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A One-Sample Analogue of a Theorem of Jurečkova

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A one-sample analogue of a theorem of Jurečkova

by

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I. INTRODUCTION.

The purpose of this note is to prove that if, for each $\nu=1,2,\ldots,X_{\nu,1},\ldots,X_{\nu,n_\nu}$ are a random sample from a distribution symmetric around o, then the signed-rank statistic

$$T_\nu(\theta) = \sum_{i=1}^{n_\nu} p_{\nu,i}\,\Psi\left(\frac{R_{|X_{\nu,i}-q_{\nu,i}\theta|}}{n_\nu+1}\right)\operatorname{sgn}(X_{\nu,i}-q_{\nu,i}\theta),$$

where $R_{|X_{\nu,i}-q_{\nu,i}\theta|}$ is the rank of $|X_{\nu,i}-q_{\nu,i}\theta|$ among $|X_{\nu,1}-q_{\nu,1}\theta|,\ldots,|X_{\nu,n_\nu}-q_{\nu,n_\nu}\theta|$, is under certain conditions on the common distribution of the $X_{\nu,i}$, on the constants $p_{\nu,i}$, $q_{\nu,i}$ and on the function Ψ, asymptotically approximately a linear function of θ in the sense that

$$\lim_{n_\nu\to\infty} P\left\{\sup_{|\theta|\leq C}\left|T_\nu(\theta)-T_\nu(0)+\theta K\sum_{i=1}^{n_\nu} p_{\nu,i}\,q_{\nu,i}\right|\geq\varepsilon\sigma(T_\nu(0))\right\}=0,$$

for every C>o and every ε>o, where K is a constant depending on the common distribution of the $X_{\nu,i}$ and on the function Ψ.

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An analogous result was proved by Jurečková [2] for the statistic

$$S_{\nu}(\theta) = \sum_{i=1}^{n_{\nu}} c_{\nu,i}\,\varphi\left(\frac{R_{X_{\nu,i} - d_{\nu,i}\theta}}{n_{\nu} + 1}\right),$$

where $R_{X_{\nu,i} - d_{\nu,i}\theta}$ is the rank of $X_{\nu,i} - d_{\nu,i}\theta$ among $X_{\nu,1} - d_{\nu,1}\theta, \ldots, X_{\nu,n_{\nu}} - d_{\nu,n_{\nu}}\theta$ and where, for each $\nu = 1,2,\ldots$, the $X_{\nu,i}$ are independently and identically distributed.

For the proof of our result some lemmas are needed which are given in section 2 ; one of these lemmas is a generalization of Theorem 5 of Lehmann [6] ; two of the lemmas are analogous to Corollary 1 and 2 of Lehmann [6] . The main results and their proofs are given in section 3.

II. SOME LEMMAS.

Let $i_1,\dots,i_n$ and $j_1,\dots,j_n$ each be a permutation of the numbers $1,\dots,n$ and let $\varepsilon_1,\dots,\varepsilon_n$, $\delta_1,\dots,\delta_n$ each be + 1 or - 1 such that $(i_k, \varepsilon_k, j_k, \delta_k)_{k=1}^n$ satisfies

$$\text{Condition } A_n : \begin{cases} 1.\ \delta_k=1 \Longrightarrow \varepsilon_k=1 \\ 2.\ \{\ell<k,\ \delta_k=1,\ j_\ell<j_k\} \Longrightarrow i_\ell<i_k \\ 3.\ \{\ell<k,\ \varepsilon_k=-1,\ j_\ell>j_k\} \Longrightarrow i_\ell>i_k \end{cases}$$

For fixed $M(1 \leqslant M \leqslant n)$ define

(2 ; 1) $\qquad a_{M,1}>a_{M,2}>\dots>a_{M,K_M}$

as the ordered values of those i_k among $i_{n-M+1}, i_{n-M+2},\dots,i_n$ for which $\varepsilon_k = +1$ and

(2 ; 2) $\qquad b_{M,1} > b_{M,2} >\dots > b_{M,L_M}$

as the ordered values of those j_k among $j_{n-M+1}, j_{n-M+2},\dots, j_n$ for which $\delta_k = +1$. Obviously, by Condition A_n.1, $K_M \geqslant L_M$; further $K_M \leqq M$.

Further define

(2 ; 3) $\qquad c_{M,1} > c_{M,2} >\dots > c_{M,M-K_M}$

as the ordered values of those i_k among $i_{n-M+1}, i_{n-M+2},\dots,i_n$ for which $\varepsilon_k = -1$ and

(2 ; 4) $\qquad d_{M,1} > d_{M,2} > \dots > d_{M,M-L_M}$

as the ordered values of those j_k among $j_{n-M+1}, j_{n-M+2},\dots,j_n$ for which $\delta_k = -1$.

<u>Lemma 2 ; 1. If $(i_k, \varepsilon_k, j_k, \delta_k)_{k=1}^n$ satisfies Condition A_n, then</u>

$$(2;5) \qquad \begin{cases} b_{M,\ell} \leqq a_{M,\ell} & \ell=1,\dots,L_M \\ c_{M,\ell} \leqq d_{M,\ell} & \ell=1,\dots,M-K_M \end{cases} \qquad M=1,\dots,n$$

<u>Proof</u> : The proof will be given in four parts.

