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THE LIMIT MAP OF A HOMOMORPHISM OF DISCRETE MÖBIUS GROUPS

by PEKKA TUKIA

1. Introduction

Let G and H be discrete groups of Möbius transformations of the closed $(n+1)$-ball $\overline{B}^{n+1}$ and let $\varphi : G \to H$ be a homomorphism. There are several situations where it has been important that there is a map $f : L(G) \to L(H)$ of the limit sets *inducing* φ, that is, f satisfies the compatibility condition

$$(1a) \qquad fg(x) = \varphi(g)f(x)$$

for $x \in L(G)$ and $g \in G$. The proofs of Nielsen's and Mostow's theorems that any isomorphism φ of two discrete Möbius groups corresponding to compact hyperbolic manifolds can be realized by a homeomorphism (if $n = 1$) or by a hyperbolic isometry of B^{n+1} (if $n > 1$) were based on the fact that in their situations there is a homeomorphism of the limit sets inducing φ. The limit set was in these cases the whole n-sphere S^n.

Thus it might be useful to make a general study of this situation. In the above mentioned occasions the map has been a homeomorphism of the limit sets but in the general situation this might be too strong a condition. Instead, we require that f is defined in a dense set A of L(G) such that A is G-*invariant*, i.e. $gA = A$ for $g \in G$. In addition, we often have a measure on L(G) and require that f is a.e. defined with respect to the measure. The measures we consider are so-called *conformal* (G)-*measures*, and a measure μ on $\overline{B}^{n+1}$ is such a measure if μ is a finite Borel measure satisfying the transformation rule

$$(1b) \qquad \mu(g(X)) = \int_X |g'|^\delta \, d\mu$$

for measurable X and $g \in G$; here $|g'|$ is the operator norm of the derivative g' of g and $\delta \geq 0$ is the *dimension* of the measure. Usually, μ will be supported by L(G) but of course we can extend by zero and regard μ as a measure of $\overline{B}^{n+1}$. A familiar example of a conformal measure is the n-dimensional Hausdorff measure on S^n satisfying (1*b*) for all Möbius transformations g. Sullivan [S1] has developed a method of Patterson [P] and shown that there is always a non-trivial measure supported by L(G) if G is non-elementary.

We say that a map f is a *limit map* of the homomorphism $\varphi : G \to H$ if f maps a non-empty G-invariant subset A of L(G) into L(H) and if f satisfies the compatibility condition ($1a$) (note that any non-empty G-invariant subset of L(G) is dense in L(G) if G is non-elementary). We often have a conformal measure μ on L(G) and f will be defined in a set of full μ-measure. The limit map will often be uniquely determined, perhaps up to a set of zero μ-measure.

There is a class of limit points having a special position in our study. This is the set of *Myrberg points* M(G) of L(G). The most convenient definition for us is that M(G) is the set of points $x \in L(G) \subset S^n$ with the property that, given distinct $u, v \in L(G)$, there is a sequence $g_i \in G$, such that $g_i(x) \to u$ and $g_i(y) \to v$ for $y \neq x$ as $i \to \infty$; the convergence will be locally uniform for $y \in \overline{B}^{n+1} \setminus \{x\}$ by Lemma 2A. If G is non-elementary, i.e. if L(G) contains more that two points, then M(G) is a dense subset of L(G) (see the discussion in Section 2).

Myrberg points were considered by P. J. Myrberg [M1] in the context of Fuchsian groups of finite volume. Myrberg's condition for these points was weaker and was essentially the same as the one in [T5] where $x \in M(G)$ if, given distinct, $u, v \in L(G)$ and $z \in B^{n+1}$, there are $g_i \in G$ such that $g_i(x) \to u$ and $g_i(z) \to v$. However, these two definitions are equivalent by Lemma 2A since obviously if $g_i(z) \to v \in S^n$ for one $z \in B^{n+1}$, then $g_i(z) \to v$ for all $z \in B^{n+1}$, as can easily be seen by the preservation of the hyperbolic metric.

Myrberg points are a subclass of the so-called *conical limit* points $\Lambda(G)$. These can be defined by the property that $x \in \Lambda(G)$ if there is a sequence $g_i \in G$ such that $g_i(x) \to u$ and, if $y \neq x$, then $g_i(y) \to v$ where u, v are distinct; here necessarily $u, v \in L(G)$. Again the convergence $g_i(y) \to v$ is locally uniform outside x by Lemma 2A. This is not the standard way to define conical limit points but it makes clear the difference of conical and Myrberg points.

The usual definition is obtained as follows. Let $x \in \Lambda(G)$. Fix $z \in S^n \setminus \{x\}$ and $z_0 \in B^{n+1}$. Let L be the hyperbolic line with endpoints z and x. Since $x \in \Lambda(G)$, there are distinct $u, v \in S^n$ and $g_i \in G$ such that $g_i(x) \to u$ and $g_i(y) \to v$ if $y \neq x$. Let L′ be the hyperbolic line with endpoints u and v. Thus g_i L approaches more and more L′ as $i \to \infty$, and obviously the hyperbolic distance $d(z_0, g_i L) \to d(z_0, L')$. Thus the distances $d(z_0, g_i L) = d(g_i^{-1}(z_0), L)$ are bounded. Furthermore, $g_i^{-1}(z_0) \to x$ since otherwise there would be a subsequence (denoted in the same manner) such that $g_i^{-1}(z_0) \to z' \neq x$. However, this is impossible since the local uniform convergence of g_i outside x implies $z_0 = \lim_{i\to\infty} g_i(g_i^{-1}(z_0)) = \lim_{i\to\infty} g_i(z') = v$, contradicting the fact that $z_0 \in B^{n+1}$ and $v \in L(G) \subset S^n$.

This argument can be reversed and we see that x is a conical limit point if and only if, given $z_0 \in B^{n+1}$ and a hyperbolic line L with endpoint x, there are $h_i \in G$ and $M > 0$ such that $h_i(z_0) \to x$ and that the distances $d(h_i(z_0), L)$ are bounded. The set of $z \in B^{n+1}$ with $d(z, L) \leqslant M$ is similar to a cone in neighborhoods of x and so we can approach x in a cone-like set; this gives the name to conical limit points.

Similarly, taking advantage of the fact that an (oriented) hyperbolic line is defined by a point $z_0 \in B^{n+1}$ and a direction α at z_0, we could show that $x \in S^n$ is a Myrberg point if and only if the following is true. Let L be a hyperbolic line with endpoint x, z_0 a point of B^{n+1}, and α a direction at z_0 such that if L′ is the hyperbolic line through z_0 with direction α, then both endpoints of L′ are in L(G). In this situation there is a sequence $g_i \in G$ such that, as $i \to \infty$, $g_i(z_0) \to x$ and $d(g_i(z_0), L) \to 0$, and such that the direction to which the derivative of g_i at z_0 maps α tends more and more towards the direction of $g_i(z)$ to x. Thus L and $g_i L'$ become closer and closer and more and more parallel at the point $g_i(z_0)$.

We can also describe the situation in the quotient $Q = B^{n+1}/G$. If $x \in \Lambda(G)$ and we travel on a hyperbolic line L with endpoint x towards x, then projecting to the quotient Q, we return infinitely often to a compact set of Q, to the set corresponding to a closed ball with center z_0. However, if $x \in M(G)$ and $L(G) = S^n$, then given any z in Q and any neighborhood U of z, we visit U infinitely often as we travel. And we come infinitely often to U from a direction arbitrarily close to a given direction.

Conical limit points have either full or zero measure for any conformal measure μ and the case of full measure occurs if and only if the *product action* $(x, y) \mapsto (g(x), g(y))$ of G is ergodic with respect to $\mu \times \mu$ ([S1], [NI]); a group action is *ergodic* if every invariant measurable set is either a nullset or the complement of a nullset. It was shown in [T5] that $\Lambda(G) \backslash M(G)$ is a nullset for any conformal measure, and hence this result is also true for Myrberg points. In particular, the Hausdorff n-measure of conical or Myrberg points is the same and is either zero or full. If the quotient B^{n+1}/G is compact, then every point of S^n is a conical limit point and hence in this case Myrberg points have full measure.

Our results fall into two types: topological and measure-theoretical. In the topological part we consider a map f of a G-invariant set A into $\overline{B}^{n+1}$ inducing a homomorphism $\varphi : G \to H$ of non-elementary groups and prove the following dichotomy (cf. Corollary 3E):

Either f is already defined or can be continuously extended to the Myrberg points so that we obtain a homeomorphism of M(G) *onto* M(H) *uniquely determined by* φ, *or* $\overline{f(U \cap A)} \supset L(H)$ *for any open set* U *intersecting* L(G).

Here the basic idea is that if U is an open set such that there are $a \in U \cap L(G)$ and $b \in L(G) \backslash \overline{U}$, then, given a Myrberg point x, there are $g_i \in G$ such that $g_i(x) \to a$ and $g_i(y) \to b$ locally uniformly on $\overline{B}^{n+1} \backslash \{x\}$. Thus if V is a neighborhood of x, some $g_i(\overline{B}^{n+1} \backslash V) \subset \overline{B}^{n+1} \backslash U$ and so $g_i^{-1} U \subset V$. It follows that $\{g_i^{-1} U\}$ is a basis of neighborhoods of x. If f is continuous at a and b and $f(a) \neq f(b)$, then the sets $f(g_i^{-1} U) = \varphi(g_i)^{-1} f U$ will converge towards a point which is, or will be, $f(x)$, provided that $f(b) \notin \overline{fU}$.

Actually, all our topological results are also valid for the so-called convergence groups. This class of groups seems to be the topological counterpart of discrete Möbius groups. We define these groups in Section 2.

In the measure-theoretical part we study the case where Myrberg or, equivalently, conical limit points have full conformal measure. In this case we can show that a measurable map f defined a.e. in L(G) and inducing a homomorphism φ is unique up to nullsets (Theorem 6B). In addition, $f(x)$ is a Myrberg point of H for a.e. x. Like in the topological part one of our results can be stated as a dichotomy (cf. Corollary 6F):

Either f can be changed in a nullset so that f is a homeomorphism of the Myrberg points or $\mu(f^{-1}[U] \cap V) > 0$ *for any open* U *intersecting* L(H) *and any open* V *intersecting* L(G).

The basic idea in the measure-theoretical part is not unlike the one in the topological part but must be supplemented by ergodicity and approximate continuity of measurable maps.

In section 7 we study the existence of f. If G is geometrically finite, then there is a canonical G-measure μ of mass 1 on L(G) (see [S1, S2] or [NI]). We will prove the following results in Theorems 7B and 7C:

If φ is any isomorphism of a geometrically finite G *onto an arbitrary discrete Möbius group* H, *then there is a limit map f of φ such that f is defined a.e. on* L(G) *with respect to the canonical* G-*measure* μ. *More generally, even if* G *is not geometrically finite, whenever the Myrberg points have full measure with respect to a conformal measure* μ, *we can show that either there exists a.e. a measurable limit map or alternatively, for a.e. x and any Stolz angle* C *at x, the set of accumulation points of* $\{ \varphi(g)(y_0) : g \in G$ *and* $g(x_0) \in C \}$ *must be the whole limit set of* H *whenever* $Gx_0 \cap C$ *is infinite.*

Above a Stolz angle is a cone-like set as defined in (7*a*); here x_0 and y_0 are arbitrary basepoints in B^{n+1}.

Our method to define $f(x)$ is to take all the elements $g_i = g_i^x \in G$ whose distance from the hyperbolic ray L_x joining 0 and x is less than a certain M and to show that $\varphi(g_i^x)(0)$ converges as $i \to \infty$ for a.e. x. The exponential growth of hyperbolic volume is mirrored in the exponential growth of the set A_r of $g \in G$ such that $d(0, g(0)) \leqslant r$. The relative number of $g \in A_r$ such that $d(0, \varphi(g)(0)) \leqslant \gamma r$ for some small $\gamma > 0$ also decreases exponentially. Hence the average hyperbolic distance of $\varphi(g)(0)$ from 0 grows at least linearly as a function of $d(0, g(0))$, allowing us to infer that the euclidean distance of $\varphi(g_i^x)(0)$ and $\varphi(g_{i+1}^x)$ is so small that $\varphi(g_i^x)$ converges as $i \to \infty$ for a.e. x.

Definitions and notation. — We let Möb(n) be the group of Möbius transformations (orientation preserving or not) of S^n; Möb(n) can be identified with the group of Möbius transformations of $\overline{B}^{n+1}$ since every Möbius transformation of S^n has a unique extension to a Möbius transformation of $\overline{B}^{n+1}$. Similarly, we can extend a Möbius transformation of S^n to a unique Möbius transformation of S^{n+1} preserving the components of $S^{n+1} \setminus S^n$. In this manner we can regard Möb(n) as a subgroup of Möb$(n+1)$ or of any Möb(m), $m > n$.

A *Möbius group* G acting on Y is a subgroup of some Möb(n) acting on a set $Y \subset \overline{B}^{n+1}$ such that GY = Y. The group G is discrete if it is a discrete subset in the natural topology of Möb(n). Since Möb$(n) \subset$ Möb(m) for $n < m$, we can regard two Möbius groups G and H as groups of Möbius transformations of the same ball $\overline{B}^{n+1}$ without sacrificing generality.

When we say that $f: A \to B$ *induces* $\varphi: G \to H$, then this includes the assumption that A and B are invariant under G and H, respectively, in addition to the validity of (1a) for $g \in G$ and $x \in A$. When we say that $f: A \to B$ is G-*compatible*, this means that f induces some homomorphism $\varphi: G \to H$ where H is a Möbius group acting on B.

We often have a map $f: X \to Y$ of two metric spaces such that there si a measure μ on X. We say that f is *measurable* if $f^{-1}U$ is measurable with respect to μ for any open $U \subset Y$. If we speak of the measure of a subset A of X, this implies that A is measurable. In particular, A is of *full measure* if A is measurable and its complement is a nullset.

The *hyperbolic metric of* B^{n+1} is d and it is given by the element of length $2\,|\,dx\,|/(1-|\,x^2\,|)$.

$L(x, y)$ = the hyperbolic line, ray or segment with endpoints x and y.

$D(x, r) = \{ z \in B^{n+1} : d(z, x) < r \}$.

$\Omega(G)$ = the ordinary set of G = the set where G acts discontinuously.

$L(G)$ = the limit of G = the complement of $\Omega(G)$.

$\Lambda(G)$ = the set of conical limit points of G.

$M(G)$ = the set of Myrberg points of G.

$B^k(x, r) = \{ z \in R^k : |\,z - x\,| < r \}$.

$B^k(r) = B^k(0, r)$ and $B^k = B^k(1)$.

S^k = the boundary of B^{k+1}.

$e_1, \ldots, e_{n+1}$ is the natural basis of R^{n+1}, $e_1 = (1, 0, \ldots, 0)$, etc.

∂ = the topological boundary.

diam = the euclidean diameter.

dist(z, V) = the euclidean distance of z from V.

2. Möbius groups, the convergence property and convergence groups

Many of our arguments are based on the *convergence property* of discrete Möbius groups. That is, given distinct g_i in a discrete Möbius group of $X = \overline{B}^{n+1}$, we can pass to a subsequence (denoted in the same manner) in such a way that there are $a, b \in X$ such that

(2a) $\quad g_i \mid X \setminus \{ a \} \to b,\quad$ and

(2b) $\quad g_i^{-1} \mid X \setminus \{ b \} \to a$

uniformly outside neighborhoods of a and b, respectively. These conditions are not independent: (2a) implies (2b) and conversely. For topological reasons a and b must be points of S^n if $X = \overline{B}^{n+1}$.

