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A. J. DE JONG

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CRYSTALLINE DIEUDONNÉ MODULE THEORY VIA FORMAL AND RIGID GEOMETRY

by A. J. DE JONG*

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Introduction

Let S be a base scheme in characteristic p. Consider the crystalline Dieudonné module functor **D** on the category of p-divisible groups over S to the category of Dieudonné crystals over S. We ask whether **D** is fully faithful or fully faithful up to isogeny over S. If so, we can ask whether **D** is even an equivalence.

The idea of associating Dieudonné crystals to p-divisible groups goes back to Grothendieck. We refer to his letter to J. Tate of 1966, his Montréal lectures in 1970 [G1] and his talk at the Nice congress [G2]. In Section 3 of [G2] Grothendieck mentions two constructions of the Dieudonné crystal; one using the exponential, another using the method of ♯-extensions. The first approach is developed in [M], the second in [MM]. In [M] the Dieudonné crystal is constructed using universal extensions by vector groups, and Messing proves the deformation theorem. In [MM] the authors prove the compa-

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rison theorem, which compares the de Rahm cohomology of an abelian scheme with the Dieudonné crystal of its p-divisible group. Open questions in the theory at that time were the full faithfulness/equivalence questions mentioned above and the problem of extending $\mathbf{D}$ to the category of finite, locally free p-groups. These questions were raised in [G1] and [MM].

In [B] and [BBM] Grothendieck's formalism of crystalline sites and crystals is developed and extended. Using this the Dieudonné crystal is defined in [BBM] as an $\mathscr{E}xt$-sheaf on the crystalline site. This definition generalizes to give the crystalline Dieudonné module functor on the category of finite, locally free p-groups.

There are partial results on the full faithfulness/equivalence problem. Berthelot and Messing [BM] prove that $\mathbf{D}$ is fully faithful on schemes having locally a p-basis. There are unpublished results of Kato proving that $\mathbf{D}$ is an equivalence over schemes smooth over perfect fields of characteristic $p > 2$. This continues work of Bloch on p-divisible formal Lie groups (1974, unpublished). Messing proved (unpublished) that the result of Kato for arbitrary characteristic follows from the results of [dJ]. However, the full faithfulness question does not have a positive answer in general; there are counterexamples given in [BM].

In this paper we extend the results of Berthelot, Bloch, Kato and Messing. The first result is an equivalence theorem.

Main Theorem **1.** — *Let $\mathfrak{X}$ be a formal scheme in characteristic p, formally smooth over* $\mathrm{Spec}(\mathbf{F}_p)$, *such that $\mathfrak{X}_{\mathrm{red}}$ is locally of finite type over a field with a finite p-basis. The crystalline Dieudonné module functor* $\mathbf{D}$ *is an equivalence over* $\mathfrak{X}$. □

In particular $\mathbf{D}$ is an equivalence over regular schemes which are of finite type over a field with a finite p-basis.

Our second main result is that $\mathbf{D}$ is fully faithful up to isogeny over schemes of finite type over a field with a finite p-basis. This assertion (Corollary 5.1.2) follows from the following slightly stronger theorem.

Main Theorem **2.** — *Let* S *be a reduced scheme of finite type over a field with a finite p-basis. Let* G_1, G_2 *be p-divisible groups over* S. *We have the following equality*:

$$\mathrm{Hom}_{\mathrm{DC/S}}(\mathbf{D}(G_2), \mathbf{D}(G_1)) = \text{torsion subgroup} \oplus \mathbf{D}(\mathrm{Hom}_S(G_1, G_2)). \quad \square$$

For certain schemes S it is known that the torsion subgroup is zero, for example if S is a locally complete intersection. If S is such a scheme and satisfies the conditions of the theorem, then the crystalline Dieudonné module functor is fully faithful.

We turn to an overview of the contents of the chapters. Chapter 1 contains some algebraic preliminaries. In Chapter 2 the crystalline Dieudonné module theory is extended to formal schemes. We start with the definition of $\mathbf{D}$ from [BBM]. We analyse what it means to have Dieudonné crystals over formal schemes. After this the fully faithfulness in the Main Theorem 1 follows from the fully faithfulness result of [BM] by a rather formal argument.

In Chapter 3 we deform Dieudonné crystals and p-divisible groups. An important technical point is the introduction of special Dieudonné modules. This allows us to deform Dieudonné crystals. We want to deform a p-divisible group given a deformation of its Dieudonné crystal. Here we use the precise description of the deformations of a p-divisible group G in terms of filtrations on its Dieudonné crystal. We can do this since $\mathbf{D}(G)$ agrees with the crystal defined in [M], as is shown in [BM].

In Chapter 4 we prove essential surjectivity in Main Theorem 1. We use induction on the dimension of $\mathfrak{X}_{red}$. The crucial step is the case $\dim(\mathfrak{X}_{red}) = 0$. Here we have to prove that $\mathbf{D}$ is essentially surjective over a field with a finite p-basis. The proof uses two ingredients: the description, in [dJ], of formal p-divisible groups in terms of Dieudonné modules, and a result on extensions of étale by multiplicative p-divisible groups.

The Main Theorem 2 is proved in Chapter 5. The proof uses ideas of Berthelot on convergent isocrystals ([B2]). By Main Theorem 1 we already know the result over the regular part of S. Over the singular part of S we get the result by induction on the dimension. Hence, we need to show that the two resulting homomorphisms glue. Using the ideas of Berthelot it is proved that they glue rigid analytically over a tube U of S. See the beginning of Chapter 5 for the definition of a tube. We are able to descend from U back to S using certain results, proved in Chapter 7, relating rigid geometry and formal geometry.

The result of Chapter 6 is that on a connected rigid analytic variety X over a discretely valued field any two points may be connected by curves on X. This result is used in Chapter 5. In Chapter 7 we describe Berthelot's functor $\mathfrak{X} \mapsto \mathfrak{X}^{rig}$. Here $\mathfrak{X}$ is a formal scheme over a complete discrete valuation ring R, where $\mathfrak{X}$ need not be of finite type over R. The result $\mathfrak{X}^{rig}$ is a rigid analytic variety over the quotient field of R. In Theorem 7.4.1 we compare bounded rigid analytic functions on $\mathfrak{X}^{rig}$ with formal functions on $\mathfrak{X}$. In Proposition 7.5.2 we prove a result on rigid descent of closed formal subschemes.

This work is strongly influenced by the work of Prof. Berthelot, indeed, this paper might not have existed had I not attended a talk by Berthelot on convergent isocrystals. I would like to thank him for stimulating conversations. Furthermore, I would like to thank Prof. Messing, Prof. Zink and R. Huber for discussions on subjects related to this paper.

1. Some algebra

In this chapter we introduce a class of formally smooth $\mathbf{F}_p$-algebras and give some of their properties. Further, we recall some notations and definitions. All rings considered are assumed commutative with 1.

1.1. *Rings with p-bases*

1.1.1. *Definition.* — A subset $\{ x_\alpha \}$ of an $\mathbf{F}_p$-algebra A is called a *p-basis* of A if A, considered as an A-module via the Frobenius of A, is a free A-module which admits as a basis the set of monomials x^{I}, where I runs over multi-indices $\mathrm{I} = (i_\alpha)$, $0 \leqslant i_\alpha < p$ and almost all i_α are zero.

This definition is taken from [BM] (Definition 1.1.1). It implies that A is reduced and is a formally smooth $\mathbf{F}_p$-algebra. If A has a p-basis then any localization $\mathrm{S}^{-1}\,\mathrm{A}$ has a p-basis and any polynomial ring $\mathrm{A}[\mathrm{T}_i]$ has a p-basis. Any field of chraracteristic p has a p-basis.

1.1.2. *Lemma.* — *For any* $\mathbf{F}_p$*-algebra* A *having a p-basis the natural augmentation of the cotangent complex of* A *over* $\mathbf{F}_p$:

$$\mathrm{L}_{\mathrm{A}/\mathbf{F}_p} \to \Omega^1_{\mathrm{A}/\mathbf{F}_p}$$

(see [Ill, II (1.2.4.1)]) is a quasi-isomorphism. (Here $\mathrm{L}_{\mathrm{A}/\mathbf{F}_p}$ *is viewed as an object of the derived category* D(A).*)*

Proof. — We first remark that for any ring A in characteristic p the Frobenius endomorphism $\mathrm{A} \to \mathrm{A}$ induces an endomorphism of $\mathrm{L}_{\mathrm{A}/\mathbf{F}_p}$ which is homotopic to the zero morphism. Indeed, the augmentation $\mathrm{P}_{\mathbf{F}_p}(\mathrm{A}) \to \mathrm{A}$ [Ill, I (1.5.5.6)] is a homotopy equivalence. Hence any endomorphism of $\mathrm{P}_{\mathbf{F}_p}(\mathrm{A})$ lifting the Frobenius endomorphism of A is homotopic to the canonical one (the one from [Ill, II 1.2]). Therefore, we may take the Frobenius morphism of $\mathrm{P}_{\mathbf{F}_p}(\mathrm{A})$ (i.e., the Frobenius endomorphism on each $\mathrm{P}_{\mathbf{F}_p}(\mathrm{A})_n$) to compute the action of Frobenius on $\mathrm{L}_{\mathrm{A}/\mathbf{F}_p}$. As $(\mathrm{L}_{\mathrm{A}/\mathbf{F}_p})_n = \Omega^1_{\mathrm{P}(\mathrm{A})_n/\mathbf{F}_p}$, this induces the zero morphism on $\mathrm{L}_{\mathrm{A}/\mathbf{F}_p}$.

Let us write for $\mathrm{A}^p \subset \mathrm{A}$ the subring of p-powers of A. Frobenius induces an isomorphism $\mathrm{A} \to \mathrm{A}^p$, hence by the remark above the inclusion $\mathrm{A}^p \subset \mathrm{A}$ induces the zero morphism $\mathrm{L}_{\mathrm{A}^p/\mathbf{F}_p} \to \mathrm{L}_{\mathrm{A}/\mathbf{F}_p}$. Consider the distinguished triangle associated to $\mathbf{F}_p \subset \mathrm{A}^p \subset \mathrm{A}$ (see [Ill, II (2.1.2.1)]):

$$\mathrm{L}_{\mathrm{A}^p/\mathbf{F}_p} \otimes_{\mathrm{A}^p} \mathrm{A} \xrightarrow{(1)} \mathrm{L}_{\mathrm{A}/\mathbf{F}_p} \xrightarrow{(2)} \mathrm{L}_{\mathrm{A}/\mathrm{A}^p} \xrightarrow{(3)} \mathrm{L}_{\mathrm{A}^p/\mathbf{F}_p} \otimes_{\mathrm{A}^p} \mathrm{A}[1].$$

We have seen that (1) induces the zero map on homology. It is easy to see that (2) induces an isomorphism $\Omega^1_{\mathrm{A}/\mathbf{F}_p} \to \Omega^1_{\mathrm{A}/\mathrm{A}^p}$. To finish the proof we have to show that (3) is injective on $\mathrm{H}_{-1}(\mathrm{L}_{\mathrm{A}/\mathrm{A}^p})$ and that $\mathrm{H}_{-i}(\mathrm{L}_{\mathrm{A}/\mathrm{A}^p}) = 0$ for all $i \geqslant 2$.

Let $\{ x_j \}_{j \in \mathrm{J}}$ be a p-basis of A. Consider the ideal I generated by the elements $\mathrm{T}_j^p - x_j^p$, $j \in \mathrm{J}$ in the polynomial ring $\mathrm{P} := \mathrm{A}^p[\mathrm{T}_j; j \in \mathrm{J}]$. It is easily shown to be Koszul ([Ill, III 3.3]). Since $\mathrm{A} \cong \mathrm{P}/\mathrm{I}$ we get by [Ill, Proposition 3.3.6] that

$$\mathrm{L}_{\mathrm{A}/\mathrm{A}^p} \cong (0 \longrightarrow \mathrm{I}/\mathrm{I}^2 \xrightarrow{\mathrm{d}_{\mathrm{P}/\mathrm{A}^p}} \Omega^1_{\mathrm{P}/\mathrm{A}^p} \otimes_{\mathrm{P}} \mathrm{A} \to 0).$$

Of course in our situation $d_{\mathrm{P}/\mathrm{A}^p} = 0$. We leave it to the reader to show by an explicit calculation that (3) induces an isomorphism

$$\mathrm{I}/\mathrm{I}^2 \to \Omega^1_{\mathrm{A}^p/\mathbf{F}_p} \otimes_{\mathrm{A}^p} \mathrm{A}.$$

(It maps $\mathrm{T}_j^p - x_j^p$ to $- \mathrm{d}x_j^p \otimes 1$.) □

1.1.3. *Lemma. — Let* A *be an* $\mathbf{F}_p$*-algebra with a finite p-basis,* $\mathrm{I} \subset \mathrm{A}$ *a finitely generated ideal. The completion* $\hat{\mathrm{A}}$ *of* A *with respect to* I *has a finite p-basis also.*

Proof. — Choose a p-basis $\{ x_1, \ldots, x_\mathrm{N} \}$ of A and generators $f_1, \ldots, f_\mathrm{M}$ of I. We denote by $x \mapsto \hat{x}$ the map $\mathrm{A} \to \hat{\mathrm{A}}$. We want to write any element $a = \lim a_n$ of $\hat{\mathrm{A}}$ (with $a_n \in \mathrm{A}/\mathrm{I}^n$) in the form

$$a = \sum_{\mathrm{J} = (j_1, \ldots, j_\mathrm{N}),\ 0 \leqslant j_\alpha < p} (a_\mathrm{J})^p\, \hat{x}^\mathrm{J}, \quad a_\mathrm{J} \in \hat{\mathrm{A}}.$$

By assumption we can write $a_n = \Sigma (a_{\mathrm{J}, n})^p\, x^\mathrm{J} \bmod \mathrm{I}^n$. We see that

$$\Sigma (a_{\mathrm{J}, n+1} - a_{\mathrm{J}, n})^p\, x^\mathrm{J} \in \mathrm{I}^n.$$

Considering the effect of

$$\left(\frac{\partial}{\partial x_1}\right)^{n_1} \cdots \left(\frac{\partial}{\partial x_\mathrm{N}}\right)^{n_\mathrm{N}}$$

on elements of the form $\Sigma\, b_\mathrm{J}^p\, x^\mathrm{J}$ and on I^n, it becomes clear that

$$(a_{\mathrm{J}, n+1} - a_{\mathrm{J}, n})^p \in \mathrm{I}^{n - p\mathrm{N}}.$$

Next, we can write any element of $\mathrm{I}^{n - p\mathrm{N}}$ (in particular the element $(a_{\mathrm{J}, n+1} - a_{\mathrm{J}, n})^p$) as

$$\sum_{|\mathrm{K}| \geqslant n/p - \mathrm{N} - (p-1)\mathrm{M}} \alpha_\mathrm{K} (f^\mathrm{K})^p, \quad \alpha_\mathrm{K} \in \mathrm{A}.$$

Here the sum runs over multi-indices $\mathrm{K} = (k_1, \ldots, k_\mathrm{M})$ with total degree

$$|\mathrm{K}| := k_1 + \ldots + k_\mathrm{M}$$

at least $c(n) := n/p - \mathrm{N} - (p-1)\,\mathrm{M}$. Writing $\alpha_\mathrm{K} = \Sigma (\alpha_{\mathrm{K}, \mathrm{J}})^p\, x^\mathrm{J}$ we see that

$$(a_{\mathrm{J}, n+1} - a_{\mathrm{J}, n})^p = \sum_\mathrm{J} \Big(\sum_{|\mathrm{K}| \geqslant c(n)} \alpha_{\mathrm{K}, \mathrm{J}} f^\mathrm{K} \Big)^p x^\mathrm{J}.$$

The only non vanishing term on the right is the one with $\mathrm{J} = 0$. Therefore, we conclude that

$$a_{\mathrm{J}, n+1} - a_{\mathrm{J}, n} \in \mathrm{I}^{c(n)}.$$

Thus, putting

$$a_J := \varprojlim a_{J,n} \bmod I^{c(n)} \in \hat{A},$$

we get

$$a = \Sigma (a_J)^p \, \hat{x}^J$$

as desired. Unicity is proved in the same manner. □

1.2. *Lifts of rings in characteristic p to characteristic* 0

1.2.1. *Definition.* — Let A be a ring of characteristic p. A *lift of* A (resp. *a lift of* A *modulo* p^n) is a p-adically complete ring $\tilde{A}$, flat over $\mathbf{Z}_p$ (resp. flat over $\mathbf{Z}/p^n\mathbf{Z}$) endowed with an isomorphism $\tilde{A}/p\tilde{A} \cong A$. A *lift of Frobenius* on such a lift $\tilde{A}$ is a ring endomorphism $\sigma : \tilde{A} \to \tilde{A}$ such that $\sigma(a) \equiv a^p \bmod p\tilde{A}$. In this situation we will also call the pair $(\tilde{A}, \sigma)$ a lift of A (resp. a lift of A modulo p^n).

1.2.2. *Lemma.* — *Suppose* A *is an* $\mathbf{F}_p$*-algebra whose cotangent complex* $L_{A/\mathbf{F}_p}$ *is quasi-isomorphic to* $\Omega^1_{A/\mathbf{F}_p}$. *If, in addition,* $\Omega^1_{A/\mathbf{F}_p}$ *is a projective* A*-module then a lift* $(\tilde{A}, \sigma)$ *of* A *exists.*

Proof. — We are going to find a lift (A_n, σ) of A modulo p^n. We argue by induction on n (the case $n = 1$ is clear). The obstruction to find A_{n+1} given A_n lies in

$$\mathrm{Ext}^2_{A_n}(L_{A_n/\mathbf{Z}/p^n\mathbf{Z}}, A)$$

by [Ill, III (2.1.3.3)]. Since $L_{A_n/\mathbf{Z}/p^n\mathbf{Z}} \otimes A \cong L_{A/\mathbf{F}_p}$ (see [Ill, II (2.3.10)]) we get

$$\mathrm{Ext}^2_{A_n}(L_{A_n/\mathbf{Z}/p^n\mathbf{Z}}, A) \cong \mathrm{Ext}^2_A(L_{A/\mathbf{F}_p}, A) \cong \mathrm{Ext}^2_A(\Omega^1_{A/\mathbf{F}_p}, A)$$

and the last group is zero by assumption. Similarly, given A_{n+1}, the obstruction to lift $\sigma : A_n \to A_n$ to A_{n+1} lies in

$$\mathrm{Ext}^1_{A_n}(L_{A_n/\mathbf{Z}/p^n\mathbf{Z}} \otimes_\sigma A_n, A).$$

(Use [Ill, III (2.2.2)]; we remark that a subscript $_0$ is missing from [Ill, III Formula (2.2.1.4)].) In this case we have the quasi-isomorphisms

$$L_{A_n/\mathbf{Z}/p^n\mathbf{Z}} \otimes_\sigma A_n \otimes A \cong L_{A_n/\mathbf{Z}/p^n\mathbf{Z}} \otimes A \otimes_\sigma A \cong L_{A/\mathbf{F}_p} \otimes_\sigma A \cong \Omega^1_{A/\mathbf{F}_p} \otimes_\sigma A$$

and we conclude in the same manner that the obstruction is zero. □

1.2.3. *Remarks.* — *a)* Combining Lemmata 1.1.2 and 1.2.2 we see that any $\mathbf{F}_p$-algebra A with a p-basis has a lift $(\tilde{A}, \sigma)$. In [BM] an explicit construction of such a lift is given.

b) Although the lift $\tilde{A}$ of the lemma is always unique up to isomorphism, this is not true in general for the pair $(\tilde{A}, \sigma)$.

1.3. *Formally smooth rings in characteristic p*

1.3.1. Let A be an $\mathbf{F}_p$-algebra and $I \subset A$ an ideal of A. We consider the following condition on the pair (A, I):

(1.3.1.1) — A is Noetherian and complete with respect to the I-adic topology on A.
— The ring A is formally smooth over $\mathbf{F}_p$ (see EGA 0_{IV} 19.3.1 or [Mat, 28.C]; we use the I-adic topology on A).
— The ring A/I contains a field with a finite p-basis and is a finitely generated algebra over this field.

We will say that an $\mathbf{F}_p$-algebra A satisfies (1.3.1.1) if there is an ideal $I \subset A$ such that (A, I) satisfies (1.3.1.1) above. Usually, we will assume that the ideal I is as big as possible, i.e., $I = \sqrt{I}$. We remark that this assumption and (1.3.1.1) uniquely determine $I \subset A$. This follows from the fact that a ring finitely generated over a field has only one adic topology for which it is complete: the discrete topology.

1.3.2. *Examples.* — Let k be a field of characteristic p with $[k : k^p] < \infty$, i.e., k has a finite p-basis.

(1.3.2.1) Any completion of a polynomial algebra $k[x_1, \ldots, x_n]$ with respect to an ideal satisfies (1.3.1.1). By Lemma 1.1.3 such a ring has a finite p-basis.

(1.3.2.2) Any finitely generated k-algebra $A = A/I$ which is a regular ring satisfies (1.3.1.1). (To see that A is formally smooth over $\mathbf{F}_p$ use EGA 0_{IV} 22.6.7.)

From Lemma 1.1.3 we see that rings of the type described in (1.3.2.1) have a finite p-basis. Hence by [BM] we know that crystals over such rings can be described in terms of modules with connections over lifts. The next lemma will imply the same for rings as in 1.3.1.

1.3.3. *Lemma.* — *Let* A *be a ring satisfying* (1.3.1.1).

a) There exists a ring B *of the form described in* (1.3.2.1) *and a surjection* $\pi : B \to A$ *such that the inverse image of an ideal of definition of* A *is an ideal of definition of* B. *We may then choose* $i : A \to B$ *such that* $\pi \circ i = \mathrm{id}_A$.

b) The homomorphism $\Omega^1_A \to \hat{\Omega}^1_A$ *(continuous differentials) is an isomorphism;* Ω^1_A *is a finite projective* A*-module.*

c) The morphism $L_{A/\mathbf{F}_p} \to \Omega^1_A$ *is a quasi-isomorphism.*

d) If we have B, i *and* π *as in a) then we can find an isomorphism*

$$B \cong A[[M]] = \prod_{n \geqslant 0} \mathrm{Sym}^n_A(M)$$

where M *is a finite projective* A*-module. We can find a lift* $(\tilde{A}, \sigma)$ *of* A (1.2.2); *as a lift of* B *we can take*

$$\tilde{B} = \tilde{A}[[\tilde{M}]],$$

for some lift $\tilde{M}$ *of the module* M. *Furthermore, we can find a lift of Frobenius* σ *on* $\tilde{B}$ *satisfying* $\sigma(\tilde{M}) \subset \mathrm{Sym}^p(\tilde{M})$ *and inducing* σ *on* $\tilde{A}$.

Proof. — Let $I \subset A$ be an ideal as in (1.3.1.1) and let $k \subset A/I$ be the field which is supposed to exist by (1.3.1.1). Take a lift $k \to A$ of the homomorphism $k \to A/I$ (this is possible, k is formally smooth over $\mathbf{F}_p$). Next we take a homomorphism

$$\varphi : k[x_1, \ldots, x_r, x_{r+1}, \ldots, x_n] \to A$$

such that the images of $x_1, \ldots, x_r$ generate A/I as a k-algebra and $\varphi(x_{r+1}), \ldots, \varphi(x_n)$ are generators of the ideal I. Let B be the completion of $k[x_1, \ldots, x_n]$ in the kernel of $k[x_1, \ldots, x_n] \to A/I$. The homomorphism φ induces a surjection $\pi : B \to A$. Let $J \subset B$ (resp. $K \subset B$) be the kernel of $B \to A/I$ (resp. $B \to A/I^2$). The ring A is formally smooth over $\mathbf{F}_p$ by (1.3.1.1), hence we can find a ring homomorphism $\psi_2 : A \to B/J^2$ fitting into the diagram:

$$\begin{array}{ccccc} A & \longrightarrow & A/I^2 & \longrightarrow & A/I \\ \downarrow{\scriptstyle \psi_2} & & \uparrow{\scriptstyle \cong} & & \uparrow{\scriptstyle \cong} \\ B/J^2 & \longrightarrow & B/K & \longrightarrow & B/J. \end{array}$$

(Remark that $J^2 \subset K \subset J$ and that ψ_2 need not be k-linear.) We see that $I^2 \subset \mathrm{Ker}(\psi_2)$, as $\psi_2(I) \subset J/J^2$, therefore we get $\psi_2 : A/I^2 \to B/J^2$. By induction on n we choose homomorphisms $\psi_n : A/I^n \to B/J^n$ $(n \geqslant 3)$, such that $\psi_n \bmod I^{n-1} = \psi_{n-1}$ (use that A is formally smooth over $\mathbf{F}_p$). We put

$$\psi_\infty = \varprojlim \psi_n : A \to B.$$

Our choice of ψ_2 implies that $\pi \circ \psi_\infty \equiv \mathrm{id}_A \bmod I^2$, so it must be an automorphism of A. If we take $i := \psi_\infty \circ (\pi \circ \psi_\infty)^{-1}$ then we see that $\pi \circ i = \mathrm{id}_A$. This proves *a)*.

The usual argument shows that, since B is formally smooth also, we must have $B \cong A[[M]]$ with M a finite (B Noetherian) projective A-module (cf. EGA 0_{IV} 19.5.3). This implies that the ideal $MB = \mathrm{Ker}(B \to A)$ is regular and that the sequence

$$0 \to M \to \Omega^1_B \otimes A \to \Omega^1_A \to 0$$

is split exact. Our result $L_{B/\mathbf{F}_p} \cong \Omega^1_B$ (Lemma 1.1.2) gives *c)* by [Ill, III 3.3.6]. Statement *b)* follows from the fact that Ω^1_B is a finite free B-module and is isomorphic to the module $\hat{\Omega}^1_B$ of continuous differentials (with respect to the J-adic topology). This fact is easily proved using the finite p-basis of B (compare with the proof of Lemma 1.1.3).

We already proved the first part of *d)*. There are no obstructions to lifting finite projective modules, so we can find the lift $\widetilde{M}$ of M. This is a finite projective $\widetilde{A}$-module; let us take a finite free module $\bigoplus \widetilde{A}.x_i$ and homomorphisms $\bigoplus \widetilde{A}.x_i \xrightarrow{s} \widetilde{M}$ and $\bigoplus \widetilde{A}.x_i \xleftarrow{t} \widetilde{M}$ with $s \circ t = \mathrm{id}_{\widetilde{M}}$. For σ we take the composition

$$\begin{array}{ccccccc} \widetilde{A}[[\widetilde{M}]] & \longrightarrow & \widetilde{A}[[x_i]] & \longrightarrow & \widetilde{A}[[x_i]] & \longrightarrow & \widetilde{A}[[\widetilde{M}]] \\ & \widetilde{m} \mapsto t(\widetilde{m}) & & x_i \mapsto x_i^p & & x_i \mapsto s(x_i) & \\ & \mathrm{id}_{\widetilde{A}} & & \sigma \text{ on } \widetilde{A} & & \mathrm{id}_{\widetilde{A}}. & \end{array}$$

This concludes the proof of Lemma 1.3.3. □

1.3.4. *Remarks.* — *a)* The lift $\tilde{A}$ of the lemma above is also complete for the $\tilde{I}$-adic topology; here $\tilde{I}$ denotes the inverse image in $\tilde{A}$ of an ideal of definition I of A.

b) The module

$$\hat{\Omega}^1_{\tilde{A}} = \text{differentials continuous for the } p\text{-adic topology on } \tilde{A} := \varprojlim \Omega^1_{\tilde{A}/p^n\tilde{A}}$$

is a finite projective $\tilde{A}$-module; it is flat over $\tilde{A}$ and we have $\hat{\Omega}^1_{\tilde{A}}/p\hat{\Omega}^1_{\tilde{A}} \cong \Omega^1_A$. To see this, use 1.3.3 *d)* to reduce to the case that A has a finite p-basis and apply [BM, 1.3.1]. More generally for any lift $\tilde{A}$ as constructed in Lemma 1.2.2 the module $\hat{\Omega}^1_{\tilde{A}}$ is a flat lift of Ω^1_A. The reader may enjoy proving this general result for himself.

c) The natural homomorphism

$$\hat{\Omega}^1_{\tilde{A}} = \hat{\Omega}^1_{(\tilde{A}, p\tilde{A})} \to \hat{\Omega}^1_{(\tilde{A}, \tilde{I})}$$

is an isomorphism. The $\tilde{A}$-module on the right denotes differentials continuous in the $\tilde{I}$-adic topology (see *a)*). This follows from 1.3.3 *b)*.

2. Crystals, formal and p-divisible groups

In this chapter we describe crystals in terms of modules with connections. We are a bit more general then [BM]. In addition we do the same for Dieudonné crystals. Finally, we define a Kodaira-Spencer map for Dieudonné crystals on certain formal schemes which measures in a certain sense how versal the crystal is.

2.1. *Crystalline sites and crystals*

2.1.1. We always work in the absolute case, with base scheme $(\Sigma, \mathscr{I}, \gamma)$ where $\Sigma = \mathrm{Spec}(\mathbf{Z}_p)$, $\mathscr{I} = p\mathcal{O}_\Sigma$ and γ denotes the canonical divided powers on $\mathscr{I}$. For any Σ-scheme T we also denote by γ the canonical divided powers on the ideal $p\mathcal{O}_T$. If S is a Σ-scheme such that p is locally nilpotent on S then we write CRIS(S/Σ) for the big fppf-crystalline site of S over $(\Sigma, \mathscr{I}, \gamma)$ (see [BBM, 1.1]). We will always endow it with the fppf-topology.

We refer to [BBM] for the notation and conventions we use regarding these sites, sheaves on them, etc. For example, recall [BBM, 1.1.3] that a sheaf $\mathscr{E}$ on CRIS(S/Σ) is given by the following data:

(2.1.1.1) a sheaf $\mathscr{E}_{(U,T,\delta)}$ (usually denoted $\mathscr{E}_T$) on the small fppf-site T_{fppf} of T for any object (U, T, δ) of CRIS(S/Σ),

(2.1.1.2) for any morphism $(u, v) : (U', T', \delta') \to (U, T, \delta)$ of CRIS(S/Σ) a sheaf homomorphism $v^{-1}(\mathscr{E}_T) \to \mathscr{E}_{T'}$.

These have to satisfy a number of obvious conditions.

Let $\mathscr{C}$ be a site, $\mathscr{A}$ a sheaf of rings on $\mathscr{C}$ and $\mathscr{F}$ a sheaf of $\mathscr{A}$-modules. We say that $\mathscr{F}$ is a *quasi-coherent sheaf of* $\mathscr{A}$*-modules* if $\mathscr{F}$ is locally on $\mathscr{C}$ isomorphic to the cokernel of a map $\mathscr{A}^{(I)} \to \mathscr{A}^{(J)}$. More precisely, for any $X \in \mathrm{Ob}\,\mathscr{C}$, there exists a covering $\{ U_i \to X \}$ of X such that $\mathscr{F}|_{U_i}$ is isomorphic to the cokernel of a morphism $\mathscr{A}^{(I)}|_{U_i} \to \mathscr{A}^{(J)}|_{U_i}$ of $\mathscr{A}$-modules. Compare [EGA, O_I 5.1.3]; see also [B, Section 1.1] for this and the following definition.

2.1.2. *Definition.* — Suppose $\mathscr{A}$ is a sheaf of rings on CRIS(S/Σ). We say that a sheaf of $\mathscr{A}$-modules $\mathscr{E}$ is a *crystal of quasi-coherent $\mathscr{A}$-modules* if the following conditions hold:

(2.1.2.1) $\mathscr{E}_T$ is a quasi-coherent sheaf of $\mathscr{A}_T$-modules on T_{fppf} for any object (U, T, δ) of CRIS(S/Σ),

(2.1.2.2) the transition homomorphisms (2.1.1.2) induce isomorphisms

$$v^{-1}(\mathscr{E}_T) \otimes_{v^{-1}(\mathscr{A}_T)} \mathscr{A}_{T'} \to \mathscr{E}_{T'}.$$

We get the definition of *crystals of finite locally free $\mathscr{A}$-modules* by replacing in (2.1.2.1) the word quasi-coherent by finite locally free. In the same manner one can also define *crystals of $\mathscr{A}$-modules of finite presentation* over CRIS(S/Σ).

2.1.3. Let X, Y be Σ-schemes on which p is locally nilpotent. Let $i : Y \to X$ be a closed immersion. To i is associated a morphism of functoriality

$$i_{\mathrm{CRIS}} = (i^*_{\mathrm{CRIS}}, i_{\mathrm{CRIS}*}) : (Y/\Sigma)_{\mathrm{CRIS}} \to (X/\Sigma)_{\mathrm{CRIS}}$$

of crystalline topoi, see [B, III 4.2.1] and [BBM, 1.1.10]. We want to study the effect of $i_{\mathrm{CRIS}*}$ on crystals of $\mathscr{O}_{Y/\Sigma}$-modules.

Let us describe $i_{\mathrm{CRIS}*}$. For any object (U, T, δ) of CRIS(X/Σ) we write $(V, D_V(T), [\ ])$ for the divided power envelope of $V := Y \times_X U$ in T, the divided powers [ ] taken compatible with those on the ideal of U in T (i.e., δ) and with γ on $p\mathscr{O}_T$. The triple $(V, D_V(T), [\ ])$ is an object of CRIS(Y/Σ). For a sheaf $\mathscr{E}$ on CRIS(Y/Σ) we have the following isomorphism

(2.1.3.1) $$i_{\mathrm{CRIS}*}(\mathscr{E})_T \cong (p_T)_*(\mathscr{E}_{D_V(T)}).$$

Here p_T is the canonical morphism $D_V(T) \to T$; it induces a functor $(p_T)_*$: Sheaves on $D_V(T)_{\mathrm{fppf}} \to$ Sheaves on T_{fppf}. This is proved exactly as in [B, IV 1.3]. We remark that for $T' \to T$ flat we have $D_{V'}(T') \cong D_V(T) \times_T T'$ (see [B, I 2.7.1]). Clearly, if $V = U$, that is, if the morphism $U \to X$ factors through Y, then $(V, D_V(T), [\ ]) = (U, T, \delta)$.

The description of i^*_{CRIS} is easier. For any object of CRIS(Y/Σ), say (U, T, δ), we denote by $i_!(U, T, \delta)$ the triple (U, T, δ) considered as an object of CRIS(X/Σ), i.e., U is considered as a scheme over X via the composition $U \to Y \to X$. For a sheaf $\mathscr{E}$ on (CRIS(X/Σ) we have

(2.1.3.2) $$i^*_{\mathrm{CRIS}}(\mathscr{E})_{(U,T,\delta)} = \mathscr{E}_{i_!(U,T,\delta)}.$$

(See [BBM, 1.1.10.3].) Thus, for any sheaf $\mathscr{E}$ on CRIS(Y/Σ) we have

(2.1.3.3) $$i^*_{\mathrm{CRIS}}\, i_{\mathrm{CRIS}*}(\mathscr{E}) \cong \mathscr{E}.$$

2.1.4. *Lemma. — In the situation of* 2.1.3, *the functor* $i_{\mathrm{CRIS}*}$ *induces an equivalence of the following categories:*

$\mathscr{C}1$. *The category of crystals of quasi-coherent (resp. finite locally free)* $\mathcal{O}_{Y/\Sigma}$*-modules on* CRIS(Y/Σ).

$\mathscr{C}2$. *The category of crystals of quasi-coherent (resp. finite locally free)* $i_{\mathrm{CRIS}*}(\mathcal{O}_{Y/\Sigma})$*-modules on* CRIS(X/Σ).

A similar statement holds for crystals of finite presentation.

Proof. — Suppose $\mathscr{E}$ is an object of $\mathscr{C}1$. Take any object (U, T, δ) of CRIS(X/Σ). The sheaf $\mathscr{E}_{D_V(T)}$ is a quasi-coherent sheaf of $\mathcal{O}_{D_V(T)}$-modules on $D_V(T)_{\mathrm{fppf}}$. By descent theory, it arises from a quasi-coherent sheaf of $\mathcal{O}_{D_V(T)}$-modules on $D_V(T)_{\mathrm{Zar}}$. The morphism $p_T : D_V(T) \to T$ is affine, hence covering T by affines we may assume there is an exact sequence

$$\mathcal{O}^{(I)}_{D_V(T)} \to \mathcal{O}^{(J)}_{D_V(T)} \to \mathscr{E}_{D_V(T)} \to 0 \tag{2.1.4.1}$$

on $D_V(T)_{\mathrm{fppf}}$. (If $\mathscr{E}$ is of finite presentation then the index sets I and J may be chosen finite; if $\mathscr{E}$ is finite locally free then we may choose $I = \emptyset$ and J finite. Adding such remarks, the rest of the proof will go through in each of the three cases.) Since the morphism $p_T : D_V(T) \to T$ is affine this gives an exact sequence

$$i_{\mathrm{CRIS}*}(\mathcal{O}_{Y/\Sigma})^{(I)}_T \to i_{\mathrm{CRIS}*}(\mathcal{O}_{Y/\Sigma})^{(J)}_T \to i_{\mathrm{CRIS}*}(\mathscr{E})_T \to 0 \tag{2.1.4.2}$$

on T_{fppf}. Therefore, we have proven that $i_{\mathrm{CRIS}*}(\mathscr{E})$ satisfies (2.1.2.1) with $\mathscr{A} = i_{\mathrm{CRIS}*}(\mathcal{O}_{Y/\Sigma})$. Let $(u, v) : (U', T', \delta') \to (U, T, \delta)$ be a morphism of CRIS(X/Σ). The crystal property of $\mathscr{E}$ implies that $\mathscr{E}_{D_{V'}(T')}$ is the pullback of $\mathscr{E}_{D_V(T)}$ via the morphism $D_{V'}(T') \to D_V(T)$. Hence we also have a sequence (2.1.4.1) over $D_{V'}(T')$ and a sequence (2.1.4.2) over T'. Thus, the sheaf $i_{\mathrm{CRIS}*}(\mathscr{E})$ satisfies (2.1.2.2) with $\mathscr{A} = i_{\mathrm{CRIS}*}(\mathcal{O}_{Y/\Sigma})$ since this is clearly true for $i_{\mathrm{CRIS}*}(\mathcal{O}_{Y/\Sigma})$. We conclude that $i_{\mathrm{CRIS}*}(\mathscr{E})$ is an object of $\mathscr{C}2$. By (2.1.3.3) the functor $i_{\mathrm{CRIS}*} : \mathscr{C}1 \to \mathscr{C}2$ is fully faithful.

Suppose $\mathscr{E} \in \mathrm{Ob}\,\mathscr{C}2$. Take any object (U, T, δ) of CRIS(Y/Σ). By assumption we can find a covering family

$$(U_i, T_i, \delta_i) \to i_!(U, T, \delta)$$

in CRIS(X/Σ) such that on each $T_{i,\mathrm{fppf}}$ there is an exact sequence

$$i_{\mathrm{CRIS}*}(\mathcal{O}_{Y/\Sigma})^{(I)}_{T_i} \to i_{\mathrm{CRIS}*}(\mathcal{O}_{Y/\Sigma})^{(J)}_{T_i} \to \mathscr{E}_{T_i} \to 0.$$

Since $U_i = U \times_T T_i$, the given map $U_i \to X$ factors through $Y \hookrightarrow X$. The system $(U_i, T_i, \delta_i) \to (U, T, \delta)$ can thus be viewed as a covering of (U, T, δ) in CRIS(Y/Σ). By (2.1.3.2) we get exact sequences

$$(\mathcal{O}_{Y/\Sigma})^{(I)}_{T_i} \to (\mathcal{O}_{Y/\Sigma})^{(J)}_{T_i} \to i^*_{\mathrm{CRIS}}(\mathscr{E})_{T_i} \to 0.$$

We conclude that $i^*_{\mathrm{CRIS}}(\mathscr{E})$ is in $\mathscr{C}1$ (verification of (2.1.2.2) is trivial). To finish the proof we need only to show that the natural homomorphism

$$\mathscr{E} \to i_{\mathrm{CRIS}*}\, i^*_{\mathrm{CRIS}}(\mathscr{E})$$

is an isomorphism. This may be checked locally on CRIS(X/Σ), hence on objects (U, T, δ) of CRIS(X/Σ) where we have an exact sequence

$$i_{\mathrm{CRIS}*}(\mathscr{O}_{Y/\Sigma})^{(I)}_{T} \to i_{\mathrm{CRIS}*}(\mathscr{O}_{Y/\Sigma})^{(J)}_{T} \to \mathscr{E}_T \to 0.$$

The description of $i_{\mathrm{CRIS}*}$ implies that this is the exact sequence

$$((p_T)_*\, \mathscr{O}_{D_V(T)})^{(I)} \to ((p_T)_*\, \mathscr{O}_{D_V(T)})^{(J)} \to \mathscr{E}_T \to 0$$

on T_{fppf}. Hence $\mathscr{E}_T \cong (p_T)_*\, \mathscr{F}$ for some quasi-coherent sheaf of $\mathscr{O}_{D_V(T)}$-modules $\mathscr{F}$ on $D_V(T)_{\mathrm{fppf}}$. By the crystal property (2.1.2.2) for $\mathscr{E}$ we must have $\mathscr{F} \cong \mathscr{E}_{D_V(T)}$ (here $(V, D_V(T), [\ ])$ is considered as an object of CRIS(X/Σ)). Since it is also true that

$$i_{\mathrm{CRIS}*}\, i^*_{\mathrm{CRIS}}(\mathscr{E})_T \cong (p_T)_*\, \mathscr{E}_{D_V(T)},$$

the proof is complete. □

The sheaf $i_{\mathrm{CRIS}*}\, \mathscr{O}_{Y/\Sigma}$ of Lemma 2.1.4 is not in general a crystal of $\mathscr{O}_{X/\Sigma}$-modules. For instance if i is the closed immersion $\mathrm{Spec}(\mathbf{F}_p) \to \mathrm{Spec}(\mathbf{F}_p[T])$ the sheaf $i_{\mathrm{CRIS}*}\, \mathscr{O}_{Y/\Sigma}$ satisfies (2.1.2.1) but not (2.1.2.2). The following proposition shows that we may replace $i_{\mathrm{CRIS}*}\, \mathscr{O}_{Y/\Sigma}$ by a crystal.

2.1.5. *Proposition.* — *In the situation* 2.1.3 *there is a sheaf of* $\mathscr{O}_{X/\Sigma}$*-algebras* $\mathscr{A}$ *on* CRIS(X/Σ) *and a homomorphism (of* $\mathscr{O}_{X/\Sigma}$*-algebras)*

(2.1.5.1) $$\mathscr{A} \to i_{\mathrm{CRIS}*}\, \mathscr{O}_{Y/\Sigma}$$

such that:

a) The sheaf $\mathscr{A}$ *is a crystal of quasi-coherent* $\mathscr{O}_{X/\Sigma}$*-modules.*

b) For any object (U, T, δ) *of* CRIS(X/Σ) *such that the morphism* U → X *is flat the homomorphism* (2.1.5.1) *induces an isomorphism*

$$\mathscr{A}_T \xrightarrow{\cong} (i_{\mathrm{CRIS}*}\, \mathscr{O}_{Y/\Sigma})_T.$$

The category of crystals of quasi-coherent $\mathscr{O}_{Y/\Sigma}$*-modules is equivalent to the category of crystals of quasi-coherent* $\mathscr{A}$*-modules. This equivalence induces an equivalence of the subcategories consisting of crystals of finite presentation (resp. the subcategories consisting of crystals of finite locally free modules).*

Proof. — We remark that *a)* and *b)* determine $\mathscr{A}$ up to unique isomorphism, hence it suffices to construct $\mathscr{A}$ in the case that X is affine. Say X = Spec(R) and Y ↪ X given by the ideal J ⊂ R.

Let us take a surjection $P := \mathbf{Z}_p[x_\alpha] \to R$ with kernel $I \subset P$ and let us denote by $(D, \bar{I}, \delta)$ the divided power envelope of I in P, the divided powers taken compatible with γ on pP. Similarly for the exact sequence

$$0 \to I(2) \to P(2) := P \otimes_{\mathbf{Z}_p} P \cong \mathbf{Z}_p[x_\alpha, y_\alpha] \to R \to 0$$

we get $(D(2), \bar{I}(2), \delta(2))$. Next, we write $\tilde{J} := \mathrm{Ker}(D \to R/J)$ and put

$$D(\tilde{J}) = \text{divided power envelope of } \tilde{J} \text{ in } D,$$
$$\text{divided powers taken compatible with } \delta \text{ on } \bar{I} \text{ and } \gamma \text{ on } pD.$$

In the same way we get $\tilde{J}(2) \subset D(2)$ and its divided power envelope $D(2, \tilde{J}(2))$.

Suppose (U', T', δ') is an affine object of $\mathrm{CRIS}(X/\Sigma)$:

$$(U', T', \delta') = (\mathrm{Spec}(B'/I'), \mathrm{Spec}(B'), \delta'_n : I' \to I').$$

By construction of D we can find a homomorphism of PD-algebras

$$\varphi : (D, \bar{I}, \delta) \to (B', I', \delta')$$

lifting the given homomorphism $R \to B'/I'$. We define $\mathscr{A}_T$ as the sheaf on T'_{fppf} associated to the B'-module $B' \otimes_D D(\tilde{J})$. To prove this does not depend on the choice of φ we remark that the natural maps

$$D(\tilde{J}) \otimes_D D(2) \to D(2, \tilde{J}(2)) \leftarrow D(2) \otimes_D D(\tilde{J})$$

are isomorphisms, see [B, I Proposition 2.8.2]. These isomorphisms satisfy an obvious cocycle condition. By construction, the sheaf $\mathscr{A}$ so defined is a crystal of quasi-coherent $\mathcal{O}_{X/\Sigma}$-modules.

