M. S. BINGHAM
K. R. PARTHASARATHY

A probabilistic Proof of Bochner's Theorem on positive definite Functions

Publications des séminaires de mathématiques et informatique de Rennes, 1966-1967
« Séminaires de probabilités et statistiques », , exp. n° 1, p. 1-13
<http://www.numdam.org/item?id=PSMIR_1966-1967____A1_0>

A PROBABILISTIC PROOF OF BOCHNER'S THEOREM ON

POSITIVE DEFINITE FUNCTIONS

by

M. S. Bingham and K. R. Parthasarathy
University of Sheffield

1. Introduction

The purpose of this paper is to give a proof of Bochner's theorem on positive definite functions in a locally compact second countable abelian group by a purely probabilistic method without using the theory of Banach algebras or general theory of representations. We reduce the problem to the Daniel-Kolmogorov construction of a stochastic process on the basis of finite dimensional distributions. The Lévy-Cramer continuity theorem is deduced as a corollary.

2. Preliminaries

Let X be a locally compact second countable abelian group and Y its character group. Y consists of all continuous homomorphisms from X into the multiplicative circle group K of complex numbers with modulus unity. We shall denote the Haar measure on Y by H and use dy to denote integration with respect to H. For any $x \in X$, $y \in Y$, we shall denote by $\langle x,y \rangle$ the value of the character y at x. We assume the fact that X and Y are character groups of each other (cf. Rudin [2]).

A complex valued function φ defined on Y is said to be positive definite if for all $y_1, y_2, \ldots, y_k \in Y$, complex numbers $c_1, c_2, \ldots, c_k$ and positive integers k, the inequality

$$\sum_{1 \leq i, j \leq k} c_i \bar{c}_j \, \varphi(y_i - y_j) \geq 0 \tag{2.1}$$

is satisfied. With these notations we have the following theorem due to Bochner [2].

Theorem 2.1 Let φ be a complex-valued positive definite function defined on Y such that φ is continuous at e and $\varphi(e) = 1$, e being the identity element of Y. Then there exists a unique probability measure μ defined on the Borel σ-field of X such that

$$\varphi(y) = \int_X \langle x, y \rangle \, d\mu(x), \quad y \in Y$$

3. The case when X is a finite-dimensional torus

In this section we shall consider the case when $X = K^r$ where K is the multiplicative circle group consisting of all complex numbers of modulus unity and K^r is the r-fold cartesian product of K. Then $Y = I^r$, where I is the additive group of all integers. Any point $x \in X$ can be represented by $\underset{\sim}{\theta} = (\theta_1, \theta_2, \ldots, \theta_r)$ where $0 \leq \theta_j < 2\pi$, $j = 1, 2, \ldots, r$. With this representation the group operation becomes coordinatewise addition modulo 2π. Any point $y \in Y$ can be represented by $\underset{\sim}{n} = (n_1, n_2, \ldots, n_r)$ where all the n_j's are integers. Then $\langle \underset{\sim}{\theta}, \underset{\sim}{n} \rangle = \exp i \sum_{j=1}^{r} n_j \theta_j$. Let φ be a positive definite function on I^r and $\varphi(\underset{\sim}{0}) = 1$. φ is automatically continuous because I^r is discrete. Even though an elementary proof of Theorem 2.1 is well known in this case, we reproduce it for the sake of completeness.

Consider the measure μ_N which is absolutely continuous with respect to the uniform distribution on K^r and which has the density function

$$f_N(\theta) = \frac{1}{N^r} \sum_{\substack{1 \leq m_j \leq N \\ 1 \leq n_j \leq N \\ j = 1, \ldots, r}} \varphi(\underset{\sim}{m} - \underset{\sim}{n}) \langle \underset{\sim}{\theta}, \underset{\sim}{n} - \underset{\sim}{m} \rangle$$

Since $\langle \underset{\sim}{\theta}, \underset{\sim}{n} - \underset{\sim}{m} \rangle = \langle \underset{\sim}{\theta}, \underset{\sim}{n} \rangle \overline{\langle \underset{\sim}{\theta}, \underset{\sim}{m} \rangle}$, it follows from (2.1) that $f_N(\underset{\sim}{\theta}) \geq 0$ for all $\underset{\sim}{\theta}$. Trivially

$$\frac{1}{(2\pi)^r} \int_0^{2\pi} \cdots \int_0^{2\pi} f_N(\underset{\sim}{\theta})\, d\theta_1\, d\theta_2 \cdots d\theta_r = 1$$