If (g_i) is a sequence such that (2a) and (2b) are true for some $a, b \in X$ we say that (g_i)

is a *convergence sequence* and b is the *attractive point* and a the *repelling point* of (g_i). Note that when passing from the sequence (g_i) to (g_i^{-1}), the attractive and repelling point are interchanged.

The convergence property makes possible to define a class of groups of homeomorphisms which somehow seems to be the topological counterpart of discrete Möbius groups. We say that a group G of homeomorphisms of a space X is a *convergence group* if any sequence of distinct elements $g_i \in G$ has a subsequence (denoted in the same manner) so that (2*a*) and (2*b*) are true for some $a, b \in X$. This notion is due to Gehring and Martin [GM] and corresponds to their definition of a discrete convergence group. It is also possible to define non-discrete convergence groups (see [GM] or [T9]) but we will here considerer only the discrete case. Gehring and Martin considered convergence groups of S^n or of $\overline{B}^{n+1}$.

All our topological theorems in Section 3 are valid also for convergence groups on a compact metric space X. Therefore we formulate them in this situation. Convergence groups occur, for instance, in connection with groups of isometries of metric hyperbolic spaces in the sense of Gromov. Therefore we believe that the extension to general spaces is not without interest, and in any case the proofs are the same. The auxiliary results we need are well-known for Möbius groups and are available for convergence groups in [GM] if $X = S^n$ or $X = \overline{B}^{n+1}$. The general case was treated in [T9]. When we say that G is a convergence group of X, it is understood that X is a compact metric space. Actually, all our arguments, like those of [T9], would be valid if X is a compact Hausdorff space such that every point has a countable basis of neighborhoods but for simplicity we assume that X is metrizable.

Many notions like the *limit set* L(G) and the *ordinary set* $\Omega(G)$ can be defined for a convergence group G of X exactly like for discrete Möbius groups of S^n and of $\overline{B}^{n+1}$. Thus $\Omega(G)$ is the subset of $x \in X$ having a neighborhood U such that $U \cap gU \neq \emptyset$ for only finitely many $g \in G$ and L(G) is the complement of $\Omega(G)$. Note that the attractive and repelling points of a convergence sequence of G are limit points.

These notions were studied in [GM] and [T9] and it was shown that they have the same properties as for discrete Möbius groups, for instance L(G) is a closed perfect set if it contains more than two points. Only in the definition of a non-elementary convergence group one must be careful. We say that a convergence group G is *non-elementary* if L(G) contains more than two points. If G is a discrete Möbius group, this gives the usual class of non-elementary Möbius groups. Since L(G) is perfect in this case, it follows that L(G) is then actually infinite. For discrete Möbius groups, but not for convergence groups, this is equivalent to the fact that G is not a finite extension of an abelian group. We refer to [GM] for a discussion of these matters.

Like elements of discrete Möbius groups, we can divide elements of a convergence group G of X into three types (cf. [GM] and [T9, Theorem 2B]: $g \in G$ is *elliptic* if it is of finite order). If g is of infinite order, then g has either one or two fixed points and in the first case g is *parabolic* and in the latter *loxodromic*. A sense-preserving loxodromic map of S^n is topologically conjugate to the map $x \mapsto 2x$ of $\overline{R}^n$, see [GM, 7.33] and [M, p. 420]. In all cases, if g is a loxodromic element of a general convergence group, it is true that $(g^i)_{i>0}$ is a convergence sequence such that the fixed points are the attractive and repelling points. This is a consequence of the convergence property, as it is easy to see that there is a neighborhood U of one of the fixed points such that $g^k \overline{U} \subset U$ for some k (cf. [T9, Lemma 2D]).

Our definitions of conical and Myrberg limit points are topological and so can be defined also for convergence groups. It is known that $M(G) \neq \emptyset$ for all discrete Möbius groups of the first kind (i.e. $L(G) = S^n$; cf. [NI, Theorem 2.2.2]; Myrberg point were called "line transitive points" in [NI]). The proof easily adapts to show that $M(G) \neq \emptyset$ for all non-elementary convergence groups; note that the theorem by Gottschalk and Hedlung on the density of fixed point pairs of loxodromic $g \in G$ on $L(G) \times L(G)$ to which Nicholls refers is available for all non-elementary convergence groups in [GM, 6.17] (if $X = S^n$) or in [T9, Theorem 2R]. It follows that for non-elementary G, M(G) is a dense subset of L(G) [T9, 2S].

We now give a criterion for finding out whether (g_i) is a convergence sequence.

Lemma **2A.** — *Let* G *be a convergence group of a compact metric space* X, *for instance a discrete Möbius group of* $\overline{B}^{n+1}$. *Let* $g_i \in G$ *be a sequence such that there are* $x, a \in X$ *as well as a sequence* $x_i \in X$ *such that* $x_i \to x$ *and* $g_i(x_i) \to a$ *as* $i \to \infty$. *Suppose in addition that there are* $b \neq a$ *in* X *and distinct* $y, z \in X$ *such that* $g_i(y) \to b$ *and* $g_i(z) \to b$ *as* $i \to \infty$. *Then* (g_i) *is a convergence sequence with* b *as the attractive point and* x *as the repelling point.*

Proof. — Suppose that g_{n_i} is a convergence subsequence. Clearly, either a or b must be the attractive point. If a is the attractive point, then both y and z must be the repelling point and this is impossible.

Thus, if g_{n_i} is any convergence subsequence, then b is the attractive and x the repelling point. Applying the convergence property, it easily follows that if (g_i) is not a convergence sequence with b as the attractive and x as the repelling point, then there are $z_i \to z' \neq x$ and a subsequence g_{n_i} such that $g_{n_i}(z_i) \to b' \neq b$. The sequence (g_{n_i}) must be infinite since either $\{g_{n_i}(y)\}$ or $\{g_{n_i}(z)\}$ is infinite. Hence we can pass to a convergence subsequence of (g_{n_i}) (denoted in the same manner). Then, as we have observed, b is the attractive point of (g_{n_i}). This is impossible since $g_{n_i}(x_i) \to a \neq b' \leftarrow g_{n_i}(z_i)$.

The next two lemmas are fairly obvious. In connection with the first lemma, it is useful to remember that if (g_i) is a convergence sequence, then the repelling point of (g_i) is the attractive point of (g_i^{-1}).

Lemma **2B.** — *Let* G *be a convergence group of a compact metric space* X. *Then* $x \in L(G)$ *if and only if there is a convergence sequence* (g_i) *whose attractive point is* x.

Lemma **2C.** — *If* G *is a convergence group of* X, *and* $A \subset X$ *contains more than two points, then* $\{h \in G : h \mid A = g \mid A\}$ *is finite for any* $g \in G$.

3. Continuous limit maps

In this section G and H are non-elementary convergence groups on compact metric spaces X and Y, respectively, for instance discrete Möbius groups on $\overline{B}^{n+1}$. In this section we study the situation where we have a homomorphism $\varphi : G \to H$ and a map f of a G-invariant set $A \subset X$ onto a H-invariant set $B \subset Y$ inducing φ (i.e. (1*a*) is true). We study what happens if f is continuous or can be extended continuously to some limit points. It turns out that if f is or can be extended continuously to a single limit point, then f is, or can be made, a homeomorphism of the Myrberg points of G onto Myrberg points of H, at least if φ is surjective.

We start with a lemma on the uniqueness of the extension to limit points. Note that if f is defined at z and if $\lim_{x \to z} f(x) = c$ exists, we do not assume that $f(z) = c$. We now have also another map h of a G-invariant set $A' \subset X$ onto an H-invariant set $B' \subset Y$.

Lemma **3A.** — *Let* $f: A \to B$ *and* $h: A' \to B'$ *be non-constant and induce* $\varphi: G \to H$. *If both* f *and* h *have a limit at* $z \in L(G)$, *then the limits are equal.*

Proof. — By Lemma 2B, we can find a convergence sequence (g_i) of G with z as the attractive point. Let y be the repelling point.

Let b be the limit of f at z. Since f is not constant, there are $u_1, u_2 \in A$ such that $f(u_1) \neq f(u_2)$. We can always replace u_k by $g(u_k)$, $g \in G$, and hence we can assume that $u_k \neq y$; we can take for g a suitable power of a loxodromic $h \in G$ as there are loxodromic elements in G not fixing u_k ([GM, 6.17] and [T9, Lemma 2Q]). Then

$$\varphi(g_i) f(u_k) = f g_i(u_k) \to b$$

since the limit exists at z, at least if $g_i(u_k) \neq z$. We can obtain by passing to a subsequence that either (α) $g_i(u_k) \neq z$ for all i and k or (β) $g_i(u_k) = z$ for all i and one k. If we have case (β), we simply replace as above u_k by $g(u_k)$ with some suitable $g \in G$. Thus we can in fact always have case (α) which we now assume.

The first consequence of the existence of the above limits for $k = 1, 2$ is that $\{\varphi(g_i)\}$ is infinite and hence we can pass to a subsequence so that $(\varphi(g_i))$ is a convergence sequence. The second consequence is that b is the attractive point of $(\varphi(g_i))$.

Let $c = \lim_{x \to z} h(x)$. A similar but simpler argument shows that c is also the attractive point of $(\varphi(g_i))$ and hence $b = c$. Here we do not pass to subsequences but start from the sequence $(\varphi(g_i))$ given above which was already a convergence sequence.

From now on φ will be surjective. Then, if f has a limit at $z \in L(G)$, the limit is not only unique but we can extend to Myrberg points as well. Although we will present a more general version later, we start with the basic situation giving the essential ideas, even at the cost of some redundancy.

Theorem **3B.** — *Let* $f: A \to B$ *induce a surjective homomorphism* $\varphi: G \to H$ *of non-elementary Möbius groups or, more generally, of non-elementary convergence groups. Assume that*

$\lim_{x \to z} f(x)$ *exists for a single point* $z \in L(G)$. *Then* f *can be extended to* $A \cup M(G)$ *in such a way that the extended map (still denoted* f*) is continuous at all points of* $M(G)$ *and satisfies*:

a) $f \mid M(G)$ *is a homeomorphism of* $M(G)$ *onto* $M(H)$ *and is uniquely determined by* φ,

b) $f(x) \neq f(y)$ *for all distinct* $x \in M(G)$ *and* $y \in A$,

c) if U *is a neighborhood of* $x \in M(G)$, *then there is a neighborhood* V *of* $f(x)$ *such that* $f^{-1} V \subset U$,

d) $\ker \varphi$ *is finite and* $g \mid L(G) = \mathrm{id}$ *if* $g \in \ker \varphi$.

Proof. — We will now prove all other points except *c)* and of *a)* we will only show that $f \mid M(G)$ is a continuous injection into $M(H)$, uniquely determined by φ. The remaining parts will be proved in connection with Theorem 3D.

Let $w = \lim_{x \to z} f(x)$. Now, if $g \in G$, then f has the limit $\varphi(g)(w)$ at $g(z)$. Since $\overline{Hw} \supset L(H)$ (cf. [GM, 6.13] and [T9, Theorem 2S]) and $L(H)$ is infinite, there are $a, b \in Gz$ such that f has limits a' and b' at a and b, respectively, such that $a' \neq b'$. In addition, we can assume that $a' \neq f(b)$ if f is defined at b.

We will now show that f has a limit at every $x \in M(G)$ and that if f is defined at x, then this limit is $f(x)$. Note that $M(G) \neq \emptyset$ (see Section 2) and hence $\overline{GA} \supset L(G) \supset M(G)$ and that $M(G)$ is dense in $L(G)$ (see [GM, 6.13] or [T9, Theorem 2S] for these results).

Pick $x \in M(G)$. By the defining property of Myrberg points, there is a sequence $g_i \in G$ such that $g_i(x) \to a$ and that $g_i(y) \to b$ if $y \neq x$ locally uniformly outside x. We can assume that $g_i(x) \neq a$ for all i which we can obtain as follows. Pick $a_k \in L(G) \setminus \{a, b\}$ such that $a_k \to a$. Use the Myrberg property to find g_{ki} such that $g_{ki}(x) \to a_i$ and $g_{ki}(y) \to b$ as $i \to \infty$ locally uniformly for any other point y. It is clear that we can take a sequence of the form $g_i = g_{ij_i}$.

Let U' and V' be neighborhoods of a' and b', respectively, such that $\overline{U}' \cap \overline{V}' = \emptyset$. If f is defined at b, set $V'' = V' \cup \{f(b)\}$, otherwise set $V'' = V'$. Since $a' \neq f(b)$, we can still have $\overline{U}' \cap \overline{V}'' = \emptyset$. If $y \neq x$, then $\varphi(g_i) f(y) = f g_i(y) \in V''$ for big i. When defining V'', we can use an arbitrary small neighborhood V' of b'. Hence, since $\mathrm{Im} f$ is infinite by non-elementariness, $\{\varphi(g_i)\}$ must be infinite and thus we can pass to a subsequence so that $\varphi(g_i))$ is a convergence sequence. It is clear that the attractive point b'' of $(\varphi(g_i))$ is in $\overline{V}''$. Let c be the repelling point of $(\varphi(g_i))$.

We claim that $\lim_{u \to x} f(u) = c$. Let W be a neighborhood of c. Let U be a neighborhood of a such that $f(A \cap (U \setminus \{a\})) \subset U'$. Now b'' and c are the repelling and attractive point of $(\varphi(g_i)^{-1})$, respectively. Since $b'' \notin \overline{U}'$, $\varphi(g_i)^{-1} U' \subset W$ for big i. Pick some i for which this is true. Let V be a neighborhood of x such that $g_i V \subset U$; since $g_i(x) \neq a$, we can assume that $a \notin g_i V$. Then

$$f(V \cap A) = \varphi(g_i)^{-1} f g_i(V \cap A) \subset \varphi(g_i)^{-1} f(A \cap (U \setminus \{a\}) \subset \varphi(g_i)^{-1} U' \subset W$$

and so f has the limit c at x. In addition, if f is defined at x, then f is continuous at x with value $f(x) = c$.

Thus we can assume that f is defined and continuous in M(G), and we will prove that $f(\mathrm{M(G)}) \subset \mathrm{M(H)}$. Let $x' \in f(\mathrm{M(G)})$. If $a', b' \in f(\mathrm{M(G)})$ are distinct, find $x, a, b \in \mathrm{M(G)}$ mapped by f onto x', a', b. The Myrberg property with respect to G, as well as the G-compatibility and continuity of f in M(G), easily imply that there is a sequence $h_i \in \mathrm{H}$ such that $h_i(x') \to a'$ and that $h_i(y') \to b'$ for any other $y \in f(\mathrm{M(G)})$ and hence for any point $y \neq x$ of $\overline{\mathrm{B}}^{n+1}$ by Lemma 2A. One easily sees by a limit process that actually here a', b' can be points of $\overline{f(\mathrm{MG})} = \mathrm{L(G)}$. It follows that $x' \in \mathrm{M(H)}$. By Lemma 3A, $f \mid \mathrm{M(G)}$ is uniquely determined by φ.

To prove *b)*, pick $a, b \in \mathrm{M(G)}$ such that $f(a) \neq f(b)$. If $x \in \mathrm{M(G)}$, pick $g_i \in \mathrm{G}$ such that $g_i(x) \to a$ and $g_i(y) \to b$ for any other point. By continuity in M(G), $fg_i(x) = \varphi(g_i) f(x) \to f(a) \neq f(b) \leftarrow \varphi(g_i) f(y) = fg_i(y)$, and so we obtain a contradiction if $f(x) = f(y)$.