The value of $i_{\mathrm{CRIS}*}(\mathcal{O}_{Y/\Sigma})$ on T'_{fppf} is the sheaf associated to the B' module $D_{B'}(J')$ = the divided power envelope of $J' := \mathrm{Ker}(B' \to (B'/I')/J(B'/I'))$ in B', the divided powers taken compatible with δ' on I' and γ on pB'. The map (2.1.5.1) is defined as the sheaf homomorphism which on T_{fppf} is determined by the homomorphism of B'-algebras

(2.1.5.2) $\quad B' \otimes_D D(\tilde{J}) \to D_{B'}(J').$

Therefore [B, I Proposition 2.8.2] gives *b)'*: assertion *b)* for (U, T, δ), where $U \to X$ is an open immersion. A slight generalization, Proposition 2.1.7 below, gives *b)* in general. However, for the proof of the last assertions of 2.1.5 we do not need 2.1.7. Indeed, we use 2.1.4 to go from $\mathcal{O}_{Y/\Sigma}$-modules to $i_{\mathrm{CRIS}*}\,\mathcal{O}_{Y/\Sigma}$-modules. Since these crystals are determined by their values on (U, T, δ)'s with $U \to X$ an open immersion (for example $T = \mathrm{Spec}(D/p^n D)$ we derive the equivalence from *b)'*). □

2.1.6. In this section the conventions and notation are those of [B, I Section 2.8.1], excepting (2.8.1) of [B], which we replace by:

(2.1.6.1) $\quad$ The ring homomorphism $B/I \to B'/I'$ is flat and $J'/I' = J/I.B'/I'$.

Thus, we are given a commutative diagram

$$\begin{array}{ccc} (B, I, \delta) & \longrightarrow & (B', I', \delta') \\ & \nwarrow \quad \nearrow & \\ & (A, I_0, \gamma) & \end{array}$$

of algebras, δ and δ' compatible with γ, $B \to B'$ is a PD-homomorphism and ideals $J \subset B$ (resp. $J' \subset B'$) containing I (resp. I'). The divided power envelope $D_B(J)$ of J in B (divided powers taken compatible with δ and γ) and similar $D_{B'}(J')$ are given.

2.1.7. *Proposition.* — *In the situation* 2.1.6, *the homomorphism*

$$D_B(J) \otimes_B B' \to D_{B'}(J')$$

is an isomorphism.

Proof. — The proof of this is the same as the proof of [B, I Proposition 2.8.2] except that one has to replace the sentence beginning on line 2 of page 55 by the following arguments: "Suppose a section x of $J'_2 \cap \mathrm{Im}(K \otimes B')$ is of the form

$$x = \sum_i (x_i^{[1]} - x_i)\, a_i \otimes \alpha_i$$

with $x_i \in J$, $a_i \in B$ and $\alpha_i \in B'$. Transferring the elements a_i to the other side of the $\otimes$-sign we may assume $a_i = 1$ for all i. The degree zero component of x lies in the degree zero component of J'_2, which is I'. Hence we see that $\Sigma_i\, x_i\, \alpha_i = 0$ in B'/I'. Flatness of B'/I' over B/I implies that there are elements $b_{ik} \in B$ and $\alpha'_k \in B'$ such that

$$c_i := \alpha_i - \Sigma\, b_{ik}\, \alpha'_k \in I'$$

and $$d_k := \Sigma\, x_i\, b_{ik} \in I.$$

(See [Mat, Theorem 1].) Thus we see that

$$\begin{aligned} x &= \sum_i (x_i^{[1]} - x_i) \otimes \alpha_i \\ &= \sum_i (x_i^{[1]} - x_i) \otimes c_i + \sum_{i,k} (x_i^{[1]} - x_i) \otimes b_{ik}\, \alpha'_k \\ &= \sum_i (x_i^{[1]} - x_i) \otimes c_i + \sum_k (d_k^{[1]} - d_k) \otimes \alpha'_k. \end{aligned}$$

Each of the terms lies in $J'_2 \cap \mathrm{Im}(K \otimes B')$." □

2.2. *Crystals and modules with connections*

2.2.1. Suppose A is an $\mathbf{F}_p$-algebra satisfying (1.3.1.1). Let $\tilde{A}$ be a lift of A and let $J \subset \tilde{A}$ be an ideal of $\tilde{A}$ such that $p^n\, \tilde{A} \subset J$ for some $n \in \mathbf{N}$. We are going to describe crystals of $\mathcal{O}_{S/\Sigma}$-modules, with $S = \mathrm{Spec}(\widetilde{A}/J)$ in terms of modules with connections. We put

(2.2.1.1) $(\hat{D}, \hat{J}, [\ ]) =$ the p-adic completion of $D_{\widetilde{A},\gamma}(J)$.

We denote by $\hat{\Omega}^1_{\widetilde{A}}$ the module of continuous differentials of $\widetilde{A}$ (cf. Remark 1.3.4). There is a natural connection

(2.2.1.2) $\nabla : \hat{D} \to \hat{D} \hat{\otimes}_{\widetilde{A}} \hat{\Omega}^1_{\widetilde{A}} \cong \hat{D} \otimes_{\widetilde{A}} \hat{\Omega}^1_{\widetilde{A}}$

compatible with $d : \widetilde{A} \to \hat{\Omega}^1_{\widetilde{A}}$ and such that

(2.2.1.3) $\nabla(j^{[n]}) = j^{[n-1]} \otimes dj \quad \forall j \in J,\ n \in \mathbf{N}.$

(See Remark 2.2.4 *d)* for the construction of ∇.)

2.2.2. *Proposition.* — *The category of crystals of quasi-coherent $\mathcal{O}_{S/\Sigma}$-modules is equivalent to the category of (p-adically) complete $\hat{D}$-modules* M *endowed with an integrable, topologically quasi-nilpotent connection*

$$\nabla : M \to M \otimes_{\widetilde{A}} \hat{\Omega}^1_{\widetilde{A}}$$

compatible with the connection (2.2.1.2) *on* $\hat{D}$.

2.2.3. *Corollary.* — *The category of crystals of quasi-coherent $\mathcal{O}_{\mathrm{Spec}(A)/\Sigma}$-modules is equivalent to the category of (p-adically) complete $\widetilde{A}$-modules* M *endowed with an integrable, topologically quasi-nilpotent connection*

$$\nabla : M \to M \otimes_{\widetilde{A}} \hat{\Omega}^1_{\widetilde{A}}.$$

2.2.4. *Remarks.* — *a)* By our conventions a complete module is separated.

b) In the above equivalences crystals of finite locally free $\mathcal{O}_{S/\Sigma}$-modules correspond to finite locally free $\hat{D}$-modules and vice versa. Similarly, crystals of $\mathcal{O}_{S/\Sigma}$-modules of finite presentation correspond to $\hat{D}$-modules locally of finite presentation.

c) The condition of topological quasi-nilpotence means the following: given any derivation $\theta \in \mathrm{Hom}_{\widetilde{A}}(\hat{\Omega}^1_{\widetilde{A}}, \widetilde{A})$ the induced endomorphism $\nabla_\theta : M \to M$ is topologically quasi-nilpotent, i.e., for any $m \in M$, there exists $n \in \mathbf{N}$ such that $\nabla^n_\theta(m) \in pM$.

d) We would like to add that, if $\mathscr{E}$ is such a crystal of quasi-coherent $\mathcal{O}_{S/\Sigma}$-modules, the corresponding module M is defined as:

$$M := \varprojlim_n \Gamma((S, \mathrm{Spec}(\hat{D}/p^n \hat{D}), [\ ]), \mathscr{E}).$$

As usual the connection comes from the identification

$$\varepsilon : M \hat{\otimes}_{\hat{D}} \hat{D}(2) \xrightarrow{\cong} \hat{D}(2) \hat{\otimes}_{\hat{D}} M$$

which we get from the crystal property of $\mathscr{E}$. Here $\hat{D}(2)$ is the p-adic completion of

$$D_{\widetilde{A}\otimes_{\mathbf{Z}_p}\widetilde{A},\gamma}(\mathrm{Ker}(\widetilde{A}\otimes_{\mathbf{Z}_p}\widetilde{A}\to\widetilde{A}/J))$$

and we remark that if $K := \mathrm{Ker}(\hat{D}(2)\to\hat{D})$ then K is a sub PD-ideal and $K/K^{[2]}\cong\hat{D}\otimes_{\widetilde{A}}\hat{\Omega}^1_{\widetilde{A}}$. If we apply this to $\mathscr{E}=\mathcal{O}_{S/\Sigma}$ we get $M\cong\hat{D}$ and in this manner we derive the connection (2.2.1.2). We leave it to the reader to prove (2.2.1.3).

e) It follows from the preceding remark that if M (resp. M′) corresponds to $\mathscr{E}$ (resp. $\mathscr{E}'$) then $M\hat{\otimes}_{\hat{D}}M'$ corresponds to $\mathscr{E}\otimes_{\mathcal{O}_{S/\Sigma}}\mathscr{E}'$.

f) Let $J'\subset\widetilde{A}$ be a second ideal and $\hat{D}'$ the corresponding divided power algebra (2.2.1.1). Assume that $J\subset J'\subset\widetilde{A}$, so that we get a closed immersion

$$i : S' := \mathrm{Spec}(\widetilde{A}/J')\to S.$$

Let $\mathscr{A}$ be the sheaf of $\mathcal{O}_{S/\Sigma}$-algebras constructed in Proposition 2.1.5 for the closed immersion i. There is a diagram (commutative up to isomorphism of functors)

(2.2.4.1)
$$\begin{array}{ccccc}
\text{crystals of q.c.} & \xrightarrow{2.1.5} & \text{crystals of q.c.} & \xrightarrow{(1)} & \text{crystals of q.c.}\\
\mathcal{O}_{S'/\Sigma}\text{-modules} & & \mathscr{A}\text{-modules} & & \mathcal{O}_{S/\Sigma}\text{-modules}\\
\downarrow{\scriptstyle d)} & & & & \downarrow{\scriptstyle d)}\\
\text{complete } \hat{D}'\text{-modules} & & \text{induced} & & \text{complete } \hat{D}\text{-modules}\\
\text{with i.t.q.n. connection} & \longrightarrow & \text{by } \hat{D}\to\hat{D}' & \longrightarrow & \text{with i.t.q.n. connection}
\end{array}$$

The functor (1) is the forgetful functor induced by the homomorphism $\mathcal{O}_{S/\Sigma}\to\mathscr{A}$ giving the structure of $\mathcal{O}_{S/\Sigma}$-algebra on $\mathscr{A}$ (Proposition 2.1.5).

g) If in *f)* we take $J' = J + p\widetilde{A}$, then $\hat{D}\cong\hat{D}'$ and the map $\mathcal{O}_{S/\Sigma}\to\mathscr{A}$ is an isomorphism. (With the notation of the proof of 2.1.5, in this case $D\cong D(\widetilde{J})$.) Thus, in this situation, all horizontal arrows in (2.2.4.1) are equivalences. Hence to prove 2.2.2 we may assume $p\widetilde{A}\subset J$.

h) The construction $\mathscr{E}\mapsto M$ is functiorial with respect to the pair $(\widetilde{A}, J)$. If $(\widetilde{A}', J')$ is another pair as in 2.2.1 and $\varphi:\widetilde{A}\to\widetilde{A}'$ is a homomorphism with $\varphi(J)\subset J'$, then φ induces a homomorphism $\hat{D}\to\hat{D}'$ and $f: S'\to S$. The module $M\hat{\otimes}_{\hat{D}}\hat{D}'$ corresponds to $f^*_{\mathrm{crys}}(\mathscr{E})$ if M corresponds to $\mathscr{E}$.

Proof of 2.2.2 *and* 2.2.3. — Let us assume 2.2.3 and deduce 2.2.2 from it. By Remark 2.2.4 part *g)* we may assume there is a closed immersion $i : S\hookrightarrow\mathrm{Spec}(A)$. The crystal of quasi-coherent $\mathcal{O}_{\mathrm{Spec}(A)/\Sigma}$-modules $\mathscr{A}$ of Proposition 2.1.5 is determined by the $\widetilde{A}$-module $\hat{D}$ with its connection ∇ (2.2.1.1): the value of $\mathscr{A}$ on the triple

$$(\mathrm{Spec}(A), \mathrm{Spec}(\widetilde{A}/p^n\widetilde{A}), \gamma)$$

is $\hat{D}/p^n\hat{D}$ by 2.1.5 *b)*. Its algebra structure is determined by the horizontal homomorphism

$$\hat{D}\hat{\otimes}_{\widetilde{A}}\hat{D}\to\hat{D}.$$

By Proposition 2.1.5, to prove 2.2.2 for S, we have to characterize crystals of quasi-coherent $\mathscr{A}$-modules as $\hat{\mathrm{D}}$-modules with a connection. However, such a crystal $\mathscr{E}$ can be seen as a crystal of quasi-coherent $\mathcal{O}_{\mathrm{Spec(A)}/\Sigma}$-modules $\mathscr{E}$ together with a morphism of $\mathcal{O}_{\mathrm{Spec(A)}/\Sigma}$-modules

$$\mathscr{A} \otimes_{\mathcal{O}_{\mathrm{Spec(A)}/\Sigma}} \mathscr{E} \to \mathscr{E}$$

satisfying the usual conditions. Using 2.2.3 this translates into a complete $\tilde{\mathrm{A}}$-module M, with a topologically quasi-nilpotent and integrable connection ∇ and a horizontal homomorphism

$$\hat{\mathrm{D}} \hat{\otimes}_{\tilde{\mathrm{A}}} \mathrm{M} \to \mathrm{M}$$

(of $\tilde{\mathrm{A}}$-modules) satisfying certain conditions; these conditions are exactly that M becomes a $\hat{\mathrm{D}}$-module and the "horizontal" implies that ∇ on M is compatible with ∇ on $\hat{\mathrm{D}}$. In this way we get 2.2.2 from 2.2.3.

To prove 2.2.3 we use Lemma 1.3.3. It produces B, $\tilde{\mathrm{B}}$ and $\tilde{\mathrm{A}} \leftarrow \tilde{\mathrm{B}}$, $\tilde{\mathrm{A}} \to \tilde{\mathrm{B}}$ such that B has a finite p-basis. The result for S = Spec(B) is [BM, Proposition 1.3.3]. Using the functoriality 2.2.4 *h)* for Spec(A) $\leftarrow$ Spec(B), Spec(A) $\to$ Spec(B) we deduce 2.2.3 in general. □

2.3. *Dieudonné crystals and Dieudonné modules*

2.3.1. Let S $\to$ Spec($\mathbf{F}_p$) be a scheme of characteristic p. If $\mathscr{E}$ is a crystal of $\mathcal{O}_{\mathrm{S}/\Sigma}$-modules we denote by $\mathscr{E}^\sigma$ its inverse image under the absolute Frobenius morphism of S.

2.3.2. *Definition.* — A *Dieudonné crystal* over S is a triple $(\mathscr{E}, f, v)$ where:

(1) $\mathscr{E}$ is a crystal of finite locally free $\mathcal{O}_{\mathrm{S}/\Sigma}$-modules;

(2) $f : \mathscr{E}^\sigma \to \mathscr{E}$ and $v : \mathscr{E} \to \mathscr{E}^\sigma$ are homomorphisms of $\mathcal{O}_{\mathrm{S}/\Sigma}$-modules such that $f \circ v = p.\mathrm{id}_{\mathscr{E}}$ and $v \circ f = p.\mathrm{id}_{\mathscr{E}^\sigma}$.

It is clear what homomorphisms of such Dieudonné crystals should be. We point out that this definition is different from [BM, 2.4] where $\mathscr{E}$ is only supposed to be of finite presentation. Since we are mainly interested in p-divisible groups, we prefer Definition 2.3.2. (But see also Remark 2.4.10.)

2.3.3. In [BBM, 3.3.6 and 3.3.10] there is constructed a crystalline Dieudonné module functor:

$$\mathbf{D} : \begin{matrix}\text{Category of } p\text{-divisible} \\ \text{groups over S}\end{matrix} \longrightarrow \begin{matrix}\text{Category of Dieudonné} \\ \text{crystal over S}\end{matrix}$$

functorial with respect to the base scheme S.

2.3.4. *Definition.* — In the situation 2.2.1, suppose given a lift of Frobenius $\sigma : \widetilde{A} \to \widetilde{A}$ such that $\sigma(J) \subset J$; in particular it induces σ on the algebra $\hat{D}$ (2.2.1.1). A *Dieudonné module* over $\hat{D}$ is a quadruple

$$(M, \nabla, F, V)$$

where:

— M is finite locally free $\hat{D}$-module;
— $\nabla : M \to M \otimes_{\widetilde{A}} \hat{\Omega}^1_{\widetilde{A}}$ is an integrable, topologically quasi-nilpotent connection;
— $F : M \hat{\otimes}_\sigma \hat{D} \to M$ and $V : M \to M \hat{\otimes}_\sigma \hat{D}$ are $\hat{D}$-linear, horizontal and such that $F \circ V = p.\mathrm{id}_M$ and $V \circ F = p.\mathrm{id}_{M \hat{\otimes}_\sigma \hat{D}}$.

Clearly, the category of Dieudonné crystals over $S = \mathrm{Spec}(\widetilde{A}/J)$ is equivalent to the category of Dieudonné modules over $\hat{D}$ (use 2.2.2 and 2.2.4 *h)*). By Definition 2.3.2 this statement makes sense only for S with $p\mathcal{O}_S = 0$, but we can extend Definition 2.3.2 to general S by relating crystals on S to crystals on $S \times \mathrm{Spec}(\mathbf{F}_p)$ (compare 2.2.4 *g)*).

2.4. *Formal schemes, p-divisible groups and Dieudonné crystals*

2.4.1. Our formal schemes will always be adic, locally Noetherian formal schemes. Hence such a formal scheme $\mathfrak{X}$ has a biggest ideal of definition $\mathcal{I} \subset \mathcal{O}_{\mathfrak{X}}$. We can write

$$\mathfrak{X} = \varinjlim \mathfrak{X}_n$$

where $\mathfrak{X}_n$ is the scheme $\mathrm{Spec}(\mathcal{O}_{\mathfrak{X}}/\mathcal{I}^n)$. The reduction $\mathfrak{X}_{\mathrm{red}}$ of $\mathfrak{X}$ is the reduced scheme $\mathfrak{X}_1 = \mathrm{Spec}(\mathcal{O}_{\mathfrak{X}}/\mathcal{I})$.

In order to have a Dieudonné module functor over formal schemes, we simply adapt our definitions.

2.4.2. *Definition.* — Let $\mathfrak{X}$ be a formal scheme.

a) A *p-divisible group* G over $\mathfrak{X}$ is a system $(G_n)_{n \geqslant 1}$ of p-divisible groups G_n over $\mathfrak{X}_n$ endowed with isomorphisms $G_{n+1}|_{\mathfrak{X}_n} \cong G_n$.

b) Suppose $\mathfrak{X} \to \mathrm{Spec}(\mathbf{F}_p)$ lies in characteristic p. A *Dieudonné crystal* $\mathcal{E}$ over $\mathfrak{X}$ is a system $(\mathcal{E}_n)_{n \geqslant 1}$ of Dieudonné crystals $\mathcal{E}_n$ over $\mathfrak{X}_n$ endowed with isomorphisms $\mathcal{E}_{n+1}|_{\mathrm{CRIS}(\mathfrak{X}_n/\Sigma)} \cong \mathcal{E}_n$.

2.4.3. It is clear that 2.3.3 extends to a functor

$$\mathbf{D} : \begin{matrix}\text{Category of } p\text{-divisible} \\ \text{groups over } \mathfrak{X}\end{matrix} \longrightarrow \begin{matrix}\text{Category of Dieudonné} \\ \text{crystals over } \mathfrak{X}\end{matrix}$$

($\mathfrak{X} \to \mathrm{Spec}(\mathbf{F}_p)$ in characteristic p). For a morphism of formal schemes $\mathfrak{X} \xrightarrow{\varphi} \mathfrak{Y}$ there are pullback functors φ^* and there is an isomorphism $\mathbf{D} \circ \varphi^* \cong \varphi^* \circ \mathbf{D}$.

2.4.4. *Lemma.* — *If* $\mathfrak{X} = \mathrm{Spf}(A)$ *is an affine formal scheme then the functor*

$$\begin{matrix} \textit{Category of } p\textit{-divisible} \\ \textit{groups over } \mathrm{Spec}(A) \end{matrix} \longrightarrow \begin{matrix} \textit{Category of } p\textit{-divisible} \\ \textit{groups over } \mathfrak{X} = \mathrm{Spf}(A) \end{matrix}$$

is an equivalence.

Proof. — Suppose G is a p-divisible group of $\mathfrak{X}$; we construct a p-divisible group over Spec(A). The group G is given by p-divisible groups G_n over $\mathfrak{X}_n = \mathrm{Spec}(A/I^n)$. The schemes $G_n[p^k] := \mathrm{Ker}(p^k : G_n \to G_n)$ are given as $\mathrm{Spec}(B_{n,k})$, where $B_{n,k}$ is a finite locally free A/I^n-algebra of rank $p^{k\text{-height}(G)}$. The maps $B_{n+1,k} \to B_{n,k}$ being surjective (see Definition 2.4.2), we see that

$$B_k := \varprojlim_n B_{n,k}$$

is a finite locally free A-algebra of rank $p^{k\text{-height}(G)}$. The p-divisible group we are looking for is $\bigcup_k \mathrm{Spec}(B_k)$. □

The corresponding assertion for crystals of quasi-coherent $\mathcal{O}_{\mathfrak{X}/\Sigma}$-modules (defined as in 2.4.2) is not true. A counterexample can be given with $\mathfrak{X} = \mathrm{Spf}(\mathbf{F}_p[[t]])$. The ring $A = \mathbf{F}_p[[t]]$ has $\{ t \}$ as a p-basis. The free rank 1 A-module M = A with operator ∇ given by $a \mapsto t^{p+1}\, \partial a/\partial t \otimes dt$ does not define a crystal on Spec(A) since $t^{p+1}\, \partial/\partial t$ is not quasi-nilpotent on A, see 2.2.4 *c)*. However, for each n, the pair (M, ∇) defines a crystal on $\mathrm{Spec}(\mathbf{F}_p[T]/(T^n))$. We prove the corresponding assertion for Dieudonné crystals only in the special case where A satisfies the condition of smoothness that was studied in 1.3. Let us give the corresponding definition for formal schemes.

2.4.5. *Definition.* — We say that a formal scheme $\mathfrak{X} \to \mathrm{Spec}(\mathbf{F}_p)$ of characteristic p has property (†) if it satisfies one of the following two equivalent conditions:

(2.4.5.1) There is a covering $\mathfrak{X} = \bigcup \mathrm{Spf}(A_i)$ by affines of $\mathfrak{X}$ and each $\mathbf{F}_p$-algebra A_i satisfies (1.3.1.1).

(2.4.5.2) The morphism $\mathfrak{X} \to \mathrm{Spec}(\mathbf{F}_p)$ is formally smooth and the scheme $\mathfrak{X}_{\mathrm{red}}$ is locally of the form $\mathrm{Spec}(\widetilde{A})$, where $\widetilde{A}$ is an algebra of finite type over a field with a finite p-basis.

2.4.6. *Remark.* — We leave it to the reader to find the definition of a formally smooth morphism of formal schemes (used in (2.4.5.2)) generalizing both [EGA, IV 17.1.1] and [EGA, 0_{IV} 19.3.1].

2.4.7. *Examples.* — Let k be a field with $\mathrm{char}(k) = p$ and $[k : k^p] < \infty$.

(2.4.7.1) Any formal scheme which is the completion of a scheme smooth over Spec(k) in a closed subscheme satisfies (†).

(2.4.7.2) Any scheme X which is locally of finite type over Spec(k) and regular satisfies (†). (See (1.3.2.2); X is considered as a formal scheme with $X = X_{\mathrm{red}}$.) In particular, we can take $X = \mathrm{Spec}(k)$.

2.4.8. Proposition. — *If* $\mathfrak{X} = \mathrm{Spf}(A)$ *with* (A, I) *as in* (1.3.1.1), *then the functor*

(2.4.8.1)
$$\begin{matrix}\textit{Category of Dieudonné} \\ \textit{crystals over } \mathrm{Spec}(A)\end{matrix} \longrightarrow \begin{matrix}\textit{Category of Dieudonné} \\ \textit{crystals over } \mathrm{Spf}(A)\end{matrix}$$

is an equivalence of categories.

Proof. — Suppose $(\widetilde{A}, \sigma)$ is a lift of A (1.3.3 and 1.2.2). Let $\widetilde{I} \subset \widetilde{A}$ denote the inverse image of I in A. Let $\hat{D}_n$ be the divided power algebra constructed in (2.2.1.1) for the ideal $J = \widetilde{I}^n \subset \widetilde{A}$; it is isomorphic to the divided power algebra $\hat{D}'_n$ constructed for the ideal $J' = \widetilde{I}^n + p\widetilde{A}$. Therefore, $\hat{D}_n$ is the ring figuring in 2.2.2 in the description of crystals on $S_n = \mathrm{Spec}(\widetilde{A}/J') = \mathrm{Spec}(A/I^n)$. The homomorphism σ induces $\sigma : \hat{D}_n \to \hat{D}_n$. Finally, we denote by $i_n : S_n \to S_{n+1}$ the closed immersion induced by $A/I^{n+1} \to A/I^n$; there is also a corresponding homomorphism $\hat{D}_{n+1} \to \hat{D}_n$.

A Dieudonné crystal on $\mathfrak{X}$ is given by Dieudonné crystals $(\mathscr{E}_n, f_n, v_n)$ on S_n, plus isomorphisms $i^*_{n, \mathrm{CRIS}}(\mathscr{E}_{n+1}) \cong \mathscr{E}_n$ of Dieudonné crystals. By 2.2.2 this translates into Dieudonné modules $(M_n, \nabla_n, F_n, V_n)$ over $\hat{D}_n$, plus isomorphisms of Dieudonné modules

(2.4.8.2) $\quad M_{n+1} \hat{\otimes} \hat{D}_n \to M_n.$

Let us consider the surjection

(2.4.8.3) $\quad \hat{D}_n \to \widetilde{A}/\widetilde{I}^n.$

Tensoring with it we get from the $\hat{D}_n$-module M_n a finite locally free $\widetilde{A}/\widetilde{I}^n$-module

$$N_n := M_n \otimes \widetilde{A}/\widetilde{I}^n.$$

On N_n there is an integrable connection $\nabla_n : N_n \to N_n \otimes \Omega^1_{\widetilde{A}/\widetilde{I}^n}$ coming from ∇_n on M_n since by (2.2.1.3) we have

$$\nabla(\mathrm{Ker}(2.4.8.3)) \subset \mathrm{Ker}(\hat{D}_n \otimes \hat{\Omega}^1_{\widetilde{A}} \to \Omega^1_{\widetilde{A}/\widetilde{I}^n}).$$

The maps F_n and V_n induce horizontal homomorphisms

$$F_n : N_n \otimes_\sigma \widetilde{A}/\widetilde{I}^n \to N_n \quad \text{and} \quad V_n : N_n \to N_n \otimes_\sigma \widetilde{A}/\widetilde{I}^n$$

satisfying $F_n V_n = p$ and $V_n F_n = p$. Finally, the isomorphisms (2.4.8.2) give homomorphisms $N_{n+1} \to N_n$ compatible with ∇, F and V, inducing isomorphisms $N_{n+1} \otimes \widetilde{A}/\widetilde{I}^n \cong N_n$.

Thus the module

$$N := \varprojlim N_n$$

is a finite locally free $\widetilde{A}$-module ($\widetilde{A}$ is complete for the $\widetilde{I}$-adic topology, Remark 1.3.4 *a)*). It is endowed with an integrable connection $\nabla = \lim \nabla_n$ ($\lim \Omega^1_{\widetilde{A}/\widetilde{I}^n} = \hat{\Omega}^1_{\widetilde{A}}$, Remark 1.3.4 *c)*). There are horizontal homomorphisms

$$F = \varprojlim F_n : N^\sigma = N \otimes_\sigma \widetilde{A} \to N$$

and

$$V = \varprojlim V_n : N \to N^\sigma = N \otimes_\sigma \widetilde{A}$$

which satisfy $FV = p$ and $VF = p$. To conclude that (N, ∇, F, V) is a Dieudonné module over $\widetilde{A}$ we still have to prove that ∇ is topologically quasi-nilpotent. The maps F an V give a complex

$$(2.4.8.4)\qquad \ldots \to N^{\sigma}/pN^{\sigma} \xrightarrow{F} N/pN \xrightarrow{V} N^{\sigma}/pN^{\sigma} \xrightarrow{F} N/pN \to \ldots$$

which is automatically exact: for example, if $x \in N/p^2 N$ is such that $V(x) = py$ for some $y \in N^{\sigma}/p^2 N^{\sigma}$, then $x \equiv F(y) \bmod pN$. For any $\theta \in \mathrm{Hom}_{\widetilde{A}}(\hat{\Omega}^1_{\widetilde{A}}, \widetilde{A})$ the operator ∇_θ is nilpotent of order at most p on N^{σ}/pN^{σ}, as this module has generators of the form $x \otimes 1$ and $\nabla_\theta(x \otimes 1) = 0$. By exactness of the complex above ∇_θ is nilpotent of order at most $2p$ on N/pN.

Conclusion: we have constructed a functor which goes from right to left in (2.4.8.1). We claim that it is quasi-inverse to (2.4.8.1). We only show how to get a functorial isomorphism of Dieudonné modules

$$(2.4.8.5)\qquad N \hat{\otimes}_{\widetilde{A}} \hat{D}_k \to M_k, \qquad \forall\, k \in \mathbf{N}$$

and leave it to the reader to deduce the claim from it. To construct (2.4.8.5) it is enough to give a homomorphism $N \to M_k$ compatible with ∇, F and V fitting in a commutative diagram:

$$\begin{array}{ccc} N & \longrightarrow & M_k \\ & \searrow \quad \swarrow & \\ & N_k & \end{array}$$

Let $f_1, \ldots, f_r \in \widetilde{A}$ be generators of the ideal $\widetilde{I}^k$. Take $x \in N$:

$$x = \varprojlim x_n \in \varprojlim N_n = N.$$

(2.4.8.6) For any l, if $n \geqslant (r(p-1) + pl)\, k$ then the element

$$y_l = \text{image of } \widetilde{x}_n \text{ in } M_k/p^l M_k$$

does not depend on the lift $\widetilde{x}_n \in M_n$ of the element $x_n \in N_n$. (The image is taken via the maps

$$M_n \to M_{n-1} \to \ldots \to M_k;$$

in particular y_l also does not depend on $n \gg 0$.)

Clearly, $N \to M_k$, $x = (x_n) \mapsto (y_l) \in \lim M_k/p^l M_k = M_k$ is the desired map.

Proof of (2.4.8.6). — Since $\widetilde{I}^k$ is generated by the r elements $f_1, \ldots, f_r$, $\widetilde{I}^n$ is generated by elements of the form

$$j = i \cdot f_1^{\alpha_1} \cdot \ldots \cdot f_r^{\alpha_r} \cdot (f_1^{\beta_1} \cdot \ldots \cdot f_r^{\beta_r})^p$$

with $i \in \widetilde{\mathrm{I}}^{n-(r(p-1)+pl)k}$, $0 \leqslant \alpha_i < p$, $0 \leqslant \beta_i$ and $\Sigma\alpha_i + p(\Sigma\beta_i) = r(p-1) + pl$. This implies $\Sigma\beta_i \geqslant l$. The image of such an element j in $\hat{\mathrm{D}}_k$ is divisible by p^l:

$$j = (p!)^{\Sigma\beta_i} \cdot i \cdot f_1^{\alpha_1} \cdot \ldots \cdot f_r^{\alpha_r} \cdot (f_1^{[p]})^{\beta_1} \cdot \ldots \cdot (f_r^{[p]})^{\beta_r} \in \hat{\mathrm{D}}_k.$$

Therefore, the image in $\hat{\mathrm{D}}_k$ of $\hat{\mathrm{J}}_n = \mathrm{Ker}(\hat{\mathrm{D}}_n \to \widetilde{\mathrm{A}}/\widetilde{\mathrm{I}}^n)$ is contained in $p^l\,\hat{\mathrm{D}}_k$. (The elements $j^{[m]}$, with j as above generate $\hat{\mathrm{J}}_n$.) This implies (2.4.8.6) since

$$\mathrm{Ker}(\mathrm{M}_n \to \mathrm{N}_n) = \hat{\mathrm{J}}_n . \mathrm{M}_n. \quad \square$$

2.4.9. *Corollary.* — *Over a formal scheme $\mathfrak{X}$ of characteristic p which satisfies* (†) *(Definition* 2.4.5*) the crystalline Dieudonné module functor* 2.4.3 *is fully faithful.*

Proof. — The question is local on $\mathfrak{X}$ so we may assume that $\mathfrak{X} = \mathrm{Spf(A)}$ with A as in (1.3.1.1). By 2.4.4 and 2.4.8 we may replace $\mathfrak{X}$ by $\mathrm{X} = \mathrm{Spec(A)}$ and the functor 2.4.3 by 2.3.3. Next, we choose an $\mathbf{F}_p$-algebra B as in Lemma 1.3.3. A formal argument shows that it suffices to demonstrate fully faithfulness of **D** over Spec(B). In this case we can apply [BM, Theorem 4.1.1]. □

2.4.10. *Remark.* — With the appropriate definitions, the functor **D** 2.4.3 is also fully faithful on finite flat group schemes over $\mathfrak{X}$ as in 2.4.5. That is, in the definition 2.3.2 of Dieudonné crystals we have to work with crystals of $\mathcal{O}_{\mathrm{S}/\Sigma}$-modules of finite presentation and in the definition 2.3.4 of Dieudonné modules we have to work with $\hat{\mathrm{D}}$-modules of finite presentation. The proof of 2.4.4 works for finite flat group schemes as well and the proof of 2.4.8 still works to show that the functor (2.4.8.1) on Dieudonné crystals of finite presentation is fully faithful. Hence, the proof of 2.4.9 shows that **D** is fully faithful on finite flat group schemes over $\mathfrak{X}$.

In the situation of Dieudonné crystals of finite presentation the proof of 2.4.8 no longer shows that (2.4.8.1) is an equivalence. Indeed, the argument to prove that ∇ on N is topologically nilpotent fails: it is no longer clear that (2.4.8.4) is exact if N is just some $\widetilde{\mathrm{A}}$-module of finite presentation. To remedy this, we introduce *truncated Dieudonné crystals of level n*: these are triples $(\mathscr{E}, f, v)$, where $\mathscr{E}$ is a crystal of finite locally free $\mathcal{O}_{\mathrm{S}/\Sigma}/p^n\,\mathcal{O}_{\mathrm{S}/\Sigma}$-modules and f and v are as usual. In the case $n = 1$ we have to add the condition that the sequence

$$(2.4.10.1) \qquad \ldots \to \mathscr{E}^\sigma \xrightarrow{f} \mathscr{E} \xrightarrow{v} \mathscr{E}^\sigma \xrightarrow{f} \mathscr{E} \to \ldots$$

be exact. The reduction modulo p : $(\mathscr{E}/p\mathscr{E}, f, v)$ of a truncated Dieudonné crystal of level n is a truncated Dieudonné crystal of level 1. By Nakayama's lemma exactness of (2.4.10.1) need only be checked in the case that S is the spectrum of a field and here we can use the same argument as was used to prove exactness of (2.4.8.4). Hence for these truncated Dieudonné crystals Proposition 2.4.8 will be true.

2.5. *Filtration and the Kodaira-Spencer map*

2.5.1. In this section $\mathfrak{X}$ will be a formal scheme over $\mathrm{Spec}(\mathbf{F}_p)$ satisfying (†) and $\mathscr{E}$ will be a Dieudonné crystal over $\mathfrak{X}$. In this situation we put

$$\mathscr{E}_{\mathfrak{X}} := \varprojlim \mathscr{E}_{\mathfrak{X}_n} := \varprojlim \mathscr{E}_{n, \mathfrak{X}_n}$$

if $\{(\mathscr{E}_n, f_n, v_n)\}_{n\geqslant 1}$ is the system defining $\mathscr{E}$. The sheaf $\mathscr{E}_{\mathfrak{X}}$ is a finite locally free $\mathcal{O}_{\mathfrak{X}}$-module. There is an integrable (quasi-nilpotent) connection

$$\nabla : \mathscr{E}_{\mathfrak{X}} \to \mathscr{E}_{\mathfrak{X}} \otimes_{\mathcal{O}_{\mathfrak{X}}} \hat{\Omega}^1_{\mathfrak{X}}. \tag{2.5.1.1}$$

Here we have put

$$\hat{\Omega}^1_{\mathfrak{X}} = \varprojlim \Omega^1_{\mathfrak{X}_n/\mathbf{F}_p}.$$

It is a finite locally free $\mathcal{O}_{\mathfrak{X}}$-module (compare 1.3.3). The connection (2.5.1.1) may be constructed locally on $\mathfrak{X}$ by 2.2.2 or by using the connections ∇_n on $\mathscr{E}_{n, \mathfrak{X}_n}$ coming from the fact that $\mathscr{E}_n$ is a crystal. Finally, there is an exact sequence (compare (2.4.8.4)):

$$\ldots \to \mathscr{E}^{(p)} \xrightarrow{f} \mathscr{E} \xrightarrow{v} \mathscr{E}^{(p)} \xrightarrow{f} \mathscr{E} \to \ldots \tag{2.5.1.2}$$

with $\mathscr{E}^{(p)}_{\mathfrak{X}} = \mathrm{Frob}^*_{\mathfrak{X}}(\mathscr{E}_{\mathfrak{X}})$ and where f (resp. v) is deduced from $(f_n)_{n\geqslant 0}$ (resp. $(v_n)_{n\geqslant 0}$). In particular the homomorphisms f and v are horizontal, for example $f(\nabla^{(p)}(s)) = \nabla(f(s))$ for a local section s of $\mathscr{E}^{(p)}_{\mathfrak{X}}$.

2.5.2. *Proposition.* — *There exists a locally free, locally direct summand* $\omega \subset \mathscr{E}_{\mathfrak{X}}$ *such that* $\omega^{(p)} \subset \mathscr{E}^{(p)}_{\mathfrak{X}}$ *is equal to* $\mathrm{Im}(v) = \mathrm{Ker}(f)$. *It is uniquely determined by these conditions.*

Proof. — For any closed point $x \in \mathfrak{X}$ the sequence (2.5.1.2) restricted to x is also exact. Indeed, this sequence is just the sequence associated to the Dieudonné crystal $\mathscr{E}|_{\mathrm{CRIS}(x/\Sigma)}$ over x. Thus $\mathrm{Im}(v)$ is a locally direct summand of $\mathscr{E}^{(p)}_{\mathfrak{X}}$. As v is horizontal, $\mathrm{Im}(v)$ is invariant under $\nabla^{(p)}$ and hence is equipped with a connection whose p-curvature [K] is zero (the p-curvature of $\nabla^{(p)}$ is zero). Thus we are done if we prove that the functor $\mathrm{Frob}^*_{\mathfrak{X}}$ induces an equivalence of categories

$$\begin{matrix}\text{finite locally} \\ \text{free } \mathcal{O}_{\mathfrak{X}}\text{-modules}\end{matrix} \longrightarrow \begin{matrix}\text{finite locally free } \mathcal{O}_{\mathfrak{X}}\text{-modules with} \\ \text{a connection whose } p\text{-curvature is zero.}\end{matrix}$$

This is [K, Theorem 5.1] in the case of schemes smooth over $\mathbf{F}_p$; exactly the same proof works for schemes having locally a finite p-basis. But we can reduce our case to this case by Lemma 1.3.3. □

2.5.3. We always denote by ω the subsheaf of $\mathscr{E}_{\mathfrak{X}}$ found in 2.5.2 and write α for the quotient: $\alpha := \mathscr{E}_{\mathfrak{X}}/\omega$. It is a finite locally free $\mathcal{O}_{\mathfrak{X}}$-module. The connection induces an $\mathcal{O}_{\mathfrak{X}}$-linear homomorphism

$$\omega \to \alpha \otimes_{\mathcal{O}_{\mathfrak{X}}} \hat{\Omega}^1_{\mathfrak{X}}. \tag{2.5.3.1}$$

We write $\Theta_{\mathfrak{X}} := \mathscr{H}om_{\mathcal{O}_{\mathfrak{X}}}(\hat{\Omega}^1_{\mathfrak{X}}, \mathcal{O}_{\mathfrak{X}})$ for the sheaf of vector fields on $\mathfrak{X}$.

2.5.4. *Definition.* — The *Kodaira-Spencer map* of the Dieudonné crystal $\mathscr{E}$ is the homomorphism of $\mathcal{O}_{\mathfrak{X}}$-modules

$$\kappa_{\mathscr{E}} : \Theta_{\mathfrak{X}} \to \mathscr{H}om_{\mathcal{O}_{\mathfrak{X}}}(\omega, \alpha)$$

deduced from (2.5.3.1). We say that $\mathscr{E}$ is *versal* if $\kappa_{\mathscr{E}}$ is a surjection.

2.5.5. If $\mathscr{E} = \mathbf{D}(G)$, the Dieudonné crystal associated to the p-divisible group G, then the filtration

$$0 \to \omega \to \mathscr{E}_{\mathfrak{X}} \to \alpha \to 0$$

is nothing but the Hodge-filtration of $\mathbf{D}(G)_{\mathfrak{X}}$ (see [BBM, 3.3.5 and 4.3.10]). We will say that G is versal if $\mathbf{D}(G)$ is versal. It follows from [BM, Corollary 3.2.11] and [M, V Theorem 1.6] that if G is versal then the restrictions of G to complete local rings of $\mathfrak{X}$ are versal deformations.

3. Deformations of Dieudonné crystals and p-divisible groups

In this chapter we construct versal deformations of Dieudonné crystals and p-divisible groups, at least over formal schemes $\mathfrak{X}$ satisfying (†).

3.1. *Deformations of Dieudonné modules*

3.1.1. *Notations.* — In this section (A_0, I_0) is a pair as in (1.3.1.1) and $(\tilde{A}_0, \sigma)$ is a lift of A_0. We are going to look at $A := A_0[[x_1, \ldots, x_n]]$ and $\tilde{A} := \tilde{A}_0[[x_1, \ldots, x_n]]$. We extend σ to $\tilde{A}$ by putting $\sigma(x_i) = x_i^p$. Further we fix a natural number $m \in \mathbf{N}$. Put

$$B := A/(x_1^m, \ldots, x_n^m) = \tilde{A}/(p, x_1^m, \ldots, x_n^m)$$

and

$$B_+ := A/(x_1^{m+1}, \ldots, x_n^{m+1}) = \tilde{A}/(p, x_1^{m+1}, \ldots, x_n^{m+1}).$$

The divided power algebra (2.2.1.1) associated to the ideal $J = (p, x_1^m, \ldots, x_n^m)$ of $\tilde{A}$ (resp. $J_+ = (p, x_1^{m+1}, \ldots, x_n^{m+1})$ of $\tilde{A}$) is denoted by $\hat{D}$ (resp. $\hat{D}_+$).

For a multi-index $I = (i_1, \ldots, i_n)$ we define

$$x^{[I/m]} := \prod_{\alpha=1}^{n} x_\alpha^{i_\alpha - m[i_\alpha/m]} \cdot (x_\alpha^m)^{[[i_\alpha/m]]}.$$

(The symbol $\lfloor \ \rfloor$ denotes the integer part function $\mathbf{R} \to \mathbf{Z}$.) It is easily seen that $\hat{\mathrm{D}}$ has the following description:

$$\hat{\mathrm{D}} = \{ \sum_{\mathrm{I}} a_{\mathrm{I}} . x^{[\mathrm{I}/m]} \mid a_{\mathrm{I}} \in \tilde{\mathrm{A}}_0 \text{ and } \lim_{|\mathrm{I}| \to \infty} a_{\mathrm{I}} = 0 \text{ in the } p\text{-adic topology} \}$$

A similar description can be given for $\hat{\mathrm{D}}_+$. From this it follows that both $\hat{\mathrm{D}}$ and $\hat{\mathrm{D}}_+$ are p-torsion free.

We introduce a (large) number of ring homomorphisms.

