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MODULI OF REPRESENTATIONS OF THE FUNDAMENTAL GROUP OF A SMOOTH PROJECTIVE VARIETY I

by Carlos T. SIMPSON

Introduction

The space of representations of the fundamental group is a natural topological invariant of a manifold. If a manifold has the additional structure of a smooth projective variety over the complex numbers, then its topological invariants can be expected to have additional structures which reflect the algebraic or analytic structure of X. The goal of Part II of this paper will be to describe the extra structures on the space of representations. The present Part I contains the main constructions needed for this description, presented in greater generality so they can be of further use.

The moduli space of representations is not a Hausdorff space, but there is a natural modification to make. The semisimplification of a representation is the direct sum of its irreducible subquotients with appropriate multiplicities. Say that two representations are *Jordan equivalent* if their semisimplifications coincide. With this equivalence relation, the moduli space becomes Hausdorff. Henceforth we will use the term *moduli space* to mean the space obtained by dividing out by this equivalence relation.

The moduli space of representations of a finitely generated group is naturally an affine algebraic variety, defined by the generators and relations. This is the incarnation we will call the *Betti* moduli space, $\mathbf{M}_B$. If the group is the fundamental group of a smooth projective variety, we will construct two other algebraic varieties with the same underlying topological moduli space. These we call the *Dolbeault* moduli space $\mathbf{M}_{Dol}$ and the *de Rham* moduli space $\mathbf{M}_{DR}$. There are natural homeomorphisms between the topological spaces underlying these three moduli space $\mathbf{M}_B$, $\mathbf{M}_{Dol}$, and $\mathbf{M}_{DR}$. These homeomorphisms do not preserve the algebraic structures, although the identification between the Betti and de Rham realizations will be complex analytic. These other algebraic structures are the additional structures obtained by virtue of the algebraic structure of the manifold.

The Dolbeault moduli space parametrizes certain holomorphic objects on X: a *Higgs bundle* [Hi1] [Si5] is a holomorphic vector bundle E together with a holomorphic map $\theta : E \to E \otimes \Omega^1_X$ such that $\theta \wedge \theta = 0$. There is a condition of semistability analogous to that for vector bundles, and $\mathbf{M}_{\mathrm{Dol}}$ is the moduli space for semistable Higgs bundles with vanishing rational Chern classes. There is a correspondence between Higgs bundles and local systems [Hi1], [Do3], [Co], [Si2], [Si5], which gives a homeomorphism between $\mathbf{M}_{\mathrm{Dol}}$ and $\mathbf{M}_{\mathrm{B}}$.

A *vector bundle with integrable connection* on X is a locally free sheaf E together with an integrable connection, in other words a map of sheaves of C vector spaces

$$\nabla : \mathrm{E} \to \mathrm{E} \otimes \Omega^1_{\mathrm{X}}$$

satisfying Leibniz's rule $\nabla(ae) = d(a)\, e + a \nabla(e)$ and the integrability condition $\nabla^2 = 0$. The de Rham space $\mathbf{M}_{\mathrm{DR}}$ is the moduli space for vector bundles with integrable connection. These correspond to representations of the fundamental group, by looking at the monodromy representation of the system of differential equations $\nabla(e) = 0$. This correspondence is complex analytic in nature, but not algebraic, and as a result, it gives a complex analytic isomorphism $\mathbf{M}_{\mathrm{DR}} \cong \mathbf{M}_{\mathrm{B}}$ which is not algebraic.

The topics mentioned so far are all treated in Part II. The present paper, Part I, treats techniques of construction of moduli spaces for coherent sheaves, vector bundles with integrable connection, Higgs bundles, and other similar objects on smooth projective schemes over complex base schemes. We use Mumford's geometric invariant theory to construct moduli spaces for sheaves provided with an action of a ring of differential operators, a general situation which covers all of these examples. This technical work in Part I is the main part of the constructions of $\mathbf{M}_{\mathrm{Dol}}$ and $\mathbf{M}_{\mathrm{DR}}$. At the end of this first paper, we discuss some analytic results which provide about half of the argument giving the homeomorphisms between the three moduli spaces (the other half being given in § 7, Part II). I have tried to keep the connections between the two parts of the paper under control, using in the second part mostly results rather than techniques from the first part. Thus, although the chapters are numbered globally—§§ 1-5 form Part I and §§ 6-11 form Part II—it should be possible to read the two parts separately.

The first version of this paper was written in the summer of 1988, and preprint versions were distributed during subsequent years. The present version has undergone a substantial revision and expansion, and I have corrected several mistakes which some people had pointed out in the earlier versions. I apologize for the fact that the organization and numbering of the paper are different from those of the preprint version.

Here are some details about the contents of Part I. In the first section, we construct the moduli space of coherent sheaves on a projective variety using Mumford's " geometric invariant theory " [Mu]. This gives the section a dual purpose: it serves to introduce the techniques of geometric invariant theory which will be used later on; and it gives a construction for coherent sheaves which implies the projectivity of the moduli space, a property which does not hold for the more general moduli spaces considered afterward.

The bulk of part I (§§ 2-4) is devoted to the construction of the moduli spaces of a general class of objects like vector bundles with integrable connections. A vector bundle with integrable connection may be considered as a module over the sheaf of rings $\mathscr{D}_{\mathrm{X}}$

of all (algebraic) differential operators on a smooth complex projective variety X. My original construction of the moduli spaces for vector bundles with integrable connection used this interpretation, and various properties of the ring $\mathscr{D}_X$. The construction presented here is a generalization, where $\mathscr{D}_X$ is replaced by any sheaf of rings Λ with properties analogous to those of $\mathscr{D}_X$ (2.1.1-2.1.6). We call Λ a *sheaf of rings of differential operators* on X, since the axioms force Λ to be generated by elements acting as derivations on the coordinate ring. The list of properties 2.1.1-2.1.6 was arrived at by looking at which properties were needed in the construction (although there is no guarantee that it is minimal). Part of the data that goes with Λ is a filtration compatible with the ring structure. The associated-graded $\mathrm{Gr}(\Lambda)$ is required to have the same left and right $\mathcal{O}_X$-module structures. The associated-graded ring in the case $\Lambda^{\mathrm{DR}} = \mathscr{D}_X$ is just the symmetric algebra on the tangent sheaf, $\mathrm{Gr}(\mathscr{D}_X) = \mathrm{Sym}^{\bullet}(TX)$. This ring is itself a sheaf of rings of differential operators (in this case it is generated by a collection of trivial derivations, so the differential operators are actually of order zero). A module over $\Lambda^{\mathrm{Dol}} = \mathrm{Sym}^{\bullet}(TX)$ is a Higgs sheaf [Hi1] [Si5]. Thus, in making our construction in this general form, we construct at once the moduli spaces of vector bundles with integrable connection, and of Higgs sheaves. There are several other examples, such as vector bundles with connections along a foliation, and vector bundles with integrable connection on a degenerating family of varieties, discussed briefly in § 2.

After discussing the axioms and some basic properties for sheaves of rings of differential operators in § 2, we define in § 3 the notions of p-semistability, μ-semistability, p-stability, and μ-stability for Λ-modules. These generalize the corresponding notions for Higgs bundles [Si5] (whereas, for vector bundles with integrable connection, semistability is automatic and stability is equivalent to irreducibility). In turn, these notions for Higgs bundles were generalizations of the corresponding notions for vector bundles or torsion-free sheaves [Mu] [Gi] [Ma1]. Then we use Hilbert schemes to construct a parameter space Q for p-semistable Λ-modules with a given Hilbert polynomial, with an action of group Sl(V).

In § 4, we use geometric invariant theory to construct a " good " quotient (in the terminology of [Gi])—in particular a universal categorial quotient of Q by the action of Sl(V). This is the *moduli space* $\mathbf{M}(\Lambda, P)$ for p-semistable Λ-modules with Hilbert polynomial P. In case $\Lambda = \mathcal{O}_X$, we recover the moduli space for coherent sheaves constructed in § 1. In both constructions, that of § 1 and that of § 4, we use Grothendieck's embedding of this Hilbert schemes into Grassmanians, hence into projective space, to obtain the linearized invertible sheaf used in applying geometric invariant theory. This is a departure from the methods which had been used by Gieseker [Gi] and Maruyama [Ma1] to construct moduli spaces for torsion-free sheaves. This departure is necessary, in § 1 because there is no good notion of exterior power of a coherent sheaf, and in § 4 because we have to include the additional structure of the action of Λ—for which we look at the Hilbert scheme of quotients of $\Lambda_r \otimes_{\mathcal{O}_X} \mathcal{O}_X(-N)^{P(N)}$ for an appropriately high level Λ_r in the filtration of Λ.

Suppose $\mathscr{W}$ is a coherent sheaf, and let $\mathbf{Hilb}(\mathscr{W}, \mathrm{P})$ denote the Hilbert scheme of quotients

$$\mathscr{W} \to \mathscr{E} \to 0$$

where $\mathscr{E}$ has Hilbert polynomial P. There is an m such that for any point of $\mathbf{Hilb}(\mathscr{W}, \mathrm{P})$, the morphism

$$\mathrm{H}^0(\mathrm{X}, \mathscr{W}(m)) \to \mathrm{H}^0(\mathrm{X}, \mathscr{E}(m))$$

is surjective, and the resulting map from the Hilbert scheme into the Grassmanian of quotients of $\mathrm{H}^0(\mathrm{X}, \mathscr{W}(m))$ is an embedding. This is Grothendieck's embedding [Gr2]. We consider geometric invariant theory for the action of $\mathrm{Sl}(\mathrm{V})$ on $\mathbf{Hilb}(\mathrm{V} \otimes \mathscr{W}, \mathrm{P})$ (where V is a finite dimensional complex vector space), using Grothendieck's embedding. This is the method used for the constructions of § 1 (with $\mathscr{W} = \mathcal{O}_{\mathrm{X}}(-\mathrm{N})$) and § 4 (with $\mathscr{W} = \Lambda_r \otimes_{\mathcal{O}_{\mathrm{X}}} \mathcal{O}_{\mathrm{X}}(-\mathrm{N})$).

In order to make the constructions more useful, we treat the relative case where S is a base scheme of finite type over $\mathrm{Spec}(\mathbf{C})$, and $\mathrm{X} \to \mathrm{S}$ is a projective morphism. The notion of sheaf of rings of differential operators works in the relative case, and our construction gives a relative moduli space $\mathbf{M}(\Lambda, \mathrm{P}) \to \mathrm{S}$. Even in the fibers, this is only a coarse moduli space, so it does not represent a functor. The categorical notion which corresponds to the universal categorical quotient constructed by geometric invariant theory, is that of *universally co-representing* a functor (see § 1, after Lemma 1.9). This property characterizes the relative moduli space. The construction is compatible with changing the base S. In particular, the fiber of $\mathbf{M}(\Lambda, \mathrm{P})$ over $s \in \mathrm{S}$ is the moduli space for Λ-modules on X_s.

The description of the points of this coarse moduli space is the same as that given by Gieseker and Maruyama for the case of torsion-free sheaves: a p-semistable Λ-module $\mathscr{E}$ has a *semisimplification* $\mathrm{gr}(\mathscr{E})$, the direct sum of the factors in its Jordan-Hölder series. Two objects are *Jordan equivalent* if $\mathrm{gr}(\mathscr{E}_1) = \mathrm{gr}(\mathscr{E}_2)$. The points of $\mathbf{M}(\Lambda, \mathrm{P})$ correspond to Jordan equivalence classes of p-semistable Λ-modules with Hilbert polynomial $\mathbf{P}$ on fibers X_s.

In order to obtain fine moduli spaces, we need to add some type of rigidification. We discuss a way to do this for Λ-modules whose Jordan-Hölder factors are locally free near a section $\xi : \mathrm{S} \to \mathrm{X}$ (we call this condition $\mathrm{LF}(\xi)$; it is slightly stronger than the requirement of being locally free along ξ), at the end of § 4. Assuming that the fibers X_s are irreducible, we obtain a *representation space* $\mathbf{R}(\Lambda, \xi, \mathrm{P})$ classifying pairs $(\mathscr{E}, \beta)$ where $\mathscr{E}$ is a p-semistable Λ-module with Hilbert polynomial P, satisfying condition $\mathrm{LF}(\xi)$, and β is a frame for $\xi^*(\mathscr{E})$. The representation space represents the appropriate functor. A group $\mathrm{Gl}(n, \mathbf{C})$ acts by change of frame, and the open subset of the moduli space $\mathbf{M}^{\mathrm{LF}(\xi)}(\Lambda, \mathrm{P})$ where condition $\mathrm{LF}(\xi)$ holds, is the good quotient of $\mathbf{R}(\Lambda, \xi, \mathrm{P})$ by the action of $\mathrm{Gl}(n, \mathbf{C})$.

In § 5, we treat some analytic questions connected with the construction. These results are to be used in part **II**. The first subject is to show that the complex analytic space $\mathbf{M}^{an}(\Lambda, P)$ is a coarse moduli space in the complex analytic category. For this, we have to discuss the corresponding type of statement for Hilbert schemes and for the parameter spaces Q for p-semistable Λ-modules; then we have to show that the complex analytic space associated to a good quotient is a universal categorical quotient in the analytic category (Proposition 5.5, which may be of independent interest).

The second subject of § 5 is the topology of $\mathbf{R}^{an}(\Lambda, \xi, P)$. We restrict the discussion to the case where Λ is a *split almost polynomial sheaf of rings of differential operators* (see the end of § 2), which nevertheless includes all of our examples, and we also assume that $X \to S$ is smooth. We obtain a criterion for convergence of a sequence of points in $\mathbf{R}^{an}(\Lambda, \xi, P)$, intrinsically stated in terms of the objects $(\mathcal{E}_i, \beta_i)$ represented by the points in the sequence. This discussion is tailored to the needs of part II, where we will apply Uhlenbeck's weak compactness theorem to obtain some continuity statements. We need to know that, under the conditions provided by Uhlenbeck's theorem, the corresponding sequences of points converge in the spaces we have constructed by algebraic geometry. The discussion is complicated by the desire to treat the relative case, where X itself need not be smooth since the base might not be smooth.

A brief description of part II

Since much of the motivation for the constructions in Part I lies in the applications in Part II, whereas Part I has to appear first for logical reasons, it seems advisable to include a brief description of the results of Part II in this introduction. This will be elaborated in the second introduction.

This second part is devoted to the moduli spaces of representations of $\pi_1(X^{top})$. The first task is to recall the classical construction of the *Betti moduli space* $\mathbf{M}_B(X, n)$, a coarse moduli space for rank n representations of the fundamental group of X^{top}. Then we apply the results of Part I to construct the *de Rham moduli space* $\mathbf{M}_{DR}(X, n)$, a coarse moduli space for rank n vector bundles with integrable connection on X, and the *Dolbeault moduli space* $\mathbf{M}_{Dol}(X, n)$, a coarse moduli space for rank n semistable Higgs bundles with Chern classes vanishing in rational cohomology. The construction of $\mathbf{M}_{DR}$ is obtained by setting Λ^{DR} to be the full sheaf of rings of differential operators $\mathcal{D}_X$. The construction of $\mathbf{M}_{Dol}$ can be obtained by thinking of a Higgs bundle as a coherent sheaf on the cotangent bundle and using the results of § 1, or by thinking of a Higgs bundle as a module over $\Lambda^{Dol} = \mathrm{Sym}^{\bullet}(TX)$ and using the results of § 4.

The three types of objects are related to each other. The Riemann-Hilbert correspondence which gives an isomorphism of associated complex analytic spaces

$$\mathbf{M}_B^{an}(X, n) \cong \mathbf{M}_{DR}^{an}(X, n).$$

The correspondence between Higgs bundles and local systems gives a homeomorphism of the underlying usual topological spaces

$$\mathbf{M}_B^{top}(X, n) \cong \mathbf{M}_{DR}^{top}(X, n) \cong \mathbf{M}_{Dol}^{top}(X, n).$$

To verify the properties of these maps, we rely on the analytic results of § 5, Part I. The terminologies Betti, de Rham and Dolbeault come from analogies between the moduli spaces, and the Betti cohomology, the algebraic de Rham cohomology, and the Dolbeault cohomology $\bigoplus_{p+q=i} H^q(X, \Omega_X^p)$ of an algebraic variety.

The moduli spaces also exist in the relative case, where X is smooth and projective over S (this is why we insist on that throughout Part I). We use the crystalline site to construct a *Gauss-Manin connection* on the relative moduli space $\mathbf{M}_{DR}(X/S, n)$ over S, a foliation transverse to the fibers corresponding to the trivialization given by the topological interpretation in terms of $\mathbf{M}_B$.

We treat the case of principal objects with linear algebraic structure group G, obtaining the Betti moduli space $\mathbf{M}_B(X, G)$, the de Rham moduli space $\mathbf{M}_{DR}(X, G)$ for principal G-bundles with integrable connection, and the Dolbeault moduli space $\mathbf{M}_{Dol}(X, G)$ for principal Higgs bundles which are semistable with vanishing rational Chern classes.

Parallel to the discussion of moduli spaces, we discuss the Betti, de Rham and Dolbeault *representation spaces* $\mathbf{R}_B(X, x, G)$, $\mathbf{R}_{DR}(X, x, G)$, and $\mathbf{R}_{Dol}(X, x, G)$. These are fine moduli spaces for objects provided with a frame for the fiber over the base point $x \in X$, constructed using the construction of the representation spaces at the end of § 4.

We discuss the local structure of the singularities of the representation spaces and moduli spaces, using the deformation theory associated to a differential graded Lie algebra developed by Goldman and Millson [GM], and a formality result gives an *iso-singularity principle*: that the singularities of the de Rham and Dolbeault moduli spaces are formally isomorphic at corresponding points.

At the end, we prove that the space of representations of the fundamental group of a Riemann surface of genus $g \geqslant 2$ is an irreducible normal variety.

Origins and acknowledgements

I would like to thank the many people whose interest and comments have contributed to the present work, and acknowledge the sources of many indispensable ideas.

The main purpose of the very first version of this paper was to construct the moduli space of Higgs bundles. These objects were introduced by N. Hitchin, and he gave an analytic construction of the moduli space for rank two Higgs bundles over a Riemann surface [Hi1]. He obtained all of the relevant properties, and went on to use this construction to calculate the Betti numbers of the space of representations, and some coherent sheaf cohomology groups of the moduli space of rank two vector bundles [Hi2]. Discussions with K. Corlette, whose work provides a key ingredient in the analytic picture [Co], K. Uhlenbeck, W. Goldman and others made clear the importance of Hitchin's work, and all of this provided the impetus to do the work presented here. Subsequently, discussions with Hitchin were very helpful, and he and Oxbury pointed out that N. Nitsure had given an algebraic construction for any rank over a Riemann surface, following the ideas of Mumford [Ni1].

Our construction of the moduli space rests on techniques developed by D. Mumford for constructing moduli spaces in algebraic geometry [Mu]. More specifically, the constructions presented here were done after studying the constructions by D. Gieseker [Gi] (for surfaces) and M. Maruyama [Ma1] [Ma2] (for higher dimensions) of moduli spaces for torsion-free sheaves. The boundedness results of Maruyama are sufficiently general that they can be applied more or less directly to new situations such as the one here (when one first looks at his papers, it is hard to appreciate the fact that he did everything as generally as possible, but when one needs a boundedness result, that turns out to be handy!). Seshadri was the first to employ the fact that the quotient of the semistable points of a projective variety is again projective, to obtain a compactification of the moduli space of stable vector bundles over a curve [Se].

One day A. Yukie, a student of Mumford, tried to explain to me how the construction of the moduli space of vector bundles would work. As he wasn't familiar with the details of Gieseker's construction, he guessed what should be the projective embedding. As best as I can remember, he explained Grothendieck's embedding. So I think that the idea to use Grothendieck's embedding for constructing moduli spaces comes from that discussion at the blackboard.

Preliminary versions of this paper contained separate constructions of the moduli space of Higgs bundles (using moduli spaces of coherent sheaves), and of the moduli space of vector bundles with integrable connection using the $\mathscr{D}_X$-module interpretation. One of the motivations for doing the construction in the more general context presented in the current version was a suggestion by P. Deligne [De2], to consider a moduli space for λ-*connections* on $X \times \mathbf{A}^1$ over $\mathbf{A}^1$, providing a deformation from the moduli space of connections ($\lambda = 1$) to the space of Higgs bundles ($\lambda = 0$). This example was discussed in [Si3]. While the moduli space of λ-connections could be constructed by an *ad hoc* adaptation of the original constructions for Higgs bundles and vector bundles with integrable connection, it was this situation which provided the impetus to realize the general construction finally presented here. More generally, Deligne has made numerous helpful suggestions in the course of the work on Higgs bundles and local systems.

The construction as presented below, as well as the part of the original preprint concerning vector bundles with integrable connection, were deeply influenced by the course on $\mathscr{D}_X$-modules given by J. Bernstein at Harvard in 1983-1984 [Be]. The formalization of the properties of the ring Λ is influenced by Bernstein's presentation of the properties of the ring $\mathscr{D}_X$. The theory of $\mathscr{D}_X$-modules is by now well known, and the list of people whose work contributed to this theory is long. I will not try to give references here, leaving the place in the reference list to Bernstein's course. The reader is invited to substitute any of a number of original papers and presentations of this theory which are available.

After preliminary versions of this paper were circulated, Nitsure constructed the moduli space of vector bundles with logarithmic connection (i.e. a connection with poles of order one along a divisor) [Ni2], refering to my construction for vector bundles with smooth connection. His construction appeared shortly after I started to consider rewriting the construction in its present generality, so he evidently had started thinking about logarithmic connections first (and it was because of his paper that I realized the example of Λ corresponding to logarithmic connections). I apologize for the unfortunately Escherian situation caused by the substantial nature of the revisions undergone by this paper since the earlier preprint versions, wherein Nitsure's space of logarithmic connections appears as an example in the final version of a preprint in his reference list. The moduli of logarithmic connections requires several additions to the construction of the moduli of smooth connections presented in the early versions of this paper, in particular the sheaves which occur are no longer automatically locally free, and the condition of p-semistability is no longer automatic. In so far as the construction presented below has these added elements incorporated, these parts must be attributed to Nitsure.

The preprint versions of this paper were full of mistakes. I would like to thank very much those who pointed them out, and in particular M. Maruyama, K. Yokogawa, N. Nitsure, and J. Le Potier. I hope that there are no old, and not too many new mistakes in the present version.

1. Moduli of coherent sheaves

This section has two purposes: to introduce the techniques of geometric invariant theory we will need later, and to construct moduli spaces of coherent sheaves. Some of the material at the end of the section is not necessary for the moduli of sheaves, but is included for later use.

First we discuss semistability and boundedness for coherent sheaves. Then we discuss various aspects of geometric invariant theory which will be used below. We give our two main lemmas about semistability in Hilbert schemes, which are applications of Mumford's criterion for semistability in Grassmanians. We use these to construct the moduli space of coherent sheaves. At the end of the section we discuss some slice theorems, then investigate the closed orbit adhering to a given orbit, giving an explicit description of limits of $\mathbf{G}_m$-orbits in Hilbert schemes, and finally give a lemma about local freeness.

All schemes will be, by convention, separated and of finite type over $\mathrm{Spec}(\mathbf{C})$. We generally denote by S a base scheme, and sometimes may tacitly assume that it is connected.

Semistability

Let X be a projective scheme over $S = \mathrm{Spec}(\mathbf{C})$ with very ample invertible sheaf $\mathcal{O}_X(1)$. For any coherent sheaf $\mathcal{E}$ on X, there is a polynomial in n with rational coefficients $p(\mathcal{E}, n)$ called the *Hilbert polynomial* of $\mathcal{E}$. It is defined by the condition that $p(\mathcal{E}, n) = \dim \mathrm{H}^0(\mathrm{X}, \mathcal{E}(n))$ for $n \gg 0$. Let $d = d(\mathcal{E})$ denote the dimension of the support of $\mathscr{E}$. It is equal to the degree of the Hilbert polynomial. The coefficient of the leading term is $r/d!$ where $r = r(\mathscr{E})$ is an integer which we will call the *rank* of $\mathscr{E}$. Denote the coefficient of the next term by $a(\mathscr{E})/(d-1)!$. Thus

$$p(\mathscr{E}, n) = rn^d/d! + an^{d-1}/(d-1)! + \dots$$

Let $\mu(\mathscr{E})$ (the *slope* of $\mathscr{E}$) denote the quotient a/r. We will call the quotient p/r the *normalized Hilbert polynomial* of $\mathscr{E}$.

With our conventions, $\mu(\mathscr{E} \otimes \mathcal{O}_X(k)) = \mu(\mathscr{E}) + k$. If $\mathscr{E} \neq 0$ then $p(\mathscr{E}, n) > 0$ for $n \gg 0$.

We make the following definitions. A coherent sheaf $\mathscr{E}$ is of *pure dimension* $d = d(\mathscr{E})$ if for any non-zero subsheaf $\mathscr{F} \subset \mathscr{E}$, we have $d(\mathscr{F}) = d(\mathscr{E})$. A coherent sheaf $\mathscr{E}$ is *p-semistable* (resp. *p-stable*) if it is of pure dimension, and if for any subsheaf $\mathscr{F} \subset \mathscr{E}$, there exists an N such that

$$\frac{p(\mathscr{F}, n)}{r(\mathscr{F})} \leqslant \frac{p(\mathscr{E}, n)}{r(\mathscr{E})}$$

(resp. $<$) for $n \geqslant N$. A coherent sheaf $\mathscr{E}$ is *μ-semistable* (resp. *μ-stable*) if it is of pure dimension d and if for any subsheaf $\mathscr{F} \subset \mathscr{E}$, we have $\mu(\mathscr{F}) \leqslant \mu(\mathscr{E})$ (resp. $<$). Note that p-semistability implies μ-semistability, whereas μ-stability implies p-stability.

If $f: X \hookrightarrow \mathbf{P}^n$ is a projective embedding with $f^*(\mathcal{O}_{\mathbf{P}^n}(1)) \cong \mathcal{O}_X(1)$, then a coherent sheaf $\mathscr{E}$ on X may be considered as a sheaf $f_* \mathscr{E}$ on $\mathbf{P}^n$. The Hilbert polynomials are the same, and the conditions of pure dimension are the same. All of the above notions are the same for the sheaf $\mathscr{E}$ on X and the sheaf $f_* \mathscr{E}$ on $\mathbf{P}^n$. Thus it suffices to discuss sheaves on $\mathbf{P}^n$ (although we give some statements in terms of sheaves on X).

Here are some elementary properties, which have the same proofs as for vector bundles. Any sheaf $\mathscr{E}$ of pure dimension d has a unique filtration

$$0 = \mathscr{E}_0 \subset \mathscr{E}_1 \subset \dots \subset \mathscr{E}_k = \mathscr{E}$$

such that the quotients $\mathscr{E}_i/\mathscr{E}_{i-1}$ are p-semistable of pure dimension d and such that the normalized Hilbert polynomials $p(\mathscr{E}_i/\mathscr{E}_{i-1})/r(\mathscr{E}_i/\mathscr{E}_{i-1})$ are strictly decreasing for large n. This filtration is called the *Harder-Narasimhan filtration*. The construction is given in a more general context in Lemma 3.1 below. There is a similar filtration for μ-semistability, although the two filtrations may not be the same. If $\mathscr{E}$ is a p-semistable sheaf of pure dimension d then there is a filtration

$$0 = \mathscr{E}_0 \subset \mathscr{E}_1 \subset \dots \subset \mathscr{E}_k = \mathscr{E}$$

such that the quotients $\mathscr{E}_i/\mathscr{E}_{i-1}$ are p-stable of pure dimension d, with the same normalized Hilbert polynomials. This filtration is not unique, but if we set $\operatorname{gr}(\mathscr{E}) = \bigoplus \mathscr{E}_i/\mathscr{E}_{i-1}$ then $\operatorname{gr}(\mathscr{E})$ is unique [Gi].

The category of p-semistable sheaves of pure dimension d with normalized Hilbert polynomial p_0 is abelian. If $\mathscr{E}$ is p-stable, then the only endomorphisms are scalars. For if f is a nonzero endomorphism, then $\ker(f)$ and $\operatorname{coker}(f)$ are zero by the p-stability of $\mathscr{E}$; thus any such f is an automorphism, so the ring of endomorphisms of $\mathscr{E}$ is a division algebra. But the only division algebra finite over $\mathbf{C}$ is $\mathbf{C}$ itself.

Boundedness

Theorem **1.1.** — *Fix* P, $d = \deg(\mathrm{P})$, *and* b. *The set of sheaves* $\mathscr{E}$ *on* X *with Hilbert polynomial* P, *such that* $\mathscr{E}$ *has pure dimension* d, *and for any subsheaf* $\mathscr{F} \subset \mathscr{E}$, $\mu(\mathscr{F}) \leqslant b$, *is bounded. In particular, the set of* μ*-semistable sheaves with Hilbert polynomial* P *is bounded.*

The proof uses the results of Maruyama on boundedness for torsion-free sheaves. We will also state a lemma that Maruyama uses, because we will need it later.

If F is a family of sheaves of pure dimension d on $\mathbf{P}^n$, and if $0 \leqslant k < d$, say that F is *k-bounded* if there exists a family F′ of sheaves on $\mathbf{P}^{n-k}$ such that F′ is bounded, and such that for every $\mathscr{E}$ contained in F, there is an open set of $(n-k)$-planes $\mathbf{P}^{n-k} \subset \mathbf{P}^n$ such that $\mathscr{E}|_{\mathbf{P}^{n-k}}$ is contained in F′. We can now state Maruyama's lemma.

Lemma **1.2.** — *For any* b, *the family* F(b) *of torsion-free sheaves* $\mathscr{E}$ *on* $\mathbf{P}^n$ *with* $r(\mathscr{E}) = r$ *fixed and* $0 \leqslant \mu(\mathscr{E}) \leqslant 1$ *such that for every subsheaf* $\mathscr{F} \subset \mathscr{E}$, *we have* $\mu(\mathscr{F}) \leqslant b$, *is* $(n-1)$*-bounded.*

Proof. — [Ma2].

From this lemma, Maruyama deduces boundedness:

Proposition **1.3.** — *Let* P *be a polynomial, and let* b *be an integer. The set of torsion-free sheaves* $\mathscr{E}$ *on* $\mathbf{P}^n$ *with* $p(\mathscr{E}, m) = \mathrm{P}(m)$ *and such that for every subsheaf* $\mathscr{F} \subset \mathscr{E}$, $\mu(\mathscr{F}) \leqslant b$, *is bounded.*

Proof. — [Ma2].