To prove *d)*, let $\mathrm{N} = \ker \varphi$ and note that if $g \in \mathrm{N}$ and $g(x) = y$, then $f(y) = \varphi(g) f(x) = f(x)$ and so $y = x$ if $x \in \mathrm{M(G)}$ by part *b)*, now known to be true. Hence $g \mid \mathrm{L(G)} = g \mid \overline{\mathrm{M(G)}} = \mathrm{id}$ for $g \in \mathrm{N}$. In addition, N is finite by Theorem 2C.

We have now almost proved Theorem 3B. The remaining points are to show that $f(\mathrm{M(G)}) = \mathrm{M(H)}$ and that f^{-1} is continuous in M(H), as well as *c)*. It turns out that in the proof of these points we need to consider a more general situation.

If Z is a topological space, we let $\mathscr{C}(\mathrm{Z})$ be the set of all closed and non-empty subsets of Z. If Z is a compact metric space, $\mathscr{C}(\mathrm{Z})$ has a natural metric ρ, the Hausdorff metric, making $\mathscr{C}(\mathrm{Z})$ into a compact metric space, and ρ is defined by

$$\rho(\mathrm{A}, \mathrm{B}) = \sup \{ d(a, \mathrm{B}), d(b, \mathrm{A}) : a \in \mathrm{A}, b \in \mathrm{B} \} \tag{3a}$$

where $d(a, \mathrm{A})$ is the distance of a from A. We will make use of this topology of $\mathscr{C}(\mathrm{Z})$ in Sections 6 and 7 but the considerations in this section are independent of any topology on $\mathscr{C}(\mathrm{Z})$.

If H is a group of homeomorphisms of Z, then H acts naturally also on $\mathscr{C}(\mathrm{Z})$ by the rule $\mathrm{A} \mapsto h\mathrm{A}$. If G is another group of homeomorphisms on a space W and $f : \mathrm{W} \to \mathscr{C}(\mathrm{Z})$ is a map, then f induces a homomorphism $\varphi : \mathrm{G} \to \mathrm{H}$ if it satisfies (1*a*) for $x \in \mathrm{W}$ and $g \in \mathrm{G}$ with this action of H on $\mathscr{C}(\mathrm{Z})$.

When we apply this, H is a convergence group on Y and we denote $\mathscr{C} = \mathscr{C}(\mathrm{Y})$. The group G will be a convergence group on X and $f : \mathrm{A} \to \mathscr{C}$ will be a map of a G-invariant set A inducing a surjective homomorphism $\varphi : \mathrm{G} \to \mathrm{H}$.

We use the following notation in connection with a map $f : \mathrm{A} \to \mathscr{C}$ when $\mathrm{V} \subset \mathrm{A}$ and $x \in \overline{\mathrm{A}}$

$$f(\mathrm{V}) = \bigcup_{z \in \mathrm{V}} f(z),$$

$$\bar{f}(x) = \bigcap_{\mathrm{U}} \overline{f(\mathrm{U} \cap \mathrm{A})} \tag{3b}$$

where the intersection is taken over all neighborhoods U of x. Thus $\bar{f}$ is defined on $\overline{\mathrm{A}}$ if f is defined on A. If a non-empty A is G-invariant, then the set of accumulation points includes L(G) when G is non-elementary and thus $\bar{f}$ is defined in L(G) in this situation. Obviously, $\bar{f}$ induces φ if f does.

The "inverse" of f as a set function is defined on fA and is

$$f^{-1}(z) = \mathrm{cl}\{x \in \mathrm{A} : z \in f(x)\}. \tag{3c}$$

If $f^{-1}(y) = \{x\}$, this means that $f(x) = \{y\}$ and that $y \in f(z)$ for no $z \neq x$. Obviously, $\overline{f^{-1}}(z) \subset \overline{f}^{-1}(z)$ for $z \in f$A, and actually the sets are equal although we will not need this latter fact.

We regard ordinary point functions f as special cases of set functions (thus, if $f(x) = \{y\}$, we identify $f(x)$ and y) and for these $f\mathrm{A} = f(\mathrm{A})$ is just the image of A. For point functions, continuity at x is equivalent to the fact that $\bar{f}(x) = \{f(x)\}$.

We first prove

Lemma **3C.** — *Suppose that* $f(z) \not\supset \mathrm{L(H)}$ *for some* $z \in \mathrm{A} \cap \mathrm{L(G)}$. *Then there is* $y \in \mathrm{A} \cap \mathrm{L(G)}$ *such that* $f(z) \cap f(y) = \emptyset$.

Proof. — Let $z \in \mathrm{A} \cap \mathrm{L(G)}$ be a point such that $f(z) \not\supset \mathrm{L(H)}$. Thus there is $w \in \mathrm{L(H)} \setminus f(z)$. Let W be a neighborhood of w such that $\mathrm{W} \cap f(z) = \emptyset$. There is $h \in \mathrm{H}$ such that $hf(z) \subset \mathrm{W}$, for instance a suitable power of any loxodromic $h \in \mathrm{H}$ whose fixed points are in W will do; such an h exists since the fixed point pairs of loxodromic $h \in \mathrm{H}$ are dense in $\mathrm{L(H)} \times \mathrm{L(H)}$ by [GM, 6.17] or [T9, Theorem 2R]. If $h = \varphi(g)$, then $hf(z) = fg(z) \subset \mathrm{W}$. Hence $f(g(z)) \cap f(z) = \emptyset$.

The next theorem concludes the proof of Theorem 3B. Note that the fact that $\overline{f^{-1}}$ is a point function on M(H) gives *c)* of Theorem 3B.

Theorem **3D.** — *Suppose that* $\bar{f}(x) \not\supset \mathrm{L(H)}$ *for a single* $x \in \mathrm{L(G)}$. *Then* $\bar{f}$ *is a point function on* M(G) *and is a homeomorphism of* M(G) *onto* M(H) *whose inverse (as a point function of* M(H) *onto* M(G)) *is* $\overline{f^{-1}}$ *and with the property that* $\bar{f}(x) \cap \bar{f}(y) = \emptyset$ *for all distinct* $x \in \mathrm{M(G)}$ *and* $y \in \bar{\mathrm{A}}$. *In particular, if f is defined at* $x \in \mathrm{M(G)}$, *then* $f(x)$ *is a point* a *and also* $\bar{f}(x) = a$.

Proof. — The set of z such that $\bar{f}(z) \not\supset \mathrm{L(H)}$ is obviously open and G-invariant by the definition (3*b*) of $\bar{f}(z)$. Hence the assumption that $\bar{f}(x) \not\supset \mathrm{L(H)}$ for a single $x \in \mathrm{L(G)}$ implies that $\bar{f}(z) \not\supset \mathrm{L(H)}$ for all $z \in \mathrm{L(G)}$, since Gz is dense in L(G) [GM, 6 13] or [T9, Theorem 2S].

Pick arbitrary $x \in \mathrm{M(G)}$. We show that $\bar{f}(x)$ is a single point.

We apply Lemma 3C to $\bar{f}$ and pick two points $a, b \in \mathrm{L(G)}$ such that $\bar{f}(a)$ and $\bar{f}(b)$ are disjoint. Pick then neighborhoods V′ and W′ of $\bar{f}(a)$ and $\bar{f}(b)$, respectively, such that $\overline{\mathrm{V}'} \cap \overline{\mathrm{W}'} = \emptyset$. By the definition of $\bar{f}$, there are neighborhoods V and W of a and b, respectively, such that $\bar{f}(\mathrm{V}) \subset \mathrm{V}'$ and $\bar{f}(\mathrm{W}) \subset \mathrm{W}'$.

By the defining property of Myrberg points, there is a sequence $g_i \in G$ such that $g_i(x) \to a$, and $g_i(y) \to b$ if $y \neq x$ is any other point. Thus, if $y \neq x$, $\bar{f}g_i(y) = \varphi(g_i)\,\bar{f}(y) \subset W'$ for big i. On the other hand, taking y near x, we can find for every i such $y \neq x$ in A such that $\varphi(g_i)\,f(y) \subset V'$; recall that x is an accumulation point of A since G is non-elementary [T9, Theorem 2S]. Hence $\{\varphi(g_i)\}_{i>0}$ is infinite. Let $h_i = \varphi(g_i)$. Then $\{h_i\}$ is infinite and it follows that we can pass to a subsequence so that (h_i) is a convergence sequence.

These same arguments also show that $\bar{f}(A \setminus \{x\})$ is infinite and if $u \in \bar{f}(A \setminus \{x\})$, then $\varphi(g_i)\,(u) = h_i(u) \subset W'$ for big i. Hence the attractive point b' of (h_i) is in $\overline{W}'$.

Similarly, if i is big, $\bar{f}(g_i(x)) \subset V'$. Now $b' \notin \overline{V}'$ is the repelling point of (h_i^{-1}), and so $h_i^{-1}\bar{f}g_i(x)$ tend towards the attractive point a' of (h_i^{-1}). Hence $\bar{f}(x) = \varphi(g_i)^{-1}\bar{f}g_i(x) = h_i^{-1}\bar{f}g_i(x)$ must be the point a'. So $\bar{f}$ must be a point function on M(G), obviously coinciding with f on the points of M(G) where f is already defined.

The definition (3b) implies that this point function is continuous on M(G). Thus, by Theorem 3B, $\bar{f} \mid M(G)$ is a continuous injection of M(G) into M(H). Actually, we can prove exactly like Theorem 3B *b)* the stronger fact that $\bar{f}(x) \cap \bar{f}(y) = \emptyset$ if $x \in M(G)$ and $y \in \bar{A}$ are distinct. We will now complement Theorem 3B and show that $\bar{f}$ defines in fact a homeomorphism of M(G) onto M(H).

Let h be the set function f^{-1} defined on fA by (3c). Then the function $\bar{h}$ of (3b) is defined on $\overline{fA} \supset L(H)$. By our assumptions, there is $x \in L(G)$ such that $\bar{f}(x) \not\supset L(G)$. Since M(G) is dense in L(G), we can assume that $x \in M(G)$. It follows that there are $y \in M(H)$ and $x \in M(G)$ such that $\bar{f}(x) \neq y$. Let U′ be a neighborhood of y such that $\bar{f}(x) \cap \overline{U}' = \emptyset$. By the definition of $\bar{f}$, there is a neighborhood U of x such that $\bar{f}(x') \cap \overline{U}' = \emptyset$ if $x' \in U$. It follows that $x \notin \bar{h}(y)$ and hence $\bar{h}(y) \not\supset L(G)$.

Obviously fA and $\overline{fA}$ are H-invariant. If φ is an isomorphism, then $\bar{h}$ clearly induces φ^{-1} and we can conclude by the first part of the proof that $\bar{h}(x)$ is a continuous point function on M(H). Obviously, $\bar{f}$ and $\bar{h}$ are inverses to each other and so the theorem is proved in this case.

If φ is not injective, then let $N = \ker \varphi$ and apply Theorem 3B *d)* to the point function $f \mid M(G)$ and obtain that N is finite and $g \mid L(G) = \mathrm{id}$ if $g \in N$. Consequently orbits Nz are either points of L(G) or finite subsets of $\Omega(G)$. It follows that X/N is a compact metrizable space on which G/N acts as a convergence group. We can regard L(G) as a subset of X/N with G/N acting on L(G). We compose φ as $G \to G/N \to H$ and f as $A \to A/N \to \mathscr{C}$. Since Theorem 3D is now known for $A/N \to \mathscr{C}$ and the isomorphism $G/N \to H$, we can conclude that it is also true for $f: A \to \mathscr{C}$ and $\varphi: G \to H$, as obviously $M(G/N) = M(H)$.

We now return to point functions. We will enhance Theorem 3B in the following manner. Let the point function f be as in Theorem 3B. Define the *cluster set* $\mathscr{C}(f, x)$ of f at x by

$$\mathscr{C}(f, x) = \{y \in Y : f^{-1}V \cap U \neq \emptyset \text{ for all neighborhoods U of } x \text{ and V of } y\} \tag{3d}$$

and this is defined for $x \in \bar{A} \supset L(G)$. This is the set function $\bar{f}$ defined by (3*b*) when we regard f as a set function. In the situation of Theorem 3B we have the following dichotomy by Theorem 3D:

Corollary **3E.** — *Either the conclusions of Theorem* 3B *are valid or the cluster set* $\mathscr{C}(f, x) \supset L(H)$ *for every* $x \in L(G)$.

Above we have basically considered the situation that we have a map which is G-compatible and continuous at some $x \in L(G)$. If f is continuous and defined at a G-invariant closed set A, then there is additional information concerning inverse images of conical limit points [MT]. Combining [MT] with the results of this section, we have

Theorem **3F.** — *Let* $f: A \to B$ *be a continuous map of a closed and non-empty set* A *inducing a surjective homomorphism* φ *of two non-elementary groups. Then* $A \supset L(G)$ *and* $f(L(G)) = L(H)$. *Furthermore,* f *maps* M(G) *homeomorphically onto* M(H) *so that* $f^{-1} M(H) = M(G)$. *If* x *is a conical limit point of* H, *then* $f^{-1}(x)$ *consists of a single conical limit point of* G *and any right inverse of* f *is continuous at* x.

Proof. — Since A is closed, $A \supset L(G)$ by [GM, 6.13] and [T9, Theorem 2S]. Similarly, since fA is closed, $L(H) \subset fA$. To see that $fL(G) = L(H)$, we can reason as follows. Pick any $x \in L(G)$. By these same references, $\overline{Gx} = L(G)$ and hence x is an accumulation point of Gx. Thus either (α) $f(x)$ is an accumulation point of $Hf(x)$ or (β) $f \mid U \cap Gx$ is constant for some neighborhood U of x. In case (α), $f(x) \in L(H)$. We will show that (β) is impossible and hence always $f(x) \in L(H)$. It follows that $fL(G) \subset L(H)$ and, being H-invariant and closed, $fL(G) = L(H)$.

To prove the impossibility of (β), we show that in this case f is constant in the whole orbit Gx. This would imply that H fixes $f(x)$ and this is impossible by non-elementariness. Let $y \in Gx$. If $y \in U$, then $f(x) = f(y)$. If not, then there is $h \in G$ such that $h(x), h(y) \in U$. We can take for h a suitable power of some loxodromic element of G whose both fixed points are in U (cf. [GM, 6.17] or [T9, Theorem 2R]). Thus $f(h(x)) = f(h(y))$ implying $\varphi(h) f(x) = \varphi(h) f(y)$ and finally $f(x) = f(y)$.

The part concerning Myrberg points follows from the results of this section. The remaining part of the theorem is a slightly stronger form of [MT, Lemma 3.4]; conical limit points were called points of approximation in [MT]. Since we can simplify the proof of [MT], we indicate it below.

If $x \in \Lambda(H)$, we pick elements $h_i \in H$ such that $h_i(x) \to a$ and $h_i(y) \to b \neq a$ for any other y. Thus h_i is a convergence sequence with x as the repelling and b as the attractive point (Lemma 2A). We can assume that h_i are distinct. Find $g_i \in G$ such that $\varphi(g_i) = h_i$. Then g_i are distinct and thus it is possible to pass to a convergence subsequence, denoted in the same manner. It is easy to see that the attractive point of (g_i) is a point of $f^{-1}(b)$. Since $fg_i[f^{-1}(x)] = h_i(x) \to a \neq b$ it follows that any point of $f^{-1}(x)$ must be the repelling point. Hence $f^{-1}(x)$ consists of just one point x' and obviously

$x' \in \Lambda(G)$ since for big i, $g_i(x')$ and $g_i(y')$, $y' \neq x'$, are in arbitrarily small neighborhoods of $f^{-1}(a)$ and $f^{-1}(b)$, respectively, and these sets are disjoint.