1. The maps $\hat{\mathrm{D}} \to \mathrm{B}$ and $\hat{\mathrm{D}}_+ \to \mathrm{B}_+$. (Reduction modulo the divided power ideals.)
2. The maps $\mathrm{B}_+ \to \mathrm{B}$ and $\hat{\mathrm{D}}_+ \to \hat{\mathrm{D}}$. (Induced by $\mathrm{J}_+ \subset \mathrm{J}$.)
3. A map $\pi : \hat{\mathrm{D}} \to \mathrm{B}_+$. It is the unique $\tilde{\mathrm{A}}_0$-linear map, such that $\pi(x_i) =$ class of x and $\pi((x_i^m)^{[l]}) = 0$ for $l \geqslant 2$. The universal property of divided power envelopes implies that π is defined by the trivial divided power structure on the square zero ideal J/J_+.
4. The maps induced by σ on $\hat{\mathrm{D}}$ and $\hat{\mathrm{D}}_+$, also denoted σ, i.e., $\sigma : \hat{\mathrm{D}} \to \hat{\mathrm{D}}$ and $\sigma : \hat{\mathrm{D}}_+ \to \hat{\mathrm{D}}_+$.
5. A homomorphism $\tau : \hat{\mathrm{D}} \to \hat{\mathrm{D}}_+$. It is defined as the σ-linear $\tilde{\mathrm{A}}_0$-module morphism such that $\tau(x_i) = x_i^p$ and such that it is a divided power homomorphism. Thus $\tau((x_i^m)^{[l]}) = x^{\{(p-1)m-1\}l} . (x_i^{m+1})^{[l]}$.
6. The maps (cf. [BBM (4.3.4.1)]) $\Phi : \mathrm{B} \to \bar{\mathrm{D}} := \hat{\mathrm{D}}/p\hat{\mathrm{D}}$ and $\Phi_+ : \mathrm{B}_+ \to \bar{\mathrm{D}}_+ := \hat{\mathrm{D}}_+/p\hat{\mathrm{D}}_+$. These are the σ-linear $\tilde{\mathrm{A}}_0$-module morphisms such that $\Phi(x_i) =$ class of x_i^p and $\Phi_+(x_i) =$ class of x_i^p.

With these definitions and notation we have the following commutative diagrams: (the bar $^-$ stands for reduction modulo p).

(3.1.1.1)

$$\begin{array}{ccccc} \hat{\mathrm{D}}_+ & \longrightarrow & \hat{\mathrm{D}} & \xrightarrow{\tau} & \hat{\mathrm{D}}_+ \\ & & \downarrow{\scriptstyle\pi} & & \downarrow \\ & & \mathrm{B}_+ & \xrightarrow{\Phi_+} & \bar{\mathrm{D}}_+ \end{array} \qquad \begin{array}{ccc} \hat{\mathrm{D}}_+ & \longrightarrow & \hat{\mathrm{D}} \\ & \searrow{\scriptstyle\sigma} & \downarrow{\scriptstyle\tau} \\ & & \hat{\mathrm{D}}_+ \end{array}$$

and

(3.1.1.2)

$$\begin{array}{ccc} \bar{\mathrm{D}}_+ & \longrightarrow & \bar{\mathrm{D}} \\ \downarrow & \swarrow{\scriptstyle\pi} & \downarrow \\ \mathrm{B}_+ & \longrightarrow & \mathrm{B} \\ \downarrow{\scriptstyle\Phi_+} & & \downarrow{\scriptstyle\Phi} \\ \bar{\mathrm{D}}_+ & \longrightarrow & \bar{\mathrm{D}} \end{array} \qquad \begin{array}{ccc} \bar{\mathrm{D}} & \longrightarrow & \mathrm{B} \\ & \searrow{\scriptstyle\bar{\sigma}} & \downarrow{\scriptstyle\Phi} \\ & & \bar{\mathrm{D}} \end{array} \qquad \begin{array}{ccc} \bar{\mathrm{D}}_+ & \longrightarrow & \mathrm{B}_+ \\ & \searrow{\scriptstyle\bar{\sigma}_+} & \downarrow{\scriptstyle\Phi_+} \\ & & \bar{\mathrm{D}}_+ \end{array}$$

In diagram (3.1.1.1) the composition $\hat{\mathrm{D}}_+ \to \hat{\mathrm{D}} \xrightarrow{\pi} \mathrm{B}_+$ is the reduction map 1. Finally, we remark that the connection ∇ on $\hat{\mathrm{D}}$ satisfies

$$\nabla(x_i^m)^{[n]} = (x_i^m)^{[n-1]} \, dx_i^m = m x_i^{m-1} (x_i^m)^{[n-1]} \, dx_i$$

and that a similar assertion holds for ∇_+. From this it is easily seen that

(3.1.1.3) $$\nabla(\mathrm{Ker}(\pi)) \subset \mathrm{Ker}(\hat{\mathrm{D}} \to \mathrm{B}) \otimes \hat{\Omega}^1_{\tilde{\mathrm{A}}}.$$

It does not seem possible to describe deformations of a Dieudonné module from $\hat{\mathrm{D}}$ to $\hat{\mathrm{D}}_+$, or even to deform them at all in some cases. Therefore we introduce the notion of a special Dieudonné module.

3.1.2. *Definition.* — A *special Dieudonné module* over $\hat{\mathrm{D}}$ is a six-tuple $(\mathrm{N}, \nabla, \mathrm{F}, \mathrm{V}, \alpha, \chi)$ where

— $(\mathrm{N}, \nabla, \mathrm{F}, \mathrm{V})$ is a Dieudonné module over $\hat{\mathrm{D}}$,
— α is a finite locally free B-module,
— $\chi : \mathrm{N} \otimes_{\hat{\mathrm{D}}} \mathrm{B} \to \alpha$ is a surjection,

these data being subject to the condition that the sequence

(3.1.2.1) $$0 \longrightarrow \mathrm{N} \xrightarrow{\mathrm{V}} \mathrm{N} \otimes_\sigma \hat{\mathrm{D}} \xrightarrow{\chi \otimes \mathrm{id}_{\hat{\mathrm{D}}}} \alpha \otimes_\Phi \bar{\mathrm{D}} \to 0$$

is exact.

Note that in (3.1.2.1) the map V is automatically injective (since $\hat{\mathrm{D}}$ is p-torsion free) and that $\chi \otimes \mathrm{id}_{\hat{\mathrm{D}}}$ is automatically surjective. Of course, the same definition gives the notion of a special Dieudonné module over $\hat{\mathrm{D}}_+$.

We will say that a homomorphism of B_+-modules

$$\chi' : \mathrm{N} \otimes_\pi \mathrm{B}_+ \to \alpha_+$$

lifts the homomorphism χ of Definition 3.1.2 if the following conditions are satisfied:

(3.1.2.2) α_+ is a finite locally free B_+-module,
(3.1.2.3) χ' is surjective,
(3.1.2.4) there is a surjection $\alpha_+ \to \alpha$ inducing an isomorphism $\alpha_+ \otimes \mathrm{B} \cong \alpha$ and making the following diagram commutative:

(3.1.2.5) $$\begin{array}{ccc} \mathrm{N} \otimes_\pi \mathrm{B}_+ & \xrightarrow{\chi'} & \alpha_+ \\ \downarrow & & \downarrow \\ \mathrm{N} \otimes \mathrm{B} & \xrightarrow{\chi} & \alpha. \end{array}$$

We remark that (3.1.2.3) follows from (3.1.2.2) and (3.1.2.4). It is important to note that given χ we can always find such a lift χ'. This is so since $\mathrm{N} \otimes_\pi \mathrm{B}_+$ is a locally free B_+-module lifting $\mathrm{N} \otimes \mathrm{B}$, hence the filtration defined by χ can be lifted.

3.1.3. *Proposition.* — *a) Given any special Dieudonné module* $(\mathrm{N}, \nabla, \mathrm{F}, \mathrm{V}, \alpha, \chi)$ *over* $\hat{\mathrm{D}}$ *and a lift* $\chi' : \mathrm{N} \otimes_\pi \mathrm{B}_+ \to \alpha_+$ *of* χ *as above, there exists a special Dieudonné module* $(\mathrm{N}_+, \nabla_+, \mathrm{F}_+, \mathrm{V}_+, \alpha_+, \chi_+)$ *over* $\hat{\mathrm{D}}_+$ *and an isomorphism of Dieudonné modules over* $\hat{\mathrm{D}}$:

$$\mathrm{N}_+ \hat{\otimes} \hat{\mathrm{D}} \to \mathrm{N}$$

such that χ_+ *is equal to the composition*

(3.1.3.1) $$N_+ \otimes B_+ \cong N_+ \otimes \hat{D} \otimes_\pi B_+ \cong N \otimes_\pi B_+ \xrightarrow{\chi'} \alpha_+.$$

The special Dieudonné module $(N_+, \nabla_+, F_+, V_+, \alpha_+, \chi_+)$ *is determined up to unique isomorphism by this.*

b) Any lift of $(N, \nabla, F, V, \alpha, \chi)$ *to a special Dieudonné module* $(N_+, \nabla_+, F_+, V_+, \alpha_+, \chi_+)$ *is determined by a lift* χ' *of* χ *as in part a).*

Proof. — Given $(N, \nabla, F, V, \alpha, \chi)$ and χ' as in *a)*, we define N_+ as the kernel of the homomorphism

(3.1.3.2) $$\begin{array}{ccc} N \hat{\otimes}_\tau \hat{D}_+ & \longrightarrow & \alpha_+ \otimes_{\Phi_+} \bar{D}_+ \\ & {\scriptstyle (3.1.1.1)}\searrow \quad \nearrow {\scriptstyle \chi' \otimes \mathrm{id}} & \\ & (N \otimes_\pi B_+) \otimes_{\Phi_+} \bar{D}_+ & \end{array}$$

It is a finite locally free $\hat{D}_+$-module by the following three facts: 1) α_+ is a finite locally free B_+-module, 2) the map (3.1.3.2) is surjective, 3) $\hat{D}_+$ is p-torsion free. The homomorphism τ is compatible with the connections ∇ and ∇_+ on $\hat{D}$ and $\hat{D}_+$, hence the connection ∇ on N induces a connection ∇'_+ on $N \hat{\otimes}_\tau \hat{D}_+$ compatible with ∇_+. If we define a connection on $\alpha_+ \otimes_{\Phi_+} \bar{D}_+$ by requiring that elements $a \otimes 1$ are horizontal, then it follows that (3.1.3.2) is compatible with these connections. In particular, we get a connection $\nabla_+ : N_+ \to N_+ \otimes \hat{\Omega}^1_{\tilde{A}}$. Tensoring the defining exact sequence

(3.1.3.3) $$0 \to N_+ \to N \hat{\otimes}_\tau \hat{D}_+ \to \alpha_+ \otimes_{\Phi_+} \bar{D}_+ \to 0$$

with $\hat{D}$ we get a commutative diagram:

(3.1.3.4) $$\begin{array}{ccccccccc} 0 & \longrightarrow & N_+ \otimes \hat{D} & \longrightarrow & N \otimes_\tau \hat{D}_+ \otimes \hat{D} & \longrightarrow & \alpha_+ \otimes_{\Phi_+} \bar{D}_+ \otimes \bar{D} & \longrightarrow & 0 \\ & & \uparrow ? & & \uparrow \cong & & \uparrow \cong & & \\ 0 & \longrightarrow & N & \xrightarrow{V} & N \otimes_\sigma \hat{D} & \longrightarrow & \alpha \otimes_\Phi \bar{D} & \longrightarrow & 0. \end{array}$$

The upper line is exact since $\hat{D}$ is p-torsion free. The isomorphisms of the right square and its commutativity are consequences of (3.1.1.1), (3.1.1.2) and (3.1.2.5). (The lower line is of course (3.1.2.1).) Thus we derive the existence of the left vertical arrow: $N \cong N_+ \otimes \hat{D}$. Using this we can write (use (3.1.1.1))

$$N \hat{\otimes}_\tau \hat{D}_+ \cong N_+ \hat{\otimes} \hat{D} \hat{\otimes}_\tau \hat{D}_+ \cong N_+ \hat{\otimes}_\sigma \hat{D}_+.$$

Fitting this into (3.1.3.3) we get

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N_+ & \longrightarrow & N \hat{\otimes}_\tau \hat{D}_+ & \longrightarrow & \alpha_+ \otimes_{\Phi_+} \bar{D}_+ & \longrightarrow & 0 \\ & & & {\scriptstyle V_+}\searrow & \downarrow \cong & \nearrow & & & \\ & & & & N_+ \hat{\otimes}_\sigma \hat{D}_+ & & & & \end{array}$$

The south-east arrow in this diagram is our definition of V_+. It is left to the reader to check that the north-east arrow is $\chi_+ \otimes \mathrm{id}_{\hat{D}_+}$ if we define χ_+ by (3.1.3.1). Hence we immediately get $(3.1.2.1)_+$ for $(N_+, \nabla_+, ?, V_+, \alpha_+, \chi_+)$. A consequence of this is that we can define F_+ as the map $x \mapsto V_+^{-1}(px)$. It is easy to see that the various morphisms written above are compatible with the corresponding connections. Hence, F_+ and V_+ are horizontal and $(N_+, \nabla_+, F_+, V_+)$ is a Dieudonné module. Finally, we have to check that $N \to N_+ \otimes \hat{D}$ is an isomorphism of Dieudonné modules. Since this involves writing down a large number of commutative diagrams, we feel this must be left to the reader.

The statement of unicity is easily proved, noting that for any solution $(N_+, \nabla_+, F_+, V_+, \alpha_+, \chi_+)$ of the problem, N_+ must be (canonically) isomorphic to the kernel of (3.1.3.2). This proves *a)*.

Part *b)* is proved in the same manner as the proof of unicity in part *a)*. □

3.1.4. *Remark.* — In the situation of 3.1.3 *a)* let

$$\omega_+ := \mathrm{Ker}(\chi' : N \otimes_\pi B_+ \to \alpha_+).$$

By (3.1.1.3) we get a Kodaira-Spencer map induced by ∇ on N:

$$\kappa : \omega_+ \to N \otimes_\pi B_+ \xrightarrow{\nabla} N \otimes B \otimes \hat{\Omega}^1_A \to \alpha_+ \otimes B \otimes \hat{\Omega}^1_A.$$

On the other hand, we get from ∇_+ a Kodaira-Spencer map

$$\kappa_+ : \omega_+ \to \alpha_+ \otimes \Omega^1_{B_+}.$$

It follows from the proof of 3.1.3 that κ and κ_+ agree as maps to $\alpha_+ \otimes B \otimes \hat{\Omega}^1_A$.

3.2. *Deformations of Dieudonné crystals and p-divisible groups*

In this section we use 3.1.3 to deform a p-divisible group given a deformation of its crystal. Further we deform any Dieudonné crystal to a versal one, at least locally on formal schemes satisfying (†). If $\mathfrak{X} \hookrightarrow \mathfrak{Y}$ is a closed immersion of formal schemes and $\mathscr{E}$ is a (Dieudonné) crystal on $\mathfrak{Y}$, we denote by $\mathscr{E}|_{\mathfrak{X}}$ the restriction of $\mathscr{E}$ to $\mathfrak{X}$, i.e., the pull-back of $\mathscr{E}$ via $\mathfrak{X} \hookrightarrow \mathfrak{Y}$.

3.2.1. *Proposition. — Suppose $\mathfrak{X} \hookrightarrow \mathfrak{Y}$ is a closed immersion of formal schemes satisfying (†), which induces an isomorphism $\mathfrak{X}_{\mathrm{red}} \cong \mathfrak{Y}_{\mathrm{red}}$. Given any p-divisible group G′ on $\mathfrak{X}$, a Dieudonné crystal $\mathscr{E}$ on $\mathfrak{Y}$ and an isomorphism $\mathbf{D}(G') \cong \mathscr{E}|_{\mathfrak{X}}$, there exists a p-divisible group G on $\mathfrak{Y}$ with $\mathbf{D}(G) \cong \mathscr{E}$.*

Proof. — Since $\mathbf{D}$ is fully faithful on $\mathfrak{Y}$ by 2.4.9, the problem is local. Hence we may assume that $\mathfrak{X} = \mathrm{Spf}(A_0)$, $\mathfrak{Y} = \mathrm{Spf}(A)$, $\mathfrak{X} \hookrightarrow \mathfrak{Y}$ is given by a surjection $A \to A_0$ and that A_0 satisfies (1.3.1.1). Exactly the same argument as used in the proof of Lemma 1.3.3 shows that there exists an isomorphism $A \cong A_0[[M]]$ (compatible with

$A \to A_0$) where M is a finite projective A_0-module. Localizing a bit more we may assume that $A \cong A_0[[x_1, \ldots, x_n]]$. Thus we are in the situation of 3.1.1. Let us use the notation introduced there.

By Lemma 2.4.4, G′ comes from a p-divisible group (denoted G_0) over $\mathrm{Spec}(A_0)$ and by 2.4.8, $\mathscr{E}$ comes from a Dieudonné crystal (also denoted $\mathscr{E}$) over Spec(A). By 2.5.2 the Dieudonné module (M, ∇, F, V) over $\widetilde{A}$ associated to $\mathscr{E}$ over $\mathfrak{Y}$ has a unique structure of a special Dieudonné module (3.1.2) $\mathbf{M} = (M, \nabla, F, V, \alpha, \chi)$ over $\widetilde{A}$.

Next, we remark that any p-divisible group G over Spec(B) (B as in 3.1.1, with m arbitrary) gives rise to a special Dieudonné module over $\hat{D}$; we write

$$\mathbf{M}(G) = (M(G), \nabla(G), F(G), V(G), \alpha(G), \chi(G))$$

for it. The Dieudonné module $(M(G), \nabla(G), F(G), V(G))$ is just the Dieudonné module associated to the Dieudonné crystal of G. The B-module $\alpha(G)$ is the module which gives rise to the sheaf $\mathscr{L}ie(G^*)$ on $S_{\mathrm{fppf}} = \mathrm{Spec}(B)_{\mathrm{fppf}}$ and $\chi(G)$ is the map induced by $\mathbf{D}(G)_S \to \mathscr{L}ie(G^*)$, the Hodge filtration, see [BBM, 3.3.5]. That $\mathbf{M}(G)$ satisfies (3.1.2.1) is a consequence of [BBM, 4.3.10].

Suppose we have such a G and an isomorphism of special Dieudonné modules:

(3.2.1.1) $\quad \mathbf{M}(G) \cong \mathbf{M} \otimes \hat{D}.$

(It is clear how $\mathbf{M}$ gives rise to a special Dieudonné module $\mathbf{M} \otimes \hat{D}$ over $\hat{D}$:

$$\mathbf{M} \otimes \hat{D} = (M \hat{\otimes} \hat{D}, \nabla \otimes 1, F \otimes 1, V \otimes 1, \alpha \otimes B, \chi \otimes \mathrm{id}_B).)$$

Claim. — There exists a lift G_+ over $\mathrm{Spec}(B_+)$ of the p-divisible group G over Spec(B), with an isomorphism

$$\mathbf{M}(G_+) \cong \mathbf{M} \otimes \hat{D}_+$$

lifting (3.2.1.1).

This is a consequence of [M, V Theorem 1.6], [Ill2] and [BM, Corollary 3.2.11]. Indeed, since the ideal of $S \hookrightarrow S_+ = \mathrm{Spec}(B_+)$ has a nilpotent divided power structure δ (with $\delta_n = 0$ for $n \geqslant 2$), we know that lifts of G are determined by lifts

$$\omega_+ \subset \mathbf{D}(G)_{(S, S_+, \delta)}$$

of $\omega_G \subset \mathbf{D}(G)_S$ (use references above). We remark that the lift G_+ determined by ω_+ will be such that $\omega_+ = \omega_{G_+}$ in $\mathbf{D}(G)_{(S, S_+, \delta)} = \mathbf{D}(G_+)_{S_+}$.

By Proposition 3.1.3 *b)* our lift $\mathbf{M} \otimes \hat{D}_+$ determines a lift

$$\chi' := \chi \otimes \mathrm{id}_{B_+} : (M \otimes \hat{D}) \otimes_\pi B_+ \to \alpha_+ = \alpha \otimes B_+$$

of $\chi \otimes \mathrm{id}_B$. Next, we use the isomorphism (3.2.1.1) to translate this into a filtration $\omega_+ \subset M(G) \otimes_\pi B_+$, which is the module associated to the sheaf $\mathbf{D}(G)_{(S, S_+, \delta)}$. It determines G_+ whose special Dieudonné module $\mathbf{M}(G_+)$ must be isomorphic to $\mathbf{M} \otimes \hat{D}_+$ by the remark above and Proposition 3.1.3. This concludes the proof of the claim above.

Thus, if we show that for $m = 1$ we have

(3.2.1.2) $\mathbf{M}(G_0) \cong \mathbf{M} \otimes \hat{D}$,

then by the claim above we can find, by induction, for all m, a p-divisible group G_m over $\mathrm{Spec}(B_m) = \mathrm{Spec}(A_0[x_1, \dots, x_n]/(x_1^m, \dots, x_n^m))$ with $G_{m+1} \otimes B_m \cong G_m$ and such that $\mathbf{D}(G_m)$ corresponds to $M \otimes \hat{D}_m$. Taking the limit $G = \lim G_m$ over Spec(A) (see Lemma 2.4.4) gives the desired p-divisible group.

To prove (3.2.1.2) we note that the isomorphism $M(G_0) \to M \otimes \hat{D}$, which is supposed to exist in the proposition, must automatically be an isomorphism of special Dieudonné module. Indeed, in the case $m = 1$, the kernel of $\chi : N \otimes_D A_0 \to \alpha$ (see Definition 3.1.2) is the set of elements $x \in N \otimes A_0$ such that $x \otimes 1 \in N \otimes A_0 \otimes_\sigma A_0 \cong N \otimes_\sigma A_0$ lies in the image of $V : N \otimes A_0 \to N \otimes_\sigma A_0$. Comparing this with Proposition 2.5.2 gives the result. □

3.2.2. *Proposition. — Let $\mathscr{E}$ be a Dieudonné crystal over an affine formal scheme* $\mathfrak{X} = \mathrm{Spf}(A_0)$. *If* A_0 *satisfies* (1.3.1.1), *then there exists a closed immersion* $\mathfrak{X} \hookrightarrow \mathfrak{Y} = \mathrm{Spf}(A)$ *and a Dieudonné crystal* $\mathscr{E}'$ *over* $\mathfrak{Y}$ *such that*

— $\mathfrak{X} \hookrightarrow \mathfrak{Y}$ *induces an isomorphism* $\mathfrak{X}_{\mathrm{red}} \cong \mathfrak{Y}_{\mathrm{red}}$,
— $\mathscr{E} \cong \mathscr{E}'|_{\mathfrak{X}}$.
— A *satisfies* (1.3.1.1),
— *the Dieudonné crystal* $\mathscr{E}'$ *is versal (Definition* 2.5.4*).*

Proof. — By abuse of notation we write α (resp. ω) for the finite locally free A_0-module corresponding to the sheaf α (resp. ω) of 2.5.3. Take generators $k_1, \dots, k_N$ of the finite locally free A_0-module

$$K := \mathrm{Hom}_{A_0}(\omega, \alpha)$$

and let

$$A = A_0[[x_1, \dots, x_N]].$$

We are in the situation of 3.1.1, the special Dieudonné module $(N, \nabla, F, V, \alpha, \chi)$ coming from $\mathscr{E}$ over $\mathfrak{X}$. It is a special Dieudonné module over $\hat{D}$ (with $m = 1$): indeed, it is of the form

$$N \cong N_0 \hat{\otimes}_{\tilde{A}_0} \hat{D}$$

where $(N_0, \nabla_0, F_0, V_0)$ is the Dieudonné module over $\tilde{A}_0$ associated to $\mathscr{E}$ over $\mathfrak{X}$. By Proposition 2.5.2 we have the exact sequence

(3.2.2.1) $0 \to \omega \to N_0 \otimes A_0 \xrightarrow{\chi_0} \alpha \to 0$

and χ is defined as the composition

$$N \otimes A_0 \cong N_0 \hat{\otimes} \hat{D} \otimes A_0 \cong N_0 \otimes A_0 \xrightarrow{\chi_0} \alpha.$$

It satisfies (3.1.2.1) by definition of ω (see 2.5.2).

In this way we also see that

(3.2.2.2) $\mathrm{N} \otimes_\pi \mathrm{B}_+ \cong \mathrm{N}_0 \hat{\otimes}_{\tilde{\mathrm{A}}_0} \hat{\mathrm{D}} \otimes_\pi \mathrm{B}_+ \cong \mathrm{N}_0 \otimes \mathrm{A}_0[x_i]/(x_i^2)$,

as $\mathrm{B}_+ = \mathrm{A}_0[x_i]/(x_i^2)$. Let us take a splitting of (3.2.2.1):

$$\mathrm{N}_0 \otimes \mathrm{A}_0 \cong \omega \oplus \alpha.$$

As a lift $\omega_+ \subset \mathrm{N}_0 \otimes \mathrm{B}_+$ of $\omega \subset \mathrm{N}_0 \otimes \mathrm{A}_0$ we take

$$\omega_+ = \begin{cases} \text{The submodule of } \mathrm{N}_0 \otimes \mathrm{B}_+ \cong \omega \otimes \mathrm{B}_+ \oplus \alpha \otimes \mathrm{B}_+ \text{ generated by} \\ \text{elements of the form} \\ \qquad \xi \otimes 1 + \Sigma_i\, k_i(\xi) \otimes x_i \in \omega \otimes \mathrm{B}_+ \oplus \alpha \otimes \mathrm{B}_+ \\ \text{where } \xi \text{ runs through } \omega. \end{cases}$$

For α_+ we take $(\mathrm{N}_0 \otimes \mathrm{B}_+)/\omega_+$. Next, we apply 3.1.3 to $\chi' : \mathrm{N}_0 \otimes \mathrm{B}_+ \to \alpha_+$, the lift of χ. This gives a special Dieudonné module $(\mathrm{N}_+, \nabla_+, \mathrm{F}_+, \mathrm{V}_+, \alpha_+, \chi_+)$ and an isomorphism $\mathrm{N}_+ \otimes \hat{\mathrm{D}} \cong \mathrm{N}$. By applying 3.1.3 again to a certain lift χ'_+ of χ_+ (it exists, see remark before Proposition 3.1.3) we get N_2 over $\hat{\mathrm{D}}_2$, a special Dieudonné module over $\hat{\mathrm{D}}_2$ ($\hat{\mathrm{D}}_+$ of 3.1.1 with $m = 2$). Repeating this *ad infinitum* we get a special Dieudonné module $(\mathrm{N}_\infty, \nabla_\infty, \mathrm{F}_\infty, \mathrm{V}_\infty, \alpha_\infty, \chi_\infty)$ over $\tilde{\mathrm{A}}_0[[\underline{x}]]$ (use 2.4.8), plus an isomorphism

(3.2.2.3) $\mathrm{N}_+ \cong \mathrm{N}_\infty \otimes \hat{\mathrm{D}}_+$

of special Dieudonné modules.

We claim that the Kodaira-Spencer map

$$\kappa_\infty : \mathrm{Hom}_{\tilde{\mathrm{A}}_0[[\underline{x}]]}(\Omega^1_{\tilde{\mathrm{A}}_0[[\underline{x}]]}, \tilde{\mathrm{A}}_0[[\underline{x}]]) \to \mathrm{Hom}_{\tilde{\mathrm{A}}_0[[\underline{x}]]}(\omega_\infty, \alpha_\infty)$$

is surjective. By (3.2.2.3) and Remark 3.1.4 we see that this map is congruent to the map derived from:

$$\omega_+ \to \mathrm{N} \otimes_\pi \mathrm{B}_+ \xrightarrow{\nabla} \mathrm{N} \otimes \mathrm{B} \otimes \hat{\Omega}^1_{\mathrm{A}} \to \alpha_+ \otimes \mathrm{B} \otimes \hat{\Omega}^1_{\mathrm{A}}.$$

The isomorphism (3.2.2.2) is horizontal, hence

$$\nabla(\xi \otimes 1 \oplus \sum_i k_i(\xi) \otimes x_i) \equiv \nabla_0(\xi) \otimes 1 + \sum_i k_i(\xi)\, dx_i.$$

It follows that $\kappa_\infty(\partial/\partial x_i) \equiv k_i \bmod (x_1, \ldots, x_{\mathrm{N}})$. Here we used that

$$\mathrm{Hom}_{\tilde{\mathrm{A}}_0[[\underline{x}]]}(\omega_\infty, \alpha_\infty) \bmod (x_1, \ldots, x_{\mathrm{N}}) \cong \mathrm{K}$$

and that $\partial/\partial x_i : \mathrm{A}_0[[\underline{x}]] \to \mathrm{A}_0[[\underline{x}]]$ is the derivation such that $\partial/\partial x_i(a) = 0$ for all $a \in \mathrm{A}_0$ and $\partial/\partial x_i(x_j) = \delta_{ij}\ \forall\, i, j \in \{1, \ldots, \mathrm{N}\}$. As $\mathrm{A}_0[[\underline{x}]]$ is complete with respect to the ideal $(x_1, \ldots, x_{\mathrm{N}})$ it follows that κ_∞ is surjective. □

3.2.3. *Remark.* — We proved something that is slightly stronger than 3.2.2. Namely, that one can take $A = A_0[[x_1, \ldots, x_N]]$ such that the elements $\partial/\partial x_i \in \Theta_{\mathfrak{Y}}$ are mapped to a generating set of $\mathcal{H}om_{\mathcal{O}_{\mathfrak{Y}}}(\omega', \alpha')$. It is also clear that we can do this simultaneously for two crystals $\mathcal{E}_1$, $\mathcal{E}_2$, i.e., such that the elements $\partial/\partial x_i$ generate the module $\mathcal{H}om_{\mathcal{O}_{\mathfrak{Y}}}(\omega_1', \alpha_1') \oplus \mathcal{H}om_{\mathcal{O}_{\mathfrak{Y}}}(\omega_2', \alpha_2')$.

3.2.4. *Corollary.* — *The conclusion of* 3.2.2 *also holds for a p-divisible group* G *over* $\mathfrak{X}$: *we can deform* G *to a versal p-divisible group* G′ *over* $\mathfrak{Y}$.

Proof. — Combine 3.2.1 with 3.2.2. □

4. Proof of Main Theorem 1

This chapter is entirely devoted to the proof of the first Main Theorem. In short it asserts that the crystalline Dieudonné module functor is an equivalence over certain smooth formal schemes.

4.1. *Statement of Main Theorem 1*

4.1.1. *Theorem.* — *If the formal scheme* $\mathfrak{X}$ *over* $\mathbf{F}_p$ *satisfies* (†) *then the functor* 2.4.3

$$\mathbf{D} : \begin{array}{c}\text{Category of } p\text{-divisible} \\ \text{groups over } \mathfrak{X}\end{array} \longrightarrow \begin{array}{c}\text{Category of Dieudonné} \\ \text{crystals over } \mathfrak{X}\end{array}$$

is an anti-equivalence. □

We already know that $\mathbf{D}$ is fully faithful (Corollary 2.4.9). It therefore suffices to construct G given a Dieudonné crystal $\mathcal{E}$ over $\mathfrak{X}$. The problem is local on $\mathfrak{X}$ and thus we may assume $\mathfrak{X} = \mathrm{Spf}(A)$, with A as in (1.3.1.1).

The construction of G is done in a number of steps. We begin with the most difficult case, the case that $A = k$, a field with a finite p-basis.

4.2. *Existence of* G *over fields*

This means that we are in the situation $\mathfrak{X} = \mathfrak{X}_{\mathrm{red}} = \mathrm{Spec}(k)$, where k is a field with a finite p-basis, i.e., $[k : k^p] < \infty$. The result will be proven in three steps. First we reduce to the case that k is separably closed. Next, we construct G in the case that $\mathcal{E}$ corresponds to a formal group G or the dual of a formal group. Finally, we do the general case over a separably closed field, using a result on extensions from 4.3.

Let (Λ, σ) be a lift of k. The Dieudonné crystal $\mathcal{E}$ determines and is determined by a Dieudonné module (M, ∇, F, V) over Λ. By 2.5.2 we also have a k-subvector space $\omega \subset \overline{M} = M/pM$ such that $\omega^{(p)} \subset \overline{M}^{(p)}$ is the image of $V : \overline{M} \to \overline{M}^{(p)}$.

4.2.1. *Reduction to the case: k is separably closed.* Let $\{x_1, \ldots, x_n\} \subset k$ be a p-basis of k. We choose a separable closure k^{sep} of k. Choose a set I indexing all field extensions of k

contained in k^{sep}. For each $i \in \mathrm{I}$ let $k_i \subset k^{\mathrm{sep}}$ denote the corresponding subfield and say that $i = \mathrm{sep}$ gives $k_{\mathrm{sep}} = k^{\mathrm{sep}}$. Choose a lift Λ_i of k_i and let $\tilde{x}_{i,1}, \ldots, \tilde{x}_{i,n} \in \Lambda_i$ be elements lifting $x_1, \ldots, x_n \in k \subset k_i$. The elements $x_1, \ldots, x_n$ form a p-basis for k_i, $\forall\, i \in \mathrm{I}$. As a lift of Frobenius on each Λ_i we take the unique lift σ such that $\sigma(\tilde{x}_{i,l}) = (\tilde{x}_{i,l})^p$, $l = 1, \ldots, n$ (see [BM, 1.2.7]). If $k_i \subset k_j$, $i, j \in \mathrm{I}$ then the homomorphism

$$\textbf{(4.2.1.1)} \qquad \alpha_{i,j} : \Lambda_i \to \Lambda_j, \qquad \tilde{x}_{i,l} \mapsto \tilde{x}_{j,l}, \quad l = 1, \ldots, n$$

lifting $k_i \hookrightarrow k_j$ (see [BM, 1.2.6]) commutes with σ. We note that

$$\textbf{(4.2.1.2)} \qquad k^{\mathrm{sep}} = k_{\mathrm{sep}} = \varinjlim_{[k_i : k] < \infty} k_i$$

and, for any $n \in \mathbf{N}$,

$$\textbf{(4.2.1.3)} \qquad \Lambda_{\mathrm{sep}}/p^n\,\Lambda_{\mathrm{sep}} = \varinjlim_{[k_i : k] < \infty} \Lambda_i/p^n\,\Lambda_i.$$

This is a consequence of [BM, 1.1.5] and (4.2.1.2).

Let $\mathscr{E}$ be our given Dieudonné crystal over $\mathrm{Spec}(k)$ and let $(\mathrm{M}, \nabla, \mathrm{F}, \mathrm{V})$ be the Dieudonné module over Λ associated to it. Suppose we have a p-divisible group G' over $\mathrm{Spec}(k^{\mathrm{sep}})$ with

$$\mathbf{D}(\mathrm{G}') \cong \mathscr{E}|_{\mathrm{CRIS}(\mathrm{Spec}(k^{\mathrm{sep}})/\Sigma)}.$$

By (4.2.1.2) the finite group scheme $\mathrm{G}'[p^n]$ is of the form $\mathrm{H} \otimes k^{\mathrm{sep}}$ where H is a finite group scheme over $\mathrm{Spec}(k_i)$ for some $i \in \mathrm{I}$ such that $[k_i : k] < \infty$. The isomorphism

$$\mathbf{D}(\mathrm{H})|_{\mathrm{CRIS}(\mathrm{Spec}(k^{\mathrm{sep}})/\Sigma)} \cong \mathbf{D}(\mathrm{G}'[p^n]) \cong \mathscr{E}/p^n\,\mathscr{E}|_{\mathrm{CRIS}(\mathrm{Spec}(k^{\mathrm{sep}})/\Sigma)}$$

is defined by an isomorphism of truncated Dieudonné modules of level n (compare Remark 2.4.10)

$$\mathrm{M}(\mathrm{H}) \otimes_{\Lambda_i} \Lambda_{\mathrm{sep}} \to (\mathrm{M}/p^n\,\mathrm{M}) \otimes_{\Lambda} \Lambda_{\mathrm{sep}}.$$

From (4.2.1.3) we see that this map is already defined over some Λ_j, with $j \in \mathrm{I}$ such that $k_i \subset k_j$ and $[k_j : k] < \infty$. The resulting isomorphism over Λ_j will be automatically an isomorphism of truncated Dieudonné modules over Λ_j since the maps $\alpha_{i,j}$ and $\alpha_{i,\mathrm{sep}}$ (4.2.1.1) are compatible with σ's and $\Omega^1_{\Lambda_j/p^n\Lambda_j} \to \Omega^1_{\Lambda_\infty/p^n\Lambda_\infty}$ is injective [BM, 1.3.1].

We conclude that for all $n \in \mathbf{N}$ there exists a finite separable extension $k \subset k'$, a finite group scheme H over $\mathrm{Spec}(k')$ and an isomorphism

$$\textbf{(4.2.1.4)} \qquad \mathbf{D}(\mathrm{H}) \cong (\mathscr{E}/p^n\,\mathscr{E})|_{\mathrm{CRIS}(\mathrm{Spec}(k')/\Sigma)}.$$

Since $\mathbf{D}$ is fully faithful over $\mathrm{Spec}(k' \otimes_k k')$ and $\mathrm{Spec}(k' \otimes_k k' \otimes_k k')$ on finite group schemes [BM, Theorem 4.1.1] we see that H descends to a group scheme G_n over $\mathrm{Spec}(k)$ with $\mathbf{D}(\mathrm{G}_n) \cong \mathscr{E}/p^n\,\mathscr{E}$ (the isomorphism (4.2.1.4) descends too). By fully faithfulness over Spec(k) we conclude that the system $(G_n)_{n \geq 1}$ gives a p-divisible group over Spec(k).

4.2.2. *The case that* F *(or* V*) is topologically nilpotent.* This means that the composition

$$F^n = F \circ F^\sigma \circ \ldots \circ F^{\sigma^{n-1}} : M^{\sigma^n} \to M$$

lands in pM for some n big enough ($M^\sigma := M \otimes_\sigma \Lambda$). By duality it suffices to treat the case that F is topologically nilpotent (see [BBM, 5.3]). We remark that in this subsection we do not need to assume that k is separably closed.

Let us define $M_1 \subset M$ by

$$M_1 := \{ m \in M \mid m \bmod p \in \omega \subset \overline{M} \}.$$

By our choice of $\omega \subset \overline{M}$ the map $M_1^\sigma \to M^\sigma$ gives an isomorphism of M_1^σ with VM. Inverting V gives us an isomorphism $M_1^\sigma \to M$. The association

$$(M, \nabla, F, V) \mapsto (M_1, \text{ the isomorphism } M_1^\sigma \cong M)$$

is functorial both in the Dieudonné module (M, ∇, F, V) and for homomorphisms $\Lambda \to \Lambda'$ commuting with σ and σ' (in particular σ itself). Thus we get $F_1 : M_1^\sigma \to M_1$ and $V_1 : M_1 \to M_1^\sigma$ from F and V and an isomorphism

$$(M_1^\sigma, F_1^\sigma, V_1^\sigma) \cong (M, F, V).$$

We put $S = \text{Spec}(\Lambda)$; it has a lift of Frobenius $f_S = \text{Spec}(\sigma)$. If G is a p-divisible group over k, then we put

$$M_S(G) = \varprojlim_n M_S(G[p^n]),$$

where M_S is the contravariant functor on truncated Barsotti-Tate groups defined in [dJ, Section 8]. By [dJ, Corollary of Theorem 9.3] we can find a p-divisible group G over Spec(k) with an isomorphism

$$M_S(G) \cong (M_1, F_1, V_1).$$

By [dJ, 7.1 and 3.5.2] we get

$$\varprojlim_n \mathbf{D}(G)_{(\text{Spec}(\Lambda/p^n\Lambda),\ \text{Spec}(k),\ \gamma)} \cong M_S(G^{(p)}) \cong M_S(G)^\sigma \cong M_1^\sigma \cong M.$$

All the isomorphisms occurring in this line are compatible with the actions of Frobenius and Verschiebung of G. Hence we see that

$$\textbf{(4.2.2.1)} \qquad (M_G, F_G, V_G) \cong (M, F, V)$$

where $(M_G, \nabla_G, F_G, V_G)$ is the Dieudonné module over Λ associated to the Dieudonné crystal $\mathbf{D}(G)$ of G. By transport of structure, (4.2.2.1) gives a second connection ∇' on M (coming from ∇_G) such that F and V are horizontal. If we show that $\nabla' = \nabla$, then $\mathscr{E} \cong \mathbf{D}(G)$ and we are done.

We remark that

$$p^n(\nabla - \nabla') = F^n V^n(\nabla - \nabla') = F^n(\nabla - \nabla')^{\sigma^n} V^n.$$

Since $(\nabla - \nabla')^{\sigma^n}$ is divisible by p^n ($\sigma : \hat{\Omega}^1_\Lambda \to \hat{\Omega}^1_\Lambda$ is divisible by p) we get

$$\nabla - \nabla' = F^n \frac{(\nabla - \nabla')^{\sigma^n}}{p^n} V^n.$$

As F is topologically nilpotent, the right hand side tends to zero as n goes to infinity.

4.2.3. *Neither* F *nor* V *is topologically nilpotent.* We assume given a separably closed field k of characteristic p, with a finite p-basis and a Dieudonné crystal $\mathscr{E}$ over Spec(k). As before (Λ, σ) is a lift of k and (M, ∇, F, V) is the Dieudonné module over Λ associated to $\mathscr{E}$. The proof of the existence of G will be completely analogous to the proof of Theorem 10.3 in [dJ].

First we claim there is a commutative diagram of Dieudonné modules over Λ:

Diagram I

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & M^{\mu} & = & M^{\mu} & & \\
 & & & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & M^{ét} & \longrightarrow & M & \longrightarrow & M_1 & \longrightarrow & 0 \\
 & & \| & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & M^{ét} & \longrightarrow & M_2 & \longrightarrow & M_{12} & \longrightarrow & 0 \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

with exact rows and columns and such that

— F is topologically nilpotent on M_{12}, M_1, M^{μ},
— V is topologically nilpotent on M_{12}, M_2, $M^{ét}$,
— F is an isomorphism on $M^{ét}$,
— V is an isomorphism on M^{μ}.

Here M_1 is the first member of $(M_1, \nabla_1, F_1, V_1)$, etc. The construction of the diagram is left to the reader, we only point out that

$$M^{ét} = \bigcap_{n \geq 1} F^n(M^{\sigma^n}) \subset M,$$

the quotient $M \to M^{\mu}$ is defined dually, $M_1 = M/M^{ét}$, $M_2 = \mathrm{Ker}(M \to M^{\mu})$ and $M_{12} = \mathrm{Ker}(M_1 \to M^{\mu})$. See also [dJ, proof of Theorem 10.3].

By 4.2.2 we already have p-divisible groups G_{12}, G_1, G_2, G^{μ} and $G^{\text{ét}}$ corresponding to $(M_{12}, \nabla_{12}, \ldots), \ldots$ The p-divisible group $G^{\text{ét}}$ is étale since $F^{\text{ét}}$ is an isomorphism. As k is separably closed we can choose an isomorphism $G^{\text{ét}} \cong (\mathbf{Q}_p/\mathbf{Z}_p)^s$ for some $s \in \mathbf{N}$. Similarly, $G^{\mu} \cong (\mathbf{G}_m[p^{\infty}])^t$ for some $t \in \mathbf{N}$.

Fully faithfulness of $\mathbf{D}$ gives us an incomplete diagram corresponding to diagram I:

Diagram II

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & G^{\mu} & = & G^{\mu} & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \leftarrow & G^{\text{ét}} & \leftarrow & ? & \leftarrow & G_1 & \leftarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \leftarrow & G^{\text{ét}} & \leftarrow & G_2 & \leftarrow & G_{12} & \leftarrow & 0. \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

(The lower line and right column are exact sequences of p-divisible groups.) For all $n \in \mathbf{N}$ we derive a diagram from diagram II by taking the kernel of multiplication by p^n:

Diagram n

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & G^{\mu}[p^n] & = & G^{\mu}[p^n] & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \leftarrow & G^{\text{ét}}[p^n] & \longleftarrow & ? & \longleftarrow & G_1[p^n] & \leftarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \leftarrow & G^{\text{ét}}[p^n] & \leftarrow & G_2[p^n] & \leftarrow & G_{12}[p^n] & \leftarrow & 0. \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

The obstruction to find a sheaf of $\mathbf{Z}/p^n\,\mathbf{Z}$-modules G_n on $\operatorname{Spec}(k)_{\text{fppf}}$ fitting into diagram n (i.e., sitting in place of ? and making the middle line and middle column exact) lies in the group

$$\operatorname{Ext}^2_{\mathbf{Z}/p^n\mathbf{Z}}(G^{\text{ét}}[p^n], G^{\mu}[p^n])$$

(extensions of sheaves of $\mathbf{Z}/p^n\,\mathbf{Z}$-modules on $\operatorname{Spec}(k)_{\text{fppf}}$). It is easy to prove that

$$\operatorname{Ext}^2_{\mathbf{Z}/p^n\mathbf{Z}}(\mathbf{Z}/p^n\,\mathbf{Z}, \mu_{p^n}) \cong H^2_{\text{fppf}}(\operatorname{Spec}(k), \mathbf{G}_m)\,[p^n] = 0.$$

Therefore this obstruction is zero. The set of all choices of such sheaves G_n is homogeneous under the group

$$\mathrm{Ext}^1_{\mathbf{Z}/p^n\mathbf{Z}}(G^{\text{ét}}[p^n], G^{\mu}[p^n]) \cong \mathrm{Ext}^1_{\mathbf{Z}/p^n\mathbf{Z}}(\mathbf{Z}/p^n\,\mathbf{Z}, \mu_{p^n})^{s.t} \cong (k^*/(k^*)^{p^n})^{s.t}.$$

Since the homomorphism $k^*/k^{*p^{n+1}} \to k^*/k^{*p^n}$ is surjective, given any G_n fitting into diagram n we can find a G_{n+1} fitting into diagram $n+1$ such that $G_n \cong G_{n+1}[p^n]$ (an isomorphism of sheaves fitting into diagram n, this isomorphism is unique: $\mu_{p^n}(k) = \{1\}$).