Proof of Theorem 1.1. — Fix a polynomial π, let d be its degree, and r be the corresponding rank. By choosing a projective embedding of X, sheaves on X may be considered as sheaves on $\mathbf{P}^n$. The slopes are the same on X or $\mathbf{P}^n$, so we may assume for the rest of the proof that $\mathrm{X} = \mathbf{P}^n$. Suppose $\mathscr{E}$ is a semistable coherent sheaf on $\mathbf{P}^n$, of pure dimension d, with Hilbert polynomial P, and such that for any subsheaf $\mathscr{F}$ we have $\mu(\mathscr{F}) \leqslant b$. In particular $\mathscr{E}$ is of pure dimension d. Let Y be the scheme-theoretic support of $\mathscr{E}$, in other words the subscheme of $\mathbf{P}^n$ defined by the ideal $\operatorname{Ann}(\mathscr{E})$. We can find a $\mathbf{P}^{n-d-1}$ which doesn't meet Y, such that every $\mathbf{P}^{n-d}$ containing it meets Y in a finite set. Let $\pi : \mathbf{P}^n - \mathbf{P}^{n-d-1} \to \mathbf{P}^d$ be the projection. Then $\pi_* \mathscr{E}$ is a coherent sheaf on $\mathbf{P}^d$. It has Hilbert polynomial P since $\mathcal{O}_{\mathrm{Y}}(1) = \pi^* \mathcal{O}_{\mathbf{P}^d}(1)$. Note that $\pi_* \mathscr{E}$ is torsion-free: otherwise there would be a subsheaf $\mathscr{F} \subset \mathscr{E}$ such that $\pi_*(\mathscr{F})$ is supported in dimension $< d$, but since $\pi : \mathrm{Y} \to \mathbf{P}^d$ is finite, $\mathscr{F}$ would then be supported in dimension $< d$, contradicting the pure dimension of $\mathscr{E}$.

In order to apply Maruyama's lemma to show that the $\pi_*(\mathscr{E})$ form a bounded family on $\mathbf{P}^d$, we have to show that there is an integer b' such that for every subsheaf $\mathscr{F} \subset \pi_*(\mathscr{E})$, $\mu(\mathscr{F}) \leqslant b'$. Consider $\pi_* \mathcal{O}_{\mathrm{Y}}$ as a sheaf of algebras on $\mathbf{P}^d$.

We claim that there is an integer m which depends only on n and r, such that $\pi_* \mathcal{O}_Y(m)$ is generated by global sections outside codimension 2. To see this, note first that the set of artinian subschemes of $\mathbf{P}^{n-d}$ of length $\leqslant r^{n-d}$ is bounded. Since $\mathcal{E}$ has pure dimension d, Y has no embedded points, so the projection $Y \to \mathbf{P}^d$ is flat in codimension 1. Thus there is an open subset $U \subset \mathbf{P}^d$ whose complement has codimension 2 such that for $u \in U$ the intersection of Y with the corresponding $\mathbf{P}^{n-d}$ has length $\leqslant r^{n-d}$. We obtain a k depending on n and r such that for any point $y \in U$, the intersection of Y with the corresponding $\mathbf{P}^{n-d}$ is generated by the sections of $\mathcal{O}(k)$ on $\mathbf{P}^{n-d}$. These sections fit together to form a bundle $\mathcal{W}$ on $\mathbf{P}^d$, and we have a morphism $\mathcal{W} \to \pi_* \mathcal{O}_Y(k)$ which is surjective over U. If we choose m so that $\mathcal{W}(m-k)$ is generated by global sections, then $\mathcal{O}_Y(m)$ is generated by global sections over U as claimed.

Suppose $\mathcal{F} \subset \pi_* \mathcal{E}$ is a subsheaf. We want to show that $\mu(\mathcal{F}) \leqslant b'$, so we may assume that $\mathcal{F}$ is μ-semistable (by replacing it with the first subsheaf in its Harder-Narasimhan filtration). We get a map $\mathcal{F} \otimes_{\mathcal{O}_{\mathbf{P}^d}} \pi_* \mathcal{O}_Y \to \pi_* \mathcal{E}$, and the image $\mathcal{G}$ is a subsheaf of $\pi_* \mathcal{O}_Y$-modules, hence by assumption, $\mu(\mathcal{G}) \leqslant b$. On the other hand, since the global sections generate $\pi_* \mathcal{O}_Y(m)$ over U, there is N and a morphism

$$\mathcal{F}^N \to \mathcal{G}(m)$$

whose image is a subsheaf $\mathcal{H}(m)$, equal to $\mathcal{G}(m)$ over U. The slope of $\mathcal{H}$ is the same as that of $\mathcal{G}$. Semistability of $\mathcal{F}$ implies that

$$\mu(\mathcal{F}) \leqslant \mu(\mathcal{G}) + m \leqslant b + m.$$

Set $b' = b + m$. Now we may apply Maruyama's boundedness result to conclude that the $\pi_* \mathcal{E}$ form a bounded family.

To complete the proof, note that there is a bundle $\mathcal{V}$ on $\mathbf{P}^d$ such that the structure of $\mathcal{O}_Y$-module of $\mathcal{E}$ is determined by the map

$$\mathcal{V} \otimes_{\mathcal{O}_{\mathbf{P}^d}} \pi_* \mathcal{E} \to \mathcal{E}.$$

For example we may assume that the fiber of $\mathcal{V}$ is the $n - d + 1$ dimensional space of global sections of $\mathcal{O}(1)$ on $\mathbf{P}^{n-d}$. The family of such maps is bounded, so this proves boundedness of the family of $\mathcal{E}$. $\square$

This proof also provides the generalization of Lemma 1.2 to sheaves:

Lemma **1.4.** — *The set of sheaves $\mathcal{E}$ on $\mathbf{P}^n$ of pure dimension d with $r(\mathcal{E}) = r$ and $0 \leqslant \mu(\mathcal{E}) \leqslant 1$, such that for any subsheaf $\mathcal{F} \subset \mathcal{E}$ we have $\mu(\mathcal{F}) \leqslant b$, is $(d-1)$-bounded.*

Proof. — Choose a generic projection π as before. The above argument shows that $\pi_* \mathcal{E}$ satisfies the hypothesis of Lemma 1.2, so for generic $\mathbf{P}^1 \subset \mathbf{P}^d$, $\pi_* \mathcal{E}|_{\mathbf{P}^1}$ is contained in a bounded family. But the restriction of $\pi_* \mathcal{E}$ to $\mathbf{P}^1$ is the same as the projection of the restriction of $\mathcal{E}$ to a generic $\mathbf{P}^{n-d+1}$. The last part of the above argument shows that if $\pi_*(\mathscr{E}|_{\mathbf{P}^{n-d+1}})$ is contained in a bounded family, then $\mathscr{E}|_{\mathbf{P}^{n-d+1}}$ is contained in a bounded family. □

We now give a lemma about k-bounded families, to be used in conjunction with Lemma 1.4. Use the notation $h^i(X, \mathscr{F})$ for the dimension of $H^i(X, \mathscr{F})$.

Lemma **1.5.** — *Suppose* F *is a k-bounded family of sheaves of pure dimension d and rank r on* $\mathbf{P}^n$, *with* $k \leqslant d - 1$. *Then there is an integer* B *such that for all* $\mathscr{E}$ *in* F *and all m, we have*

$$h^0(\mathbf{P}^n, \mathscr{E}(m)) \leqslant \begin{cases} 0 & \text{if } m + B \leqslant 0 \\ r(m+B)^d/d! & \text{if } m + B \geqslant 0. \end{cases}$$

Proof. — The proof is by induction on k. It is trivial if $k = 0$ because a 0-bounded family is bounded. Suppose F is a k-bounded family of sheaves on $\mathbf{P}^n$, so by definition we have a $(k-1)$-bounded family F′ of sheaves on $\mathbf{P}^{n-1}$. If $\mathscr{E}$ is in F then there is a generic hyperplane $Y \cong \mathbf{P}^{n-1}$ such that $\mathscr{E}|_Y$ is contained in F′. We have exact sequences

$$0 \to \mathscr{E}(m-1) \to \mathscr{E}(m) \to \mathscr{E}|_Y(m) \to 0.$$

Let B′ be the constant given for the family F′. From these exact sequences we get $h^0(\mathbf{P}^n, \mathscr{E}(m)) - h^0(\mathbf{P}^n, \mathscr{E}(m-1)) = 0$ for $m \leqslant -B'$, so in fact $h^0(\mathbf{P}^n, \mathscr{E}(m)) = 0$ for $m \leqslant -B'$. Furthermore,

$$h^0(\mathbf{P}^n, \mathscr{E}(m)) - h^0(\mathbf{P}^n, \mathscr{E}(m-1)) \leqslant r(m+B')^{d-1}/(d-1)!$$

for $m \geqslant -B'$. In general there is a constant C depending on d such that if $f(m)$ is a function satisfying $f(0) = 0$ and

$$f(m) - f(m-1) \leqslant m^{d-1}/(d-1)!$$

for $m \geqslant 0$, then $f(m) \leqslant (m+C)^d/d!$. We may take $B = B' + C$. □

The relative case

Suppose that S is a scheme of finite type over Spec($\mathbf{C}$). Suppose $X \to S$ is a projective scheme over S, with very ample $\mathcal{O}_X(1)$. For every closed point $s \in S$, the fiber X_s is a projective scheme over Spec($\mathbf{C}$) with very ample $\mathcal{O}_{X_s}(1)$.

We make the following notational convention about base changes. Suppose $X \to S$ and $S' \to S$ are schemes over S (usually, X is the family we are considering and S′ is a new base scheme). Then denote the fiber product by $X' = X \times_S S'$. Do the same for a double prime.

We extend the definition of semistability to the relative case by imposing flatness and then considering the fibers X_s separately. Fix a polynomial P of degree d. A *p-semistable sheaf* $\mathscr{E}$ *on* X/S *with Hilbert polynomial* P is a coherent sheaf $\mathscr{E}$ on X, flat over S, such that for each closed point $s \in S$, $\mathscr{E}_s$ is a p-semistable sheaf of pure dimension d and Hilbert polynomial P on the fiber X_s.

Corollary **1.6.** — *The set of possible sheaves* $\mathscr{E}_s$ *which are semistable of pure dimension* d *and Hilbert polynomial* P *on fibers* X_s, *collected over all* $s \in S$, *is bounded.*

Proof. — Embed $X \subset \mathbf{P}^n_S$ using $\mathcal{O}_X(1)$. Then semistability on a fiber X_s is the same as semistability on $\mathbf{P}^n$. The set of semistable sheaves on $\mathbf{P}^n$ with Hilbert polynomial P is bounded, and the set of subschemes $X_s \subset \mathbf{P}^n$ which occur is bounded. □

Corollary **1.7.** — *There is an integer* B *depending only on* r *and* d, *such that if* $X \to S$ *is a projective morphism with relatively very ample* $\mathcal{O}_X(1)$ *and if* $\mathscr{E}$ *is a* μ*-semistable sheaf of pure dimension* d *and rank* r *on a fiber* X_s, *then*

$$h^0(X_s, \mathscr{E}(k)) \leqslant \begin{cases} 0 & \text{if } \mu(\mathscr{E}) + k + B \leqslant 0 \\ r(\mu(\mathscr{E}) + k + B)^d/d! & \text{if } \mu(\mathscr{E}) + k + B \geqslant 0 \end{cases}$$

for any k.

Proof. — Combine Lemmas 1.4 and 1.5 to obtain the statement for sheaves on $\mathbf{P}^n$. Embedding $X \subset \mathbf{P}^n \times S$, μ-semistable sheaves on X_s correspond to μ-semistable sheaves on $\mathbf{P}^n$. □

A useful fact in connection with questions of boundedness is the following result of Grothendieck. It provides a finiteness statement for the set of Hilbert polynomials of certain saturated subsheaves of a given sheaf.

Proposition **1.8.** — *Suppose* $\mathscr{F}$ *is a given sheaf on* X, *flat over* S. *Suppose that the fibers* $\mathscr{F}_s$ *over closed points have pure dimension* d. *Fix a number* b. *The family of saturated subsheaves* $\mathscr{G}_s \subset \mathscr{F}_s$ *in fibers over geometric points* $s \to S$, *such that* $\mu(\mathscr{G}_s) \geqslant b$, *is bounded.*

Proof. — This follows from Grothendieck's statement (which is for the quotient sheaves $\mathscr{F}_s/\mathscr{G}_s$), Lemma 2.6 of [Gr2]. □

The notation $H^i(X/S, \mathscr{F})$

Use the following notation for direct images and higher direct images. Suppose $f: X \to S$ is a proper morphism, and $\mathscr{F}$ is a coherent sheaf on X. Then we will denote by $H^i(X/S, \mathscr{F})$ the coherent sheaf $R^i f_*(\mathscr{F})$ on S.

Lemma **1.9.** — *Given any family of sheaves* $\mathscr{F}_s$ *on fibers* X_s *which, collected over all fibers, is bounded, then there is an* M *which works uniformly for the following property. For any* $S' \to S$ *and any sheaf* $\mathscr{F}$ *on* $X' = X \times_S S'$ *which is flat over* S', *such that all of the fibers* $\mathscr{F}_s$ *over closed points* $s \in S'$ *appear in our bounded family, and for any* $m \geqslant M$, $H^i(X'/S', \mathscr{F}(m)) = 0$ *for* $i \geqslant 0$, *and* $H^0(X'/S', \mathscr{F}(m))$ *is locally free and commutes with base change. This means that if* $f: S'' \to S'$, *then* $f^* H^0(X'/S', \mathscr{F}(m)) \cong H^0(X''/S'', f^* \mathscr{F}(m))$.

Proof. — These results, due to Grothendieck [Gr1], are conveniently collected in Chapter 0 of [Mu]. □

There is a similar result for $\mathscr{F}$ which are not flat, giving the vanishing and compatibility with base change. However, M will depend on $\mathscr{F}$ and the $H^0(X'/S', \mathscr{F}(m))$ will not necessarily be locally free.

Geometric invariant theory

We begin by describing a notion similar to but weaker than the notion of representing a functor. This property will characterize our moduli spaces. Suppose $Y^\natural$ is a functor from the category of S-schemes to the category of sets. Suppose Y is a scheme over S, and $\varphi : Y^\natural(S') \to Y(S')$ is a natural transformation of functors. We say that Y *corepresents* the functor $Y^\natural$ if, for every S-scheme W and natural transformation of functors $\psi : Y^\natural \to W$, there is a unique morphism of schemes $f : Y \to W$ giving a factorization $\psi = f \circ \varphi$. Suppose $V \to Y$ is a morphism of schemes. Define the fiber product of functors

$$V \times_Y Y^\natural(S') \stackrel{\text{def}}{=} V(S') \times_{Y(S')} Y^\natural(S').$$

We say that Y *universally corepresents the functor* $Y^\natural$ if, for every morphism of schemes $V \to Y$, V corepresents the functor $V \times_Y Y^\natural$.

The property that Y corepresents $Y^\natural$ serves to characterize Y uniquely up to unique isomorphism. The property of universally corepresenting a functor is more flexible, for example it allows us to define group actions.

A morphism of functors $g : Y_1^\natural \to Y_2^\natural$ is a *local isomorphism* if it induces an isomorphism of sheafifications in the étale topology. More concretely, this means that if S' is an S-scheme, and $u, v \in Y_1^\natural(S')$ such that $g(u) = g(v)$, then there is a surjective étale morphism $S'' \to S'$ such that $u|_{S''} = v|_{S''}$; and if $w \in Y_2^\natural(S')$ then there are a surjective étale morphism $S'' \to S'$ and $u \in Y_1^\natural(S'')$ such that $g(u) = w|_{S''}$. Suppose $g : Y_1^\natural \to Y_2^\natural$ is a local isomorphism. If Y is a scheme, and $h_1 : Y_1^\natural \to Y$ is a morphism, then there is a unique morphism $h_2 : Y_2^\natural \to Y$ such that $h_2 g = h_1$. The condition that (Y, h_1) corepresents $Y_1^\natural$ is equivalent to the condition that (Y, h_2) corepresents $Y_2^\natural$. If $Z \to Y$ is a morphism of schemes, then the fiber product morphism $g \times_Y Z$ is also a local isomorphism. Hence the condition that (Y, h_1) universally corepresents $Y_1^\natural$ is equivalent to the condition that (Y, h_2) universally corepresents $Y_2^\natural$.

For the first part of our discussion of geometric invariant theory, we will treat the case that the base scheme is $S = \operatorname{Spec}(\mathbf{C})$. Suppose G is a reductive algebraic group, and suppose Z is a scheme on which G acts. Define the quotient functor of schemes S' over $\operatorname{Spec}(\mathbf{C})$,

$$Y^\natural(S') \stackrel{\text{def}}{=} Z(S')/G(S').$$

If Y is a scheme with a morphism $\varphi : Z \to Y$ invariant under the action of G, this induces a morphism $Y^\natural \to Y$. The scheme Y is a *categorical quotient* if it corepresents the quotient functor $Y^\natural$, and a *universal categorical quotient* if it universally corepresents the quotient functor. These definitions are equivalent to those of [Mu].

Some terminology which has become common after [Mu] and [Gi] is the following. The morphism $\varphi : Z \to Y$ is a *good quotient* if it is a universal categorical quotient, the map φ is affine (i.e. the inverse image of an affine open set is affine), and the quotient Y is quasiprojective. The first two conditions imply that if $U = \mathrm{Spec}(A) \subset Y$ then $\varphi^{-1}(U) = \mathrm{Spec}(B)$ where B is an A-algebra with action of G, and $A = B^G$ is the subring of invariants. Here is a well-known property which characterizes the points of a good quotient.

Lemma **1.10.** — *Suppose $\varphi : Z \to Y$ is a good quotient. If V_1 and V_2 are two disjoint G-invariant closed sets in Z, then the images $\varphi(V_1)$ and $\varphi(V_2)$ are disjoint. The closed points $y \in Y$ are in one to one correspondence with the closed orbits in Z. If z is a closed point in Z, its image is the point y corresponding to the (unique) closed orbit in the closure of the orbit of Z.*

Proof. — This is implicit in [Mu], and was pointed out explicitly by Seshadri [Se] [Se2]. □

Suppose that a reductive group G acts on a scheme Z. Suppose $\mathscr{L}$ is an invertible sheaf (line bundle) on Z, with action of G. (In other words, there is an isomorphism between the two pullbacks of $\mathscr{L}$ to $G \times Z$, and the three pullbacks of this isomorphism to $G \times G \times Z$ form a commutative triangle.) Mumford makes the following definitions [Mu]. A point $z \in Z$ is *semistable* if there exists n and a G-invariant section $f \in H^0(Z, \mathscr{L}^{\otimes n})^G$ such that $f(z) \neq 0$, and $Z_{f\neq 0} \stackrel{\mathrm{def}}{=} \{ x : f(x) \neq 0 \}$ is affine. A point is *properly stable* if there exists f as above, such that the orbits of G in $Z_{f\neq 0}$ are closed there, and the stabilizer of z is finite. The open subsets of properly stable and semistable points are denoted $Z^s \subset Z^{ss} \subset Z$.

Proposition **1.11.** — *With the same notations, there exists a good quotient $\varphi : Z^{ss} \to Y$, and an open set $Y^s \subset Y$ such that the inverse image of Y^s is Z^s, and the quotient $Y^s = Z^s/G$ is a universal geometric quotient. There is an ample invertible sheaf $\mathscr{L}_Y$ on Y with $\varphi^* \mathscr{L}_Y = \mathscr{L}$. If Z is projective and $\mathscr{L}$ is ample, then Y is projective.*

Proof. — Theorem 1.10 of [Mu]. Mumford mentions the last statement in a subsequent remark; for a proof, see [Se]. □

Conversely, if $\varphi : Z \to Y$ is a good quotient, we can choose an ample line bundle $\mathscr{L}_Y$ on Y and set $\mathscr{L} = \varphi^*(\mathscr{L}_Y)$. This is an invertible sheaf with action of G, with respect to which all points of Z are semistable. The quotient Y is that given by the proposition. This is Converse 1.12 of [Mu].

In Chapter 2 of [Mu], Mumford presents a numerical criterion for stability or semistability in a projective scheme. This can be described briefly as saying that if Z is projective, then a point z is semistable for the action of G if and only if it is semistable for the action of every one-parameter subgroup $G_m \subset G$. Semistability for the action of a one-parameter subgroup is then analyzed in terms of the weights of the action on coordinates of the point.

One important abstract component of this analysis is the following statement.

Proposition **1.12.** — *Suppose* V *and* Z *are schemes on which* G *acts, with* V *projective. Suppose* $\mathscr{L}$ *is a very ample invertible sheaf on* Z. *Suppose* $f: V \to Z$ *is a* G*-invariant closed embedding. Then* $V^{ss} = f^{-1}(Z^{ss})$ *and* $V^{s} = f^{-1}(Z^{s})$, *where semistability and proper stability are measured with respect to* $\mathscr{L}$ *and* $f^*(\mathscr{L})$.

Proof. — This is Theorem 1.19 of [Mu]. □

The relative case

Suppose $Z \to S$ is a scheme over a base. Suppose that a reductive algebraic group G acts on Z, acts trivially on S, and preserves the morphism $Z \to S$. If we can take a categorical quotient of Z by the action of G, the quotient will map to S by its universal mapping property. Our only task is to compare the notions of semistability in Z and in the fibers over geometric points.

Lemma **1.13.** — *Suppose* $Z \to S$ *is projective, and* $\mathscr{L}$ *is a relatively very ample invertible sheaf with action of* G. *If* $t \to S$ *is a geometric point, then the semistable points of the fiber are those which are semistable in the total space,* $(Z_t)^{ss} = (Z^{ss})_t$; *and the same holds for the properly stable points.*

Proof. — This was proved in [Se] (it follows easily from Proposition 1.12 stated above). □

Grassmanians

Let V and W be vector spaces, and let $\mathbf{Grass}(V \otimes W, a)$ denote the Grassmanian of quotients of dimension a of the vector space $V \otimes W$. The group $Sl(V)$ acts on $\mathbf{Grass}(V \otimes W, a)$. There is a canonical $Sl(V)$-invariant projective embedding of $\mathbf{Grass}(V \otimes W, a)$, given by a very ample invertible sheaf $\mathscr{L}$ which can be described as follows. Over a point represented by a quotient

$$V \otimes W \to B \to 0,$$

the fiber of $\mathscr{L}$ is the line $\wedge^a B$.

Proposition **1.14.** — *A point* $V \otimes W \to U \to 0$ *in* $\mathbf{Grass}(V \otimes W, a)$ *is semistable (resp. properly stable) for the action of* $Sl(V)$ *and invertible sheaf* $\mathscr{L}$ *if and only if, for all nonzero proper subspaces* $H \subset V$, *we have* $\operatorname{Im}(H \otimes W) \neq 0$ *and*

$$\frac{\dim(H)}{\dim(\operatorname{Im}(H \otimes W))} \leqslant \frac{\dim(V)}{\dim(U)}$$

(resp. $<$*). Hcre* $\operatorname{Im}(H \otimes W)$ *denotes the image in the quotient* U.

Proof. — [Mu], Proposition 4.3. □

Suppose $\mathscr{W}$ is a coherent sheaf on S. There is a projective scheme $\mathbf{Grass}(\mathscr{W}, a)$ over S representing the functor which to each $f: S' \to S$ associates the set of quotients

$f^* \mathscr{W} \to \mathscr{F} \to 0$ with $\mathscr{F}$ locally free of rank a on S'. If $\mathscr{W}_1$ is locally free and $\mathscr{W}_1 \to \mathscr{W} \to 0$ is a surjection, then $\mathbf{Grass}(\mathscr{W}, a)$ may be constructed as a closed subscheme of $\mathbf{Grass}(\mathscr{W}_1, a)$. The fiber of $\mathbf{Grass}(\mathscr{W}, a)$ over $s \in \mathrm{S}$ is $\mathbf{Grass}(\mathscr{W}_s, a)$. There is a relatively ample invertible sheaf $\mathscr{L}$ restricting to the one described above on the fibers.

Suppose $\mathscr{W}$ is a coherent sheaf on S and V is a finite-dimensional complex vector space. Then we obtain a projective scheme $\mathbf{Grass}(\mathrm{V} \otimes \mathscr{W}, a)$ over S, again with action of Sl(V) and very ample linearized invertible sheaf $\mathscr{L}$. According to Lemma 1.13, a point lying over $s \in \mathrm{S}$ is semistable if and only if it satisfies the criterion of Proposition 1.14 in its fiber $\mathbf{Grass}(\mathrm{V} \otimes \mathscr{W}_s, a)$.

Hilbert schemes

Mumford's criterion for semistability in Grassmanians can be generalized to a criterion for semistability in Hilbert schemes. To do this, we use the embeddings of the Hilbert scheme in Grassmanians which were given by Grothendieck in the course of his construction [Gr2].

Suppose X is a projective scheme over S. Fix a relatively very ample invertible sheaf $\mathcal{O}_{\mathrm{X}}(1)$.

Fix a polynomial $\mathrm{P}(n)$ with these properties. Suppose $\mathscr{W}$ is a coherent sheaf on X. Grothendieck constructs the *Hilbert scheme* $\mathbf{Hilb}(\mathscr{W}, \mathrm{P})$ parametrizing quotients

$$\mathscr{W} \to \mathscr{F} \to 0$$

with Hilbert polynomial P [Gr2]. More precisely, the Hilbert scheme represents a functor as follows. For any connected scheme $\sigma : \mathrm{S}' \to \mathrm{S}$, the S'-valued points of $\mathbf{Hilb}(\mathscr{W}, \mathrm{P})$ are the isomorphism classes of quotients on $\mathrm{X} \times_{\mathrm{S}} \mathrm{S}'$,

$$\sigma^*(\mathscr{W}) \to \mathscr{F} \to 0$$

where $\mathscr{F}$ is flat over S' and $p(\mathscr{F}, n) = \mathrm{P}(n)$. The fiber of $\mathbf{Hilb}(\mathscr{W}, \mathrm{P})$ over a closed point $s \in \mathrm{S}$ is $\mathbf{Hilb}(\mathscr{W}_s, \mathrm{P})$.

Grothendieck proved that $\mathbf{Hilb}(\mathscr{W}, \mathrm{P})$ is projective over S, and his construction gives some explicit projective embeddings. There is an M such that for any $m \geqslant \mathrm{M}$ we get an embedding

$$\psi_m : \mathbf{Hilb}(\mathscr{W}, \mathrm{P}) \to \mathbf{Grass}(\mathrm{H}^0(\mathrm{X}/\mathrm{S}, \mathscr{W}(m)), \mathrm{P}(m))$$

as follows. We may choose M so that $\mathrm{H}^0(\mathrm{X}/\mathrm{S}, \mathscr{W}(m))$ is compatible with base change $\mathrm{S}' \to \mathrm{S}$. The set of quotients represented by points of the Hilbert scheme is bounded and the quotient sheaves are flat, so we may choose M such that for any quotient $\mathscr{W} \to \mathscr{F}$ on $\mathrm{X}' = \mathrm{X} \times_{\mathrm{S}} \mathrm{S}'$ represented by an S'-valued point of the Hilbert scheme, $\mathrm{H}^0(\mathrm{X}'/\mathrm{S}', \mathscr{F}(m))$ is locally free of rank $\mathrm{P}(m)$. Furthermore we may assume that for any such quotient, if we let $\mathscr{G}$ be the kernel, we have $\mathrm{H}^1(\mathrm{X}'/\mathrm{S}', \mathscr{G}(m)) = 0$ (to see this, reduce to the case where $\mathscr{W}$ is flat over S by taking a projective embedding of X and a surjection to $\mathscr{W}$; in this case $\mathscr{G}$ is flat and we can apply Lemma 1.9). Thus we get a surjection of locally free sheaves on S′

$$H^0(X'/S', \mathscr{W}(m)) \to H^0(X'/S', \mathscr{F}(m)) \to 0,$$

in other words an S′-valued point of $\mathbf{Grass}(H^0(X/S, \mathscr{W}(m)), P(m))$. This functor corresponds to a morphism which we denote ψ_m from the Hilbert scheme to the Grassmanian. Grothendieck shows that, after increasing M some more, the ψ_m are closed embeddings [Gr2]. Let $\mathscr{L}_m$ denote the very ample invertible sheaf on $\mathbf{Hilb}(\mathscr{W}, P)$ which is the pullback of the canonical invertible sheaf on the Grassmanian by the embedding ψ_m. Over an S′-valued point represented by a quotient $\mathscr{W} \to \mathscr{F}$, the restriction of the invertible sheaf $\mathscr{L}_m$ is canonically identified with the invertible sheaf $\wedge^{P(m)} H^0(X'/S', \mathscr{F}(m))$.

Suppose now that $\mathscr{W}$ is a coherent sheaf on X, flat over S, and that V is a finite-dimensional vector space. The group Sl(V) acts on $\mathbf{Hilb}(V \otimes \mathscr{W}, P)$, and on the invertible sheaves $\mathscr{L}_m$. The group action preserves the map to S. Since the Hilbert scheme is proper over S, we can describe the sets of properly stable and semistable points with respect to the group action and invertible sheaf, by restricting to fibers over geometric points. Suppose $s \to S$ is a geometric point. We analyze stability of a point in $\mathbf{Hilb}(V \otimes \mathscr{W}_s, P)$. The results are stated in our two main lemmas. For brevity of notation, assume that $s = S = \mathrm{Spec}(\mathbf{C})$.

Lemma **1.15.** — *There exists* M *such that for* $m \geqslant M$, *the following holds. Suppose* $V \otimes \mathscr{W} \to \mathscr{F} \to 0$ *is a point in* $\mathbf{Hilb}(V \otimes \mathscr{W}, P)$. *For any subspace* $H \subset V$, *let* $\mathscr{G}$ *denote the subsheaf of* $\mathscr{F}$ *generated by* $H \otimes \mathscr{W}$. *Suppose that* $p(\mathscr{G}, m) > 0$ *and*

$$\frac{\dim(H)}{p(\mathscr{G}, m)} \leqslant \frac{\dim(V)}{P(m)}$$

(resp. $<$*) for all nonzero proper subspaces* H. *Then the point is semistable (resp. properly stable) with respect to the linearized invertible sheaf* $\mathscr{L}_m$ *and the action of* Sl(V).