It is a general topological fact that if f is a continuous map of a compact space, then any right inverse of f is continuous at a point w such that $f^{-1}(w)$ is a single point.

In contrast to Theorem 3F, note, however, that it is not necessary that $f(x) \in \Lambda(H)$ even if $x \in \Lambda(G)$. For a counterexample, see the remark in the next section.

4. Examples: Kleinian groups of S^2

We conclude the topological part of our study by giving a couple of examples on how to apply our theorems to some Kleinian groups of S^2 defined as discrete Möbius groups of S^2.

Example 1. — Let H be a non-elementary Kleinian group of S^2 whose ordinary set $\Omega(H)$ contains an invariant component U, i.e. $gU = U$ for all $g \in G$. If U is simply connected, there is a conformal homeomorphism f from the unit disk $D = B^2 \subset R^2$ onto U, the group $G = f^{-1} Hf$ is a Fuchsian group and $\varphi(g) = fgf^{-1}$ is an isomorphism $G \to H$. Since H is non-elementary, L(H) has positive conformal capacity [M2] and hence f has the radial limit a.e. in $\partial D = S^1$ with respect to the linear measure of S^1 [NE]. We denote the radial limit by f_r.

If H is finitely generated, then U/H is a finite Riemann surface and hence the Fuchsian group G is of the first kind. In addition, G is geometrically finite since for Fuchsian groups finite generation is equivalent to geometrical finiteness. Thus in the finitely generated case Theorem 7C would also imply the existence of an a.e. defined map f_φ onto L(H) inducing φ. By Theorem 6B the radial limit f_r and f_φ coincide a.e. with respect to the linear measure. However, we concentrate on the map f_r and thus H can be any non-elementary Kleinian group of S^2 with a simply connected invariant domain U.

We will now apply results of Section 3 to the map f_r. Here the measure on S^1 is the linear measure.

Theorem **4A.** — *Suppose that f_r is not injective. Then the radial limit f_r exists at all Myrberg points of* G *and the map defined by f and f_r is a homeomorphism of* $B^2 \cup M(G)$ *onto* $U \cup M(H)$.

If H *is finitely generated, f_r is always injective outside a nullset of* S^1.

Proof. — Suppose that f_r exists at distinct points $a, b \in S^1$ such that $f_r(a) = f_r(b)$. Let L be the hyperbolic line joining a and b in B^2. Then $S = fL \cup \{f_r(a)\}$ is a topological circle. Let X and Y be its complementary domains and let X′ and Y′ be the components of $B^2 \setminus L$ such that $fX' \subset X$ and $fY' \subset Y$.

Since H is non-elementary, L(H) contains more than one point and hence at least one of the components, say X, contains a point x of L(H). Then $x \notin \overline{Y}$. Let y be any point on $\partial Y' \setminus \overline{L}$. If we define $\bar{f}$ as in (3*b*), then $\bar{f}(y) \subset \overline{Y}$ and hence $x \notin \bar{f}(y)$. Thus $\bar{f}(y) \not\supset L(H)$ and Theorem 3D implies that $\bar{f}$ is a point function on M(G) which is a homeomorphism of M(G) onto M(H). The definition (3*b*) of $\bar{f}$ implies that the map defined by f and $\bar{f}$ on $B^2 \cup M(G)$ is a homeomorphism of $B^2 \cup M(G) \to U \cup M(G)$. This implies the first part of the theorem since obviously f_r and $\bar{f}$ coincide on M(G).

Thus if f_r is not injective, it maps M(G) homeomorphically onto M(H). Since M(G) has full measure in the finitely generated case, we can conclude that in this case f_r is always injective outside a nullset.

A parabolic element $h \in H$ is *accidental* if $g = \varphi^{-1}(h) \in G$ is loxodromic; such parabolic elements always exist unless H is quasi-Fuchsian or degenerate (i.e. the invariant component U is $\Omega(G)$) [MA, Theorem 4]. In this case the radial limit exists at both fixed points of g and the limit is the fixed point of h. Thus in this case the conclusion of Theorem 4A is true and we can prove

Corollary **4B.** — *Suppose that* H *is finitely generated and not degenerate. Then the conclusions of Theorem* 4A *are true.*

Proof. — As we have observed, there are accidental parabolic elements unless H is degenerate or quasi-Fuchsian. Thus, what we have said implies the theorem if we observe that in case H is quasi-Fuchsian, f can be extended to a homeomorphism $\bar{B}^2 \to \bar{U}$ (see [T6, Corollary 3.5.1]).

If the radial limit f_r does not exist at $x \in S^1$, we can define the radial cluster set as a generalization of the radial limit as

$$\tilde{f}_r(x) = \overline{L(0, x)} \cap L(H)$$

where $L(0, x)$ is the hyperbolic ray with endpoints 0 and x. We can now generalize Theorem 4A as

Theorem **4C.** — *If there are distinct* $x, y \in S^1$ *such that* $\tilde{f}_r(x) \cap \tilde{f}_r(y) \neq \emptyset$ *and* $\tilde{f}_r(x) \cup \tilde{f}_r(y) \neq L(H)$, *then the conclusions of Theorem* 4A *hold true.*

Proof. — The theorem is proved exactly like Theorem 5A if we know that, when L is the hyperbolic line joining x and y, then $fL \cup \tilde{f}_r(x) \cup \tilde{f}_r(y)$ divides S^2 into two components. We prove this as follows.

We can assume that H is degenerate, i.e. U is the ordinary set $\Omega(H)$ of H, cf. Corollary 4B above. Note that $\tilde{f}_r(x)$ and $\tilde{f}_r(y)$ are connected (each of them is the intersection of a descending sequence of connected compact sets) and hence so is $\tilde{f}_r(x) \cup \tilde{f}_r(y)$. Also the complement of $\tilde{f}_r(x) \cup \tilde{f}_r(y)$ must be connected since otherwise the complement of L(H) could not be connected contrary to the degeneracy.

It follows that $S^2 \backslash (\tilde{f}_r(x) \cup \tilde{f}_r(y))$ is homeomorphic to the euclidean plane and hence, if we collapse $\tilde{f}_r(x) \cup \tilde{f}_r(y)$ to a point, S^2 remains homeomorphic to S^2. In this collapse $fL \cup \tilde{f}_r(x) \cup \tilde{f}_r(y)$ becomes a Jordan curve whose complement has two components. Consequently, $S^2 \backslash (fL \cup \tilde{f}_r(x) \cup \tilde{f}_r(y))$ also has two components.

Remark. — If H is geometrically finite, then f can be extended to continuous map $f: \overline{B}^2 \to \overline{U}$ (cf. [T4, Section 4G] where we have discussed this). If $h = \varphi(g) \in H$ is an accidental parabolic element, then f maps the fixed points a and b of g onto the fixed point v of h. Since $a, b \in \Lambda(G)$ but $v \notin \Lambda(H)$, this provides an example of a situation where a conical limit point is not mapped onto a conical limit point by a continuous limit map.

Example 2. — As another, somewhat strange and unusual, example we mention that Cannon and Thurston [CT] have shown that there is a finitely generated Fuchsian group G with limit set $L(G) = S^1$ and a Kleinian group H of S^2 with limit set $L(H) = S^2$ and a continuous map $f: S^1 \to S^2$ inducing an isomorphism $\varphi : G \to H$. Here f must be surjective and so f is a continuous map of 1-dimensional sphere onto 2-dimensional sphere and hence f cannot be injective. By Theorem 3F, f is still a homeomorphism of M(G) onto M(H) and in addition, $M(G) = f^{-1} M(H)$. Thus the non-injectivity of f is concentrated on non-Myrberg points.

We note that both M(G) and M(H) have full linear or planar measure on S^1 or on S^2, respectively. For M(G) this follows from [M1]. The group H is geometrically tame [B, Theorem A]. Hence [T, 9.9.3] implies that the geodesic flow is ergodic on the tangent space of B^{n+1}/H, and this is equivalent to the fact that conical limit points have full measure, cf. [S1] or [NI, Theorem 8.3.5]. Hence also M(H) has full measure, cf. the Introduction.

Thus f maps homeomorphically the subset $M(G) = f^{-1} M(H)$ of S^1 of full linear measure onto the subset M(H) of full planar measure of S^2. Corollary 3D of [T4] adds the following feature. There is a subset A of M(G) of full linear measure such that the planar measure of fA vanishes.

5. Shadows and conformal measures

From now on, with the exception of Theorem 7A, we will consider only Möbius groups and study properties of limit maps in the presence of conformal measures. We will present here general results needed later.

Lemma **5A.** — *Let* $V \subset \overline{B}^{n+1}$ *and* $z \in \overline{B}^{n+1}$. *Suppose that* $g \in \text{Möb}(n)$ *and that there is* $c > 0$ *such that the euclidean distance* $\text{dist}(z, V) > c$ *and* $\text{dist}(g(z), gV) > c$. *Then there is* $M = M(c) > 1$ *such that, for all* $x, y \in V$,

$$1/M < \frac{|g'(x)|}{|g'(y)|} \leqslant M.$$

Proof. — This can easily proved using the fact that

$$\frac{1}{|x-y|^2} = \frac{|g'(x)|\,|g'(y)|}{|g(x)-g(y)|^2}$$

for all x, y, cf. [NI, eq. (1.3.2)]. Consequently

$$\frac{|z-x|^2}{|z-y|^2} = \frac{|g'(z)|\,|g'(y)|\,|g(z)-g(x)|^2}{|g'(z)|\,|g'(x)|\,|g(z)-g(y)|^2}$$

implying the lemma for $M = 4c^{-2}$.

In section 6 we will consider "shadows" of the hyperbolic open disks $D(x, r)$ of radius r and center x. Let $\pi : B^{n+1}\setminus\{0\} \to S^n$ be the map

(5a) $\qquad \pi(x) =$ the projection of x from 0 to S^n,

that is, x is on the hyperbolic ray from 0 to $\pi(x)$. The *shadow* of the disk $D(x, r)$ from 0 is

(5b) $\qquad S(x, r) = \pi(D(x, r)\setminus\{0\}).$

We will now prove some simple lemmas for this situation. First we note that if $x \in B^{n+1}$, then a simple calculation shows that

(5c) $$\frac{1-|x|}{2} < e^{-d(0,x)} = \frac{1-|x|}{1+|x|} \leqslant 1-|x|.$$

Lemma **5B.** — *For every $r > 0$, there is $c = c(r) > 1$ such that the euclidean diameters of $D(x, r)$ and $S(x, r)$ are in the interval $[c^{-1} e^{-d(0,x)}, ce^{-d(0,x)}]$ for all $x \in B^{n+1}$.*

Proof. — To prove this, it is easiest first to consider the situation in the halfspace model $H^{n+1} = R^n \times (0, \infty)$ of the hyperbolic space. Let $z = (y, t) \in H^{n+1}$ where $y \in R^n$ and $t > 0$. Let $\pi_0 : H^{n+1} \to R^n$ be the projection $z = (y, t) \mapsto y$ and let $S_0(z, r) = \pi_0(D(z, r))$. Let $\rho(z, r)$ be the euclidean diameter diam $D(z, r)$ which is also the euclidean diameter of $S_0(z, r)$. Since there is a euclidean similarity of H^{n+1} mapping z onto an arbitrary $w \in H^{n+1}$, the number $c_0 = \rho(z, r)/t$ depends only on r and not on z.

We now obtain the present case as follows. We can assume that $x \in B^{n+1}$ is on the ray $L(-e_{n+1}, 0)$ joining $-e_{n+1}$ and 0. Let g be a Möbius transformation $\overline{B}^{n+1} \to \overline{H}^{n+1}$ such that $g(0) = e_{n+1}$, $g(e_{n+1}) = \infty$ and $g(-e_{n+1}) = 0$. Let $g(x) = z = (0, t) \in L(0, e_{n+1})$. In small neighborhoods of $-e_{n+1}$, g is more and more like a similarity which implies that $\rho(z, r)/t = c_0$ and diam $D(x, r)/(1-|x|)$ are almost the same if $|x|$ is close to 1. By continuity, they are in bounded ratio for all $|x|$. By (5c), they are in bounded ratio if we replace $(1-|x|)$ by $e^{-d(0,x)}$.

For diam$(S(x, r))$ we still need the observation that diam$(S(x, r))$/diam$(D(x, r))$ tends to 1 as $|x| \to 1$. This and continuity imply that they are in bounded ratio for all x.

We will need the following information on the conformal measure of shadows.

Lemma **5C.** — *Let G be a discrete Möbius group of $\overline{B}^{n+1}$ and let μ be a non-trivial conformal G-measure of dimension δ supported by L(G). Let $x \in B^{n+1}$. Then there is $M_0 > 0$ such that for every $M \geqslant M_0$ there is $C > 1$ such that for $g \in G$*

(5d) $$C^{-1} e^{-\delta d(0,g(x))} \leqslant \mu(S(g(x), M)) \leqslant Ce^{-\delta d(0,g(x))}.$$

Proof. — This is a consequence of [NI, Theorem 4.3.2], and of the preceding lemma. Nicholls proved that the inequalities are true except possibly for finitely many $g \in G$ but we can remove this exception by making M_0 and C big enough. Alternatively, we could adapt Lemma 2C of [T5] for the half-space H^{n+1} to the ball B^{n+1}; this would give the lemma for all conformal measures of S^n without assuming that the support is L(G).

Usually we have a conformal measure μ on A such that the product action of G on $A \times A$ is ergodic with respect to $\mu \times \mu$. Let $f: A \to Y$ be a measurable map which induces a homomorphism $\varphi : G \to H$. Here H need not be a Möbius group. We take Y to be a metric space and H can be any group of Borel maps of H.

Lemma **5D.** — *Suppose that* G *acts ergodically on* $A \times A$ *with respect to* $\mu \times \mu$. *If* $f: A \to Y$ *is measurable and induces* $\varphi : G \to H$, *then no pre-image of a point under f has positive μ-measure unless f is a.e. constant. In particular, if* H *is a Möbius group on* $Y \subset \overline{B}^{n+1}$ *and* φG *is non-elementary, then no pre-image of a point has positive μ-measure.*

Proof. — Suppose that $\mu(f^{-1}(a)) > 0$ for some a. Then $B = Gf^{-1}(x)$ is a G-invariant set of positive measure and since the ergodic action of G on $A \times A$ implies that G also acts ergodically on A, it follows that B equals A up to a nullset. Let $b = \varphi(g)(a)$ for some $g \in G$. Then $\mu(f^{-1}(b)) = \mu(gf^{-1}(a)) > 0$. If $b \neq a$, $G[(f \times f)^{-1}(a, a)]$ and $G[(f \times f)^{-1}(a, b)$ are disjoint G-invariant subsets of $A \times A$ of positive measure. By the ergodicity of the product action, this is impossible and hence $b = a$ and so $(\varphi G)\, a = \{ a \}$. Thus f is constant on the set B of full measure and, if G is a Möbius group on Y, then φG is elementary [GM, 6.13].

6. Properties of measurable limit maps

We will now turn our attention to the situation that we have a discrete Möbius group G and a conformal G-measure μ on a G-invariant set X. We usually assume that G is non-elementary and that the product action of G on $X \times X$ is ergodic. Then obviously $\mu(X \setminus L(G)) = 0$ and so we can assume that $\overline{X} = L(G)$. Let us recall from the introduction that the product action is ergodic if and only if a.e. $x \in X$ is a Myrberg point. However, at first the assumption of ergodicity is not needed.

