \to \infty]{} 0$$

4) If $\omega < \eta$, then $\pi(\omega) \le \pi(\eta)$ with equality if and only if there is no point of Ω_α between ω and η.

Let k be minimal with $\omega_k < \eta_k$. Then

$$\pi(\eta) - \pi(\omega) = (\eta_k - \omega_k)\beta_k - \sum_{j=k+1}^{\infty} \omega_j \beta_j + \sum_{j=k+1}^{\infty} \eta_j \beta_j$$

$$\ge \beta_k - \sum_{j=k+1}^{\infty} \omega_j \beta_j \ge 0$$

because of 2) with equality everywhere if $\eta_j = 0$ for $j \ge k + 1$, $\eta_k - \omega_k = 1$, and $\omega_{k+1}\, \omega_{k+2} \ldots$ Maximal.

5) π is continuous.

if $\omega < \eta$ and $\omega_j = \eta_j$ for $1 \leq j \leq k$, then $\pi(\eta) - \pi(\omega) \leq \sum_{j=k+1}^{\infty} \eta_j \beta_j \leq \beta_k$. By 1), π is continuous.

6) Suppose that $[0,1] - \pi(\Omega_\alpha) \neq \emptyset$. Since π is continuous, $\pi(\Omega_\alpha)$ is compact and $[0,1] - \pi(\Omega_\alpha)$ is open in $[0,1]$. Thus there exists an interval $[a,b]$ with $0 \leq a < b \leq 1$, $a, b \in \pi(\Omega_\alpha)$, $(a,b) \cap \pi(\Omega_\alpha) = \emptyset$. Choosing ω maximal and η minimal with $\pi(\omega) = a$ and $\pi(\eta) = b$, 4) yields a contradiction.

7) Suppose $\omega \in \Omega_\alpha$ is non - critical. Then

$$\psi_\alpha(\pi(\omega)) = \alpha + \sum_{k=1}^{\infty} \omega_k \beta_k = (\omega_1+1)\beta_1 + \sum_{k=2}^{\infty} \omega_k \beta_k = \pi(\psi_\alpha(\omega)).$$

If ω is k - critical for some $k \geq 1$, then by 2)

$$\begin{aligned}\psi_\alpha(\pi(\omega)) &= \sum_{j=1}^{k} \hat{\omega}_j \beta_j + \beta_k + \sum_{j=k+1}^{\infty} \omega_j \beta_j \\ &= 1 - \beta_k [1 - \alpha_{k+1}] + \beta_k + \sum_{j=k+1}^{\infty} \omega_j \beta_j \\ &= 1 + \beta_{k+1} + \sum_{j=k+1}^{\infty} \omega_j \beta_j \\ &= \sum_{j=k+2}^{\infty} \omega_j \beta_j + (\omega_{k+1} + 1)\beta_{k+1} \mod 1 \\ &= \pi(\psi_\alpha(\omega)).\end{aligned}$$

The proof is finished, because 4), 5), 6) and 7) imply a), b), c) and d).

If $x \in [0,1]$, then we call a sequence in $\pi^{-1}(x)$ an <u>α - expansion</u> of x. Like decimal expansions, the α - expansion is unique except for a countable number of points.

§ 4 - Quasi - ergodic measures. -

Suppose T is an invertible bimeasurable transformation of the measurable space $(Y, \mathcal{Y})$. A probability measure μ on $(Y, \mathcal{Y})$ is called

non singular if for any $F \in \mathcal{Y}$,

$$\mu(F) = 0 \iff \mu(TF) = 0,$$

quasiergodic if for any $F \in \mathcal{Y}$ with $TF = F$,

$$\mu(F) = 0 \text{ or } 1$$

These properties obviously depend only on the measure class of μ. If $\alpha \in (0,1)$ is irrational, examples of non singular quasiergodic measure classes on (X, Ψ_α) are given by the Lebesgue measure class and by discrete measures whose sets of positivity are single Ψ_α- orbits. Until now, no other examples have been found.

Proposition 4. -

For any irrational α, there exist measures μ_p $(0 < p < 1)$ definde on X such that

a) each μ_p is continuous

b) $\mu_p \perp \mu_q$ if $p \neq q$

c) each μ_p is non singular and quasi ergodic on (X, Ψ_α)

Moreover, the measures μ_p can be given by a simple construction on Ω_α.

Proof. -

For $\alpha = \{n_k\}$ we set

$$\Omega' = \prod_{k=1}^{\infty} \{0, \dots, n_k - 2\}$$

Then, $\Omega' \subseteq \Omega_\alpha$ and since infinitely many n_k are greater than 2, Ω' is really an infinite product. For $0 < p < 1$, let m_p be the product measure on Ω' obtained from the discrete measures $\{p, 1 - p\}$ on $\{0,1\}$ placed at those components for which $n_k \geq 3$.

1) m_p is quesiergodic on $(\Omega_\alpha, \varphi_\alpha)$.

Suppose that $E \subsetneq \Omega_\alpha$ and $\varphi_\alpha E = E$. It follows from the definition of φ_α that $\omega \in \Omega_\alpha$ and $\eta \in \Omega_\alpha$ are in the same φ_α - orbit, iff $\{i \mid \omega_i \neq \eta_i\}$ is finite. But then $E \cap \Omega'$ is measurable with respect to the σ - algebra on Ω' generated by the components greater than n, i. e. $E \cap \Omega'$ is in the tail field of Ω'. By the zero one law, $m_p(E) = 0$ or 1.

2) For constants $c_n > 0$ with $\sum_{n \in \mathbb{Z}} c_n = 1$, set

$$m'_p := \sum_{n \in \mathbb{Z}} c_n \varphi_\alpha^n m_p.$$

Then the probability measure m'_p is obviously non singular on $(\Omega_\alpha, \varphi_\alpha)$ and remains queqi - ergodic, since $\varphi_\alpha(E) = E$ and $m_p(E) = 1$ imply $\varphi_\alpha^n m_p(E) = 1$ for each n.

3) We have $m'_p \perp m'_q$ for $p \neq q$.

For $0 < p < 1$ let

$$S_p := \{ \omega \in \Omega_\alpha \mid r_0(\omega) = p \text{ and } r_1(\omega) = 1 - p \},$$

where

$$r_i(\omega) = \lim_{n \to \infty} \frac{\# \text{ of } i \text{ among } \omega_1, \ldots, \omega_n}{n}, \quad i = 0, 1.$$

Then $m_p(S_p) = 1$ and because φ_α applied to $\omega \in \Omega_\alpha$ changes only a finite number of coordinates, we have $\varphi_\alpha(S_p) = S_p$. Thus $m'_p(S_p) = 1$ and $S_p \cap S_q = \emptyset$ implies $m'_p \perp m'_q$, if $p \neq q$.

4) Setting $\mu_p = \pi(m'_p)$, proposition 3 yields the desired result.

There is also a proof of existence of nonsingular quasiergodic measures which are continuous and admit no finite invariant aquivalent measure. The proof works for any strictly ergodic system (X, φ).¥ (Oral communication from W. KRIEGER).

¥ and uses a category argument.