Clearly, this proves that we can find at least a sheaf $G' = \bigcup G_n$ fitting into the diagram II. Since the sheaves G_n are representable, as they are extensions of representable sheaves, it follows that G' is a p-divisible group. The set of isomorphism classes of such G' fitted into diagram II is homogeneous under the group

$$\mathrm{Ext}^1_{p\text{-div. gr.}}(G^{\text{ét}}, G^{\mu}) \cong \mathrm{Ext}^1_{p\text{-div. gr.}}(\mathbf{Q}_p/\mathbf{Z}_p, \mathbf{G}_m[p^{\infty}])^{s.t}$$

and the corresponding group for diagram I is

$$\mathrm{Ext}^1_{\mathrm{DC}}(\mathbf{D}(G^{\mu}), \mathbf{D}(G^{\text{ét}})) \cong \mathrm{Ext}^1_{\mathrm{DC}}(\mathbf{D}(\mathbf{G}_m[p^{\infty}]), \mathbf{D}(\mathbf{Q}_p/\mathbf{Z}_p))^{s.t}.$$

To produce G it therefore suffices to prove Proposition 4.3.6 below.

4.3. *Extensions of étale p-groups by multiplicative p-groups*

In this section we prove the analogs of [dJ] 8.7, 8.8 and 8.9 in the crystalline setting.

Let S be a scheme of characteristic p. We introduce the following categories: for any $n \in \mathbf{N}$ we put

$\mathrm{BT}(n)_S$ = the category of truncated Barsotti-Tate groups of level n over S

and $\mathrm{DC}(n)_S$ = the category of truncated Dieudonné crystals of level n over S (see Remark 2.4.10).

In both categories it is clear what a short exact sequence is and there are corresponding extension groups. The crystalline Dieudonné module functor

$$\mathbf{D} : \mathrm{BT}(n)_S \to \mathrm{DC}(n)_S$$

(see [BBM, 3.3.6 and 3.3.10]) transforms short exact sequences into short exact sequences; this follows from [BBM, 4.3.1].

4.3.1. *Lemma. — There exists a natural injection*

$$\Gamma(S, \mathcal{O}_S^*)/\Gamma(S, \mathcal{O}_S^*)^{p^n} \to \mathrm{Ext}^1_{\mathrm{BT}(n)_S}(\mathbf{Z}/p^n\,\mathbf{Z}_S, \mu_{p^n,S})$$

which is an isomorphism if $\mathrm{H}^1_{\mathrm{fppf}}(S, \mathbf{G}_m)\,[p] = (0)$.

Proof. — This is well known (cf. [dJ, Lemma 8.7]). Let us describe the extension $E_n(g)$ associated to a section $g \in \Gamma(S, \mathcal{O}_S^*)$. The sheaf $E_n(g)$ is defined as follows:

$$E_n(g)\,(U) = \{(f, m) \in \Gamma(U, \mathcal{O}_U^* \times \mathbf{Z}_U) \mid f^{p^n} = g^m\}/\langle (g, p^n) \rangle$$

for $U \in \mathrm{Ob}\mathcal{S}ch/S$. The maps which define the structure of extension on $E_n(g)$ are

$$\mu_{p^n}(U) \ni f \mapsto (f, 0) \in E_n(g)\,(U)$$

and $$E_n(g) \ni (f, m) \mapsto \bar{m} \in \mathbf{Z}/p^n\,\mathbf{Z}(U). \quad \square$$

In [BM, 2.1 and 2.2] the Dieudonné crystals of $\mu_{p^n, S}$ and $\mathbf{Z}/p^n\,\mathbf{Z}_S$ are determined. The result is as follows:

$$\mathbf{D}(\mathbf{Z}/p^n\,\mathbf{Z}_S) \cong (\mathcal{O}_{S/\Sigma}/p^n\,\mathcal{O}_{S/\Sigma}, 1, p)$$

and $$\mathbf{D}(\mu_{p^n, S}) \cong (\mathcal{O}_{S/\Sigma}/p^n\,\mathcal{O}_{S/\Sigma}, p, 1).$$

The identifications are given by (2.1.3.1) and (2.2.3.1) of [BM].

Suppose that $S = \mathrm{Spec}(A)$ and A is a ring with a p-basis. Let $(\tilde{A}, \sigma)$ be a lift of A, put $A_n = \tilde{A}/p^n\,\tilde{A}$ and $\Omega_n = \Omega^1_{A_n}$. For any $n \in \mathbf{N}$ we define an abelian group W_n as follows:

$$W_n := \{(\eta, a) \in \Omega_n \times A_n \mid d\eta = 0, \sigma(\eta) = da + p\eta\}/\{(db, \sigma(b) - pb) \mid b \in A_n\}.$$

4.3.2. *Lemma.* — *There is natural isomorphism*

$$W_n \to \mathrm{Ext}^1_{DC(n)_S}(\mathbf{D}(\mu_{p^n, S}), \mathbf{D}(\mathbf{Z}/p^n\,\mathbf{Z}_S)).$$

Proof. — For each pair $(\eta, a) \in \Omega_n \times A_n$ with $d\eta = 0$ and $\sigma(\eta) = da + p\eta$ we define an extension of truncated Dieudonné modules of level n:

$$0 \to (A_n, d_{A_n/\mathbf{Z}}, 1, p) \to M(\eta, a) \to (A_n, d_{A_n/\mathbf{Z}}, p, 1) \to 0.$$

It is defined by the following formulae:

$$\textbf{(4.3.2.1)} \qquad \begin{cases} M(\eta, a) = A_n.1 \oplus A_n.\tilde{1} & \\ \nabla 1 = 0 & \nabla \tilde{1} = 1 \otimes \eta \\ F1^\sigma = 1 & F\tilde{1}^\sigma = a.1 + p.\tilde{1} \\ V1 = p.1^\sigma & V\tilde{1} = -a.1^\sigma + \tilde{1}^\sigma. \end{cases}$$

The reader can easily convince himself that this defines indeed an extension of Dieudonné modules and that any such extension can be written in this form. Also, it is clear that all possible isomorphisms of such extensions are the isomorphisms (with $b \in A_n$)

$$\begin{aligned} M(\eta + db, a + \sigma(b) - pb) &\to M(\eta, a) \\ 1 &\mapsto 1 \\ \tilde{1} &\mapsto b.1 + \tilde{1}. \end{aligned}$$

Finally, it is also easy to show that

$$[M(\eta, a)] + [M(\eta', a')] = [M(\eta + \eta', a + a')].$$

(Addition of extension classes.)

Therefore our lemma follows from the description of crystals over S in terms of $\widetilde{A}$-modules with connections [BM, 1.3.3] and the description of $\mathbf{D}(\mu_{p^n})$ and $\mathbf{D}(\mathbf{Z}/p^n\mathbf{Z})$ given above. □

4.3.3. *Remark.* — A similar result holds in the situation of Definition 2.3.4.

We define a homomorphism

$$A^*/(A^*)^{p^n} \to W_n \tag{4.3.4}$$

by the rule

$$\text{class of } g \in A^* \mapsto \text{class of } (\mathrm{dlog}(\tilde{g}), \log(\sigma(\tilde{g})\, \tilde{g}^{-p})).$$

Here $\tilde{g} \in A_n^*$ is an element mapping to $g \in A^*$. Since $\sigma(\tilde{g}) \equiv \tilde{g}^p \bmod pA_n$, the product $\sigma(\tilde{g})\,\tilde{g}^{-p}$ can be written as $1 + p\delta$ and we can take its logarithm

$$\log(1 + p\delta) = p\delta - \frac{1}{2}(p\delta)^2 + \frac{1}{3}(p\delta)^3 - \ldots \in A_n.$$

It is easy to prove that the element of W_n so defined does not depend on the lift $\tilde{g}$ (use that any other lift $\tilde{g}'$ can be written as $\tilde{g}.(1 + p\delta)$).

4.3.5. *Lemma.* — *The following diagram is commutative:*

$$\begin{array}{ccc} A^*/(A^*)^{p^n} & \xrightarrow{\quad 4.3.4 \quad} & W_n \\ \downarrow{\scriptstyle 4.3.1} & & \downarrow{\scriptstyle 4.3.2} \\ \mathrm{Ext}^1_{\mathrm{BT}(n)_S}(\mathbf{Z}/p^n\mathbf{Z}_S, \mu_{p^n,S}) & \xrightarrow{\mathbf{D}} & \mathrm{Ext}^1_{\mathrm{DC}(n)_S}(\mathbf{D}(\mu_{p^n,S}), \mathbf{D}(\mathbf{Z}/p^n\mathbf{Z}_S)). \end{array}$$

Proof. — Suppose (U, T, δ) is an object of $\mathrm{CRIS}(S/\Sigma)$ and $g \in A^*$ is lifted to $\tilde{g} \in \Gamma(T, \mathcal{O}_T^*)$. To this we will associate an extension of sheaves on $\mathrm{CRIS}(U/T)$:

$$0 \to \mathcal{O}_{U/T} \to \mathcal{U}(\tilde{g}) \to \underline{E_n(g)_U} \to 0$$

We recall that for a sheaf $\mathcal{F}$ on the big fppf-site of S, we write $\underline{\mathcal{F}}$ for the sheaf on $\mathrm{CRIS}(S/\Sigma)$ defined by $\underline{\mathcal{F}}(U, T, \delta) := \mathcal{F}(U)$; see [BBM, page 15] for this notation. The extension $\mathcal{U}(\tilde{g})$ is defined as the push-out by $\log : (1 + \mathcal{I}_{U/T})^* \to \mathcal{O}_{U/T}$ of an extension

$$0 \to (1 + \mathcal{I}_{U/T})^* \to \mathcal{V}(\tilde{g}) \to \underline{E_n(g)_U} \to 0.$$

Here the sheaf $\mathcal{V}(\tilde{g})$ is defined as follows: for $(U', T', \delta') \in \mathrm{Ob}\,\mathrm{CRIS}(U/T)$ the sections of $\mathcal{V}(\tilde{g})$ over (U', T', δ') are:

$$\{(f, m) \in \Gamma(T', \mathcal{O}_{T'}^* \times \mathbf{Z}_{T'}) \mid \text{the image } \bar{f} \text{ of } f \text{ in } \mathcal{O}_{U'}^* \text{ satisfies } \bar{f}^{p^n} = g^m\}/\langle(\tilde{g}, p^n)\rangle.$$

It is clear how $\mathcal{V}(\tilde{g})$ becomes an extension as indicated above.

The extension $\mathscr{U}(\tilde{g})$ defines a section $\tilde{1}(\tilde{g})$ of the sheaf

$$\mathscr{E}xt^1_{U/T}(\underline{E_n(g)_U}, \mathcal{O}_{U/T}) \cong \mathbf{D}(E_n(g)_U)|_{CRIS(U/T)}.$$

Under the map

$$\mathbf{D}(E_n(g)_U)_T \to \mathbf{D}(\mu_{p^n})_T \cong \mathcal{O}_T/p^n\,\mathcal{O}_T$$

the element $\tilde{1}(\tilde{g})$ maps to 1; this is a consequence of our definitions and the definition of [BM, (2.2.3.1)].

In a similar way the sheaf $\mathscr{X}(a)$, defined for $a \in \Gamma(T, \mathcal{O}_T)$ on CRIS(U/T) by

$$\Gamma((U', T', \delta'), \mathscr{X}(a)) := \{(f, m) \in \Gamma(T', \mathcal{O}_{T'} \times \mathbf{Z}_{T'})\}/\langle(a, p^n)\rangle$$

is an extension

$$0 \to \mathcal{O}_{U/T} \to \mathscr{X}(a) \to \underline{\mathbf{Z}/p^n\,\mathbf{Z}_U} \to 0$$

and corresponds to the section a of

$$\mathbf{D}(\mathbf{Z}/p^n\,\mathbf{Z})_{U/T} = \mathscr{E}xt^1_{U/T}(\underline{\mathbf{Z}/p^n\,\mathbf{Z}_U}, \mathcal{O}_{U/T}) \cong \mathcal{O}_{U/T}/p^n\,\mathcal{O}_{U/T}.$$

Suppose $\tilde{g}_1$ and $\tilde{g}_2 \in \Gamma(T, \mathcal{O}_T^*)$ are two sections lifting the image of g in $\Gamma(U, \mathcal{O}_U^*)$. We claim that the difference $\tilde{1}(\tilde{g}_1) - \tilde{1}(\tilde{g}_2)$ is given by the class of the extension $\mathscr{X}(\log(\tilde{g}_1\,\tilde{g}_2^{-1}))$ in

$$\mathbf{D}(\mathbf{Z}/p^n\,\mathbf{Z})_T \subset \mathbf{D}(E_n(g))_T.$$

To prove this we consider the pullback

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}_{U/T} \oplus \mathcal{O}_{U/T} & \longrightarrow & P & \longrightarrow & \underline{E_n(g)_U} & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow{\scriptstyle (1,-1)} & & \\ 0 & \longrightarrow & \mathcal{O}_{U/T} \oplus \mathcal{O}_{U/T} & \longrightarrow & \mathscr{U}(\tilde{g}_1) \oplus \mathscr{U}(\tilde{g}_2) & \longrightarrow & \underline{E_n(g)_U} \oplus \underline{E_n(g)_U} & \longrightarrow & 0 \end{array}$$

and we construct a homomorphism

$$\alpha : P \to \mathscr{X}(\log(\tilde{g}_1\,\tilde{g}_2^{-1}))$$

which proves the equality (we leave it to the reader to check this). If

$$(U', T', \delta') \in \mathrm{Ob}\ \mathrm{CRIS}(U/T) \quad \text{and} \quad (f_i, m_i) \in \Gamma(T', \mathcal{O}_{T'}^*) \times \mathbf{Z}$$

is such that $\bar{f}_i^{p^n} = g^{m_i}$ for $i = 1, 2$ and $l \in \mathbf{Z}$ is such that $\bar{f}_1 . g^l = \bar{f}_2^{-1}$, $m_1 + lp^n = -m_2$ then we associate to this the pair

$$(\log(f_1\,f_2\,\tilde{g}_1^l), m_1 + lp^n) \in \Gamma(T', \mathcal{O}_{T'}) \times \mathbf{Z}$$

(and hence a section of $\mathscr{X}(\log(\tilde{g}_1 \tilde{g}_2^{-1}))$. We again leave it to the reader to check that this induces a homomorphism α as desired using the definitions of $\mathscr{V}(\tilde{g}_i)$, etc.; the crucial point is that to the pairs (f_1, m_1) and $(f_2 . \tilde{g}_2^{-1}, m_2 - p^n)$ is associated the pair

$$(\log(f_1 f_2 \tilde{g}_2^{-1} \tilde{g}_1^{l+1}), m_1 + lp^n + p^n) = (\log(\tilde{g}_1 \tilde{g}_2^{-1}), p^n) + (\log(f_1 f_2 \tilde{g}_1^{l}), m_1 + lp^n).$$

Finally, we are able to prove the commutativity of the square of the lemma. Choose a lift $\tilde{g} \in A_n^*$ of g as in the definition of 4.3.4. We have to check that the element $\tilde{1}(\tilde{g}) \in \mathbf{D}(E_n(g))_{(S, \mathrm{Spec}(A_n), \gamma)}$ satisfies formulae (4.3.2.1) with $\eta = \mathrm{dlog}(\tilde{g})$ and $a = \log(\sigma(\tilde{g})\, \tilde{g}^{-p})$. This is checked using the above; for example the property that $\nabla \tilde{1}(\tilde{g}) = \mathrm{dlog}(\tilde{g}).1$ follows from the fact that

$$\mathrm{pr}_1^{-1}(\tilde{1}(\tilde{g})) - \mathrm{pr}_2^{-1}(\tilde{1}(\tilde{g})) = \log(\mathrm{pr}_1^*(\tilde{g}).\mathrm{pr}_2^*(\tilde{g})^{-1}).1;$$

here pr_1 and pr_2 are the projection morphisms

$$\mathrm{Spec}(A_n \otimes_{\mathbf{Z}} A_n/I^2) \rightrightarrows \mathrm{Spec}(A_n)$$

of the first infinitesimal neighbourhood of the diagonal embedding of $\mathrm{Spec}(A_n)$. (See also the proof of [dJ, Proposition 8.9].) □

We note that the obvious homomorphisms $W_n \to W_{n-1}$ are induced by taking the kernel of p^{n-1} on an extension of $\mathbf{D}(\mu_{p^n})$ by $\mathbf{D}(\mathbf{Z}/p^n\mathbf{Z})$. Let us write

$$W_n \supset W_n' := \{ x \in W_n \mid \exists y \in W_{n+1} \text{ whose image in } W_n \text{ is } x \}.$$

Since any element of $A^*/(A^*)^{p^n}$ can be lifted to an element of $A^*/(A^*)^{p^{n+1}}$ it follows that the image of 4.3.4 lies in W_n'.

4.3.6. *Proposition.* — *a) If* A *is a field of characteristic* p *the map* 4.3.4

$$A^*/(A^*)^{p^n} \to W_n'$$

is an isomorphism for all $n \in \mathbf{N}$.

b) Over a field k *of characteristic* p *the crystalline Dieudonné module functor* $\mathbf{D}$ *induces a bijection of extension groups*

$$\mathrm{Ext}^1_{p\text{-div. gr.}}(\mathbf{Q}_p/\mathbf{Z}_p, \mathbf{G}_m[p^\infty]) \cong \mathrm{Ext}^1_{\mathrm{DC}}(\mathbf{D}(\mathbf{G}_m[p^\infty]), \mathbf{D}(\mathbf{Q}_p/\mathbf{Z}_p)).$$

Proof. — Part *b)* is a consequence of part *a)* since the map of Ext-groups is just the map ($k = A$)

$$\varprojlim A^*/(A^*)^{p^n} \to \varprojlim W_n \cong \varprojlim W_n'.$$

Injectivity of $A^*/(A^*)^{p^n} \to W_n'$ is a consequence of fully faithfulness of $\mathbf{D}$ on Spec(A), but is also easy to prove directly: if $(\tilde{g}^{-1}\, d\tilde{g}, \mathrm{dlog}(\sigma(\tilde{g})\, \tilde{g}^{-p})) = (db, \sigma(b) - pb)$, then $b \in pA_n$ (look at second factor), hence $d\tilde{g} \in pA_n$, hence $\tilde{g} = h^p(1 + pf)$, hence $b \in p^2 A_n$, etc.

Surjectivity is proved as follows. Since the kernel of the map $W'_n \to W'_1$ can be identified with W'_{n-1}, we only need to prove surjectivity for $n = 1$. Let us take $(\eta, a) \in \Omega_2 \times A_2$ with $d\eta = 0$ and $\sigma(\eta) = da + p\eta$. We can write $a = a_1^p + pa_2$ since $da = \sigma(\eta) - p\eta \in p\Omega_2$. Changing (η, a) by $(da_1, \sigma(a_1) - pa_1)$ we may assume that $a \in pA_2 : a = pa'$. We can now divide the equation $\sigma(\eta) = da + p\eta$ by p to get

$$\frac{1}{p}\,\sigma(\eta) = da' + \eta \in \Omega_1.$$

This equation can be viewed as a condition on the 1-form $\eta \in \Omega_1 = \Omega^1_A$ without referring to the lift (A_2, σ): the equation says that the relation

$$\sum_i f_i^p\, g_i^{p-1}\, dg_i \equiv \sum_i f_i\, dg_i \pmod{dA}$$

holds in Ω^1_A/dA for any presentation $\eta = \Sigma f_i\, dg_i$ of η. We have to prove that any $\eta \in \Omega^1_A$ satisfying this is a logarithmic differential: $\eta = g^{-1}\, dg$ for some $g \in A^*$.

A relation of the type above is realised by a relation

$$\sum_i f_i^p\, g_i^{p-1}\, dg_i = \sum_i f_i\, dg_i + \sum_j d\alpha_j + \sum_l \beta_l(d(\gamma_l + \delta_l) - d\gamma_l - d\delta_l) + \sum_m \varepsilon_m(d\zeta_m\, \theta_m - \zeta_m\, d\theta_m - \theta_m\, d\zeta_m)$$

in $A[dA]$. Clearly, there exists a subfield $k \subset A$ finitely generated over $\mathbf{F}_p$ containing all the elements $f_i, g_i, \alpha_j, \beta_l, \gamma_l, \delta_l, \varepsilon_m, \zeta_m, \theta_m$. We have reduced the problem to the case of a field k finitely generated over $\mathbf{F}_p$. The 1-form η comes from a 1-form $\eta \in \Gamma(X, \Omega^1_X)$ of some variety X smooth over $\mathbf{F}_p$ with $\mathbf{F}_p(X) \cong k$. By assumption $d\eta = 0$ and η lies in the kernel of the map (see [Ill3, § 2])

$$W^* - C : Z\Omega^1_{X/\mathbf{F}_p} \to \Omega^1_{X/\mathbf{F}_p}.$$

Hence by [Ill3, 2.1.17] it is a logarithmic differential on an open part of X. □

4.4. *Construction of* G *in the reduced case*

In this section we assume that $\mathfrak{X} = \mathfrak{X}_{\mathrm{red}}$. Hence, we may assume that $\mathfrak{X} = S = \mathrm{Spec}(A)$ is affine and of finite type over a field k with a finite p-basis. Since S is formally smooth over $\mathrm{Spec}(\mathbf{F}_p)$, it is a regular scheme. Also we may and do assume that S is irreducible (i.e., connected). We argue by induction on dim S; the case dim S = 0 is 4.2.

Let there be given a Dieudonné crystal $(\mathscr{E}, f, v)$ over S. By 4.2 we have a p-divisible group G_K over Spec(K), where K is the function field of S (it is finitely generated over k, hence it also has a finite p-basis). This p-divisible group is such that

$$\mathbf{D}(G_K) \cong (\mathscr{E}, f, v)\,|_{\mathrm{CRIS}(\mathrm{Spec}(K)/\Sigma)}. \tag{4.4.1}$$

Fix $n \in \mathbf{N}$. There exists a nonempty Zariski open $U \subset S$ and a finite locally free group scheme G_n over U with $G_n \times_U \operatorname{Spec}(K) \cong G_K[p^n]$. Suppose (A_n, σ) is a lift of A modulo p^n. A lift of K modulo p^n is:

$$\varinjlim_{f \in A_n,\, f \bmod p \neq 0} A_n\left[\frac{1}{f}\right]$$

with the natural action of σ on it. Using the description of crystals in terms of modules with connection 2.2.3 and arguing as in 4.2.1 we see that we may assume that the isomorphism (4.4.1) modulo p^n exists over U (after possibly shrinking U a bit):

$$\textbf{(4.4.2)} \qquad \mathbf{D}(G_n) \cong (\mathscr{E}/p^n\, \mathscr{E})\,|_{\mathrm{CRIS}(U/\Sigma)}.$$

Using the fully faithfulness of $\mathbf{D}$ ([BBM, 4.2.6] or Remark 2.4.10) we may even suppose that U is the biggest Zariski open subscheme of S such that a finite locally free group scheme G_n over U exists, endowed with an isomorphism (4.4.2).

Let $T = S\backslash U$ with the reduced closed subscheme structure. The regular locus $T^{\mathrm{reg}} \subset T$ is open dense in T. We are going to extend G_n to $U \cup T^{\mathrm{reg}} = S\backslash \operatorname{Sing}(T)$ together with the isomorphism (4.4.2). This contradicts the maximality of U unless $T = \emptyset$, which is what we wanted to prove. Indeed, then we have G_n over S with (4.4.2) for all n and hence by fully faithfulness ([BBM, 4.2.6] or Remark 2.4.10) a p-divisible group $G = (G_n)_{n \geqslant 1}$.

Again using the fully faithfulness of $\mathbf{D}$ on finite locally free group schemes, the problem of extending G_n is a local one. Thus we may assume that T is a regular scheme and that

$$T = \operatorname{Spec}(A/I) \to S = \operatorname{Spec}(A)$$

is defined by an ideal I generated by a regular sequence $f_1, \ldots, f_c \in A$. We put $\mathfrak{Y} = \operatorname{Spf}(\hat{A})$ where

$$\hat{A} := \varprojlim A/I^n \cong A/I[[f_1, \ldots, f_c]].$$

(To find such an isomorphism compare with Lemma 1.3.3 and recall that A/I is regular.)

By our induction hypothesis ($\dim T < \dim S$), we can find a p-divisible group H″ over T with $\mathbf{D}(H'') \cong \mathscr{E}|_{\mathrm{CRIS}(T/\Sigma)}$. By Proposition 3.2.1 we can find a p-divisible group H′ over $\mathfrak{Y}$ with $\mathbf{D}(H') \cong \mathscr{E}|_{\mathfrak{Y}}$. By Lemma 2.4.4 this comes from a p-divisible group H over $Y := \operatorname{Spec}(\hat{A})$ with

$$\textbf{(4.4.3)} \qquad \mathbf{D}(H) \cong \mathscr{E}|_{\mathrm{CRIS}(\operatorname{Spec}(\hat{A})/\Sigma)}.$$

4.4.4. *Lemma. — The functor* $\mathbf{D}$ *is fully faithful on finite locally free group schemes on each of the schemes*

$$Y_i := \operatorname{Spec}(\hat{A}[1/f_i]) \quad \textit{and} \quad Y_{ij} := \operatorname{Spec}(\hat{A}[1/f_i f_j]).$$

Proof. — Since we have the isomorphism $\hat{A} \cong A/I[[f_1, \ldots, f_c]]$ we may use Lemma 1.3.3 to replace A/I by a ring B with a finite p-basis. The ring $B[[f_1, \ldots, f_c]]$

has a finite p-basis by Lemma 1.1.3. Hence so does $B[[f_1, \ldots, f_c]][1/f_i]$ and $B[[f_1, \ldots, f_c]][1/f_i f_j]$. It suffices to apply 4.1.1 of [BM]. □

The morphisms $Y_i \to S$ and $Y_{ij} \to S$ factor through $U \subset S$ and by the lemma we get isomorphisms

$$(4.4.5) \qquad \alpha_i : H[p^n]_{Y_i} \cong G_{n, Y_i}$$

from the isomorphisms

$$\mathbf{D}(H[p^n]_{Y_i}) \cong \mathscr{E}/p^n \mathscr{E}|_{\mathrm{CRIS}(Y_i/\Sigma)} \cong \mathbf{D}(G_{n, Y_i}).$$

These isomorphisms agree on Y_{ij} after pullback with $Y_{ij} \to Y_i$ and $Y_{ij} \to Y_j$. By Proposition 4.6.2, we get a unique finite locally free group scheme G'_n over S with isomorphisms $G'_{n, U} \cong G_n$ and $G'_{n, Y} \cong H[p^n]$ and inducing (4.4.5).

Suppose $T_n \hookrightarrow S_n$ is a lift of $T \hookrightarrow S$ modulo $p^n : S_n = \mathrm{Spec}(A_n)$, A_n is a lift of A modulo p^n and $T_n = \mathrm{Spec}(A_n/(\widetilde{f_1}, \ldots, \widetilde{f_c}))$ where $\widetilde{f_i} \in A_n$ lifts $f_i \in A$. We also get $Y_{i, n}$ and $Y_{ij, n}$ in a similar manner. Crystals killed by p^n on T (resp. S, Y_i or Y_{ij}) are described by $\mathcal{O}_{T_n}$-modules (resp. $\mathcal{O}_{S_n}$, $\mathcal{O}_{Y_{i,n}}$ or $\mathcal{O}_{Y_{ij,n}}$-modules) with connections, hence Proposition 4.6.2 now gives us an isomorphism of crystals

$$\mathbf{D}(G'_n) \cong \mathscr{E}/p^n \mathscr{E}$$

agreeing with (4.4.2) and (4.4.3). It is an isomorphism of Dieudonné crystals since this was true of (4.4.2). This concludes the case 4.4.

4.5. *Existence of* G *in the general case*

We proceed as in 4.4 by induction on the dimension of $\mathfrak{X}_{\mathrm{red}}$. The case that $\dim \mathfrak{X}_{\mathrm{red}} = 0$ is a combination of 4.2 and Proposition 3.2.1.

As noted in 4.1 we may assume that $\mathfrak{X} = \mathrm{Spf}(A)$ is affine and that A is of type (1.3.1.1). Using Lemma 1.3.3 we can replace A by a complete ring of type (1.3.2.1), which has a finite p-basis (this does not change the dimension of $\mathfrak{X}_{\mathrm{red}}$!). Suppose $I \subset A$ is the biggest ideal of definition of A. The scheme $\mathrm{Spec}(A/I)$ is a reduced scheme of finite type over a field, hence its regular locus is a dense open subset. Thus we can choose an element $f \in A$, $f \notin I$ such that $\mathrm{Spec}(A/I[1/f])$ is regular and such that $\dim \mathrm{Spec}(A/I + fA) < \dim \mathrm{Spec}(A/I)$. We put

$$A\{1/f\} = \varprojlim A/I^n[1/f]$$

and

$$\hat{A} = \varprojlim A/f^n A, \qquad \hat{I} = I.\hat{A}.$$

Suppose we are given a Dieudonné crystal $\mathscr{E}'$ on $\mathfrak{X} = \mathrm{Spf}(A)$. By 4.4 we get a p-divisible group G''_1 over $\mathrm{Spec}(A/I[1/f])$ and by Proposition 3.2.1 it can be deformed to a p-divisible group G'_1 over $\mathrm{Spf}(A\{1/f\})$ with

$$(4.5.1) \qquad \mathbf{D}(G'_1) \cong \mathscr{E}'|_{\mathrm{Spf}(A\{1/f\})}.$$

By our induction hypothesis we get a p-divisible group G'_2 over $\mathrm{Spf}(\hat{A})$ with

$$\mathbf{D}(G'_2) \cong \mathscr{E}'|_{\mathrm{Spf}(\hat{A})}. \tag{4.5.2}$$

However, using Lemma 2.4.4 and Proposition 2.4.8 we may assume $\mathscr{E}'$ comes from a Dieudonné crystal $\mathscr{E}$ over Spec(A), G'_1 (resp. G'_2) comes from a p-divisible group G_1 (resp. G_2) over $\mathrm{Spec}(A\{1/f\})$ (resp. $\mathrm{Spec}(\hat{A})$) and the isomorphism (4.5.1) (resp. (4.5.2)) is defined over $\mathrm{Spec}(A\{1/f\})$ (resp. $\mathrm{Spec}(\hat{A})$).

By Lemma 1.1.3 the ring $\hat{A}$ has a finite p-basis. Hence $\hat{A}[1/f]$ has a finite p-basis, hence the ring

$$B := \varprojlim(\hat{A}/\hat{I}^n)\,[1/f]$$

has a finite p-basis (Lemma 1.1.3 again). Clearly, we have homomorphisms $\hat{A} \to B$ and $A\{1/f\} \to B$. By fully faithfulness over Spec(B) ([BBM, 4.1.1]) and (4.5.1) and (4.5.2) we get an isomorphism

$$G_{1,\,\mathrm{Spec(B)}} \cong G_{2,\,\mathrm{Spec(B)}}. \tag{4.5.3}$$

Reducing everything modulo a fixed power n of I we get $G_{1,n}$ over $\mathrm{Spec}(A/I^n[1/f])$, $G_{2,n}$ over $\mathrm{Spec}(\hat{A}/\hat{I}^n)$ and

$$G_{1,n} \times \mathrm{Spec}(\hat{A}/\hat{I}^n[1/f]) \cong G_{2,n} \times \mathrm{Spec}(\hat{A}/\hat{I}^n[1/f]). \tag{4.5.4}$$

By Proposition 4.6.2 we get a p-divisible group H_n over $\mathrm{Spec}(A/I^n)$ agreeing with $G_{1,n}$ over $\mathrm{Spec}(A/I^n[1/f])$, agreeing with $G_{2,n}$ over $\mathrm{Spec}(\hat{A}/\hat{I}^n)$ and inducing the isomorphism (4.5.4). Taking the limit for $n \to \infty$ (Lemma 2.4.4) gives a p-divisible group H over Spec(A), isomorphic to G_1 over $\mathrm{Spec}(A\{1/f\})$, isomorphic to G_2 over $\mathrm{Spec}(\hat{A})$ and inducing (4.5.3).

To prove that $\mathbf{D}(H) \cong \mathscr{E}$ we use the description of crystals in terms of modules with connections ([BM, 1.3.3] and 2.2.2) over suitable lifts of A, $\hat{A}$, ... modulo p^n. The argument is similar to the last part of the proof of 4.4 and is left to the reader.

4.6. *A remark on descent*

Only included for quick reference.

Suppose A is a Noetherian ring and $I \subset A$ is an ideal. Put $X = \mathrm{Spec}(A)$, $Z = \mathrm{Spec}(A/I)$, $U = X \backslash Z$, $Y = \mathrm{Spec}(\hat{A})$ where

$$\hat{A} = \varprojlim A/I^n,$$

and $U' = Y\backslash Z = U \times_X Y$. We have the morphisms

$$p_1 : U' \to U \quad \text{and} \quad p_2 : U' \to Y.$$

We consider the following category:

(4.6.1) Objects are triples $(\mathscr{E}_U, \mathscr{E}_Y, \alpha)$ where

— $\mathscr{E}_U$ (resp. $\mathscr{E}_Y$) is a coherent sheaf of $\mathcal{O}_U$(resp. $\mathcal{O}_Y$)-modules,

— α is an isomorphism

$$\alpha : p_1^*(\mathscr{E}_U) \xrightarrow{\sim} p_2^*(\mathscr{E}_Y).$$

Morphisms $(f_U, f_Y) : (\mathscr{E}_U, \mathscr{E}_Y, \alpha) \to (\mathscr{E}'_U, \mathscr{E}'_Y, \alpha')$ are module homomorphisms $f_U : \mathscr{E}_U \to \mathscr{E}'_U$ and $f_Y : \mathscr{E}_Y \to \mathscr{E}'_Y$ compatible with α and α'.

4.6.2. *Proposition.* — *The natural functor*

$$F : \textit{coherent } \mathcal{O}_X\textit{-modules} \to (4.6.1)$$

is an equivalence of categories.

Proof. — For any $(\mathscr{E}_U, \mathscr{E}_Y, \alpha) \in \mathrm{Ob}\,(4.6.1)$ put

$$\Gamma((\mathscr{E}_U, \mathscr{E}_Y, \alpha)) := \{(s_U, s_Y) \in \Gamma(U, \mathscr{E}_U) \times \Gamma(Y, \mathscr{E}_Y) \mid \alpha(p_1^*(s_U)) = p_2^*(s_Y)\}.$$

It is easy to show that for a coherent $\mathcal{O}_X$-module $\mathscr{E}$ we have $\Gamma(X, \mathscr{E}) = \Gamma(F(\mathscr{E}))$. Thus F is fully faithful. Returning to our object $(\mathscr{E}_U, \mathscr{E}_Y, \alpha)$ of (4.6.1), it is easy to find a finitely generated submodule $M \subset \Gamma((\mathscr{E}_U, \mathscr{E}_Y, \alpha))$ such that $\widetilde{M}_U \to \mathscr{E}_U$ is an isomorphism ($\widetilde{M}$ is the sheaf on X associated to the A-module M). Then $\mathscr{E}_Y/\widetilde{M}_Y$ will be a coherent sheaf of $\mathcal{O}_Y$-modules with zero restriction on $U' \subset Y$, hence killed by some power of $\hat{I} = I.\hat{A}$. Thus $(0, \mathscr{E}_Y/\widetilde{M}_Y, 0)$ is in the essential image of F. To show that $(\mathscr{E}_U, \mathscr{E}_Y, \alpha)$ is in the essential image of F also we note that F identifies Ext-groups. If N is a finitely generated A-module annihilated by some power of I, then the natural map

$$\mathrm{Ext}^1_A(N, M) \to \mathrm{Ext}^1_A(N, M \otimes \hat{A})$$

is an isomorphism. □

5. Proof of Main Theorem 2

In this chapter we prove the second main theorem. It implies that the crystalline Dieudonné module functor is fully faithful up to isogeny over schemes of finite type over a field with a finite p-basis.

The proof of the theorem relies on ideas of Berthelot on convergent isocrystals. Let me try to indicate briefly two of the main ideas of the article [B2]. See also [B3].

Take a reduced scheme S of finite type over a field k of characteristic $p > 0$. Choose a Cohen ring R for the field k. Consider a formal scheme $\mathfrak{X}$, formally smooth over Spf(R), whose underlying reduced scheme $\mathfrak{X}_{\mathrm{red}}$ is equal to S. Note that we do not assume that $\mathfrak{X}$ is of finite type over Spf(R). For example we can construct $\mathfrak{X}$ by taking a closed immersion of S into a scheme X which is smooth over Spec(R) and taking $\mathfrak{X}$

to be the completion of X along S. To the formal scheme $\mathfrak{X}$ we associate a smooth rigid analytic space $\mathfrak{X}^{\mathrm{rig}}$ (see Chapter 7 or [B2, 0.2.6]). If $\mathfrak{X}$ is the completion of X as above, this is just the admissible open subvariety of the rigid analytic variety X^{rig} consisting of points $x \in X^{\mathrm{rig}}$ specializing to points of S. Such a rigid analytic variety $U = \mathfrak{X}^{\mathrm{rig}}$ is called a tube of S. Any two tubes U_1, U_2 of S are dominated by a third tube U and we may assume that the morphisms $U \to U_i$ are fibrations whose fibres are open unit balls. Therefore we get a cohomology theory of varieties S in characteristic p by associating to S the cohomology of one of its tubes.

Berthelot constructs a functor from the category of Dieudonné crystals over S to the category of convergent F-isocrystals over U: these are finite, locally free sheaves of $\mathcal{O}_U$-modules endowed with an integrable connection and an action of F (suitably defined). This last category is independent of the tube U chosen. Berthelot shows that this functor is fully faithful up to isogeny.

5.1. *Statement of Main Theorem 2*

5.1.1. *Theorem. — Let k be a field with a finite p-basis. Suppose that* S *is a* reduced *scheme of finite type over* Spec(k). *For any two p-divisible groups* G_1, G_2 *over* S *we have*

$$\text{(5.1.1.1)} \qquad \mathrm{Hom}_{DC/S}(\mathbf{D}(G_2), \mathbf{D}(G_1)) = \text{torsion subgroup} \oplus \mathbf{D}(\mathrm{Hom}_S(G_1, G_2)).$$

That is, for any $\varphi : \mathbf{D}(G_2) \to \mathbf{D}(G_1)$ *there is a unique* $\psi : G_1 \to G_2$ *over* S *such that* $\varphi - \mathbf{D}(\psi)$ *is torsion.* □

The proof of this theorem is long and complicated and will occupy most of the rest of this chapter. But first let us deduce a corollary of it.

5.1.2. *Corollary. — Let* S *be a scheme of characteristic p which has a finite open covering* $S = \bigcup_i U_i$ *such that each* U_i *is of finite type over a field* k_i *which has a finite p-basis. The crystalline Dieudonné module functor* $\mathbf{D}$ (2.3.3) *is fully faithful up to isogeny over* S.

Proof. — Suppose G_1, G_2 are p-divisible groups over S. We have to show that $\mathbf{D}$ induces a bijection

$$\text{(5.1.2.1)} \qquad \mathrm{Hom}_S(G_1, G_2) \otimes \mathbf{Q} \cong \mathrm{Hom}_{DC/S}(\mathbf{D}(G_2), \mathbf{D}(G_1)) \otimes \mathbf{Q}.$$

Here DC/S stands for the category of Dieudonné crystals over S (2.3.2). Of course we may replace S by any of the U_i; thus the reduction S′ of S satisfies the assumptions of 5.1.1. On the other hand, it is well known that the maps

$$\mathrm{Hom}_S(G_1, G_2) \otimes \mathbf{Q} \to \mathrm{Hom}_{S'}(G_{1,S'}, G_{2,S'}) \otimes \mathbf{Q}$$

and

$$\mathrm{Hom}_{DC/S}(\mathbf{D}(G_2), \mathbf{D}(G_1)) \otimes \mathbf{Q} \to \mathrm{Hom}_{DC/S'}(\mathbf{D}(G_{2,S'}), \mathbf{D}(G_{1,S'})) \otimes \mathbf{Q}$$

are isomorphisms. In both cases this can be seen using that some power of the Frobenius morphism of S factors through S′. Consequently 5.1.2 follows from 5.1.1. □

5.2. *Formal setup*

Take S, G_1 and G_2 as in the theorem. If we prove equality in (5.1.1.1) over the members of a finite open cover of S then equality in (5.1.1.1) follows for S. Thus we may and do assume that S is connected, affine and of finite type over a field k with a finite p-basis. The proof of 5.1.1 is by induction on dim(S). The case dim(S) = 0 follows from Theorem 4.1.1. Our induction hypothesis is now that we have proven 5.1.1 for all schemes T as in 5.1.1 with dim T < dim S.

Suppose we are given a homomorphism of Dieudonné crystals over S:

(5.2.1) $\varphi : \mathbf{D}(G_2) \to \mathbf{D}(G_1).$

We are looking for an element

$$\psi \in \mathrm{Hom}_S(G_1, G_2)$$

such that $\varphi - \mathbf{D}(\psi)$ is torsion. If we solve this problem for

$$\begin{array}{ccc} \varphi' : \mathbf{D}(G_1 \times_S G_2) & \longrightarrow & \mathbf{D}(G_1 \times_S G_2) \\ \downarrow\cong & & \downarrow\cong \\ \mathbf{D}(G_1) \oplus \mathbf{D}(G_2) & \longrightarrow & \mathbf{D}(G_1) \oplus \mathbf{D}(G_2) \end{array}$$

given by the matrix

$$\begin{pmatrix} \mathbf{D}(\mathrm{id}_{G_1}) & \varphi \\ 0 & \mathbf{D}(\mathrm{id}_{G_2}) \end{pmatrix},$$

then we have also solved the problem for φ (just put ψ equal to

$$G_1 \to G_1 \times_S G_2 \xrightarrow{\psi'} G_1 \times_S G_2 \to G_2).$$

Thus we may assume that φ is an isomorphism.

Let us choose a closed immersion $S \hookrightarrow \mathbf{A}^n_k$ for some $n \in \mathbf{N}$. We consider the formal scheme $\mathfrak{X}_0$ which is the completion of $\mathbf{A}^k_n$ along S. Thus $\mathfrak{X}_0 = \mathrm{Spf}(A)$ is an affine formal scheme, A is of type (1.3.2.1) and $S \cong (\mathfrak{X}_0)_{red}$. By [Ill2, Theorem 4.4] we can deform G_1 (resp. G_2) to a p-divisible group $H'_{1,0}$ (resp. $H'_{2,0}$) over $\mathfrak{X}_0$ (see 2.4.2). By Corollary 3.2.4 (see also Remark 3.2.3) we can find a natural number N, such that we can deform $H'_{1,0}$ and $H'_{2,0}$ over the closed immersion

$$\mathfrak{X}_0 = \mathrm{Spf}(A) \to \mathfrak{Y}_0 = \mathrm{Spf}(A[[x_1, \ldots, x_N]])$$

to p-divisible groups $H_{1,0}$ and $H_{2,0}$ over $\mathfrak{Y}_0$ which are simultaneously versal: the Kodaira-Spencer maps (see 2.5.4) of $\mathbf{D}(H_{1,0})$ and $\mathbf{D}(H_{2,0})$ induce a surjection

(5.2.2) $$\bigoplus_{i=1}^{N} \mathcal{O}_{\mathfrak{Y}_0} \frac{\partial}{\partial x_i} \to \mathcal{H}om_{\mathcal{O}_{\mathfrak{Y}_0}}(\omega_{1,0}, \alpha_{1,0}) \oplus \mathcal{H}om_{\mathcal{O}_{\mathfrak{Y}_0}}(\omega_{2,0}, \alpha_{2,0}).$$

Here we have written $\omega_{i,0}$ and $\alpha_{i,0}$ for the sheaves ω and α constructed in Section 2.5.3 for the Dieudonné crystal $\mathbf{D}(H_{i,0})$ over $\mathfrak{Y}_0$.

Let us take a lift $(\mathcal{O}, \sigma)$ of k and put K equal to the quotient field of $\mathcal{O}$. Next, we take a lift $(\widetilde{A}, \sigma)$ of A over $(\mathcal{O}, \sigma)$, i.e., such that there is a homomorphism $\mathcal{O} \to \widetilde{A}$ compatible with σ. (This is easy to find by lifting $\mathbf{A}_k^n$ to $\mathbf{A}_{\mathcal{O}}^n$ and taking the completion of $\mathbf{A}_{\mathcal{O}}^n$ along $S \subset \mathbf{A}_k^n \subset \mathbf{A}_{\mathcal{O}}^n$.) As a " lift " of $\mathfrak{Y}_0$ we take the formal scheme

$$\mathfrak{Y} := \mathrm{Spf}(\widetilde{A}[[x_1, \ldots, x_N]]).$$

We have a lift of Frobenius $\sigma : \mathfrak{Y} \to \mathfrak{Y}$ given by σ on $\widetilde{A}$ and $\sigma(x_i) = x_i^p$. Finally, we again use [Ill2, Theorem 4.4] to lift $H_{1,0}$ (resp. $H_{2,0}$) to a p-divisible group H_1 (resp. H_2) over $\mathfrak{Y}$. (This can be done as follows: $H_{i,0}$ may be considered as a p-divisible group over $\mathrm{Spec}(A[[\underline{x}]])$ by Lemma 2.4.4, then lift $H_{i,0}$ successively to $\mathrm{Spec}(\widetilde{A}[[\underline{x}]]/p^l\,\widetilde{A}[[\underline{x}]])$ using the theorem of Illusie.)