An easy computation shows that, for $\underset{\sim}{j} = (j_1, j_2, \ldots, j_r)$, we have

$$\frac{1}{(2\pi)^r} \int f_N(\underset{\sim}{\theta}) \langle \underset{\sim}{\theta}, \underset{\sim}{j} \rangle \, d\underset{\sim}{\theta} = \varphi(\underset{\sim}{j}) \prod_{k=1}^{r} \left(1 - \frac{|j_k|}{N}\right), \quad \text{if } |j_k| < N, \; k = 1, 2, \ldots, r.$$

$$= 0 \quad \text{otherwise}$$

Since any set of probability measures on K^r is weakly conditionally compact, $\{\mu_N\}$ has a convergent subsequence. Let μ be any limit of $\{\mu_N\}$. Then

$$\int \langle \underset{\sim}{\theta}, \underset{\sim}{j} \rangle d\mu = \lim_N \int \langle \underset{\sim}{\theta}, \underset{\sim}{j} \rangle d\mu_N = \varphi(\underset{\sim}{j})$$

for all $\underset{\sim}{j}$. This establishes the existence of a μ with the required property. Uniqueness follows from the well known fact that any continuous function on K^r is a uniform limit of linear combinations of functions of the form $\langle \underset{\sim}{\theta}, \underset{\sim}{j} \rangle$. This completes the proof.

Before proceeding to the proof of Theorem 2.1 in the general case we shall mention a simple corollary. To this end we need some notations. For any two abstract sets A and B, we shall denote by A^B the set of all functions defined on B and taking values in A. Then K^Y is the space of all functions defined on Y and taking values in K. Any element $\underset{\sim}{\theta} \in K^Y$ can be represented as a function $\underset{\sim}{\theta}(y)$ where $0 \leq \underset{\sim}{\theta}(y) < 2\pi$ for all $y \in Y$. Let π_y denote the projection map $\underset{\sim}{\theta} \longrightarrow \underset{\sim}{\theta}(y)$ from K^Y onto K. Let $\underset{\sim}{\mathfrak{S}}$ denote the smallest σ-field of subsets of K^Y with respect to which all the π_y are measurable. We shall denote by $\mathcal{N}$ the character group of K^Y. $\mathcal{N}$ is the set of all integer valued functions $\underset{\sim}{n}(y)$ defined on Y and vanishing outside a finite subset. We shall denote by $\underset{\sim}{0}$ the function which is identically 0. With these notations we have the following corollary.

<u>Corollary 3.1</u> Let $\psi(\underset{\sim}{n})$, $\underset{\sim}{n} \in \mathcal{N}$ be any positive definite function defined on $\mathcal{N}$ and $\psi(\underset{\sim}{0}) = 1$. Then there exists a unique probability measure

P on $\mathcal{S}$ such that

$$\int \left[\exp\ i \sum_{y \in Y} \underset{\sim}{n}(y)\, \underset{\sim}{\theta}(y)\right] dP(\underset{\sim}{\theta}) = \psi(\underset{\sim}{n})$$

for all $\underset{\sim}{n} \in \mathcal{N}$.

<u>Proof</u> For any finite subset $F \subseteq Y$, let

$$\mathcal{N}_F = \left\{ \underset{\sim}{n} : \underset{\sim}{n} \in \mathcal{N},\ \underset{\sim}{n}(y) = 0 \text{ if } y \notin F \right\}$$

When ψ is restricted to $\mathcal{N}_F$ we obtain a positive definite function on I^F. Hence by the validity of Theorem 2.1 for K^F, we deduce the existence of a measure P_F on the Borel σ-field of K^F, satisfying the property

$$\int_{K^F} \left[\exp\left(i \sum_{y \in F} \underset{\sim}{n}(y)\, \underset{\sim}{\theta}(y)\right)\right] dP_F(\underset{\sim}{\theta}) = \psi(\underset{\sim}{n}),\ \underset{\sim}{n} \in \mathcal{N}_F.$$

It follows from the uniqueness of P_F for every F that if $F_1 \supseteq F_2$ are two finite subsets of Y, the measure P_{F_1} induces P_{F_2} through the natural projection from K^{F_1} onto K^{F_2} obtained by dropping the yth coordinate whenever $y \notin F_2$. In other words the family of measures $\{P_F : F \subseteq Y, F \text{ finite}\}$ is consistent. Hence, by the Daniel-Kolmogorov theorem [1] there exists a unique probability measure P on $\mathcal{S}$ with the required properties.