1) The lemma is true for $M = 1$ and any $n \geqq 1$. To prove this, notice that by Condition A_n.1 it is sufficient to prove that

$$(2\,;\,6)\qquad \begin{cases} j_n \leqq i_n & \text{if } \delta_n = 1 \\ j_n \geqq i_n & \text{if } \varepsilon_n = -1 \end{cases}$$

This can be seen as follows.

$$(2\,;\,7)\qquad \begin{cases} j_n = (\# \text{ of } j_k \leqq j_n) = n - (\# \text{ of } j_k > j_n) \\ i_n = (\# \text{ of } i_k \leqq i_n) = n - (\# \text{ of } i_k > i_n) \end{cases}$$

By Condition A_n.2

$$(2\,;\,8)\qquad (\# \text{ of } j_k \leq j_n) \leqq (\# \text{ of } i_k \leq i_n) \text{ if } \delta_n = 1$$

and by Condition A_n.3

$$(2\,;\,9)\qquad (\# \text{ of } j_k > j_n) \leqq (\# \text{ of } i_k > i_n) \text{ if } \varepsilon_n = -1$$

2) If the lemma is true for some (n,M) then the lemma is true for $(n+1, M)$. To see this consider, for some $n \geqq 1$, $(i_k, \varepsilon_k, j_k, \delta_k)_{k=1}^{n+1}$ satisfying Condition A_{n+1}. From $(i_k, \varepsilon_k, j_k, \delta_k)_{k=1}^{n+1}$ derive $(i'_k, \varepsilon_k, j'_k, \delta_k)_{k=2}^{n+1}$, satisfying Condition A_n, as follows. Let

$$(2\,;\,10)\qquad \begin{cases} r_k = \text{rank of } i_k \text{ among } (i_1, i_k) \\ s_k = \text{rank of } j_k \text{ among } (j_1, j_k) \end{cases} \qquad k = 2,\ldots,n+1$$

and let

$$(2\,;\,11)\qquad \begin{cases} i'_k = i_k - (r_k-1) \\ j'_k = j_k - (s_k-1) \end{cases} \qquad k = 2,\ldots,n+1$$

Then $i'_2,\ldots,i'_{n+1}$ and $j'_2,\ldots,j'_{n+1}$ are each permutations of the numbers $1,\ldots,n$ and from

$$(2\,;\,12)\qquad \begin{cases} i_k < i_\ell \iff i'_k < i'_\ell \\ j_k < j_\ell \iff j'_k < j'_\ell \end{cases} \qquad k,\ell = 2,\ldots,n+1$$

it then follows that $\{i'_k, \varepsilon_k, j'_k, \delta_k\}_{k=2}^{n+1}$ satisfies Condition A_n.

For fixed $M \leqslant n$ let $a'_{M,\ell}$, $b'_{M,\ell}$, $c'_{M,\ell}$, $d'_{M,\ell}$, L'_M and K'_M be defined, as in (2 ; 2) - (2 ; 4), for $(i'_k, \epsilon_k, j'_k, \delta_k)_{k=n+2-M}^{n+1}$ and let $a_{M,\ell}$, $b_{M,\ell}$, $c_{M,\ell}$, $d_{M,\ell}$, K_M and L_M be so defined for

$(i_k, \epsilon_k, j_k, \delta_k)_{k=n+2-M}^{n+1}$; then $L_M = L'_M$ and $K_M = K'_M$. Assuming the lemma to be true for (n,M) we have

$$(2\ ;\ 13)\qquad \begin{cases} b'_{M,\ell} \leqq a'_{M,\ell} & \ell = 1,\dots,L_M \\ c'_{M,\ell} \leqslant d'_{M,\ell} & \ell = 1,\dots,M - K_M \end{cases}$$

Now let ℓ_o be the number of $b_{M,\ell} > j_1$; then by (2 ; 11)

$$(2\ ;\ 14)\qquad b'_{M,\ell} = \begin{cases} b_{M,\ell} - 1 & \ell = 1,\dots,\ell_o \\ b_{M,\ell} & \ell = \ell_o + 1,\dots,L_M. \end{cases}$$

Let k_o be the number of $a_{M,\ell} > i_1$; then by (2 ; 11)

$$(2\ ;\ 15)\qquad a'_{M,\ell} = \begin{cases} a_{M,\ell} - 1 & \ell = 1,\dots,k_o \\ a_{M,\ell} & \ell = k_o + 1,\dots,K_M. \end{cases}$$

Further, by Condition $A_{n+1}.2$, $\ell_o \leqq k_o$. From (2 ; 13) - (2 ; 15) it then follows that

$$(2\ ;\ 16)\qquad b_{M,\ell} \leqq a_{M,\ell} \qquad \ell = 1,\dots,L_M.$$

The proof that

$$(2\ ;\ 17)\qquad c_{M,\ell} \leqq d_{M,\ell} \qquad \ell = 1,\dots,M - K_M$$

is analogous, using Condition $A_{n+1}.3$.

3) If the lemma is true for some $n \geqslant 2$ with $M = n-1$, then the lemma is true for the same n with $M = n$. This can be seen as follows. Assuming the lemma to be true for $M = n-1$ we have

$$(2\ ;\ 18)\qquad \begin{cases} b_{n-1,\ell} \leqslant a_{n-1,\ell} & \ell = 1,\dots,L_{n-1} \\ c_{n-1,\ell} \leqslant d_{n-1,\ell} & \ell = 1,\dots,n-1-K_{n-1} \end{cases}$$

and it will be proved that

$$(2\ ;\ 19)\qquad \begin{cases} 1.\ b_{n,\ell} \leqslant a_{n,\ell} & \ell = 1,\dots,L_n \\ 2.\ c_{n,\ell} \leqslant d_{n,\ell} & \ell = 1,\dots,n-K_n \end{cases}$$

The following three cases can be distinguished

a) $\delta_1=\varepsilon_1=-1$. Then $L_n = L_{n-1}$, $K_n = K_{n-1}$, $b_{n,\ell} = b_{n-1,\ell}(\ell=1,\dots,L_n)$ and $a_{n,\ell}= a_{n-1,\ell}(\ell=1,\dots,K_n)$, so that (2 ; 19.1) is obvious. Further $(a_{n,\ell},\ \ell=1,\dots,K_n\ ,\ c_{n,\ell}\ ,\ \ell=1,\dots,n-K_n)$ and
$(b_{n,\ell}\ ,\ \ell=1,\dots,L_n\ ,\ d_{n,\ell}\ ,\ \ell=1,\dots,n-L_n)$ are each permutations of the numbers $1,\dots,n$ so that (2 ; 19.2) follows from (2 ; 19.1)

b) $\delta_1=-1$, $\varepsilon_1=1$. Then $L_n = L_{n-1}$, $K_n = K_{n-1}+1$, $b_{n,\ell} = b_{n-1,\ell}(\ell=1,\dots,L_n)$ and $c_{n,\ell} = c_{n-1,\ell}(\ell=1,\dots,n-K_n)$. To prove (2 ; 19.1) let k_o be the number of $a_{n-1,\ell}(\ell=1,\dots,K_{n-1})$ larger than i_1 ; then

$$(2\ ;\ 20)\qquad a_{n,\ell} = \begin{cases} a_{n-1,\ell} & \ell = 1,\dots,k_o \\ i_1 & \ell = k_o+1 \\ a_{n-1,\ell-1} & \ell = k_o+2,\dots,K_n \end{cases}$$