The value $\mathbf{D}(H_i)_{\mathfrak{Y}}$ of the Dieudonné crystal $\mathbf{D}(H_i)$ over $\mathfrak{Y}$ (defined by a suitable inverse limit as in section 2.5) is endowed with a canonical integrable, topologically quasi-nilpotent connection (compare 2.2.3)

$$\text{(5.2.3)} \qquad \nabla : \mathbf{D}(H_i)_{\mathfrak{Y}} \to \mathbf{D}(H_i)_{\mathfrak{Y}} \otimes_{\mathcal{O}_{\mathfrak{Y}}} \hat{\Omega}^1_{\mathfrak{Y}}.$$

By [BBM, 3.3.5] there is a Hodge filtration

$$0 \to \omega_i \to \mathbf{D}(H_i)_{\mathfrak{Y}} \to \alpha_i \to 0$$

reducing to the filtration $\omega_{i,0} \subset \mathbf{D}(H_{i,0})_{\mathfrak{Y}_0}$ modulo $p\mathcal{O}_{\mathfrak{Y}}$. Hence, we see that (5.2.3) still induces a surjection:

$$\text{(5.2.4)} \qquad \bigoplus_{i=1}^{N} \mathcal{O}_{\mathfrak{Y}} \frac{\partial}{\partial x_i} \to \mathscr{H}om_{\mathcal{O}_{\mathfrak{Y}}}(\omega_1, \alpha_1) \oplus \mathscr{H}om_{\mathcal{O}_{\mathfrak{Y}}}(\omega_2, \alpha_2).$$

We also have canonical horizontal isomorphisms

$$\sigma^* \mathbf{D}(H_i)_{\mathfrak{Y}} \cong \mathbf{D}(\sigma^* H_i)_{\mathfrak{Y}} \cong \mathbf{D}(\mathrm{Frob}^*_{\mathfrak{Y}_0}(H_{i,0}))_{(\mathfrak{Y}_0, \mathfrak{Y}, \gamma)},$$

hence Frobenius and Verschiebung induce horizontal homomorphisms

$$F : \sigma^* \mathbf{D}(H_i)_{\mathfrak{Y}} \to \mathbf{D}(H_i)_{\mathfrak{Y}} \quad \text{and} \quad V : \mathbf{D}(H_i)_{\mathfrak{Y}} \to \sigma^* \mathbf{D}(H_i)_{\mathfrak{Y}}$$

satisfying the usual relations.

5.3. *Rigid analytic setup*

To the formal scheme $\mathfrak{Y}$ there is associated a rigid analytic space $Y = \mathfrak{Y}^{\mathrm{rig}}$ over K. See [B2, 0.2.6], see also chapter 7. It is a smooth rigid analytic variety over K, since it is an open subspace of $\mathbf{A}_K^{n+N, \mathrm{rig}}$ (see above). To the finite locally free sheaves ω_i, α_i and $\mathbf{D}(H_i)_{\mathfrak{Y}}$ over $\mathfrak{Y}$ there are associated finite locally free sheaves of $\mathcal{O}_Y$-modules ω_i^{rig}, α_i^{rig} and $\mathbf{D}(H_i)^{\mathrm{rig}}$ on Y. See 7.1.11. The lift σ on $\mathfrak{Y}$ induces a general morphism

$\sigma : Y \to Y$ over the continuous homomorphism $\sigma : K \to K$. See 7.2.6. Frobenius and Verschiebung on $\mathbf{D}(H_i)_{\mathfrak{Y}}$ induce

$$F : \sigma^* \mathbf{D}(H_i)^{rig} \to \mathbf{D}(H_i)^{rig} \quad \text{and} \quad V : \mathbf{D}(H_i)^{rig} \to \sigma^* \mathbf{D}(H_i)^{rig}.$$

In addition, the connection ∇ (5.2.3) induces an integrable connection (see 7.1.12)

$$\nabla^{rig} : \mathbf{D}(H_i)^{rig} \to \mathbf{D}(H_i)^{rig} \otimes_{\mathcal{O}_Y} \Omega_Y.$$

Again we have that surjectivity of (5.2.4) implies surjectivity of the " Kodaira-Spencer " maps of $0 \to \omega_i^{rig} \to \mathbf{D}(H_i)^{rig} \to \alpha_i^{rig} \to 0$ induced by ∇^{rig}, i.e., the following homomorphism is surjective:

$$\textbf{(5.3.1)} \qquad \bigoplus_{i=1}^{N} \mathcal{O}_Y \frac{\partial}{\partial x_i} \to \mathcal{H}om_{\mathcal{O}_Y}(\omega_1^{rig}, \alpha_1^{rig}) \oplus \mathcal{H}om_{\mathcal{O}_Y}(\omega_2^{rig}, \alpha_2^{rig}).$$

We claim that $\mathbf{D}(H_i)^{rig}$ is canonically isomorphic to the convergent isocrystal $\mathbf{D}(G_i)^{an}$ over Y associated to the Dieudonné crystal $\mathbf{D}(G_i)$ on CRIS(S/Σ). See [B2, 2.4.1]. This is easy to prove and left to the reader. In fact we only need this to get the homomorphism of sheaves of $\mathcal{O}_Y$-modules

$$\varphi^{an} : \mathbf{D}(H_2)^{rig} = \mathbf{D}(G_2)^{an} \to \mathbf{D}(G_1)^{an} = \mathbf{D}(H_1)^{rig},$$

constructed in [B2, 2.4]; it is deduced from φ by functoriality of the construction of an associated convergent isocrystal. Let us recall its construction.

The Dieudonné crystal $\mathbf{D}(H_{i,0})$ is given by a Dieudonné module $(M_i, \nabla_i, F_i, V_i)$ over $\widetilde{A}[[\underline{x}]]$ (see 2.2.3). The closed immersion $S \hookrightarrow \mathfrak{Y}_0 \hookrightarrow \mathfrak{Y}$ is given by an ideal $I \subset \widetilde{A}[[\underline{x}]]$. We remark that I is the biggest ideal of definition of $\widetilde{A}[[\underline{x}]]$ and that $p\widetilde{A}[[\underline{x}]] + x_1 \widetilde{A}[[\underline{x}]] + \ldots + x_N \widetilde{A}[[\underline{x}]] \subset I$. Consider the divided power algebra $(\hat{D}, \hat{I}, [\ ])$ constructed for the pair $I \subset \widetilde{A}[[\underline{x}]]$ in (2.2.1.1). The Dieudonné module defined by G_i is the Dieudonné module $(M_i \hat{\otimes} \hat{D}, \nabla_i \otimes 1 + 1 \otimes \nabla, F_i \otimes 1, V_i \otimes 1)$ over $\hat{D}$. Thus our φ comes from an isomorphism

$$\textbf{(5.3.2)} \qquad \varphi : M_2 \hat{\otimes} \hat{D} \to M_1 \hat{\otimes} \hat{D}$$

of Dieudonné modules.

Let us consider the ring B_n of 7.1.1 constructed starting out with $I \subset \widetilde{A}[[\underline{x}]]$. It is the p-adic completion of $\widetilde{A}[[\underline{x}]]\,[I^n/p]$. As in 7.1.3 the space $Y = \mathfrak{Y}^{rig}$ is the increasing union $Y = \bigcup V_n$ of the affinoid subvarieties $V_n = Sp(C_n) = Sp(B_n \otimes_{\mathcal{O}} K)$. Berthelot [B2, (2.4.1.2)] constructs a homomorphism of $\widetilde{A}[[\underline{x}]]$-albegras

$$\textbf{(5.3.3)} \qquad \rho : \hat{D} \to B_1$$

mapping $i^{[n]}$ to $i^n/n!$ for all $i \in I$. To prove that it exists take generators $f_1, \ldots, f_r$ of $I \subset \widetilde{A}[[\underline{x}]]$ and recall that any element x of $\hat{D}$ can be written as

$$x = \sum_{M=(m_1,\ldots,m_r)} a_M f_1^{[m_1]} f_2^{[m_2]} \ldots f_r^{[m_r]}$$

with $a_M \in \tilde{A}[[\underline{x}]]$ and $a_M \to 0$ p-adically if $|M| \to \infty$. Thus the sum

$$\rho(x) := \sum_M a_M \frac{1}{m_1!\, m_2! \ldots m_r!} f_1^{m_1} \ldots f_r^{m_r}$$

converges in B_1 since $f^m/m! \in B_1$ for $f \in I$. We also remark that since $\sigma(I) \subset I^p + p\tilde{A}[[\underline{x}]]$ it induces homomorphisms

(5.3.4) $$\sigma : B_n \to B_{pn}, \quad n = 1, 2, 3, \ldots$$

We conclude that $\sigma(\mathrm{Sp}(C_{n+1})) \subset \mathrm{Sp}(C_n)$, i.e., $\sigma(V_{n+1}) \subset V_n$.

The construction of φ^{an} is as follows. Using (5.3.2) and ρ we get an isomorphism

(5.3.5) $$M_2 \hat{\otimes}_{\tilde{A}[[\underline{x}]]} B_1 \cong M_2 \hat{\otimes} \hat{D} \hat{\otimes}_\rho B_1 \to M_1 \hat{\otimes} \hat{D} \hat{\otimes}_\rho B_1 \cong M_1 \hat{\otimes}_{\tilde{A}[[\underline{x}]]} B_1.$$

Hence, we get an isomorphism $M_2 \hat{\otimes}_{\tilde{A}[[\underline{x}]]} C_1 \to M_2 \hat{\otimes}_{\tilde{A}[[\underline{x}]]} C_1$ or, equivalently, an isomorphism

$$\varphi_1 : \mathbf{D}(H_2)^{rig}|_{V_1} \to \mathbf{D}(H_1)^{rig}|_{V_1}.$$

This isomorphism is horizontal for ∇^{rig} and compatible with F_i^{rig} and V_i^{rig} since φ is so. Next we define

$$\varphi_n : \mathbf{D}(H_2)^{rig}|_{V_n} \to \mathbf{D}(H_1)^{rig}|_{V_n}$$

by induction on n as follows:

(5.3.6) $$\varphi_{n+1} := \frac{1}{p} (F_1^{rig} \text{ on } \mathbf{D}(H_1)^{rig})|_{V_{n+1}} \circ \sigma^*(\varphi_n) \circ (V_2^{rig} \text{ on } \mathbf{D}(H_2)^{rig})|_{V_{n+1}}.$$

Thus it is $1/p$ times the composition

$$\mathbf{D}(H_2)^{rig}|_{V_{n+1}} \xrightarrow{V_2^{rig}} \sigma^* \mathbf{D}(H_2)^{rig}|_{V_{n+1}} \xrightarrow{\sigma^*(\varphi_n)} \sigma^* \mathbf{D}(H_1)^{rig}|_{V_{n+1}} \xrightarrow{F_1^{rig}} \mathbf{D}(H_1)^{rig}|_{V_{n+1}}.$$

We have used $\sigma(V_{n+1}) \subset V_n$ to get $\sigma^*(\varphi_n)$ over V_{n+1}. Remark that since φ_n is compatible with F and V we get $\varphi_{n+1}|_{V_n} = \varphi_n$. It follows easily that φ_{n+1} is horizontal for ∇^{rig} and compatible with F_i^{rig} and V_i^{rig} since φ_n is so. The compatible system of maps $(\varphi_n)_{n \geq 1}$ defines our isomorphism

(5.3.7) $$\varphi^{an} := \varinjlim \varphi_n : \mathbf{D}(H_2)^{rig} \to \mathbf{D}(H_1)^{rig}.$$

It is horizontal and compatible with F and V. We remark that φ^{an} is the unique (horizontal) map $\mathbf{D}(H_2)^{rig} \to \mathbf{D}(H_1)^{rig}$ extending φ_1. (If χ where a second such map then $\varphi^{an} - \chi$ would be a morphism of finite locally free sheaves over the connected smooth rigid analytic space Y (see 7.4) zero on a nonempty open subset V_1 of Y, hence zero.) In particular, φ^{an} does not depend on the choice of σ on $\mathfrak{Y}$.

5.4. *Relation between rigid points and* φ

5.4.1. *Situation.* — Here $\mathcal{O} \subset \mathcal{O}'$ is a finite extension of complete discrete valuation rings, $K' = \mathcal{O}' \otimes K$ is the quotient field of $\mathcal{O}'$. Further, $x : \mathrm{Spf}(\mathcal{O}') \to \mathfrak{Y}$ is a morphism of formal schemes over $\mathrm{Spf}(\mathcal{O})$. To this there is associated a morphism of rigid analytic spaces over K (see § 7):

(5.4.1.1) $\quad x^{\mathrm{rig}} : \mathrm{Sp}(K') \to Y.$

Its image consists of one point, denoted $x^{\mathrm{rig}} \in Y$ by abuse of notation. Any point of Y occurs as x^{rig} for some x as above (see 7.1.10). Let $x_0 = \mathrm{Spec}(k') = \mathrm{Spf}(\mathcal{O}')_{\mathrm{red}}$ be the unique point of $\mathrm{Spf}(\mathcal{O}')$. The morphism x induces

(5.4.1.2) $\quad x_{\mathrm{red}} : x_0 = \mathrm{Spec}(k') \to \mathfrak{Y}_{\mathrm{red}} = S.$

This induces a morphism of crystalline topoi $(x_0/\Sigma)_{\mathrm{CRIS}} \to (S/\Sigma)_{\mathrm{CRIS}}$. Hence we get, putting $G_{i,x_0} := x_{\mathrm{red}}^*(G_i)$, an isomorphism of Dieudonné crystals over $\mathrm{CRIS}(x_0/\Sigma)$:

$$x^*_{\mathrm{red,\,CRIS}}(\varphi) : \mathbf{D}(G_{2,x_0}) \to \mathbf{D}(G_{1,x_0}).$$

By Theorem 4.1.1 (k' is a finite extension of k, hence has a finite p-basis also) we get a corresponding isomorphism of p-divisible groups

$$\psi_{x_0} : G_{1,x_0} \to G_{2,x_0}.$$

Let us take a uniformizer $\pi \in \mathcal{O}'$. If $\ell \in \mathbf{N}$ is large enough then $\pi^\ell \mathcal{O}'$ has a divided power structure δ. Put $U = \mathrm{Spec}(\mathcal{O}'/\pi^\ell \mathcal{O}')$ and $T = \mathrm{Spf}(\mathcal{O}')$. The triple (U, T, δ) can be considered as an ind-object of $\mathrm{CRIS}(U/\Sigma)$. By obstruction theory and since $x_0 \hookrightarrow U$ is a thickening of x_0, the map $p^\ell \psi_{x_0}$ lifts (uniquely) to a homomorphism

$$\widetilde{p^\ell \psi_{x_0}} : H_{1,U} \to H_{2,U}.$$

It induces a homomorphism of $\mathcal{O}'$-modules

(5.4.1.3) $\quad \chi := \mathbf{D}(\widetilde{p^\ell \psi_{x_0}})_{(U,T,\delta)} : \mathbf{D}(H_{2,U})_{(U,T,\delta)} \to \mathbf{D}(H_{1,U})_{(U,T,\delta)}.$

Note that we have the identification of $\mathcal{O}'$-modules

(5.4.1.4) $\quad \mathbf{D}(H_{i,U})_{(U,T,\delta)} \cong M_i \otimes_{\widetilde{A}[[\underline{x}]]} \mathcal{O}'.$

On the other hand we have a canonical isomorphism of K′-vector spaces

(5.4.1.5) $\quad (x^{\mathrm{rig}})^* \, \mathbf{D}(H_i)^{\mathrm{rig}} \cong M_i \otimes_{\widetilde{A}[[\underline{x}]]} K'.$

If $x^{\mathrm{rig}} \in V_n$ then it is given by $x^\# : \widetilde{A}[[\underline{x}]] \to B_n \to \mathcal{O}'$ and thus

$$(x^{\mathrm{rig}})^* \, \mathbf{D}(H_i)^{\mathrm{rig}} \cong M_i \,\hat{\otimes}_{\widetilde{A}[[\underline{x}]]} C_n \otimes_{C_n} K' \cong M_i \otimes_{\widetilde{A}[[\underline{x}]]} K'.$$

We claim that we have the following equality

(5.4.1.6) $\chi \otimes \mathrm{id}_{K'} = p^\ell (x^{\mathrm{rig}})^* (\varphi^{\mathrm{an}})$

as homomorphisms $M_2 \otimes K' \to M_1 \otimes K'$, where we used the identifications (5.4.1.4) and (5.4.1.5).

We only need to prove (5.4.1.6) for the $\ell \in \mathbf{N}$ such that $\pi^\ell \mathcal{O}' = p\mathcal{O}'$. Indeed, if $\ell' > \ell$ then $\chi' = p^{\ell' - \ell}\chi$. We remark that, as $x^\#(I) \subset \pi\mathcal{O}'$ and $\sigma(I) \subset I^p + p\tilde{A}[[\underline{x}]]$, we get $x^\# \sigma^\ell(I) \subset p\mathcal{O}'$ and hence a commutative diagram

$$\begin{array}{ccc} \tilde{A}[[\underline{x}]] & \xrightarrow{\sigma^\ell} & \tilde{A}[[\underline{x}]] \\ \downarrow & & \downarrow{\scriptstyle x^\#} \\ \hat{D} & \xrightarrow{\quad\tau\quad} & \mathcal{O}' \end{array}$$

where $\tau : (\hat{D}, \hat{I}, [\ ]) \to (\mathcal{O}', p\mathcal{O}', \gamma)$ is a PD-homomorphism. We also have $x^\#(I^\ell) \subset p\mathcal{O}'$, hence $x^\#$ factorises as $\tilde{A}[[\underline{x}]] \to B_\ell \to \mathcal{O}'$. We remark finally that the maps τ and $B_\ell \to \mathcal{O}'$ fit into a commutative diagram:

$$\text{(5.4.1.7)} \qquad \begin{array}{ccc} \hat{D} & \xrightarrow{\rho} & B_1 \\ \downarrow{\scriptstyle \tau} & & \downarrow{\scriptstyle \sigma^\ell} \\ \mathcal{O}' & \longleftarrow & B_\ell \end{array}$$

The ℓ^{th}-iterate Frob_U^ℓ of the Frobenius morphism of U factors through x_0 as $U \xrightarrow{s} x_0 \overset{i}{\hookrightarrow} U$. Thus we see that $\widetilde{p^\ell \psi_{x_0}}$ is the composition:

$$H_{1,U} \xrightarrow{(F_{H_1})^\ell} (\mathrm{Frob}_U^\ell)^* H_{1,U} \xrightarrow{s^*(\psi_{x_0})} (\mathrm{Frob}_U^\ell)^* H_{2,U} \xrightarrow{(V_{H_2})^\ell} H_{2,U}.$$

Since $\mathbf{D}(\psi_{x_0}) = \varphi|_{\mathrm{CKIS}(x_0/\Sigma)}$ we see that χ is equal to the composition

$$\begin{array}{ccccccc} M_2 \otimes_{\tilde{A}[[\underline{x}]]} \mathcal{O}' & \xrightarrow{V_2^\ell} & M_2^{\sigma^\ell} \otimes_{\tilde{A}[[\underline{x}]]} \mathcal{O}' & & M_1^{\sigma^\ell} \otimes_{\tilde{A}[[\underline{x}]]} \mathcal{O}' & \xrightarrow{F_1^\ell} & M_1 \otimes_{\tilde{A}[[\underline{x}]]} \mathcal{O}' \\ & & \| & & \| & & \\ & & M_1 \otimes \hat{D} \otimes_\tau \mathcal{O}' & \xrightarrow{\varphi \otimes \mathrm{id}_{\mathcal{O}'}} & M_2 \otimes \hat{D} \otimes_\tau \mathcal{O}'. & & \end{array}$$

Working through the inductive procedure used to define φ_ℓ, we see that $p^\ell \varphi_\ell$ on $M_2 \otimes B_\ell$ is defined as the composition

$$\begin{array}{ccccccc} M_2 \otimes_{\tilde{A}[[\underline{x}]]} B_\ell & \xrightarrow{V_2^\ell} & M_2^{\sigma^\ell} \otimes_{\tilde{A}[[\underline{x}]]} B_\ell & & M_1^{\sigma^\ell} \otimes_{\tilde{A}[[\underline{x}]]} B_\ell & \xrightarrow{F_1^\ell} & M_1 \otimes_{\tilde{A}[[\underline{x}]]} B_\ell \\ & & \| & & \| & & \\ & & M_2 \otimes \hat{D} \otimes_\rho B_1 \otimes_{\sigma^\ell} B_\ell & \xrightarrow{\varphi \otimes \mathrm{id} \otimes \mathrm{id}} & M_1 \otimes \hat{D} \otimes_\rho B_1 \otimes_{\sigma^\ell} B_\ell. & & \end{array}$$

The equality (5.4.1.6) is now a consequence of the description of $\varphi_\ell = \varphi^{\mathrm{an}}|_{V_\ell}$ and χ given above and the commutativity of (5.4.1.7).

5.4.2. *Lemma.* — *In Situation* 5.4.1.

a) The following two statements are equivalent:

(i) *There exists a morphism* $\psi_x : H_{1,\mathcal{O}'} = x^*(H_1) \to x^*(H_2) = H_{2,\mathcal{O}'}$ *of p-divisible groups over* $\mathcal{O}'$ *lifting* ψ_{x_0}.

(ii) *There exists a morphism* $\alpha : H_{1,\mathcal{O}'} \to H_{2,\mathcal{O}'}$ *of p-divisible groups over* $\mathcal{O}'$ *such that*

$$(x^{\mathrm{rig}})^* (\varphi^{\mathrm{an}}) = \mathbf{D}(\alpha)_{\mathcal{O}'} \otimes \mathrm{id}_{K'}.$$

If these conditions are fulfilled then we have $\psi_x = \alpha$.

b) The following two statements are equivalent:

(i) *For some* $n \in \mathbf{N}$ *there exists a homomorphism* $\alpha : H_{1,\mathcal{O}'} \to H_{2,\mathcal{O}'}$ *of p-divisible groups over* $\mathcal{O}'$ *lifting* $p^n \psi_{x_0}$.

(ii) *The homomorphism* $(x^{\mathrm{rig}})^* (\varphi^{\mathrm{an}})$ *maps* $(x^{\mathrm{rig}})^* (\omega_2^{\mathrm{rig}})$ *into* $(x^{\mathrm{rig}})^* (\omega_1^{\mathrm{rig}})$.

Proof. — We choose an $\ell \in \mathbf{N}$ so big that $\pi^\ell \mathcal{O}'$ has a nilpotent divided power structure.

Let us prove *b)*. "(ii) $\Rightarrow$ (i)": By (5.4.1.6) we know that the homomorphism $\chi = \mathbf{D}(\widetilde{p^\ell \psi_{x_0}})_{(U,T,\delta)}$ maps $\omega_{H_{2,\mathcal{O}'}}$ into $\omega_{H_{1,\mathcal{O}'}}$. By [BM, 3.2.11], and [M, V Theorem 1.6] this implies that $\widetilde{p^\ell \psi_{x_0}}$ lifts to a homomorphism over $\mathcal{O}'$. "(i) $\Rightarrow$ (ii)": If α lifts $p^n \psi_{x_0}$ then $p^{\ell-n} \alpha$ lifts $p^\ell \psi_{x_0}$, hence also lifts $\widetilde{p^\ell \psi_{x_0}}$ (we may take ℓ large enough so that $\ell - n$ is positive). By (5.4.1.6) and the references above, we now see that $(x^{\mathrm{rig}})^* (\varphi^{\mathrm{an}})$ maps $(x^{\mathrm{rig}})^* (\omega_2^{\mathrm{rig}})$ into $(x^{\mathrm{rig}})^* (\omega_1^{\mathrm{rig}})$.

Let us prove *a)*. "(i) $\Rightarrow$ (ii)": If ψ_x lifts ψ_{x_0} then $p^\ell \psi_x$ lifts $p^\ell \psi_{x_0}$ and $\widetilde{p^\ell \psi_{x_0}}$. Hence, by (5.4.1.6), we get $p^\ell (x^{\mathrm{rig}})^* (\varphi^{\mathrm{an}}) = \mathbf{D}(p^\ell \psi_x)_{\mathcal{O}'} \otimes \mathrm{id}_{K'} = p^\ell \mathbf{D}(\psi_x)_{\mathcal{O}'} \otimes \mathrm{id}_{K'}$. Dividing by p^ℓ on both sides and putting $\alpha = \psi_x$ gives (ii). "(ii) $\Rightarrow$ (i)": If we have α as in (ii) then the conditions of *b)* (ii) are satisfied. Hence, there is an $\alpha' : H_{1,\mathcal{O}'} \to H_{2,\mathcal{O}'}$ lifting $p^n \psi_{x_0}$ with $\mathbf{D}(\alpha')_{\mathcal{O}'} \otimes \mathrm{id}_{K'} = p^n (x^{\mathrm{rig}})^* (\varphi^{\mathrm{an}})$ (see proof of part *b)*). Thus we see that $\mathbf{D}(\alpha' - p^n \alpha)_{\mathcal{O}'} = 0$, hence $\alpha' = p^n \alpha$. Since α' lifts $p^n \psi_{x_0}$ it follows that α lifts ψ_{x_0}. □

5.4.3. *Claim.* — There exists a closed formal subscheme $\mathfrak{Z} \subset \mathfrak{Y}$, which is formally smooth over $\mathrm{Spf}(\mathcal{O})$ (2.4.6), with $\mathfrak{Z}_{\mathrm{red}} = \mathfrak{Y}_{\mathrm{red}} = S$ and there exists an isomorphism of p-divisible groups over $\mathfrak{Z}$

$$\psi : H_1|_{\mathfrak{Z}} \to H_2|_{\mathfrak{Z}}$$

such that the homomorphism of $\mathcal{O}_{\mathfrak{Z}^{\mathrm{rig}}}$-modules

$$\mathbf{D}(\psi)^{\mathrm{rig}} : \mathbf{D}(H_2)^{\mathrm{rig}}|_{\mathfrak{Z}^{\mathrm{rig}}} = \mathbf{D}(H_2|_{\mathfrak{Z}})^{\mathrm{rig}} \to \mathbf{D}(H_1|_{\mathfrak{Z}})^{\mathrm{rig}} = \mathbf{D}(H_1)^{\mathrm{rig}}|_{\mathfrak{Z}^{\mathrm{rig}}}$$

is equal to $\varphi^{\mathrm{an}}|_{\mathfrak{Z}^{\mathrm{rig}}}$. The closed formal subscheme $\mathfrak{Z} \subset \mathfrak{Y}$ is characterized by the following property:

(5.4.3.1) Suppose $x : \mathrm{Spf}(\mathcal{O}') \to \mathfrak{Y}$ is as in 5.4.1. It factorizes through $\mathfrak{Z}$ if and only if the equivalent conditions of 5.4.2 *a)* hold.

In 5.5 we show that this implies the theorem. We will prove the claim in two steps. First, in 5.6, we prove it in the case that we already know 5.1.1 is true. This produces a closed formal subscheme $\mathfrak{Z}_1$ in the formal open subscheme of $\mathfrak{Y}$ whose underlying reduced scheme is the regular locus of S and a closed formal subscheme $\mathfrak{Z}_2$ in the formal scheme which is the completion of $\mathfrak{Y}$ along the singular locus of S, by Theorem 4.1.1 resp. induction. Using Proposition 7.5.2 and some analytic geometry we get $\mathfrak{Z} \subset \mathfrak{Y}$ by gluing $\mathfrak{Z}_1$ and $\mathfrak{Z}_2$.

5.5. *The claim implies the theorem*

Let us prove that our theorem follows from 5.4.3. The isomorphism ψ induces an isomorphism over $\mathfrak{Z}_{\text{red}} = S$:

$$\psi|_S : G_1 \to G_2$$

The difference $\delta := \varphi - \mathbf{D}(\psi|_S) \in \text{Hom}_{DC/S}(\mathbf{D}(G_2), \mathbf{D}(G_1))$ has

$$\delta^{an}|_{\mathfrak{Z}^{rig}} = \varphi^{an}|_{\mathfrak{Z}^{rig}} - \mathbf{D}(\psi|_{\mathfrak{Z}_{red}})|_{\mathfrak{Z}^{rig}} = \varphi^{an}|_{\mathfrak{Z}^{rig}} - \mathbf{D}(\psi)^{rig} = 0.$$

It is now a consequence of [B2, Theorem 2.4.2] that δ is torsion. This is seen as follows. Since δ^{an} is the unique horizontal extension of $\delta^{an}|_{\mathfrak{Z}^{rig}}$ we also have $\delta^{an} = 0$. We let $\tau : B_1 \to \hat{D}/p$-torsion be the homomorphism which is σ-linear with respect to $\tilde{A}[[\underline{x}]]$ such that $\tau(i/p) = (p-1)!\, i^{[p]}$, for all $i \in I$. This is well-defined as B_1 has no p-torsion. The homomorphism $\sigma : \hat{D} \to \hat{D}$ ($\hat{D}$ as in 5.3) can be factored modulo p-torsion as

$$\hat{D}/p\text{-torsion} \xrightarrow{\rho} B_1 \xrightarrow{\tau} \hat{D}/p\text{-torsion}.$$

The element

$$\delta \in \text{Hom}_{\tilde{A}[[\underline{x}]]}(M_2, M_1) \otimes_{\tilde{A}[[\underline{x}]]} \hat{D}$$

satisfies $p\delta = F_1 \circ \sigma_*(\delta) \circ V_2$ (it is a homomorphism of Dieudonné modules). Hence

$$p\delta = F_1 \circ \tau_*\, \rho_*(\delta) \circ V_2 = 0 \in \text{Hom}_{\tilde{A}[[\underline{x}]]}(M_2, M_1) \otimes_{\tilde{A}[[\underline{x}]]} \hat{D}/p\text{-torsion}$$

since $\delta^{an}|_{V_1} = 0$ implies $\rho_*(\delta) = 0$. Thus δ is torsion.

5.6. *Proof of* 5.4.3 *in case* 5.1.1 *is known*

The assumption that 5.1.1 holds is true for example if S is a regular scheme (Theorem 4.1.1) or if S is replaced by a scheme T with smaller dimension (this is our induction hypothesis). In this case we have a homomorphism of p-divisible groups over S

$$\psi_S : G_1 \to G_2$$

such that $\mathbf{D}(\psi_S) - \varphi$ is torsion. It follows that $\mathbf{D}(\psi_S)^{an} = \varphi^{an}$.

We define a closed formal subscheme $\mathfrak{Z} \subset \mathfrak{Y}$ as follows. Let $\mathfrak{Y}_n$ denote the n^{th} infinitesimal neighbourhood of S in $\mathfrak{Y}$. By obstruction theory, the homomorphism $p^n \psi_S$ lifts to a homomorphism

$$\widetilde{p^n \psi_S} : H_{1, \mathfrak{Y}_n} \to H_{2, \mathfrak{Y}_n}.$$

We put

$$\begin{aligned}\mathfrak{Z}_n = {} & \text{closed subscheme of } \mathfrak{Y}_n \text{ defined by} \\ & \text{the vanishing of the morphism } H_1[p^n]_{\mathfrak{Y}_n} \xrightarrow{\widetilde{p^n \psi_S}} H_{2, \mathfrak{Y}_n}.\end{aligned}$$

It is easy to see that any morphism of schemes $f : T \to \mathfrak{Y}_n$ factors through $\mathfrak{Z}_n$ if and only if $f^*(\psi_S)$ over $T \times_{\mathfrak{Y}_n} S$ lifts to a homomorphism over T. Thus we have $\mathfrak{Z}_n = \mathfrak{Y}_n \cap \mathfrak{Z}_{n+1}$ (scheme theoretic intersection) and therefore we get

$$\mathfrak{Z} := \varinjlim \mathfrak{Z}_n \subset \varinjlim \mathfrak{Y}_n = \mathfrak{Y}.$$

By the very definition of $\mathfrak{Z}$ the homomorphism ψ_S lifts to

$$\psi : H_1|_{\mathfrak{Z}} \to H_2|_{\mathfrak{Z}}.$$

In addition, (5.4.3.1) is clear from the definition of $\mathfrak{Z}$. Also if $x : \mathrm{Spf}(\mathcal{O}') \to \mathfrak{Z}$ is such a morphism then $x^*(\psi)$ is a lift of ψ_{x_0}. Hence, we get $(x^{\text{rig}})^* \, \mathbf{D}(\psi)^{\text{rig}} = (x^{\text{rig}})^* \, (\varphi^{\text{an}})$ for all points x^{rig} of $\mathfrak{Z}^{\text{rig}}$ (see 5.4.2). Thus, we see that we will have $\mathbf{D}(\psi)^{\text{rig}} = \varphi^{\text{an}}|_{\mathfrak{Z}^{\text{rig}}}$ if we can prove that $\mathfrak{Z}$ is formally smooth over $\mathrm{Spf}(\mathcal{O})$, for in that case $\mathfrak{Z}^{\text{rig}}$ will be a smooth rigid analytic space (see 7.1.12) and hence homomorphisms of locally free $\mathcal{O}_{\mathfrak{Z}^{\text{rig}}}$-modules are determined by their values at points of $\mathfrak{Z}^{\text{rig}}$.

Let $J_{\mathfrak{Z}} \subset \mathcal{O}_{\mathfrak{Y}}$ denote the ideal sheaf of $\mathfrak{Z}$ in $\mathfrak{Y}$. The obstruction to lift ψ to the first infinitesimal neighbourhood of $\mathfrak{Z}$ in $\mathfrak{Y}$ lies in

$$\mathcal{H}om_{\mathcal{O}_{\mathfrak{Y}}}(\omega_2, \alpha_1) \otimes_{\mathcal{O}_{\mathfrak{Y}}} J_{\mathfrak{Z}}/J_{\mathfrak{Z}}^2$$

by [Ill2, Theorem 4.4]; here we can also use [M, V Theorem 1.6]. By a standard argument, it follows that $\mathfrak{Z} \subset \mathfrak{Y}$ is locally given by (at most) dd^* equations, i.e., $J_{\mathfrak{Z}}$ is locally generated by (at most) dd^* sections. Here $d = \dim(G_1) = \dim(G_2) = \mathrm{rk}(\omega_2)$ and $d^* = \dim(G_1^t) = \dim(G_2^t) = \mathrm{rk}(\alpha_1)$.

Let us show that in any closed point $s \in S$ these equations define a linear subspace $\bar{J}$ of $\mathfrak{m}_s/(\mathfrak{m}_s^2 + p\mathcal{O}_{\mathfrak{Y}, s})$ of dimension at least dd^*. Here $\mathcal{O}_{\mathfrak{Y}, s}$ is the local ring of $\mathfrak{Y}$ at the point $s \in S = \mathfrak{Y}_{\text{red}}$ and $\mathfrak{m}_s \subset \mathcal{O}_{\mathfrak{Y}, s}$ is its maximal ideal. Put $R = \mathcal{O}_{\mathfrak{Y}, s}/(\mathfrak{m}_s^2 + p\mathcal{O}_{\mathfrak{Y}, s} + \bar{J})$; thus $\mathrm{Spec}(R) \to \mathfrak{Y}$ is a closed immersion factoring through $\mathfrak{Z} \subset \mathfrak{Y}$. Our ψ over $\mathfrak{Z}$ gives a homomorphism $H_{1, R} \to H_{2, R}$ and hence a horizontal homomorphism of Dieudonné modules over R compatible with filtrations. Thus we get a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \omega_2 \otimes R & \longrightarrow & M_2 \otimes R & \longrightarrow & \alpha_2 \otimes R & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \omega(\psi) \otimes R} & & \downarrow{\scriptstyle \mathbf{D}(\psi) \otimes R} & & \downarrow{\scriptstyle \alpha(\psi) \otimes R} & & \\ 0 & \longrightarrow & \omega_1 \otimes R & \longrightarrow & M_1 \otimes R & \longrightarrow & \alpha_1 \otimes R & \longrightarrow & 0 \end{array} \qquad (5.6.1)$$

and $\mathbf{D}(\psi) \otimes \mathrm{R}$ is horizontal for $\nabla_i : \mathrm{M}_i \otimes \mathrm{R} \to \mathrm{M}_i \otimes \Omega^1_{\mathrm{R}}$. By our choice of $\mathfrak{Y} = \mathrm{Spf}(\tilde{\mathrm{A}}[[\underline{x}]])$ there is a homomorphism

$$\Omega^1_{\mathrm{R}} \to \Omega^1_{\mathrm{R}/\tilde{\mathrm{A}}}$$

and it is clear that $k(s) \otimes \Omega^1_{\mathrm{R}/\tilde{\mathrm{A}}}$ is generated by the elements $1 \otimes \mathrm{d}x_i$. All the relations between these elements are determined as follows: write any element $f \in \bar{\mathrm{J}}$ as $f \equiv \Sigma a_i x_i + g$, $a_i \in k(s)$ with $g \in \mathfrak{m}_s \cap \tilde{\mathrm{A}}$, then the element $\Sigma a_i \otimes \mathrm{d}x_i$ is zero in $k(s) \otimes \Omega^1_{\mathrm{R}/\tilde{\mathrm{A}}}$. Recall that we have the Kodaira-Spencer map:

$$\kappa_i(\partial/\partial x_j) : \omega_i \otimes k(s) \to \alpha_i \otimes k(s)$$

for $j = 1, \ldots, \mathrm{N}$ and $i = 1, 2$. The composition

$$\omega_i \otimes \mathrm{R} \to \mathrm{M}_i \otimes \mathrm{R} \xrightarrow{\nabla_i} \mathrm{M}_i \otimes \Omega^1_{\mathrm{R}} \to \alpha_i \otimes \Omega^1_{\mathrm{R}} \to \alpha_i \otimes k(s) \otimes \Omega^1_{\mathrm{R}/\tilde{\mathrm{A}}}$$

is a map $\omega_i \otimes k(s) \to \alpha_i \otimes k(s) \otimes \Omega^1_{\mathrm{R}/\tilde{\mathrm{A}}}$. Using the definition of κ_i it is easy to see that under this map we have:

$$\eta \mapsto \text{class of } \sum_j \kappa_i(\partial/\partial x_j)(\eta) \otimes \mathrm{d}x_j.$$

At this point the commutativity of 5.6.1 implies that the homomorphism $\omega_2 \otimes k(s) \to \alpha_1 \otimes k(s) \otimes \Omega^1_{\mathrm{R}/\tilde{\mathrm{A}}}$ given by

$$\eta \mapsto \sum_j (\alpha(\psi)(\kappa_2(\partial/\partial x_j)(\eta)) - \kappa_1(\partial/\partial x_j)(\omega(\psi)(\eta))) \otimes \mathrm{d}x_j$$

must be zero. Taking a basis for ω_2 and a basis for α_1 we see that this gives dd^* elements in $\bigoplus k(s)\, \mathrm{d}x_j$ which should be zero in $\Omega^1_{\mathrm{R}/\tilde{\mathrm{A}}} \otimes k(s)$. Our assumption that (5.2.4) is surjective implies that these dd^* elements are $k(s)$-linearly independent. Hence, $\dim_{k(s)} \bar{\mathrm{J}} \geqslant dd^*$.

However, we have proved something which is even slightly stronger, namely: there exist dd^* elements $f_1, \ldots, f_{dd^*} \in \widehat{\mathcal{O}_{\mathfrak{Y}, s}} \cong \tilde{\mathrm{A}}^\wedge[[x_1, \ldots, x_{\mathrm{N}}]]$ generating the ideal $\mathrm{J}^\wedge$ of $\mathfrak{Z}$ in $\tilde{\mathrm{A}}^\wedge[[x_1, \ldots, x_{\mathrm{N}}]]$ such that modulo $(\mathfrak{m}_s^\wedge)^2$ the linear terms in the x_j of the f_i are linearly independent. The superscript $^\wedge$ refers to $\mathfrak{m}_s$-adic completion. We leave it to the reader to show that this implies that

$$\tilde{\mathrm{A}}^\wedge[[x_1, \ldots, x_{\mathrm{N}}]]/\mathrm{J}^\wedge \cong \tilde{\mathrm{A}}^\wedge[[y_1, \ldots, y_{\mathrm{N}-dd^*}]].$$

Thus this algebra is formally smooth over $\mathcal{O}$ (since $\tilde{\mathrm{A}}^\wedge$ is so). We conclude that the algebra $\Gamma(\mathfrak{Z}, \mathcal{O}_{\mathfrak{Z}}) = \tilde{\mathrm{A}}[[\underline{x}]]/\mathrm{J}$ is formally smooth over $\mathcal{O}$ since all its complete local rings at maximal ideals are so.

5.7. *Construction of the analytic closed subvariety* Z

To start with, we define $\mathrm{Z} \subset \mathrm{Y}$ as a *set* as follows:

(5.7.1)
$$\mathrm{Z} = \left\{ \begin{array}{l} x^{\mathrm{rig}} \in \mathrm{Y} \text{ such that the morphism } x : \mathrm{Spf}(\mathcal{O}') \to \mathfrak{Y} \\ \text{corresponding to it (see 7.1.10 and 5.4.1) satisfies} \\ \text{the equivalent conditions of Lemma 5.4.2 part } a). \end{array} \right\}$$

On the other hand we have an *analytic closed subvariety* $W \subset Y$ whose points x^{rig} are those such that x satisfies the equivalent conditions of Lemma 5.4.2 part *b)*. It is defined as follows:

$$\textbf{(5.7.2)} \qquad W = \left\{ \begin{array}{l} \text{The analytic closed subvariety of Y defined} \\ \text{by the vanishing of the homomorphism} \\ \omega_2^{\mathrm{rig}} \to \mathbf{D}(H_1)^{\mathrm{rig}}/\omega_1^{\mathrm{rig}} \cong \alpha_1^{\mathrm{rig}} \text{ induced by } \varphi^{\mathrm{an}}. \end{array} \right\}$$

Thus it is clear from Lemma 5.4.2 that $Z \subset W$ as a set.

5.7.3. *Lemma. — The rigid analytic closed subvariety* $W \subset Y$ *is smooth and all components of it have codimension* dd^* *in* Y.

Proof. — It is immediately clear from (5.7.2) that W is locally defined by (at most) dd^* equations in Y. The argument to prove that it is locally smooth of codimension dd^* is exactly the same as the proof of the corresponding fact for $\mathfrak{Z}$ in 5.6. (Use that φ^{an} is horizontal and that (5.3.1) is surjective.) □

Let us take a point $z_0 \in Z \subset W$. Recall that Y was defined as the countable union $Y = \bigcup_n V_n$ of affinoid varieties V_n over K. Let us decompose the affinoid variety $W \cap V_n$ into its connected (hence irreducible) components:

$$\textbf{(5.7.4)} \qquad W \cap V_n = W_{n,1} \cup W_{n,2} \cup \ldots \cup W_{n,i_n}.$$

Each $W_{n,i}$ is a connected smooth affinoid variety and the cover (5.7.4) is admissible. We define

$$\textbf{(5.7.5)} \qquad Z' = \bigcup_{(n,i) \text{ with } z_0 \in W_{n,i}} W_{n,i}$$

(we only take those $W_{n,i}$ which contain z_0). It is a connected rigid analytic variety since it is the increasing union of connected affinoids. Its complement W' in W can be written as

$$\textbf{(5.7.6)} \qquad W' = \bigcup_{(n,i)} W_{n,i},$$

where the union runs over those pairs (n, i) such that $W_{n,i} \not\subset Z'$. (We remark that this is not equivalent to $z_0 \notin W_{n,i}$ in general.) Since the covering

$$\textbf{(5.7.7)} \qquad W = \bigcup_n W \cap V_n = \bigcup_{(n,i)} W_{n,i}$$

is an admissible affinoid cover of W, it follows that both Z' and W' are admissible open subvarieties of W and that (5.7.5) and (5.7.6) define admissible affinoid coverings of Z' and W'. It also follows that $W = Z' \cup W'$ is an admissible open cover of W (it can be refined to (5.7.7)). Thus, we finally conclude that Z' is an analytic closed subvariety of W. Since W is analytically closed in Y, we also get that $Z' \subset Y$ is analytically closed.

By our definition of Z', any point $z \in Z'$ can be connected by curves in Z' to z_0 (see Chapter 6).

5.7.8. *Proposition.* — *a) For any morphism* $f: C \to Z'$ *of a reduced normal connected 1-dimensional affinoid variety* C *to* Z' *we have:*

$$\exists\, c \in C, \qquad f(c) \in Z \Leftrightarrow \mathrm{Im}(f: C \to W) \subset Z.$$

b) Any two points z_1, z_2 *of* Z *can be connected by curves in* Z, *i.e., there exist morphisms* $f_i: C_i \to Y$ *of connected affinoid curves* C_i *to* Y *such that* $\mathrm{Im}(f_i: C_i \to Y) \subset Z$ *(set theoretically) and connecting* z_1 *to* z_2 *as in chapter* 6.

5.7.9. *Corollary.* — *As subsets of* Y *we have* $Z = Z'$. *Thus* Z *has the natural structure of a smooth closed subvariety of codimension* dd^* *of* Y.