There are similarities between the method of Section 3 for the continuous case and the present situation but basically the problem is that if f is not continuous, it may be that $x_i \to x$ but not that $f(x_i) \to f(x)$. The methods used to circumvent this difficulty are ergodicity and approximate continuity. We need also to consider the situation where $f(x)$ can be a more general object such as a closed subset of $\overline{B}^{n+1}$.

We recall the definition of approximate continuity. Let A and X be metric spaces whose metric is denoted by d. Suppose in addition that μ is a Borel measure on A and let $B(a, r) = \{ x \in X : d(x, a) < r \}$. A map $f: X \to Y$ is measurable if the inverse image of every open set is measurable and f is approximately continuous at a point $x \in A$, if for every $\varepsilon > 0$ there is $r_0 > 0$ such that whenever $0 < r < r_0$,

$$\frac{\mu(\{y \in B(x, r) : d(f(y), f(x)) > \varepsilon\})}{\mu(B(x, r))} < \varepsilon.$$

We include in the definition of approximate continuity at a point x the assumption that $\mu(U) > 0$ for every neighborhood U of x. Later f will always be defined at a subset of a euclidean space and the metric will be the euclidean metric. In addition, μ will be a Borel measure and hence Borel regular and X will be a separable metric space. Hence we can conclude by [FE, 2.9.13] that a measurable map $f: X \to Y$ is approximately continuous a.e. in X.

We start with a general situation and study the uniqueness of measurable maps inducing a homomorphism φ. It turns out that at conical limit points approximate limits depend only on φ if they exist. We first prove a general lemma.

In the following lemma, we have a discrete Möbius group $G \subset \text{Möb}(n)$, a G-invariant $X \subset \overline{B}^{n+1}$ and a non-trivial conformal G-measure μ on X, a metric space Y and a measurable map $f: X \to Y$ inducing a homomorphism $\varphi: G \to H$ (i.e. (1*a*) is true for $x \in X$ and $g \in G$) where H is a group of Borel maps of Y.

Lemma **6A.** — *Suppose that f is approximately continuous at a point $x \in X$. If (g_i) is a congervence sequence of G with x as the repelling and b as the attractive point, and if $g_i(x) \to a \neq b$ as $i \to \infty$, then $(\varphi(g_i))$ has a subsequence (denoted in the same manner) such that $\varphi(g_i)^{-1} f(z) \to f(x)$ as $i \to \infty$ for $z \in X'$ where $X' \subset X \setminus \{b\}$ has full μ-measure in $X \setminus \{b\}$. If H is a Möbius group such that φG is non-elementary, then fX' is infinite.*

Proof. — Let L be the hyperbolic line joining a and b. Pick $w \in L$. Let P_w be the hyperbolic n-plane intersecting L orthogonally at w and let V_w be the component of $B^{n+1} \setminus P_w$ such that $a \in \overline{V}_w$.

Let L′ be the hyperbolic line with endpoints x and $-x$; note that $x \in L(G) \subset S^n$ since x is the repelling point of (g_i). Since $g_i(x) \to a$, and x and b are the repelling and attractive point of (g_i), respectively, it follows that $g_i(-x) \to b$ and so $g_i L' \to L$. Hence, if w_i is the orthogonal projection of w (in hyperbolic geometry) onto $g_i L'$, then $w_i \to w$. It follows that if P_i is the hyperbolic n-plane orthogonal to $g_i L'$ at w_i, then, at least for big i, there is a component V_i of $B^{n+1} \setminus P_i$ such that $a \in \overline{V}_i$. In addition, $V_i \to V_w$.

Now $g_i^{-1} \overline{V}_i$ form a basis of closed neighborhoods of x of the form $\overline{B}^{n+1}(x, r) \cap \overline{B}^{n+1}$. Since f is approximately continuous at x, it follows that we can pass to subsequence so that if

$$V_i' = \{u \in V_i : |f(g_i^{-1}(u)) - f(x)| < 2^{-i}\},$$

then $\mu(g_i^{-1}[V_i \setminus V_i']) / \mu(g_i^{-1} V_i) < 2^{-i}$. Applying Lemma 5A to the sets $g_i^{-1} V_i$ and maps g_i with $z = -x$, we obtain that the maps $g_i^{-1} \mid V_i$ preserve the ratios of μ-measures of sets, up to multiplication by a constant which is bounded away from 0 and ∞. It follows that for some $c > 0$,

$$\mu(V_i') > \mu(V_i)(1 - c2^{-i}).$$

Since $V_i \to V_w$, we can again pass to a subsequence so that $\mu(V_w \setminus V_i) < 2^{-i}$. It follows that, passing once more to a subsequence, we can find a subset V_w' of V_w of full measure such that every $z' \in V_w'$ is in V_i' beginning from some i, i.e. $|f(g_i^{-1}(z')) - f(x)| < 2^{-i}$. It follows that $\varphi(g_i)^{-1} f(z') \to f(x)$.

To obtain that this is true in a subset of $X \setminus \{b\}$ of full measure, we choose a sequence $w_k \in L$ such that $w_k \to b$ and choose for each w_k appropriate subsequences $(g_{ki})_{i>0}$ so that $\varphi(g_{ki}^{-1}) f(z) \to f(x)$ as $i \to \infty$ when $z \in X_k'$ and X_k' has full measure in V_{w_k}. We choose g_{ki} so that the sequence for w_{k+1} is a subsequence of the sequence for w_k. Having chosen these subsequences, we pass to a diagonal type subsequence (g_i) such that $\varphi(g_i) f(z) \to f(x)$ when $z \in \bigcup_{k>0} X_k'$ and this has full measure in $X \setminus \{b\}$.

Suppose then that φG is a non-elementary Möbius group. Observe that $X' \cup \{b\}$ has full measure in X. Hence $X' \cup \{b\}$ has a G-invariant subset $X'' \neq \emptyset$ having full measure in X. Since φG is non-elementary, it follows that $\overline{fX''} \supset L(\varphi G)$ (cf. [GM, 6.13]). Also since φG is non-elementary, $L(\varphi G)$ is actually infinite [GM, 4.5] and hence both fX'' and fX' must be infinite.

Remark. — We could allow f to be any map, possibly not measurable, if we define approximate continuity at x using the outer measure μ^* corresponding to μ. In addition, μ need not be Borel regular: it suffices that all open sets are measurable.

This same remark applies to the first part of the next lemma (up to the last sentence). Of course, for the last sentence we need f_i to be measurable and μ Borel regular in order to have that f_i are a.e. approximately continuous.

Theorem **6B.** — *Let μ be a conformal G-measure on X and let $X_1, X_2 \subset X$ be G-invariant. Let $f_i : X_i \to \overline{B}^{n+1}$ be measurable maps inducing a surjective homomorphism $\varphi : G \to H$ of non-elementary Möbius groups of $\overline{B}^{n+1}$. If both f_1 and f_2 are approximately continuous at a conical limit point $x \in X_1 \cap X_2$ of G, then $f_1(x) = f_2(x)$. In particular, if $X_1 = X_2 = X$ and if the product action of G on $X \times X$ is ergodic with respect to μ, then $f_1 = f_2$ a.e. in X.*

Proof. — Since $x \in \Lambda(G)$, there is a sequence $g_i \in G$ such that $g_i(x) \to a$ and $g_i(y) \to b$ for any other y and where $a \neq b$. By Lemma 2A, (g_i) is a convergence sequence with b as the attractive point and x as the repelling point. Hence, by the preceding lemma, there is a subset X' of $X_1 \setminus \{b\}$ of full measure such that, after passing to a subsequence, $\varphi(g_i)^{-1} f_1(x') \to f_1(x)$ for $x' \in X'$. In addition we know that fX' is infinite and hence there are $x_1, x_2 \in X'$ such that $f_1(x_1) \neq f_1(x_2)$.

Since $\varphi(g_i)^{-1} f_1(x_k) \to f_1(x)$ for two points $f_1(x_1)$ and $f_1(x_2)$, we can first conclude that $\{\varphi(g_i)\}$ is infinite and then, after passing to a subsequence so that $(\varphi(g_i))$ is a convergence sequence, that $f_1(x)$ must be the attractive point of $(\varphi(g_i)^{-1})$ and hence the repelling point of $(\varphi(g_i))$.

Exactly the same argument shows that $f_2(x)$ is also the repelling point of $(\varphi(g_i))$. Here were start from the subsequence obtained in the end of the last paragraph so that $(\varphi(g_i))$ is already a convergence sequence. Hence $f_1(x) = f_2(x)$.

Finally, we note that the ergodicity of the product action is equivalent to the fact that conical limit points have full measure, as explained in the Introduction. Everything is now proved.

We now come to the part using ergodicity. From now on until the end of this section, G is a discrete non-elementary Möbius group and μ is a non-trivial conformal G-measure such that the product action of G is ergodic on $X \times X$. As we have remarked, then a.e. $x \in X$ is a Myrberg point and hence we can assume that $\overline{X} = L(G)$. Observe that in this situation every open non-empty subset of $\overline{X}$ has positive measure.

The next lemma combines ergodicity and the convergence property of Möbius groups.

Lemma **6C.** — *Let* $W \subset X \times X$ *be measurable and have positive* $\mu \times \mu$*-measure. Then there is* $Z \subset X \times X$ *of full* $\mu \times \mu$*-measure such that if* $z = (x, y) \in Z$, *then there is a convergence sequence* (g_i) *of* G *such that* $g_i(z) \in W$ *for all* i *and that* x *is the repelling point of* (g_i) *and that the attractive point is* b *such that* $(a, b) \in \overline{W}$ *for some* a.

Proof. — Since there are no atoms (follows from Lemma 5D if we set $f = \mathrm{id}$ in it), the set W has a subset W_0 of positive measure such that $W_0 \subset U \times V$ where $\overline{U} \cap \overline{V} = \emptyset$. It is not difficult to see by ergodicity that there is a subset $Z \subset X \times X$ of full measure such that if $z = (x, y) \in Z$, then there is a sequence of distinct $g_i \in G$ with $g_i(z) \in W_0 \subset U \times V$. Since g_i are distinct, it is possible to pass to a convergence subsequence. Hence we can assume that (g_i) is a convergence sequence. Suppose that $g_i(x) \to a$ and $g_i(y) \to b$ as $i \to \infty$. Then $a \in \overline{U}$ and $b \in \overline{V}$ and hence $a \neq b$. This is compatible with the convergence property only if either

1° x is the repelling and b the attractive point, or

2° y is the repelling and a the attractive point.

Let Z_W be the set of points $(x, y) \in X \times X$ such that there is a sequence of distinct $g_i \in G$ such that $g_i(x, y) \in W$. Thus we can express Z_W, up to a nullset, as the union of sets $Z_1 = Z_1(W)$ and $Z_2 = Z_2(W)$ defined by the requirement that Z_i is the set of points $z = (x, y) \in Z_W$ for which there is a convergence sequence (g_i) such that we have the situation of i^0 with the additional information (if we have case 1°) that there is a such that $(a, b) \in \overline{W}$ or (if we have case 2°) that there is b such that $(a, b) \in \overline{W}$.

Clearly, all the sets Z_W and Z_i are G-invariant and since $Z_1 \cup Z_2 = Z_W$, at least one of them must have full measure by ergodicity. If $Z_1(W)$ has, we are done. Otherwise we note that $Z_1(W)$ is a nullset and so is $Z_1(W')$ for any measurable $W' \subset W$. It follows that $Z_2(W')$ must have full measure for any $W' \subset W$ of positive measure. Furthermore, if $W'' \subset X \times X$ has positive measure, then there are by ergodicity $g \in G$ and $W' \subset W$ of positive measure such that $gW' \subset W''$. Since $Z_2(W')$ has full measure, it follows that also $Z_2(W'')$ has. The conclusion is that $Z_2(W)$ has full measure for any W of positive measure.

If $A \subset X \times X$, set $A^* = \{(u, v) : (v, u) \in A\}$. Since $Z_1(W) = Z_2(W^*)^*$, it follows that also $Z_1(W)$ has full measure for any W of positive measure.

In Lemma 6C we had only the group G acting on X with ergodic action on $X \times X$. Now we take another group H acting on a separable metric space Y. Although G was a Möbius group, we first assume of H that it is only a group of Borel maps of Y. Let $f: X \to Y$ be a measurable map which induces a surjective homomorphism $\varphi : G \to H$. This means that (1*a*) is true. We define for $x \in \overline{X} = L(G)$ the *essential cluster set* of f at x by

$$(6a) \qquad \mathscr{A}(f, x) = \{y \in Y : \mu(f^{-1} V \cap U) > 0 \text{ for every neighborhood U of } x \text{ and V of } y\}.$$

Theorem **6D.** — *There is a* G*-invariant subset* $A \subset X$ *of full* μ*-measure such that for every* $x \in A$, *there is a subset* $A_x \subset X \backslash \{x\}$ *of full measure such that* f *is approximately continuous on* A *and on each* A_x *and with the following property. Let* $x \in A$ *and* $y \in A_x$ *and let* $a, b \in \overline{X} = L(G)$, $u \in \mathscr{A}(f, a)$ *and* $v \in \mathscr{A}(f, b)$. *Then there is a sequence* $g_i \in G$ *of distinct elements such that*

$$(6b) \qquad g_i(x) \to a, \quad g_i(y) \to b, \quad \varphi(g_i) f(x) \to u \quad \text{and} \quad \varphi(g_i) f(y) \to v$$

as $i \to \infty$. *In addition,* (g_i) *is a convergence sequence whose repelling and attractive points are* x *and* b, *respectively, and* $\varphi(g_i)^{-1} f(z) \to f(x)$ *when* z *varies in subset* A' *of* A *of full measure (possibly depending on* x *and on the chosen sequence* g_i*). Furthermore,* fA *and* fA' *are infinite.*

If H *is a discrete Möbius group on* $Y \subset \overline{B}^{n+1}$, *then* H *is non-elementary,* $fA \subset M(H)$ *and* $(\varphi(g_i))$ *is a convergence sequence whose repelling and attractive points are* $f(x)$ *and* v, *respectively.*

Remark. — If $Y = \overline{B}^{n+1}$ and $H \subset$ Möb(n), it follows from the convergence property, that we can actually take above $A_x = X \backslash f^{-1}[f(x)]$ at least if we do not require approximate continuity on A_x. We know by Lemma 5D that $\mu(f^{-1}[f(x)]) = 0$.

Proof. — We find countable open covers U_{ij}, $j \in I_i$, of $X \times X$ and V_{ik}, $k \in J_i$, of $Y \times Y$ such that $d(U_{ij}) < 1/i$ and $d(V_{ik}) < 1/i$ when d is the product metric. Let I be the set of (i, j, k) such that $U_{ijk} = U_{ij} \cap (f \times f)^{-1} (V_{ik})$ has positive $\mu \times \mu$-measure. Let B_{ijk} be the subset Z of full measure of $X \times X$ which is given by the preceding lemma with respect to $W = U_{ijk}$ and set $B = \bigcap_{(i,j,k) \in I} B_{ijk}$. Thus B has full measure.