If $L_n \leqslant k_o \leqslant K_{n-1}$ then (2 ; 19.1)is immediate. If $o \leqslant k_o < L_n = L_{n-1}$, then (2 ; 19.1) is immediate for $\ell=1,\dots,k_o$. Further

$$(2\ ;\ 21)\qquad b_{n,k_o+1} = b_{n-1,k_o+1} \leqslant a_{n-1,k_o+1} < i_1 = a_{n,k_o+1}$$

and for $\ell= k_o+2,\dots,L_n$

$$(2\ ;\ 22)\qquad b_{n,\ell} = b_{n-1,\ell} \leqslant a_{n-1,\ell} = a_{n,\ell+1} \leqslant a_{n,\ell}$$

The proof of (2 ; 19.2) is analogous.

c) $\delta_1=\varepsilon_1=1$. Then $L_n = L_{n-1}+1$, $K_n = K_{n-1}+1$, $c_{n,\ell} = c_{n-1,\ell}(\ell=1,\dots,n-K_n)$ and $d_{n,\ell}=d_{n-1,\ell}(\ell=1,\dots,n-L_n)$ so that (2 ; 19.2) is obvious. Further (see a)) (2 ; 19.1) follows from (2 ; 19.2)

4) The lemma now follows by induction on M. According to part 1 of the proof, the lemma is true for $M = 1$ and any $n \geqslant 1$. Let M_o be an integer $\geqslant 1$ and assume the lemma is true for $M = M_o$ and any $n \geqslant M_o$, then it will be proved that the lemme is true for $M = M_o+1$ and any $n \geqslant M_o+1$. This can be seen as follows. According to the induction hypothesis the lemma is true for $n = M_o+1$ and $M = M_o$; according to part 3 of the proof this implies the truth for $n = M_o+1$ and $M = M_o+1$; according to part

2 of the proof this implies the truth for $M = M_o+1$ and any $n \geqslant M_o+1$.

Q. E. D.

In Lemma 2 ; 1 it was shown that Condition A_n is sufficient for (2 ; 5) to hold for each $M = 1,\dots,n$. For (2 ; 5) to hold for a particular value of M it is obviously sufficient that $(i_k, \varepsilon_k, j_k, \delta_k)_{k=1}^n$ satisfies

$$\text{Condition } A_{n,M} \quad \begin{cases} \text{For each } k \geqslant n-M+1 \\ 1.\ \delta_k=1 \Longrightarrow \varepsilon_k=1 \\ 2.\ \text{for each} \quad \ell \leqslant k-1\ (\delta_k=1, j_\ell<j_k) \longrightarrow i_\ell<i_k \\ 3.\ \text{for each} \quad \ell \leqslant k-1\ (\varepsilon_k=-1, j_\ell>j_k) \Longrightarrow i_\ell>i_k \end{cases}$$

Further, if $(i_k,\varepsilon_k,j_k,\delta_k)_{k=1}^n$ satisfies Condition $A_{n,M}$ for $M = M_o$ then $(i_k,\varepsilon_k,j_k,\delta_k)_{k=1}^n$ satisfies Condition $A_{n,M}$ for all $M \leqslant M_o$, which proves the following lemma.

<u>Lemma 2 ; 2. If $(i_k,\varepsilon_k,j_k,\delta_k)_{k=1}^n$ satisfies Condition $A_{n,M}$ for $M = M_o$, then</u>

$$(2\ ;\ 23) \quad \begin{cases} a_{M,\ell} \leq b_{M,\ell} & \ell=1,\dots,L_M \\ c_{M,\ell} \leq d_{M,\ell} & \ell=1,\dots,M-K_M \end{cases} \quad 1 \leqslant M \leqslant M_o$$

<u>Lemma 2 ; 3. If h is nondecreasing and nonnegative and if $(i_k,\varepsilon_k,j_k,\delta_k)_{k=1}^n$ satisfies Condition $A_{n,M}$ for $M = M_o$, then</u>

$$(2\ ;\ 24) \quad \begin{cases} \displaystyle\sum_{\substack{\ell=n+1-M \\ \varepsilon_\ell>0}}^{n} h(i_\ell) \geqslant \sum_{\substack{\ell=n+1-M \\ \delta_\ell>0}}^{n} h(j_\ell) \\ \displaystyle\sum_{\substack{\ell=n+1-M \\ \varepsilon_\ell<0}}^{n} h(i_\ell) \leqslant \sum_{\substack{\ell=n+1-M \\ \delta_\ell<0}}^{n} h(j_\ell) \end{cases} \quad 1 \leqslant M \leqslant M_o$$

<u>Proof</u> : Because h is nondecreasing, it follows from Lemma 2 ; 2 that, for $1 \leqslant M \leqslant M_o$,

$$(2\ ;\ 25)\qquad \begin{cases} 1.\ h(b_{M\ell}) \leq h(a_{M,\ell}) & \ell=1,\dots,L_M \\ 2.\ h(c_{M,\ell}) \leq h(d_{M,\ell}) & \ell=1,\dots,M-K_M. \end{cases}$$