Proof. — For large m the Hilbert scheme is embedded into $\mathbf{Grass}(V \otimes W, P(m))$ where $W = H^0(\mathscr{W}(m))$. For all points in the Hilbert scheme, and all subspaces H, the sheaves $\mathscr{G}$ run over a bounded family. Let $\mathscr{K}$ denote the kernel

$$0 \to \mathscr{K} \to H \otimes \mathscr{W} \to \mathscr{G} \to 0.$$

Then the $\mathscr{K}$ also run over a bounded family. In particular we may choose M large enough so that for $m \geqslant M$, $h^0(\mathscr{G}(m)) = p(\mathscr{G}, m)$ and $h^1(\mathscr{K}(m)) = 0$ for all such $\mathscr{G}$ and $\mathscr{K}$. Let $\mathrm{Im}(H \otimes W)$ denote the image of $H \otimes W$ in $H^0(\mathscr{G}(m)) \subset H^0(\mathscr{F}(m))$. Twist the previous exact sequence by $\mathcal{O}_X(m)$ and take the long exact sequence of cohomology, to get an exact sequence

$$H \otimes W \to H^0(\mathscr{G}(m)) \to H^1(\mathscr{K}(m)).$$

The third term vanishes so this gives $\dim(\mathrm{Im}(H \otimes W)) = p(\mathscr{G}, m)$. Now apply Proposition 1.14 to conclude the proof. □

Lemma **1.16.** — *There exists* M *such that for* $m \geqslant \mathrm{M}$, *if* $\mathscr{F}$ *is the quotient sheaf represented by a point of* $\mathbf{Hilb}(\mathrm{V} \otimes \mathscr{W}, \mathrm{P})$ *which is semistable with respect to* $\mathscr{L}_m$ *and the action of* $\mathrm{Sl}(\mathrm{V})$, *then the following property holds. For any nonzero subspace* $\mathrm{H} \subset \mathrm{V}$, *let* $\mathscr{G} \subset \mathscr{F}$ *be the subsheaf generated by* $\mathrm{H} \otimes \mathscr{W}$. *Then* $r(\mathscr{G}) > 0$ *and*

$$\frac{\dim(\mathrm{H})}{r(\mathscr{G})} \leqslant \frac{\dim(\mathrm{V})}{r(\mathscr{F})}.$$

Proof. — Again, set $\mathrm{W} = \mathrm{H}^0(\mathscr{W}(m))$. Suppose that

$$\frac{\dim(\mathrm{H})}{r(\mathscr{G})} > \frac{\dim(\mathrm{V})}{r(\mathscr{F})},$$

or possibly that $\dim(\mathrm{H}) > 0$ but $r(\mathscr{G}) = 0$. Then for $m \gg 0$, we get

$$\frac{\dim(\mathrm{H})}{p(\mathscr{G}, m)} > \frac{\dim(\mathrm{V})}{\mathrm{P}(m)}.$$

Since the possibilities for H range over a bounded family, we can choose M so that for any such H and any $m \geqslant \mathrm{M}$, the previous inequality holds. Furthermore we may assume that $h^0(\mathscr{G}(m)) = p(\mathscr{G}, m)$ and $h^1(\mathscr{K}(m)) = 0$ where $\mathscr{K}$ is the kernel as before. Then $\dim(\mathrm{Im}(\mathrm{H} \otimes \mathrm{W})) = p(\mathscr{G}, m)$ and we may apply Proposition 1.14 to obtain a contradiction. □

Remark. — In the situation of the lemma, suppose $\mathscr{F} \to \mathscr{K} \to 0$ is a quotient sheaf. Let $\mathrm{H} \subset \mathrm{V}$ be the kernel of the map $\mathrm{V} \to \mathrm{H}^0(\mathrm{Hom}(\mathscr{W}, \mathscr{K}))$, and let J be the image. We have $\dim(\mathrm{H}) = \dim(\mathrm{V}) - \dim(\mathrm{J})$. If $\mathscr{G}$ is the subsheaf generated by $\mathrm{H} \otimes \mathscr{W}$, then $\mathscr{G}$ maps to zero in $\mathscr{K}$, so $r(\mathscr{G}) \leqslant r(\mathscr{F}) - r(\mathscr{K})$. Thus the conclusion of the lemma implies that

$$\frac{\dim(\mathrm{J})}{r(\mathscr{K})} \geqslant \frac{\dim(\mathrm{V})}{r(\mathscr{F})}.$$

By a similar argument for $\mathscr{K} = \mathscr{F}$, we can conclude that the map $\mathrm{V} \to \mathrm{H}^0(\mathrm{Hom}(\mathscr{W}, \mathscr{F}))$ is injective.

Moduli of semistable sheaves

Suppose S is a scheme of finite type over $\mathrm{Spec}(\mathbf{C})$ and suppose $\mathrm{X} \to \mathrm{S}$ is projective. We will consider the functor $\mathbf{M}^{\natural}(\mathcal{O}_{\mathrm{X}}, \mathrm{P})$ which associates to any S-schemes S′ the set of semistable sheaves $\mathscr{E}$ on X′/S′, of pure dimension d, with Hilbert polynomial P. If $\mathscr{E}$ is such a sheaf, and if $f: \mathrm{S}'' \to \mathrm{S}'$ is a further base change, then $f^* \mathscr{E}$ is such a sheaf on X″/S″—this gives the functoriality of $\mathbf{M}^{\natural}(\mathcal{O}_{\mathrm{X}}, \mathrm{P})$. (Recall our conventions $\mathrm{X}' = \mathrm{X} \times_{\mathrm{S}} \mathrm{S}'$ and $\mathrm{X}'' = \mathrm{X} \times_{\mathrm{S}} \mathrm{S}''$.) We will construct a *moduli space* $\mathbf{M}(\mathcal{O}_{\mathrm{X}}, \mathrm{P})$ which universally corepresents this functor.

Fix a large number N. Let $\mathscr{W} = \mathcal{O}_{\mathrm{X}}(-\mathrm{N})$ and $\mathrm{V} = \mathbf{C}^{\mathrm{P}(\mathrm{N})}$. For an S-scheme S′, the set of S′-valued points in $\mathbf{Hilb}(\mathrm{V} \otimes \mathscr{W}, \mathrm{P})$ may be described as the set of pairs $(\mathscr{E}, \alpha)$

where $\mathscr{E}$ is a coherent sheaf on $X' = X \times_S S'$, flat over S' with Hilbert polynomial P, and $\alpha : V \otimes \mathcal{O}_{S'} \to H^0(X'/S', \mathscr{E}(N))$ is a morphism such that the sections in the image of α generate $\mathscr{E}(N)$.

Let $Q_1 \subset \mathbf{Hilb}(V \otimes \mathscr{W}, P)$ denote the open set where the sheaf $\mathscr{E}$ has pure dimension d and is p-semistable (the openness of the semistability condition is proved in a more general context in the next chapter). The set of p-semistable sheaves on the fibers, with Hilbert polynomial P, is bounded. Thus we can assume that N is chosen large enough so that: every p-semistable sheaf with Hilbert polynomial P appears as a quotient corresponding to a point of Q_1; for any p-semistable sheaf with Hilbert polynomial P, the $H^0(X'/S', \mathscr{E}(N))$ is locally free over S' of rank P(N); and formation of the H^0 commutes with further base extension. Set Q_2 equal to the open set in Q_1 where α is an isomorphism. Then Q_2 represents the functor which associates to an S-scheme S' the set of pairs $(\mathscr{E}, \alpha)$ where $\mathscr{E}$ is a p-semistable sheaf on X' with Hilbert polynomial P, and $\alpha : V \otimes \mathcal{O}_{S'} \cong H^0(X'/S', \mathscr{E}(N))$.

We will also fix M large, and consider $m \geqslant M$. We may assume that for each such m there is an embedding ψ_m of $\mathbf{Hilb}(V \otimes \mathscr{W}, P)$ in a Grassmanian, corresponding to a very ample line bundle $\mathscr{L}_m$. The group Sl(V) acts on $\mathbf{Hilb}(V \otimes \mathscr{W}, P)$ and the line bundle $\mathscr{L}_m$. The open subset Q_2 is preserved by the action.

Let $Q_2^\natural$ and $\mathrm{Sl}(V)^\natural$ denote the functors represented by Q_2 and Sl(V) respectively. The second of these is a functor in groups acting on $Q_2^\natural$. Form the quotient functor $Q_2^\natural/\mathrm{Sl}(V)^\natural$, associating to an S-scheme S' the quotient set $Q_2(S')/\mathrm{Sl}(V)(S')$. There is a natural morphism of functors $Q_2 \to \mathbf{M}^\natural(\mathcal{O}_X, P)$ invariant under the group action, so we get a natural morphism $\varpi : Q_2^\natural/\mathrm{Sl}(V)^\natural \to \mathbf{M}^\natural(\mathcal{O}_X, P)$. As will be explained in the proof of Theorem 1.21 below, ϖ is a local isomorphism. Thus, to construct a scheme $\mathbf{M}(\mathcal{O}_X, P)$ universally corepresenting the functor $\mathbf{M}^\natural(\mathcal{O}_X, P)$ it suffices to construct a scheme universally corepresenting the quotient $Q_2^\natural/\mathrm{Sl}(V)^\natural$, that is to say a universal categorical quotient of Q_2 by the action of Sl(V). We will construct the moduli space as a good quotient $\mathbf{M}(\mathcal{O}_X, P) = Q_2/\mathrm{Sl}(V)$.

We would like to get a criterion for semistability of points in the Hilbert scheme with respect to the action of Sl(V) and linearized invertible sheaf $\mathscr{L}_m$ for large N and $m \geqslant M(N)$. However, there may be a problem having to do with irreducible components of the Hilbert scheme parametrizing sheaves which are not of pure dimension (I don't know if this problem really exists). To get around this we make the following definition. Let d be the degree of the polynomial P. Define $\mathbf{Hilb}(V \otimes \mathscr{W}, P, d)$ to be the closure in $\mathbf{Hilb}(V \otimes \mathscr{W}, P)$ of the set of points such that the quotient sheaf $\mathscr{E}$ is of pure dimension d. Note that $Q_2 \subset \mathbf{Hilb}(V \otimes \mathscr{W}, P, d)$.

Lemma **1.17.** (Cf. [Gi].) — *If $\mathscr{E}$ is the quotient sheaf represented by a point of* $\mathbf{Hilb}(V \otimes \mathscr{W}, P, d)$, *let $\mathscr{Z}$ denote the coherent subsheaf of sections supported in dimension $\leqslant d - 1$. Then there is a sheaf $\mathscr{E}'$ of pure dimension d, with Hilbert polynomial* P, *and an inclusion*

$$0 \to \mathscr{E}/\mathscr{Z} \to \mathscr{E}'.$$

Proof. — There exists a curve S, a point $0 \in S$ and a morphism $S \to \mathbf{Hilb}(V \otimes \mathscr{W}, P)$ such that 0 goes to the point corresponding to $\mathscr{E}$, and the generic point η of S maps into the set where the quotient is of pure dimension d. Let E be the quotient sheaf on $\mathbf{P}^n_S$, so $E_0 = \mathscr{E}$. Let $Y \subset \mathbf{P}^n \times \{0\}$ be the support of $\mathscr{Z}$. Let $U = \mathbf{P}^n_S - Y$, and let $j : U \to \mathbf{P}^n_S$ be the inclusion. Let $E' = j_*(E|_U)$. The sections of E′ are the meromorphic sections of E which are regular over η and at the generic point of the support of E_0. Since E is flat over S, no subsheaf can have support lying over 0, so the condition that the fiber E_η has pure dimension d means that E has pure dimension $d+1$. The open set U restricted to the support of E is the complement of a subset of codimension 2, so purity of the locus of poles implies that E′ is coherent. It also has no S-torsion, and $\dim(S) = 1$ so E′ is flat over S. Let $\mathscr{E}' = E'_0$. It has Hilbert polynomial P. There is a map $E \to E'$ and hence $\mathscr{E} \to \mathscr{E}'$; this latter is an isomorphism on the support of $\mathscr{E}$ intersected with U. Finally we claim that $\mathscr{E}'$ has pure dimension d. Note that $\mathscr{E}'|_U$ has pure dimension d. Choose $m \gg 0$ so that $\pi_* E'(m)$ is locally free, such that its fiber $\pi_*(E'(m))_0$ is the space of global sections of $E'_0(m)$ (where π is the projection from $\mathbf{P}^n$ to S). We may replace $\mathscr{E}'$ by $\mathscr{E}'(m)$, in order that any subsheaf supported in dimension $d-1$ will have a section. Suppose h_0 is a non-zero section of $\mathscr{E}'$ supported in dimension $d-1$. Then it extends to a section h of E′, and $h_0|_U = 0$, in other words $h|_U$ vanishes on U_0. Therefore there is a section h' of $E|_U$ such that $h = th'$ on U. By definition of E′, h' is a section of E′ everywhere. But E′ has pure dimension $d+1$, so $h = th'$ everywhere. Thus $h_0 = 0$, a contradiction which proves the claim. The lemma follows. □

Lemma **1.18.** — *There exists* N_0 *such that for all* $N \geqslant N_0$, *the following is true. Suppose* $\mathscr{E}$ *is a p-semistable sheaf on a fiber* X_s, *with Hilbert polynomial* P. *Then for all subsheaves* $\mathscr{F} \subset \mathscr{E}$, *we have*

$$\frac{h^0(\mathscr{F}(N))}{r(\mathscr{F})} \leqslant \frac{P(N)}{r(\mathscr{E})}$$

and if equality holds then

$$\frac{p(\mathscr{F}, m)}{r(\mathscr{F})} = \frac{P(m)}{r(\mathscr{E})}$$

for all m.

Proof. — Since the set of possibilities for $\mathscr{E}$ is bounded, we may fix one. Let $\mu = \mu(\mathscr{E})$. Suppose $\mathscr{F}$ is a subsheaf. Let $r = r(\mathscr{F})$. Let $\mathscr{F}_i$ denote the terms in its Harder-Narasimhan filtration, let $\mathscr{Q}_i = \mathscr{F}_i/\mathscr{F}_{i-1}$, let $r_i = r(\mathscr{Q}_i)$, and let $\mu_i = \mu(\mathscr{Q}_i)$. We have $h^0(\mathscr{F}(N)) \leqslant \Sigma_i\, h^0(\mathscr{Q}_i(N))$. Also we have $\mu_i \leqslant \mu$, and $\Sigma_i\, r_i = r$. Now by Corollary 1.7 we have

$$h^0(\mathscr{Q}_i(N)) \leqslant \begin{cases} 0 & \text{if } \mu_i + N + B \leqslant 0 \\ r_i(\mu_i + N + B)^d/d! & \text{if } \mu_i + N + B \geqslant 0. \end{cases}$$

Let $\nu = \min(\mu_i)$. Then we get

$$h^0(\mathscr{F}(\mathrm{N})) \leqslant \begin{Bmatrix} 0 & \text{if } \mu + \mathrm{N} + \mathrm{B} \leqslant 0 \\ (r-1)\,(\mu + \mathrm{N} + \mathrm{B})^d/d! & \text{if } \mu + \mathrm{N} + \mathrm{B} \geqslant 0 \end{Bmatrix} + \begin{Bmatrix} 0 & \text{if } \nu + \mathrm{N} + \mathrm{B} \leqslant 0 \\ (\nu + \mathrm{N} + \mathrm{B})^d/d! & \text{if } \nu + \mathrm{N} + \mathrm{B} \geqslant 0 \end{Bmatrix}.$$

For any A there is a $\mathrm{C} \geqslant \mathrm{A}$ such that if $\nu \leqslant \mu - \mathrm{C}$ then

$$\begin{Bmatrix} 0 & \text{if } \mu + \mathrm{N} + \mathrm{B} \leqslant 0 \\ (r-1)\,(\mu + \mathrm{N} + \mathrm{B})^d/d! & \text{if } \mu + \mathrm{N} + \mathrm{B} \geqslant 0 \end{Bmatrix} + \begin{Bmatrix} 0 & \text{if } \nu + \mathrm{N} + \mathrm{B} \leqslant 0 \\ (\nu + \mathrm{N} + \mathrm{B})^d/d! & \text{if } \nu + \mathrm{N} + \mathrm{B} \geqslant 0 \end{Bmatrix} \leqslant r(\mathrm{N} - \mathrm{A})^d/d!$$

for $\mathrm{N} \geqslant \mathrm{C}$. We can also choose A such that $(\mathrm{N} - \mathrm{A})^d/d! < \dfrac{1}{r(\mathscr{E})} p(\mathscr{E}, \mathrm{N})$ for $\mathrm{N} \geqslant \mathrm{A}$. Let C be the number given above, and assume $\mathrm{N}_0 \geqslant \mathrm{C}$. Then if $\nu \leqslant \mu - \mathrm{C}$, we have

$$\frac{h^0(\mathscr{F}(\mathrm{N}))}{r(\mathscr{F})} < \frac{p(\mathscr{E}, \mathrm{N})}{r(\mathscr{E})}$$

for $\mathrm{N} \geqslant \mathrm{N}_0$. On the other hand the set of saturations $\mathscr{F}^{\mathrm{sat}}$ of subsheaves $\mathscr{F} \subset \mathscr{E}$ such that $\nu \geqslant \mu - \mathrm{C}$ is bounded; thus we may choose N_0 such that for $\mathrm{N} \geqslant \mathrm{N}_0$, $h^0(\mathscr{F}^{\mathrm{sat}}(\mathrm{N})) = p(\mathscr{F}^{\mathrm{sat}}, \mathrm{N})$ for any such $\mathscr{F}$. Furthermore the set of Hilbert polynomials which occur is finite. For each of them, we have the comparison of polynomials

$$\frac{p(\mathscr{F}^{\mathrm{sat}}, m)}{r(\mathscr{F})} \leqslant \frac{p(\mathscr{E}, m)}{r(\mathscr{E})},$$

meaning a comparison of values for large m. But since there are only finitely many polynomials involved, we may choose N_0 such that for any $\mathrm{N} \geqslant \mathrm{N}_0$,

$$\frac{p(\mathscr{F}^{\mathrm{sat}}, \mathrm{N})}{r(\mathscr{F})} \leqslant \frac{p(\mathscr{E}, \mathrm{N})}{r(\mathscr{E})}.$$

We may also arrange so that if the equality

$$\frac{p(\mathscr{F}^{\mathrm{sat}}, \mathrm{N})}{r(\mathscr{F})} = \frac{p(\mathscr{E}, \mathrm{N})}{r(\mathscr{E})}$$

holds for any one $\mathrm{N} \geqslant \mathrm{N}_0$, then equality holds for all N. Now $h^0(\mathscr{F}(\mathrm{N})) \leqslant h^0(\mathscr{F}^{\mathrm{sat}}(\mathrm{N}))$, and we can choose N_0 so that if $\mathrm{N} \geqslant \mathrm{N}_0$ then $\mathscr{F}^{\mathrm{sat}}(\mathrm{N})$ is generated by global sections. Thus if $h^0(\mathscr{F}^{\mathrm{sat}}(\mathrm{N})) = h^0(\mathscr{F}(\mathrm{N}))$ then $\mathscr{F} = \mathscr{F}^{\mathrm{sat}}$. We get the desired statements. □

Theorem **1.19.** — *Fix a polynomial* P *of degree d. There exist* N *and* M *such that for* $m \geqslant \mathrm{M}$, *the following is true. A point* $\mathscr{E}$ *in* $\mathbf{Hilb}(\mathrm{V} \otimes \mathscr{W}, \mathrm{P}, d)$ *is semistable (resp. properly stable) for the action of* $\mathrm{Sl}(\mathrm{V})$ *with respect to the embedding determined by m, if and only if the quotient* $\mathscr{E}$ *is a p-semistable (resp. p-stable) coherent sheaf of pure dimension d and the map* $\mathrm{V} \to \mathrm{H}^0(\mathscr{E})$ *is an isomorphism.*

Proof. — First the " if " direction. Suppose we have a point in $\mathbf{Hilb}(\mathrm{V} \otimes \mathscr{W}, \mathrm{P}, d)$ such that $\mathscr{E}$ is a p-semistable sheaf and the map

$$\mathrm{V} \to \mathrm{H}^0(\mathscr{E}(\mathrm{N}))$$

is an isomorphism. By Lemma 1.15 what remains to be proved is that we may choose $\mathrm{M}(\mathrm{N})$ such that for $m \geqslant \mathrm{M}$, and for any subsheaf $\mathscr{F} \subset \mathscr{E}$ generated by sections of $\mathscr{E}(\mathrm{N})$,

$$\frac{h^0(\mathscr{F}(\mathrm{N}))}{p(\mathscr{F}, m)} \leqslant \frac{\mathrm{P}(\mathrm{N})}{\mathrm{P}(m)}$$

(resp. $<$ if $\mathscr{E}$ is stable). We may choose N_0 such that for any $\mathrm{N} \geqslant \mathrm{N}_0$ and any point in $\mathbf{Hilb}(\mathrm{V} \otimes \mathscr{W}, \mathrm{P})$ representing a semistable sheaf $\mathscr{E}$, the conclusion of Lemma 1.18 holds for $\mathscr{E}$. Once N is fixed, the set of subsheaves $\mathscr{F}$ generated by sections of $\mathscr{E}(\mathrm{N})$ is bounded, so the set of polynomials $p(\mathscr{F}, m)$ is finite. These polynomials all have first term $r(\mathscr{F})\, m^d$, so we may make M big enough so that for the $\mathscr{F}$ where

$$\frac{h^0(\mathscr{F}(\mathrm{N}))}{r(\mathscr{F})} < \frac{\mathrm{P}(\mathrm{N})}{r(\mathscr{E})}$$

we get the desired conclusion. (This completes the stable case.) For those $\mathscr{F}$ where equality holds, we have

$$\frac{p(\mathscr{F}, m)}{r(\mathscr{F})} = \frac{\mathrm{P}(m)}{r(\mathscr{E})}.$$

Thus in this case also we get the desired conclusion,

$$\frac{h^0(\mathscr{F}(\mathrm{N}))}{p(\mathscr{F}, m)} \leqslant \frac{\mathrm{P}(\mathrm{N})}{\mathrm{P}(m)}.$$

This completes the proof of the first half of Theorem 1.19.

Now we turn to the second half, the " only if " direction. Suppose that $\mathrm{N} \geqslant \mathrm{N}_0$. We will choose N_0 as we go along. Let $\mathrm{M} = \mathrm{M}(\mathrm{N})$ be as in Lemma 1.16. Suppose

$$\mathrm{V} \otimes \mathscr{W} \to \mathscr{E} \to 0$$

is a point of $\mathbf{Hilb}(\mathrm{V} \otimes \mathscr{W}, \mathrm{P}, d)$ which is semistable with respect to the embedding determined by some $m \geqslant \mathrm{M}$. Let $\mathscr{E}'$ be the subsheaf of $\mathscr{E}$ of sections supported in dimension $d-1$, and let $\mathcal{E}'$ be the sheaf of pure dimension d given by Lemma 1.17. The remark following Lemma 1.16 implies that for any quotient $\mathcal{E}' \to \mathcal{G} \to 0$ we have

$$\frac{h^0(\mathcal{G}(N))}{r(\mathcal{G})} \geqslant \frac{P(N)}{r(\mathcal{E})}.$$

Let $\mathcal{G}$ be the μ-semistable quotient of $\mathcal{E}'$ with the smallest $\mu = \mu(\mathcal{G})$, in other words the last step in the Harder-Narasimhan filtration. Apply Corollary 1.7 to the sheaf $\mathcal{G}(N)$ to conclude that

$$\frac{P(N)}{r(\mathcal{E})} \leqslant \frac{h^0(\mathcal{G}(N))}{r(\mathcal{G})} \leqslant (\mu(\mathcal{G}) + N + B)^d/d!.$$

Since P is fixed, there is a C and we may choose N_0 so that for $N \geqslant N_0$,

$$\frac{P(N)}{r(\mathcal{E})} \geqslant (N-C)^d/d!.$$

Therefore $N - C \leqslant \mu(\mathcal{G}) + N + B$, so $\mu(\mathcal{G}) \geqslant -B-C$. Note that B and C are independent of N, and recall that the Hilbert polynomial of $\mathcal{E}'$ is equal to P. Therefore by Theorem 1.1 the sheaves $\mathcal{E}'$ remain in a bounded family independent of N. In particular we may increase N_0 so that for $N \geqslant N_0$, $h^0(\mathcal{E}'(N)) = P(N)$ and $\mathcal{E}'(N)$ is generated by global sections. Now applying the remark following Lemma 1.16 to the quotient sheaf $\mathcal{E}/\mathcal{T}$, we get $h^0(\mathcal{E}/\mathcal{T}(N)) \geqslant P(N)$. Since $\mathcal{E}/\mathcal{T}$ is included in $\mathcal{E}'$, the sections of $\mathcal{E}/\mathcal{T}(N)$ generate $\mathcal{E}'$, so $\mathcal{E}/\mathcal{T} = \mathcal{E}'$. But since the Hilbert polynomials of $\mathcal{E}$ and $\mathcal{E}'$ are the same, the Hilbert polynomial of $\mathcal{T}$ must be zero, so $\mathcal{T} = 0$. Thus $\mathcal{E} = \mathcal{E}'$ is of pure dimension d, and remains in a bounded family independent of N. Furthermore, $h^0(\mathcal{E}(N)) = P(N)$. Lemma 1.16 implies that the map

$$V \to H^0(\mathcal{E}(N))$$

is injective, so by counting dimensions it is an isomorphism. It remains to be shown that $\mathcal{E}$ is p-semistable. For each element of the bounded family of $\mathcal{E}$'s which is not semistable we may choose a quotient $\mathcal{G}$ such that

$$\frac{p(\mathcal{G},k)}{r(\mathcal{G})} < \frac{P(k)}{r(\mathcal{E})}$$

for large k. We may assume that the $\mathcal{G}$ remain in a bounded family, and hence we may increase N_0 so that for $N \geqslant N_0$,

$$\frac{p(\mathcal{G},N)}{r(\mathcal{G})} < \frac{P(N)}{r(\mathcal{E})}$$

and furthermore $h^0(\mathcal{G}(N)) = p(\mathcal{G}, N)$. But this now contradicts the conclusion of the remark following Lemma 1.16, so there are no $\mathcal{E}$ which are not semistable. This completes the proof of the statement about semistable points.

Suppose, under the same circumstances as above, that $\mathcal{E}$ is not p-stable. Let $\mathcal{F} \subset \mathcal{E}$ be a nonzero proper subsheaf with the same normalized Hilbert polynomial.

Put $U = H^0(\mathscr{E}(m))$ and $W = H^0(\mathscr{W}(m)) = H^0(\mathcal{O}_X(m-N))$, and set $H = H^0(\mathscr{F}(N)) \subset V$. As the set of possibilities for $\mathscr{F}$ is bounded, we may assume that m and N are big enough so that the image of $H \otimes W$ is equal to $H^0(\mathscr{F}(m)) \subset U$, and

$$\frac{\dim(H)}{h^0(\mathscr{F}(m))} = \frac{P(N)}{h^0(\mathscr{E}(m))}.$$

By the criterion of Proposition 1.14, $\mathscr{E}$ maps under ψ_m to a point which is not properly stable in the Grassmanian. By Proposition 1.12, $\mathscr{E}$ is not a properly stable point in the Hilbert scheme. This completes the proof of the statement about stability. □

Corollary **1.20.** — *The scheme* Q_2 *constructed at the start of this section is equal to the set of semistable points of* $\mathbf{Hilb}(V \otimes \mathscr{W}, P, d)$ *under the action of* $\mathrm{Sl}(V)$. *The open subset* Q_2^s *parametrizing p-stable sheaves is equal to the set of properly stable points under the action of* $\mathrm{Sl}(V)$.

Proof. — Note that Q_2 is an open subscheme of $\mathbf{Hilb}(V \otimes \mathscr{W}, P)$ contained in $\mathbf{Hilb}(V \otimes \mathscr{W}, P, d)$. Mumford proves that there is an open subset of semistable points for the group action [Mu]. The previous theorem implies that the points of these subsets are the same, so they are equal. The same argument holds for the properly stable points. □

Theorem **1.21.** — *Let* $\mathbf{M}(\mathcal{O}_X, P) \stackrel{\text{def}}{=} Q_2/\mathrm{Sl}(P(N))$ *be the good quotient given by the construction of* [Mu], *applied to the group action on* $\mathbf{Hilb}(V \otimes \mathscr{W}, P, d)$.

(1) There exists a natural transformation $\varphi : \mathbf{M}^\natural(\mathcal{O}_X, P) \to \mathbf{M}(\mathcal{O}_X, P)$ *such that* $\mathbf{M}(\mathcal{O}_X, P)$ *universally corepresents* $\mathbf{M}^\natural(\mathcal{O}_X, P)$.

(2) $\mathbf{M}(\mathcal{O}_X, P)$ *is a projective scheme.*

(3) The points of $\mathbf{M}(\mathcal{O}_X, P)$ *represent the equivalence classes of semistable sheaves under the relation that* $\mathscr{E}_1 \sim \mathscr{E}_2$ *if* $\mathrm{gr}(\mathscr{E}_1) = \mathrm{gr}(\mathscr{E}_2)$.

(4) There is an open subset $\mathbf{M}^s(\mathcal{O}_X, P) \subset \mathbf{M}(\mathcal{O}_X, P)$, *with inverse image equal to* Q_2^s, *whose points represent isomorphism classes of p-stable sheaves. Locally in the étale topology on* $\mathbf{M}^s(\mathcal{O}_X, P)$ *there is a universal sheaf* $\mathscr{F}^{\mathrm{univ}}$ *such that if* $\mathscr{F}$ *is an element of* $\mathbf{M}^\natural(\mathcal{O}_X, P)$ (S') *whose fibers* $\mathscr{F}_s$ *are p-stable, then the pull-back of* $\mathscr{F}^{\mathrm{univ}}$ *via* $S' \to \mathbf{M}^s(\mathcal{O}_X, P)$ *is isomorphic to* $\mathscr{F}$ *after tensoring with the pull-back of a line bundle on* S'.

(5) If $x \in \mathbf{M}^s(\mathcal{O}_X, P)$ *is a point such that* Q_2^s *is smooth at the inverse image of* x, *then* $\mathbf{M}^s(\mathcal{O}_X, P)$ *is smooth at* x.