Proof of the corollary. — By the remark above the proposition we see that 5.7.8 part *a)* implies $Z' \subset Z$. Part *b)* implies that any point $z \in Z$ may be connected by curves in Z to $z_0 \in Z' \subset Z$. Of course these curves then map into W and hence they must lie in the connected component of W containing z_0. This is Z', hence $z \in Z'$. □

Proof of 5.7.8 *a)*. — The morphism of rigid analytic spaces $C \to Z' \to Y$ is given by a morphism

$$\mathfrak{f}: \mathfrak{C} \to \mathfrak{Y} \tag{5.7.10}$$

of $FS_{\mathcal{O}}$ where $\mathfrak{C}$ is a model for C (see 7.1.6 and 7.1.7). Thus $\mathfrak{C}$ is an affine formal scheme, flat and of finite type over $\mathrm{Spf}(\mathcal{O})$, say $\mathfrak{C} = \mathrm{Spf}(R)$. Since C is normal, we may normalize $\mathfrak{C}$ and hence assume that R is normal. Since $\dim(C) = 1$, we see that $T = \mathrm{Spec}(R)$ is an excellent normal two-dimensional scheme. Its singular points lie in $T_0 = \mathrm{Spec}(R/pR) \subset T$. By resolution of singularities of two-dimensional schemes [L], we see there exists a blow up T' of T such that T' is a regular scheme. This blow up is done in a subscheme of codimension two of T contained in $T_0 \subset T$ (Remark C on page 155 of [L]). We can blow up T' some more in closed points to reach the situation where all (reduced) irreducible components $T'_{0,i}$ of $T'_0 = V(p) \subset T'$ are regular (1-dimensional) schemes and the intersections of $T_{0,i}$ with $T_{0,j}$ are transversal for all $i \neq j$, i.e., such that $T_{0,i} \cap T_{0,j}$ (scheme theoretic intersection) is a reduced zero-dimensional scheme.

Let us consider the completion $\mathfrak{C}'$ of the scheme T' along its special fibre T'_0. It is a formal scheme of finite type over $\mathrm{Spf}(\mathcal{O})$ and it is well-known that the morphism $\mathfrak{C}' \to \mathfrak{C}$ induces an isomorphism $\mathfrak{C}'^{\mathrm{rig}} \to \mathfrak{C}^{\mathrm{rig}}$ of rigid analytic varieties over K. (See [R] or [BL, Theorem 4.1].) Hence, we may replace $\mathfrak{C}$ by $\mathfrak{C}'$ and assume that: $\mathfrak{C}_{\mathrm{red}} = \bigcup_i \mathfrak{C}_{\mathrm{red},i}$ where each $\mathfrak{C}_{\mathrm{red},i}$ is a regular 1-dimensional scheme of finite type over Spec (k) and the intersections of these are transversal. (Of course $\mathfrak{C}$ is no longer an affine formal scheme.)

These assumptions imply that the restriction

$$\varphi|_{\mathrm{CRIS}(\mathfrak{C}_{\mathrm{red},i}/\Sigma)}: \mathbf{D}(G_2)|_{\mathrm{CRIS}(\mathfrak{C}_{\mathrm{red},i}/\Sigma)} \to \mathbf{D}(G_1)|_{\mathrm{CRIS}(\mathfrak{C}_{\mathrm{red},i}/\Sigma)}$$

((5.7.10) induces $\mathfrak{C}_{\mathrm{red},i} \to \mathfrak{Y}_{\mathrm{red}} = S$) comes from a homomorphism of p-divisible groups (by Theorem 4.1.1) $G_1|_{\mathfrak{C}_{\mathrm{red},i}} \to G_2|_{\mathfrak{C}_{\mathrm{red},i}}$. Since the intersections are reduced,

$\mathbf{D}$ is fully faithful over $\mathfrak{C}_{\mathrm{red},i} \cap \mathfrak{C}_{\mathrm{red},j}$ (Theorem 4.1) and hence these homomorphisms glue to give

$$\psi_{\mathfrak{C}_{\mathrm{red}}} : \mathrm{G}_1|_{\mathfrak{C}_{\mathrm{red}}} \to \mathrm{G}_2|_{\mathfrak{C}_{\mathrm{red}}}.$$

As the closed immersion $\mathfrak{C}_{\mathrm{red}} \to \mathrm{Spec}(\mathcal{O}_{\mathfrak{C}}/p^2\,\mathcal{O}_{\mathfrak{C}}) =: \mathfrak{C}_2$ is nilpotent, there exists an $\ell \in \mathbf{N}$ such that $p^\ell\,\psi_{\mathfrak{C}_{\mathrm{red}}}$ lifts to a homomorphism

$$\widetilde{p^\ell\,\psi_{\mathfrak{C}_{\mathrm{red}}}} : \mathrm{H}_1|_{\mathfrak{C}_2} \to \mathrm{H}_1|_{\mathfrak{C}_2}.$$

Let us consider the triple $\Delta := (\mathfrak{C}_2, \mathfrak{C}, \gamma)$ as an ind-object of $\mathrm{CRIS}(\mathfrak{C}_2/\Sigma)$. Thus $\widetilde{p^\ell\,\psi_{\mathfrak{C}_{\mathrm{red}}}}$ induces a homomorphism

$$\textbf{(5.7.11)} \qquad \mathbf{D}(\widetilde{p^\ell\,\psi_{\mathfrak{C}_{\mathrm{red}}}})_\Delta : \mathbf{D}(\mathrm{H}_2)_\Delta \cong \mathbf{D}(\mathrm{H}_{2,\mathfrak{C}})_{\mathfrak{C}} \to \mathbf{D}(\mathrm{H}_{1,\mathfrak{C}})_{\mathfrak{C}} \cong \mathbf{D}(\mathrm{H}_1)_\Delta.$$

For any $x : \mathrm{Spf}(\mathcal{O}') \to \mathfrak{C} \to \mathfrak{Y}$ the pullback $x^*(\widetilde{p^\ell\,\psi_{\mathfrak{C}_{\mathrm{red}}}})$ is the unique lift of $p^\ell\,\psi_{x_0}$ over $\mathrm{Spec}(\mathcal{O}'/p^2\,\mathcal{O}')$. Hence, by (5.4.1.6), we see that

$$(x^{\mathrm{rig}})^*\,((5.7.11)^{\mathrm{rig}}) = ((\mathfrak{f} \circ x)^{\mathrm{rig}})^*\,(\psi^{\mathrm{an}}).$$

Recall that the morphism $f : \mathfrak{C}^{\mathrm{rig}} \to \mathrm{Y}$ factors through $\mathrm{Z}' \subset \mathrm{W}$. Hence by the equation just proved we see that (5.7.11) preserves the Hodge filtrations defined by $\mathrm{H}_{i,\mathfrak{C}}$. Thus by [BM, Corollary 2.3.11] and [M, V Theorem 1.6] we get a morphism of p-divisible groups (the divided power structure on $p^2\,\mathcal{O}_{\mathfrak{C}}$ is nilpotent)

$$\alpha : \mathrm{H}_{1,\mathfrak{C}} \to \mathrm{H}_{2,\mathfrak{C}}$$

which lifts $p^\ell\,\psi_{\mathrm{red}}$.

The scheme $\mathrm{H}_i[p^\ell]_{\mathfrak{C}}$ is a finite flat group scheme over $\mathfrak{C}$. Thus the associated rigid analytic variety $\mathrm{N}_i = \mathrm{H}_i[p^\ell]_{\mathfrak{C}}^{\mathrm{rig}}$ is a group variety over $\mathrm{C} = \mathfrak{C}^{\mathrm{rig}}$, which is finite flat over C. Since C lives in characteristic 0, the structure morphism $\mathrm{N}_i \to \mathrm{C}$ must be étale. Of course α induces $\alpha^{\mathrm{rig}} : \mathrm{N}_1 \to \mathrm{N}_2$. Now note that for all $c \in \mathrm{C}$ we have

$$f(c) \in \mathrm{Z} \Leftrightarrow \mathrm{Ker}(\alpha^{\mathrm{rig}})_c = \mathrm{N}_{1,c}.$$

Also $\mathrm{Ker}(\alpha^{\mathrm{rig}})$ is a closed analytic subvariety of N_1, hence is finite étale over C. To prove the assertion *a)* of Proposition 5.7.8 it suffices to note that for a finite étale morphism of rigid analytic spaces the cardinality of the fibres is locally constant on the base. □

Proof of 5.7.8 *part b)*. — We first prove that any two points z_1, z_2 of Z with $sp(z_1) = sp(z_2)$ may be connected by curves in Z. Here sp is the specialization morphism $\mathrm{Y} \to \mathfrak{Y}_{\mathrm{red}} = \mathrm{S}$ (see 7.1.10). To see this, we use that by 7.2.5 we have

$$sp^{-1}(s) \cong (\mathfrak{Y}_s^\wedge)^{\mathrm{rig}}, \qquad s \in \mathrm{S} \text{ closed point.}$$

Since $s = \mathrm{Spec}(k(s))$ is a regular scheme of finite type over k we get by 5.6 a solution $\mathfrak{Z}_s \subset \mathfrak{Y}_s^\wedge$ of the problem posed in 5.4.3; by (5.4.3.1) we have

$$sp^{-1}(s) \cap Z = \mathfrak{Z}_s^{\mathrm{rig}}.$$

(We remark that all the proporties of the data S, G_1, G_2, H_1, H_2, φ, $\mathfrak{Y}$, κ_i, etc., used in 5.6 still hold over the completion $\mathfrak{Y}_s^\wedge$ of $\mathfrak{Y}$. For example $\mathfrak{Y}_s^\wedge = \mathrm{Spf}(\widetilde{A}^\wedge[[x_1, \ldots, x_N]])$ and $\widetilde{A}^\wedge$ is a formally smooth $\mathcal{O}$-algebra, $(5.2.4)_s^\wedge$ is surjective over $\mathfrak{Y}_s^\wedge$ as it is the completion of (5.2.4), and so on.)

Thus $\mathfrak{Z}_s \cong \mathrm{Spf}(R)$ where R is a formally smooth, local complete $\mathcal{O}$-algebra. By Proposition 7.3.6 $\mathfrak{Z}_s^{\mathrm{rig}}$ is connected. Since it is a countable increasing union of affinoids, it is also the countable increasing union of connected affinoids. Applying Proposition 6.1.1 gives that any two points in $sp^{-1}(s) \cap Z$ can be connected by curves in Z.

To finish the proof of 5.7.8 *b)* let us take two closed points s_1, s_2 of S. We have to show that we may connect $sp^{-1}(s_1) \cap Z$ to $sp^{-1}(s_2) \cap Z$ by curves in Z. Since S is connected (see assumption in 5.2), we can connect s_1 to s_2 by curves in S: we can find a sequence of connected regular affine 1-dimensional schemes C_i of finite type over $\mathrm{Spec}(k)$ and morphisms

$$f_i : C_i \to S, \quad i = 1, \ldots, n$$

which connect s_1 to s_2:

1) $s_1 \in \mathrm{Im}(C_1 \to S)$, $s_2 \in \mathrm{Im}(C_n \to S)$,
2) $\mathrm{Im}(C_i \to S) \cap \mathrm{Im}(C_{i+1} \to S) \neq \emptyset$, $i = 1, \ldots, n-1$.

(We leave it to the reader to construct C_i and f_i; the hypothesis of regularity is trivial to establish: just take normalizations.) Therefore it suffices to connect the sets $sp^{-1}(f(c)) \cap Z$ by curves in Z if we are given a single such curve $f : C \to S$ as above.

To do this write $C = \mathrm{Spec}(R)$. The morphism f is given by a k-algebra homomorphism $A \to R$. Choose a lift $\widetilde{R}$ of R (see 1.2.2, 1.3.3 and (1.3.2.2)) and lift $\widetilde{A} \to \widetilde{R}$ of $A \to R$ (this is possible since $\widetilde{A}$ is formally smooth over $\mathbf{Z}_p$). We remark that $\widetilde{A} \to \widetilde{R}$ defines an $\mathcal{O}$-algebra structure on $\widetilde{R}$, compatible with the k-algebra structure on R. We extend this map to a homomorphism

$$\textbf{(5.7.12)} \qquad \widetilde{A}[[x_1, \ldots, x_N]] \to \widetilde{R}[[x_1, \ldots, x_N]]$$

in the obvious manner. This gives a morphism

$$\varepsilon : \mathfrak{Y}' = \mathrm{Spf}(\widetilde{R}[[x_1, \ldots, x_N]]) \to \mathrm{Spf}(\widetilde{A}[[x_1, \ldots, x_N]]) = \mathfrak{Y}.$$

At this point we have to check that all the assumptions on the data $\varepsilon^* H_1$, $\varepsilon^* H_2$, $\varepsilon_{\mathrm{red}}^* G_1$, $\varepsilon_{\mathrm{red}}^* G_2$, $\varepsilon_{\mathrm{red}}^*(\varphi)$, $\mathfrak{Y}'$ which are needed in 5.6 for the proof of 5.4.3 hold: By construction $\mathfrak{Y}' \to \mathrm{Spf}(\mathcal{O})$ is formally smooth and $\mathfrak{Y}'_{\mathrm{red}}$ is regular. Surjectivity of (5.2.4) over $\mathfrak{Y}'$ follows from our choice of (5.7.12). We also remark that we have $(\varepsilon^{\mathrm{rig}})^*(\varphi^{\mathrm{an}}) = \varepsilon^*_{\mathrm{red,CRIS}}(\varphi)^{\mathrm{an}}$; this is just the functoriality of the construction $(.)^{\mathrm{an}}$. Applying 5.6 to the situation over $\mathfrak{Y}'$ we get a closed formal subscheme $\mathfrak{Z}' \subset \mathfrak{Y}'$, with $\mathfrak{Z}'_{\mathrm{red}} = \mathfrak{Y}'_{\mathrm{red}} = C$, which is formally smooth over $\mathrm{Spf}(\mathcal{O})$. Thus $\mathfrak{Z}' = \mathrm{Spf}(\tilde{R}[[x_1, \ldots, x_N]]/J)$ and $\tilde{R}[[x_1, \ldots, x_N]]/J$ is a regular, hence normal, ring without nontrivial idempotents (since R has none). By 7.4.1 we see that $\Gamma((\mathfrak{Z}')^{\mathrm{rig}}, \mathcal{O}^0)$ has no nontrivial idempotents, hence $(\mathfrak{Z}')^{\mathrm{rig}}$ is connected. Thus any two points in $(\mathfrak{Z}')^{\mathrm{rig}}$ may be connected by curves in $(\mathfrak{Z}')^{\mathrm{rig}}$. (Arguments as before using Proposition 6.1.1.)

It follows from the characterisation of $\mathfrak{Z}' \subset \mathfrak{Y}'$ in 5.4.3 that the composition $(\mathfrak{Z}')^{\mathrm{rig}} \subset (\mathfrak{Y}')^{\mathrm{rig}} \to \mathfrak{Y}^{\mathrm{rig}} = Y$ maps into Z (set theoretically). Since we have the diagram

$$\begin{array}{ccccc} (\mathfrak{Z}')^{\mathrm{rig}} & \longrightarrow & (\mathfrak{Y}')^{\mathrm{rig}} & \longrightarrow & \mathfrak{Y}^{\mathrm{rig}} \\ \downarrow{\scriptstyle sp_{\mathfrak{Z}'}} & & \downarrow{\scriptstyle sp_{\mathfrak{Y}'}} & & \downarrow{\scriptstyle sp_{\mathfrak{Y}}} \\ C & \longrightarrow & C & \xrightarrow{f} & S \end{array}$$

and since $sp_{\mathfrak{Z}'}^{-1}(c) \neq \emptyset$ for all $c \in C$, we get the desired result. □

5.8. *End of the proof of* 5.4.3

Let us take an element $\bar{f} \in \Gamma(S, \mathcal{O}_S)$ such that:

1) $\dim V(\bar{f}) < \dim S$,
2) $S \setminus V(\bar{f})$ is a regular scheme.

This is possible as S is an affine reduced scheme of finite type over a field. We put $T = V(\bar{f})_{\mathrm{red}}$ equal to the reduced closed subscheme of S underlying $V(\bar{f})$. Further we take $U = S \setminus T$ the complement of T in S and regular by our choice of $\bar{f}$. Take $f \in \tilde{A}$ lifting $\bar{f} \in \Gamma(S, \mathcal{O}_S)$. As in 7.5.1 we put

$$\mathfrak{T} = \mathfrak{Y}^{\wedge}_{T} = \text{completion of } \mathfrak{Y} \text{ along } T,$$

$$\mathfrak{U} = \text{open formal subscheme of } \mathfrak{Y} \text{ with } \mathfrak{U}_{\mathrm{red}} = U.$$

It is clear from our choice of $f \in \tilde{A}$ that

$$\mathfrak{T} = \mathrm{Spf}(\tilde{A}^{\wedge}[[x_1, \ldots, x_N]]), \qquad \tilde{A}^{\wedge} = \varprojlim \tilde{A}/f^n \tilde{A}$$

and
$$\mathfrak{U} = \mathrm{Spf}(\tilde{A}\{1/f\}[[x_1, \ldots, x_N]]).$$

As was argued above (in the proof of 5.7.8 *b)*), the essential properties of G_i, H_i, φ, $\mathfrak{Y}$ used in 5.6 hold for $G_i|_T$, $H_i|_{\mathfrak{T}}$, ... and $G_i|_U$, $H_i|_{\mathfrak{U}}$, ... By our induction hypothesis 5.1.1 holds for T and 5.1.1 holds of U since it is regular (Theorem 4.1.1). Thus we get pairs:

$$(\mathfrak{Z}_T \subset \mathfrak{T}, \psi_T : H_1|_{\mathfrak{Z}_T} \to H_2|_{\mathfrak{Z}_T}) \quad \text{and} \quad (\mathfrak{Z}_U \subset \mathfrak{U}, \psi_U : H_1|_{\mathfrak{Z}_U} \to H_2|_{\mathfrak{Z}_U})$$

as in 5.4.3. Hence by our definition of $Z \subset Y$ above, we see that

$$\mathfrak{Z}_T^{rig} = Z \cap \mathfrak{X}^{rig} \text{ (resp. } \mathfrak{Z}_U = Z \cap \mathfrak{U}^{rig})$$

as rigid analytic closed subvarieties of $\mathfrak{X}^{rig}$ (resp. $\mathfrak{U}^{rig}$). Hence we may now apply 7.5.2 to get a closed formal subscheme $\mathfrak{Z} \subset \mathfrak{Y}$ with $\mathfrak{Z} \cap \mathfrak{X} = \mathfrak{Z}_T$ and $\mathfrak{Z} \cap \mathfrak{U} = \mathfrak{Z}_U$. It follows that $\mathfrak{Z}$ is formally smooth over $\mathrm{Spf}(\mathcal{O})$ since this was true of both $\mathfrak{Z}_T$ and $\mathfrak{Z}_U$.

Therefore, if we write $\mathfrak{Z} = \mathrm{Spf}(R)$ and argue as in the proof of Lemma 1.3.3, we see that we can choose an isomorphism

$$\tilde{A}[[x]] \cong R[[y_1, \ldots, y_{dd^*}]].$$

By Lemma 1.3.3 we can choose a lift of Frobenius σ_R on R and use it to get a new lift of Frobenius σ' on $\tilde{A}[x]]$ compatible with $\tilde{A}[[x]] \to R$. Since the new choice of the lift of Frobenius σ' on $\mathfrak{Y}$ does not change φ^{an}, we may suppose that σ fixes $\mathfrak{Z} \subset \mathfrak{Y}$. Thus $M_i \otimes_{\tilde{A}[[x]]} R$ has the natural structure of a Dieudonné module over R (since $M_i^\sigma \otimes R = (M_i \otimes R)^{\sigma_R}$).

We claim that the homomorphism of $\mathcal{O}_{\mathfrak{Z}^{rig}} = \mathcal{O}_Z$-modules

$$\varphi^{an}|_Z : \mathbf{D}(H_2)^{rig}|_Z \to \mathbf{D}(H_2)^{rig}|_Z$$

has matrix coefficients lying in $\Gamma(Z, \mathcal{O}_Z^0)$; these matrix coefficients have to be computed with respect to bases of M_1 and M_2. (Recall that $\mathbf{D}(H_i)^{rig} = M_i \otimes \mathcal{O}_Y$.) To see that our claim is true we use that $\varphi^{an}|_{\mathfrak{Z}_T^{rig}} = \mathbf{D}(\psi_T)^{rig}$ and $\varphi^{an}|_{\mathfrak{Z}_U^{rig}} = \mathbf{D}(\psi_U)^{rig}$, so that the absolute values of these matrix coefficients is $\leqslant 1$ in points lying in $\mathfrak{Z}_T^{rig} \cup \mathfrak{Z}_U^{rig} = \mathfrak{Z}^{rig}$. Therefore, we can use that

$$\Gamma(Z, \mathcal{O}_Z^0) = \Gamma(\mathfrak{Z}^{rig}, \mathcal{O}_{\mathfrak{Z}^{rig}}^0) = \Gamma(\mathfrak{Z}, \mathcal{O}_{\mathfrak{Z}})$$

by Theorem 7.4.1 to see that $\varphi^{an}|_Z$ comes from a homomorphism

$$\varphi' : \mathbf{D}(H_2)_{\mathfrak{Z}} \to \mathbf{D}(H_1)_{\mathfrak{Z}}.$$

Since φ^{an} is horizontal, we get that $\varphi^{an}|_{\mathfrak{Z}^{rig}}$ is horizontal, which implies that φ' must be horizontal (see 7.1.12). Since φ^{an} commutes with F and V, we get that $\varphi^{an}|_{\mathfrak{Z}^{rig}}$ commutes with F and V (here we use that $\sigma(\mathfrak{Z}) \subset \mathfrak{Z}$) which gives that φ' must commute with F and V. In this way we see that φ' comes from a homomorphism of Dieudonné modules $\varphi' : M_2 \otimes R \to M_1 \otimes R$. By Theorem 4.1.1 therefore, we get a homomorphism of p-divisible groups $\psi_0 : H_1|_{\mathrm{Spec}(R/pR)} \to H_2|_{\mathrm{Spec}(R/pR)}$. Note that φ' is also compatible with the Hodge filtrations defined by $H_i|_{\mathfrak{Z}}$ (by definition of Z), hence ψ_0 lifts to a homomorphism

$$\psi : H_1|_{\mathfrak{Z}} \to H_2|_{\mathfrak{Z}}$$

of p-divisible groups over $\mathfrak{Z}$ such that $\mathbf{D}(\psi)_R = \varphi'$ and hence $\mathbf{D}(\psi)^{rig} = \varphi^{an}|_{\mathfrak{Z}^{rig}}$. (If $p = 2$ one has to argue as in the proof of 5.7.8 *a)*.) This concludes the proof of 5.4.3.

6. Connected rigid analytic varieties

Let $\mathcal{O}$ be a complete discrete valuation ring, K its quotient field, $\pi \in \mathcal{O}$ a uniformizer and k the residue field: $k = \mathcal{O}/\pi\mathcal{O}$. In this section we will prove that connected rigid analytic varieties are also path-connected in a certain sense. For the general definitions and notation concerning rigid analytic varieties we refer to [BGR].

6.1. *Formulation of the result*

For any rigid analytic variety X over K we will call a *curve* in X a nonconstant morphism $C \to X$ of a connected 1-dimensional affinoid variety C over K into X. We will say that two points $x_0, x_1 \in X$ can be *connected by curves in* X, if there exists a sequence of curves in X

$$C_i \to X, \quad i = 1, \dots, n$$

such that:

1) $x_0 \in \mathrm{Im}(C_1 \to X)$, $x_1 \in \mathrm{Im}(C_n \to X)$,
2) $\mathrm{Im}(C_i \to X) \cap \mathrm{Im}(C_{i+1} \to X) \neq \emptyset$, $i = 1, \dots, n-1$.

6.1.1. *Proposition. — Let* X *be a quasi-compact rigid analytic variety. If* X *is connected then any two points in* X *can be connected by curves in* X. □

The proof of this will occupy the rest of this chapter.

There is an obvious reduction to the case that X is a reduced, irreducible, normal affinoid variety. Say $X = \mathrm{Sp}(A)$ has dimension d. We argue by induction on d. The case $d = 1$ is clear, thus we may assume that $d \geq 2$.

Let us take a finite surjective morphism

$$f : X \to \mathbf{B}^d = \mathrm{Sp}(T_d)$$

of X onto the unit ball of dimension d (see [BGR, Corollary 6.1.2/2]).

6.2. *Reduction to the case that f is generically étale*

We claim that we may assume that f is generically étale, i.e., that there exists a Zariski open $U \subset \mathbf{B}^d$ such that $f : f^{-1}(U) \to U$ is flat and unramified. (Unramified is equivalent to $\Omega^1_{f^{-1}(U)/U} = (0)$.) Since f is generically flat by [Mat, Theorem 53], we see that this is equivalent to the assertion that the extension of quotient fields $Q(T_d) \subset Q(A)$ is separable. Hence, if $\mathrm{char}(K) = 0$ then f is automatically generically étale. If $\mathrm{char}(K) = p$ then there exists a field L,

$$Q(T_d) \subset L \subset Q(A),$$

such that $Q(T_d) \subset L$ is separable and $L \subset Q(A)$ is purely inseparable. Let A′ be the normal closure of T_d in L; it is equal to $A \cap L$ and it is an affinoid algebra over K, see [BGR, 6.1.2/4 and 6.1.1/6]. The morphism f factorizes as

$$X \xrightarrow{i} X' = \mathrm{Sp}(A') \xrightarrow{f'} \mathbf{B}^d.$$

The map i has the property that $\mathcal{O}_{X'} \subset i_* \mathcal{O}_X$ is purely inseparable in the sense that for any $s \in i_* \mathcal{O}_X$ there exists an $n \in \mathbf{N}$ such that $s^{p^n} \in \mathcal{O}_{X'}$. This implies that i induces a bijection on points and that for any fibre product

$$\begin{array}{ccc} C = C' \times_{X'} X & \longrightarrow & X \\ \downarrow & & \downarrow \\ C' & \longrightarrow & X' \end{array}$$

the same holds for $C \to C'$. Thus, if $C' \to X'$ is a curve in X′, then $C \to X$ is a curve in X. Hence we may replace X by X′ and f by f' and in this way get the situation that f is generically étale.

6.3. *Reduction to the case $\tilde{f}$ generically étale*

Let us write $\mathcal{O}\{\underline{x}\}$ or $\mathcal{O}\{x_1, \ldots, x_d\}$ for the π-adic completion of $\mathcal{O}[x_1, \ldots, x_d]$. The homomorphism $T_d \to A$ induces a finite homomorphism

$$\mathcal{O}\{x_1, \ldots, x_d\} = T_d^0 \to A^0$$

(see [BGR, Corollary 6.4.1/6]). To this situation we associate some numerical invariants:

$$\begin{aligned} n(A^0/T_d^0) &= n(A/T_d) \\ &= \text{the multiplicity of } (\pi) \subset \mathcal{O}\{\underline{x}\} \text{ in the discriminant of } A^0 \\ &\qquad\qquad\qquad\qquad \text{over } T_d^0 \end{aligned}$$

and $\qquad \deg(A^0/T_d^0) = \deg(A/T_d) = \text{the degree of } T_d \to A = [Q(A) : Q(T_d)].$

The discriminant of A^0 over T_d^0 is defined: $T_d^0 \to A^0$ is finite and generically étale, T_d^0 is a regular ring and A^0 is normal as X is normal. Further, let η denote the generic point of $\mathrm{Spec}(k[x_1, \ldots, x_d])$ and let $\eta_1, \ldots, \eta_r$ denote the generic points of $\mathrm{Spec}(\tilde{A})$. (Recall that $\tilde{A} := A^0/A^{\infty} = A^0/\sqrt{\pi A^0}$.) The inverse image of the point $\eta \in \mathrm{Spec}(k[x_1, \ldots, x_d]) \subset \mathrm{Spec}(T_d^0)$ in $\mathrm{Spec}(A^0)$ is $\{\eta_1, \ldots, \eta_r\}$. The residue fields $k(\eta_1), \ldots, k(\eta_r)$ are finite extensions of the field $k(\eta) = k(x_1, \ldots, x_d)$. We want to reduce to the case that all the field extensions $k(\eta) \subset k(\eta_i)$ are separable. Of course if $\mathrm{char}(k) = 0$ there is nothing to prove, so let us assume for the moment that $\mathrm{char}(k) = p$.

Note that if K′ is a finite extension of K and Y is a rigid analytic variety over K′ then Y can be viewed as a rigid analytic variety over K also. If Y is connected then Y seen as a rigid analytic variety over K is connected also. Thus if we can find a surjective morphism of K-varieties $Y \to X$ and the assertion of the proposition is true for Y over K′, then the assertion is true for X (over K) also.

Suppose that the field extensions $k(\eta) \subset k(\eta_i)$, $i < s$ are separable for some $s \in \{1, \ldots, r\}$. There exists a purely inseparable finite field extension k' of k and a number $\ell \in \mathbf{N}$ such that the field extension

$$\begin{array}{ccc} k(\eta) = k(x_1, \ldots, x_d) & \subset & k'(y_1, \ldots, y_d) \\ x_i & \mapsto & y_i^{p^\ell} \end{array}$$

makes $k(\eta) \subset k(\eta_s)$ separable; this is meant to signify that the residue field of the local ring

$$k'(y_1, \ldots, y_d) \otimes_{k(\eta)} k(\eta_s)$$

is a finite separable extension of $k'(y_1, \ldots, y_d)$.

6.3.1. *Lemma.* — *There exists an extension of complete discrete valuation rings* $\mathcal{O} \subset \mathcal{O}'$ *with ramification index 1, such that the extensions of quotient fields* $K \subset K'$ *is separable and such that* $\mathcal{O}'/\pi\mathcal{O}' \cong k'$.

Proof. — It suffices to do the case that $k' \cong k[\alpha]/(\alpha^p - a)$. Take $\mathcal{O}' = \mathcal{O}[\alpha]/(\alpha^p - \tilde{a})$ if $\mathrm{char}(K) = 0$ and $\mathcal{O}' = \mathcal{O}[\alpha]/(\alpha^p - \pi\alpha - \tilde{a})$ if $\mathrm{char}(K) = p$. □

We consider the ring extension ($\mathcal{O}'$ as in 6.3.1)

$$\begin{array}{lcll} \mathcal{O}\{x_1, \ldots, x_d\} & \to & \mathcal{O}'\{y_1, \ldots, y_d\} & \\ x_i & \mapsto & y_i^{p^\ell} & \text{if } \mathrm{char}(K) = 0, \\ x_i & \mapsto & y_i^{p^\ell} + \pi y_i & \text{if } \mathrm{char}(K) = p. \end{array} \tag{6.3.2}$$

It defines a finite flat morphism $\mathrm{Spec}(\mathcal{O}'\{\underline{y}\}) \to \mathrm{Spec}(\mathcal{O}\{\underline{x}\})$ which is generically étale. Let us put

$$B' := K'\langle y_1, \ldots, y_d \rangle \otimes_{K\langle x_1, \ldots, x_d \rangle} A.$$

It is an affinoid algebra over K′, reduced as $K\langle \underline{x} \rangle \subset K'\langle \underline{y} \rangle$ is generically étale. Any component of Spec(B′) dominates Spec (A) as $A \to B'$ is flat. The map $K'\langle \underline{y} \rangle \to B'$ is generically étale since it is a base change of $K\langle \underline{x} \rangle \to A$. Let B be the normalization of B′.

If Sp(B) is not connected, say $\mathrm{Sp}(B) = X_1 \cup X_2$ with X_1 connected then we have a diagram of rigid analytic spaces over K:

$$\begin{array}{ccccc} X_1 & \longrightarrow & X_1 \cup X_2 & \longrightarrow & \mathbf{B}^d_{K'} \\ & \searrow & \downarrow & & \downarrow \\ & & X & \longrightarrow & \mathbf{B}^d_K. \end{array}$$

Since $X_1 \to X$ is finite and dominant, it is surjective. Hence we may replace X by X_1, seen as a rigid analytic variety over K′ (see remark above). In this case the degree of $X_1 \to \mathbf{B}^d_{K'}$ is smaller than the degree of $X \to \mathbf{B}^d$.

Suppose that Sp(B) is connected. We note that we have an inclusion

$$\mathcal{O}'\{y_1, \ldots, y_d\} \otimes_{\mathcal{O}\{x_1, \ldots, x_d\}} A^0 \subset B^0. \tag{6.3.3}$$

This gives

$$n(B^0/\mathcal{O}'\{\underline{y}\}) \leqslant n(A^0/\mathcal{O}\{\underline{x}\}). \tag{6.3.4}$$

(Use the fact that formation of discriminant commutes with base extension.) If equality holds in (6.3.4) then (6.3.3) must be an isomorphism generically along $V(\pi)$. More precisely, let η' be the generic point of $\mathrm{Spec}(k'[y_1, \ldots, y_d])$ and let $C_{\eta'}$ be the local ring of η' in the ring $\mathcal{O}'\{\underline{y}\}$; if equality holds in (6.3.4), then (6.3.3) $\otimes\, C_{\eta'}$ is an isomorphism. In particular, in that case, the generic points of $\mathrm{Spec}(\widetilde{B})$ can be numbered $\eta'_1, \ldots, \eta'_r$ and have residue fields:

$$\begin{aligned} k'(\eta'_i) &\cong \text{residue field of the local ring} \\ &\quad k'(\eta') \otimes_{k(\eta)} k(\eta_i) = k'(y_1, \ldots, y_d) \otimes_{k(x_1, \ldots, x_d)} k(\eta_i). \end{aligned}$$

Thus we see by our choice of k' and ℓ that the fields $k'(\eta'_1), \ldots, k'(\eta'_s)$ are separable over $k'(\eta')$. In other words, we have increased s by 1.

Continuing in this manner, we see that either $\deg(A/T_d)$ decreases to 1, or $n(A/T_d)$ decreases to zero, or s becomes equal to r. In all cases we will have $s = r$, i.e., all the field extensions $k(\eta) \subset k(\eta_i)$ are separable.

6.4. *Rest of the proof*

Before we proceed, let us prove the proposition for $X = \mathbf{B}^d$. We have a projection onto the d^{th} factor $\mathrm{pr}_d : \mathbf{B}^d \to \mathbf{B}$, which has a section $s : \mathbf{B} \to \mathbf{B}^d$. For any point $x \in \mathbf{B}^d$ the fibre $F := \mathrm{pr}_d^{-1}(\mathrm{pr}_d(x)) \subset \mathbf{B}^d$ is a connected $(d-1)$-dimensional variety. Hence, by our induction hypothesis, x may be connected by curves in F to $s(\mathrm{pr}_d(x))$. Since $s : \mathbf{B} \to \mathbf{B}^d$ is a curve in $\mathbf{B}^d$, we get the desired result. The proof for general X follows the same pattern.

First we construct the morphism $X \to \mathbf{B}$ replacing pr_d. To this end we consider the morphism of schemes

$$\tilde{f} : \widetilde{X} = \mathrm{Spec}(\widetilde{A}) \to \mathrm{Spec}(k[x_1, \ldots, x_d]) = \mathbf{A}_k^d$$

induced by f. We remark that $\widetilde{X}$ is equidimensional of dimension d. Let us decompose $\widetilde{X}$ into its irreducible components:

$$\widetilde{X} = Z_1 \cup Z_2 \cup \ldots \cup Z_r.$$

Let $U \subset \mathbf{A}_k^d$ be the Zariski open subset over which $\tilde{f}$ is étale. By our result that the extensions $k(\eta) \subset k(\eta_i)$ are separable we know that U is not empty. The schemes $\tilde{f}^{-1}(U) \cap Z_i$ are connected étale coverings of U, hence determined by $\pi_1(U)$-sets with transitive $\pi_1(U)$-actions.

For any partitioning $\{1, \ldots, r\} = I \cup J$, $I \neq \emptyset$, $I \neq \emptyset$ and $I \cap J = \emptyset$ we put

$$Z_{I,J} = (\bigcup_{i \in I} Z_i) \cap (\bigcup_{j \in J} Z_j).$$

Note that the admissible open subvariety V of X defined by

$$X \supset V := \{ x \in X \text{ such that } sp(x) \notin Z_{I,J} \}$$

is disconnected (*sp* is the specialization mapping, see [BGR, 7.1.5] or our 7.1.10). This is so since V is the rigid analytic variety associated to a disconnected formal scheme over $\mathrm{Spf}(\mathcal{O})$, namely the open formal subscheme $\mathcal{U}$ of $\mathrm{Spf}(A^0)$ whose reduction is equal to $\tilde{X} \backslash Z_{I,J}$, which is disconnected by construction of $Z_{I,J}$. This implies that

(6.4.1) $$\mathrm{codim}_{\tilde{X}} Z_{I,J} = 1.$$

Indeed, if this codimension were bigger, then any function $g \in \Gamma(V, \mathcal{O}_V)$ would extend uniquely to an analytic function on the whole of X, by [Bart, Satz 3.5] or [Lü2, Satz 2]. In particular a nontrivial idempotent on V would extend to a nontrivial idempotent on X, in contradiction with the assumption that X is connected.

Let $S \subset \mathrm{Spec}(A^0)$ be the Zariski closed subset of primes $\mathfrak{p} \subset A^0$ such that the local ring $A^0_{\mathfrak{p}}$ is not Cohen-Macaulay. To see that S is closed, we remark that by [Mat, Theorem 46] $A^0_{\mathfrak{p}}$ is Cohen-Macaulay if and only if $A^0_{\mathfrak{p}}$ is flat over $\mathcal{O}\{\underline{x}\}_{\mathfrak{q}}$, $\mathfrak{q} = \mathfrak{p} \cap \mathcal{O}\{\underline{x}\}$. Hence, the complement of S is the set of primes $\mathfrak{p}$ for which $A^0_{\mathfrak{p}}$ is flat over $\mathcal{O}\{\underline{x}\}_{\mathfrak{q}}$, which is an open set by [Mat, Theorem 53]. Since A^0 is normal, we conclude that the codimension of S in $\mathrm{Spec}(A^0)$ is a least 3 (by Serre's criterium: normal $\Leftrightarrow$ (S_2) and (R_1) [Mat, Theorem 39]). This implies that the codimension of $\tilde{S} := S \cap \tilde{X}$ is at least 2.

In the same manner the singular locus T of $\mathrm{Spec}(A^0)$ is closed of codimension at least 2. Thus $\tilde{T} := T \cap \tilde{X}$ has codimension at least 1 in $\tilde{X}$.

Let us consider a general hyperplane

$$\mathbf{A}^d_k \supset H = \{(x_1, \ldots, x_d) \in \mathbf{A}^d_k \mid \Sigma a_i x_i = a\}$$

for a_i, $a \in k$. We claim that if we choose a_i, $a \in k$ sufficiently general then

(6.4.2) $\tilde{f}^{-1}(H) \cap Z_i$ is irreducible, generically reduced and not contained in $\tilde{T}$,

(6.4.3) for any partitioning $\{1, \ldots, r\} = I \cup J$, $I \neq \emptyset$, $I \neq \emptyset$ and $I \cap J = \emptyset$ the intersection $\tilde{f}^{-1}(H) \cap Z_{I,J}$ has a component of dimension $d - 2$ not contained in $\tilde{S}$.

Of course this might not be possible if k is finite; in that case it will be possible after a finite extension $k \subset k'$, which is harmless for our purposes (replace $\mathcal{O}$ by $\mathcal{O}'$, K by K', etc.).

Condition (6.4.3) is easy to satisfy since the codimension of $\widetilde{S}$ in $\widetilde{X}$ is 2 and the codimension of $Z_{I,J}$ in $\widetilde{X}$ is 1. The schemes $\widetilde{f}^{-1}(H) \cap Z_i$ are generically reduced as soon as $H \cap U$ is nonempty. The irreducibility of $\widetilde{f}^{-1}(H) \cap Z_i$ in condition (6.4.2) follows from [J]. Avoiding $\widetilde{T}$ is no problem since its codimension is 1 and hence so is the codimension of its image in $\mathbf{A}^d_k$.

Next, we take any lifts $\widetilde{a}_i, \widetilde{a} \in \mathcal{O}$ of the elements $a_i, a \in k$. Write

$$h = \Sigma \widetilde{a}_i x_i - \widetilde{a} \in \mathcal{O}\{x_1, \ldots, x_d\} \subset A^0.$$

I claim that the affinoid space

$$\mathrm{Sp}(A/hA) = \{ x \in X \mid h(x) = 0 \}$$

is connected. If not then the scheme $\mathrm{Spec}(A^0/hA^0)$ is reducible, say

$$\mathrm{Spec}(A^0/hA^0) = T_1 \cup T_2, \qquad \dim(T_1) = \dim(T_2) = d$$

(recall that $\dim(\mathrm{Spec}(A^0)) = d + 1$!) and $(T_1 \cap T_2)_{\mathrm{red}} \subset V(\pi)$ (otherwise $\mathrm{Sp}(A/hA)$ would still be connected). By the construction of h we know that we must have

$$(T_1 \cap V(\pi))_{\mathrm{red}} = \bigcup_{i \in I} \widetilde{f}^{-1}(H) \cap Z_i$$

for a certain subset $I \subset \{1, \ldots, r\}$ and similarly for some $J \subset \{1, \ldots, r\}$

$$(T_2 \cap V(\pi))_{\mathrm{red}} = \bigcup_{j \in J} \widetilde{f}^{-1}(H) \cap Z_j.$$

Clearly, $I \cup J = \{1, \ldots, r\}$. Let ξ_i be the generic point of $\widetilde{f}^{-1}(H) \cap Z_i$. The local ring $A^0_{\xi_i}$ of $\mathrm{Spec}(A^0)$ at ξ_i is regular (6.4.2) and has dimension 2. Let $t \in A^0_{\xi_i}$ be the irreducible element of $A^0_{\xi_i}$ defining $Z_i \cap \mathrm{Spec}(A^0_{\xi_i})$; it is also the generator of the kernel of $A^0_{\xi_i} \to \widetilde{A}_{\xi_i}$. Since $\widetilde{f}^{-1}(H) \cap Z_i$ is generically reduced for all i (6.4.2), we get that

$$A^0_{\xi_i}/(t, h) \cong k(\xi_i).$$

Thus $\{t, h\}$ is a regular system of parameters for $A^0_{\xi_i}$ ([Mat, 12.J]) and hence $(A^0/hA^0)_{\xi_i} = A^0_{\xi_i}/hA^0_{\xi_i}$ is a regular ring. This implies that only one component of $\mathrm{Spec}(A^0/hA^0)$ passes through ξ_i, hence $I \cap J = \emptyset$.

Let $\mathfrak{q} \subset A^0$ be the prime ideal corresponding to a generic point of a component of $\widetilde{f}^{-1}(H) \cap Z_{I,J}$. Remark that $\mathfrak{q} \in T_1$ and $\mathfrak{q} \in T_2$. By our assumption (6.4.3) above, we can choose this $\mathfrak{q}$ such that $A^0_{\mathfrak{q}}$ is Cohen-Macaulay of dimension 3. Since $h \in \mathfrak{q}A^0_{\mathfrak{q}}$ is not a zero divisor, the local ring $R := A^0_{\mathfrak{q}}/hA^0_{\mathfrak{q}}$ is Cohen-Macaulay too. Thus

$\dim(R) = \operatorname{depth}(R) = 2$. By [Hart, Proposition 2.1], we see that $\operatorname{Spec}(R)\setminus\{\mathfrak{m}\}$ is connected ($\mathfrak{m}$ is the maximal ideal of R). Thus the decomposition of Spec(R) as

$$\operatorname{Spec}(R) = (T_1 \cap \operatorname{Spec}(R)) \cup (T_2 \cap \operatorname{Spec}(R)) = T'_1 \cup T'_2$$

cannot have $T'_1 \cap T'_2 = \{\mathfrak{m}\}$. However, $T'_1 \cap T'_2 \subset \operatorname{Spec}(R) \cap V(\pi) = \operatorname{Spec}(R/\pi R)$ consists of $\mathfrak{m}$ and a number of prime ideals corresponding to the generic points of $\tilde{f}^{-1}(H) \cap Z_i$. Since $I \cap J = \emptyset$ we get $T'_1 \cap T'_2 = \{\mathfrak{m}\}$ a contradiction.

Conclusion: the morphism

$$h : X \to \mathbf{B}$$

has a connected fibre $h^{-1}(0)$ of dimension smaller than d. By our induction hypothesis, we can connect any two points in $h^{-1}(0)$ by curves in $h^{-1}(0)$. Therefore, it suffices to connect any point $x \in X$ to a point of $h^{-1}(0)$ via curves in X. To do this, remark that the morphism h factorizes as

$$X \xrightarrow{f} \mathbf{B}^d \xrightarrow{h'} \mathbf{B}$$

by the very definition of h. It is also clear from the definition of our h that we may assume that $h' = \mathrm{pr}_d$ by changing coordinates on $\mathbf{B}^d$. Given our point $x \in X$ with $f(x) = (x_1, \ldots, x_d)$ we put

$$C := \{(y_1, \ldots, y_d) \in \mathbf{B}^d \mid y_1 = x_1, \ldots, y_{d-1} = x_{d-1}\}.$$

It is an irreducible curve in $\mathbf{B}^d$, the map $\mathrm{pr}_d : C \to \mathbf{B}$ is surjective. The inverse image $f^{-1}(C) \subset X$ is a finite union of irreducible curves in $X : f^{-1}(C) = \bigcup C_i$. For each of these curves the morphism $f : C_i \to C$ is finite hence surjective, hence $h : C_i \to \mathbf{B}$ is surjective. Since one of these curves contains the point x we are done.

7. Formal schemes and rigid analytic geometry; Berthelot's construction

Let $\mathcal{O}$ be a complete discrete valuation ring, K its quotient field, $\pi \in \mathcal{O}$ a uniformizer and k the residue field: $k = \mathcal{O}/\pi\mathcal{O}$. In [B2], Berthelot has constructed a functor which associates to a formal scheme over $\operatorname{Spf}(\mathcal{O})$ a rigid analytic space over K. This construction is more general than that of Raynaud [R] since the formal schemes are not necessarily of finite type over $\operatorname{Spf}(\mathcal{O})$. Indeed, the source category for this functor is defined as follows.