From (2 ; 25.1) and the fact that h is non negative it follows that, for $1 \leq M \leq M_o$,

$$(2\ ;\ 26)\qquad \sum_{\substack{\ell=n+1-M\\ \delta_\ell>o}}^{n} h(j_\ell) = \sum_{\ell=1}^{L_M} h(b_{M,\ell}) \leq \sum_{\ell=1}^{L_M} h(a_{M,\ell}) \leq \sum_{\ell=1}^{K_M} h(a_{M,\ell}) = $$

$$= \sum_{\substack{\ell=n+1-M\\ \varepsilon_\ell>o}}^{n} h(i_\ell)$$

From (2 ; 25.2) and the fact that h is non negative it follows that, for $1 \leq M \leq M_o$,

$$(2\ ;\ 27)\qquad \sum_{\substack{\ell=n+1-M\\ \varepsilon_\ell<o}}^{n} h(i_\ell) = \sum_{\ell=1}^{M-K_M} h(c_{M,\ell}) \leq \sum_{\ell=1}^{M-K_M} h(d_{M,\ell}) \leq \sum_{\ell=1}^{M-L_M} h(d_{M,\ell}) =$$

$$= \sum_{\substack{\ell=n+1-M\\ \delta_\ell<o}}^{n} h(j_\ell). \qquad \text{Q. E. D.}$$

<u>Remark</u>. In the two special cases, where $\delta_k=1$ for all k or $\varepsilon_k=-1$ for all k, Lemma 2 ; 1 reduces to Theorem 5 of Lehmann [6]. Further, in each of these special cases, Lemma 2 ; 3 is analogous to Corollary 1 of Lehmann [6].

<u>Lemma 2 ; 4. Let α_1 , α_2 ,...., α_n be n numbers satisfying</u>

$$(2\ ;\ 28)\qquad o \leq \alpha_1 \leq \dots \leq \alpha_n ,$$

<u>let h be non decreasing and non negative and let $(i_k , \varepsilon_k , j_k , \delta_k)_{k=1}^{n}$ satisfy</u>

$$(2\ ;\ 29)\qquad \begin{cases} 1.\ (\delta_k=1\ ,\ \alpha_k > o) \Longrightarrow \varepsilon_k=1 \\ 2.\ (\delta_k=1\ ,\ \alpha_k > o\ ,\ \ell<k\ ,\ j_\ell < j_k) \Longrightarrow i_\ell < i_k \\ 3.\ (\varepsilon_k=-1\ ,\ \alpha_k > o\ ,\ \ell< k\ ,\ j_\ell > j_k) \Longrightarrow i_\ell > j_k \end{cases}$$

then

$$(2\ ;\ 30)\qquad \sum_{k=1}^{n} \alpha_k\, \varepsilon_k\, h(i_k) \geqslant \sum_{k=1}^{n} \alpha_k\, \delta_k\, h(j_k).$$

Proof : The following proof is analogous to Lehmann's proof of his Corollary 2 in [6].

(2 ; 30) is obviously true if $\alpha_k = o$ for all $k = 1,\dots,n$, so in the following it will be supposed that $\alpha_k > o$ for at least one k. Further, since h is non negative,

$$\sum_{\ell=1}^{n} h(\ell) \geqslant o \quad\text{and}\quad \sum_{\ell=1}^{n} h(\ell) = o \text{ if and only if } h(\ell) = o \text{ for all } \ell = 1,\dots,n,$$

in which case (2 ; 30) is obvious. In the following it will be supposed that $\sum_{\ell=1}^{n} h(\ell) > o$.

Let $o \leqslant \beta_1 < \beta_2 < \dots < \beta_T$ be the different values of $\alpha_1,\dots,\alpha_n$ and let $n_t (t=1,\dots,T)$ be the number of α_k equal β_t. Further let

$N_t = \sum_{s=1}^{t} n_s \quad (t=1,\dots,T)$ and $N_o = o$. Con ider the random variables X and Y each taking the values $(-\beta_T , -\beta_{T-1} ,\dots, -\beta_1 , \beta_1,\dots, \beta_{T-1} , \beta_T)$ with

$$(2\ ;\ 31)\quad \begin{cases} 1.\ P(X \leqslant -\beta_s) = \dfrac{\sum\limits_{\substack{\ell=N_{s-1}+1 \\ \varepsilon_\ell<o}}^{N_T} h(i_\ell)}{\sum\limits_{\ell=1}^{n} h(\ell)} \\[2ex] 2.\ P(X \leqslant \beta_s) = 1 - \dfrac{\sum\limits_{\substack{\ell=N_s+1 \\ \varepsilon_\ell>o}}^{N_T} h(i_\ell)}{\sum\limits_{\ell=1}^{n} h(\ell)} \end{cases} \qquad s=1,\dots,T$$

and

$$(2\ ;\ 32)\quad \begin{cases} 1.\ P\,(Y \lesseqgtr -\beta_s) = \dfrac{\sum\limits_{\substack{\ell=N_{s-1}+1 \\ \delta_\ell<0}}^{N_T} h(j_\ell)}{\sum\limits_{\ell=1}^{n} h(\ell)} \\[2em] 2.\ P(Y \lesseqgtr \beta_s) = 1 - \dfrac{\sum\limits_{\substack{\ell=N_s+1 \\ \delta_\ell>0}}^{N_T} h(j_\ell)}{\sum\limits_{\ell=1}^{n} h(\ell)} \end{cases} \qquad s = 1,\ldots,T$$

where, if $\beta_T = 0$, $P(X \leqslant 0)$ and $P(Y \leqslant 0)$ are defined by (2 ; 31.2) and (2 ; 32.2) respectively.