4. The general case

Before proceeding to the proof of Theorem 2.1 we need a simple lemma.

Lemma 4.1 Let $f(y)$ be a complex-valued Haar-measurable function satisfying the following conditions:

(1) $$|f(y)| = 1 \quad \text{for all } y \in Y$$

(2) $$f(y) \cdot f(y') = f(y + y') \quad \text{a.e. } y, y' \ (H \times H).$$

Then there exists a unique $x \in X$ such that

$$f(y) = \langle x, y \rangle \quad \text{a.e. } y \ (H).$$

Proof Consider the set function

$$\mu(C) = \int_C f(y)\, dy$$

for all Haar-measurable sets contained in a compact set. If $C + y'$ denotes the translate of C by y', we have from the invariance of Haar measure under translations,

$$\begin{aligned} \mu(C + y') &= \int_{C+y'} f(y)\, dy \\ &= \int_C f(y + y')\, dy \\ &= f(y')\, \mu(C) \quad \text{a.e. } y' (H). \end{aligned}$$

Choose and fix a compact set C such that $\mu(C) \neq 0$. Then

$$f(y) = \frac{\mu(C+y)}{\mu(C)} \quad \text{a.e. } y \ (H).$$

We shall now prove that $\mu(C + y)$ is continuous in y. Indeed, we have from condition (1) of the lemma

$$\begin{aligned} |\mu(C+y_1) - \mu(C+y_2)| &= \left| \int \chi_{C+y_1}(y) f(y)\,dy - \int \chi_{C+y_2}(y) f(y)\,dy \right| \\ &\leq \int \left| \chi_{C+y_1}(y) - \chi_{C+y_2}(y) \right| dy \\ &= \int \left| \chi_{C+y_1-y_2}(y) - \chi_C(y) \right| dy \end{aligned}$$

where χ_C denotes the indicator function of C. The last term on the right hand side of the above inequalities is simply the Haar measure of $(C + y_1 - y_2) \Delta C$ which tends to zero as $y_1 - y_2$ approaches the identity.

Thus $f(y)$ is almost everywhere equal to a continuous function $g(y)$. It follows from the conditions of the lemma that $|g(y)| = 1$ for all $y \in Y$ and $g(y).g(y') = g(y + y')$ for all $y, y' \in Y$. In other words g is a character on Y. Hence, by the duality theorem, there is a unique point $x \in X$ such that $g(y) = \langle x, y \rangle$ for all y. This completes the proof of the lemma.

<u>Proof of Theorem 2.1</u> Adopting the notations described before the statement of corollary 3.1, we introduce the function ψ defined on $\mathcal{N}^\rho$ by

$$\psi(\underset{\sim}{n}) = \varphi\Big(\sum_{y\in Y} \underset{\sim}{n}(y)y\Big)$$

The positive definiteness of φ in Y implies the positive definiteness of ψ in $\mathcal{N}$. Further $\psi(\underset{\sim}{0}) = \varphi(e) = 1$. Thus, by corollary 3.1, there exists a probability measure P on $\mathfrak{F}$ such that

$$\int_{K^Y} \exp\Big[i \sum_{y\in Y} \underset{\sim}{n}(y)\underset{\sim}{\theta}(y)\Big] dP(\underset{\sim}{\theta}) = \psi(\underset{\sim}{n}) \quad , \quad \underset{\sim}{n} \in \mathcal{N}$$

Consider the probability space $(K^Y, \overline{\mathfrak{F}}, P)$ where $\overline{\mathfrak{F}}$ denotes the P-completion of $\mathfrak{F}$. Treating the elements of Y as a time variable, define the stochastic process

$$z(y,\underset{\sim}{\theta}) = \exp i\underset{\sim}{\theta}(y) \qquad , \quad y\in Y \ , \underset{\sim}{\theta} \in K^Y .$$