Let now $(x, y) \in B$, $a, b \in \overline{X}$, $u \in \mathscr{A}(f, a)$ and $v \in \mathscr{A}(f, b)$. We claim that there is a convergence sequence (g_i) of G with x as the repelling and b as the attractive point and satisfying (6*b*). To see this, find for each i indexes j_i and k_i such that if $U_i = U_{ij_i}$,

$U'_i = U_{ij_ik_i} \subset U_i$ and $V_i = V_{ik_i}$, then $(a, b) \in U_i \supset U'_i$ and $(u, v) \in V_i$. Let $(x, y) \in B$. Then, for each i, there is a convergence sequence $g_{ij}, j > 0$, such that $(g_{ij}(x), g_{ij}(y)) \in U'_i$, and hence $(\varphi(g_{ij})\,(f(x)),\ \varphi(g_{ij})\,(f(y))) \in V_i$, and x is the repelling point and the attractive point is a point b' such that $(a', b') \in \overline{U}'_i \subset \overline{U}_i$ for some a'.

It is clear that we can choose for each i a (big) index m_i such that the sequence $g_i = g_{im_i}$ is a convergence sequence and has x as the repelling point and the attractive point is b'' such that $(a'', b'') \in \bigcap_{i>0} \overline{U}_i = \{(a, b)\}$ for some a''. Thus the attractive point must be b. In addition, $g_i(x, y) \in U'_i \subset U_i$, implying the first two limits of (6*b*) since $d(U_i) < 1/i$ and $(a, b) \in U_i$. Similarly, the last two limits of (6*b*) follow from the fact that $(fg_i(x), fg_i(y)) = (\varphi(g_i) f(x), \varphi(g_i) f(y))$ is in the set V_i containing (u, v) with diameter $< 1/i$.

We now apply the Fubini theorem to the set B. Thus there is a set $A \subset X$ of full measure such that for every $x \in A$ there is a measurable $A_x \subset X$ of full measure such that $\{(x, y) : x \in A, y \in A_x\}$ has full $\mu \times \mu$-measure in B. We can assume in addition that f is approximately continuous in the sets A and A_x. Since G is discrete and hence countable, we can assume that they are G-invariant. These sets A and A_x satisfy the first paragraph of the lemma except for the assertion concerning A′ to be proved now.

Since the product action is ergodic and G non-elementary, there are no atoms (follows from Lemma 5D if we set $f = \mathrm{id}$ in it). Hence, if $a \neq b$, Lemma 6A implies, after passing to a subsequence, that there is $A' \subset A$ of full measure such that $\varphi(g_i)^{-1} f(z) \to f(x)$ if $z \in A'$. If $a = b$, we find sequences $a_k \to a$, $b_k \to b$ such that $a_k \neq b_k$, and $u_k \in \mathscr{A}(f, a_k)$ and $v_k \in \mathscr{A}(f, b_k)$ such that $u_k \to u$ and $v_k \to v$. We choose for each k a sequence g_{ki} as above. It is not difficult to see that there is a diagonal type sequence g_i of g_{ki}'s which satisfies everything above, including the claim involving A′.

The sets fA' and fA are infinite as follows by Lemma 5D since each $f^{-1}\,w$ is nullset. Similarly, fA_x is infinite for every $x \in A'$ as we need to know below.

Assume then that $Y \subset \overline{B}^{n+1}$ and that $H \subset \text{Möb}(n)$. Since fA' is infinite, it follows that $f(x)$ must be the attractive point of $(\varphi(g_i)^{-1})$. Similarly, the infinitiness of fA_x implies that v is the attractive point of $(\varphi(g_i))$. It follows that $f(x)$ and v are the repelling and attractive point of $(\varphi(g_i))$, respectively.

Finally, we show that in this case H is non-elementary and that $fA \subset M(H)$. Let $z \in fA$. We show that if $u, v \in \overline{fA}$ are distinct, then there are $h_i \in H$ such that $h_i(z) \to u$ and $h_i(w) \to v$ for any other point $w \in \overline{B}^{n+1}$. Clearly, it suffices to prove this for $u, v \in fA$. Find $x, a, b \in A$ mapped to z, u, v by f. Now f is approximately continuous in A, and so $u \in \mathscr{A}(f, a)$ and $v \in \mathscr{A}(f, b)$ and hence, as we have seen, there are convergence sequences (g_i) and $(\varphi(g_i))$ of G and H, respectively, satisfying (6*b*) for $y \in A_x$, and such that $f(x)$ is the repelling point and v the attractive point of $(\varphi(g_i))$. So $\varphi(g_i)\,(f(x)) \to u$ and $\varphi(g_i)\,(y') \to v$ if $y \neq f(x)$. This proves our claim with $h_i = \varphi(g_i)$.

Thus any point $v \in fA$ is the attractive point of some convergence sequence of H. Hence $fA \subset L(H)$ and so L(H), like fA, must be infinite. It follows that H is non-elementary. Thus $\overline{fA} = L(H)$ and we have shown that every $z \in fA$ is in M(H).

We now apply this in the situation where the space Y is the set $\mathscr{C}(\overline{B}^{n+1})$ of all closed and non-empty subsets of $\overline{B}^{n+1}$ and $H \subset \text{Möb}(n)$ is discrete and non-elementary. Recall from Section 3 that Möb(n) acts naturally on $\mathscr{C}(\overline{B}^{n+1})$ which is compact in the Hausdorff metric given by $(3a)$.

Corollary **6E.** — *Let* $f: X \to \mathscr{C}(\overline{B}^{n+1})$ *be a measurable map inducing a surjective map* $\varphi: G \to H$. *Suppose that* $f(x) \not\supset L(H)$ *in a set of positive measure. Then* f *is a point function on the subset* $A \subset X$ *of full measure given by Theorem* 6D.

Proof. — By Lemma 3C there are two points $a, b \in A$ such that $f(a) \cap f(b) = \emptyset$. Then $u = f(a) \in \mathscr{A}(f, a)$ and $v = f(b) \in \mathscr{A}(f, b)$ by approximate continuity in A. Let $x \in A$. Find the sequence $g_i \in G$ given by Theorem 6D such that $(6b)$ is true for $y \in A_x$. Since there is a subset $A' \subset A$ of full measure such that $\varphi(g_i)^{-1} f(z') \to f(x)$ for $z' \in A'$, there are points z_1 and z_2 such that $\varphi(g_i)^{-1} f(z_k) \to f(x)$ as $i \to \infty$ and that $f(z_1) \cap f(z_2) = \emptyset$. The latter condition is true by the approximate continuity for some z_1 near a and z_2 near b. In the present situation $\{ \varphi(g_i)^{-1}) \}$ must also be infinite.

Pass to a subsequence so that $(\varphi(g_i)^{-1})$ is a convergence sequence whose attractive and repelling point are a' and b', respectively. Since $b' \notin f(z_1) \cap f(z_2) = \emptyset$, one of the sets $f(z_k)$, say $f(z_1)$ does not contain b'. Thus the sequence of sets $\varphi(g_i)^{-1} f(z_1) \to \{ a' \}$. Hence $f(x) = \{ a' \}$. So f is actually a point function on A.

Let us compare the essential cluster set $\mathscr{A}(f, x)$ of $(6a)$ to the cluster set $\mathscr{C}(f, x)$ defined by $(3d)$ when f is a measurable map $X \to Y$. Clearly, $\mathscr{C}(f, x) \supset \mathscr{A}(f, x)$ for every x. On the other hand, it is also easy to see using the fact that open sets have a countable basis that there is $X' \subset X$ of full measure such that if $U \subset X$ and $V \subset Y$ are open, then $V \cap f(X' \cap U) \neq \emptyset$ if and only if $\mu(f^{-1}[V] \cap U) > 0$. In addition, X' can be taken to be G-invariant. It follows that $\mathscr{C}(f \mid X', x) = \mathscr{A}(f, x)$ for every $x \in X'$.

Actually, we can define $\mathscr{A}(f, x)$ for any map f if we use the outer measure μ^* corresponding to μ in the definition $(6a)$ of $\mathscr{A}(f, x)$. As above, there is G-invariant $X' \subset X$ of full measure such that $\mathscr{C}(f \mid X', x) = \mathscr{A}(f', x)$ if $x \in X'$. Hence we need not assume measurability in the next analogy of Corollary 3E, obtained by applying Corollary 3E to the map $f|X'$. Recall that if the product action of G is ergodic on $X \times X$, as we assume, then $X = M(G)$ modulo nullsets. Hence the set where $\mathscr{A}(f, x)$ is defined is $L(G)$. As above, H is a discrete and non-elementary Möbius group.

Corollary **6F.** — *Let* $f: X \to \overline{B}^{n+1}$ *be a map inducing a surjective homomorphism* $\varphi: G \to H$. *Either* f *coincides outside a nullset with a homeomorphism of* $M(G)$ *onto* $M(H)$ *or the essential cluster set* $\mathscr{A}(f, x) \supset L(H)$ *for all* $x \in L(G)$.

Proof. — We only remark that since the set of x's such that $\mathscr{A}(f, x) \supset L(H)$ is closed, we can have that the second alternative is true for all $x \in \overline{X}$ instead of a.e. $x \in X$. Note that if $\mathscr{A}(f, x) \not\supset L(H)$ a.e., then, as we have seen, we could replace X by a set of full measure so that all the conclusions of Theorem 3B are valid and for this conclusion we do not need to assume ergodicity for the action of G on $X \times X$.

We can now combine Corollary 6F and Theorem 6D and show that in the situation there is a form of the Myrberg property also involving the map f. We include it since the non-continuous case seems rather strange and this result might be useful in unraveling the properties of such a situation, including the question whether it can exist.

Corollary **6G.** — *Let the situation be as in Corollary* 6F *except that f needs to be measurable. Either there is a homeomorphism* $M(G) \to M(H)$ *of the Myrberg points inducing* φ *and coinciding with f a.e. or we have the following situation. There is a set* $A \subset X$ *of full measure and for every* $x \in A$ *there is a set* A_x *of full measure such that the following is true. Let* $x \in A$ *and* $y \in A_x$. *Let* $a, b \in L(G)$ *and* $u, v \in L(H)$. *Then there is a convergence sequence* $g_i \in G$ *such that*

$$g_i(x) \to a, \quad g_i(y) \to b, \quad \varphi(g_i)f(x) \to u \quad \text{and} \quad \varphi(g_i)f(y) \to v.$$

In addition, x and $f(x)$ are the repelling points of the convergence sequences (g_i) *and* $(\varphi(g_i))$, *respectively, b and v being the attractive points.*

Remarks. — 1. A continuous limit map is necessarily injective in $M(G)$ by Theorem 3B. A non-continuous limit map need not be as is shown by the example in [T4, Section 4F]; in this example the homomorphism which the limit map induces is not injective. Whether there exist non-injective limit maps inducing an isomorphism is not known. However we can recall the result [T4, Theorem 3B], concerning injectivity of f, where we proved that if G acts ergodically on $A \times A$ and if $f: A \to L(H)$ is any map (measurable or not) which induces a homomorphism $\varphi : G \to H$, then either f is injective outside a nullset or it has "locally dense image" which implies that $\mu^*(f^{-1}[fV] \cap U) > 0$ for any open sets U and V of A of positive measure. Here μ^* is the outer measure corresponding to μ.

We can add to this dichotomy the following observation. Let F be the map $X \to \mathscr{C}(B^{n+1})$ such that $F(x) = \overline{f^{-1}[f(x)]}$ which induces $\mathrm{id} : G \to G$. If F is a measurable, we can apply Corollary 6E to F and obtain that either $\overline{f^{-1}[f(x)]} \supset L(G)$ for a.e. x or f is injective outside a nullset.

2. We can complement the dichotomies of Sections 3 and 6 by recalling Theorem 3C of [T4]. It follows from this theorem that if f is a measurable injection of $M(G)$ into $\overline{B}^{n+1}$ such that f^{-1} is also measurable and such that f induces φ, then there is the dichotomy that either f is a.e. the restriction of a Möbius transformation or f is singular in the sense that f maps a set of full μ-measure onto a set of zero ν-measure for any conformal H-measure ν.

7. The existence of the limit map

We first remark that if $\varphi : G \to H$ is an isomorphism of geometrically finite groups, then there is a fairly good picture of the limit map inducing φ. If the groups are convex cocompact, i.e. do not contain parabolics, then there is a homeomorphism $L(G) \to L(H)$ inducing φ. This was an essential observation in the proof of Mostow's rigidity theorem. More generally, this is true if both φ and φ^{-1} preserve parabolic elements of rank 1 [T2, Theorem 3.3]. Even without the condition on parabolic elements, there is a map $f: L(G) \to L(H)$ which is continuous at all points $x \in L(G)$ except when x is fixed by some rank-1 parabolic $g \in G$ of such that $\varphi(g)$ is loxodromic. This follows by a theorem of Floyd [FL] as was noted in [T3, Corollary].

We will prove later in this section that there is a limit map inducing an isomorphism of a geometrically finite G onto an arbitrary discrete H. Before it we examine the general situation.

A natural way to define the limit map f_φ for an isomorphism $\varphi : G \to H$ would be the following. Take base points $x_0, y_0 \in B^{n+1}$, for instance $x_0 = 0 = y_0$, such that x_0 is fixed by no $g \in G \setminus \{ \mathrm{id} \}$. Thus there is a natural map $f_\varphi : Gx_0 \to Hx_0$ of the orbits such that $f_\varphi(g(x_0)) = \varphi(g)\ (x_0)$ and this is a bijection if y_0 is fixed by no $h \in H \setminus \{ \mathrm{id} \}$. If we can extend f_φ continuously to a point $x \in L(G)$, it is natural to define $f_\varphi(x)$ to be this continuous extension. The extension, and its existence, is independent of the choice of x_0 and y_0.

This method works if G and H are geometrically finite without rank-1 parabolics [T2]. However, in the general case, there is no guarantee that there is such a continuous extension. However, we have the dichotomy that either it exists at all Myrberg points M(G) of G, or, the cluster set $\mathscr{C}(f_\varphi, x)$ of f_φ (cf. (3*d*)) is the whole L(H) at every $x \in L(G)$. Note that $\mathscr{C}(f_\varphi, x)$ is contained in the set of accumulation points of Hy_0, that is $\mathscr{C}(f_\varphi, x) \subset L(H)$ [T9, Theorem 2S].

Thus we have the following theorem. Like our other topological theorems it is actually valid for convergence groups. Thus G and H can be convergence groups of compact metric spaces X and Y, respectively, φ any isomorphism $G \to H$ and $x_0 \in \Omega(G)$ any point not fixed by any $g \in G \setminus \{ \mathrm{id} \}$ and y_0 can be any point of $\Omega(H)$. The map f_φ is defined as above and we have

Theorem **7A.** — *Either the cluster set* $\mathscr{C}(f_\varphi, x)$ *of* f *is* L(H) *for all* $x \in L(G)$ *or* f_φ *can be extended to a continuous map* $Gx_0 \cup M(G) \to Hy_0 \cup M(H)$ *inducing* φ *such that the extension to* M(G) *is a homeomorphism of* M(G) *onto* M(H); *if* f_φ *is a bijection of* Gx_0 *onto* Hy_0, *then we have a homeomorphism of* $Gx_0 \cup M(G)$ *onto* $Hy_0 \cup M(H)$.

Conversely, if there is a map $f: A \to Y$ *of a non-empty* G*-invariant set* A *of* X *inducing* φ *such that* $a = \lim_{z \to x} f(z)$ *exists at some* $x \in L(G)$, *then also* $\lim_{z \to x} f_\varphi(z) = a$.

Proof. — By Corollary 3E, we need only prove the last part. By Theorem 3B, it suffices to prove that if $f: M(G) \to M(H)$ is continuous and induces φ and has the limit a at $x \in L(G)$, then also f_φ has the limit a at x.