Proof of parts (1), (2) and (3). — For part (1) it suffices to show that the natural transformation from the quotient functor $Q_2^\natural/\mathrm{Sl}(V)^\natural$ to $\mathbf{M}^\natural(\mathcal{O}_X, P)$ is a local isomorphism. Note first of all that the boundedness results and our choice of N, together with the possibility of choosing local frames for the direct images $H^0(X'/S', \mathscr{E}(N))$, imply that the natural transformation $Q_2^\natural/\mathrm{Gl}(V)^\natural \to \mathbf{M}^\natural(\mathcal{O}_X, P)$ is a local isomorphism. On the other hand, the center $\mathbf{G}_m \subset \mathrm{Gl}(V)$ acts trivially on Q_2 so the action descends to an action of $\mathrm{PGl}(V)$ on Q_2. The morphism of group schemes $\mathrm{Gl}(V) \to \mathrm{PGl}(V)$ is locally surjective in the étale topology, so the morphism of quotient functors

$Q_2^\natural/\mathrm{Gl}(V)^\natural \to Q_2^\natural/\mathrm{PGl}(V)^\natural$ is a local isomorphism. The morphism of group schemes $\mathrm{Sl}(V) \to \mathrm{PGl}(V)$ is locally surjective in the étale topology, so the morphism of quotient functors $Q_2^\natural/\mathrm{Sl}(V)^\natural \to Q_2^\natural/\mathrm{PGl}(V)^\natural$ is a local isomorphism. We obtain a commutative diagram where three sides are local isomorphisms. Taking the sheafifications, we obtain a commutative diagram where three sides are isomorphisms, so the fourth side is an isomorphism. In the original diagram, the fourth side is a local isomorphism $Q_2^\natural/\mathrm{Sl}(V)^\natural \to \mathbf{M}^\natural(\mathcal{O}_X, P)$. The discussion before Lemma 1.10 implies that the natural transformation φ exists. As $\mathbf{M}(\mathcal{O}_X, P)$ is a universal categorical quotient of Q_2 by $\mathrm{Sl}(V)$, it universally corepresents the quotient functor $Q_2^\natural/\mathrm{Sl}(V)^\natural$. Therefore $(\mathbf{M}(\mathcal{O}_X, P), \varphi)$ universally corepresents $\mathbf{M}^\natural(\mathcal{O}_X, P)$, for property (1).

The closed subset $\mathbf{Hilb}(V \otimes \mathcal{W}, P, d)$ of the Hilbert scheme is projective [Gr2], so the good quotient of its set of semistable points is projective [Se] [Se2]. Thus $\mathbf{M}(\mathcal{O}_X, P)$ is projective, proving (2).

To prove (3) using Lemma 1.10 (Seshadri's result of [Se] [Se2]), we have to verify that for semistable sheaves $\mathcal{E}$ and $\mathcal{F}$, the closures of the corresponding orbits in Q_2 intersect if and only if $\mathrm{gr}(\mathcal{E}) = \mathrm{gr}(\mathcal{F})$. This proof comes from [Gi]. First of all given an extension $0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0$ we can find a family of extensions $\mathcal{E}_t$ of $\mathcal{E}''$ by $\mathcal{E}'$, parametrized by $t \in \mathbf{A}^1$, such that for each $t \neq 0$ the extension is isomorphic to the given one, and for $t = 0$ the extension is trivial. To do this, base change the extension $\mathcal{E}$ to $X \times \mathbf{A}^1$ and then form a new extension of sheaves on $X \times \mathbf{A}^1/\mathbf{A}^1$ by pulling back via $t : \mathcal{E}'' \to \mathcal{E}''$. Applying this repeatedly to a semistable $\mathcal{E}$, using the extensions in the Jordan-Hölder filtration, we find that the orbit corresponding to $\mathrm{gr}(\mathcal{E})$ is in the closure of the orbit corresponding to $\mathcal{E}$. So if $\mathrm{gr}(\mathcal{E}) = \mathrm{gr}(\mathcal{F})$ then the closures of the orbits of $\mathcal{E}$ and $\mathcal{F}$ intersect.

Conversely the orbit corresponding to $\mathrm{gr}(\mathcal{E})$ is closed. For this it suffices to consider the fiber over one point $s \in S$. Suppose $\mathcal{E}$ is a p-semistable sheaf on X_s such that $\mathrm{gr}(\mathcal{E}) \cong \mathcal{E}$. Suppose T is a curve and $t_0 \in T$ is a closed point, and suppose that $\mathcal{F}$ is a p-semistable sheaf on $X_s \times T$ over T such that $\mathcal{F}_t \cong \mathcal{E}$ for $t \neq t_0$. If $\mathcal{E}_i$ is a stable component of $\mathcal{E}$ then by semicontinuity, there are at least as many maps from $\mathcal{E}_i$ to $\mathcal{F}_{t_0}$ as to $\mathcal{E}$; since $\mathcal{F}_{t_0}$ is p-semistable, this implies that $\mathcal{F}_{t_0}$ is a direct sum of copies of $\mathcal{E}_i$ with the same multiplicities as $\mathcal{E}$, so $\mathcal{F}_{t_0} = \mathcal{E}$. This completes the proof of (3). □

We will prove (4) and (5) below, after discussing Luna's étale slice theorem.

Slice theorems

Suppose Z is a scheme over $S = \mathrm{Spec}(\mathbf{C})$ with relatively very ample $\mathcal{L}$. Suppose a reductive group G acts on Z and $\mathcal{L}$.

Proposition **1.22**. — *Suppose* Z *is projective over* S. *Suppose* $z \in Z^{ss}$ *is a semistable point such that the orbit* Gz *is closed in* Z^{ss}. *Then the stabilizer* $H = \mathrm{Stab}(z) \subset G$ *is reductive.*

Proof. — This is due to Matsushima [Mt]. □

Proposition **1.23** (Luna's étale slice theorem). — *Suppose z is a point such that the orbit* Gz *is closed in* Z^{ss}. *Let* $H = \mathrm{Stab}(z)$. *There is a locally closed affine* H-*stable subscheme* $V \subset Z^{ss}$ *passing through z, such that the* G-*invariant morphism* $\psi : G \times^H V \to Z$ *is étale. The image* U *of* ψ *is an affine open set such that* $U = \varphi^{-1}\varphi(U)$ *where* $\varphi : Z^{ss} \to Z^{ss}/G$ *is the projection to the quotient. Furthermore the morphism of good quotients* $V/H \to Z^{ss}/G$ *is étale.*

Proof. — See [Lu]. □

In the situation of the proposition, we can choose an H-invariant complement $H^{\perp} \subset G$ containing the identity e, such that the map $H^{\perp} \times H \to G$ is étale. (This can be seen by applying the above theorem to the case of the action of H on G.) Then the proposition says that the morphism $H^{\perp} \times V \to Z$ is étale.

Suppose Z is a scheme with invertible sheaf $\mathscr{L}$ and action of G on Z and $\mathscr{L}$. We will discuss the set of properly stable points Z^s. Mumford constructs a *universal geometric quotient* $Y = Z^s/G$ (see [Mu] for the definition; among other things, it means that the points of the quotient correspond exactly to the orbits). The action of G on Z^s is said to be *proper* if the map $\Psi : G \times Z^s \to Z^s \times Z^s$ defined by $\Psi(g, z) = (gz, z)$ is proper. The action is *free* if Ψ is a closed immersion.

Lemma **1.24.** — *The action of* G *on* Z^s *is proper. Suppose furthermore that for any* $z \in Z^s$, $\mathrm{Stab}(z) = \{e\}$. *Then the action of* G *on* Z^s *is free, the morphism* $Z^s \to Y$ *is faithfully flat, and* $\Psi : G \times_Y Z^s \cong Z^s \times_Y Z^s$ *is an isomorphism. The scheme* Z^s *is a principal* G-*bundle over* Y *in the étale topology.*

Proof. — Much of this follows from [Mu], and the remainder is Corollary 1 in [Lu]. □

Proof of (4) and (5) in Theorem 1.21. — The stabilizer in PGl(V) of the point of Q_2^s corresponding to a p-stable sheaf, is $\{e\}$. By the above result, the set of properly stable points Q_2^s is a principal PGl(V)-bundle over $\mathbf{M}^s(\mathcal{O}_X, P)$ in the étale topology. This implies that a universal family exists étale locally. The existence of slices implies that at points where Q_2^s is smooth, the quotient is smooth. □

Limits of orbits

We now make a general observation about limits of orbits in the set of semistable points (reducing to the case of $\mathbf{G}_m$), followed by an explicit analysis of the limits of $\mathbf{G}_m$-orbits in Hilbert schemes.

Suppose Z is a projective scheme over $S = \mathrm{Spec}(\mathbf{C})$ with very ample $\mathscr{L}$. Suppose a reductive group G acts on Z and $\mathscr{L}$.

Lemma **1.25.** — *Suppose* $z \in Z^{ss}$. *Let* $B \subset Z^{ss}$ *be the unique closed orbit in the closure of the orbit of z. Then there is a one parameter subgroup* $\lambda : \mathbf{G}_m \to G$ *such that* $\lim_{t\to 0} \lambda(t)\, z$ *is a point of* B.

Proof. — This is essentially a statement from the proof of Proposition 2.3 in Chapter 2 of [Mu]. We have to make some preliminary reductions. We may assume that $Z = \mathbf{P}(V)$, with G acting linearly on V. Choose a point y in the orbit B. Let $H = \mathrm{Stab}(y)$, and choose a linear H-stable subspace $\mathbf{P}(W) \subset \mathbf{P}(V)$ passing through y, complementary to the tangent space of B—this gives a slice near y as described in Proposition 1.23. Suppose $y(r)$ is a family of points approaching y as $r \to 0$. Using the map $H^{\perp} \times \mathbf{P}(W) \to \mathbf{P}(V)$, which is étale at (e, y), we can write $y(r) = g(r)\, w(r)$ for $g(r) \to e$ in $H^{\perp}$ and $w(r) \to y$ in $\mathbf{P}(W)$. Since y is in the closure of Gz, we can choose $y(r) \in Gz$, and hence $w(r) \in Gz$. Thus y appears in the closure of $(Gz) \cap \mathbf{P}(W)$.

Choose a point $z' \in (Gz) \cap \mathbf{P}(W)$ (and note that $Gz = Gz'$). We claim that Hz' is dense in $(Gz') \cap \mathbf{P}(W)$. The map

$$H^{\perp} \times (Hz') \to Gz'$$

has surjective differential at (e, z'). On the other hand, the inverse image of Gz' under the map

$$H^{\perp} \times \mathbf{P}(W) \to \mathbf{P}(V)$$

is equal to $H^{\perp} \times (Gz' \cap \mathbf{P}(W))$, so the map

$$H^{\perp} \times (Gz' \cap \mathbf{P}(W)) \to Gz'$$

is étale locally at (e, z'). These imply that the inclusion $Hz' \subset (Gz' \cap \mathbf{P}(W))$ has surjective differential at z', so it is surjective locally at z'. Hence the orbit Hz' is dense in $(Gz' \cap \mathbf{P}(W))$ as claimed. From this claim and the preceding discussion, the point y is in the closure of the orbit Hz'. We have reduced to the case where H acts on $\mathbf{P}(W)$ and y is a fixed point. Let L_y denote the line in W corresponding to y. The group H acts on L_y by a character $\chi : H \to \mathbf{C}^*$. Change the representation W by tensoring with χ^{-1}. This does not change the action on $\mathbf{P}(W)$, but now we may assume that H fixes the points in L_y. Choose an H-invariant complement $U = L_y^{\perp} \subset W$, and a point $\tilde{y} \in L_y$. Projection from $U \times \{\tilde{y}\}$ to $\mathbf{P}(W)$ gives an isomorphism of U (with its linear action) with an invariant affine neighborhood of y. We have $z' \in U$, for otherwise Hz' would be contained in the complement of U and could not contain y in its closure. Now z' is a point in the linear space U, and the origin is contained in the closure of Hz'. Hence we may apply the statement (*ii*) used in the proof of Proposition 2.3 of [Mu] to obtain a one parameter subgroup $\lambda' : \mathbf{G}_m \to H$ such that $\lim_{t \to 0} \lambda'(t)\, z' = 0$ in U. Let $g \in G$ be an element with $gz = z'$. Then $\lambda(t) = g^{-1} \lambda'(t)\, g$ is a one-parameter subgroup of G with $\lambda(t)\, z = g^{-1} \lambda'(t)\, z'$, hence

$$\lim_{t \to 0} \lambda(t)\, z = g^{-1} y \in B.$$

This proves the lemma. □

Remark. — We will use the lemma in the following way. Suppose $A \subset Z^{ss}$ is a G-invariant locally closed subset, and suppose $z \in A$. In order to prove that

$$\overline{Gz} \cap A = \overline{Gz} \cap Z^{ss},$$

it suffices to prove that for every one-parameter subgroup $\lambda : \mathbf{G}_m \to G$, if the limit $y = \lim_{t \to 0} \lambda(t)\, z$ is in Z^{ss}, then this limit y is in A. For by the lemma, this implies that the unique closed orbit B in the closure of Gz, is contained in A. Then $D = \overline{Gz} \cap (Z^{ss} - A)$ is a closed G-invariant subvariety of $\overline{Gz} \cap Z^{ss}$ which does not contain B, hence D must be empty.

In order to apply this technique, we must be able to analyze explicitly the limits of orbits under actions of $\mathbf{G}_m$. We will treat the case of Hilbert schemes. Suppose $\mathscr{W}$ is a coherent sheaf of $\mathcal{O}_X$-modules, and V is a vector space, and fix a Hilbert polynomial P. The group $Gl(V)$ acts on $\mathbf{Hilb}(V \otimes \mathscr{W}, P)$. Suppose we have a map $\lambda : \mathbf{G}_m \to Gl(V)$, in other words an action of $\mathbf{G}_m$ on the vector space V. Suppose

$$p : V \otimes \mathscr{W} \to \mathscr{F} \to 0$$

is a point in $\mathbf{Hilb}(V \otimes \mathscr{W}, P)$. Then for each $t \in \mathbf{G}_m$ we get a point $\lambda(t)\, p = p \circ (t \otimes 1)$. The Hilbert scheme is proper, and by the valuative criterion of properness, there is a unique limit point $\lim_{t \to 0} \lambda(t)\, p$. We describe this limit point explicitly.

The vector space V decomposes under the action of $\mathbf{G}_m$ as a direct sum $V = \bigoplus_\alpha V_\alpha$ where $t \in \mathbf{G}_m$ acts on V_α by multiplication by t^α. This gives rise to a filtration F of V defined by

$$F_\beta V = \bigoplus_{\alpha \geqslant \beta} V_\alpha.$$

The direct sum decomposition gives an isomorphism $V \cong \mathrm{Gr}^F_\bullet(V)$. From our point $p : V \otimes \mathscr{W} \to \mathscr{F}$ we obtain a filtration $F_\bullet$ of $\mathscr{F}$, setting $F_\beta \mathscr{F} = p(F_\beta V \otimes \mathscr{W})$. Then we get a surjection

$$\mathrm{Gr}^F_\bullet(V) \otimes \mathscr{W} \to \mathrm{Gr}^F_\bullet(\mathscr{F}) \to 0,$$

and composing this with the isomorphism $V \cong \mathrm{Gr}^F(V)$, we get a point

$$q : V \otimes \mathscr{W} \to \mathrm{Gr}^F_\bullet(\mathscr{F}) \to 0$$

of the Hilbert scheme $\mathbf{Hilb}(V \otimes \mathscr{W}, P)$.

Lemma **1.26**. — *We have* $q = \lim_{t \to 0} \lambda(t)\, p$.

Proof. — We will use the embedding ψ_m for some $m \gg 0$. Let $W = H^0(\mathscr{W}(m))$, and for our quotient $\mathscr{F}$, let $U = H^0(\mathscr{F}(m))$. Then $p' = \psi_m(p)$ is the point

$$p' : V \otimes W \to U \to 0$$

in $\mathbf{Grass}(V \otimes W, P(m))$. Let $F_\alpha U$ denote the image of $(F_\alpha V) \otimes W$. Note that if we choose m large enough, then $F_\alpha U = H^0(F_\alpha \mathscr{F}(m))$. The associated graded of p' gives a morphism

$$\mathrm{Gr}^F_\bullet(V) \otimes W \to \mathrm{Gr}^F_\bullet(U) \to 0,$$

and again composing this with the isomorphism $V \cong \mathrm{Gr}^F_\bullet(V)$, a point

$$q' : V \otimes W \to \mathrm{Gr}^F_\bullet(U) \to 0$$

of $\mathbf{Grass}(V \otimes W, P(m))$. This point q' is equal to $\psi_m(q)$. The embedding ψ_m is invariant under the action of $\mathbf{G}_m$, so it suffices to prove that $\lim_{t\to 0} \lambda(t)\, p' = q'$. Let j denote the canonical embedding of $\mathbf{Grass}(V \otimes W, P(m))$ into the projective space of quotient lines of $\wedge^{P(m)}(V \otimes W)$. Let $L = \wedge^{P(m)} U$, and $p'' = j(p') = \wedge^{P(m)} p'$. The embedding j is invariant under $\mathbf{G}_m$, so it suffices to find the limit of $\lambda(t)\, p''$. There are induced filtrations $F_\bullet$ on $\wedge^{P(m)}(V \otimes W)$ and L, and

$$F_\alpha L = p''(F_\alpha \wedge^{P(m)}(V \otimes W)).$$

This may be checked by choosing a basis compatible both with the quotient U and the filtration $F_\bullet$ of $V \otimes W$. Again, the associated graded q'' of p'' is equal to the image $j(q')$ of the associated graded of p'. The filtration of $\wedge^{P(m)}(V \otimes W)$ is deduced from the action of $\mathbf{G}_m$ in the same manner as before. There is only one step of the filtration of L, corresponding to the smallest α such that L is a quotient of $F_\alpha(\dots)$. The associated graded q'' corresponds to the morphism $\mathrm{Gr}^F_\alpha(\dots) \to L$. One can check by using coordinates which diagonalize the action of $\lambda(t)$ that $\lim_{t\to 0} \lambda(t)\, p'' = q''$. This completes the proof. □

A criterion for local freeness

Lemma **1.27.** — *Suppose* F *is a coherent sheaf on* X *which is flat over* S, *and such that the restrictions* $F|_{X_s}$ *to the fibers are locally free. Then* F *is locally free.*

Proof. — Localizing, we may suppose that $S = \mathrm{Spec}(A)$ and $X = \mathrm{Spec}(B)$ where $A \subset B$ are noetherian local $\mathbf{C}$-algebras with maximal ideals $\mathbf{m}_A$ and $\mathbf{m}_B$ respectively. Let $I = B.\mathbf{m}_A$. Suppose F is a finitely generated B-module which is flat over A, and F/IF is free over B/I. Choose a collection of elements $f_1, \dots, f_n$ such that their images form a basis for the free module F/IF. We obtain an exact sequence

$$0 \to K \to B^n \to F \to 0;$$

surjectivity follows from Nakayama's lemma and we denote by K the kernel. Tensoring with $A/\mathbf{m}_A$ over A yields

$$0 \to K/IK \to (B/I)^n \to F/IF \to 0,$$

which remains exact because F is flat over A. But this implies that $K/IK = 0$, and since K is finitely generated, Nakayama's lemma implies that $K = 0$. Thus F is a free module. □

2. Sheaves of rings of differential operators

Suppose S is a noetherian scheme over $\mathbf{C}$, and $f: X \to S$ is a scheme of finite type over S. A *sheaf of rings of differential operators on* X *over* S is a sheaf of (not necessarily commutative) $\mathcal{O}_X$-algebras Λ over X, with a filtration $\Lambda_0 \subset \Lambda_1 \subset \ldots$ which satisfies the following properties.

2.1.1. $\Lambda = \bigcup_{i=0}^{\infty} \Lambda_i$ and $\Lambda_i . \Lambda_j \subset \Lambda_{i+j}$.

2.1.2. The image of the morphism $\mathcal{O}_X \to \Lambda$ is equal to Λ_0.

2.1.3. The image of $f^{-1}(\mathcal{O}_S)$ in $\mathcal{O}_X$ is contained in the center of Λ.

2.1.4. The left and right $\mathcal{O}_X$-module structures on $\mathrm{Gr}_i(\Lambda) \stackrel{\text{def}}{=} \Lambda_i/\Lambda_{i-1}$ are equal.

2.1.5. The sheaves of $\mathcal{O}_X$-modules $\mathrm{Gr}_i(\Lambda)$ are coherent.

2.1.6. The sheaf of graded $\mathcal{O}_X$-algebras $\mathrm{Gr}(\Lambda) \stackrel{\text{def}}{=} \bigoplus_{i=0}^{\infty} \mathrm{Gr}_i(\Lambda)$ is generated by $\mathrm{Gr}_1(\Lambda)$ in the sense that the morphism of sheaves

$$\mathrm{Gr}_1(\Lambda) \otimes_{\mathcal{O}_X} \ldots \otimes_{\mathcal{O}_X} \mathrm{Gr}_1(\Lambda) \to \mathrm{Gr}_i(\Lambda)$$

is surjective.

This definition is an abstraction of the well-known properties of the sheaf of rings of differential operators $\mathcal{D}_{X/S}$, as presented in [Be] and elsewhere.

In this section we will point out some elementary properties and give some examples. The first remark is that Λ has two structures of sheaf of $\mathcal{O}_X$-modules, coming from multiplication on the left and on the right. On the associated graded ring $\mathrm{Gr}(\Lambda)$ these two structures are, by hypothesis, equal.

Lemma **2.2.** — *The subsheaves Λ_i are coherent sheaves of $\mathcal{O}_X$-modules under both the left and the right structures. The full sheaf Λ is quasicoherent for both structures.*

Proof. — Treat, for example, the left module structure. By induction on i, we may suppose that Λ_{i-1} is coherent. Then Λ_i is an extension of $\mathrm{Gr}_i(\Lambda)$ by Λ_{i-1}, both of which are coherent. An extension of coherent $\mathcal{O}_X$-modules is again coherent, so Λ_i is coherent. The sheaf Λ is a union of coherent subsheaves, so it is quasicoherent. The case of the right module structure is the same. □

Note that Λ_0 is a sheaf of quotient rings of $\mathcal{O}_X$, so there is a closed subscheme $X_0 \subset X$ with $\mathcal{O}_{X_0} = \Lambda_0$. Suppose $U \subset X$ is an affine open subset. If $a \in \Lambda_0(U)$ and $v \in \Lambda_1(U)$ then the commutator $[v, a] = va - av$ is in $\Lambda_0(U)$ (by 2.1.4). One has the formula

$$[v, ab] = a[v, b] + [v, a]\, b,$$

so the function $a \mapsto [v, a]$ is a derivation. It acts trivially on sections of $\mathcal{O}_S$ (by 2.1.3), so it is a derivation of $\mathcal{O}_{X_0}$ over $\mathcal{O}_S$. The derivation is trivial if $v \in \Lambda_0$, so it only depends on the class of v in $\mathrm{Gr}_1(\Lambda)$. By the universal property characterizing the module of differentials $\Omega^1_{X_0/S}$, there is a unique morphism

$$\sigma : \mathrm{Gr}_1(\Lambda) \to \mathrm{Hom}(\Omega^1_{X_0/S}, \mathcal{O}_{X_0})$$

such that $\sigma(v)\ (da) = [v, a]$. We call σ the *symbol* of Λ. Note that this formula can be rewritten to read

$$va = av + \sigma(v)\ (da);$$

thus it can be used to pass elements of Λ_0 through elements of Λ_1 at the expense of adding additional terms of lower order.

Suppose $S = \mathrm{Spec}(A)$ and $X = \mathrm{Spec}(B)$ are affine. Then Λ is the quasicoherent sheaf associated to the B-algebra $L = \Lambda(X)$, and there is a filtration $L_0 \subset L_1 \subset \ldots \subset L$. These satisfy properties analogous to the previous properties for Λ:

2.3.1. $L = \bigcup_{i=0}^{\infty} L_i$ and $L_i . L_j \subset L_{i+j}$.

2.3.2. The image of B in L is equal to L_0.

2.3.3. The image of A is contained in the center of L.

2.3.4. The resulting left and right B-module structures on $\mathrm{Gr}_i(L) = L_i/L_{i-1}$ are equal.

2.3.5. The B-modules $\mathrm{Gr}_i(L)$ are finitely generated.

2.3.6. The B-algebra $\mathrm{Gr}(L) = \bigoplus_{i=0}^{\infty}$ is generated by $\mathrm{Gr}_1(L)$: the morphism

$$\mathrm{Gr}_1(L) \otimes_B \ldots \otimes_B \mathrm{Gr}_1(L) \to \mathrm{Gr}_i(L)$$

is surjective.

Conversely, a B-algebra with filtration satisfying these properties gives rise to a ring of differential operators.

Lemma **2.4.** — *Given a* B-*algebra* L *with filtration satisfying* 2.3.1-2.3.6, *let* Λ_l *and* Λ_r *denote the associated sheaves of* $\mathcal{O}_X$-*modules obtained by localizing with respect to the left and right structures. There is a unique isomorphism* $\Lambda_l \cong \Lambda_r$ *of sheaves compatible with the restriction maps* $L \to \Lambda_l(U)$ *and* $L \to \Lambda_r(U)$ *and compatible with the left and right* B-*module structures of* $\Lambda_l(U)$ *and* $\Lambda_r(U)$. *Identify* Λ_l *and* Λ_r *using this isomorphism, and call the resulting sheaf* Λ. *There is a unique morphism* $\Lambda \otimes_{\mathcal{O}_X} \Lambda \to \Lambda$ *of left and right* $\mathcal{O}_X$-*modules extending the multiplication of* L. *The left and right subsheaves associated to* L_i *are equal, giving a filtration*

$$\Lambda_0 \subset \Lambda_1 \subset \ldots \subset \Lambda.$$

Finally, the sheaf of $\mathcal{O}_X$-*algebras* Λ, *provided with this filtration, satisfies the hypotheses* 2.1.1-2.1.6 *for being a sheaf of rings of differential operators.*

Proof. — This proof comes from the one given by J. Bernstein in his course [Be] for the sheaf of rings $\mathcal{D}_X$. Suppose $M \subset B$ is a multiplicative subset. The left localisation $M^{-1} L$ is a left M^{-1} B-module and a right B-module. Note that $M^{-1} L$ is the union of $M^{-1} L_i$. We claim that M acts invertibly on the right, namely that for any $v \in M^{-1} L$ and $m \in M$, $vm = 0 \Rightarrow v = 0$ and there exists $u \in M^{-1} L$ such that $v = um$. Suppose $v \in M^{-1} L_i$, and write $v = n^{-1} w$ for $w \in L_i$. Write w as a sum of terms of the form $w_1 w_2 \ldots w_i + z$ where $w_j \in L_1$ and $z \in L_{i-1}$. Then

$$w_1 \ldots w_i m = (w_1 \ldots w_{i-1}) (mw_i + \sigma(w_i) (dm)).$$

By induction we obtain $w_1 \ldots w_i m = mw_1 \ldots w_i + z'$ where $z' \in L_{i-1}$. Thus $m^{-1} vm = (mn)^{-1} wm = (mn)^{-1} mw + (mn)^{-1} y$ with $y \in L_{i-1}$. Hence

$$m^{-1} vm - v \in M^{-1} L_{i-1}.$$

Now suppose that $vm = 0$ in $M^{-1} L$ (and hence also in $M^{-1} L_i$). By the previous formula we get $v \in M^{-1} L_{i-1}$. Continuing by induction on i we obtain $v = 0$. Furthermore, we have $v = y + m^{-1} vm$ where $y \in M^{-1} L_{i-1}$. By induction we may write $y = xm$, hence $v = (x + m^{-1} v) m$. This completes the proof of the claim.

The fact that M acts invertibly on the right implies that the natural map $M^{-1} L \to M^{-1} LM^{-1}$ is an isomorphism. The same argument works for the right localisation LM^{-1}, and we obtain the isomorphisms

$$LM^{-1} \cong M^{-1} LM^{-1} \cong M^{-1} L.$$

Note that the previous argument also gives

$$L_i M^{-1} \cong M^{-1} L_i M^{-1} \cong M^{-1} L_i.$$

The sheaf associated to L by left localisation is given by $\Lambda_l(U) = M^{-1} L$ where M is the multiplicative system of elements of B which are invertible on U. The sheaf associated to L by right localisation is $\Lambda_r(U) = LM^{-1}$. The above isomorphisms give $\Lambda_l(U) \cong \Lambda_r(U)$. This is compatible with the filtrations. We obtain the desired isomorphisms (the statements about uniqueness are left to the reader). Finally the multiplication in L gives

$$M^{-1} L \otimes_B LM^{-1} \to M^{-1} LM^{-1}.$$

If M acts invertibly on the right of the left hand side in the tensor product and on the left in the right hand side of the tensor product, then the tensor product is the same as that taken over $M^{-1} B$. We obtain the desired multiplication $\Lambda \otimes_{\mathcal{O}_X} \Lambda \to \Lambda$. The required properties follow from 2.3.1-2.3.6. $\square$

We define base change for Λ. Suppose $g : S' \to S$ is a morphism of noetherian schemes over **C**. Let $X' = X \times_S S'$, and let p_1 and p_2 denote the two projections. Set

$$\Lambda'_l = \mathcal{O}_{X'} \otimes_{p_1^{-1} \mathcal{O}_X} p_1^{-1}(\Lambda)$$

and $$\Lambda'_r = p_1^{-1}(\Lambda) \otimes_{p_1^{-1} \mathcal{O}_X} \mathcal{O}_{X'},$$

where the first formula uses the left module structure of Λ and the second formula uses the right module structure.

Lemma **2.5.** — *There exists a unique isomorphism of sheaves of rings* $\Lambda'_l \cong \Lambda'_r$ *which restricts to the identity on* $p_1^*(\Lambda)$ *and* $p_2^*(\mathcal{O}_{S'})$.

Proof. — It suffices to prove this in the case where X, S and S′ are affine. Keep the same notation as before and let $S' = \mathrm{Spec}(A')$. Put $B' = B \otimes_A A'$, so $X' = \mathrm{Spec}(B')$. Note that Λ'_l and Λ'_r are both equal to the sheaves associated to the B′-algebras $L' = L \otimes_A A' = A' \otimes_A L$. The left and right tensor products are equal in this case, since the image of A is in the center of L (thus the left and right A-module structures are the same). With the previous discussion of the affine case, this gives the desired isomorphism. □

Let Λ' denote either Λ'_l or Λ'_r, each identified with the other via the isomorphism given in the lemma. Set

$$\Lambda'_{i,l} = \mathcal{O}_{X'} \otimes_{p_1^{-1}\mathcal{O}_X} p_1^{-1}(\Lambda_i),$$

and

$$\Lambda'_{i,r} = p_1^{-1}(\Lambda_i) \otimes_{p_1^{-1}\mathcal{O}_X} \mathcal{O}_{X'}.$$

Lemma **2.6.** — *The images of* $\Lambda'_{i,l}$ *and* $\Lambda'_{i,r}$ *in* Λ' *are equal. Call these images* Λ'_i. *Then* Λ' *with the filtration given by the* Λ'_i *satisfies the conditions for a sheaf of rings of differential operators on* X′ *with respect to* S′.