If $\beta_1 > 0$, condition (2 ; 29) reduces to Condition A_n and from Lemma 2 ; 3 it then follows that

$$(2\ ;\ 33)\qquad P(X \leqslant x) \leqslant P(Y \leqslant x) \text{ for all } x.$$

If $\beta_1 = 0$, condition (2 ; 29) is Condition $A_{n,M}$ for $M = N_T - N_1 = n-n_1$, so that in this case (2 ; 24) holds for $M \leqslant n-n_1$, which proves (2 ; 33)

From (2 ; 33) it follows that

$$(2\ ;\ 34)\qquad \mathcal{E}X \geqslant \mathcal{E}Y,$$

which is equivalent to

$$(2\ ;\ 35)\quad \sum_{s=1}^{T} \beta_s \sum_{\ell=N_{s-1}+1}^{N_s} \varepsilon_\ell h(i_\ell) \geqslant \sum_{s=1}^{T} \beta_s \sum_{\ell=N_{s-1}+1}^{N_s} \delta_\ell h(j_\ell),$$

which is equivalent to

$$(2\ ;\ 36)\quad \sum_{\ell=1}^{n} \alpha_\ell\, \varepsilon_\ell h(i_\ell) \geqslant \sum_{\ell=1}^{n} \alpha_\ell\, \delta_\ell\, h(j_\ell) \qquad \text{Q. E. D.}$$

III, MAIN RESULTS.

Let, for each $\nu = 1, 2, \ldots,$ $X_{\nu,1}, \ldots, X_{\nu,n_\nu}$ be independently and identically distributed random variables with common distribution function $F(x)$ satisfying

$$(3\,;\,1)\quad \begin{cases} 1.\ F(x) \text{ has an absolutely continuous density } f(x) \\ 2.\ \int_0^1 \varphi_F^2(u)\,du < \infty, \text{ where } \varphi_F(u) = -\dfrac{f'(F^{-1}(u))}{f(F^{-1}(u))} (0 \leq u \leq 1) \\ \quad \text{and where } f' \text{ is the derivative of } f \\ 3.\ f(x) = f(-x) \quad \text{for all } x. \end{cases}$$

Let $\Psi(u)$ $(0 \leq u \leq 1)$ be a function satisfying

$$(3\,;\,2)\quad \begin{cases} 1.\ \Psi(u) \text{ can be written as the sum of two functions } \Psi_1(u) \\ \quad \text{and } \Psi_2(u) \text{ where } \Psi_1(u) \text{ is non decreasing and non negative} \\ \quad \text{and } \Psi_2(u) \text{ is non increasing and non positive} \\ 2.\ \int_0^1 \Psi_i^2(u)\,du < \infty\ (i = 1, 2) \text{ and } \int_0^1 \Psi^2(u)\,du > 0 \end{cases}$$

Let $p_{\nu,1}, \ldots, p_{\nu,n_\nu}$ and $q_{\nu,1}, \ldots, q_{\nu,n_\nu}$ be vectors of constants satisfying

$$(3\,;\,3)\quad \begin{cases} 1.\quad \sum_{i=1}^{n_\nu} p_{\nu,i}^2 > 0 \\ 2.\quad \lim_{\nu\to\infty} \dfrac{\max_{1\leq i\leq n_\nu} p_{\nu,i}^2}{\sum_{i=1}^{n_\nu} p_{\nu,i}^2} = 0, \end{cases}$$

$$(3\,;\,4)\quad \begin{cases} 1.\quad \sum_{i=1}^{n_\nu} q_{\nu,i}^2 \leq M \quad \text{for some positive number } M \text{ independent of } \nu \\ 2.\ \lim_{\nu\to\infty} \max_{1\leq i\leq n_\nu} q_{\nu,i}^2 = 0 \end{cases}$$

and, for each $\nu = 1, 2, \ldots,$ either

(3 ; 5) $$\begin{cases} 1.\ p_{\nu,i}\ q_{\nu,i} \gtrless o & \text{for all } i = 1, \ldots, n_\nu \\ 2.\ (|p_{\nu,i}| - |p_{\nu,i'}|)\ (|q_{\nu,i}| - |q_{\nu,i'}|) \gtrless o & \text{for all } i,i' = 1, \ldots, n_\nu \end{cases}$$

or,

(3 ; 6) $$\begin{cases} 1.\ p_{\nu,i}\ q_{\nu,i} \lesseqgtr o & \text{for all } i = 1, \ldots, n_\nu \\ 2.\ (|p_{\nu,i}| - |p_{\nu,i'}|)\ (|q_{\nu,i}| - |q_{\nu,i'}|) \gtrless o & \text{for all } i,i' = 1, \ldots, n_\nu \end{cases}$$

Let $R_{|X_{\nu,i} - q_{\nu,i}\theta|}$ be the rank of $|X_{\nu,i} - q_{\nu,i}\theta|$ among

$|X_{\nu,1} - q_{\nu,1}\theta|, \ldots, |X_{\nu,n_\nu} - q_{\nu,n_\nu}\theta|$, let

(3 ; 7) $$\operatorname{sgn} u = \begin{cases} 1 & \text{if } u > o \\ -1 & \text{if } u < o \end{cases}$$

and let

(3 ; 8) $$T_\nu(\theta) = \sum_{i=1}^{n_\nu} p_{\nu,i}\, \Psi\left(\frac{R_{|X_{\nu,i} - q_{\nu,i}\theta|}}{n_\nu + 1}\right) \operatorname{sgn}(X_{\nu,i} - q_{\nu,i}\theta).$$

<u>Theorem 3 ; 1. If $F(x)$ is continuous, if $\Psi(u)$ is non decreasing and non negative then, for each $\nu = 1, 2, \ldots,$ $T_\nu(\theta)$ is with probability one a non increasing step function of θ if (3 ; 5) holds and a non decreasing step function of θ if (3 ; 6) holds</u> :

<u>Proof</u> : In the proof the index ν will be omitted. The proof will be given for the case that (3 ; 5) holds. The result for the case that (3 ; 6) holds is then obvious.

If $F(x)$ continuous, $T(\theta)$ is, with probability one, not well defined only for those values of θ satisfying $\theta = -\frac{X_i}{q_i}$ for some i with $q_i \neq o$ and for those values of θ satisfying $|X_i - q_i\theta| = |X_{i'} - q_{i'}\theta|$ for some pair (i,i') with $q_i \neq o$ or $q_{i'} \neq o$. These values of θ where $T(\theta)$ is not well defined, define a finite number of intervals for θ within each of which $T(\theta)$ is independent of θ.