If f_φ does not have the limit a at x, then there is a sequence $(g_i) \in G$ such that $g_i(x_0) \to x$ but that $\varphi(g_i)\ (y_0) \not\to a$. Since $x_0 \in \Omega(G)$, the set of g_i's is infinite, and hence we can assume that (g_i) is a convergence sequence. Necessarily x is the attractive point of (g_i) since otherwise x_0 would have to be the repelling point and this is impossible since $x_0 \notin L(G)$.

Let $h_i = \varphi(g_i)$. Since x is the attractive point of (g_i), it follows that $g_i(z) \to x$ if $z \in A \setminus \{ b \}$ where b is the repelling point of (g_i). Thus $f(g_i(z)) = h_i(f(z)) \to a$ if $z \in A \setminus \{ b \}$. Since H is non-elementary and $fA \neq \emptyset$, the H-invariant set fA is infinite and we can conclude that $h_i(z') \to a$ for an infinite number of points z'. Thus $\{ h_i \}$ is infinite and if we pass to a subsequence (denoted in the same manner), we see that the attractive point of (h_i) is a.

Since $y_0 \in \Omega(H)$, y_0 is not the repelling point of (h_i). Thus $h_i(y_0) \to a$ contrary to the assumption. Our claim is proved.

If G is geometrically finite and H is any discrete Möbius group, we will obtain the limit map by a procedure which resembles the taking of the radial limit. We explain this below for general discrete G.

We concentrate on the conical limit point set $\Lambda(G)$ of G. Now the characterization of $\Lambda(G)$ by means of approach in a Stolz angle is appropriate. We define a *Stolz angle* at $x \in S^n$ to be a set of the form

$$\{ z \in B^{n+1} : d(z, S) \leqslant m \} \tag{7a}$$

where S is a hyperbolic ray with endpoint x, the other endpoint lying in B^{n+1}. Conical limit points can be defined as the set of $x \in S^n$ such that there is a Stolz angle C at x and $z \in B^{n+1}$ such that $Gz \cap C$ is infinite. Another form of this definition is that there are $g_i \in G$ such that $g_i(z) \to x$ and that, if $z \in B^{n+1}$ and L is a hyperbolic line with endpoint x, then the distances $d(g_i(z), L)$ are bounded. For the equivalence of these definitions and our original definition, see the discussion in the Introduction (cf. also [BM]).

Characterization using the Stolz angle gives the following characterization by means of the shadows. Recall from section 5 that the shadow $S(x, M)$ of the open hyperbolic disk $D(x, M)$ was the projection from 0 of $D(x, M)$ onto S^n, cf. (5*b*). Using this notion, we can characterize the conical limit point set $\Lambda(G)$ as the set of points $z \in S^n$ which are in the shadow of $D(g(x_0), M)$ for infinitely many $g \in G$ for some M which may depend on x and on the base point x_0. Thus if

$$\Lambda(x_0, M) = \{ x \in S^n : x \in S(g(x_0), M) \text{ for infinitely many } g \in G \},$$

then $\Lambda = \Lambda(G) = \bigcup_{M>0} \Lambda(x_0, M)$ (from now on we omit the group G, which is fixed, from the notation of conical limit points). The set $\Lambda(x_0, M)$ need not be G-invariant but if we set

$$\widetilde{\Lambda}(x_0, M) = \bigcap_{M' > M} \Lambda(x_0, M'),$$

for $M \geqslant 0$, we obtain a G-invariant set. It is not difficult to see that Myrberg points are a subset of $\Lambda(x_0, M)$ for all $M > 0$ and all x_0 such that x_0 is on a hyperbolic line joining two points of L(G). Hence, for such x_0 and all $M \geqslant 0$, $\widetilde{\Lambda}(x_0, M)$ and $\Lambda(x_0, M)$ have full μ-measure for a G-measure μ if conical limit points have (cf. the Introduction).

If $x \in \Lambda(x_0, M)$, we can associate to x in a canonical manner a sequence $g_i = g_i[M, x_0, x]$ so that

$$(7b) \qquad \{ g_i : i > 0 \} = \{ g \in G : x \in S(g(x_0), M) \}.$$

This does not fix the order of g_i but this is not so important since we are interested only of the convergence of $\varphi(g_i)(0)$. Anyway, we set that if $|g_i(0)| < |g_j(0)|$, then $i < j$. This fixes the order up to finite indeterminacy since there is a fixed $N = N(M, x_0)$ so that at most N disks $D(g(x_0), M)$, $g \in G$, can intersect. In any case, if the limit

$$(7c) \qquad f_{Mx_0}(x) = \lim_{i \to \infty} \varphi(g_i[M, x_0, x])(y_0)$$

exists, it does not depend on the ordering nor does it depend on the basepoint $y_0 \in B^{n+1}$ though it might depend on x_0.

If $M' > M + d(x_0, x'_0)$, $(g_i[M, x_0, x])$ is a subsequence of $(g_i[M', x'_0, x])$ (up to the finite indeterminacy of indexes). It follows from this observation that if $x \in \widetilde{\Lambda}(x_0, 0)$ is fixed and if the limit $(7c)$ exists for all M and for some x_0, then it exists for all M and for all x_0 such that $x \in \Lambda(x_0, M)$ and the limit does not depend on x_0 nor on M.

More generally, we define the set functions

$$F_{Mx_0}(x) = \operatorname{acc}\{ \varphi(g_i[M, x_0, x])(y_0) : i > 0 \}$$

where acc is the set of accumulation points (it does not depend on y_0). Then F_{Mx_0} is a set function defined for $x \in \Lambda(x_0, M)$ with values in $\mathscr{C}(L(H))$, the family of all non-empty and closed subsets of $L(H)$. It need not induce φ but if we set

$$\widetilde{F}_{Mx_0} = \bigcap_{M' > M} F_{Mx_0}$$

for $x \in \widetilde{\Lambda}(x_0, M)$ when $M \geqslant 0$, then $\widetilde{F}_{Mx_0}$ induces φ as a map $\widetilde{\Lambda}(x_0, 0) \to \mathscr{C}(L(H))$ in the sense of Section 3.

We can now state a dichotomy similar to Theorem 7A.

Theorem **7B.** — *Let* $\varphi : G \to H$ *be an isomorphism of two discrete Möbius groups. Suppose that there is a conformal* G*-measure* μ *on* $L(G)$ *such that the product action of* G *is ergodic with respect to* $\mu \times \mu$. *Then there is* $A \subset L(G)$ *of full measure such that either* (α) *there is a measurable map* $f : A \to M(H)$ *inducing* φ *such that the limit* $(7c)$ *exists for all* $M > 0$ *whenever* $x \in \Lambda(x_0, M) \cap A$ *and equals* $f(x)$, *or* (β) $\widetilde{F}_{Mx_0}(x) = L(H)$ *for all* $M > 0$ *whenever* $x \in \widetilde{\Lambda}(x_0, M) \cap A$.

Conversely, suppose that there is a measurable map of a G*-invariant set* $A \subset L(G)$ *of positive measure into* $\overline{B}^{n+1}$ *inducing* φ. *If* f *is approximately continuous at a point* $x \in \Lambda(x_0, M)$ *with respect to* μ, *then the limit* $(7c)$ *exists and is equal to* $f(x)$. *This part of the theorem is true even if the product action is not ergodic.*

Remark. — If the product action is ergodic, then $M(G)$ has full measure and so we can assume that $A \subset M(G)$. Recall that if x_0 is on a hyperbolic joining two points of $L(G)$, then $M(G) \subset \Lambda(x_0, M)$ for every $M > 0$. Thus if $L(G) = S^n$, then $M(G) \subset \Lambda(x_0, M)$ for every $x_0 \in B^{n+1}$ and $M > 0$.

If we have above case (β), and $x \in \Lambda(x_0, M')$ for some $M' < M$, then $F_{Mx_0} \supset L(H)$ since $F_{Mx_0}(x) \supset \widetilde{F}_{M' x_0}$. Thus in this case we can replace $\widetilde{F}_{Mx_0}(x)$ by $F_{Mx_0}(x)$ in (β). The problem is that it seems perfectly possible that $x \in \Lambda(M, x_0)$ but that $x \notin \Lambda(M', x_0)$ for no $M' < M$. However, as we have seen, if x_0 is on a hyperbolic line joining two points of $L(G)$, then $M(G) \subset \Lambda(x_0, M)$ for any M. Thus in this case we can replace $\widetilde{F}_{Mx_0}(x)$ by $F_{Mx_0}(x)$ in (β).

Proof. — We first prove the last part. Let $x \in \Lambda(x_0, M)$. Let $g_i = g_i[M, x_0, x]$. We claim that the limit (7c) exists and is $f(x)$ if f is approximately continuous at x. If this is not true we can pass to a subsequence, denoted in the same manner, so that $\varphi(g_i)(y_0) \to c \neq f(x)$. We derive a contradiction from this.

Let $L = L(x, -x)$ be the hyperbolic line joining x and $-x$. Then $d(g_i(x_0), L) < M$ and hence $d(x_0, g_i^{-1} L) < M$. Thus by passing to a subsequence we can assume that $g_i^{-1}(x) \to u$ and $g_i^{-1}(-x) \to v$ where $u \neq v$. In addition, a geometric argument easily shows that $g_i^{-1}(w) \to v$ as $i \to \infty$ for any $w \in L$. Hence, by Lemma 2A, (g_i^{-1}) is a convergence sequence with v as the attractive and x as the repelling point. We can now apply Lemma 6A and find a subset A' of $A \setminus \{v\}$ of full measure such that $\varphi(g_i) f(z) \to f(x)$ for $z \in A'$.

In addition, by Lemma 6A, fA' is infinite. Hence we can pick distinct $u_i \in A'$, $i \leqslant 3$, such that also $f(u_i)$ are distinct. If $a, b, c \in S^n$ are distinct, define

$$p(a, b, c) = \text{the orthogonal projection (in hyperbolic geometry) of } c \text{ onto } L(a, b).$$

Since the existence and value of (7c) do not depend on y_0, we can take y_0 to be the point $p(f(u_1), f(u_2), f(u_3))$. Let $h_i = \varphi(g_i)$. Obviously p commutes with Möbius transformations, and hence $h_i(y_0) = p(h_i f(u_1), h_i f(u_2), h_i f(u_3)) \to f(x)$ since $h_i f(u_k) \to f(x)$ for all k as $i \to \infty$. This contradicts the assumption that $h_i(y_0) \to c \neq f(x)$.

Having proved the last part, we note that if there are $x_0 \in B^{n+1}$ and $M > 0$ such that $\widetilde{F}_{Mx_0}(x) \not\supset L(H)$ in a set of positive measure, we obtain by Corollary 6E, that $\widetilde{F}_{Mx_0}$ equals a point function $f\colon A \to M(H)$ in a set $A = A_{x_0}$ of full measure. We can assume that $f \mid A$ is approximately continuous. The first part of the proof implies that in fact $F_{M' x_0}(x) = f(x)$ for all $M' > 0$ whenever $x \in A \cap \Lambda(x_0, M')$. Thus either we have case (α) or $\widetilde{F}_{Mx_0}(x) = L(H)$ for $x \in A$ where A has full measure. To begin with, A may depend on M and x_0 but passing to a countable intersection of sets of full measure, we can obtain that A is independent of x_0 and M.

We will now prove the existence of the limit map if H is any discrete Möbius group when G is geometrically finite, that is, G has a finite sided fundamental domain (for a more precise definition see e.g. [T2]). We will not use directly the existence of a finite sided fundamental domain but rather the following.

Let B_G be the *hyperbolic convex hull* of L(G), that is the smallest (hyperbolically) convex and closed subset of B^{n+1} such that $\overline{B}_G \supset L(G)$. The simplest geometrically finite groups are such groups G for which B_G/G is compact. In this case G is called *convex cocompact*. If a geometrically finite group G is not convex cocompact, then G contains parabolic elements. In this case we can still find a smaller set $B'_G \subset B_G$ such that B'_G/G is compact by removing certain horoballs at parabolic fixed points. This is explained in detail later but it is this decomposition of B_G into a part with compact quotient and ends corresponding to parabolic fixed points which allows us to prove the theorem in the non-compact case.

By Sullivan [S2], there is a canonical conformal measure μ on L(G) of mass 1 whose dimension is the so-called exponent of convergence δ_G of G. It is up to multiplication by a constant the only conformal measure of dimension δ_G on L(G). The exponent of convergence is positive by [S1, Corollary 2]. If there are parabolic elements in G, we need also to know that no parabolic fixed point is an atom of μ (cf. [S2] or [NI, 3.5.10]). Since in this situation a limit point is a conical limit point unless it is fixed by a parabolic $g \in G$, $\Lambda(G)$ has full measure and hence the product action of G is ergodic and so in this regard we have the situation of Theorem 7B.

The dimension of a conformal measure giving non-zero measure to $\Lambda(G)$ must be δ_G by [S1, Theorem 21]. Since the complement of $\Lambda(G)$ is the set P of parabolic fixed points of G, a conformal measure of dimension $\delta \neq \delta_G$ would have to be supported by the countable set of parabolic fixed points of G. Hence the canonical conformal measure seems to be the only reasonable conformal measure on L(G) (up to multiplication by a constant) for geometrically finite G. For convex cocompact groups it is not only the only reasonable measure but even (up to multiplication by a constant) the only conformal measure supported by L(G).

Theorem **7C.** — *Let* G *be a non-elementary and geometrically finite group and let* μ *be the canonical conformal measure of dimension* δ *on* L(G). *Then every isomorphism* φ *of* G *onto another discrete Möbius group is induced by a measurable map* $f: A \to L(H)$ *where* $A \subset L(G)$ *has full* μ*-measure in* L(G). *The map* f *is uniquely determined by* φ *up to* μ*-nullsets.*

Remark. — If μ is not a multiple of the canonical conformal measure, then, as we saw above, the mass of μ is concentrated on the countable set of parabolic fixed points of G. If x is fixed by a parabolic $g \in G$, then $\varphi(g)$ is parabolic or loxodromic and choosing $f(x)$ to be a fixed point of $\varphi(g)$, we obtain an a.e. defined measurable map f inducing φ.

From now on μ is the canonical conformal measure. By Theorem 6B, f is essentially unique if it exists since the action of G is ergodic on $L(G) \times L(G)$ [S2]. Thus it suffices to prove by Theorem 7B:

Lemma **7D.** — *There is a set* $A \subset L(G)$ *of full* μ*-measure such that the limit* (7c) *exists for all* $M > 0$ *whenever* $x \in A \cap \Lambda(x_0, M)$.

Proof. — Clearly, we can fix $x_0 = y_0 = 0$ and prove that the limit (7*c*) exists in a subset of full measure of $\Lambda(x_0, M)$ if M exceeds a lower bound. This will imply the existence of the limit (7*c*) a.e. for arbitrary $M > 0$ and whenever $x \in \Lambda(x_0, M)$.

The proof is simpler for convex cocompact groups, and we first present the proof in this case and later give the modifications for the non-compact case.

We can assume that $0 \in B_G$. Since B_G/G is compact, there is $M > 0$ such that if $z \in B_G$, then $GD(z, M) \supset B_G$ where $D(x, r)$ is the open hyperbolic disk with radius r and center x. From now on we assume that M is so big that this condition is met.

Let $A(r, M)$ be the annulus

$$A(r, M) = D(0, r + M)\setminus D(0, r). \tag{7d}$$

The basic idea of the proof is to compare the growth of the number of points of G0 in $A(r, M)$ and of H0 in $D(0, \gamma r)$ for suitable (small) $\gamma > 0$. In the first case we need an exponential lower bound and in the latter an exponential upper bound as a function of r. A consequence will be that the average hyperbolic distance from 0 of points of $\{ \varphi(g)\ (0) : g \in G, g(0) \in A(r, M)\}$ is $\geqslant \alpha r$ for some $\alpha > 0$. It will follow that if $h_i = \varphi(g_i[M, 0, x])$, then $| h_{i+1}(0) - h_i(0)|$ is on the average so small that (7*c*) converges. Actually, we only use exponential growth for H explicitly (cf. (7*e*) below), the exponential growth for G is coded into the properties of μ.