Proof. — Again, we may reduce to the affine case; keep the previous notation. Let $L' = L \otimes_A A'$ and let L'_i be the image of $L_i \otimes_A A'$ in L'. The L'_i are finitely generated; since B′ is a noetherian **C**-algebra, this implies that L'_i is finitely presented. Thus L′ with filtration L'_i satisfies the conditions listed above (the remaining conditions are easily verified). The image of $\Lambda'_{i,l}$ (resp. $\Lambda'_{i,r}$) in Λ' is equal to the subsheaf associated to L'_i by left (resp. right) localisation. The proof of Lemma 2.4 shows that the left and right localisations of the L'_i are equal. Lemma 2.4 implies that Λ' with filtration given by Λ'_i satisfies the conditions for a sheaf of rings of differential operators. □

Λ-modules

Suppose $f: X \to S$ are as above, and Λ is a sheaf of rings of differential operators. For the purposes of this paper we define a *Λ-module* to be a sheaf $\mathcal{E}$ of left Λ-modules on X such that $\mathcal{E}$ is coherent with respect to the resulting structure of sheaf of $\mathcal{O}_X$-modules.

We note how this behaves under base change.

Lemma **2.7.** — *Suppose* $S' \to S$ *is a morphism of noetherian schemes over* **C**. *Suppose* $\mathcal{E}$ *is a Λ-module on* X. *Then the pullback* $\mathcal{E}' = p_1^*(\mathcal{E})$ *has a natural structure of* Λ'*-module on* X′.

Proof. — This structure will be characterised as the unique one compatible with the existing $p_1^{-1}(\Lambda)$-module structure of $p_1^{-1}(\mathcal{E})$ and the $\mathcal{O}_{X'}$-module structure of $\mathcal{E}'$. With this in mind, it suffices to treat the case where X, S and S′ are affine. Keeping the same notation as above, the Λ-module $\mathcal{E}$ is the sheafification of an L-module E such that E is finitely presented over B. The pullback E' is the sheafification of $E \otimes_A A'$, and this has a natural structure of $L' = L \otimes_A A'$-module. □

Lemma **2.8.** — *Suppose* $U \subset X$ *is an affine subset. Then* $\Lambda(U)$ *is generated as a ring by* $\Lambda_1(U)$.

Proof. — To prove that $\Lambda(U)$ is generated by $\Lambda_1(U)$ it suffices to prove that $\mathrm{Gr}(\Lambda(U))$ is generated as a $\Lambda_0(U)$-algebra by $\mathrm{Gr}_1(\Lambda(U))$. Since U is affine, $\mathrm{Gr}(\Lambda)(U) = \mathrm{Gr}(\Lambda(U))$. Our hypothesis says that

$$\mathrm{Gr}_1(\Lambda) \otimes_{\Lambda_0} \dots \otimes_{\Lambda_0} \mathrm{Gr}_1(\Lambda) \to \mathrm{Gr}_i(\Lambda)$$

is a surjective map of sheaves. But it is a map of coherent sheaves of $\mathcal{O}_X$-modules, so the fact that U is affine implies that the map

$$\mathrm{Gr}_1(\Lambda)(U) \otimes_{\Lambda_0(U)} \dots \otimes_{\Lambda_0(U)} \mathrm{Gr}_1(\Lambda)(U) \to \mathrm{Gr}_i(\Lambda)(U)$$

is surjective. This implies that $\mathrm{Gr}(\Lambda(U))$ is generated by $\mathrm{Gr}_1(\Lambda(U))$. □

Corollary **2.9.** — *Suppose* $\mathcal{E}$ *is a* Λ*-module and* $\mathcal{F} \subset \mathcal{E}$ *is a coherent subsheaf of* $\mathcal{O}_X$*-modules such that* $\Lambda_1 . \mathcal{F} \subset \mathcal{F}$. *Then* $\mathcal{F}$ *has a unique structure of* Λ*-module compatible with the* Λ*-module structure of* $\mathcal{E}$.

Proof. — If this structure exists, it is unique since $\mathcal{F}(U) \subset \mathcal{E}(U)$ for any open set U. To show that it exists, we must show that if $x \in \mathcal{F}(U)$ and $v \in \Lambda(U)$ then $vx \in \mathcal{F}(U)$. It suffices to show this locally, so we may assume that U is affine. By the lemma, $\Lambda(U)$ is generated by $\Lambda_1(U)$. Thus v is a sum of terms of the form $v_1 \dots v_k$ with $v_j \in \Lambda_1(U)$. By hypothesis, $v_k x \in \mathcal{F}(U)$, and continuing inductively we obtain $v_1 \dots v_k x \in \mathcal{F}(U)$. Thus $vx \in \mathcal{F}(U)$. □

The split almost polynomial case

Suppose that X is flat over S. We say that a sheaf of rings of differential operators Λ on X over S is *almost polynomial* if $\Lambda_0 = \mathcal{O}_X$, $\mathrm{Gr}_1(\Lambda)$ is locally free over $\mathcal{O}_X$, and the graded ring $\mathrm{Gr}(\Lambda)$ is the symmetric algebra on $\mathrm{Gr}_1(\Lambda)$. We say that Λ is *polynomial* if, furthermore, $\Lambda \cong \mathrm{Gr}(\Lambda)$.

The property that Λ is almost polynomial is preserved under base change. In fact, if $f : S' \to S$ is a morphism, and if Λ' is the sheaf of rings of differential operators on $X' = X \times_S S'$ obtained by base change, then $\mathrm{Gr}(\Lambda') = f^* \mathrm{Gr}(\Lambda)$. To see this, note that since X is flat over S, all of the $\mathrm{Gr}_i(\Lambda)$ are flat over S. Thus the maps from $f^*(\Lambda_i)$ to $f^*(\Lambda_j)$ for $j > i$ are injective, so $\Lambda'_i = f^*(\Lambda_i)$ and f^* preserves the associated graded.

A *split almost polynomial sheaf of rings of differential operators* Λ is an almost polynomial sheaf of rings of differential operators Λ together with a morphism

$$\zeta : \mathrm{Gr}_1(\Lambda) \to \Lambda_1$$

of left $\mathcal{O}_X$-modules splitting the projection from Λ_1 to $\mathrm{Gr}_1(\Lambda)$.

We will give an explicit description of the split almost polynomial sheaves of rings Λ in terms of triples (H, δ, γ). Before giving the statement of the relation with rings Λ, we describe the objects H, δ and γ, and give some properties and constructions.

2.10.1. Suppose H is a locally free sheaf of $\mathcal{O}_X$-modules on X.

2.10.2. Suppose $\delta : \mathcal{O}_X \to H$ is a derivation over $\mathcal{O}_S$.

2.10.3. Put $\Lambda_1^{H,\delta} \stackrel{\text{def}}{=} H^* \oplus \mathcal{O}_X$, and $\Lambda_0^{H,\delta} = \mathcal{O}_X \subset \Lambda_1^{H,\delta}$. Give $\Lambda_1^{H,\delta}$ the left $\mathcal{O}_X$-module structure of the direct sum, and a twisted right $\mathcal{O}_X$-module structure by the formula

$$(\lambda, a)\, b = (b\lambda, a + \lambda(\delta(b))).$$

Note that the left and right $\mathcal{O}_X$-module structures on $\Lambda_1^{H,\delta}/\Lambda_0^{H,\delta} \cong H^*$ are the same.

2.10.4. Put $K \stackrel{\text{def}}{=} \wedge^2 H$ with the same left and right module structure.

2.10.5. Suppose $\gamma : H \to K \otimes_{\mathcal{O}_X} \Lambda_1^{H,\delta}$ is a morphism of right $\mathcal{O}_X$-modules such that the composition with the projection into $K \otimes_{\mathcal{O}_X} H^*$ is equal to the canonical map $H \to \wedge^2 H \otimes_{\mathcal{O}_X} H^*$.

2.10.6. From this data define a bracket

$$\{\ ,\ \}_\gamma : H^* \otimes_{\mathbf{C}} H^* \to H^*$$

by the formula

$$\{a, b\}_\gamma . u = (a \wedge b).(\gamma(u)) - (\delta(a.u)).b + (\delta(b.u)).a - (a.u)\, b + (b.u)\, a,$$

where the periods indicate pairings between dual spaces. Here a and b are sections of H^*, u is a section of H, and the right hand side, *a priori* a section of $\Lambda_1^{H,\delta}$, is in fact a section of $\Lambda_0^{H,\delta} = \mathcal{O}_X$. The formula is $\mathcal{O}_X$-linear in u so it defines a section $\{a, b\}_\gamma$ of H^*.

2.10.7. The bracket is antisymmetric and satisfies a Leibniz formula:

$$\{a, yb\}_\gamma = (\delta(y).a)\, b + y\{a, b\}_\gamma.$$

2.10.8. Assume that γ has the property that the resulting bracket satisfies the Jacobi identity:

$$\{\{a, b\}_\gamma, c\}_\gamma + \{\{b, c\}_\gamma, a\}_\gamma + \{\{c, a\}_\gamma, b\}_\gamma = 0.$$

Remark. — The data of the bracket satisfying properties 2.10.7 is equivalent to the data of the map γ satisfying the properties 2.10.5. The map γ can be recovered from the bracket using the formula 2.10.6.

Theorem **2.11.** — *Suppose* (Λ, ζ) *is a split almost polynomial sheaf of rings of differential operators. Then there exists a unique triple* (H, δ, γ) *satisfying* 2.10.1-2.10.8 *and isomorphism* $\eta : \mathrm{Gr}_1(\Lambda) \cong H^*$, *such that* δ *corresponds to the symbol and the bracket* $\{\ ,\ \}_\gamma$ *gives the commutator of elements under the isomorphism* $\Lambda_1 \cong H^* \oplus \mathcal{O}_X = \Lambda_1^{H,\delta}$ *given by the splitting.*

Suppose (H, δ, γ) *is a triple satisfying properties* 2.10.1-2.10.8. *Then there is a split almost polynomial sheaf of rings of differential operators* $(\Lambda^{H,\delta,\gamma}, \zeta)$ *together with isomorphism* $\eta : \mathrm{Gr}_1(\Lambda^{H,\delta,\gamma}) \cong H^*$ *such that* δ *corresponds to the symbol and* γ *corresponds to the commutator of elements under the isomorphism* $\Lambda_1^{H,\delta,\gamma} \cong H^* \oplus \mathcal{O}_X = \Lambda_1^{H,\delta}$ *given by the splitting. If* (Λ, ζ) *is any other split almost polynomial sheaf of rings of differential operators corresponding to* (H, δ, γ) *under the previous paragraph, then there is a unique isomorphism* $\Lambda \cong \Lambda^{H,\delta,\gamma}$ *compatible with the splittings and the isomorphisms* η.

Proof. — For the first paragraph, set $H = \mathrm{Gr}_1(\Lambda)^*$. The derivation δ comes from the symbol defined before. The splitting gives an isomorphism $\Lambda_1 \cong H^* \oplus \mathcal{O}_X$ compatible with the direct sum structure of left $\mathcal{O}_X$-module and the twisted structure of right $\mathcal{O}_X$-module defined in 2.10.3. To define the bracket, let A denote the image of $\Lambda_1 \otimes_{\mathcal{O}_X} H^*$ in Λ_2. If a and b are in $H^* \subset \Lambda_1$ then ab and ba are contained in A. Thus the commutator $ab - ba$ is an element of $A \cap \Lambda_1 = H^*$. Define the bracket $\{a, b\}_\gamma$ to be equal to this commutator. This satisfies the Leibniz rule, so it comes from a map γ, and it satisfies the Jacobi identity. This gives the required triple (H, δ, γ).

For the statement of the second paragraph, it suffices to treat the question locally on X, so we may assume that H is a free $\mathcal{O}_X$-module. Fix a basis $\lambda_1, \ldots, \lambda_k$ for H^*. Define $\Lambda^{H,\delta,\gamma}$ to be the quasicoherent left $\mathcal{O}_X$-module freely generated by the monomials $\lambda_1^{i_1} \ldots \lambda_k^{i_k}$. Then use the derivation δ to define the commutators of λ_i with sections of $\mathcal{O}_X$, and use the bracket $\{\ ,\ \}_\gamma$ to define the commutators of λ_i and λ_j. Use these commutators to define the multiplication law on $\Lambda^{H,\delta,\gamma}$ (they give the rules for passing things through each other in order to rearrange any expression into a sum of monomial products ordered as above). The Leibniz rule 2.10.7 and the Jacobi identity 2.10.8 insure that the multiplication is well defined independent of the rearrangement procedure, and that the ring axioms are satisfied. The associated graded algebra is just a polynomial algebra, since the commutators only introduce terms of lower order. □

Lemma **2.12.** — *Suppose* (H, δ, γ) *is a triple satisfying* 2.10.1-2.10.7. *The transpose of the map* γ *is an inclusion* $\gamma^T : K^* \hookrightarrow \Lambda_1 \otimes_{\mathcal{O}_X} H^*$ *of left* $\mathcal{O}_X$*-modules, such that the composition*

$$K^* \hookrightarrow \Lambda_1 \otimes_{\mathcal{O}_X} H^* \to H^* \otimes_{\mathcal{O}_X} H^*$$

is equal to the canonical isomorphism $K^* \cong \wedge^2 H^*$. *The image of* γ^T *is the kernel of the map* $\Lambda_1 \otimes_{\mathcal{O}_X} H^* \to \Lambda_2$. *Let*

$$w_1 : \Lambda \to H \otimes_{\mathcal{O}_X} \Lambda$$

denote the transpose of the multiplication $\Lambda \otimes_{\mathcal{O}_X} H^* \to \Lambda$. *Let*

$$w_2 : K \otimes_{\mathcal{O}_X} \Lambda_1 \otimes_{\mathcal{O}_X} \Lambda \to K \otimes_{\mathcal{O}_X} \Lambda$$

denote the map given by multiplication of the last two factors. The composition $w_2 \gamma w_1$ *is equal to zero.*

Proof. — The transpose is defined by the formula $\gamma^{\mathrm{T}}(u).a = u.\gamma(a)$ for $u \in \mathrm{K}^*$ and $a \in \mathrm{H}$. The composition of γ^{T} with the projection into $\mathrm{H}^* \otimes_{\mathcal{O}_{\mathrm{X}}} \mathrm{H}^*$ is the transpose of the composition of γ with the projection into $\mathrm{K} \otimes_{\mathcal{O}_{\mathrm{X}}} \mathrm{H}^*$. This second composition is, by hypothesis, the canonical map, so the composition of γ^{T} with the projection is the canonical inclusion. In particular, this implies that γ^{T} is injective. For the last two statements, it is convenient to calculate in terms of a local frame $\lambda_1, \ldots, \lambda_k$ for H^*, the dual frame $h_1, \ldots, h_k$ for H, and Kronecker's delta. If we write

$$\{\lambda_i, \lambda_j\}_\gamma = \sum_{\ell=1}^{k} \Gamma_{ij}^{\ell} \lambda_\ell,$$

then the map γ is

$$\gamma(h_\ell) = \sum_{i,j} (h_i \wedge h_j)(\Gamma_{ij}^{\ell} + \delta_{i\ell}\lambda_j - \delta_{j\ell}\lambda_i).$$

We get

$$\gamma^{\mathrm{T}}(\lambda_i \wedge \lambda_j) = \sum_{\ell} \Gamma_{ij}^{\ell} \otimes \lambda_\ell + \lambda_j \otimes \lambda_i - \lambda_i \otimes \lambda_j.$$

From the description of Λ given in the second part of the proof of Theorem 2.11, the image of γ^{T} is the kernel of the map $\Lambda_1 \otimes_{\mathcal{O}_{\mathrm{X}}} \mathrm{H}^* \to \Lambda_2$. Finally, we have $w_1(u) = \Sigma_i h_i \otimes \lambda_i u$, so

$$\begin{aligned} w_2 \gamma w_1(u) &= \sum_{i,j,\ell} (h_i \wedge h_j) \otimes (\Gamma_{ij}^{\ell} + \delta_{i\ell}\lambda_j - \delta_{j\ell}\lambda_i)\lambda_\ell u \\ &= \sum_{i,j} (h_i \wedge h_j) \otimes (\sum_{\ell} \Gamma_{ij}^{\ell}\lambda_\ell + \lambda_j\lambda_i - \lambda_i\lambda_j) u \\ &= 0. \quad \square \end{aligned}$$

Suppose that Λ is a split almost polynomial sheaf of rings of differential operators corresponding to a triple $(\mathrm{H}, \delta, \gamma)$ as in Theorem 2.11. Suppose $\mathscr{E}$ is a Λ-module. We obtain a morphism of sheaves

$$\varphi : \mathscr{E} \to \mathrm{H} \otimes_{\mathcal{O}_{\mathrm{X}}} \mathscr{E}$$

by the formula

$$\lambda.\varphi(e) = \zeta(\lambda)(e)$$

for $\lambda \in \mathrm{H}^*$. Recall that the splitting $\zeta : \mathrm{H}^* \to \Lambda_1$ is a morphism of left $\mathcal{O}_{\mathrm{X}}$-modules, so

$$(a\lambda).\varphi(e) = \zeta(a\lambda)(e) = a\zeta(\lambda)(e) = a(\lambda.\varphi(e))$$

for sections e of $\mathscr{E}$ and a of $\mathcal{O}_{\mathrm{X}}$. Thus $\varphi(e)$ gives an $\mathcal{O}_{\mathrm{X}}$-linear function from H^* to $\mathscr{E}$, hence an element of $\mathrm{H} \otimes_{\mathcal{O}_{\mathrm{X}}} \mathscr{E}$ as required. This morphism φ satisfies the *Leibniz formula* with respect to the derivation δ, that is $\varphi(ae) = a\varphi(e) + \delta(a) \otimes e$ for sections e of $\mathscr{E}$ and a of $\mathcal{O}_{\mathrm{X}}$.

Any such map φ satisfying the Leibniz formula determines a map

$$\varphi' : \Lambda_1 \otimes_{\mathcal{O}_{\mathrm{X}}} \mathscr{E} \to \mathscr{E}$$

in a natural way. Composing with γ gives a map

$$\varphi'\gamma : \mathrm{H} \otimes_{\mathcal{O}_{\mathrm{X}}} \mathscr{E} \to \mathrm{K} \otimes_{\mathcal{O}_{\mathrm{X}}} \mathscr{E}.$$

If $(\mathscr{E}, \varphi)$ comes from a Λ-module then the composed map $\varphi' \gamma \varphi : \mathscr{E} \to \mathrm{K} \otimes_{\mathscr{O}_\mathrm{X}} \mathscr{E}$, is equal to zero. This follows from the last statement of the previous lemma, by tensoring with $\mathscr{E}$ over Λ. The following lemma gives a converse.

Lemma **2.13.** — *Suppose $\mathscr{E}$ is a coherent sheaf of $\mathscr{O}_\mathrm{X}$ modules and*

$$\varphi : \mathscr{E} \to \mathrm{H} \otimes_{\mathscr{O}_\mathrm{X}} \mathscr{E}$$

is a morphism of sheaves satisfying the Leibniz rule $\varphi(ae) = a\varphi(e) + \delta(a) \otimes e$. This gives a morphism $\varphi' : \Lambda_1 \otimes_{\mathscr{O}_\mathrm{X}} \mathscr{E} \to \mathscr{E}$. There exists a structure of Λ-module for $\mathscr{E}$ which gives rise to the map φ if and only if the composition

$$\varphi' \gamma \varphi : \mathscr{E} \to \mathrm{K} \otimes_{\mathscr{O}_\mathrm{X}} \mathscr{E}$$

is equal to zero; and the Λ-module structure is uniquely determined.

Proof. — By Theorem 2.11, we may assume that $\Lambda = \Lambda^{\mathrm{H}, \delta, \gamma}$. The question is local, so we may assume that H is free and that Λ is constructed by using the relations given by commutators coming from δ and γ. The map φ gives an action of H^* on $\mathscr{E}$. The Leibniz rule insures that this is compatible with the commutators coming from δ, while the condition $\varphi' \gamma \varphi = 0$ is equivalent to the condition that this action is compatible with the commutators coming from $\{\ ,\ \}_\gamma$ (see the proof of the previous lemma). The Λ-module structure is uniquely determined by the action of Λ_1. □

By this lemma, if (Λ, ζ) is a split almost polynomial sheaf of rings of differential operators, then the notion of Λ-module is the same as the notion of a pair $(\mathscr{E}, \varphi)$ where $\mathscr{E}$ is a coherent sheaf of $\mathscr{O}_\mathrm{X}$-modules and $\varphi : \mathscr{E} \to \mathrm{H} \otimes_{\mathscr{O}_\mathrm{X}} \mathscr{E}$ is a morphism satisfying Leibniz's rule, such that the composition $\varphi' \gamma \varphi$ is equal to zero. This last condition may be paraphrased as saying that φ gives an action of H^* on $\mathscr{E}$ compatible with the bracket $\{\ ,\ \}_\gamma$ (which is the commutator in Λ).

The main examples

Vector bundles with integrable connection. — Suppose $\mathrm{X} \to \mathrm{S}$ is a smooth morphism. The main example of a split almost polynomial sheaf of rings of differential operators is the sheaf of differential operators itself, $\Lambda = \mathscr{D}_{\mathrm{X/S}}$ [Be]. The filtration is Λ_i equal to the sheaf of differential operators of order $\leqslant i$ on X/S —the associated graded ring $\mathrm{Gr}(\Lambda)$ is isomorphic to the symmetric algebra $\mathrm{Sym}^\bullet(\mathrm{T(X/S)})$ [Be]. We have $\mathrm{H} = \Omega^1_{\mathrm{X/S}}$, δ is the canonical derivation, and the bracket is just the commutator of vector fields. The map

$$\gamma : \Omega^1_{\mathrm{X/S}} \to \Omega^2_{\mathrm{X/S}} \otimes_{\mathscr{O}_\mathrm{X}} \Lambda_1$$

may be written in local coordinates $x_1, \ldots, x_k$ as

$$\gamma(u) = du \otimes 1 + \sum_{i=1}^{k} (u \wedge dx_i) \otimes \frac{\partial}{\partial x_i}.$$

The $\mathcal{O}_X$-coherent Λ-modules are automatically locally free over $\mathcal{O}_X$ [Be]. A Λ-module thus consists of a locally free sheaf $\mathcal{E}$ with a *connection* $\nabla : \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{O}_X} \Omega^1_{X/S}$ satisfying Leibniz's rule and the *integrability condition* $\nabla' \gamma \nabla = 0$, habitually written $\nabla^2 = 0$. If necessary, use the superscript Λ^{DR} to denote this example.

Higgs bundles. — The other main example we will use is

$$\Lambda^{\mathrm{Higgs}} \stackrel{\mathrm{def}}{=} \mathrm{Gr}(\Lambda^{DR}) = \mathrm{Sym}^{\bullet}(T(X/S)).$$

Again $H = \Omega^1_{X/S}$ but the symbol is zero and γ is defined as above without the differential. More generally, if V is any locally free sheaf, we can put $\Lambda^V = \mathrm{Sym}^{\bullet}(V^*)$. The Λ^V-modules are *Hitchin pairs* $(\mathcal{E}, \varphi)$ where $\mathcal{E}$ is a coherent sheaf of $\mathcal{O}_X$-modules and $\varphi : \mathcal{E} \to V \otimes_{\mathcal{O}_X} \mathcal{E}$ is a morphism with $\varphi' \gamma \varphi = 0$, a condition habitually written $\varphi \wedge \varphi = 0$. This notion was introduced for locally free sheaves of rank one in [Hi1]. The condition $\varphi \wedge \varphi = 0$ was introduced in [Si2]. In case $V = \Omega^1_{X/S}$ and $\mathcal{E}$ is locally free, we call these *Higgs bundles*. We sometimes denote Λ^{Higgs} by Λ^{Dol}.

Other examples

We mention here several other examples. These will not be treated further in Part II, but some might be treated in future papers, and some have already been treated by other authors. The possibility of considering these examples was the reason for rewriting the construction of moduli spaces in the general context of a sheaf of rings of operators Λ. Our constructions of the next two sections give moduli spaces for semistable Λ-modules in all cases.

Connections along a foliation. — Suppose $\mathcal{F} \subset X$ is a smooth holomorphic foliation of a smooth projective variety X. Define $\Lambda^{\mathcal{F}}$ to be the sheaf of rings of differential operators along the leaves of the foliation. If $T(\mathcal{F}) \subset T(X)$ is the tangent bundle to the foliation, then $\Lambda^{\mathcal{F}}$ is a split almost polynomial sheaf of rings with $H \cong T^*(\mathcal{F})$ and K giving the usual integrability condition. The $\Lambda^{\mathcal{F}}$-modules are sheaves E provided with an integrable connection $\nabla : E \to E \otimes_{\mathcal{O}_X} T^*(\mathcal{F})$ along the leaves of the foliation.

Deformation to the associated graded. — Suppose Λ is a split almost polynomial sheaf of rings of operators on X over S. Let τ denote the linear coordinate on $\mathbf{A}^1$. Define Λ^R on $X \times \mathbf{A}^1$ over $S \times \mathbf{A}^1$ to be the subsheaf of $p_1^*(\Lambda)$ generated by sections of the form $\Sigma \tau^i \lambda_i$ for λ_i sections of Λ_i. This is again a split almost polynomial sheaf of rings of operators. There is an action of $\mathbf{C}^*$ covering the action on $\mathbf{A}^1$. For any $t_0 \neq 0$, the fiber $\Lambda^R|_{X \times \{t_0\}}$ is naturally isomorphic to Λ. On the other hand, $\Lambda^R|_{X \times \{0\}}$ is isomorphic to the associated graded $\mathrm{Gr}(\Lambda)$, which is polynomial. This was the reason for our choice of terminology " almost polynomial ".

This construction probably also works for any Λ, although it may be that Λ should satisfy some additional hypotheses in order for the resulting Λ^R to satisfy properties 2.1.1-2.1.6.

In the split almost polynomial case, suppose Λ corresponds to $(\mathrm{H}, \delta, \gamma)$; then Λ^{R} corresponds to $(p_1^*(\mathrm{H}), \tau\,\delta, \gamma_\tau)$ where γ_τ is the map corresponding to the bracket $\tau\{\ ,\ \}_\gamma$.

τ-connections. — If we apply the previous constructions to $\Lambda = \Lambda^{\mathrm{DR}}$, we obtain a family Λ^{R} which is a deformation from Λ^{DR} to Λ^{Dol}. A Λ^{R}-module on $\mathrm{X} \times \mathbf{A}^1$ over $\mathrm{S} \times \mathbf{A}^1$ consists of a sheaf E with an operator

$$\nabla : \mathrm{E} \to \Omega^1_{\mathrm{X} \times \mathbf{A}^1/\mathrm{S} \times \mathbf{A}^1} \otimes_{\mathcal{O}_{\mathrm{X} \times \mathbf{A}^1}} \mathrm{E}$$

satisfying the Leibniz rule $\nabla(ae) = a\nabla(e) + \tau\, d(a)\, e$ where t is the coordinate on $\mathbf{A}^1$, as well as the usual integrability condition $\nabla^2 = 0$. This notion of "τ-connection" was suggested to me by P. Deligne; it motivated the more general construction of deformation to the associated graded. The constructions of subsequent chapters will give a moduli space over $\mathrm{S} \times \mathbf{A}^1$ whose fiber over $\mathrm{S} \times \{t_0\}$ is equal to the moduli space for vector bundles with integrable connection ($t_0 \neq 0$), and whose fiber over $\mathrm{S} \times \{0\}$ is the moduli space of Higgs bundles. This moduli space of τ-connections was suggested by Deligne.

Logarithmic connections. — Suppose $\mathrm{X} \to \mathrm{S}$ is smooth and $\mathrm{D} \subset \mathrm{X}$ is a divisor with relative normal crossings [Del]. Let $\Omega^1_{\mathrm{X}/\mathrm{S}}(\log \mathrm{D})$ denote the sheaf of logarithmic differentials, and let $\mathrm{T}(\mathrm{X}/\mathrm{S})(\log \mathrm{D}) \subset \mathrm{T}(\mathrm{X}/\mathrm{S})$ denote the subsheaf of the tangent sheaf dual to $\Omega^1_{\mathrm{X}/\mathrm{S}}(\log \mathrm{D})$. Let $\Lambda^{\mathrm{DR}, \log \mathrm{D}}$ denote the split almost polynomial ring of differential operators with $\mathrm{H} = \Omega^1_{\mathrm{X}/\mathrm{S}}(\log \mathrm{D})$, δ the usual derivation, and with integrability condition equal to the usual one outside of D. Then a $\Lambda^{\mathrm{DR}, \log \mathrm{D}}$-module is the same thing as a sheaf E with logarithmic relative connection

$$\nabla : \mathrm{E} \to \Omega^1_{\mathrm{X}/\mathrm{S}}(\log \mathrm{D}) \otimes_{\mathcal{O}_\mathrm{X}} \mathrm{E}$$

satisfying the usual Leibniz rule and integrability condition. These objects were considered by N. Nitsure in [Ni2]. He constructed the moduli space for them.

Connections on degenerating families. — Suppose S is a smooth curve, $s \in \mathrm{S}$ is a closed point, and suppose $f : \mathrm{X} \to \mathrm{S}$ is a projective morphism which is smooth except over s. Suppose that the total space X is smooth, and that the inverse image $\mathrm{D} = f^{-1}(s)$ is a disjoint union of smooth divisors meeting with normal crossings. Let $\mathrm{T}(\mathrm{X}/\mathrm{S}) \subset \mathrm{T}(\mathrm{X})$ denote the subbundle of tangent vectors which are tangent to the fibers of f. It is equal to the usual relative tangent bundle where f is smooth. Let $\mathrm{H} = \mathrm{T}^*(\mathrm{X}/\mathrm{S})$ denote the dual. Set

$$\Omega^1_{\mathrm{X}/\mathrm{S}}(\log \mathrm{D}) \overset{\mathrm{def}}{=} \frac{\Omega^1_\mathrm{X}(\log \mathrm{D})}{f^*(\Omega^1_\mathrm{S}(\log s))}.$$

This is a locally free sheaf equal to $\Omega^1_{\mathrm{X}/\mathrm{S}} \otimes_{\mathcal{O}_\mathrm{X}} \mathcal{O}_\mathrm{X}(\mathrm{D})$ outside of the singular points of f. We have

$$\mathrm{H} = \Omega^1_{\mathrm{X}/\mathrm{S}}(\log \mathrm{D}) \otimes_{\mathcal{O}_\mathrm{X}} \mathcal{O}_\mathrm{X}(-\mathrm{D}).$$

In particular, H is locally free. Note that there is an H-valued derivation of $\mathcal{O}_X$ corresponding to differentiation in the directions tangent to the fibers. There is an integrability condition equal to the usual one over the smooth points. We obtain a split almost polynomial ring of differential operators $\Lambda^{\mathrm{DR,\,sing}}$. A $\Lambda^{\mathrm{DR,\,sing}}$-module is a sheaf E with operator

$$\nabla : E \to H \otimes_{\mathcal{O}_X} E$$

giving an integrable connection outside of the singularities of f. Over $S - \{s\}$ this $\Lambda^{\mathrm{DR,\,sing}}$ is the same as the relative Λ^{DR}, and the notion of $\Lambda^{\mathrm{DR,\,sing}}$-module is the same as the notion of vector bundle with relative integrable connection. Note that any $\Lambda^{\mathrm{DR,\,sing}}$-module which is flat over S is automatically locally free away from the singularities of f, but need not be locally free at the singular points of f (the crossing points of D).