For all $y_1, y_2, \ldots, y_k \in Y$, integers $n_1, n_2, \ldots, n_k$ and positive integers k, we have

$$(4.1) \qquad E \prod_{j=1}^{k} z(y_j, \underset{\sim}{\theta})^{n_j} = \varphi\Big(\sum_{j=1}^{k} n_j y_j\Big)$$

where E denotes expectation with respect to P. In particular,

$$(4.2) \qquad E\, z(y,\underset{\sim}{\theta})\, z(y',\underset{\sim}{\theta})\, z(y+y',\underset{\sim}{\theta})^{-1} = 1 \quad , \quad y, y' \in Y .$$

Since the random variable within the expectation sign above is of modulus unity, it follows that

$$(4.3) \qquad z(y+y',\underset{\sim}{\theta}) = z(y,\underset{\sim}{\theta})\, z(y',\underset{\sim}{\theta}) \quad \text{a.e. } \underset{\sim}{\theta} \ (P)$$

for every pair $y, y' \in Y$. We also have

$$(4.4) \qquad E\, z(y,\underset{\sim}{\theta}) = \varphi(y) \quad , \; y \in Y \; .$$

From (4.1) an easy computation gives

$$(4.5) \qquad E\left| z(y,\underset{\sim}{\theta}) - z(y',\underset{\sim}{\theta})\right|^2 = 2 - \varphi(y-y') - \varphi(y'-y)$$

Since $\varphi(e) = 1$ and φ is continuous at the identity, the above equation shows that $z(y, \underset{\sim}{\theta})$ is stochastically continuous in y. Therefore, by a theorem similar to Theorem 2.6, page 61 of Doob [1], we may assume that $z(y, \underset{\sim}{\theta})$ is $H \times P$-measurable. By applying Fubini's theorem to the function $\left| z(y, \underset{\sim}{\theta})\, z(y', \underset{\sim}{\theta}) - z(y + y', \underset{\sim}{\theta})\right|$ in the space $C \times C \times K^Y$ for every compact $C \subseteq Y$, we conclude that there exists a set $A \subseteq K^Y$ such that $P(A) = 1$, and for every $\underset{\sim}{\theta} \in A$,

$$z(y,\underset{\sim}{\theta})\, z(y',\underset{\sim}{\theta}) = z(y+y',\underset{\sim}{\theta}) \quad \text{a.e. } y, y' \ (H \times H)$$

It now follows from lemma 4.1 that there exists an element $x(\underset{\sim}{\theta}) \in X$ such that

$$(4.6) \qquad \langle x(\underset{\sim}{\theta}), y\rangle = z(y,\underset{\sim}{\theta}) \qquad \text{a.e. } y \ (H)$$

for every $\underset{\sim}{\theta} \in A$. Define $x(\underset{\sim}{\theta}) = e$, the identity of X, for every $\theta \notin A$. Since $\langle x(\underset{\sim}{\theta}), y \rangle$ is continuous in y for every $\underset{\sim}{\theta}$, it follows that $\langle x(\underset{\sim}{\theta}), y \rangle$ is P-measurable in $\underset{\sim}{\theta}$ for every $y \in Y$. Since the smallest σ-field with respect to which all the characters are measurable, is the Borel σ-field in X, it follows that $x(\underset{\sim}{\theta})$ is an X-valued random variable defined on $(K^Y, \overline{\underset{\sim}{\mathcal{S}}}, P)$. Let μ be the distribution of $x(\underset{\sim}{\theta})$ in X.

Using Fubini's theorem, we deduce from equation (4.6) that

$$\int \langle x(\underset{\sim}{\theta}), y \rangle \, dP(\underset{\sim}{\theta}) = \int z(y, \underset{\sim}{\theta}) \, dP(\underset{\sim}{\theta}) \quad \text{a.e. } y \ (H)$$

Since the left hand integral in the above equation is simply $\int \langle x, y \rangle \, d\mu(x)$, equation (4.4) gives

$$\int \langle x, y \rangle \, d\mu(x) = \varphi(y) \quad \text{a.e. } y \ (H) \,. \tag{4.7}$$

Further, (4.4) and (4.5) imply the continuity of φ at all points of Y. Hence we have

$$\int \langle x, y \rangle \, d\mu(x) = \varphi(y) \,, \quad y \in Y \,. \tag{4.8}$$