7.0.1. *Definition.* — We write $FS_{\mathcal{O}}$ for the category of locally Noetherian adic formal schemes $\mathfrak{X}$ over $\operatorname{Spf}(\mathcal{O})$ whose reduction $\mathfrak{X}_{\mathrm{red}}$ is a scheme locally of finite type over $\operatorname{Spec}(k)$. Morphisms are morphisms of formal schemes over $\operatorname{Spf}(\mathcal{O})$.

The target category for the functor is simply the category of rigid analytic varieties over K (see [BGR, 9.3]); this category will be denoted Rig_K, the subcategory of affinoids over K is written Aff_K. Berthelot's functor [B2, 0.2.6] will be denoted $(\cdot)^{\mathrm{rig}} : \mathrm{FS}_{\mathcal{O}} \to \mathrm{Rig}_K$, $\mathfrak{X} \mapsto \mathfrak{X}^{\mathrm{rig}}$.

7.1. *Berthelot's construction for affine formal schemes*

Let $\mathfrak{X} = \mathrm{Spf}(A)$ be an affine formal scheme over $\mathrm{Spf}(\mathcal{O})$. Let $I \subset A$ be the biggest ideal of definition of A. (Recall that by our conventions on formal schemes the topological ring A is Noetherian and adic.) Thus $\mathfrak{X}$ is an object of $\mathrm{FS}_{\mathcal{O}}$ if and only if A/I is a finitely generated k-algebra. We remark that this is also equivalent to the fact that A is the quotient of the ring $A_{n,m} := \mathcal{O}\{x_1, \ldots, x_n\}[[y_1, \ldots, y_m]]$ for some $n, m \in \mathbf{N}$. Let us assume that A satisfies these conditions.

It is clear that the rigid space associated to the formal scheme $\mathrm{Spf}(A_{n,m})$ should be $\mathbf{B}^n \times D^m$, where $\mathbf{B}$ is the closed unit disc and D is the open unit disc over K. If we take as coordinates $x_1, \ldots, x_n$ on $\mathbf{B}^n$ and as coordinates $y_1, \ldots, y_m$ on D^m then we can view any element f of $A_{n,m}$ as a (bounded) analytic function on $\mathbf{B}^n \times D^m$, also denoted f by abuse of notation. Suppose that $g_1, \ldots, g_r \in A_{n,m}$ generate the kernel of the surjection $A_{n,m} \to A$. Again it is clear that $\mathfrak{X}^{\mathrm{rig}} \subset \mathbf{B}^n \times D^m$ should be defined as the closed analytic subvariety given by the vanishing of the analytic functions $g_1, \ldots, g_r$. Although this gives a definition of $\mathfrak{X}^{\mathrm{rig}}$ which works, we proceed in a somewhat more functorial manner below. It can be shown using 7.1.7 below that both descriptions give isomorphic rigid analytic spaces.

7.1.1. Let A be as above. Take $n \in \mathbf{N}$. Let us write $A[I^n/\pi]$ for the subring of $A \otimes_{\mathcal{O}} K$ generated by the image of $A \to A \otimes_{\mathcal{O}} K$ and the elements i/π, $i \in I^n$. We define the ring $B_n = B_n(A)$ as the I-adic completion of $A[I^n/\pi]$, i.e., the completion of $A[I^n/\pi]$ with respect to the ideal $IA[I^n/\pi]$. Further, we introduce the notation $C_n = C_n(A) := B_n \otimes_{\mathcal{O}} K$. There are continuous homomorphisms $B_{n+1} \to B_n$ induced by the inclusions $A[I^{n+1}/\pi] \hookrightarrow A[I^n/\pi]$. This gives $A \otimes_{\mathcal{O}} K$-algebra homomorphisms $C_{n+1} \to C_n$. Furthermore the construction that associates to A the direct system $\{B_n\}_{n \geqslant 1}$ (resp. $\{C_n\}_{n \geqslant 1}$) is a functor on the category of $\mathcal{O}$-algebras as above.

7.1.2. *Lemma. — With notation as above.*

a) The ring homomorphism $A \otimes_{\mathcal{O}} K \to C_n$ *is flat.*

b) The K*-algebra* C_n *is an affinoid algebra over* K.

c) The morphism $\mathrm{Sp}(C_n) \to \mathrm{Sp}(C_{n+1})$ *identifies* $\mathrm{Sp}(C_n)$ *with an affinoid subdomain of* $\mathrm{Sp}(C_{n+1})$.

Proof. — Since $A[I^n/\pi]$ is a Noetherian ring, the homomorphism $A[I^n/\pi] \to B_n$ is flat, hence (by base extension) $A \otimes_{\mathcal{O}} K = A[I^n/\pi] \otimes_{\mathcal{O}} K \to C_n$ is flat.

It is clear that $I^n A[I^n/\pi] \subset \pi A[I^n/\pi] \subset IA[I^n/\pi]$, thus B_n is also the π-adic completion of $A[I^n/\pi]$. Suppose $f_1, \ldots, f_N \in I^n$ are generators of I^n; then there is a surjection

$$A/I[x_1, \ldots, x_N] \to A[I^n/\pi]/IA[I^n/\pi] \cong B_n/IB_n.$$
$$x_i \mapsto \text{class of } f_i/\pi.$$

We conclude that B_n/IB_n is a k-algebra of finite type as A/I was assumed to be so. It follows that B_n is an $\mathcal{O}$-algebra topologically of finite type and that $C_n \cong B_n \otimes_{\mathcal{O}} K$ is an affinoid algebra over K.

The obvious inclusion $A[I^{n+1}/\pi] \subset A[I^n/\pi]$ induces a continuous homomorphism $B_{n+1} \to B_n$. We get a surjection

$$B_{n+1}\{x_1, \ldots, x_N\} \to B_n$$
$$x_i \mapsto \text{image of } f_i/\pi \text{ in } B_n$$

and this induces a surjection $C_{n+1}\langle x_1, \ldots, x_N\rangle \to C_n$. In its kernel are certainly the elements $\pi x_i - f_i$. It induces an isomorphism

$$C_{n+1}\langle x_1, \ldots, x_N\rangle/(\pi x_i - f_i) \cong C_n.$$

This follows from the fact that the map

$$A[I^{n+1}/\pi][x_1, \ldots, x_N]/(\pi x_i - f_i) \to A[I^n/\pi]$$

is an isomorphism modulo π-torsion. □

7.1.3. *Definition.* — The rigid analytic space $\mathfrak{X}^{\text{rig}}$ associated to the affine formal scheme $\mathfrak{X} = \text{Spf}(A)$ is defined as the (increasing) admissible union of the affinoid spaces $\text{Sp}(C_n)$. In a formula: $\mathfrak{X}^{\text{rig}} := \bigcup_n \text{Sp}(C_n)$.

Thus $\mathfrak{X}^{\text{rig}}$ is a separated K-analytic space (see [BGR, 9.6.1/7]). If

$$\varphi : \mathfrak{Y} = \text{Spf}(A') \to \mathfrak{X}$$

is a morphism of affine objects of $FS_{\mathcal{O}}$ then we get a morphism of rigid analytic spaces $\varphi^{\text{rig}} : \mathfrak{Y}^{\text{rig}} \to \mathfrak{X}^{\text{rig}}$. This is clear from 7.1.1. The next lemma shows that our functor agrees with Raynaud's functor (see [R] or [BL, Theorem 4.1]) in the case that $\mathfrak{X}$ is of finite type over $\text{Spf}(\mathcal{O})$.

7.1.4. *Lemma.* — *Let* $\mathfrak{X} = \text{Spf}(A)$ *as above.*

a) The rigid analytic variety $\mathfrak{X}^{\text{rig}}$ *depends only on* $A' = A/\pi$*-torsion; more precisely, the morphism* $\varphi : \mathfrak{Y} := \text{Spf}(A') \to \mathfrak{X}$ *induces an isomorphism* $\varphi^{\text{rig}} : \mathfrak{Y}^{\text{rig}} \to \mathfrak{X}^{\text{rig}}$.

b) If the ideal πA *is an ideal of definition of* A *or equivalently, if* $\mathfrak{X} \to \text{Spf}(\mathcal{O})$ *is of finite type* [EGA, I 10.13.3], *then* $\mathfrak{X}^{\text{rig}} = \text{Sp}(A \otimes_{\mathcal{O}} K)$.

Proof. — Trivial. □

We would like to show that the rigid analytic space $\mathfrak{X}^{\text{rig}}$ represents a functor, at least on the category of affinoid spaces. Before we do so let us show that any rigid analytic space is determined by the functor it represents on Aff_K.

7.1.5. *Lemma. — The functor*

$$\text{Rig}_K \to \text{Funct}(\text{Aff}_K^0, \textit{Sets})$$

which maps X *to* $h_X(\cdot) := \text{Mor}_{\text{Rig}_K}(\cdot, X)$ *is fully faithful.*

Proof. — Suppose that X, Y are rigid analytic varieties and $\alpha : h_X \to h_Y$ is a morphism of functors. By substituting Sp(K′) in $h_X(\cdot)$, where K′ runs through all finite extensions of K we find a map of sets $\beta : X \to Y$. Let $i : U \to X$ define an affinoid subdomain of X and consider the morphism $\alpha(i) : U \to Y$. As α is a transformation of functors the map $\alpha(i)$ agrees with β on $U \subset X$. A similar compatibility holds in case we have a composition $V \to U \to X$ of open immersions of affinoids. Therefore β comes from a unique morphism $X \to Y$, which agrees with $\alpha(i)$ on U, for any $i : U \to X$ as above. □

7.1.6. Suppose that Y is an affinoid rigid analytic over K. A *model* for Y is an affine formal scheme $\mathfrak{Y}$, flat and of finite type over $\text{Spf}(\mathcal{O})$ endowed with an isomorphism $\mathfrak{Y}^{\text{rig}} \cong Y$. Such a model always exists. A morphism of models $\mathfrak{Y}_1$ and $\mathfrak{Y}_2$ is a morphism $\varphi : \mathfrak{Y}_1 \to \mathfrak{Y}_2$ of formal schemes over $\text{Spf}(\mathcal{O})$ compatible with the given isomorphisms $\mathfrak{Y}_i \cong Y$. For any two models $\mathfrak{Y}_1$, $\mathfrak{Y}_2$ of Y there is a third $\mathfrak{Y}$ lying over both of them, i.e., such that there are maps of models $\mathfrak{Y} \to \mathfrak{Y}_1$ and $\mathfrak{Y} \to \mathfrak{Y}_2$. In addition, if $\psi : Y' \to Y$ is a morphism of affinoids and $\mathfrak{Y}$ is a model of Y then there exists a model $\mathfrak{Y}'$ of Y′ and a morphism $\varphi : \mathfrak{Y}' \to \mathfrak{Y}$ such that $\varphi^{\text{rig}} = \psi$.

7.1.7. *Proposition. — By the functoriality of* $(\cdot)^{\text{rig}}$ *we a get a map*

$$\text{(7.1.7.1)} \qquad \varinjlim_{\text{models } \mathfrak{Y} \text{ of } Y} \text{Mor}_{\text{FS}_{\mathcal{O}}}(\mathfrak{Y}, \mathfrak{X}) \to \text{Mor}_{\text{Rig}_K}(Y, \mathfrak{X}^{\text{rig}}).$$

Both sides are in a natural way contravariant functors on Aff_K; *the above is an isomorphism of functors. This property determines* $\mathfrak{X}^{\text{rig}}$ *up to unique isomorphism in view of Lemma* 7.1.5.

Proof. — The remarks made in 7.1.6 show that the left hand side of (7.1.7.1) defines a functor and that (7.1.7.1) is a transformation of functors. We will construct the inverse to (7.1.7.1). Suppose that Y = Sp(B) and that $Y \to \mathfrak{X}^{\text{rig}}$ is given by $Y \to \text{Sp}(C_n)$ for some $n \in \mathbf{N}$. Take a surjection $\mathcal{O}\{x_1, \ldots, x_N\} \to B_n$ and extend the composition $\mathcal{O}\{x_1, \ldots, x_N\} \to B_n \to B$ to a surjection $K\langle x_1, \ldots, x_N, x_{N+1}, \ldots, x_M\rangle \to B$. The image of $\mathcal{O}\{x_1, \ldots, x_M\}$ in B is a subring $R \subset B$ which is flat and of topologically finite type over $\mathcal{O}$ and is such that $R \otimes K = B$. Hence $\mathfrak{Y} := \text{Spf}(R)$ is a model of $\mathfrak{Y}$. The composition

$$\mathfrak{Y} = \text{Spf}(R) \to \text{Spf}(B_n) \to \text{Spf}(A) = \mathfrak{X}$$

gives an element of the left hand side of (7.1.7.1). We leave it to the reader to prove that this defines an inverse to (7.1.7.1). □

7.1.8. It is clear from our construction of $\mathfrak{X}^{\mathrm{rig}}$ ($\mathfrak{X}$ as above) that there is a canonical homomorphism

$$(7.1.8.1) \qquad A \otimes_{\mathcal{O}} K = \Gamma(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}) \otimes_{\mathcal{O}} K \to \Gamma(\mathfrak{X}^{\mathrm{rig}}, \mathcal{O}_{\mathfrak{X}^{\mathrm{rig}}}).$$

The image of A under this homomorphism lies in fact in the sheaf $\mathcal{O}^0_{\mathfrak{X}^{\mathrm{rig}}}$ of functions whose absolute value is bounded by 1:

$$\Gamma(U, \mathcal{O}^0_{\mathfrak{X}^{\mathrm{rig}}}) = \{ f \in \Gamma(U, \mathcal{O}_{\mathfrak{X}^{\mathrm{rig}}}) \mid |f(x)| \leqslant 1 \ \forall\, x \in U \}.$$

This is clear from the fact that the homomorphism $A \to C_n$ factors as

$$A \to B_n \to B_n \otimes K \cong C_n$$

and that $B_n \subset C_n^0 =$ power bounded elements of C_n. (See [BGR, 6.2.3/1].) Thus we get a homomorphism

$$(7.1.8.2) \qquad \Gamma(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}) \to \Gamma(\mathfrak{X}^{\mathrm{rig}}, \mathcal{O}^0_{\mathfrak{X}^{\mathrm{rig}}}).$$

7.1.9. *Lemma. — Let* $\mathfrak{X} = \mathrm{Spf}(A)$ *as above. There is a bijection functorial in* A *between the following two sets:*

1. *Maximals ideals* $\mathfrak{m} \subset A \otimes_{\mathcal{O}} K$.
2. *Points* x *of* $\mathfrak{X}^{\mathrm{rig}}$.

If the point $x(\mathfrak{m}) \in \mathfrak{X}^{\mathrm{rig}}$ *corresponds to* $\mathfrak{m} \subset A \otimes_{\mathcal{O}} K$ *then there exists a canonical homomorphism of local rings*

$$(A \otimes_{\mathcal{O}} K)_{\mathfrak{m}} \to \mathcal{O}_{\mathfrak{X}^{\mathrm{rig}}, x(\mathfrak{m})}$$

compatible with (7.1.8.1). *This homomorphism induces an isomorphism on completions.*

Proof. — To establish the bijection one might use the bijection (7.1.7.1) and argue as in [BL, 3.4]. (See also 7.1.10 below.) However, we also give another argument.

We may assume that A is $\mathcal{O}$-flat. Take a point $x \in \mathfrak{X}^{\mathrm{rig}}$. It corresponds to a maximal ideal $\mathfrak{p} \subset C_n$ for some n big enough. Consider the homomorphism

$$\varphi : A \otimes_{\mathcal{O}} K \to C_n/\mathfrak{p}.$$

The field on the right is a finite extension of K; hence it is finite algebraic over $\mathrm{Im}(\varphi)$. We also have that the image of φ is dense, since $A \otimes K$ is dense in C_n. Thus we get that φ is surjective, in other words that $\mathfrak{m} := \mathrm{Ker}(\varphi) \subset A \otimes K$ is a maximal ideal with the same residue field as $\mathfrak{p}$ in C_n.

On the other hand, suppose $\mathfrak{m} \subset A \otimes K$ is a maximal ideal. The prime ideal $\mathfrak{q} = A \cap \mathfrak{m}$ of A is maximal among the prime ideals of A not containing π. This implies that $\sqrt{\mathfrak{q} + \pi A}$ is a maximal ideal of A. We see from this that $A/\mathfrak{q}$ is a local ring of dimension 1 and that its residue field is a finite extension of k (by our assumption on A).

Thus $\mathcal{O} \to A/\mathfrak{q}$ is a finite flat ring homomorphism. The normalization $\mathcal{O}'$ of the ring $A/\mathfrak{q}$ is a complete discrete valuation ring finite over $\mathcal{O}$. Let us denote the valuation by $|\ |: \mathcal{O}' \to \mathbf{R}_+$. The image of $I \subset A$ in $\mathcal{O}'$ lands in the maximal ideal of $\mathcal{O}'$, hence there exists an $n \in \mathbf{N}$ such that $|i| \leqslant |\pi|^{1/n}\ \forall\, i \in I$ (use that I is finitely generated). This implies that $|i| \leqslant |\pi|\ \forall\, i \in I^n$. This divisibility property allows us to define a ring homomorphism

$$\textbf{(7.1.9.1)} \qquad \begin{aligned} A[I^n/\pi] &\to \mathcal{O}' \\ i/\pi &\mapsto (\text{image of } i \text{ in } \mathcal{O}')/\pi. \end{aligned}$$

Again by the inequality above, this extends to a homomorphism $B_n \to \mathcal{O}'$ and finally to a homomorphism

$$C_n = B_n \otimes_{\mathcal{O}} K \to \mathcal{O}' \otimes_{\mathcal{O}} K = A/\mathfrak{q} \otimes_{\mathcal{O}} K = (A \otimes_{\mathcal{O}} K)/\mathfrak{m}.$$

This homomorphism is surjective since it extends the homomorphism $A \otimes K \to A/\mathfrak{q} \otimes K$. Hence, its kernel is a maximal ideal $\mathfrak{p} \subset C_n$.

We leave it to the reader to show that these constructions define mutually inverse bijections as indicated in the lemma.

Suppose $\mathfrak{m} \subset A \otimes K$, $\mathfrak{q} \subset A$ and $\mathfrak{p} \subset C_n$ correspond to each other in the sense explained above. Take generators $g_1, \ldots, g_s \in A$ of $\mathfrak{q}$. We claim that $\mathfrak{p} = g_1 C_n + \ldots + g_s C_n$. If we prove this then we have shown: the homomorphism

$$A_{\mathfrak{q}} = (A \otimes_{\mathcal{O}} K)_{\mathfrak{m}} \to (C_n)_{\mathfrak{p}}$$

is local, identifies residue fields (see above), is flat (Lemma 7.1.2) and unramified. Thus it induces an isomorphism on completions. By [BGR, 7.3.2/3] this gives the last assertion of the lemma.

Let us take generators $f_1, \ldots, f_N \in A$ of the ideal I^n. Since $\mathcal{O}'$ is the normalization of $A/\mathfrak{q}$, any element of $\mathcal{O}'$ satisfies a monic equation with coefficients in $A/\mathfrak{q}$. In particular we can find such an equation for the image of f_i/π in $\mathcal{O}'$. This implies that we can find equations

$$\textbf{(7.1.9.2)} \qquad (f_i)^d = a_{i,1}\, \pi (f_i)^{d-1} + a_{i,2}\, \pi^2 (f_i)^{d-2} + \ldots + a_{i,d}\, \pi^d + \Sigma \lambda_i^j g_j$$

with $a_{i,\ell} \in A$, $\lambda_i^j \in A$.

The ring $C_n/g_1 C_n + \ldots + g_s C_n$ is an affinoid algebra over $A/\mathfrak{q} \otimes K = (A \otimes K)/\mathfrak{m}$. The elements f_i/π give an affinoid generating system of $C_n/(g_1, \ldots, g_s)$ over $A/\mathfrak{q} \otimes K$ (they generate $A[I^n/\pi]$ over A). The equations (7.1.9.2) give that these elements are integral over $A/\mathfrak{q} \otimes K$, hence by [BGR, 6.3.2/2] $C_n/(g_1, \ldots, g_s)$ is finite over $A/\mathfrak{q} \otimes K$. As $A \otimes K$ is dense in C_n, we must have that $A/\mathfrak{q} \otimes K$ is dense in $C_n/(g_1, \ldots, g_s)$. Thus we must have equality. □

7.1.10. Let us look at morphisms of $FS_{\mathcal{O}}$ of the form $\mathrm{Spf}(\mathcal{O}') \to \mathfrak{X}$, where $\mathcal{O} \subset \mathcal{O}'$ is a finite extension of discrete valuation rings. A second such morphism $\mathrm{Spf}(\mathcal{O}'') \to \mathfrak{X}$ is said to be equivalent to $\mathrm{Spf}(\mathcal{O}') \to \mathfrak{X}$ if there exists a commutative diagram

$$\begin{array}{ccc} \mathrm{Spf}(\mathcal{O}''') & \longrightarrow & \mathrm{Spf}(\mathcal{O}') \\ \downarrow & & \downarrow \\ \mathrm{Spf}(\mathcal{O}'') & \longrightarrow & \mathfrak{X} \end{array}$$

where $\mathcal{O} \subset \mathcal{O}'''$ is also a finite extension of discrete valuation rings. It follows from the above that points of $\mathfrak{X}^{\mathrm{rig}}$ correspond bijectively with equivalence classes of such $\mathrm{Spf}(\mathcal{O}') \to \mathfrak{X}$.

There is a specialization mapping

$$sp : \mathfrak{X}^{\mathrm{rig}} \to \mathfrak{X}.$$

It maps the point $x \in \mathfrak{X}^{\mathrm{rig}}$ corresponding to $\mathfrak{m} \subset A \otimes K$ and $\mathfrak{q} = \mathfrak{m} \cap A \subset A$ to the unique maximal ideal of A containing π an $\mathfrak{q}$. If the point x corresponds to the equivalence class of $\varphi : \mathrm{Spf}(\mathcal{O}') \to \mathfrak{X}$ as above then $sp(x) = \varphi(\text{unique point of } \mathrm{Spf}(\mathcal{O}'))$. The map sp can be viewed as a morphism of ringed sites, see [B2, 0.2.6]. We remark that if $Z \subset \mathfrak{X}_{\mathrm{red}}$ is locally closed then $sp^{-1}(Z)$ is an admissible open subvariety of $\mathfrak{X}^{\mathrm{rig}}$. For example, if Z is closed, defined by the ideal $(g_1, g_2, \ldots, g_s)$ of A then

$$sp^{-1}(Z) = \{ x \in \mathfrak{X}^{\mathrm{rig}} \mid |g_i(x)| < 1 \ \forall\, i = 1, \ldots, s \}.$$

(Use (7.1.8.1) to view g_i as a function on $\mathfrak{X}^{\mathrm{rig}}$.) This is an admissible open subset by [BGR, 9.1.4/5].

7.1.11. Suppose $\mathfrak{X} = \mathrm{Spf}(A)$ as above. There is a functor from coherent $\mathcal{O}_{\mathfrak{X}}$-modules to coherent $\mathcal{O}_{\mathfrak{X}^{\mathrm{rig}}}$-modules; let us denote this functor by $\mathfrak{F} \mapsto \mathfrak{F}^{\mathrm{rig}}$. If $\mathfrak{F}$ is defined by the finite A-module M [EGA, I 10.10.5] then the sheaf $\mathfrak{F}^{\mathrm{rig}}$ is defined by the sequence of modules $M \otimes_A C_n \cong M \hat{\otimes}_A C_n$. It is clear that if $\mathfrak{F}$ if finite locally free then $\mathfrak{F}^{\mathrm{rig}}$ is finite locally free. There is an obvious map $\Gamma(\mathfrak{X}, \mathfrak{F}) \to \Gamma(\mathfrak{X}^{\mathrm{rig}}, \mathfrak{F}^{\mathrm{rig}})$ extending (7.1.8.1).

7.1.12. In this subsection we assume that $[k : k^p] < \infty$. On any $\mathfrak{X}$ as above we have the sheaf of continuous differentials defined as follows

$$\hat{\Omega}^1_{\mathfrak{X}} := \varprojlim \Omega^1_{\mathfrak{X}_n}.$$

It is a coherent sheaf of $\mathcal{O}_{\mathfrak{X}}$-modules. This can be seen by writing $\mathscr{A}$ as a quotient of $\mathcal{O}\{x_1, \ldots, x_n\}[[y_1, \ldots, y_m]]$ for some n, m and using the assumption that $[k : k^p] < \infty$.

Let us define the sheaf of differentials on an affinoid space $Y = \mathrm{Sp}(B)$ over K. Take any model $\mathfrak{Y} = \mathrm{Spf}(R)$ of Y. We put $\Omega_Y := (\hat{\Omega}^1_{\mathfrak{Y}})^{\mathrm{rig}}$. We leave it to the reader to prove that this is the sheaf associated to the module of continuous differentials $\hat{\Omega}^1_{B/Z}$ of B (see [EGA, 0_{IV} 20.4]). Thus the result is independent of the choice of the model and formation of it commutes with taking affinoid open subdomains of Y. By gluing we define the sheaf of differentials Ω_Y for any rigid analytic variety Y over K. We remark that the sheaf of differentials on Sp(K) is nontrivial in general: it has rank $[k : k^p]$ if char(K) $= 0$ and rank $[k : k^p] + 1$ if char(K) $= p$.

At this point we can state in more generality that for any formal scheme $\mathfrak{X}$ as above there is a canonical isomorphism

$$(\hat{\Omega}^1_{\mathfrak{X}})^{\mathrm{rig}} \cong \Omega_{\mathfrak{X}^{\mathrm{rig}}}.$$

This follows from the fact that the module $\hat{\Omega}^1_{B_n}$ is equal to the completion of $\Omega^1_{A[I^n/\pi]}$ and that this is equal to Ω^1_A up to π-torsion.

Finally, suppose the formal scheme $\mathfrak{X}$ is formally smooth over Spf($\mathcal{O}$) (see 2.4.3). In this case $\hat{\Omega}^1_{\mathfrak{X}}$ is finite locally free. Let the coherent sheaf of $\mathcal{O}_{\mathfrak{X}}$-modules $\mathfrak{F}$ be endowed with an integrable connection $\nabla : \mathfrak{F} \to \mathfrak{F} \otimes \hat{\Omega}^1_{\mathfrak{X}}$. It is now clear from the above that the resulting rigid analytic sheaf $\mathfrak{F}^{\mathrm{rig}}$ comes equipped with an integrable connection

$$\nabla^{\mathrm{rig}} : \mathfrak{F}^{\mathrm{rig}} \to \mathfrak{F}^{\mathrm{rig}} \otimes_{\mathcal{O}_{\mathfrak{X}^{\mathrm{rig}}}} \Omega_{\mathfrak{X}^{\mathrm{rig}}}.$$

7.1.13. Let us give a more precise description of the rings $A[I^n/\pi]$ and B_n. To give it, we choose generators $f_1, \ldots, f_r \in A$ of the ideal I. Consider the polynomial ring $R := A/\pi A[x_1, \ldots, x_r]$ and the homogeneous ideal $J \subset R$ generated by homogeneous polynomials $P(x_1, \ldots, x_r) \in R$ with

$$P(f_1, \ldots, f_r) = 0 \text{ in } A/\pi A.$$

It is generated by finitely many homogeneous polynomials $\overline{P}_1, \ldots, \overline{P}_s$ with degrees $d_1, \ldots, d_s$. For any $i = 1, \ldots, s$ we choose a homogeneous lift $P_i \in A[x_1, \ldots, x_r]$ of $\overline{P}_i \in R$. By our definition of J we can write $P_i(f_1, \ldots, f_r) = \pi q_i$, for some $q_i \in A$. Put $c := \max\{d_1, \ldots, d_s\}$. For any $n \geqslant c$ we consider the subring

$$A[T_M; \mid M \mid = n] \subset A[x_1, \ldots, x_r],$$

generated by monomials of degree n: for any multi-index $M = (m_1, \ldots, m_r)$ of total degree $\mid M \mid = m_1 + \ldots + m_r = n$ it has one variable T_M and these are subject to the relations $T_{M_1} T_{M_2} = T_{M_3} T_{M_4}$ if $M_1 + M_2 = M_3 + M_4$. We define a homomorphism

$$\varphi : A[T_M; \mid M \mid = n] \to A[I^n/\pi]$$

by putting $\varphi(T_M) = f^M/\pi = (1/\pi) \prod_i f_i^{m_i}$. It is surjective. If x^J is a monomial of degree $n - d_i$ then $x^J P_i(x_1, \ldots, x_r)$ is homogeneous of degree n and hence corresponds to an element

$$L_{J,i} := x^J P_i(x_1, \ldots, x_r) \in A[T_M; \mid M \mid = n].$$

We remark that it is linear in the variables T_M. The elements $L_{J,i} - \pi q_i f^J$ of the ring $A[T_M; \mid M \mid = n]$ are clearly in the kernel of φ.

7.1.13.1. *Lemma.* — *If* A *has no* π*-torsion then the elements* $L_{J,i} - \pi q_i f^J$ *generate the kernel of* φ *as an ideal.*

Proof. — If $F(T_M) \in \mathrm{Ker}(\varphi)$, write $F(T_M) = F_0 + F_1(T_M) + \ldots + F_d(T_M)$ with F_i homogeneous of degree i in T_M, then it is seen that $F_d(x^M)$ is in J. Thus it follows that we can reduce F to a polynomial of lower degree modulo the elements $L_{J,i} - \pi q_i f^J$. Details are left to the reader. □

We conclude that in this case

$$A[I^n/\pi] \cong A[T_M; \mid M \mid = n]/(L_{J,i} - \pi q_i f^J).$$

In particular we derive from this the existence of a surjective ring homomorphism

$$\text{(7.1.13.2)} \qquad \beta_n : A[I^n/\pi] \to A/I^{n-c}.$$

It is defined as the A-algebra homomorphism with $\beta_n(T_M) = 0\ \forall\ M$. It exists:

$$\beta_n(L_{J,i} - \pi q_i f^J) = -\pi q_i f^J \in I^{n-c},$$

since $|J| = n - d_i \geqslant n - c$. These homomorphisms induce homomorphisms

$$\beta_n : B_n \to A/I^{n-c}$$

and are compatible for varying n: we have commutative diagrams

$$\text{(7.1.13.3)} \qquad \begin{array}{ccc} B_n & \longrightarrow & A/I^{n-c} \\ \downarrow & & \downarrow \\ B_{n-1} & \longrightarrow & A/I^{n-c-1}. \end{array}$$

7.2. *Berthelot's functor for general formal schemes*

In this section we construct $\mathfrak{X}^{\mathrm{rig}}$ for general $\mathfrak{X}$ in $\mathrm{FS}_{\mathcal{O}}$. This is done by gluing $\mathfrak{U}^{\mathrm{rig}}$ for affine open formal subschemes $\mathfrak{U} \subset \mathfrak{X}$. It is possible by the proposition below.

7.2.1. *Proposition.* — *Let* $\varphi : \mathfrak{Y} \to \mathfrak{X}$ *be a morphism of affine formal schemes of* $\mathrm{FS}_{\mathcal{O}}$.

a) Suppose $\mathfrak{U} \subset \mathfrak{X}$ *is an affine open formal subscheme of* $\mathfrak{X}$ *with underlying reduced scheme* $U := \mathfrak{U}_{\mathrm{red}} \subset \mathfrak{X}_{\mathrm{red}}$. *The morphism* $\mathfrak{U}^{\mathrm{rig}} \to \mathfrak{X}^{\mathrm{rig}}$ *induces an isomorphism of* $\mathfrak{U}^{\mathrm{rig}}$ *onto the open analytic subvariety* [BGR, p. 354] $sp^{-1}(U) \subset \mathfrak{X}^{\mathrm{rig}}$.

b) Suppose $\mathfrak{X} = \bigcup \mathfrak{U}_i$ *is an affine covering of* $\mathfrak{X}$. *The covering* $\mathfrak{X}^{\mathrm{rig}} = \bigcup \mathfrak{U}_i^{\mathrm{rig}}$ *is admissible.*

c) If φ *is of finite type* [EGA, I 10.13.3] *then* φ^{rig} *is quasi-compact.*

d) If φ *is finite (resp. a closed immersion) then so is* φ^{rig}.

Proof. — Suppose $\mathfrak{U}$ is the affine open formal subscheme $\mathrm{Spf}(A\{1/f\})$ of $\mathfrak{X} = \mathrm{Spf}(A)$. In this case $sp^{-1}(U)$ is clearly equal to $\{x \in \mathfrak{X}^{\mathrm{rig}}; |f(x)| \geqslant 1\}$ and by Lemma 7.2.2 below we have $\mathfrak{U}^{\mathrm{rig}} \cong \bigcup_n \mathrm{Sp}(C_n\langle 1/f\rangle)$. Hence *a)* follows in this case. Next, suppose the covering $\mathfrak{X} = \bigcup \mathfrak{U}_i$ is given by affine open $\mathfrak{U}_i$ of the form $\mathfrak{U}_i = \mathrm{Spf}(\mathrm{A}\{1/f_i\})$. Finitely many f_i generate the unit ideal of A, say $f_1, \ldots, f_{\mathrm{N}}$ suffice. Thus for any affinoid subdomain $\mathrm{V} \subset \mathfrak{X}^{\mathrm{rig}}$, the intersections $\mathrm{V} \cap \mathfrak{U}_i^{\mathrm{rig}}$ for $i = 1, \ldots, \mathrm{N}$ are affinoid and cover V. It follows that the covering $\mathfrak{X}^{\mathrm{rig}} = \bigcup \mathfrak{U}_i^{\mathrm{rig}}$ is admissible. The general case of *a)* and *b)* follow from a formal argument using that a basis for the topology of $\mathfrak{X}$ is given by the subschemes $\mathrm{Spf}(\mathrm{A}\{1/f\})$.

Assertions *c)* and *d)* are a direct consequence of Lemma 7.2.2 below. □

7.2.2. *Lemma. — If* A *is as above and* $\mathrm{A} \to \mathrm{A}'$ *is topologically of finite type (i.e., satisfies the conditions of* [EGA, $\mathrm{O_I}$ 7.5.5]*) then*

$$\mathrm{Spf}(\mathrm{A}')^{\mathrm{rig}} \cong \bigcup_n \mathrm{Sp}(\mathrm{C}_n \hat{\otimes}_{\mathrm{A}} \mathrm{A}') = \bigcup_n \mathrm{Sp}(\mathrm{K} \otimes_{\mathcal{O}} \mathrm{B}_n \hat{\otimes}_{\mathrm{A}} \mathrm{A}').$$

More precisely, the inverse image of $\mathrm{Sp}(\mathrm{C}_n)$ *in* $\mathrm{Spf}(\mathrm{A}')^{\mathrm{rig}}$ *is equal to the affinoid space* $\mathrm{Sp}(\mathrm{C}_n \hat{\otimes}_{\mathrm{A}} \mathrm{A}')$.

Proof. — First note that $\mathrm{C}_n \hat{\otimes}_{\mathrm{A}} \mathrm{A}'$ is an affinoid K-algebra, since it is of topologically finite type over C_n. Second, $\mathrm{Sp}(\mathrm{C}_n \hat{\otimes}_{\mathrm{A}} \mathrm{A}') \to \mathrm{Sp}(\mathrm{C}_{n+1} \hat{\otimes}_{\mathrm{A}} \mathrm{A}')$ defines an affinoid subdomain:

$$\begin{aligned} \mathrm{C}_n \hat{\otimes}_{\mathrm{A}} \mathrm{A}' &\cong \mathrm{C}_{n+1} \langle x_1, \ldots, x_{\mathrm{N}} \rangle / (\pi x_i - f_i) \hat{\otimes}_{\mathrm{A}} \mathrm{A}' \\ &\cong (\mathrm{C}_{n+1} \hat{\otimes}_{\mathrm{A}} \mathrm{A}') \langle x_1, \ldots, x_{\mathrm{N}} \rangle / (\pi x_i - f_i). \end{aligned}$$

(Notation as in the proof of Lemma 7.1.2.) Let us define $\mathrm{Y}' := \bigcup \mathrm{Sp}(\mathrm{C}_n \hat{\otimes}_{\mathrm{A}} \mathrm{A}')$. It is a separated rigid analytic space over K. There is a morphism of rigid analytic spaces $\mathrm{Y}' \to \mathrm{Spf}(\mathrm{A}')^{\mathrm{rig}}$: for each n there is a morphism

$$\mathrm{Spf}(\mathrm{B}_n \hat{\otimes}_{\mathrm{A}} \mathrm{A}') \to \mathrm{Spf}(\mathrm{A}')$$

hence a morphism

$$\mathrm{Sp}(\mathrm{C}_n \hat{\otimes}_{\mathrm{A}} \mathrm{A}') = \mathrm{Spf}(\mathrm{B}_n \hat{\otimes}_{\mathrm{A}} \mathrm{A}')^{\mathrm{rig}} \to \mathrm{Spf}(\mathrm{A}')^{\mathrm{rig}}.$$

Finally, we have to show that any morphism $\psi : \mathrm{W} \to \mathrm{Spf}(\mathrm{A}')^{\mathrm{rig}}$, W an affinoid K-variety, factors as $\mathrm{W} \to \mathrm{Y}' \to \mathrm{Spf}(\mathrm{A}')^{\mathrm{rig}}$ (see 7.1.5). We use (7.1.7) that ψ comes from a morphism of formal schemes: $\mathfrak{M} = \mathrm{Spf}(\mathrm{R}) \to \mathrm{Spf}(\mathrm{A}')$, where $\mathfrak{M}$ is a model of W. This morphism is given by a continuous ring homomorphism $\mathrm{A}' \to \mathrm{R}$. On the other hand, the composition $\mathrm{W} \to \mathrm{Spf}(\mathrm{A}')^{\mathrm{rig}} \to \mathrm{Spf}(\mathrm{A})^{\mathrm{rig}}$ comes from a continuous ring homomorphism $\mathrm{C}_n \to \mathrm{R} \otimes \mathrm{K}$ for some n. Of course these have to agree on A, hence we get $\mathrm{C}_n \hat{\otimes}_{\mathrm{A}} \mathrm{A}' \to \mathrm{R} \otimes \mathrm{K}$ as desired. □

7.2.3. Let us construct $\mathfrak{X}^{\mathrm{rig}}$ for a separated [EGA, I 10.15.1] formal scheme $\mathfrak{X} \in \mathrm{Ob}\,\mathrm{FS}_{\mathcal{O}}$. Choose a covering $\mathfrak{X} = \bigcup \mathfrak{U}_i$ by affine formal schemes $\mathfrak{U}_i$. The intersections $\mathfrak{U}_{ij} = \mathfrak{U}_i \cap \mathfrak{U}_j$ are affine also. Thus by Proposition 7.2.1 we see that $\mathfrak{U}_{ij}^{\mathrm{rig}}$ is an open analytic subvariety of both $\mathfrak{U}_i^{\mathrm{rig}}$ and $\mathfrak{U}_j^{\mathrm{rig}}$. Therefore, by [BGR, 9.3.2], we can define $\mathfrak{X}^{\mathrm{rig}}$ as the pasting of the $\mathfrak{U}_i^{\mathrm{rig}}$ along $\mathfrak{U}_{ij}^{\mathrm{rig}}$.

For a general $\mathfrak{X}$ we use the same procedure, using that we already have the definition of $\mathfrak{U}_{ij}^{\mathrm{rig}}$ for the rigid analytic space associated to the separated but not necessarily affine formal schemes $\mathfrak{U}_{ij}$. We leave the verifications to the reader.

7.2.4. *Proposition. — Suppose* $\varphi : \mathfrak{X} \to \mathfrak{Y}$ *is a morphism of* $\mathrm{FS}_{\mathcal{O}}$.

a) $\mathfrak{X}^{\mathrm{rig}}$ *is a quasi-separated rigid analytic space; if* $\mathfrak{X}$ *is separated then* $\mathfrak{X}^{\mathrm{rig}}$ *is separated.*

b) If $\mathfrak{X}$ *is flat over* $\mathrm{Spf}(\mathcal{O})$ *and nonempty then* $\mathfrak{X}^{\mathrm{rig}}$ *is nonempty.*

c) If $\mathfrak{X}$ *is a formally reduced (resp. normal) formal scheme (see the proof for definitions) then* $\mathfrak{X}^{\mathrm{rig}}$ *is reduced (resp. normal).*

d) If φ *is an open immersion then* φ^{rig} *is an open immersion* [BGR, p. 354].

e) If φ *is a closed immersion (resp. a finite morphism) then so is* φ^{rig}.

f) If φ *is a morphism of finite type then* φ^{rig} *is a quasi-compact.*

g) If $\mathfrak{Z} \to \mathfrak{Y}$ *is a second morphism in* $\mathrm{FS}_{\mathcal{O}}$ *with target* $\mathfrak{Y}$ *then*

(7.2.4.1) $$(\mathfrak{X} \times_{\mathfrak{Y}} \mathfrak{Z})^{\mathrm{rig}} \cong \mathfrak{X}^{\mathrm{rig}} \times_{\mathfrak{Y}^{\mathrm{rig}}} \mathfrak{Z}^{\mathrm{rig}}.$$

(Here the fibre products are taken in the category of rigid analytic spaces [BGR, 9.3.5/2] *and in the category of formal schemes* [EGA, I 10.7.3]. *We remark that* $\mathfrak{X} \times_{\mathfrak{Y}} \mathfrak{Z}$ *is in* $\mathrm{FS}_{\mathcal{O}}$.*)*

h) If φ *is a separated morphism of formal schemes* [EGA, I 10.15.1] *then* φ^{rig} *is separated.*

Proof. — We say that $\mathfrak{X}$ is formally reduced (resp. normal) if it can be covered by affines Spf(A) with A reduced (resp. normal). This is equivalent to the condition that all complete local rings of $\mathfrak{X}$ are reduced (resp. normal), since our rings A are excellent. Thus *c)* follows from Lemma 7.1.9.

Part *b)* is left to the reader. Part *d)* follows from the definition. Parts *e)* and *f)* follow from 7.2.1.

The diagonal morphism $\Delta : \mathfrak{X} \to \mathfrak{X} \times_{\mathcal{O}} \mathfrak{X}$ is of finite type. Indeed, it is clearly adic and $\Delta_{\mathrm{red}} : \mathfrak{X}_{\mathrm{red}} \to (\mathfrak{X} \times_{\mathcal{O}} \mathfrak{X})_{\mathrm{red}} \cong \mathfrak{X}_{\mathrm{red}} \times_k \mathfrak{X}_{\mathrm{red}}$ is of finite type (use that Δ_{red} is an immersion and that $(\mathfrak{X} \times_{\mathcal{O}} \mathfrak{X})_{\mathrm{red}}$ is locally Noetherian, next apply [EGA, I 6.3.5]). Thus it follows from *f)* and *g)* that $\mathfrak{X}^{\mathrm{rig}}$ is quasi-separated.

We claim that a morphism $\varphi : \mathfrak{X} \to \mathfrak{Y}$ in $\mathrm{FS}_{\mathcal{O}}$ is separated if and only if the diagonal morphism $\Delta : \mathfrak{X} \to \mathfrak{X} \times_{\mathfrak{Y}} \mathfrak{X}$ is a closed immersion. To see that Δ is a closed immersion if φ is separated we use Lemma 10.14.4 of [EGA, I]. First, $\mathfrak{X} \times_{\mathfrak{Y}} \mathfrak{X}$ is locally Noetherian and second $\Delta(\mathfrak{X}) \subset \mathfrak{X} \times_{\mathfrak{Y}} \mathfrak{X}$ is a closed subset (by definition). For the U_α occurring in the Lemma we take $\mathrm{Spf}(\mathrm{A} \hat{\otimes}_{\mathrm{B}} \mathrm{A}) \subset \mathfrak{X} \times_{\mathfrak{Y}} \mathfrak{X}$, where Spf(A) is an affine open formal subscheme of $\mathfrak{X}$ mapping into the open affine Spf(B) of $\mathfrak{Y}$. The reverse implication is trivial. We conclude that *h)* follows from *g)* and *e)*.