More generally. — Suppose X and S are smooth and $f: X \to S$ is a morphism. Suppose $V \subset TX$ is a subsheaf of the tangent sheaf such that $df|_V = 0$, such that V is preserved under commutator of vector fields, and such that V is locally free. Then we can set $H = V^*$, and we obtain δ and γ satisfying the axioms 2.10.1-2.10.8. Theorem 2.11 gives a sheaf of rings of differential operators Λ. Most of the above examples are of this form.

3. Semistable Λ-modules

Suppose X is projective over $S = \mathrm{Spec}(\mathbf{C})$, and Λ is a sheaf of rings of differentia operators. A Λ-module $\mathcal{E}$ is of *pure dimension d* if the underlying $\mathcal{O}_X$-coherent sheaf is of pure dimension d. The Hilbert polynomial, the rank, and the slope of $\mathcal{E}$ are defined to be those of the underlying sheaf; we keep the same notations for these as in the previous section.

A Λ-module $\mathcal{E}$ is *p-semistable* (resp. *p-stable*) if it is of pure dimension, and if for any sub-Λ-module $\mathcal{F} \subset \mathcal{E}$ with $0 < r(\mathcal{F}) < r(\mathcal{E})$, there exists an N such that

$$\frac{p(\mathcal{F}, n)}{r(\mathcal{F})} \leqslant \frac{p(\mathcal{E}, n)}{r(\mathcal{E})}$$

(resp. $<$) for $n \geqslant N$. A Λ-module $\mathcal{E}$ is μ*-semistable* (resp. μ*-stable*) if it is of pure dimension and if for any sub-Λ-module $\mathcal{F} \subset \mathcal{E}$ with $0 < r(\mathcal{F}) < r(\mathcal{E})$, we have $\mu(\mathcal{F}) \leqslant \mu(\mathcal{E})$ (resp. $<$). These definitions are the same as in the previous section, except that we only consider subsheaves $\mathcal{F}$ preserved by the action of Λ. As before, p-semistability implies μ-semistability, whereas μ-stability implies p-stability.

Fix the dimension d of support of sheaves we are considering. If $\mathcal{F}$ is a sheaf supported in dimension d, let $\mathcal{F}_{\mathrm{tor}}$ denote the subsheaf of sections supported in dimension $\leqslant d - 1$. It is a coherent subsheaf, itself supported in dimension $d - 1$. If $\mathcal{E}$ is a sheaf of pure dimension d on X, and if $\mathcal{V}$ is a subsheaf, define its *saturation* $\mathcal{V}^{\mathrm{sat}}$ to be the inverse image in $\mathcal{E}$ of the subsheaf $(\mathcal{E}/\mathcal{V})_{\mathrm{tor}} \subset \mathcal{E}/\mathcal{V}$.

If $\mathcal{F}$ is a Λ-module, then $\mathcal{F}_{\text{tor}}$ is a sub-Λ-module. If $\mathcal{E}$ is a Λ-module and $\mathcal{V} \subset \mathcal{E}$ is a sub-Λ-module, then the saturation $\mathcal{V}^{\text{sat}}$ is also preserved by Λ. To see this, note that $(\mathcal{E}/\mathcal{V})$ is a Λ-module, the subsheaf $(\mathcal{E}/\mathcal{V})_{\text{tor}}$ is preserved by Λ, the quotient by this subsheaf is a Λ-module, and the kernel of the map from $\mathcal{E}$ to this quotient is a sub-Λ-module. This kernel is $\mathcal{V}^{\text{sat}}$.

The rank stays the same, and the slope and normalized Hilbert polynomial can only increase upon going to the saturation, so in the definitions of semistability and stability it suffices to consider saturated subsheaves.

Lemma **3.1.** — *Suppose $\mathcal{E}$ is a Λ-module on* X *over* Spec(**C**). *There is a unique filtration (called the p*-Harder-Narasimhan filtration) *by submodules preserved by* Λ,

$$0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \ldots \subset \mathcal{E}_k = \mathcal{E}$$

such that the quotients $\mathcal{E}_i/\mathcal{E}_{i-1}$ are p-semistable Λ-modules, of pure dimension d, with strictly decreasing normalized Hilbert polynomials.

Proof. — The construction is the same as in the well known case of vector bundles. The set of possible slopes of subsheaves of $\mathcal{E}$ is bounded above, so in particular the set of slopes of subsheaves preserved by Λ is bounded above. Let μ_1 be the largest such slope. Then, by Proposition 1.8, the set of saturated subsheaves with slope μ_1 is bounded. In particular, the set of possible normalized Hilbert polynomials of such subsheaves is finite. Let p_1 be the largest normalized Hilbert polynomial (in the lexicographic order by highest coefficient first) of a sub-Λ-module of $\mathcal{E}$ with slope μ_1. If $\mathcal{G}_1$ and $\mathcal{G}_2$ are saturated subsheaves with normalized Hilbert polynomial p_1 then they are both p-semistable. Hence the kernel of the map into $\mathcal{E}$ is a subsheaf of $\mathcal{G}_1 \oplus \mathcal{G}_2$ of normalized Hilbert polynomial less than or equal to p_1. Thus the image, equal to the sum of $\mathcal{G}_1$ and $\mathcal{G}_2$, has normalized Hilbert polynomial greater than or equal to p_1—and by maximality of p_1, it is equal. This image is saturated, otherwise the saturation would be a subsheaf with larger normalized Hilbert polynomial. By repeating this construction, we may find a saturated subsheaf $\mathcal{G}$ of maximal rank among those which are preserved by ∇ and have maximal normalized Hilbert polynomial p_1. This is the first step in the p-Harder-Narasimhan filtration; and the rest of the filtration may be constructed inductively by starting again with the quotient $\mathcal{E}/\mathcal{G}$. $\square$

There is a similar *μ-Harder-Narasimhan filtration* where the quotients are μ-semistable with strictly decreasing slopes. The construction is the same as the previous one (except that one must take the saturation of each subsheaf constructed along the way).

The category of p-semistable Λ-modules of given normalized Hilbert polynomial p_0 is abelian. This is because all morphisms between such objects are strict: given $f: \mathcal{F} \to \mathcal{G}$, the image must have normalized Hilbert polynomial at once $\geqslant p_0$ and $\leqslant p_0$, hence equal to p_0—and the same for the saturation of the image—but then the image and its saturation, having the same Hilbert polynomial, must be equal. The cokernel and kernel of f are torsion-free Λ-modules with normalized Hilbert polynomial p_0.

If $\mathscr{E}$ is a p-stable Λ-module, then $\mathrm{End}(\mathscr{E})$ consists only of scalars. To prove this, note that all endomorphisms of $\mathscr{E}$ except zero are automorphisms. For if $f \in \mathrm{End}(\mathscr{E})$, then the image of f is a subsheaf preserved by Λ of normalized Hilbert polynomial p_0; by stability, the image is either zero or all of $\mathscr{E}$. If the image is all of $\mathscr{E}$, then the rank of the kernel is equal to zero (by additivity of Hilbert polynomials in exact sequences), so f is an automorphism. Therefore $\mathrm{End}(\mathscr{E})$ is a division algebra. But the only division algebra finite over $\mathbf{C}$ is $\mathbf{C}$ itself, so $\mathrm{End}(\mathscr{E})$ consists entirely of scalars.

Jordan equivalence

Suppose that $\mathscr{E}$ is a p-semistable Λ-module. Then there exists a unique filtration by saturated subsheaves preserved by the operators, such that the quotients are direct sums of p-stable Λ-modules with the same normalized Hilbert polynomials. To construct the first step in the filtration, suppose there are two submodules which are direct sums of p-stable Λ-modules of the appropriate normalized Hilbert polynomial. The direct sum of the two is a semisimple object in the abelian category of p-semistable Λ-modules, so their sum inside $\mathscr{E}$ (which is a quotient of the direct sum) is also semisimple, i.e. a direct sum of p-stable Λ-modules of the correct normalized Hilbert polynomial. Hence there is a maximal possible first step of the filtration; take that and proceed inductively to construct the rest of the filtration. Define $\mathrm{gr}(\mathscr{E})$ to be the direct sum of the quotients in this filtration. Say that two p-semistable Λ-modules $\mathscr{E}_1$ and $\mathscr{E}_2$ are *Jordan equivalent* if $\mathrm{gr}(\mathscr{E}_1) \cong \mathrm{gr}(\mathscr{E}_2)$.

Boundedness

We will apply the boundedness results of the previous section to show that the set of μ-semistable Λ-modules with a given Hilbert polynomial is bounded. We first give some preparatory lemmas which will be useful later on, too.

Lemma **3.2.** — *Suppose $\mathscr{E}$ is a Λ-module, of pure dimension d and rank r on* X. *Suppose $\mathscr{F} \subset \mathscr{E}$ is any subsheaf, not necessarily preserved by Λ. Let $\mathscr{G} \subset \mathscr{E}$ be the saturation of the image of the morphism $\Lambda_r \otimes_{\mathcal{O}_X} \mathscr{F} \to \mathscr{E}$. Then $\mathscr{G}$ is a subsheaf preserved by Λ.*

Proof. — Let $\mathscr{E}_i$ denote the image of $\Lambda_i \otimes_{\mathcal{O}_X} \mathscr{F}$ in $\mathscr{E}$. Let $\mathscr{E}_i^{\mathrm{sat}}$ denote its saturation. Let r_i be the rank of $\mathscr{E}_i$ (also equal to the rank of $\mathscr{E}_i^{\mathrm{sat}}$). We have $r_{i+1} \geqslant r_i$, and $0 \leqslant r_i \leqslant r$. Hence, there is an i in the interval $0 \leqslant i \leqslant r$ such that $r_i = r_{i+1}$. In particular, the Hilbert polynomials of $\mathscr{E}_i^{\mathrm{sat}}$ and $\mathscr{E}_{i+1}^{\mathrm{sat}}$ have the same leading coefficient. This implies that the quotient $\mathscr{E}_{i+1}^{\mathrm{sat}}/\mathscr{E}_i^{\mathrm{sat}}$ is supported in dimension $\leqslant d-1$. But this quotient is a subsheaf of $\mathscr{E}/\mathscr{E}_i^{\mathrm{sat}}$, and the fact that $\mathscr{E}_i^{\mathrm{sat}}$ is saturated implies that $\mathscr{E}/\mathscr{E}_i^{\mathrm{sat}}$ has no nonzero sections supported in dimension $\leqslant d-1$. Thus $\mathscr{E}_{i+1}^{\mathrm{sat}} = \mathscr{E}_i^{\mathrm{sat}}$. Now, since $\Lambda_{i+1} = \Lambda_1 . \Lambda_i$, the image of $\Lambda_1 \otimes_{\mathcal{O}_X} \mathscr{E}_i$ is contained in $\mathscr{E}_{i+1}$. In particular, the image of $\Lambda_1 \otimes_{\mathcal{O}_X} \mathscr{E}_i$ is contained in $\mathscr{E}_i^{\mathrm{sat}}$. Suppose u is a section of $\mathscr{E}_i^{\mathrm{sat}}$ over an open set U, and suppose $\lambda \in \Lambda_1(U)$. Let $U' \subset U$ be the complement of the support of $(\mathscr{E}/\mathscr{E}_i)_{tor}$ (so it is the complement of a subset of dimension $\leqslant d-1$). Then $u|_{U'}$ is a section of $\mathscr{E}_i$, so $\lambda u|_{U'}$ is a section of $\mathscr{E}_i^{sat}$. Thus λu projects to a section of $\mathscr{E}/\mathscr{E}_i^{sat}$ supported on $U - U'$, in other words supported in dimension $\leqslant d-1$. As $\mathscr{E}_i^{sat}$ is saturated, this implies that λu projects to zero in $\mathscr{E}/\mathscr{E}_i^{sat}$, in other words $\lambda u \in \mathscr{E}_i^{sat}(U)$. We have shown that $\mathscr{E}_i^{sat}$ is preserved by Λ_1. Corollary 2.9 implies that it is preserved by Λ, so in particular all of the $\mathscr{E}_j$ are contained in $\mathscr{E}_i^{sat}$. As $\mathscr{E}_i \subset \mathscr{E}_r$, this implies that $\mathscr{E}_r^{sat} = \mathscr{E}_i^{sat}$. Thus $\mathscr{E}_r^{sat}$, which is the sheaf called $\mathscr{G}$ in the statement of the lemma, is preserved by Λ. □

Lemma **3.3.** — *Let m denote a number such that* $\mathrm{Gr}_1(\Lambda) \otimes_{\mathcal{O}_X} \mathcal{O}_X(m)$ *is generated by global sections. Then for any μ-semistable Λ-module $\mathscr{E}$ of pure dimension d and rank r, and any subsheaf $\mathscr{F} \subset \mathscr{E}$ (not necessarily preserved by Λ), we have $\mu(\mathscr{F}) \leqslant \mu(\mathscr{E}) + mr$. In other words, $\mu(\mathscr{F}) - \mu(\mathscr{E})$ is bounded above in a way depending only on r and Λ.*

Proof. — We may assume that $\mathscr{F} \subset \mathscr{E}$ is the first step in the μ-Harder-Narashiman filtration of $\mathscr{E}$, in particular that $\mathscr{F}$ is μ-semistable. Let $\mathscr{E}_i$ be the image of $\Lambda_r \otimes_{\mathcal{O}_X} \mathscr{F}$ in $\mathscr{E}$. Let $\mathscr{G}$ be the saturation of $\mathscr{E}_r$; by the previous lemma, $\mathscr{G}$ is a sub-Λ-module. By the assumption of μ-semistability, $\mu(\mathscr{G}) \leqslant \mu(\mathscr{E})$. The slope increases when taking the saturation, so we have

$$\mu(\mathscr{E}_r) \leqslant \mu(\mathscr{E}).$$

We have surjections of coherent sheaves

$$\mathrm{Gr}_1(\Lambda) \otimes_{\mathcal{O}_X} (\mathscr{E}_i/\mathscr{E}_{i-1}) \to \mathscr{E}_{i+1}/\mathscr{E}_i \to 0,$$

and this tensor product is a tensor product of $\mathcal{O}_X$-modules where the left and right structures coincide. As m is chosen so that $\mathrm{Gr}_1(\Lambda) \otimes_{\mathcal{O}_X} \mathcal{O}_X(m)$ is generated by global sections, we obtain a finite dimensional $\mathbf{C}$-vector space A and a surjection

$$A \otimes_{\mathbf{C}} (\mathscr{E}_i/\mathscr{E}_{i-1}) \otimes_{\mathcal{O}_X} \mathcal{O}_X(-m) \to \mathscr{E}_{i+1}/\mathscr{E}_i \to 0.$$

Let α_i be the smallest slope of any quotient $\mathcal{O}_X$-module of $\mathscr{E}_i$. Note that we may assume that this quotient $\mathscr{Q}_i$ is a μ-semistable sheaf of slope α_i. The quotient $\mathscr{Q}_{i+1}$ of $\mathscr{E}_{i+1}$ either has a nontrivial subsheaf which is a quotient of $\mathscr{E}_i$, in which case $\alpha_i \leqslant \alpha_{i+1}$, or is a quotient of $A \otimes_{\mathbf{C}} (\mathscr{E}_i/\mathscr{E}_{i-1}) \otimes_{\mathcal{O}_X} \mathcal{O}_X(-m)$, in which case $\alpha_i - m \leqslant \alpha_{i+1}$. Since $\mathscr{E}_0 = \mathscr{F}$ is a μ-semistable sheaf, we have $\alpha_0 = \mu(\mathscr{F})$. The inequalities above give

$$\alpha_r \geqslant \alpha_0 - rm.$$

On the other hand, $\mathscr{E}_r^{sat} = \mathscr{G}$, so

$$\alpha_r \leqslant \mu(\mathscr{E}_r) \leqslant \mu(\mathscr{G}).$$

Lemma 3.2 and the condition that $\mathscr{E}$ is a μ-semistable Λ-module imply that $\mu(\mathscr{G}) \leqslant \mu(\mathscr{E})$. We conclude that

$$\mu(\mathscr{F}) = \alpha_0 \leqslant \mu(\mathscr{E}) + mr.$$

This proves the lemma. □

Corollary **3.4.** — *The set of* μ*-semistable* Λ*-modules on* X *with a given Hilbert polynomial* P *is bounded.*

Proof. — Together with the above lemma, Theorem 1.1 immediately implies that the set of sheaves of $\mathcal{O}_X$-modules underlying μ-semistable Λ-modules with Hilbert polynomial P, is bounded. To complete the proof, it suffices to note that the structure of Λ-module of $\mathcal{E}$ is determined by the morphism $\Lambda_1 \otimes_{\mathcal{O}_X} \mathcal{E} \to \mathcal{E}$. As $\mathcal{E}$ runs through a bounded family, the family of such morphisms is bounded. □

The relative case

Suppose now that S is a scheme of finite type over $\mathrm{Spec}(\mathbf{C})$, that $X \to S$ is projective (with fixed relatively very ample $\mathcal{O}_X(1)$), and that Λ is a sheaf of rings of differential operators on X over S. For each geometric point $s \to S$ (with $s \cong \mathrm{Spec}(\mathbf{C})$), let $X_s = X \times_S s$ and let Λ_s be the sheaf of rings of differential operators on X_s obtained by base change (Lemmas 2.5 and 2.6).

A Λ-module $\mathcal{E}$ on X is *p-semistable* (resp. μ*-semistable*, *p-stable*, or μ*-stable*) if $\mathcal{E}$ is flat over S, and if the restrictions $\mathcal{E}_s$ to the geometric fibers X_s are of pure dimension d and p-semistable (resp. μ-semistable, p-stable or μ-stable) Λ_s-modules, all with the same Hilbert polynomials (this last condition is inserted in case S is not connected).

Proposition **3.5.** — *The set of* μ*-semistable* Λ_s*-modules on geometric fibers* X_s *with a given Hilbert polynomial* P *is bounded.*

Proof. — Apply Theorem 1.1 as in Corollary 1.6, using the bound given by Lemma 3.3. We just have to verify that the same number m works for all geometric points of S. We claim that $\mathrm{Gr}_1(\Lambda_s)$ is a quotient of the fiber $\mathrm{Gr}_1(\Lambda) \otimes_{\mathcal{O}_S} \mathcal{O}_s$. For this it suffices to work in an affine open subset (keeping the notations of 2.3, $S = \mathrm{Spec}(A)$, $X = \mathrm{Spec}(B)$, and Λ corresponds to L). Let $\mathbf{C}_s$ denote the field of complex numbers considered as an A-algebra corresponding to the point s, and let $M = L \otimes_A \mathbf{C}_s$. We have surjections

$$L_i \otimes_A \mathbf{C}_s \to M_i$$

and hence $(L_1/L_0) \otimes_A \mathbf{C}_s \to M_1/M_0$. Thus $\mathrm{Gr}_1(M)$ is a quotient of $\mathrm{Gr}_1(L) \otimes_A \mathbf{C}_s$, the claimed statement. Continuing with the proof of the proposition, in order to have $\mathrm{Gr}_1(\Lambda_s)(m)$ generated by global sections, it suffices to have $\mathrm{Gr}_1(\Lambda)\,(m) \otimes_{\mathcal{O}_S} \mathbf{C}_s$ generated by global sections. Since S is of finite type over $\mathbf{C}$ and $\mathrm{Gr}_1(\Lambda)$ is a coherent sheaf on X, there exists such an m. This shows that m may be choosen uniformly for all geometric fibers. □

Corollary **3.6.** — *There is a number* N_0 *depending on* Λ *and* P *such that for any* $N \geqslant N_0$, *any* $S' \to S$, *and any* μ*-semistable* Λ*-module* $\mathcal{E}$ *with Hilbert polynomial* P *on* $X' = X \times_S S'$, *we have that* $H^i(X'/S', \mathscr{E}(N)) = 0$ *for* $i > 0$, $H^0(X'/S', \mathscr{E}(N))$ *is locally free of rank* $P(N)$ *on* S', *formation of* $H^0(X'/S', \mathscr{E}(N))$ *commutes with further base change, and the map*

$$H^0(X'/S', \mathscr{E}(N)) \otimes_{\mathcal{O}_{S'}} \mathcal{O}_X(-N) \to \mathscr{E} \to 0$$

is surjective.

Proof. — By the boundedness established above, this is an immediate consequence of Lemma 1.9. □

Lemma 3.7. — *Suppose that* $\mathscr{E}$ *is a* Λ*-module on* X, *flat over* S. *There is an open subset* $U \subset S$ *such that for any geometric point* $s \to S$ *the restriction* $\mathscr{E}_s$ *is* μ*-semistable (resp.* μ*-stable,* p*-semistable, or* p*-stable) if and only if* $s \in U$.

Proof. — We use a method which U. Bhosle has attributed to Ramanan. Let $\mu_0 = \mu(\mathscr{E}_s)$ (we may assume that S is connected, so these slopes are independent of s). By Proposition 1.8 the set of Hilbert polynomials of quotients of restrictions $\mathscr{E}_s \to \mathscr{G}_s \to 0$ with $\mu(\mathscr{G}_s) \leqslant \mu_0$ and $\mathscr{G}_s$ of pure dimension d, is finite. Let Σ denote the set of such polynomials corresponding to quotients with $\mu(\mathscr{G}_s) < \mu_0$ (resp. $\mu(\mathscr{G}_s) \leqslant \mu_0$ or the appropriate inequalities for normalized Hilbert polynomials). For $q \in \Sigma$, let $H_q \to S$ denote the Hilbert scheme of quotients

$$\mathscr{E} \to \mathscr{G} \to 0$$

flat with Hilbert polynomials q over the base. There is a universal quotient $\mathscr{G}^{\text{univ}}$ on $X \times_S H_q$; let

$$0 \to \mathscr{F}^{\text{univ}} \to \mathscr{E}^{\text{univ}} \to \mathscr{G}^{\text{univ}} \to 0$$

denote the kernel. For any H_q-scheme $S' \to H_q$ we can pull back to obtain an exact sequence

$$0 \to \mathscr{F}' \to \mathscr{E}' \to \mathscr{G}' \to 0$$

of $\mathcal{O}_{X'}$-modules on $X' = X \times_S S'$. The middle term is the Λ'-module obtained by base change from $\mathscr{E}$. Let $(\Lambda_1)'$ denote the pullback of Λ_1 to X'; recall that $\Lambda'_1 \subset \Lambda'$ is a quotient of $(\Lambda_1)'$. From the Λ'-module structure of $\mathscr{E}'$ we obtain a morphism

$$a(S') : (\Lambda_1)' \otimes_{\mathcal{O}_{X'}} \mathscr{F}' \to \mathscr{G}',$$

and $\mathscr{G}'$ is a quotient Λ'-module if and only if this morphism is zero. However, this morphism is pulled back from a universal morphism

$$a^{\text{univ}} : (\Lambda_1)^{\text{univ}} \otimes_{\mathcal{O}_{X \times_S H_q}} \mathscr{F}^{\text{univ}} \to \mathscr{G}^{\text{univ}}$$

over H_q. Since $\mathscr{G}^{\text{univ}}$ is flat over H_q, the condition on $S' \to H_q$ that the morphism $a(S')$ is zero, is represented by a closed subscheme $Z_q \subset H_q$ (see the proof of Theorem 3.8

below). Let $D \subset S$ denote the union over the finite set of $q \in \Sigma$ of the images of Z_q in S. The Hilbert schemes H_q are proper over S, so images of the closed subsets Z_q are closed in S. Thus D is closed in S.

The open subset $U = S - D$ is the one required by the lemma. Suppose $s \to S$ is a geometric point such that $\mathscr{E}_s$ is not μ-semistable (resp. not μ-stable, not p-semistable, or not p-stable). Then there exists a quotient $\mathscr{E}_s \to \mathscr{G}_s$ of Λ_s-modules contradicting (semi)-stability. As discussed above, we may suppose that $\mathscr{G}_s$ is of pure dimension d (in other words, the kernel $\mathscr{F}_s$ is saturated). Hence the Hilbert polynomial of $\mathscr{G}_s$ is one of the $q \in \Sigma$. The quotient $\mathscr{G}_s$ corresponds to a point in one of the Hilbert schemes H_q. This point lies in the subset Z_q because $\mathscr{G}_s$ is a quotient Λ-module. Hence the image of the geometric point s is contained in $D \subset S$. Conversely if $s \to D$ is a geometric point in D then it can be lifted to a geometric point of some Z_q. We obtain a quotient Λ_s-module $\mathscr{G}_s$ of $\mathscr{E}_s$ whose Hilbert polynomial is of a nature to contradict (semi)-stability. This proves that the points of D are exactly those not satisfying our condition for inclusion in U. □

A parametrizing scheme for p-semistable Λ-modules

Theorem **3.8.** — *Fix a polynomial* P, *and let* $N \geqslant N_0$ *for the* N_0 *given by Corollary* 3.6. *The functor which associates to each* S-*scheme* S′ *the set of isomorphism classes of pairs* $(\mathscr{E}, \alpha)$, *where* $\mathscr{E}$ *is a p-semistable* Λ-*module with Hilbert polynomial* P *on* $X' = X \times_S S'$, *and*

$$\alpha : (\mathcal{O}_{S'})^{p(N)} \xrightarrow{\cong} H^0(X'/S', \mathscr{E}(N)),$$

is representable by a quasiprojective scheme Q *over* S. *For any* $k \geqslant 1$, *the morphism from* Q *to the Hilbert scheme of quotients of* $\Lambda_k \otimes_{\mathcal{O}_X} \mathcal{O}_X(N)^{P(N)}$ *with Hilbert polynomial* P, *given by the* Λ-*module structure of the universal object* $\mathscr{E}^{\mathrm{univ}}$, *is a locally closed embedding.*

Proof. — Before beginning the proof, we make a general remark. Suppose S′ is an S-scheme of finite type, and E and F are coherent sheaves of $\mathcal{O}_{X'}$-modules on X′. Suppose that E is flat over S′ (but F need not be flat). Suppose that $\varphi : F \to E$ is a morphism of coherent sheaves. Then there is a closed subscheme $T \subset S'$ such that for any $f : S'' \to S'$, $f^*(\varphi) = 0$ as a morphism of sheaves on $X'' = X' \times_{S'} S''$ if and only if f factors through $f : S'' \to T$. To prove this we may choose a surjection $\mathcal{O}_{X'}(-m)^k \to F \to 0$ with $m \gg 0$. This remains a surjection when pulled back to X″. One is reduced to the case where $F = \mathcal{O}_{X'}(-m)^k$. A morphism to E is then just a k-tuple of sections of $E(m)$. For m big enough, the direct image of $E(m)$ is locally free on S′ and compatible with all base-changes. The subscheme is then just the intersection of the subschemes defined by the corresponding sections of this locally free direct image.

Fix $N \geqslant N_0$ as in Corollary 3.6. Let $Q_1 \to S$ be the Hilbert scheme of quotients

$$\Lambda_k \otimes_{\mathcal{O}_X} \mathcal{O}_X(-N)^{P(N)} \to \mathscr{E} \to 0$$

with Hilbert polynomial p. For each S-scheme $f: S' \to S$, the set $Q_1(S')$ of morphisms $e: S' \to Q_1$ over S is equal to the set of isomorphism classes of quotients

$$\zeta : f^*(\Lambda_k \otimes_{\mathcal{O}_X} \mathcal{O}_X(-N)^{P(N)}) \to \mathcal{E} \to 0$$

on X'/S' such that $\mathcal{E}$ is flat over S', and for each $s \in S'$, the Hilbert polynomial of $\mathcal{E}_s$ is p. Given e we denote the corresponding quotient by $\mathcal{E}$.

We have a morphism $\mathcal{O}_X(-N)^{P(N)} \to \Lambda_k \otimes_{\mathcal{O}_X} \mathcal{O}_X(-N)^{P(N)}$, and if $f: S' \to S$ we obtain

$$\mathcal{O}_{X'}(-N)^{P(N)} \to f^*(\Lambda_k \otimes_{\mathcal{O}_X} \mathcal{O}_X(-N)^{P(N)}).$$

If $e \in Q_1(S')$ we say that e satisfies *condition* Q2 if the composed map

$$\mathcal{O}_{X'}(-N)^{P(N)} \to f^*(\Lambda_0 \otimes_{\mathcal{O}_X} \mathcal{O}_X(-N)^{P(N)}) \to \mathcal{E}$$

is surjective, and for each closed point $s \in S'$, the resulting map $\mathbf{C}^{P(N)} \to H^0(X_s, \mathcal{E}_s(N))$ is injective. There exists a unique open subscheme $Q_2 \subset Q_1$ representing this condition, in other words, such that $e \in Q_2(S')$ if and only if e satisfies condition Q2.

If $f: S' \to S$ then each $f^*(\Lambda_j)$ has a structure of right $\mathcal{O}_{X'}$-module, and we have

$$f^*(\Lambda_j \otimes_{\mathcal{O}_X} \mathcal{O}_X(-N)^{P(N)}) \cong f^*(\Lambda_j) \otimes_{\mathcal{O}_{X'}} \mathcal{O}_{X'}(-N)^{P(N)}.$$

Note also that there are natural maps $f^*(\Lambda_j) \to f^*(\Lambda_k)$ compatible with the left and right $\mathcal{O}_{X'}$-module structures, for $j \leqslant k$.

Suppose $e \in Q_2(S')$. For any $j \leqslant k$ we obtain a map

$$f^*(\Lambda_j) \otimes_{\mathcal{O}_{X'}} \mathcal{O}_{X'}(-N)^{P(N)} \to \mathcal{E}.$$

Let $\mathcal{B}$ denote the kernel

$$0 \to \mathcal{B} \to \mathcal{O}_{X'}(-N)^{P(N)} \to \mathcal{E} \to 0.$$

We say that e satisfies *condition* Q3 if the map obtained from the previous ones by composition,

$$\psi_e : f^*(\Lambda_1) \otimes_{\mathcal{O}_{X'}} \mathcal{B} \to \mathcal{E},$$

is zero. We claim that there exists a closed subscheme $Q_3 \subset Q_2$ such that $e \in Q_3(S')$ if and only if e satisfies Q3. To see this, note that $\mathcal{B}$ is the pullback of a universal kernel $\mathcal{B}^{\mathrm{univ}}$ over $X \times_S Q_2$. Let $f^{\mathrm{univ}}: Q_2 \to S$ denote the projection. Then

$$f^*(\Lambda_1) \otimes_{\mathcal{O}_{X'}} \mathcal{B} = e^*((f^{\mathrm{univ}})^* (\Lambda_1) \otimes_{\mathcal{O}_{X \times_S Q_2}} \mathcal{B}^{\mathrm{univ}})$$

and $\psi_e = e^*(\psi^{\mathrm{univ}})$ where

$$\psi^{\mathrm{univ}} : (f^{\mathrm{univ}})^* (\Lambda_1) \otimes_{\mathcal{O}_{X \times_S Q_2}} \mathcal{B}^{\mathrm{univ}} \to \mathcal{E}^{\mathrm{univ}}$$

is a morphism to the universal quotient $\mathcal{E}^{\mathrm{univ}}$ over $X \times_S Q_2$. Note that $\mathcal{E}^{\mathrm{univ}}$ is flat over Q_2. Our general discussion from the start of the proof applies: there is a closed subscheme $Q_3 \subset Q_2$ such that for $e \in Q_2(S')$ we have $e^*(\psi^{\mathrm{univ}}) = 0$ if and only if $e \in Q_3(S')$.