Now consider two values θ_1 and θ_2 of θ for which $T(\theta)$ is well defined and let $\theta_1 < \theta_2$. Then it will be proved that $T(\theta_1) \geqslant T(\theta_2)$. Without loss of generality the X_i can be numbered in such a way that $|p_1| \leqslant \dots \leqslant |p_n|$ Then, by (3 ; 5.2), $|q_1| \leqslant \dots \leqslant |q_n|$. Write $T(\theta)$ as

$$(3\ ;\ 9) \qquad T(\theta) = \sum_{k=1}^{n} |p_k| \Upsilon\left(\frac{{}^R|X_k - q_k\theta|}{n+1}\right) \operatorname{sgn} p_k(X_k - q_k\theta),$$

where, for $p_k = o$, $\operatorname{sgn} p_k(X_k - q_k\theta)$ is defined as 1.

Now apply Lemma 2 ; 4 with, for $k = 1, \dots, n$

$$(3\ ;\ 10) \qquad \begin{cases} \alpha_k = |p_k| \\ \varepsilon_k = \operatorname{sgn} p_k(X_k - q_k\theta_1) & \delta_k = \operatorname{sgn} p_k(X_k - q_k\theta_2) \\ i_k = {}^R|X_k - q_k\theta_1| & j_k = {}^R|X_k - q_k\theta_2| \end{cases}$$

Then $T(\theta_1) \geqslant T(\theta_2)$ if (2 ; 29) is satisfied. That (2 ; 29) is satisfied can be seen from the following steps a), b) and c)

a) (2 ; 29.1) is identical with

$$\left\{p_k(X_k - q_k\,\theta_2) > o,\ p_k \neq o\right\} \Longrightarrow p_k(X_k - q_k\,\theta_1) > o$$

which follows immediately from (3 ; 5.1) and

$$p_k(X_k - q_k\,\theta_1) = p_k(X_k - q_k\,\theta_2) + p_k\,q_k\,(\theta_2 - \theta_1)$$

b) (2 ; 29.2) is identical with

$$\left\{p_k(X_k - q_k\,\theta_2) > o,\ p_k \neq o,\ \ell < k,\ |X_\ell - q_\ell\theta_2| < |X_k - q_k\theta_2|\right\} \Longrightarrow$$

$$|X_\ell - q_\ell\theta_1| < |X_k - q_k\,\theta_1|$$

This can be seen as follows. We have

$$-\frac{p_k}{|p_k|}(X_k - q_k\,\theta_2) < X_\ell - q_\ell\,\theta_2 < \frac{p_k}{|p_k|}(X_k - q_k\,\theta_2)$$

so that, using (3 ; 5),

$$X_\ell - q_\ell\,\theta_1 < \frac{p_k}{|p_k|}(X_k - q_k\,\theta_1) + (\theta_2 - \theta_1)\,(q_\ell - \frac{p_k}{|p_k|}q_k)$$

$$= \frac{p_k}{|p_k|}(X_k - q_k\,\theta_1) + (\theta_2 - \theta_1)\,(q_\ell - |q_k|) \leqq$$

$$\leqq \frac{p_k}{|p_k|}(X_k - q_k\,\theta_1)$$

also

$$X_\ell - q_\ell\,\theta_1 > -\frac{p_k}{|p_k|}(X_k - q_k\,\theta_1) + (\theta_2 - \theta_1)\,(q_\ell + \frac{p_k}{|p_k|}q_k)$$

$$= -\frac{p_k}{|p_k|}(X_k - q_k\,\theta_1) + (\theta_2 - \theta_1)\,(q_\ell + |q_k|) \geqq$$

$$\geqq -\frac{p_k}{|p_k|}(X_k - q_k\,\theta_1),$$

so that $|X_\ell - q_\ell\,\theta_1| \leqq |X_k - q_k\,\theta_1|$.

c) (2 ; 29.3) is identical with

$$\left\{p_k(X_k - q_k\,\theta_2) < 0,\ p_k \neq 0,\ \ell < k,\ |X_\ell - q_\ell\,\theta_2| > |X_k - q_k\,\theta_2|\right\} \Longrightarrow |X_\ell - q_\ell\,\theta_1| > |X_k - q_k\,\theta_1|.$$

The proof of this is analogous to that for (2 . 29.2). Q. E. D.

A special case of Theorem 3 ; 1 with $\Psi(u) = u$ and $p_{\nu,i} = q_{\nu,i}$ $(i = 1,\dots,n_\nu)$ was proved by Koul ([5], Lemma 2 ; 2).

<u>Theorem 3 ; 2. If (3 ; 1) - (3 ; 4) and (3 ; 5) or (3 ; 6) are satisfied then</u>

$$(3\ ;\ 11)\qquad \lim_{\nu\to\infty} P\left\{\sup_{|\theta|\leq C}\left|T_\nu(\theta) - T_\nu(0) + \theta K \sum_{i=1}^{n_\nu} p_{\nu,i}\,q_{\nu,i}\right| > \varepsilon\sigma(T_\nu(0))\right\} = 0,$$

where $K = \int_0^1 \Psi(u)\,\varphi_F(\frac{u+1}{2})\,du$.

<u>Proof</u> : The index ν will be omitted in the proof. It is sufficient to prove the theorem for the case where $\Psi_2(u) = 0$ for all u. Further the proof will be given for the case where (3 ; 5) holds ; the result for

the case where (3 ; 6) holds is then obvious.