Finally, we have to prove *g)*. By functoriality of Berthelot's construction, there are morphisms:

$$(\mathfrak{X} \times_{\mathfrak{Y}} \mathfrak{Z})^{\mathrm{rig}} \to \mathfrak{X}^{\mathrm{rig}} \quad \text{and} \quad (\mathfrak{X} \times_{\mathfrak{Y}} \mathfrak{Z})^{\mathrm{rig}} \to \mathfrak{Z}^{\mathrm{rig}}$$

agreeing as maps to $\mathfrak{Y}^{\mathrm{rig}}$. Thus we get the morphism

$$(\mathfrak{X} \times_{\mathfrak{Y}} \mathfrak{Z})^{\mathrm{rig}} \to \mathfrak{X}^{\mathrm{rig}} \times_{\mathfrak{Y}^{\mathrm{rig}}} \mathfrak{Z}^{\mathrm{rig}}.$$

To check that it is an isomorphism we may assume that our formal schemes are affine. In this case the fact that (7.2.4.1) is an isomorphism follows easily from 7.1.5 and 7.1.7. Indeed, by 7.1.7, a morphism of an affinoid variety W into $\mathfrak{X}^{\mathrm{rig}} \times_{\mathfrak{Y}^{\mathrm{rig}}} \mathfrak{Z}^{\mathrm{rig}}$ is given by morphisms

$$\mathfrak{M} \to \mathfrak{X} \quad \text{and} \quad \mathfrak{M} \to \mathfrak{Z}$$

agreeing as morphisms to $\mathfrak{Y}$, of a suitable model $\mathfrak{M}$ of W. Hence we get $\mathfrak{M} \to \mathfrak{X} \times_{\mathfrak{Y}} \mathfrak{Z}$ which induces $W = \mathfrak{M}^{\mathrm{rig}} \to (\mathfrak{X} \times_{\mathfrak{Y}} \mathfrak{Z})^{\mathrm{rig}}$. Thus we have

$$\mathrm{Mor}_{\mathrm{Rig}_K}(W, (\mathfrak{X} \times_{\mathfrak{Y}} \mathfrak{Z})^{\mathrm{rig}}) \cong \mathrm{Mor}_{\mathrm{Rig}_K}(W, (\mathfrak{X}^{\mathrm{rig}} \times_{\mathfrak{Y}^{\mathrm{rig}}} \mathfrak{Z}^{\mathrm{rig}})).$$

The result now follows from 7.1.5. □

7.2.5. *Lemma. — Let $\mathfrak{X}$ be an object of* $\mathrm{FS}_{\mathcal{O}}$ *and suppose* $Z \subset \mathfrak{X}_{\mathrm{red}}$ *is a closed subscheme. The completion* $\mathfrak{X}_Z^\wedge$ *of* $\mathfrak{X}$ *along* Z *is an object of* $\mathrm{FS}_{\mathcal{O}}$ *also and the morphism* $(\mathfrak{X}_Z^\wedge)^{\mathrm{rig}} \to \mathfrak{X}^{\mathrm{rig}}$ *induced by* $\mathfrak{X}_Z^\wedge \to \mathfrak{X}$ *is an open immersion inducing an isomorphism of* $(\mathfrak{X}_Z^\wedge)^{\mathrm{rig}}$ *with the open subvariety* $sp^{-1}(Z)$ *of* $\mathfrak{X}^{\mathrm{rig}}$.

Proof. — See [B2, 0.2.7]. In the case that $\mathfrak{X}$ is affine this can also be seen using 7.1.7. Suppose that W is an affinoid variety over K and that a morphism $f: W \to sp^{-1}(Z)$ is given. This comes from a morphism $\varphi : \mathfrak{M} \to \mathfrak{X}$ of a model $\mathfrak{M}$ of W. We must have $\varphi(\mathfrak{M}_{\mathrm{red}}) \subset Z$, since the specialization mapping $sp : W \to \mathfrak{M}_{\mathrm{red}}$ is surjective [BGR, 7.1.5/4]. Hence φ factors as $\mathfrak{M} \to \mathfrak{X}_Z^\wedge \to \mathfrak{X}$. Thus f is a composition $W \to (\mathfrak{X}_Z^\wedge)^{\mathrm{rig}} \to \mathfrak{X}^{\mathrm{rig}}$. Using that $sp^{-1}(Z)$ is admissible open (7.1.10) it follows from 7.1.5 that $(\mathfrak{X}_Z^\wedge)^{\mathrm{rig}} \cong sp^{-1}(Z)$. □

7.2.6. Suppose that $\sigma : \mathcal{O} \to \mathcal{O}'$ is a homomorphism of complete discrete valuation rings, coming from an extension of quotient fields $K \subset K'$ such that the topology on K is induced by that of K′. In this case there is a base field extension functor $X \mapsto X \hat{\otimes} K'$ defined for quasi-separated analytic varieties X over K. See [BGR, 9.3.6]. Of course there is also a base change functor $\mathrm{FS}_{\mathcal{O}} \to \mathrm{FS}_{\mathcal{O}'}$, given simply by $\mathfrak{X} \mapsto \mathfrak{X} \times_{\mathrm{Spf}(\mathcal{O})} \mathrm{Spf}(\mathcal{O}')$. It results easily from the definitions that there is a canonical isomorphism

$$(\mathfrak{X} \times_{\mathrm{Spf}(\mathcal{O})} \mathrm{Spf}(\mathcal{O}'))^{\mathrm{rig}} \cong \mathfrak{X}^{\mathrm{rig}} \hat{\otimes} K'.$$

Given an analytic variety Y over K′ and a quasi-separated analytic variety X over K we define a general morphism $f: Y \to X$ over σ to be a morphism $Y \to X \hat{\otimes} K'$ of analytic varieties over K. For motivation, see [JP]. We remark that such a general morphism gives rise to pullback functors f^* on sheaves and coherent sheaves. Furthermore, if the sheaves of differentials on X and Y are defined (see 7.1.12), then there is a canonical homomorphism $f^* \Omega_X \to \Omega_Y$.

Finally, suppose we are given formal schemes $\mathfrak{Y}$ of $\mathrm{FS}_{\mathcal{O}'}$ and $\mathfrak{X}$ of $\mathrm{FS}_{\mathcal{O}}$ and a morphism of formal schemes $\varphi : \mathfrak{Y} \to \mathfrak{X}$ lying over the morphism $\mathrm{Spf}(\sigma) : \mathrm{Spf}(\mathcal{O}) \to \mathrm{Spf}(\mathcal{O}')$. It follows from the above that φ induces a general morphism $\varphi^{\mathrm{rig}} : \mathfrak{Y}^{\mathrm{rig}} \to \mathfrak{X}^{\mathrm{rig}}$.

7.3. *Complete local rings and Berthelot's construction*

7.3.1. *Situation.* — Here A and B are complete local Noetherian flat $\mathcal{O}$-algebras such that $A/\mathfrak{m}_A$ and $B/\mathfrak{m}_B$ are finite extensions of $k = \mathcal{O}/\pi\mathcal{O}$. We assume that A has no zero divisors. In addition, we are given a finite injective homomorphism $\varphi : A \to B$. We define the degree of φ as $\deg(\varphi) = \dim_{Q(A)} Q(A) \otimes_A B$. Here $Q(A)$ is the quotient field of the ring A.

The formal schemes Spf(A) and Spf(B) are objects of $FS_{\mathcal{O}}$. Let us put $X := \mathrm{Spf}(A)^{\mathrm{rig}}$ and $Y := \mathrm{Spf}(B)^{\mathrm{rig}}$. The map $\mathrm{Spf}(\varphi) : \mathrm{Spf}(B) \to \mathrm{Spf}(A)$ induces a morphism of rigid analytic spaces $\varphi^{\mathrm{rig}} : Y \to X$. It is a finite morphism by Proposition 7.2.1.

7.3.2. *Lemma.* — *a) There is an analytically closed, nowhere dense subset* $Z \subset X$ *such that the morphism*

$$\varphi^{\mathrm{rig}} : Y \backslash (\varphi^{\mathrm{rig}})^{-1}(Z) \to X \backslash Z$$

is finite flat of degree equal to $\deg(\varphi)$.

b) If A *and* B *are normal local rings then one can choose* Z *such that* $\mathrm{codim}_X Z \geqslant 2$.

Proof. — Consider a presentation of B as an A-module:

$$A^{m_1} \xrightarrow{T} A^{m_0} \to B \to 0.$$

The ideal $J \subset A$ generated by the $(m_2 - \deg(\varphi))$-minors of T is nonzero by assumption. We claim that $Z := \mathrm{Spf}(A/J)^{\mathrm{rig}} \subset X$ works. By Proposition 7.2.1 it is a closed subvariety of X and using Lemma 7.1.9 it is easy to see that Z is nowhere dense in X (since $J \neq 0$). This proves *a)*.

Let us prove the claim. Lemma 7.2.2 implies the equality $Y = \bigcup_n \mathrm{Sp}(C_n \otimes_A B)$. It is clear that if $g \in J$ then $B[1/g] \cong (A[1/g])^{\deg(\varphi)}$ as an $A[1/g]$-module. Hence, $(C_n \otimes_A B)\,[1/g] \cong (C_n[1/g])^{\deg(\varphi)}$. Since $Z \cap \mathrm{Sp}(C_n)$ is given by $\mathrm{Sp}(C_n/JC_n)$ (again 7.2.2) the claim is proven.

If A and B are normal and $\mathfrak{p} \subset A$ has height 1 then $A_{\mathfrak{p}}$ (and $B_{\mathfrak{q}}$ for any $\mathfrak{q} \subset B$ lying over $\mathfrak{p}$) is regular. Hence by [Mat, Theorem 46] the homomorphism $A_{\mathfrak{p}} \to B \otimes A_{\mathfrak{p}} = \prod B_{\mathfrak{q}}$ is finite flat. Thus $\mathfrak{p} \notin V(J) \subset \mathrm{Spec}(A)$. Conclusion: if A and B are normal then $\mathrm{codim}_{\mathrm{Spec}(A)} V(J) \geqslant 2$ hence $\mathrm{codim}_X Z \geqslant 2$. (Use Lemma 7.1.9.) This proves *b)*. □

7.3.3. *Lemma.* — *Suppose* A *and* B *are normal. Any element* $f \in \Gamma(Y, \mathcal{O}_Y)$ *satisfies an equation*

$$T^{\deg(\varphi)} + b_1 T^{\deg(\varphi)-1} + \ldots + b_{\deg(\varphi)} = 0$$

with $b_i \in \Gamma(X, \mathcal{O}_X)$. *If* $f \in \Gamma(Y, \mathcal{O}_Y^0)$ *then we may take* $b_i \in \Gamma(X, \mathcal{O}_X^0)$.

Proof. — Take $Z \subset X$ as in part *b)* of 7.3.2. The sheaf $\mathscr{E} := \varphi_*^{\mathrm{rig}}(\mathcal{O}_Y)|_{X \setminus Z}$ is a finite locally free sheaf of $\mathcal{O}_{X \setminus Z}$-modules of rank $\deg(\varphi)$. Hence the endomorphism T of $\mathscr{E}$ given by multiplication by f satisfies an equation

$$T^{\deg(\varphi)} + b_1 T^{\deg(\varphi)-1} + \ldots + b_{\deg(\varphi)} = 0$$

with $b_i \in \Gamma(X \setminus Z, \mathcal{O}_{X \setminus Z})$. The functions b_i extend to analytic functions $b_i \in \Gamma(X, \mathcal{O}_X)$ on X by [Lü, Theorem 1.6 part II] (X is normal by Proposition 7.2.4). This proves the first statement.

Now let $f \in \Gamma(Y, \mathcal{O}_Y^0)$. We have to prove $b_i \in \Gamma(X, \mathcal{O}_X^0)$. To do this take any affinoid subvariety $W \subset X$ and put $W' = (\varphi^{\mathrm{rig}})^{-1}(W) \subset Y$. It is an affinoid subvariety of Y. We consider the ring extension

$$R := \Gamma(W, \mathcal{O}_W^0) \subset R' := \Gamma(W', \mathcal{O}_{W'}^0).$$

The extension $R \subset R'$ is finite by [BGR, 6.4.1/6]. Since X and Y are normal, R and R′ are normal $\mathcal{O}$-algebra's topologically of finite type. By the result of the lemma above we have

$$\deg(\varphi) = \dim_{Q(R)} Q(R) \otimes_R R'.$$

Thus an argument similar to the proof of the lemma gives us an ideal $J \subset R$ with $\operatorname{codim} V(J) \geqslant 2$ and such that $\operatorname{Spec}(R') \to \operatorname{Spec}(R)$ is finite flat of degree $\deg(\varphi)$ outside $V(J) \subset \operatorname{Spec}(R)$. Thus the element f (seen as an element of R′ via $\Gamma(Y, \mathcal{O}_Y^0) \to \Gamma(W', \mathcal{O}_{W'}^0) = R'$) satisfies an equation

$$T^{\deg(\varphi)} + b'_1 T^{\deg(\varphi)-1} + \ldots + b'_{\deg(\varphi)} = 0$$

with $b'_i \in \Gamma(\operatorname{Spec}(R) \setminus V(J), \mathcal{O}_{\operatorname{Spec}(R)}) = \Gamma(\operatorname{Spec}(R), \mathcal{O}_{\operatorname{Spec}(R)}) = R$. Comparing these b'_i with the b_i above, we see that they must be equal as elements of $\Gamma(W, \mathcal{O}_W)$. This concludes the proof of the lemma. □

7.3.4. *Lemma.* — *If* $A = \mathcal{O}[[x_1, \ldots, x_n]]$ *then the homomorphism* (7.1.8.2)

$$\mathcal{O}[[x_1, \ldots, x_n]] \to \Gamma(X, \mathcal{O}_X^0)$$

is an isomorphism. In this case there is an isomorphism $X \cong D^n$ *of* X *with the n-dimensional open unit ball* D^n *with coordinates* $x_1, \ldots, x_n$.

Proof. — A power-series $\Sigma a_I x^I$ is bounded by 1 on the open unit ball if and only if all $|a_I| \leqslant 1$, i.e., $a_I \in \mathcal{O}$. Thus we need only to find an isomorphism $X \to D^n$. Remark that $A = \mathcal{O}[[x_1, \ldots, x_n]]$ is the completion of $R := \mathcal{O}\{x_1, \ldots, x_n\}$ in the maximal ideal $(x_1, \ldots, x_n)$. Since clearly $\operatorname{Spf}(R)^{\mathrm{rig}} \cong \mathbf{B}^n$, it follows from Lemma 7.2.5 that $X \cong D^n$ (see also 7.1.10). □

7.3.5. *Lemma.* — *If* B *is as in* 7.3.1 *and* B *is normal, then* $Y = \mathrm{Spf}(B)^{\mathrm{rig}}$ *is a connected rigid analytic variety.*

Proof. — Choose a finite injective homomorphism

$$\varphi : A = \mathcal{O}[[x_1, \ldots, x_d]] \to B$$

(to find it argue as in [Mat, p. 212]). Consider the maximal separable extension $Q(A) \subset L$ contained in $Q(B)$. Let $B' = B \cap L$ = normal closure of A in L. It is a local ring which is a finite extension of A. The morphism $Y \to X = \mathrm{Spf}(A)^{\mathrm{rig}}$ factors as $Y \to Y' \to X$ with $Y' = \mathrm{Spf}(B')^{\mathrm{rig}}$. In case $\mathrm{char}(K) = 0$, we have $Y = Y'$. In case $\mathrm{char}(K) = p$, the morphism $i : Y \to Y'$ is finite and purely inseparable: for any local section s of $i_*(\mathcal{O}_Y)$ there exists a number n such that s^{p^n} lies in $\mathcal{O}_{Y'}$. Thus the morphism $Y \to Y'$ is an isomorphism on underlying Zariski-topological spaces. (It is even an isomorphism on underlying G-topological spaces.) Hence it suffices to prove that Y' is connected. We may therefore assume that $Q(A) \subset Q(B)$ is finite separable.

Let us choose a finite Galois extension $Q(A) \subset L$ containing $Q(B)$. The normal closure B' of A in L is a local ring, finite over A (since A is Nagata [Mat, 31.C]) and contains B. Clearly, we may replace B by B' (since $\mathrm{Spf}(B')^{\mathrm{rig}} \to \mathrm{Spf}(B)^{\mathrm{rig}}$ is surjective). Hence, we may assume $Q(A) \subset Q(B)$ finite Galois, say with group $G = \mathrm{Gal}(Q(B)/Q(A))$.

The groups G acts by automorphisms on B over A. Hence on Y over X. We remark that there exists a Zariski open $U \subset \mathrm{Spec}(A)$ such that the inverse image $V \subset \mathrm{Spec}(B)$ is a principal homogeneous G-space over U. In fact we can take $U = \mathrm{Spec}(A) \setminus$ the discriminant locus of $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$. Hence, the same argument as in the proof of Lemma 7.3.2 shows that there is a Zariski open $U \subset X$ such that $(\varphi^{\mathrm{rig}})^{-1}(U)$ is a principal homogeneous G-space over U.

For any connected component $Z \subset Y$ the morphism

$$\varphi^{\mathrm{rig}}|_Z : Z \to X$$

is finite (as the composition $Z \to Y \to X$). As it is also generically flat (by 7.3.2) its image must be a connected component of X. Since X is connected (7.3.4) we see that $Z \to X$ is surjective. Consequently, Y has only finitely many (connected) components, say $Y = Z_1 \cup \ldots \cup Z_r$. Let $Z = Z_1$ and put

$$G \supset H := \{ g \in G \mid g(Z) = Z \}.$$

Claim. — We have $\# H = \deg(\varphi^{\mathrm{rig}}|_Z : Z \to X)$ and the natural homomorphism

$$\Gamma(X, \mathcal{O}_X^0) \to \Gamma(Z, \mathcal{O}_Z^0)^H$$

is an isomorphism.

If we have this, then we are through. Indeed, the homomorphism

$$B \to \Gamma(Y, \mathcal{O}_Y^0) \to \Gamma(Z, \mathcal{O}_Z^0)$$

is H-invariant. By the claim we would get a homomorphism

$$B^H \to \Gamma(Z, \mathcal{O}_Z^0)^H = \Gamma(X, \mathcal{O}_X^0) = A$$

of A-algebras. This contradicts Galois theory unless $H = G$, i.e., $Z = Y$.

To prove the claim consider a point $x \in U \subset X$ in the Zariski open subset of X constructed above. The group G acts transitively on the finite set $(\varphi^{\mathrm{rig}})^{-1}(\{x\})$, hence it acts transitively on the components $Z_1, \ldots, Z_r$, hence $\#H = \#G/r$. We must also have $\deg(\varphi^{\mathrm{rig}}|_{Z_i}) = \deg(\varphi^{\mathrm{rig}}|_Z)$ (transitivity again) and hence

$$\#G = \deg(\varphi) = \sum_i \deg(\varphi^{\mathrm{rig}}|_{Z_i}) = r.\deg(\varphi^{\mathrm{rig}}|_Z).$$

Any H-invariant function on Z can be uniquely extended to a G-invariant function on Y. Therefore, it suffices to show the map

$$\Gamma(X, \mathcal{O}_X^0) \to \Gamma(Y, \mathcal{O}_Y^0)^G$$

is a bijection. Take an $f \in \Gamma(Y, \mathcal{O}_Y^0)$ which is G-invariant. The morphism $(\varphi^{\mathrm{rig}})^{-1} U \to U$ is a quotient morphism, hence $f|_{(\varphi^{\mathrm{rig}})^{-1} U}$ comes from a unique $g \in \Gamma(U, \mathcal{O}_U^0)$. Since it is bounded, it comes from a unique $g \in \Gamma(X, \mathcal{O}_X^0)$ by [Lü, Theorem 1.6 part I] (by Lemma 7.3.4 $X \cong D^d$ is absolutely normal). This proves the claim. □

Finally, we come to the main result of this subsection.

7.3.6. *Proposition. — If* B *is as in* 7.3.1 *and* B *is normal, then the homomorphism* (7.1.8.2)

$$B \to \Gamma(\mathrm{Spf}(B)^{\mathrm{rig}}, \mathcal{O}^0) = \Gamma(Y, \mathcal{O}_Y^0)$$

is bijective.

Proof. — Choose a finite injective homomorphism

$$\varphi : A = \mathcal{O}[[x_1, \ldots, x_d]] \to B.$$

Choose $f \in \Gamma(Y, \mathcal{O}_Y^0)$. From Lemma 7.3.3 and Lemma 7.3.4 we see that f satisfies an equation

$$T^n + a_1 T^{n-1} + \ldots + a_n = 0$$

with $a_i \in A = \mathcal{O}[[\underline{x}]]$. We will show that the assumption $f \notin B$ leads to a contradiction with Lemma 7.3.5.

The ring $B[f] \subset \Gamma(Y, \mathcal{O}_Y^0)$ is reduced (Y is reduced). Since Y is irreducible (for the Zariski topology) as it is connected and normal, we see that $B[f]$ has no zero divisors. By the equation above we see that $B[f]$ is a finite B-algebra. The normal closure

$$B[f]^{\mathrm{nor}} \subset Q(B[f])$$

is a finite B-algebra, since B is Nagata. Any element $g \in B[f]^{\mathrm{nor}}$ can be seen as a meromorphic function on Y which is integral over $B[f] \subset \Gamma(Y, \mathcal{O}_Y^0)$, hence $g \in \Gamma(Y, \mathcal{O}_Y^0)$.

Therefore we have $B[f]^{nor} \subset \Gamma(Y, \mathcal{O}^0_Y)$. Arguing as before, we see that $B[f]^{nor}$ has no zero divisors, thus (being a finite B-algebra) it is a local B-algebra. We consider the morphism

$$\psi : Y' := \mathrm{Spf}(B[f]^{nor})^{rig} \to Y$$

induced by $B \subset B[f]^{nor}$. We claim that ψ has a section $s : Y \to Y'$. This is seen as follows. Take any affinoid subdomain $U \subset Y$. Put $R := \Gamma(U, \mathcal{O}^0_U)$. The homomorphism

$$B[f]^{nor} \to \Gamma(Y, \mathcal{O}^0_Y) \to \Gamma(U, \mathcal{O}^0_U) = R$$

induces a morphism $\mathrm{Spf}(R) \to \mathrm{Spf}(B[f]^{nor})$ and hence

$$U = \mathrm{Spf}(R)^{rig} \to \mathrm{Spf}(B[f]^{nor})^{rig} = Y'.$$

This defines $s|_U$. It is a section of ψ since the composition $B \to B[f]^{nor} \to \Gamma(Y, \mathcal{O}^0_Y)$ is the map (7.1.8.2). Since Y and Y′ are reduced, normal and connected and since ψ is finite surjective, we see that $s(Y)$ must be a connected component of Y′. Lemma 7.3.5 thus implies that $Y' = s(Y)$. By Lemma 7.3.2 the degree of ψ is $\dim_{Q(B)} Q(B) \otimes_B B[f]^{nor}$. Thus we get a contradiction unless this degree is 1, i.e., $f \in B$. □

7.4. *Analytic functions and formal functions*

In this section we prove that bounded rigid analytic functions on $\mathfrak{X}^{rig}$ come from functions on $\mathfrak{X}$.

7.4.1. *Theorem. — Let $\mathfrak{X}$ be a formal scheme in $FS_{\mathcal{O}}$. If $\mathfrak{X}$ is $\mathcal{O}$-flat and (formally) normal then the homomorphism* (7.1.8.2)

$$\Gamma(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}) \to \Gamma(\mathfrak{X}^{rig}, \mathcal{O}^0_{\mathfrak{X}^{rig}})$$

is an isomorphism.

Proof. — We may assume $\mathfrak{X} = \mathrm{Spf}(A)$ is affine. Let us write $X = \mathfrak{X}^{rig}$. It is constructed (7.1.3) as the union:

$$X = \bigcup_n \mathrm{Sp}(C_n) = \bigcup_n \mathrm{Sp}(B_n \otimes_{\mathcal{O}} K).$$

Take $f \in \Gamma(X, \mathcal{O}^0_X)$. Its restriction to $\mathrm{Sp}(C_n)$ is an element $f_n \in C_n$ which is integral over B_n, i.e., it lies in the integral closure B'_n of B_n in $C_n = B_n \otimes_{\mathcal{O}} K : f_n \in B'_n$.

Take a maximal ideal $\mathfrak{m} \subset B_n$. Put $\mathfrak{m}' = \mathfrak{m} \cap A$; it is a maximal ideal of A ($B_n/\mathfrak{m}$ is a finite extension of k). Consider the commutative diagram of $\mathcal{O}$-algebras:

$$\text{(7.4.1.1)} \qquad \begin{array}{ccccc} A & \longrightarrow & B_n & \longrightarrow & B'_n \\ \downarrow & & \downarrow & & \downarrow \\ A^{\wedge}_{\mathfrak{m}'} & \longrightarrow & (B_n)^{\wedge}_{\mathfrak{m}} & \longrightarrow & (B'_n)^{\wedge}_{\mathfrak{m}B'_n}. \end{array}$$

Here $(\cdot)^{\wedge}_{\mathfrak{p}}$ denotes completion with respect to $\mathfrak{p}$. This diagram gives rise to a commutative diagram of rigid analytic spaces by applying the functor $(\cdot)^{\text{rig}}$. We remark that $A^{\wedge}_{\mathfrak{m}'}$ is normal, so that we can apply 7.3.6 to f restricted to $\mathrm{Spf}(A^{\wedge}_{\mathfrak{m}'})^{\text{rig}} \subset X$. This gives $f_{\mathfrak{m}'} \in A^{\wedge}_{\mathfrak{m}'}$. The image of f_n and $f_{\mathfrak{m}'}$ in the ring $(B'_n)^{\wedge}_{\mathfrak{m}B'_n}$ coincide (they give the same rigid analytic functions). Thus we conclude that f_n lies in $B'_n \cap (B_n)^{\wedge}_{\mathfrak{m}}$. Write f_n as a/b, with $a, b \in B_n$, then we see that $a \in b(B_n)^{\wedge}_{\mathfrak{m}}$, hence $a \in bB_{n,\mathfrak{m}}$ (by faithful flatness of $B_{n,\mathfrak{m}} \subset (B_n)^{\wedge}_{\mathfrak{m}}$). Thus $f_n \in B_{n,\mathfrak{m}}$ $\forall\, \mathfrak{m} \in \Omega(B_n)$. This gives that $f_n \in B_n$.

At this point we use the homomorphisms $\beta_n : B_n \to A/I^{n-c}$ (7.1.13.3). The element

$$\hat{f} = \varprojlim \beta_n(f_n) \in \varprojlim A/I^{n-c} = A$$

lies in A and in this way we get an inverse to the homomorphism $A \to \Gamma(X, \mathcal{O}^0_X)$. Finally, we have to show that if $\hat{f} = 0$ then $0 = f \in \Gamma(X, \mathcal{O}^0_X)$. This is so since

$$\mathrm{Ker}(\beta_n) . B_{n_0} \subset I^{n-n_0} B_{n_0}$$

if $n, n_0 > c$. This follows easily from the explicit description of the rings B_n in 7.1.13. □

7.4.2. *Remark.* — The most general hypothesis on $\mathfrak{X}$ under which 7.4.1 is true are the following: $\mathfrak{X}$ should be $\mathcal{O}$-flat and for any open affine formal subscheme Spf(A) of $\mathfrak{X}$ the ring A should be integrally closed in the ring $A \otimes_{\mathcal{O}} K$. This fact will not be used in this paper.

7.5. *Rigid descent of closed formal subschemes*

7.5.1. Suppose $\mathfrak{X}$ is an object of $FS_{\mathcal{O}}$. Let $T \subset \mathfrak{X}_{\text{red}}$ be a closed subscheme and let $U = \mathfrak{X}_{\text{red}} \setminus T$. We define

$$\mathfrak{T} = \mathfrak{X}^{\wedge}_{T} = \text{completion of } \mathfrak{X} \text{ along } T,$$

$$\mathfrak{U} = \text{open formal subscheme of } \mathfrak{X} \text{ with } \mathfrak{U}_{\text{red}} = U,$$

$$X = \mathfrak{X}^{\text{rig}}.$$

We remark that $\mathfrak{T}^{\text{rig}} \cong sp^{-1}(T)$ is an open subvariety of X by Lemma 7.2.5. Similarly, $\mathfrak{U}^{\text{rig}} \cong sp^{-1}(U)$ is an open subvariety of X by 7.2.1. In fact X is the disjoint union of $sp^{-1}(T)$ and $sp^{-1}(U)$ but the covering $X = sp^{-1}(T) \cup sp^{-1}(U)$ is not admissible in general.

Let us consider triples $(\mathfrak{Z}_T, \mathfrak{Z}_U, Z)$ of the following kind

(7.5.1.1) $\mathfrak{Z}_T$ (resp. $\mathfrak{Z}_U$) is an $\mathcal{O}$-flat closed formal subscheme of $\mathfrak{T}$ (resp. $\mathfrak{U}$) and $Z \subset X$ is a closed analytic subvariety of X

satisfying the following condition

(7.5.1.2) We have the equality $(\mathfrak{Z}_T)^{\text{rig}} = Z \cap sp^{-1}(T)$ (resp. $(\mathfrak{Z}_U)^{\text{rig}} = Z \cap sp^{-1}(U)$) as closed analytic subvarieties of $\mathfrak{T}^{\text{rig}} \cong sp^{-1}(T)$ (resp. $\mathfrak{U}^{\text{rig}} \cong sp^{-1}(U)$).

For each $\mathcal{O}$-flat closed formal subscheme $\mathfrak{Z} \subset \mathfrak{X}$ we get a triple $(\mathfrak{Z} \cap \mathfrak{T}, \mathfrak{Z} \cap \mathfrak{U}, \mathfrak{Z}^{\text{rig}})$ as in (7.5.1.1) satisfying (7.5.1.2). The proposition below shows that the converse is true also. The idea is that in some sense the "covering" of $\mathfrak{X}$ given by

$$sp^{-1}(\mathrm{T}) \sqcup sp^{-1}(\mathrm{U}) \rightrightarrows \mathrm{X} \sqcup \mathfrak{U} \sqcup \mathfrak{T} \to \mathfrak{X}$$

is sufficiently good to allow descent for closed subvarieties.

7.5.2. *Proposition.* — *Any triple* (7.5.1.1) $(\mathfrak{Z}_{\mathrm{T}}, \mathfrak{Z}_{\mathrm{U}}, \mathrm{Z})$ *satisfying* (7.5.1.2) *comes from an* $\mathcal{O}$*-flat closed formal subscheme* $\mathfrak{Z} \subset \mathfrak{X}$ *as described above.*

Proof. — We may assume that $\mathfrak{X} = \mathrm{Spf}(\mathrm{A})$ is affine. We may assume that A has no π-torsion. Let $\mathrm{I} \subset \mathrm{A}$ be the biggest ideal of definition of A and suppose that $\mathrm{T} \subset \mathrm{Spec}(\mathrm{A}/\mathrm{I})$ is given by the ideal $\mathrm{I} + g_1 \mathrm{A} + \ldots + g_r \mathrm{A}$ of A. (We assume that $g_i \notin \mathrm{I}$ for all i.) We put

$$\mathrm{A}^\wedge := \varprojlim \mathrm{A}/(g_1 \mathrm{A} + \ldots + g_r \mathrm{A})^n \cong \varprojlim \mathrm{A}/(\mathrm{I} + g_1 \mathrm{A} + \ldots + g_r \mathrm{A})^n$$

and for $i = 1, \ldots, r$:

$$\mathrm{A}\{1/g_i\} := \varprojlim \mathrm{A}/\mathrm{I}^n[1/g_i].$$

These are flat A-algebras. We note that we may assume T is reduced so that $\mathrm{I} + g_1 \mathrm{A}^\wedge + \ldots + g_r \mathrm{A}^\wedge$ is the biggest ideal of definition of $\mathrm{A}^\wedge$. We have $\mathfrak{T} = \mathrm{Spf}(\mathrm{A}^\wedge)$ and $\mathfrak{U} = \bigcup \mathrm{Spf}(\mathrm{A}\{1/g_i\})$, so that $\mathfrak{Z}_{\mathrm{T}}$ and $\mathfrak{Z}_{\mathrm{U}}$ are given by ideals

$$\mathrm{J}^\wedge \subset \mathrm{A}^\wedge \quad \text{and} \quad \mathrm{J}_i \subset \mathrm{A}\{1/g_i\}.$$

These ideals are such that $\mathrm{A}^\wedge/\mathrm{J}^\wedge$ and $\mathrm{A}\{1/g_i\}/\mathrm{J}_i$ are π-torsion free (and of course $\mathrm{J}_i \mathrm{A}\{1/g_i g_j\} = \mathrm{J}_j \mathrm{A}\{1/g_i g_j\}$).

In 7.1.3 we defined X as the increasing union $\mathrm{X} = \bigcup_n \mathrm{V}_n$ of affinoid varieties V_n:

$$\mathrm{V}_n := \mathrm{Sp}(\mathrm{C}_n) = \mathrm{Sp}(\mathrm{B}_n \otimes_{\mathcal{O}} \mathrm{K}) = \mathrm{Spf}(\mathrm{B}_n)^{\text{rig}}.$$

The closed subvariety $\mathrm{Z} \cap \mathrm{V}_n$ is defined by an ideal $\mathrm{I}_n \subset \mathrm{B}_n$; we define I_n as the kernel of the homomorphism

$$\mathrm{B}_n \to \Gamma(\mathrm{Z} \cap \mathrm{V}_n, \mathcal{O}_{\mathrm{Z}}).$$

Thus it is clear that $\mathrm{B}_n/\mathrm{I}_n$ is π-torsion free.

Recall that in 7.1.13 we defined surjective ring homomorphisms $\beta_n : B_n \to A/I^{n-c}$. We will prove that the ideals $\beta_n(I_n) \subset A/I^{n-c}$ satisfy the following two statements (at least after replacing c by a sufficiently large constant):

1) $\beta_n(I_n)\, A^\wedge/(IA^\wedge)^{n-c} = J^\wedge \bmod (IA^\wedge)^{n-c}$,
2) $\beta_n(I_n)\, A\{1/g_i\}/(IA\{1/g_i\})^{n-c} = J_i \bmod (IA\{1/g_i\})^{n-c}$.

If we have this then we are done: 1) and 2) imply that $\beta_{n+1}(I_{n+1}) \bmod I^{n-c} = \beta_n(I_n)$, i.e., the maps $\beta_{n+1}(I_{n+1}) \to \beta_n(I_n)$ are surjective. Thus the limit

$$J := \varprojlim \beta_n(I_n) \subset \varprojlim A/I^{n-c} = A$$

is an ideal of A such that $JA^\wedge = J^\wedge$ and $JA\{1/g_i\} = J_i$. Hence we can take $\mathfrak{Z} = \mathrm{Spf}(A/J)$.

To prove 1) and 2) we put

$$B_n^\wedge := \varprojlim_{\ell} B_n/(g_1 B_n + \ldots + g_r B_n)^\ell \cong B_n \hat{\otimes}_A A^\wedge$$

and
$$B_n\{1/g_i\} := \varprojlim_{\ell} B_n/\pi^\ell[1/g_i] \cong B_n \hat{\otimes}_A A\{1/g_i\}.$$

It is clear that β_n induces homomorphisms

$$\beta_n^\wedge : B_n^\wedge \to A^\wedge/(IA^\wedge)^{n-c}$$

and
$$\beta_n\{1/g_i\} : B_n\{1/g_i\} \to A\{1/g_i\}/(IA\{1/g_i\})^{n-c}.$$

Furthermore it is easily seen that

$$\beta_n(I_n)\,.\,A^\wedge/(IA^\wedge)^{n-c} = \beta_n^\wedge(I_n B_n^\wedge)$$

and that

$$\beta_n(I_n)\,.\,A\{1/g_i\}/(IA\{1/g_i\})^{n-c} = \beta_n\{1/g_i\}(I_n B_n\{1/g_i\}).$$

By our choice of I_n we get the equality

$$\mathrm{Spf}(B_n/I_n)^{\mathrm{rig}} = V_n \cap Z$$

thus we also have (use 7.2.4 part *g)* and 7.2.5)

$$\mathrm{Spf}(B_n^\wedge/I_n B_n^\wedge)^{\mathrm{rig}} = V_n \cap sp^{-1}(T) \cap Z = V_n \cap \mathrm{Spf}(A^\wedge)^{\mathrm{rig}} \cap Z$$

and (use 7.2.1)

$$\mathrm{Spf}(B_n\{1/g_i\}/I_n B_n\{1/g_i\})^{\mathrm{rig}} = V_n \cap \{|g_i| \geqslant 1\} \cap Z$$
$$= V_n \cap \mathrm{Spf}(A\{1/g_i\})^{\mathrm{rig}} \cap Z.$$

At this point we are able to prove the inclusions $\supset$ of 1) and 2). For 1), take $h \in J^\wedge$. The image of h in $B_n^\wedge$ lies in $I_n B_n^\wedge$: it is zero as a function on $V_n \cap sp^{-1}(T) \cap Z$ and $B_n^\wedge / I_n B_n^\wedge \cong (B_n/I_n)^\wedge$ is π-torsion free. Thus $\beta_n(h) \equiv h$ lies in $\beta_n^\wedge(I_n B_n^\wedge)$. For 2), take $h \in J_i$. The image of h in $B_n\{1/g_i\}$ lies in $I_n B_n\{1/g_i\}$: it is zero as a function on $V_n \cap \mathrm{Spf}(A\{1/g_i\})^{\mathrm{rig}} \cap Z$ and $B_n\{1/g_i\}/I_n B_n\{1/g_i\} \cong (B_n/I_n)\{1/g_i\}$ is π-torsion free. Thus $\beta_n(h) \equiv h$ lies in $\beta_n\{1/g_i\}(I_n B_n\{1/g_i\})$.

To prove the other inclusion, we need that the construction of 7.1.13, which associates to an $\mathcal{O}$-algebra A the system $(B_n, \beta_n)_{n \geqslant 1}$, is a functor. That is, if $A \to A'$ is a homomorphism of such algebras then there exists a $c \in \mathbf{N}$ and commutative diagrams:

$$\begin{array}{ccc} B_n = B_n(A) & \longrightarrow & B_n(A') \\ \downarrow \beta_n & & \downarrow \beta_n' \\ A/I^{n-c} & \longrightarrow & A'/(I')^{n-c}. \end{array}$$

This is easy to see using the explicit constructions in 7.1.1 and 7.1.13. Thus we have a commutative diagram (for some $c \in \mathbf{N}$ independent of n):

$$\begin{array}{ccccc} B_n\{1/g_i\} & \cong & B_n(A\{1/g_i\}) & \longrightarrow & B_n(A\{1/g_i\}/J_i) \\ \downarrow \beta_n\{1/g_i\} & & \downarrow \beta_n & & \downarrow \beta_n \\ (A/I^{n-c})\{1/g_i\} & \cong & A\{1/g_i\}/(IA\{1/g_i\})^{n-c} & \longrightarrow & A\{1/g_i\}/(J_i + (IA\{1/g_i\})^{n-c}). \end{array}$$

Also it is clear that $I_n B_n\{1/g_i\}$ lies in the kernel of the upper horizontal arrow of this diagram; an element of I_n gives the zero function of $\mathrm{Spf}(B_n(A\{1/g_i\}/J_i))^{\mathrm{rig}} \subset Z \cap V_n$. Thus we see that the inclusion $\subset$ holds in 2) for some constant c independent of n.

It is more tricky to prove the inclusion $\subset$ in 1). To do it we note that we have the following equalities of open subvarieties of V_n:

$$\begin{aligned} \mathrm{Spf}(B_n^\wedge)^{\mathrm{rig}} &= \{ x \in V_n \mid |g_i(x)| < 1 \} \\ &= V_n \cap \mathrm{Spf}(A^\wedge)^{\mathrm{rig}} \\ &= \bigcup\nolimits_N (V_n \cap \mathrm{Spf}(B_N(A^\wedge))^{\mathrm{rig}}) \\ &= \bigcup\nolimits_N (\mathrm{Spf}(B_n)^{\mathrm{rig}} \cap \mathrm{Spf}(B_N(A^\wedge))^{\mathrm{rig}}) \subset \bigcup\nolimits_N \mathrm{Spf}(B_N)^{\mathrm{rig}} \end{aligned}$$

Thus we see that $\mathrm{Spf}(B_n^\wedge)^{\mathrm{rig}}$ may be written as the union of the rigid spaces associated to the $\mathcal{O}$-algebras

$$B_n \hat{\otimes}_{B_N} B_N(A^\wedge).$$

The inclusions are given by the homomorphisms

$$B_n^\wedge \cong B_n \hat{\otimes}_A A^\wedge \to B_n \hat{\otimes}_{B_N} B_N(A^\wedge).$$

These fit into the following commutative diagram (for some c independent of n, N):

$$\begin{array}{ccc}
B_n^\wedge & \xrightarrow{\quad \beta_n^\wedge \quad} & A^\wedge/(IA^\wedge)^{n-c} \\
\downarrow \cong & & \downarrow \cong \\
B_n \hat{\otimes}_A A^\wedge & \xrightarrow{\quad \beta_n \otimes \mathrm{id}_{A^\wedge} \quad} & (A/I^{n-c}) \otimes_A A^\wedge \\
\downarrow & & \downarrow \\
B_n \hat{\otimes}_{B_N} B_N(A^\wedge) & \longrightarrow & A/I^{n-c} \otimes_{A/I^{N-c}} A^\wedge/(IA^\wedge + g_1 A^\wedge + \ldots + g_r A^\wedge)^{N-c} \\
\downarrow & & \downarrow \\
B_n \hat{\otimes}_{B_N} B_N(A^\wedge/J^\wedge) & \longrightarrow & A^\wedge/J^\wedge + I^{n-c} + (IA^\wedge + g_1 A^\wedge + \ldots + g_r A^\wedge)^{N-c}.
\end{array}$$

In the same way as before we see that $I_n B_n^\wedge$ maps to zero under the left vertical arrows. Hence we see that

$$B_n^\wedge(I_n B_n^\wedge) \subset J^\wedge + I^{n-c} + (IA^\wedge + g_1 A^\wedge + \ldots + g_r A^\wedge)^{N-c}$$

for all $N > c$. This proves the inclusion $\subset$ in 1). The proof of 7.5.2 is complete. □

REFERENCES

[Bart] W. Bartenwerfer, Einige Fortsetzungssätze in der p-adischen Analysis, *Math. Ann.*, **185** (1970), 191-210.

[B] P. Berthelot, *Cohomologie cristalline des schémas de charactéristique $p > 0$*, Springer Lecture Notes, **407**, 1974.

[B2] P. Berthelot, *Cohomologie rigide et cohomologie rigide à support propre*, Version privisoire du 9-08-1991.

[B3] P. Berthelot, Géométrie rigide et cohomologie des variétés algébriques de caractéristique p, *Bull. Soc. Math. Fr., Mém.*, **23** (1986), 7-32.

[BBM] P. Berthelot, L. Breen, W. Messing, *Théorie de Dieudonné cristalline II*, Springer Lecture Notes, **930**, 1982.

[BM] P. Berthelot, W. Messing, Théorie de Dieudonné cristalline III, in *The Grothendieck Festschrift I*, Progress in Mathematics, **86**, Birkhäuser (1990), 171-247.

[BGR] S. Bosch, U. Güntzer, R. Remmert, *Non-archimedian analysis*, Grundlehren, **261**, Springer Verlag, 1984.

[BL] S. Bosch, W. Lütkebohmert, Formal and rigid geometry, *Math. Ann.*, **295** (1993), 291-317.

[G1] A. Grothendieck, *Groupes de Barsotti-Tate et cristaux de Dieudonné*, Séminaire de mathématiques supérieures, Département de mathématiques, Université de Montréal, Les Presses de l'Université de Montréal, 1974.

[G2] A. Grothendieck, Groupes de Barsotti-Tate et cristaux de Dieudonné, *Actes, Congrès international des mathématiciens*, 1970, t. 1, p. 431-436.

[EGA] A. Grothendieck, J. Dieudonné, *Éléments de géométrie algébrique I, II, III, IV*, Publ. Math. I.H.E.S., **4, 8, 11, 17, 20, 24, 28, 32** (1961-1967).

[Hart] R. Hartshorne, Complete intersections and connectedness, *Amer. J. Math.*, **84** (1962), 495-508.

[Ill] L. Illusie, *Complexe cotangent et déformations I, II*, Springer Lecture Notes, **239** (1971), **283** (1972).

[Ill2] L. Illusie, Déformations de groupes de Barsotti-Tate, d'après A. Grothendieck, in *Séminaire sur les pinceaux arithmétiques : la conjecture de Mordell*, Astérisque, **127** (1985), 151-198.

[Ill3] L. Illusie, Complexe de de Rham-Witt et cohomologie cristalline, *Ann. Scient. Éc. Norm. Sup.*, **12** (1979), 501-661.

[dJ] A. J. de Jong, Finite locally free group schemes in characteristic p and Dieudonné modules, *Invent. Math.*, **114** (1993), 89-137.

[JP] A. J. de Jong, M. van der Put, *Étale cohomology of rigid analytic spaces*, Preprint of the University of Groningen, W-9506.

[J] J.-P. JOUANOLOU, *Théorèmes de Bertini et applications*, Progress in Mathematics **42,** Birkhäuser, 1983.

[K] N. KATZ, Nilpotent connections and the monodromy theorem: applications of a result of Turittin, *Publ. Math. I.H.E.S.*, **35** (1971), 175-232.

[L] J. LIPMAN, Desingularization of two-dimensional schemes, *Ann. of Math.*, **107** (1978), 151-207.

[Lü] W. LÜTKEBOHMERT, Der Satz von Remmert-Stein in der nichtarchimedischen Funktionentheorie, *Math. Z.*, **139** (1974), 69-84.

[Lü2] W. LÜTKEBOHMERT, Fortsetzbarkeit k-meromorpher Funktionen, *Math. Ann.*, **220** (1976), 273-284.

[Mat] H. MATSUMURA, *Commutative algebra*, Mathematics lecture note series, **56,** The Benjamin/Cummings Publishing Company, Inc., 1980.

[MM] B. MAZUR, W. MESSING, *Universal extensions and one-dimensional crystalline cohomology*, Springer Lecture Notes, **370** (1974).

[M] W. MESSING, *The crystals associated to Barsotti-Tate groups: with applications to abelian schemes*, Springer Lecture Notes, **264** (1972).

[R] M. RAYNAUD, Géométrie analytique rigide d'après Tate, Kiehl, *Table ronde d'analyse non archimédienne*, Bull. Soc. Math. Fr. Mém., **39/40** (1974), 319-327.

Department of Mathematics
Harvard University
One Oxford Street
Cambridge, MA 02138-2901
U.S.A.

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