For $e \in Q_3(S')$, the morphism

$$f^*(\Lambda_1) \otimes_{\mathcal{O}_{X'}} \mathcal{O}_{X'}(-N)^{P(N)} \to \mathcal{E}$$

factors through a morphism

$$f^*(\Lambda_1) \otimes_{\mathcal{O}_{X'}} \mathcal{E} \to \mathcal{E},$$

and this factorization is the pullback of a universal one on Q_3. For any j we obtain a morphism

$$\mu_j : f^*(\Lambda_1) \otimes_{\mathcal{O}_{X'}} \ldots \otimes_{\mathcal{O}_{X'}} f^*(\Lambda_1) \otimes_{\mathcal{O}_{X'}} \mathcal{E} \to \mathcal{E}.$$

Let $\mathcal{K}_j$ denote the kernel of the surjection

$$\Lambda_1 \otimes_{\mathcal{O}_X} \ldots \otimes_{\mathcal{O}_X} \Lambda_1 \to \Lambda_j \to 0.$$

Note that $\mathcal{K}_j$ is a left and right $\mathcal{O}_X$-module, and one can form the base changes $f^*(\mathcal{K}_j)$ which will be left and right $\mathcal{O}_{X'}$-modules, in the same way as described in the previous section. The general properties of interchanging things in tensor products imply that $f^*(\Lambda_1) \otimes_{\mathcal{O}_{X'}} \ldots \otimes_{\mathcal{O}_{X'}} f^*(\Lambda_1) = f^*(\Lambda_1 \otimes_{\mathcal{O}_X} \ldots \otimes_{\mathcal{O}_X} \Lambda_1)$. For any $f : S' \to S$ there is an exact sequence

$$f^*(\mathcal{K}_j) \xrightarrow{\nu_j} f^*(\Lambda_1) \otimes_{\mathcal{O}_{X'}} \ldots \otimes_{\mathcal{O}_{X'}} f^*(\Lambda_1) \to f^*(\Lambda_j) \to 0.$$

Say that e satisfies *condition* Q4(j) if the composition

$$\mu_j \nu_j : f^*(\mathcal{K}_j) \otimes_{\mathcal{O}_{X'}} \mathcal{E} \to \mathcal{E}$$

is equal to zero. Again, this morphism is pulled back from a universal one over Q_3, and the universal $\mathcal{E}^{\mathrm{univ}}$ is flat over Q_3, so there are closed subscheme $Q_{4,j} \subset Q_3$ such that $e \in Q_{4,j}$ if and only if e satisfies condition Q4(j). Let $Q_{4,\infty}$ denote the intersection of all of these closed subschemes in Q_3. As S is noetherian and Q_3 is quasiprojective over S, Q_3 is noetherian. Thus $Q_{4,\infty}$ is again a closed subscheme, and we have $e \in Q_{4,\infty}(S')$ if and only if e satisfies all of the conditions Q4(j).

Suppose $e \in Q_{4,\infty}(S')$. We obtain factorizations

$$f^*(\Lambda_1) \otimes_{\mathcal{O}_{X'}} \ldots \otimes_{\mathcal{O}_{X'}} f^*(\Lambda_1) \otimes_{\mathcal{O}_{X'}} \mathcal{E} \to f^*(\Lambda_j) \otimes_{\mathcal{O}_{X'}} \mathcal{E} \to \mathcal{E}.$$

The resulting morphisms

$$\varphi_j : f^*(\Lambda_j) \otimes_{\mathcal{O}_{X'}} \mathcal{E} \to \mathcal{E}$$

are compatible with the morphisms

$$f^*(\Lambda_j) \to f^*(\Lambda_i)$$

for $i \geqslant j$. Recall that Λ'_j is the image of $f^*(\Lambda_j)$ in $f^*(\Lambda)$. The compatibility for all $i \geqslant j$ implies that the morphisms φ_j factor through morphisms

$$\varphi'_j : \Lambda'_j \otimes_{\mathcal{O}_{X'}} \mathscr{E} \to \mathscr{E},$$

which are compatible as j varies, to give

$$\varphi' : \Lambda' \otimes_{\mathcal{O}_{X'}} \mathscr{E} \to \mathscr{E}.$$

In the case of the φ'_j and φ', note that these are no longer obtained by pullback from a universal example over $Q_{4,\infty}$. From the fact that our morphisms were defined by combining several times the morphism $f^*(\Lambda_1) \otimes_{\mathcal{O}_{X'}} \mathscr{E} \to \mathscr{E}$, we obtain a compatibility: the two morphisms

$$f^*(\Lambda_i) \otimes_{\mathcal{O}_{X'}} f^*(\Lambda_j) \otimes_{\mathcal{O}_{X'}} \mathscr{E} \to \mathscr{E},$$

the first one obtained by multiplying into $f^*(\Lambda_{i+j})$ first and then acting on $\mathscr{E}$, the second obtained by acting successively on $\mathscr{E}$, are equal. The same holds for the two maps

$$\Lambda'_i \otimes_{\mathcal{O}_{X'}} \Lambda'_j \otimes_{\mathcal{O}_{X'}} \mathscr{E} \to \mathscr{E},$$

so we obtain an action of Λ' on $\mathscr{E}$. Let $\Lambda^{Q_{4,\infty}}$ denote the base change of Λ via $Q_{4,\infty} \to S$. The universal $\mathscr{E}^{\mathrm{univ}}$ over $X \times_S Q_{4,\infty}$ is a $\Lambda^{Q_{4,\infty}}$-module. If $e \in Q_{4,\infty}$ then the Λ'-module structure of the resulting $\mathscr{E}$ is obtained by base change from the $\Lambda^{Q_{4,\infty}}$-module structure of $\mathscr{E}^{\mathrm{univ}}$.

Finally, suppose $e \in Q_{4,\infty}(S')$. Say that e satisfies *condition* Q5 if the morphism

$$\zeta' : f^*(\Lambda_k) \otimes_{\mathcal{O}_{X'}} \mathcal{O}_{X'}(-N)^{P(N)} \to \mathscr{E}$$

obtained from the Λ'-module structure is equal to the original morphism ζ obtained from the inclusion $e \in Q_1(S')$ in the Hilbert scheme. This condition is again of the form that the pullback of a map is equal to zero, because the morphism ζ' is pulled back from a universal example over $Q_{4,\infty}$. Thus there is a closed subset $Q_5 \subset Q_{4,\infty}$ such that e satisfies condition Q5 if and only if $e \in Q_5(S')$. If we denote by Λ^{Q_5} the base change of Λ via $g : Q_5 \to S$, then the universal $\mathscr{E}^{\mathrm{univ}}$ on $X \times_S Q_5$ has a structure of Λ^{Q_5}-module, and the morphism

$$\xi : g^*(\Lambda_k) \otimes_{\mathcal{O}_{Q_5}} \mathcal{O}_{Q_5}(-N)^{P(N)} \to \mathscr{E}^{\mathrm{univ}}$$

coming from the map to the Hilbert scheme Q_1 is also given by the Λ^{Q_5}-module structure of $\mathscr{E}^{\mathrm{univ}}$.

Suppose $e \in Q_5(S')$. We say that e is *semistable* if the corresponding Λ'-module $\mathscr{E}$ restricts to p-semistable Λ_s-modules $\mathscr{E}_s$ on the fibers over closed points $s \in S'$. According to Lemma 3.7, there is an open subset $Q \subset Q_5$ such that e is semistable if and only if $e \in Q(S')$. This will be the parameter scheme required for the theorem.

Let Λ^Q denote the base-change of Λ to Q via $Q \to S$, and let $\mathscr{E}^Q$ denote the universal Λ^Q-module on Q. We have a natural morphism

$$\alpha^Q : (\mathcal{O}_Q)^{P(N)} \to H^0(X \times_S Q/Q, \mathscr{E}^Q(N)).$$

Since the restrictions of $\mathscr{E}$ to fibers over closed points are semistable, Corollary 3.6 applies: the relative $H^0(X \times_S Q/Q, \mathscr{E}^Q(N))$ is locally free of rank $P(N)$ and compatible with all base changes. Condition Q2 implies that α^Q is injective on the fibers over closed points $s \in Q$. Thus α^Q is an isomorphism. We obtain a universal object $(\mathscr{E}^Q, \alpha^Q)$ over Q, giving rise to the same type of object $(\mathscr{E}, \alpha)$ under any base change $e : S' \to Q$.

Suppose $f : S' \to S$ is an S-scheme of finite type, and suppose $(\mathscr{E}, \alpha)$ is a pair of the type envisioned in the statement of the theorem. By Corollary 3.6, the morphism α induces a surjection

$$\mathcal{O}_{X'}(-N)^{P(N)} \to \mathscr{E} \to 0,$$

and the Λ'-module structure of $\mathscr{E}$ induces a map

$$f^*(\Lambda_k) \otimes_{\mathcal{O}_{X'}} \mathcal{O}_{X'}(-N)^{P(N)} \to \mathscr{E},$$

so we obtain a point $e \in Q_1(S')$. It is evident that $e \in Q(S')$. The pullback $e^*(\mathscr{E}^Q, \alpha^Q)$ is isomorphic to $(\mathscr{E}, \alpha)$.

We obtain maps in both directions between the functor considered in the theorem and the functor $S' \mapsto Q(S')$. The fact that $e^*(\mathscr{E}^Q, \alpha^Q) \cong (\mathscr{E}, \alpha)$ means that one of the compositions is the identity. On the other hand, if $e \in Q(S')$ then the construction of the previous paragraph, applied to $e^*(\mathscr{E}, \alpha)$, gives back e. This isomorphism of functors completes the proof that Q is the desired parameter scheme.

We obtain a single morphism from Q into the Hilbert scheme Q_1: the morphism obtained in a natural way from $(\mathscr{E}^Q, \alpha^Q)$ is the same as the inclusion $Q \subset Q_1$ obtained from the construction of Q. By construction, this is a locally closed embedding. This completes the proof of the theorem. □

4. Invariant theory for Λ-modules

Let $X \to S$ be a projective morphism, and Λ a sheaf of rings of differential operators on X over S. In this section we will construct moduli spaces for p-semistable Λ-modules with a fixed Hilbert polynomial.

Lemma **4.1.** — *There is an integer* B *depending on* Λ, r, *and* d, *such that if* $\mathscr{E}$ *is a* μ*-semistable* Λ*-module of pure dimension* d *and rank* r *on a fiber* X_s, *then*

$$h^0(X_s, \mathscr{E}(k)) \leqslant \begin{cases} 0 & \textit{if } \mu(\mathscr{E}) + k + B \leqslant 0 \\ r(\mu(\mathscr{E}) + k + B)^d/d! & \textit{if } \mu(\mathscr{E}) + k + B \geqslant 0 \end{cases}$$

for any k.

Proof. — Let

$$0 = \mathscr{F}_0 \subset \mathscr{F}_1 \subset \ldots \subset \mathscr{F}_j = \mathscr{E}$$

denote the μ-Harder-Narasimhan filtration for the sheaf $\mathscr{E}$ (not the Λ-module!). Let $\mathscr{G}_i = \mathscr{F}_i/\mathscr{F}_{i-1}$ denote the quotients; they are μ-semistable sheaves of pure dimension d. Let $\mu_i = \mu(\mathscr{G}_i)$ and $r_i = r(\mathscr{G}_i)$. By Lemma 3.3, there is a number b depending only on r and Λ such that $\mu_i \leqslant \mu(\mathscr{E}) + b$. By Corollary 1.7, there is a number B_0 such that

$$h^0(X_s, \mathscr{G}_i(k)) \leqslant \begin{cases} 0 & \text{if } \mu_i + k + B_0 \leqslant 0 \\ r_i(\mu_i + k + B_0)^d/d! & \text{if } \mu_i + k + B_0 \geqslant 0. \end{cases}$$

On the other hand,

$$h^0(X_s, \mathscr{E}(k)) \leqslant \sum_{i=1}^{j} h^0(X_s, \mathscr{G}_i(k)).$$

Setting $B = B_0 + b$, we obtain the desired estimate (since $r = \Sigma r_i$). □

Lemma **4.2.** — *There exists* N_0 *depending on* Λ *and* P *such that for all* $N \geqslant N_0$, *the following is true. Suppose* $\mathscr{E}$ *is a p-semistable* Λ*-module with Hilbert polynomial* P *on a fiber* X_s. *Then for all sub-*Λ*-modules* $\mathscr{F} \subset \mathscr{E}$, *we have*

$$\frac{h^0(\mathscr{F}(N))}{r(\mathscr{F})} \leqslant \frac{P(N)}{r(\mathscr{E})}$$

and if equality holds then

$$\frac{p(\mathscr{F}, m)}{r(\mathscr{F})} = \frac{P(m)}{r(\mathscr{E})}$$

for all m.

Proof. — The same as the proof of Lemma 1.8, but refering to the previous lemma in place of Corollary 1.7. □

Fix a polynomial P and let r denote the corresponding rank. Fix N as required by this lemma and Corollary 3.6, so as to work for all p-semistable Λ-modules of rank less than or equal to r and normalized Hilbert polynomials equal to P/r, on fibers X_s. Let $Q \to S$ denote the parameter scheme constructed in Theorem 3.8. We would like to construct a universal categorical quotient of Q by the action of $Sl(p(N))$.

Let $\mathscr{W} = \Lambda_r \otimes_{\mathcal{O}_X} \mathcal{O}_X(-N)$, and let $V = \mathbf{C}^{P(N)}$. Theorem 3.8 gives a locally closed embedding $Q \subset \mathbf{Hilb}(\mathscr{W} \otimes V, P)$. Let $\mathscr{L}_m$ denote the line bundle on $\mathbf{Hilb}(\mathscr{W} \otimes V, P)$ corresponding to the projective embedding ψ_m described in the paragraphs before Lemma 1.15.

Lemma **4.3.** — *There is an* M *such that for any* $m \geqslant M$, *the subscheme* $Q \subset \mathbf{Hilb}(\mathscr{W} \otimes V, P)$ *is contained in the set* $\mathbf{Hilb}^{ss, m}(\mathscr{W} \otimes V, P)$ *of semistable points of* Q *relative to the line bundle* $\mathscr{L}_m$.

Proof. — Since Q is noetherian, it suffices to fix a closed point q and find M so that $q \in \mathbf{Hilb}^{ss, m}(\mathscr{W} \otimes V, P)$ for $m \geqslant M$. Let $s \in S$ be the image of q, and restrict everything to X_s for the rest of the proof. Suppose $\mathscr{W} \otimes V \to \mathscr{E} \to 0$ is the quotient represented by $q \in Q$. We apply the criterion of Lemma 1.15. Suppose $H \subset V$. Let $\mathscr{F}$ be the subsheaf of $\mathscr{E}$ generated by $\mathscr{W} \otimes H$, and let $\mathscr{G}$ be the saturation of $\mathscr{F}$. By the construction of Q, $\mathscr{G}$ is the saturation of the image of the morphism $\Lambda_r \otimes_{\mathcal{O}_X} \mathscr{H} \to \mathscr{E}$ where $\mathscr{H}$ is the image of the map $\mathcal{O}_X(-N) \otimes H \to \mathscr{E}$. By Lemma 3.2, $\mathscr{G}$ is a sub-Λ-module of $\mathscr{E}$.

Apply Lemma 4.2 to $\mathscr{G}$, using the hypothesis that $\mathscr{E}$ is p-semistable. Since $V \cong H^0(\mathscr{E}(N))$, we have $H \subset H^0(\mathscr{F}(N)) \subset H^0(\mathscr{G}(N))$. We find that

$$\frac{\dim(H)}{r(\mathscr{G})} \leqslant \frac{h^0(\mathscr{G}(N))}{r(\mathscr{G})} \leqslant \frac{P(N)}{r(\mathscr{E})}.$$

Note that $r(\mathscr{F}) = r(\mathscr{G})$ and $\dim(V) = P(N)$. There are two cases, depending on whether there is inequality or equality between the first and last entries. Suppose that inequality holds,

$$\frac{\dim(H)}{r(\mathscr{F})} < \frac{P(N)}{r(\mathscr{E})}.$$

The set of possibilities for H is bounded, so the set of possibilities for $\mathscr{F}$ is bounded. Thus we may choose M large enough so that for $m \geqslant M$, $p(\mathscr{F}, m)$ is approximated by $r(\mathscr{F})\, m^d$ (up to a bounded multiple of m^{d-1}). The same may be assumed for $P(m)$. So in this case of inequality, for $m \geqslant M$,

$$\frac{\dim(H)}{p(\mathscr{F}, m)} < \frac{P(N)}{p(\mathscr{E}, m)}.$$

Thus the criterion of Lemma 1.15 for semistability in the Hilbert scheme holds. Suppose instead that we have equality,

$$\frac{\dim(H)}{r(\mathscr{G})} = \frac{P(N)}{r(\mathscr{E})}.$$

By Lemma 4.2, this implies that $p(\mathscr{G}, m)/r(\mathscr{G}) = P(m)/r(\mathscr{E})$ for all m. Thus $\mathscr{G}$ is itself a p-semistable Λ-module with the same normalized Hilbert polynomial as $\mathscr{E}$. By our choice of N, $\mathscr{G}(N)$ is generated by global sections. Equality also implies that $H = H^0(\mathscr{F}(N)) = H^0(\mathscr{G}(N))$. Therefore $\mathscr{F} = \mathscr{G}$, and since the normalized Hilbert polynomials of $\mathscr{F}$ and $\mathscr{E}$ are equal, we get

$$\frac{\dim(H)}{p(\mathscr{F}, m)} = \frac{P(N)}{P(m)}$$

for all m. So the criterion of Lemma 1.15 is satisfied in the case of equality too. This shows that for $m \geqslant M$, $q \in \mathbf{Hilb}^{ss, m}(\mathscr{W} \otimes V, P)$. $\square$

For the rest of the argument, fix an m so that we get a map

$$\mathrm{Q} \to \mathbf{Hilb}^{ss, m}(\mathscr{W} \otimes \mathrm{V}, \mathrm{P})/\mathrm{Sl}(\mathrm{V}).$$

We want to show that its image is a locally closed subset representing a good quotient of Q by Sl(V).

Lemma **4.4.** — *The closure in* $\mathbf{Hilb}^{ss, m}(\mathscr{W} \otimes \mathrm{V}, \mathrm{P})$ *of any* Sl(V)*-orbit in* Q *is itself contained in* Q.

Proof. — Suppose a is a point in Q, corresponding to a point

$$a : \mathscr{W} \otimes \mathrm{V} \to \mathscr{E} \to 0$$

of $\mathbf{Hilb}^{ss, m}(\mathscr{W} \otimes \mathrm{V}, \mathrm{P})$. Suppose

$$b : \mathscr{W} \otimes \mathrm{V} \to \mathscr{E}' \to 0$$

is a point in the closure of the orbit of a. By the argument described in the remark after Lemma 1.25, we may assume that there is a one-parameter subgroup $\varphi : \mathbf{G}_m \to \mathrm{Sl}(\mathrm{V})$ such that $b = \lim_{t \to 0} \varphi(t)\, a$. Now apply the discussion of limit points of $\mathbf{G}_m$ orbits. There is a filtration $\mathrm{F}_\bullet$ of V and an isomorphism $\mathrm{V} \cong \mathrm{Gr}^{\mathrm{F}}_{\bullet}(\mathrm{V})$ coming from the one parameter subgroup, $\varphi(t)$ acting on $\mathrm{Gr}^{\mathrm{F}}_{\alpha}(\mathrm{V})$ by t^{α}. This induces a filtration $\mathscr{F}_{\beta} = a(\mathscr{W} \otimes \mathrm{F}_{\beta}(\mathrm{V}))$ on $\mathscr{E}$. The previous discussion shows that $\mathscr{E}' = \mathrm{Gr}^{\mathscr{F}}_{\bullet}$, and the quotient map b is equal to the associated-graded of a.

Define a new filtration of $\mathscr{E}$ by letting $\mathscr{H}_{\beta}$ be the saturation of $\mathscr{F}_{\beta}$. By Lemma 3.2, $\mathscr{H}_{\beta}$ are sub-Λ-modules of $\mathscr{E}$. Let $\mathscr{G}_{\beta}$ be the image of $\mathrm{Gr}^{\mathscr{F}}_{\beta}$ in $\mathrm{Gr}^{\mathscr{H}}_{\beta}$. Let $p_{\beta} : \mathscr{E}' \to \mathrm{Gr}^{\mathscr{H}}_{\beta}$ be the composition of the natural map $\mathrm{Gr}^{\mathscr{F}}_{\beta'} \to \mathrm{Gr}^{\mathscr{H}}_{\beta}$ with the projection on the βth factor. Note that $\mathrm{Gr}^{\mathscr{H}}_{\beta}$ is a Λ-module. The map

$$\mathscr{W} \otimes \mathrm{Gr}^{\mathrm{F}}_{\beta}\,\mathrm{V} \to \mathrm{Gr}^{\mathscr{H}}_{\beta}$$

is a subquotient of the map a, so it is compatible with the action of Λ on the right and the partial action of Λ on the right. Therefore the composed map

$$p_{\beta}\, b : \mathscr{W} \otimes \mathrm{V} \to \mathrm{Gr}^{\mathscr{H}}_{\beta}$$

is given by the splitting $\mathrm{V} \cong \mathrm{Gr}^{\mathrm{F}}_{\bullet}\,\mathrm{V}$, the map $\mathrm{Gr}^{\mathrm{F}}_{\beta}\,\mathrm{V} \to \mathrm{H}^0(\mathrm{Gr}^{\mathscr{H}}_{\beta}(\mathrm{N}))$, and the action of Λ. Let $\mathrm{U}_{\beta} \subset \mathrm{V}$ be the kernel of the map $\mathrm{V} \to \mathrm{H}^0(\mathscr{G}_{\beta}(\mathrm{N}))$ induced by $p_{\beta}\, b$. In particular, $\dim(\mathrm{U}_{\beta}) \geqslant \mathrm{P}(\mathrm{N}) - h^0(\mathscr{G}_{\beta}(\mathrm{N}))$. Let $\mathscr{V}_{\beta} \subset \mathscr{E}'$ denote the subsheaf generated by $\mathscr{W} \otimes \mathrm{U}_{\beta}$. This subsheaf maps to zero in $\mathrm{Gr}^{\mathscr{H}}_{\beta}$, hence it maps to zero in $\mathscr{G}_{\beta}$. Thus $r(\mathscr{V}_{\beta}) \leqslant r(\mathscr{E}') - r(\mathscr{G}_{\beta})$. We are assuming that b is in $\mathbf{Hilb}^{ss, m}(\mathscr{W} \otimes \mathrm{V}, \mathrm{P})$, so by Lemma 1.16 we have

$$\frac{\dim(\mathrm{U}_{\beta})}{r(\mathscr{V}_{\beta})} \leqslant \frac{\mathrm{P}(\mathrm{N})}{r(\mathscr{E})}.$$

Comparing this with the previous inequalities gives

$$\frac{h^0(\mathscr{G}_\beta(\mathrm{N}))}{r(\mathscr{G}_\beta)} \geqslant \frac{\mathrm{P}(\mathrm{N})}{r(\mathscr{E})}.$$

Similarly, by applying Lemma 1.16 to the kernel of the map

$$\mathrm{V} \to \mathrm{H}^0(\bigoplus_\beta \mathscr{G}_\beta(\mathrm{N}))$$

induced by the quotient $\bigoplus p_\beta\, b$, we find that this kernel is zero. Therefore the map $\mathrm{V} \to \mathrm{H}^0(\mathscr{E}'(\mathrm{N}))$ is injective.

Now we claim that each $\mathscr{H}_\beta$ is a p-semistable sub-Λ-module of $\mathscr{E}$ with normalized Hilbert polynomial equal to the normalized Hilbert polynomial of $\mathscr{E}$. We prove this by induction on β. Note that the filtration $\mathscr{H}_\beta$ is decreasing. Suppose that this claim is known for $\mathscr{H}_{\beta+1}$. Then $\mathscr{E}/\mathscr{H}_{\beta+1}$ is a p-semistable Λ-module with the appropriate normalized Hilbert polynomial, of rank less than or equal to $r(\mathscr{E})$, and $\mathrm{Gr}^{\mathscr{H}}_\beta \subset \mathscr{E}/\mathscr{H}_{\beta+1}$. By our choice of N, Lemma 4.2 applies:

$$\frac{h^0(\mathrm{Gr}^{\mathscr{H}}_\beta(\mathrm{N}))}{r(\mathrm{Gr}^{\mathscr{H}}_\beta)} \leqslant \frac{\mathrm{P}(\mathrm{N})}{r(\mathscr{E})},$$

and if equality holds then $\mathrm{Gr}^{\mathscr{H}}_\beta$ has the same normalized Hilbert polynomial as $\mathscr{E}$. But $\mathscr{G}_\beta \subset \mathrm{Gr}^{\mathscr{H}}_\beta$, and the ranks are the same since the quotient $\mathrm{Gr}^{\mathscr{H}}_\beta/\mathscr{G}_\beta$ is supported in dimension $d-1$. The inequality of the previous paragraph shows that equality holds here, and furthermore $\mathrm{H}^0(\mathscr{G}_\beta(\mathrm{N})) = \mathrm{H}^0(\mathrm{Gr}^{\mathscr{H}}_\beta(\mathrm{N}))$. Therefore $\mathrm{Gr}^{\mathscr{H}}_\beta$ has the same normalized Hilbert polynomial as $\mathscr{E}$ and $\mathscr{E}/\mathscr{H}_{\beta+1}$. Since $\mathrm{Gr}^{\mathscr{H}}_\beta$ is a sub-Λ-module of $\mathscr{E}/\mathscr{H}_{\beta+1}$, it is p-semistable. This in turn implies that $\mathscr{H}_\beta$ is p-semistable with the required normalized Hilbert polynomial, completing the inductive proof of the claim.

By our choice of N, all of the sheaves $\mathrm{Gr}^{\mathscr{H}}_\beta(\mathrm{N})$ are generated by global sections. The incidental fact from the previous paragraph, that $\mathrm{H}^0(\mathscr{G}_\beta(\mathrm{N})) = \mathrm{H}^0(\mathrm{Gr}^{\mathscr{H}}_\beta(\mathrm{N}))$, implies that $\mathscr{G}_\beta = \mathrm{Gr}^{\mathscr{H}}_\beta$. Thus $\mathrm{Gr}^{\mathscr{F}}_\bullet \to \mathrm{Gr}^{\mathscr{H}}_\bullet$ is surjective. This implies that $\mathscr{F}_\beta = \mathscr{H}_\beta$ (one shows by induction on β that $\mathscr{F}_\beta \to \mathscr{H}_\beta$ is surjective). Therefore the $\mathscr{F}_\beta$ are p-semistable sub-Λ-modules of $\mathscr{E}$ with the appropriate normalized Hilbert polynomial. The associated-graded $\mathscr{E}' = \mathrm{Gr}^{\mathscr{F}}_\beta$ is a p-semistable Λ-module with the normalized Hilbert polynomial of $\mathscr{E}$. The map $\mathscr{W} \otimes \mathrm{V} \to \mathrm{Gr}^{\mathscr{H}}_\bullet$ is compatible with Λ (it is obtained as the associated-graded of the map a, through a splitting of the filtration on V which does not affect the Λ structure), but this map is the same as the map b via $\mathrm{Gr}^{\mathscr{H}}_\beta \cong \mathscr{E}'$. Thus the map b is compatible with the Λ-module structure of $\mathscr{E}'$. Finally, by our choice of N, we have $h^0(\mathscr{E}'(\mathrm{N})) = \mathrm{P}(\mathrm{N}) = \dim(\mathrm{V})$, so the injection $\mathrm{V} \hookrightarrow \mathrm{H}^0(\mathscr{E}'(\mathrm{N}))$ is an isomorphism. And $\mathscr{E}'(\mathrm{N})$ is generated by global sections, so b restricts to a surjection

$$\Lambda_0 \otimes \mathscr{O}_{\mathrm{X}}(-\mathrm{N}) \otimes \mathrm{V} \to \mathscr{E}' \to 0.$$

Therefore b represents a point in the subscheme Q. This proves the lemma. □

Lemma **4.5.** — *Let* $\varphi : \mathbf{Hilb}^{ss, m}(\mathscr{W} \otimes V, P) \to \mathbf{Hilb}^{ss, m}(\mathscr{W} \otimes V, P)/\mathrm{Sl}(V)$ *denote the good quotient given by* [Mu]. *The image* $\varphi(Q)$ *is a locally closed subset.*

Proof. — The proofs of Theorems 1.1 and 1.10 in Mumford's book [Mu] imply that if $B \subset \mathbf{Hilb}^{ss, m}(\mathscr{W} \otimes V, P)$ is a closed Sl(V)-invariant subset, then $\varphi(B)$ is closed. In our situation, let $\bar{Q}$ denote the closure of Q in $\mathbf{Hilb}^{ss, m}(\mathscr{W} \otimes V, P)$. Note that Q and $\bar{Q}$ are Sl(V)-invariant. Thus the image $\varphi(\bar{Q})$ is closed. Let $B = \bar{Q} - Q$. It is closed and Sl(V)-invariant, so $\varphi(B)$ is closed. We claim that

$$\varphi(Q) = \varphi(\bar{Q}) - \varphi(B).$$

Note that the right side is certainly contained in the left. To prove equality, suppose to the contrary that there were closed points $a \in Q$ and $b \in B$ such that $\varphi(a) = \varphi(b)$. Let $O(a)$ and $O(b)$ denote the orbits of a and b respectively. One of the properties of the quotient φ is that two closed points in $\mathbf{Hilb}^{ss, m}(\mathscr{W} \otimes V, P)$ have the same image if and only if the closures of their orbits intersect. Thus

$$\overline{O(a)} \cap \overline{O(b)} \neq \emptyset.$$

On the other hand, B is closed, so $\overline{O(b)} \subset B$, and the previous lemma stated exactly that $\overline{O(a)} \subset Q$. But B is the complement of Q, so this is a contradiction. This completes the proof that $\varphi(Q) = \varphi(\bar{Q}) - \varphi(B)$. Therefore the image of Q is a locally closed subset. □

Corollary **4.6.** — *With respect to the induced line bundle* $\mathscr{L}_m|_Q$, *all of the points of* Q *are semistable. The image* $\varphi(Q)$ *has a structure of locally closed subscheme such that* $Q \to \varphi(Q)$ *is the good quotient given by* [Mu].