The proof is analogous to the proof of Jurečková of her Theorem 3 ; 1 in [2] . As in her case it can be supposed without loss of generality that $\sum_{i=1}^{n} p_i^2 = 1$ and it can be seen, using the result of Hájek and Šidák ([1] , Theorem 1. 7) that it is sufficient to prove

$$\lim_{\nu \to \infty} P\left\{ \sup_{|\theta| \leq C} \left| T(\theta) - T(o) + \theta K \sum_{i=1}^{n} p_i q_i \right| > \varepsilon \right\} = o$$

As in Jureckova's proof and using the results of Hájek and Šidák ([1] , section VI. 2. 5) it can be proved that for any fixed set of points $\theta_1, \ldots, \theta_r$

$$\lim_{\nu \to \infty} P\left\{ \left| T(\theta_i) - T(o) + \theta_i K \sum_{i=1}^{n} p_i q_i \right| \leq \varepsilon \text{ for all } i = 1 \ldots r \right\} = 1$$

Further, for a fixed $C > o$, choosing $\theta_1, \ldots, \theta_r$ with

$$-C = \theta_1 < \theta_2 < \ldots < \theta_{r-1} < \theta_r = C$$

and

$$K \, | \theta_{i+1} - \theta_i | \leq \frac{1}{2} \varepsilon \frac{1}{\sqrt{M}},$$

where M is the constant in (3 ; 4), it can be seen, as in Jurečková's proof and using theorem 3;1 above, that

$$\left\{ \left| T(\theta_i) - T(o) + \theta_i K \sum_{i=1}^{n} p_i q_i \right| \leq \frac{\varepsilon}{2} \text{ for all } i = 1, \ldots, r \right\} \Longrightarrow$$

$$\sup_{|\theta| \leq C} \left| T(\theta) - T(o) + \theta K \sum_{i=1}^{n} p_i q_i \right| \leq \varepsilon$$

Q. E. D.

The conditions on the $p_{\nu,i}$ and $q_{\nu,i}$ in Theorem 3 ; 2 can be weakened as follows. (see also Jurečková [2] , Remark, page 1897). For every sequence of pairs of vectors $(p_{\nu,1}, \ldots, p_{\nu,n_\nu})$, $(q_{\nu,1}, \ldots, q_{\nu,n_\nu})$ it is possible to find a sequence of quadruplets of vectors $(p_{\nu,1}^{(\ell)}, \ldots, p_{\nu,n_\nu}^{(\ell)})$, $\ell = 1, 2, 3, 4$ such that for each $\nu = 1, 2, \ldots$

$$(3\ ;\ 12)\quad \begin{cases} 1.\ \ p_{\nu,i} = \sum_{\ell=1}^{4} p_{\nu,i}^{(\ell)} & i = 1,\ldots, n_\nu \\ 2.\ \ p_{\nu,i}^{(\ell)}\, q_{\nu,i} \gtreqless o \text{ for } \ell = 1,\ 2,\ i = 1,\ldots,n_\nu \\ \quad p_{\nu,i}^{(\ell)}\ \ q_{\nu,i} \lesseqgtr o \ \text{ for } \ell = 3,\ 4,\ i = 1,\ldots,\ n_\nu \\ 3.\ \ (|p_{\nu,i}^{(\ell)}| - |p_{\nu,i'}^{(\ell)}|)\ (|q_{\nu,i}| - |q_{\nu,i'}|) \gtreqless o \ \ \ell = 1,2,3,4 \\ \qquad\qquad \text{and } i,\ i' = 1,\ldots,\ n_\nu \end{cases}$$

That this is possible can be seen as follows. For every pair of vectors $(p_{\nu,1},\ldots, p_{\nu,n_\nu})$, $(q_{\nu,1},\ldots, q_{\nu,n_\nu})$ one can find $\alpha_{\nu,1}\ ,\ldots, \alpha_{\nu,n_\nu}$, $\beta_{\nu,1}\ ,\ldots, \beta_{\nu,n_\nu}$ such that $p_{\nu,i} = \alpha_{\nu,i} + \beta_{\nu,i}$ and

$$\left.\begin{array}{l} (\alpha_{\nu,i} - \alpha_{\nu,i'})\ (|q_{\nu,i}| - |q_{\nu,i'}|) \gtreqless o \\ (\beta_{\nu,i} - \beta_{\nu,i'})\ (|q_{\nu,i}| - |q_{\nu,i'}|) \leqq o \end{array}\right. \quad \text{for all } i,i'=1,\ldots,n_\nu$$

Further one can find $\gamma_\nu \gtreqless o$ such that $\alpha_{\nu,i} + \gamma_\nu \gtreqless o$, $\beta_{\nu,i} - \gamma_\nu \lesseqgtr o$ for all $i = 1,\ldots, n_\nu$. By taking $p'_{\nu,i} = \alpha_{\nu,i} + \gamma_\nu$, $p''_{\nu,i} = \beta_{\nu,i} - \gamma_\nu$ one has found $p'_{\nu,1}\ ,\ldots, p'_{\nu,n_\nu}$ and $p''_{\nu,1}\ ,\ldots, p''_{\nu\ ,n_\nu}$ such that $p_{\nu,i} = p'_{\nu,i} + p''_{\nu,i}$, $p'_{\nu,i} \gtreqless o$, $p''_{\nu,i} \lesseqgtr o$ $(i = 1,\ldots,n_\nu)$ and

$$\left.\begin{array}{l} (|p'_{\nu,i}| - |p'_{\nu,i'}|)\ (|q_{\nu,i}| - |q_{\nu,i'}|) \gtreqless o \\ (|p''_{\nu,i}| - |p''_{\nu,i'}|)\ (|q_{\nu,i}| - |q_{\nu,i'}|) \gtreqless o \end{array}\right. \quad \text{all } i,i' = 1,..,n_\nu$$