We can assume that 0 is not fixed by any $h \in H$, $h \neq \mathrm{id}$. Hence there is $m > 0$ such that $D(0, m) \cap D(g(0), m) = \emptyset$ for any $g \in H\setminus\{ \mathrm{id} \}$. This and the exponential growth of the hyperbolic volume imply that the number $N_H(r)$ of elements $h \in H$ such that $h(0) \in D(0, r)$ satisfies the inequality

$$N_H(r) \leqslant A_H\, e^{\beta r} \tag{7e}$$

for some positive A_H and β (we could take $\beta = n + 1$).

The next step in the proof is to consider the measure of the "shadows" $S(g(0), M)$ of the balls $D(g(0), m)$ on S^n, cf. (5*b*). By Lemma 5C, there is $C > 1$ such that

$$C^{-1}\, e^{-\delta d(0, g(0))} \leqslant \mu(S(g(0), M)) \leqslant Ce^{-\delta d(0, g(0))}, \tag{7f}$$

provided that M is bigger than the constant M_0 of Lemma 5C. We can assume this.

Choose $\gamma > 0$. We want to estimate the measure of the union of the shadows $S(g(0), M)$ when g is in the set

$$G_k = \{ g \in G : g(0) \in A(kM, M) \text{ and } \varphi(g)\ (0) \in D(0, k\gamma) \}.$$

Set

$$X_k = \bigcup_{g \in G_k} S(g(0), M)$$

and

$$Y_k = \bigcup_{j \geqslant k} X_j.$$

We know by (7*e*) that there are at most $A_H e^{\beta\gamma k}$ elements in G_k. Hence, using (7*f*), we obtain the estimate $\mu(X_k) \leqslant A_H C e^{(\beta\gamma - M\delta)k}$. Since $\delta > 0$, we can choose γ so small that $\beta\gamma - M\delta < 0$ and now

$$(7g) \qquad \mu(Y_k) \leqslant A_H C e^{(\beta\gamma - M\delta)k} \frac{1}{1 - e^{\beta\gamma - M\delta}}.$$

Thus if we finally set

$$Z = \bigcap_{k>0} Y_k = \bigcap_{k>0} \bigcup_{j \geqslant k,\, g \in G_j} S(g(0), M),$$

then Z is a μ-nullset.

Notice that so far we have not made use of the fact that G is convex cocompact: Z is a nullset for arbitrary discrete G and any $M > 0$.

We can now define the limit map f for $x \in \Lambda(0, M)\setminus Z$. Here Z depends on M but of course it will follow that there is a μ-nullset Z_0 such that the limit (7*c*) exists whenever $x \in \Lambda(x_0, M)\setminus Z_0$ for any M and x_0.

Let $g_i = g_i[M, 0, x]$ and $h_i = \varphi(g_i)$. To prove that the limit (7*c*) exists for a.e. x we first note that $D(g_i(0), M)$ and $D(g_{i+1}(0), M)$ intersect and hence $d(g_i(0), g_{i+1}(0)) \leqslant 2M$ and so $g_i^{-1} g_{i+1}$ and hence also $h_i^{-1} h_{i+1}$, vary in a finite set. It follows that there is $s > 0$ such that $d(h_i(0), h_{i+1}(0)) < s$ for all i. Let k_i be the number such that $g_i(0) \in A(k_i M, M)$. If $x \in L(G)\setminus Z$, then there is k such that $x \notin Y_k$. This means that $d(0, h_i(0)) \geqslant \gamma k_i$ if $k_i \geqslant k$. Thus, by Lemma 5B, there is $c = c(s)$ such that

$$(7h) \qquad | h_{i+1}(0) - h_i(0) | < c e^{-\gamma k_i}$$

if $k_i \geqslant k$ and $x \notin Y_k$. Finally, we note that the length of the hyperbolic segment $L_M = L(0, x) \cap A(kM, M)$ is M. Furthermore, L_M is contained entirely in B_G whose G-quotient is compact. It easily follows that the number of i such that $g_i(0) \in A(kM, M)$ is bounded by a number $N = N_M$. Thus $k_i \geqslant \dfrac{i - N}{N}$. If $x \notin Y_k$, then the estimate (7*h*) is valid for $i \geqslant kN + N = r_k$ and hence

$$(7i) \qquad \sum_{i \geqslant r_k} | h_{i+1}(0) - h_i(0) | \leqslant \sum_{i \geqslant r_k} c e^{-\gamma(i-N)/N} < \infty.$$

Thus indeed in this case the points $h_i(0)$ converge towards a point y which must be a point of L(H) since y is in the accumulation set of H0.

The non-compact case. — We now give the modifications if the group is not convex cocompact, that is, there are parabolic elements. Here we need some knowledge on the behavior of the group near parabolic fixed points.

Let P be the set of points fixed by some parabolic $g \in G$. A *horoball* based at $v \in S^n$ is an open ball $B \subset B^{n+1}$ such that ∂B is tangent to S^n at v and a *complete set of horoballs* B_v, $v \in P$, is a disjoint set of horoballs B_v where B_v is based at v and such that

$g(B_v) = B_{g(v)}$ for $g \in G$. By [T1, Lemma B] there is a complete set of horoballs satisfying: if we set $B'_G = B_G \backslash (\bigcup_{v \in P} B_v)$, then the quotient $B'_G \backslash G$ is compact. We assume that the number M above is so chosen that $GD(z, M) \supset B'_G$ for any $z \in B'_G$. We can still define the sets X_k, Y_k and Z as above and Z is still a μ-nullset.

We assume that $0 \in B'_G$. Suppose that $x \in L(G) \backslash P$, and define $g_i = g_i[M, 0, x] \in G$ and $h_i = \varphi(g_i)$ as earlier. The problem is that it may happen that $D(g_i(0), M)$ and $D(g_{i+1}(0), M)$ do not always touch. This can happen when $L(0, x)$ dives deeply into some B_v after touching $D(g_i(0), M)$. We will show that we can define a μ-nullset Z_ρ such that if $x \notin Z_\rho$, then $L(0, x)$ dives into the horoballs in a controlled manner. In addition, we will define another nullset Z' whose definition is similar to Z and show that (7*c*) converges if the limit point $x \notin P \cup Z \cup Z' \cup Z_\rho$. Since P, the set of points fixed by some parabolic $g \in G$, is countable and there are no atoms, $P \cup Z \cup Z' \cup Z_\rho$ is a nullset.

The set Z_ρ is defined by means of smaller horoballs $B'_v \subset B_v$. If $t \in (0, 1)$, and B is a horoball based at v, we let $tB \subset B$ be the horoball based at v such that the hyperbolic distance of $\partial B \backslash \{v\}$ and $\partial(tB) \backslash \{v\}$ is $|\log t|$. Let d_v be the euclidean diameter of B_v. We set for $\rho > 0$

$$B'_v = |\log d_v|^{-\rho} B_v.$$

Since $0 \notin B_v$ for all v, $d_v < 1$ and B'_v are well-defined. A point $x \in S^n$ is in shadow of B'_v if the hyperbolic ray $L(0, x)$ intersects B'_v. The set Z_ρ is the set of points which are in the shadow of infinitely many B'_v, $v \in P$. It is a nullset if ρ is big enough by [T7, Lemma 3A and Remark 1 after it]. We fix some ρ such that Z_ρ is a nullset.

The set Z' is defined by means of the horoballs B_v as follows. Let $q_v \in \partial B_v$ be the point closest to 0 (with respect to the euclidean metric) and let p_v be the number such that $q_v \in A(p_v M, M)$. Thus P is the disjoint union of P_k's where $P_k = \{v \in P : p_v = k\}$. Define

$$P'_k = \{v \in P_k : D(g(0), M) \cap \partial B_v \neq \emptyset \text{ for some } g \in G \text{ such that } d(\varphi(g)(0), 0) < \gamma k\}.$$

Let $S(B_v) = \pi(B_v)$, π as in (5*b*), be the shadow of B_v. We now estimate the measure of $\cup\{S(B_v) : v \in P'_k\}$ quite like above in the proof that Z was a nullset. Since the number of horoballs B_v whose euclidean diameter exceeds a given $\varepsilon > 0$ is finite, there is a number N' such that each $D(0, M)$ can touch at most N' horoballs B_v, $v \in P$, and this is true also for $D(g(0), M)$. Hence we have by (7*e*) the estimate

$$\operatorname{card}(P'_k) \leqslant N' A_H e^{\beta\gamma k}$$

when $\operatorname{card}(P'_k)$ is the number of elements of P'_k.

It is geometrically evident that there is $M' > 0$ such that $S(B_v) \subset S(q_v, M')$. Since $q_v \in L(0, v)$, $q_v \in B'_G$ and hence q_v is in $D(g(0), M)$ for some $g \in G$ and so $S(B_v) \subset S(g(0), M' + M)$. Thus we can conclude by Lemma 5C that there is $c' > 0$

such that $\mu(S(B_v)) \leqslant \mu(S(g(0), M' + M)) \leqslant c' e^{-(p_v - 1)M}$ since $d(0, g(0)) \geqslant (p_v - 1) M$. Hence there is $c'' > 0$ such that $\mu(\bigcup_{v \in P'_k} S(B_v)) \leqslant c'' N' A_H e^{(\beta\gamma - \delta M)k}$. Consequently, if $\beta\gamma - M\delta < 0$, and this we can assume,

$$Z' = \bigcap_{k>0} \bigcup_{j \geqslant k,\, v \in P'_j} S(B_v)$$

is a μ-nullset.

We claim that if $x \in L(G)$ and x is not in the μ-nullset $P \cup Z \cup Z' \cup Z_\rho$, then (7*c*) converges. By the definitions of the sets Z, Z' and Z_ρ, there is $x' \in L(0, x)$ such that the ray $L(x', x)$ does not intersect any B'_v, $v \in P$, nor any $D(g(0), M)$ such that $g \in \bigcup_{k>0} G_k$, nor B_v such that $v \in \bigcup_{k>0} P'_k$. Choose i_0 such that $D(g_i(0), M)$ does not intersect $L(0, x')$ if $i \geqslant i_0$. We now consider only $i \geqslant i_0$.

If $D(g_i(0), M) \cap D(g_{i+1}(0), M) \neq \emptyset$, we still have the estimate (7*h*). If not, the situation is as follows.

Let $L_x = L(0, x)$ be the hyperbolic ray with endpoints 0 and x. The balls $D(g_i(0), M)$ and $D(g_{i+1}(0), M)$ as well as L_x intersect some ∂B_v in such a way that B_v is "between" $D(g_i(0), M)$ and $D(g_{i+1}(0), M)$. As above, let k_i be the number such that $g_i(0) \in A(k_i M, M)$. Then $d(0, q_v) \leqslant (k_i + 1) M$ and hence, in view (5*c*),

$$e^{-(k_i+1)M} \leqslant d_v = \mathrm{diam}(B_v).$$

This allows the following estimate. Now, L_x does not intersect $B'_v = |\log d_v|^{-\rho} B_v$ but intersects B_v at points which we denote by a and b so that a is closer to 0. Hence a simple calculation, which we have done in [T7, Lemma 4A *a)*] shows that the hyperbolic length of the (non-hyperbolic) geodesic of $\partial B_v \cap B_G$ joining a and b is not more than

$$(7j) \qquad 2\,|\log d_v|^\rho \leqslant 2[(k_i + 1) M]^\rho.$$

Now a is the first point where L_x intersects ∂B_v. Then $a \in D(g_i(0), M)$ and a is on the part of ∂B_v which is "visible" from 0. Hence, if as above q_v is the point of ∂B_v where L_v intersects ∂B_v, then $d(q_v, a)$ is bounded by an absolute constant and hence $d(q_v, g_i(0))$ is bounded by some $r = r_M$. It follows that there is an integer $p = p_M$ such that $B_v \cap D(0, (k_i - p) M) = \emptyset$. Hence $k_i - p < p_v$ and, since $v \notin P'_k$,

$$(7k) \qquad d(0, \varphi(g)\,(0)) \geqslant (k_i - p)\,\gamma$$

whenever $g \in G$ and $d(g(0), M) \cap B_v \neq \emptyset$.

Since $(\partial B_v \cap B_G)/G$ is compact (cf. [T7, Lemma 2A]), we can find a sequence $f_0 = g_i, f_1, \ldots, f_q = g_{i+1}$ of elements of G such that

$$D(f_j(0), M'') \cap D(f_{j+1}(0), M'') \cap (\partial B_v \cap B_G) \neq \emptyset$$

for some $M'' > 0$ and that $f_{j+1} = f_j \sigma_j$ where $\sigma_j \in G$ and σ_j vary in a finite set. In addition the number q is proportional to the length of the geodesic of ∂B_v (i.e. a geodesic as a subset of B_v) joining the point $a \in D(g_i(0), M)$ and the point $b \in D(g_{i+1}(0), M)$

and hence by (7j), there is $c_0 > 0$ such that $q \leqslant c_0 k_i^\rho$. Since each $d(f_j(0), \mathrm{M})$ intersects B_v, (7k) implies that $d(0, f_j(0)) \geqslant (k_i - p)\gamma$ and hence we have the situation of (7h) and so (7h) is valid with a new constant c_1 if we substitute $k_i - p$ for k_i and f_j for h_i. Adding up, we have the estimate

$$(7l) \qquad | h_i(0) - h_{i+1}(0) | \leqslant c_0 c_1 k_i^\rho e^{(p-k_i)\gamma}.$$

As in the first part, there is N such that $k_i \geqslant (i - \mathrm{N})/\mathrm{N}$. The convergence of (7$c$) follows.

Remark. — It is apparent from our estimates that if a rather modest estimate on the growth of $d(0, \varphi(g)(0))$ as a function of $d(0, g(0))$ is available, then the limit (7c) exists. For instance if G is convex cocompact and if, for big values of $d(0, g(0))$,

$$(7m) \qquad d(0, \varphi(g)(0)) \geqslant \log d(0, g(0))^\alpha,$$

where $\alpha > 1$, then instead of (7h) we have $| h_{i+1}(0) - h_i(0) | < cd(0, g(0))^{-\alpha}$. Since $d(0, g_i(0))$ has a lower bound which is proportional to i, this guarantees the convergence of (7i) and hence of (7c). Furthermore, the limit map will be continuous in this case.

If G is geometrically finite and contains parabolic elements, then we obtain the convergence if $\alpha > \rho + 1$ for $x \in \mathrm{L(G)} \setminus (\mathrm{P} \cup \mathrm{Z}_\rho)$; this follows since now (7l) is true with suitable constants if we substitute in it $d(0, g(0))^{-\alpha} \approx k_i^{-\alpha}$ for $e^{(p-k_i)\gamma}$.

If *both* G and H are convex cocompact, then the map $f_\varphi : \mathrm{G0} \to \mathrm{H0}$ is pseudo-isometric, so that $d(0, \varphi(g)(0))/d(0, g(0))$ is bounded away from 0 and ∞ for big enough $d(0, g(0))$. The proof of the existence of the limit map in Mostow's rigidity theorem was based on the pseudo-isometric condition. We have seen that if G is convex cocompact (and H arbitrary discrete), the pseudo-isometric condition is far stronger than is necessary for a continuous extension to the limit set.

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