In order to prove the uniqueness of μ, choose and fix a dense subset $\{y_k\}$, $k = 1, 2, \ldots$ of Y. Consider the map

$$T : x \to \left(\langle x, y_1 \rangle, \langle x, y_2 \rangle, \ldots \ldots \right)$$

of X into the space K^∞ which is a countable product of K. Since characters are continuous and separate points, it follows that τ is a one-one and continuous map of X into K^{∞}. Since X is a countable union of compact sets, it follows by considering the restriction of τ to these compact sets that τ^{-1} is measurable. In other words τ is a Borel isomorphism between X and the σ-compact set $\tau(X)$. If μ and ν are two measures satisfying (4.8), then $\mu\tau^{-1}$ and $\nu\tau^{-1}$ are measures in K^{∞} with the same finite dimensional distributions. Further $\mu\tau^{-1}$ and $\nu\tau^{-1}$ are concentrated in $\tau(X)$. Therefore $\mu = \nu$. This completes the proof of Theorem 2.1.

5. The continuity theorem

For any probability measure μ defined on the Borel σ-field of X, the function φ defined on Y by the equation

$$\varphi(y) = \int \langle x, y \rangle \, d\mu(x)$$

is called the characteristic function of μ. It is easy to show that φ is continuous and positive definite. Bochner's theorem asserts that every continuous positive definite function arises in this way. Further the correspondence $\mu \longleftrightarrow \varphi$ is one-one. The uniform continuity of the characters implies the uniform continuity of φ. Further if a sequence of probability measures μ_n converges weakly to μ, then the corresponding characteristic functions φ_n converge uniformly over compact sets to the characteristic function φ of μ. The following theorem asserts a much stronger version of the converse.

<u>Theorem 5.1</u> Let $\{\mu_n\}$, $n = 1, 2, \ldots$ be a sequence of probability measures defined on the Borel σ-field of X and $\{\varphi_n\}$, $n = 1, 2, \ldots$ be the corresponding characteristic functions. If φ_n converges pointwise to a function φ which is continuous at the identity of Y, then φ is the characteristic function of a probability measure μ and μ_n converges weakly to μ.

<u>Proof</u> First of all it is obvious that φ is a positive definite function on Y and $\varphi(e) = 1$. Hence by Theorem 2.1 there exists a measure μ with φ as its characteristic function.

In order to prove the weak convergence consider the map τ described in the last paragraph of section 4. τ maps X into the space K^{∞}. It is clear that the characteristic function of $\mu_n \tau^{-1}$ converges to the characteristic function of $\mu \tau^{-1}$. Since the set of all probability measures in K^{∞} is weakly compact, it follows that $\mu_n \tau^{-1}$ converges weakly to $\mu \tau^{-1}$ in K^{∞}.

Choose and fix a sequence $\{C_j\}$ of pairwise disjoint Borel subsets of X whose union is X and such that the boundary of each C_j has μ-measure 0 and $\overline{C}_j$ is compact. Then for any closed set $F \subseteq X$, we have

$$(5.1) \qquad \overline{\lim_{n \to \infty}}\, \mu_n(F) = \overline{\lim_{n \to \infty}} \sum_{j=1}^{\infty} \mu_n(F \cap C_j)$$

$$\leq \sum_{j=1}^{\infty} \overline{\lim_{n \to \infty}}\, \mu_n(F \cap \overline{C}_j) .$$

Since τ is a Borel isomorphism which maps compact subsets of X onto closed subsets of K^{∞} and $\mu_n \tau^{-1}$ converges weakly to $\mu \tau^{-1}$, we have

$$\text{(5.2)} \qquad \overline{\lim_{n \to \infty}} \mu_n(F \cap \overline{C_j}) = \overline{\lim_{n \to \infty}} \mu_n \tau^{-1}\left[\tau(F \cap \overline{C_j})\right]$$
$$\leq \mu \tau^{-1}\left[\tau(F \cap \overline{C_j})\right]$$
$$= \mu(F \cap \overline{C_j}) = \mu(F \cap C_j)$$

Combining (5.1) and (5.2) we have

$$\overline{\lim_{n \to \infty}} \mu_n(F) \leq \mu(F) .$$

Since F is an arbitrary closed set, it follows that μ_n converges weakly to μ. This completes the proof.

REFERENCES

[1] J. L. Doob, Stochastic Processes. John Wiley & Sons, New York, 1953.

[2] W. Rudin, Fourier Analysis on Groups. Interscience Pub. Co., New York, 1962.