Further, if $p_{\nu,1}\ ,\ldots, p_{\nu,n_\nu}$ and $q_{\nu,1}\ ,\ldots, q_{\nu,n_\nu}$ satisfy the condition that the $p_{\nu,i}$ all have the same sign and

$$(3\ ;\ 13)\qquad (|p_{\nu,i}| - |p_{\nu,i'}|)\ (|q_{\nu,i}| - |q_{\nu,i'}|) \gtreqless o \quad \text{all } i,i'=1,..,n_\nu$$

then one can find $p'_{\nu,1}\ ,\ldots, p'_{\nu,n_\nu}$, $p''_{\nu,1}\ ,\ldots, p''_{\nu,n_\nu}$ such that

$$(3\ ;\ 14)\quad \begin{cases} 1.\ p_{\nu,i} = p'_{\nu,i} + p''_{\nu,i} \\ 2.\ p'_{\nu,i}\, q_{\nu,i} \cdot p_{\nu,i} \geqslant 0\ ,\ p''_{\nu,i}\, q_{\nu,i} \cdot p_{\nu,i} \leqslant 0 \qquad i=1,\ldots,n_\nu \\ 3.\ (|p'_{\nu,i}| - |p'_{\nu,i'}|)\,(|q_{\nu,i}| - |q_{\nu,i'}|) \geqslant 0 \\ \quad (|p''_{\nu,i}| - |p''_{\nu,i'}|)\,(|q_{\nu,i}| - |q_{\nu,i'}|) \geqslant 0. \end{cases}$$

This can be done as follows. Suppose, without loss of generality, $|q_{\nu\ i}| \leqslant |q_{\nu,i+1}|$ $i = 1,\ldots, n_\nu - 1$ and take

$$p'_{\nu,i} = 2i\ p_{\nu,i}\ \frac{q_{\nu,i}}{|q_{\nu,i}|} \qquad p''_{\nu,i} = \left[1 - 2i\ \frac{q_{\nu,i}}{|q_{\nu,i}|}\right] p_{\nu,i}\ ,$$

where $\frac{q_{\nu\ i}}{|q_{\nu,i}|}$ is taken as 1 if $q_{\nu,i} = 0$. Then

$$p'_{\nu,i}\ q_{\nu,i}\ p_{\nu i} = 2i\ p^2_{\nu,i}\ |q_{\nu,i}| \geqslant 0$$

$$p''_{\nu,i}\ q_{\nu,i}\ p_{\nu i} = \left[q_{\nu,i} - 2i\ |q_{\nu,i}|\right]\ p^2_{\nu,i} \leqq 0$$

Further, using (3 ; 13) ,

$$|p'_{\nu,i+1}| - |p'_{\nu,i}| = (2i+1)\ |p_{\nu,i+1}| - 2i\ |p_{\nu,i}| \geqq |p_{\nu,i}| \geqq 0$$

and, again using (3 ; 13) ,

$$|p''_{\nu,i+1}| - |p''_{\nu,i}| \geqslant |p_{\nu,i}| \left\{\left|1 - (2i+2)\ \frac{q_{\nu,i+1}}{|q_{\nu,i+1}|}\right| - \left|1-2i\ \frac{q_{\nu,i}}{|q_{\nu,i}|}\right|\right\} \geqslant 0\ ,$$

because $\left|1-2i\ \frac{q_{\nu,i}}{|q_{\nu,i}|}\right|$ is non decreasing in i.

Further it is clear that, if $p_{\nu,1}, \ldots, p_{\nu,n_\nu}$ satisfies $\sum_{i=1}^{n_\nu} p^2_{\nu,i} > 0$ for each ν(condition 3 ; 3.1), then, for each ν, there exists an ℓ (ℓ= 1, 2, 3, 4) such that $\sum_{i=1}^{n_\nu} \left\{p^{(\ell)}_{\nu,i}\right\}^2 > 0$. Also, if $p_{\nu,i}$ is written as $\sum_{\ell=1}^{4} p^{(\ell)}_{\nu,i}$, $T_\nu(\theta)$ can be written as the sum of four statistics and (3 ; 11) remains true, if it is true for each of these four statistics and

(3;15) $$\sum_{\ell=1}^{4} \sum_{i=1}^{n_\nu} \{p_{\nu,i}^{(\ell)}\}^2 \leq M_1 \sum_{i=1}^{n_\nu} p_{\nu,i}^2$$

for some M_1 independent of ν. Further (3;11) is true for each of these four **statistics** if (3;1),(3;2) and (3;4) are satisfied and the $p_{\nu,i}^{(\ell)}$ satisfy (3;12) and

(3 ; 16)

1. for at least one ℓ

$$\sum_{i=1}^{n_\nu} \left\{p_{\nu,i}^{(\ell)}\right\}^2 > 0 \qquad \text{for each } \nu$$

2. For each ℓ for which 1. is not satisfied

$$\sum_{i=1}^{n_\nu} \left\{p_{\nu,i}^{(\ell)}\right\}^2 = 0 \qquad \text{for each } \nu$$

3. For each ℓ for which 1. is satisfied

$$\lim_{\nu \to \infty} \frac{\max\limits_{1 \leq i \leq n_\nu} \left\{p_{\nu,i}^{(\ell)}\right\}^2}{\sum_{i=1}^{n_\nu} \left\{p_{\nu,i}^{(\ell)}\right\}^2} = 0$$

This proves the following theorem.

Theorem 3 ; 3 : If (3 ; 1) ; (3 ; 2) and (3 ; 4) are satisfied, **if** there exist $p_{\nu,1}^{(\ell)}, \ldots, p_{\nu,n_\nu}^{(\ell)}$ (ℓ=1, 2, 3, 4) such that (3;12),(3;15) and (3;16) are satisfied then (3 ; 11) holds.

A special case of Theorem 3 ; 2 with $p_{\nu,i} = q_{\nu,i} = \frac{1}{\sqrt{n_\nu}}$ was used by Kraft and van Eeden [3] , [4] to find the asymptotic properties of linearized estimates based on signed ranks for the one sample location problem .

Koul [5] proves a theorem analogous to Theorem 3 ; 2 for the p variate case where $R_{|X_{\nu,i} - q_{\nu i}\theta|}$ is replaced by

$R_{|X_{\nu,i} - \sum_{j=1}^{p} q_{\nu,ij}\theta_j|}$. He considers the case where $p_{\nu,i} = q_{\nu ij}$

for some j and all $i = 1, \ldots, n_\nu$; further in his case $\Psi(u) = u$ and his conditions on F are stronger than (3 ; 1)

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