Proof. — Let A denote the inverse image in $\mathbf{Hilb}^{ss, m}(\mathscr{W} \otimes V, P)$ of $\varphi(Q)$. The map $A \to Q/\mathrm{Sl}(V)$ is an affine map representing a universal categorical quotient. If Q were not a closed subset of A, then A would intersect the subset B refered to in the previous proof, and this would contradict that proof's assertion that $\varphi(B)$ does not meet $\varphi(Q)$. Therefore Q is closed in A. This implies that $\varphi : Q \to \varphi(Q)$ is an affine map. There is a line bundle which we also denote by $\mathscr{L}_m$ on the quotient $\mathbf{Hilb}^{ss, m}(\mathscr{W} \otimes V, P)/\mathrm{Sl}(V)$, which pulls back to $\mathscr{L}_m$ on $\mathbf{Hilb}^{ss, m}(\mathscr{W} \otimes V, P)$. For any point $q \in Q$, there is a section u of $\mathscr{L}_m^{\otimes k}$ on the quotient $\mathbf{Hilb}^{ss, m}(\mathscr{W} \otimes V, P)/\mathrm{Sl}(V)$ such that u does not vanish at $\varphi(q)$, and such that the subset where $u \neq 0$ is affine. We may also assume that u vanishes on the closed set $\varphi(B)$. Therefore the subset of $\varphi(Q)$ where $u \neq 0$ is a closed (hence affine) subset of the whole affine set where $u \neq 0$. Since φ is affine, this implies that the subset of Q where $u \neq 0$ is an affine subset. By definition then, every point q of Q is semistable with respect to the line bundle $\mathscr{L}_m$. Thus a universal categorical quotient of Q by Sl(V) exists, with the properties described in [Mu]. We get a map from this quotient to $\mathbf{Hilb}^{ss, m}(\mathscr{W} \otimes V, P)/\mathrm{Sl}(V)$. On the affine open subsets defined by sections of the line bundle such as described above, suppose f is a function on the affine subset of Q, invariant under $\mathrm{Sl}(V)$. Then since $\mathrm{Sl}(V)$ is reductive, we can lift this to an invariant function f on the affine subset of $\mathbf{Hilb}^{ss, m}(\mathscr{W} \otimes V, P)$. This then descends to a function on $\mathbf{Hilb}^{ss, m}(\mathscr{W} \otimes V, P)/\mathrm{Sl}(V)$. This shows that the functions on the quotient of Q come from functions on open subsets of $\mathbf{Hilb}^{ss, m}(\mathscr{W} \otimes V, P)/\mathrm{Sl}(V)$. Therefore the quotient of Q is included as a locally closed subscheme of $\mathbf{Hilb}^{ss, m}(\mathscr{W} \otimes V, P)/\mathrm{Sl}(V)$. □

Let $\mathbf{M}^{\natural}(\Lambda, P)$ denote the functor of schemes over S which associates to $S' \to S$ the set of isomorphism classes of p-semistable Λ'-modules on X' over S' with Hilbert polynomial P.

Theorem **4.7.** — *Let* $\mathbf{M}(\Lambda, P) = Q/\mathrm{Sl}(V)$ *be the good quotient. There is a morphism of functors* $\varphi : \mathbf{M}^{\natural}(\Lambda, P) \to \mathbf{M}(\Lambda, P)$ *such that* $(\mathbf{M}(\Lambda, P), \varphi)$ *universally corepresents the functor* $\mathbf{M}^{\natural}(\Lambda, P)$. *The following properties are satisfied.*

(1) $M(\Lambda, P)$ *is a quasiprojective variety.*

(2) *The geometric points of* $\mathbf{M}(\Lambda, P)$ *represent the equivalence classes of p-semistable* Λ*-modules with Hilbert polynomial* P *on fibers* X_s, *under the relation of Jordan equivalence* ($\mathscr{F}_1 \sim \mathscr{F}_2$ *if* $\mathrm{gr}(\mathscr{F}_1) = \mathrm{gr}(\mathscr{F}_2)$). *The equivalence class of a* Λ*-module* $\mathscr{F}$ *on* X_s *corresponds to the point* $\varphi(\mathscr{F})$ *in the fiber of* $\mathbf{M}(\Lambda, P)$ *over* $s \in S$.

(3) *The closed orbits in* Q *are exactly those corresponding to semisimple objects,* $\mathscr{F}$ *with* $\mathrm{gr}(\mathscr{F}) \cong \mathscr{F}$.

(4) *There is an open subset* $\mathbf{M}^s(\Lambda, P) \subset \mathbf{M}(\Lambda, P)$ *whose points represent isomorphism classes of p-stable* Λ*-modules. Locally in the étale topology on* $\mathbf{M}^s(\Lambda, P)$ *there is a universal* Λ*-module* $\mathscr{F}^{\mathrm{univ}}$ *such that if* $\mathscr{F}$ *is an element of* $\mathbf{M}^{\natural}(\Lambda, P)(S')$ *whose fibers* $\mathscr{F}_s$ *are stable, then the pull-back of* $\mathscr{F}^{\mathrm{univ}}$ *via* $S' \to \mathbf{M}^s(\Lambda, P)$ *is isomorphic to* $\mathscr{F}$ *after tensoring with the pull-back of a line bundle on* S'.

Proof. — The proof is the same as the proof of Theorem 1.21. For part (1) note that the good quotients of [Mu] are quasiprojective, but our quotient will not in general be projective. Although not stated in Theorem 1.21, statement (3) here is given by the proof of statement (3) there. □

The representation spaces

In this section we will rigidify the moduli functors by including the data of a frame along a section $\xi : S \to X$. For this, we consider Λ-modules which are locally free as sheaves of $\mathcal{O}_X$-modules near the section ξ. We construct *representation spaces* which parametrize pairs $(\mathscr{E}, \beta)$ where $\mathscr{E}$ is a semistable Λ-module which is locally free along ξ, and $\beta : \xi^*(\mathscr{E}) \cong \mathcal{O}_S^n$. These representation spaces will be fine moduli spaces, representing the appropriate functors. The terminology comes from the analogy (or correspondence, in the case of Λ^{DR}) with representations of the fundamental group.

We make the following hypotheses for the rest of this section. Suppose $X \to S$ is projective, flat, and has irreducible fibers. Suppose $\xi : S \to X$ is a section. Fix a polynomial P of degree equal to the relative dimension $d = \dim(X/S)$. Let $\deg(X)$ be the rank of $\mathcal{O}_X$ (it is the degree of the projective embedding determined by $\mathcal{O}_X(1)$), and let r be the rank corresponding to the polynomial P. Let $n = r/\deg(X)$. A sheaf of pure dimension d and rank r on a fiber X_s is a torsion-free sheaf whose rank, in the usual sense, is n.

Suppose $\mathcal{F}$ is a p-semistable Λ-module with Hilbert polynomial P on X over S. We say that $\mathcal{F}$ *satisfies condition* $\mathrm{LF}(\xi)$ if for every closed point $s \in S$, $\mathrm{gr}(\mathcal{F}_s)$ is locally free as a sheaf of $\mathcal{O}_{X_s}$-modules at $\xi(s)$. We have the following properties.

4.8.1. Suppose $\mathcal{F}$ is a p-semistable Λ-module on X satisfying condition $\mathrm{LF}(\xi)$. Then $\mathcal{F}$ is locally free as a sheaf of $\mathcal{O}_X$-modules along the section ξ. This follows from Lemma 1.27.

4.8.2. After base change by $S' \to S$, there is a section which we also denote by $\xi : S' \to X'$. The condition $\mathrm{LF}(\xi)$ or its negation are preserved by base change.

4.8.3. Suppose that $\mathcal{F}$ is a p-semistable Λ-module on X over S, satisfying condition $\mathrm{LF}(\xi)$. Consider a closed point $s \in S$. Suppose that $\mathcal{G}$ is a sub-Λ-module of $\mathcal{F}_s$ such that the normalized Hilbert polynomial of $\mathcal{G}$ is the same as the normalized Hilbert polynomial of $\mathcal{F}_s$. Then $\mathcal{G}$ and $\mathcal{F}_s/\mathcal{G}$ satisfy $\mathrm{LF}(\xi(s))$. This is because $\mathrm{gr}(\mathcal{G})$ and $\mathrm{gr}(\mathcal{F}_s/\mathcal{G})$ are direct summands in $\mathrm{gr}(\mathcal{F}_s)$.

4.8.4. Let Q denote the parameter space given by Theorem 3.8. Let $Q^{\mathrm{lf}(\xi)}$ denote the open subset of Q parametrizing Λ-modules $\mathcal{F}$ which are locally free along ξ, and let $Q^{n\mathrm{lf}(\xi)} = Q - Q^{\mathrm{lf}(\xi)}$ be the complementary closed subset. Both of these sets are invariant under the action of $\mathrm{Sl}(V)$. Let $Q^{\mathrm{LF}(\xi)}$ denote the subset of Q parametrizing Λ-modules $\mathcal{F}$ satisfying condition $\mathrm{LF}(\xi)$. It is contained in $Q^{\mathrm{lf}(\xi)}$ by 4.8.1. Let $\varphi : Q \to M(\Lambda, P)$ denote the quotient of Theorem 4.7. Since condition $\mathrm{LF}(\xi(s))$ depends only on the $\mathrm{gr}(\mathcal{F}_s)$, and two points q and q' in Q map to the same point in $\mathbf{M}(\Lambda, P)$ if and only if $\mathrm{gr}(\mathcal{F}_q^{\mathrm{univ}}) \cong \mathrm{gr}(\mathcal{F}_{q'}^{\mathrm{univ}})$, there is a subset $\mathbf{M}^{\mathrm{LF}(\xi)}(\Lambda, P) \subset \mathbf{M}(\Lambda, P)$ such that $Q^{\mathrm{LF}(\xi)} = \varphi^{-1}(\mathbf{M}^{\mathrm{LF}(\xi)}(\Lambda, P))$. A point $y \in \mathbf{M}(\Lambda, P)$ is contained in $\mathbf{M}^{\mathrm{LF}(\xi)}(\Lambda, P)$ if and only if $\varphi^{-1}(y)$ is contained in $Q^{\mathrm{lf}(\xi)}$ (note that if the closed orbit lying over y is contained in $Q^{\mathrm{lf}(\xi)}$ then y satisfies condition $\mathrm{LF}(\xi)$). But the image of the closed $\mathrm{Sl}(V)$-invariant subset $Q^{n\mathrm{lf}(\xi)}$ by the good quotient φ is closed [Mu]. Thus $\mathbf{M}^{\mathrm{LF}(\xi)}(\Lambda, P) = \mathbf{M}(\Lambda, P) - \varphi(Q^{n\mathrm{lf}(\xi)})$ is open. This implies that $Q^{\mathrm{LF}(\xi)}$ is open, and implies that the condition $\mathrm{LF}(\xi)$ is an open condition on the base S.

4.8.5. The morphism $\varphi : Q^{\mathrm{LF}(\xi)} \to \mathbf{M}^{\mathrm{LF}(\xi)}(\Lambda, P)$ is a universal categorical quotient, and thus $\mathbf{M}^{\mathrm{LF}(\xi)}(\Lambda, P)$ universally co-represents the functor $\mathbf{M}^{\mathrm{LF}(\xi), \natural}(\Lambda, P)$ which associates to an S-scheme S' the set of p-semistable Λ-modules $\mathcal{E}$ on X'/S' with Hilbert polynomial P, satisfying condition $\mathrm{LF}(\xi)$.

Lemma **4.9.** — *Suppose that* $\mathcal{E}$ *and* $\mathcal{F}$ *are* p*-semistable* Λ*-modules with the same normalized Hilbert polynomial on* X/S, *both satisfying condition* $\mathrm{LF}(\xi)$. *Suppose that* $f : \mathcal{E} \to \mathcal{F}$ *is an endomorphism of* Λ*-modules such that* $\xi^*(f) = 0$. *Then* $f = 0$.

Proof. — First consider the case when $S = \mathrm{Spec}(\mathbf{C})$. Then $\ker(f)$ is a sub-Λ-module of $\mathscr{E}$ and $\mathrm{im}(f)$ is a sub-Λ-module of $\mathscr{F}$, so the normalized Hilbert polynomials of $\ker(f)$ and $\mathrm{im}(f)$ are less than or equal to the normalized Hilbert polynomial of $\mathscr{F}$. The exact sequence

$$0 \to \ker(f) \to \mathscr{F} \to \mathrm{im}(f) \to 0$$

implies that the normalized Hilbert polynomials of $\ker(f)$ and $\mathrm{im}(f)$ must be equal to the normalized Hilbert polynomial of $\mathscr{F}$. Then $\mathrm{gr}(\mathscr{F}) = \mathrm{gr}(\ker(f)) \oplus \mathrm{gr}(\mathrm{im}(f))$ and $\mathrm{gr}(\mathscr{E}) = \mathrm{gr}(\mathrm{im}(f)) \oplus \mathrm{gr}(\mathrm{coker}(f))$. By condition LF, this implies that $\ker(f)$ and $\mathrm{im}(f)$, and $\mathrm{coker}(f)$ are locally free at $\xi(s)$. From the condition that $\xi^*(f) = 0$ it follows that $\mathrm{im}(f) = 0$ in an open neighborhood of $\xi(s)$. Since $\mathrm{im}(f)$ has pure dimension $d = \dim(X)$ and X is irreducible, this implies that $\mathrm{im}(f) = 0$.

We now treat the general case. Suppose, in the situation of the lemma, that $f \neq 0$. Then by Krull's theorem, we may find a base change $p : S' \to S$ where S' is the spectrum of an artinian local $\mathbf{C}$-algebra of finite type, such that $p^*(f) \neq 0$. We may then replace S by S'. In this way, we have reduced to the case (which we assume from now on) that $S = \mathrm{Spec}(R)$ with R an artinian local $\mathbf{C}$-algebra of finite type. Let $\mathbf{m}$ denote the maximal ideal of R. We may choose k and ℓ such that

$$f : \mathbf{m}^k \mathscr{E} \to \mathscr{F}/\mathbf{m}^l \mathscr{F}$$

is nonzero, but such that the maps

$$f : \mathbf{m}^{k+1} \mathscr{E} \to \mathscr{F}/\mathbf{m}^l \mathscr{F}$$

and
$$f : \mathbf{m}^k \mathscr{E} \to \mathscr{F}/\mathbf{m}^{l-1} \mathscr{F}$$

are zero. Thus we get a nonzero map

$$f_{kl} : \mathbf{m}^k \mathscr{E}/\mathbf{m}^{k+1} \mathscr{E} \to \mathbf{m}^{l-1} \mathscr{F}/\mathbf{m}^l \mathscr{F}.$$

The source and target are Λ-modules on X, and the map f_{kl} is a morphism of Λ-modules on X, because R is contained in the center of Λ.

Since $\mathscr{E}$ and $\mathscr{F}$ are flat over S, the source and target of f_{kl} are, respectively, $(\mathscr{E}/\mathbf{m}\mathscr{E}) \otimes_{R/\mathbf{m}} (\mathbf{m}^k/\mathbf{m}^{k+1})$ and $(\mathscr{F}/\mathbf{m}\mathscr{F}) \otimes_{R/\mathbf{m}} (\mathbf{m}^{l-1}/\mathbf{m}^l)$. Since R is contained in the center of Λ, these identifications of the source and target are compatible with the Λ-module structure. Thus the source and target of f_{kl} are Λ-modules on $X_0 = X \times_{\mathrm{Spec}(R)} \mathrm{Spec}(R/\mathbf{m})$, and f_{kl} is a morphism. The tensor product formulas show that the source and target are actually direct sums of $\mathscr{E}_0 = \mathscr{E} \otimes_R (R/\mathbf{m})$ and $\mathscr{F}_0 = \mathscr{F} \otimes_R (R/\mathbf{m})$. In particular, the source and target of f_{kl} are p-semistable Λ-modules on X_0 satisfying condition LF(ξ). We have $\xi^*(f_{kl}) = 0$, so the lemma applied to X_0 over $\mathrm{Spec}(\mathbf{C})$ (proved in the first paragraph) implies that $f_{kl} = 0$. This contradiction completes the proof. □

The universal object $\mathscr{F}^{\mathrm{univ}}$ on $\mathrm{X} \times_{\mathrm{S}} \mathrm{Q}^{\mathrm{LF}(\xi)}$ is locally free along the universal section $\xi : \mathrm{Q}^{\mathrm{LF}(\xi)} \to \mathrm{X} \times_{\mathrm{S}} \mathrm{Q}^{\mathrm{LF}(\xi)}$. The action of the group $\mathrm{Gl}(\mathrm{V})$ on Q preserves $\mathrm{Q}^{\mathrm{LF}(\xi)}$ and it lifts to an action on the universal object $\mathscr{F}^{\mathrm{univ}}$. We obtain a locally free sheaf $\xi^*(\mathscr{F}^{\mathrm{univ}})$ over $\mathrm{Q}^{\mathrm{LF}(\xi)}$, with action of $\mathrm{Gl}(\mathrm{V})$. Let $\mathrm{T} \to \mathrm{Q}^{\mathrm{LF}(\xi)}$ be the frame bundle of $\xi^*(\mathscr{F}^{\mathrm{univ}})$. It is a principal $\mathrm{Gl}(n, \mathbf{C})$-bundle over $\mathrm{Q}^{\mathrm{LF}(\xi)}$ representing the functor which associates to any S-scheme S′, the set of all triples $(\mathscr{E}, \alpha, \beta)$ where: $\mathscr{E}$ is a p-semistable Λ-module with Hilbert polynomial P on X′/S′, satisfying condition $\mathrm{LF}(\xi)$; $\alpha : \mathscr{O}_{\mathrm{S}'} \otimes_{\mathbf{C}} \mathrm{V} \xrightarrow{\cong} \mathrm{H}^0(\mathrm{X}'/\mathrm{S}', \mathscr{F}(\mathrm{N}))$; and $\beta : \xi^*(\mathscr{E}) \xrightarrow{\cong} \mathscr{O}^n_{\mathrm{S}'}$. The group $\mathrm{Gl}(\mathrm{V}) \times \mathrm{Gl}(n, \mathbf{C})$ acts on T, compatibly with the action of $\mathrm{Gl}(\mathrm{V})$ on $\mathrm{Q}^{\mathrm{LF}(\xi)}$. The action can be defined by the action on the functor.

Remark. — The center $\mathbf{G}_m \subset \mathrm{Gl}(\mathrm{V})$ acts trivially on Q, because a scaling of the frame α can be removed by a scalar endomorphism of $\mathscr{F}$. However, this no longer works when we go to the frame bundle T. Only a diagonal $\mathbf{G}_m \subset \mathrm{Gl}(\mathrm{V}) \times \mathrm{Gl}(n, \mathbf{C})$ acts trivially.

Recall that $\mathscr{L}_m$ denotes the very ample line bundle on $\mathrm{Q}^{\mathrm{LF}(\xi)}$ induced by a projective embedding ψ_m. We may choose a linearization of the action of $\mathrm{Gl}(\mathrm{V})$ on $\mathscr{L}_m^{\otimes b}$ in such a way that the center $\mathbf{G}_m$ acts trivially. For example, a power $\mathscr{L}_m^{\otimes b}$ descends to a line bundle on the categorical quotient $\mathbf{M}^{\mathrm{LF}(\xi)} = \mathrm{Q}^{\mathrm{LF}(\xi)}/\mathrm{Sl}(\mathrm{V})$. Since the map $\mathrm{Q}^{\mathrm{LF}(\xi)} \to \mathbf{M}^{\mathrm{LF}(\xi)}$ is $\mathrm{Gl}(\mathrm{V})$-invariant, the trivial action on $\mathscr{L}_m^{\otimes b}$ on $\mathbf{M}^{\mathrm{LF}(\xi)}$ pulls back to a linearization of $\mathscr{L}_m^{\otimes b}$ on $\mathrm{Q}^{\mathrm{LF}(\xi)}$. All points of $\mathrm{Q}^{\mathrm{LF}(\xi)}$ are semistable points for this action of $\mathrm{Gl}(\mathrm{V})$. The line bundle $\mathscr{L}_m^{\otimes b}$ on $\mathrm{Q}^{\mathrm{LF}(\xi)}$ has a linearization with respect to the group $\mathrm{Gl}(\mathrm{V}) \times \mathrm{Gl}(n)$, where the second factor acts trivially. Let L denote the pullback of the $\mathrm{Gl}(\mathrm{V}) \times \mathrm{Gl}(n)$-linearized bundle $\mathscr{L}_m^{\otimes b}$ to T.

Theorem **4.10.** — *Every point of* T *is stable for the action of* $\mathrm{Gl}(\mathrm{V})$ *with respect to the linearized line bundle* L, *and the action of* $\mathrm{Gl}(\mathrm{V})$ *on* T *is free. The resulting geometric quotient* $\mathbf{R}(\Lambda, \xi, \mathrm{P}) \overset{\mathrm{def}}{=} \mathrm{T}/\mathrm{Gl}(\mathrm{V})$ *represents a functor—it parametrizes pairs* $(\mathscr{F}, \beta)$ *where* $\mathscr{F}$ *is a* p*-semistable* Λ*-module on* X′ *over* S′ *with Hilbert polynomial* P *on* X′ *over* S′, *satisfying condition* $\mathrm{LF}(\xi)$, *and* $\beta : \xi^*(\mathscr{F}) \cong \mathscr{O}^n_{\mathrm{S}'}$ *is a frame. The group* $\mathrm{Gl}(n, \mathbf{C})$ *acts on* $\mathbf{R}(\Lambda, \xi, \mathrm{P})$, *and every point is semistable for this action (with respect to the linearized line bundle obtained from* L*). The good quotient* $\mathbf{R}(\Lambda, \xi, \mathrm{P})/\mathrm{Gl}(n)$ *is naturally equal to the moduli space* $\mathbf{M}^{\mathrm{LF}(\xi)}(\Lambda, \mathrm{P})$. *The closed orbits in* $\mathbf{R}(\Lambda, \xi, \mathrm{P})$ *correspond to the* Λ*-modules which are direct sums of* p*-stable ones. The subset* $\mathbf{R}^s(\Lambda, \xi, \mathrm{P})$ *of properly stable points for the action of* $\mathrm{Sl}(n, \mathbf{C})$ *is exactly the set of points corresponding to* p*-stable* Λ*-modules.*

Proof. — The projection $\pi : \mathrm{T} \to \mathrm{Q}^{\mathrm{LF}(\xi)}$ is an affine map, since T is the frame bundle for a locally free sheaf. All of the points of $\mathrm{Q}^{\mathrm{LF}(\xi)}$ are semistable for the action of $\mathrm{Gl}(\mathrm{V})$ and linearized line bundle $\mathscr{L}_m^{\otimes b}$. Thus if $q \in \mathrm{T}$ is any point, then there is a section $\sigma \in \mathrm{H}^0(\mathrm{Q}^{\mathrm{LF}(\xi)}, \mathscr{L}_m^{\otimes ab})$ which is $\mathrm{Gl}(\mathrm{V})$-invariant, such that $(\mathrm{Q}^{\mathrm{LF}(\xi)})_{\sigma \neq 0}$ is affine, and such that $\sigma(\pi(q)) \neq 0$. Now σ pulls back to an invariant section of $\mathrm{L}^{\otimes a}$ which doesn't vanish at q. The fact that π is an affine map means that the set $T_{\sigma \neq 0} = \pi^{-1}(Q^{LF(\xi)}_{\sigma \neq 0})$ is affine. Thus q is a semistable point of T. To prove that every point of T is stable, we must show that the orbits are closed and the stabilizers are finite. The points of T lying over $s \in S$ consist of triples (E, α, β) with $\alpha : H^0(X_s, E(N)) \cong V$ and $\beta : E_{\xi(s)} \cong \mathbf{C}^n$. If $h : (E, \alpha, \beta) \cong (E, g\alpha, \beta)$, then h restricts to the identity at $\xi(s)$, so $h = 1$ by Lemma 4.9, $g\alpha = \alpha$, and hence $g = 1$. Thus the stabilizer of (E, α, β) is equal to $\{ 1 \}$. In particular, the dimensions of the orbits are all the same. Thus no orbit can be contained in the closure of another orbit, so the orbits are closed. This proves that all points are stable. By [Mu], a geometric quotient $\mathbf{R}(\Lambda, \xi, P) = T/Gl(V)$ exists.

To show that the action is free, note that the map $Gl(V) \times T \to T \times_{\mathbf{R}(\Lambda, \xi, P)} T$ sending (g, q) to $(g(q), q)$ is proper, by Corollary 2.5 of [Mu]. Suppose S' is an S-scheme, $q, q' \in T(S')$ and $g, g' \in Gl(V)(S')$, such that $(g(q), q) = (g'(q'), q')$. Then $q = q'$. Let (E, α, β) be the triple corresponding to the point q. Then the fact that $g(q) = g'(q)$ means that there is an isomorphism of triples $h : (E, \alpha, \beta) \cong (E, g^{-1} g' \alpha, \beta)$. Then $\xi^*(h) = 1$, so $h = 1$ by Lemma 4.9. Thus $g^{-1} g' = 1$. Thus the map

$$Gl(V) \times T \to T \times_{\mathbf{R}(\Lambda, \xi, P)} T$$

is an inclusion of functors. A proper map which is an inclusion of functors is a closed immersion. This means that the action is free, hence T is a principal Gl(V) bundle over $\mathbf{R}(\Lambda, \xi, P)$ [Mu].

There is a universal Λ-module $\mathscr{F}^{univ}$ on $X \times_S T$ over T, and the action of Gl(V) on T lifts to an action on $\mathscr{F}^{univ}$. This gives descent data, so $\mathscr{F}^{univ}$ descends to a universal p-semistable Λ-module $\mathscr{F}^{R}$ with Hilbert polynomial P on $X \times_S \mathbf{R}(\Lambda, \xi, P)$ over $\mathbf{R}(\Lambda, \xi, P)$. Furthermore, we have a frame $\beta : \xi^*(\mathscr{F}) \cong \mathcal{O}^n_{\mathbf{R}(\Lambda, \xi, P)}$.

The scheme $\mathbf{R}(\Lambda, \xi, P)$ together with the universal object $(\mathscr{F}^{R}, \beta^{R})$ represents the required functor (which we denote by $\mathbf{R}^{\natural}(\Lambda, \xi, P)$). If q_1 and q_2 are points in $\mathbf{R}(\Lambda, \xi, P)(S')$ such that $q_1^*(\mathscr{F}^{R}, \beta^{R}) \cong q_2^*(\mathscr{F}^{R}, \beta^{R})$ then locally on S' we can lift to points $(\mathscr{F}_i, \alpha_i, \beta_i)$ of T with $(\mathscr{F}_1, \beta_1) \cong (\mathscr{F}_2, \beta_2)$. Then α_1 and α_2 differ by a change of frame given by $g \in Gl(V)(S')$. Thus the lifted points differ by the action of Gl(V), and the invariance of the projection π implies that the original points q_1 and q_2 were equal. Thus the morphism $\mathbf{R}(\Lambda, \xi, P) \to \mathbf{R}^{\natural}(\Lambda, \xi, P)$ is an injection of functors. If $(\mathscr{F}, \beta) \in \mathbf{R}^{\natural}(\Lambda, \xi, P)(S')$ and if $q \in \mathbf{R}(\Lambda, \xi, P)(S')$ are such that $q^*(\mathscr{F}^{R}, \beta^{R}) \cong (\mathscr{F}, \beta)$, then this isomorphism is unique by Lemma 4.9. This implies that the functor $\mathbf{R}^{\natural}(\Lambda, \xi, P)$ is a sheaf in the étale topology. If $(\mathscr{F}, \beta) \in \mathbf{R}^{\natural}(\Lambda, \xi, P)(S')$ then, locally in the étale (or Zariski) topology of S', we can choose a frame α to get a point $(\mathscr{F}, \alpha, \beta)$ in T. This point projects to a point $q \in \mathbf{R}(\Lambda, \xi, P)(S')$ with $q^*(\mathscr{F}^{R}, \beta^{R}) \cong (\mathscr{F}, \beta)$. This shows that the morphism $\mathbf{R}(\Lambda, \xi, P) \to \mathbf{R}^{\natural}(\Lambda, \xi, P)$ is a local isomorphism. Since both the source and the target are sheaves in the étale topology, this is an isomorphism of functors. Thus $\mathbf{R}(\Lambda, \xi, P)$ represents the functor.

The line bundle L descends to an ample line bundle $\mathscr{L}$ on $\mathbf{R}(\Lambda, \xi, P)$ with linearization of the action of $Gl(n, \mathbf{C})$. Note that this action comes from the trivial action on

$L = \pi^* \mathscr{L}_m^{\otimes b}$ which is due to the action of $\mathrm{Gl}(n, \mathbf{C})$ along the fibers of π. If r is any point in $\mathbf{R}(\Lambda, \xi, \mathrm{P})$, we may lift it to a point q in T, and choose a section $\sigma \in \mathrm{H}^0(\mathrm{T}, \mathrm{L}^{\otimes a})$ which doesn't vanish at q, such that $\mathrm{T}_{\sigma \neq 0}$ is affine and σ is invariant under Gl(V). Recall from above that we could choose σ to be a pull back of a section on $\mathrm{Q}^{\mathrm{LF}(\xi)}$, so we may assume that σ is also invariant under $\mathrm{Gl}(n, \mathbf{C})$. The section σ descends to a section σ' of $(\mathscr{L})^{\otimes a}$, and $\mathbf{R}(\Lambda, \xi, \mathrm{P})_{\sigma \neq 0} = \mathrm{T}_{\sigma \neq 0}/\mathrm{Gl}(\mathrm{V})$ is affine [Mu]. Note that σ' is invariant under $\mathrm{Gl}(n, \mathbf{C})$, and does not vanish at r. Thus every point r in $\mathbf{R}(\Lambda, \xi, \mathrm{P})$ is semistable with respect to the line bundle $\mathscr{L}$ and the action of $\mathrm{Gl}(n, \mathbf{C})$. By [Mu] we may form the good quotient $\mathrm{M}' = \mathbf{R}(\Lambda, \xi, \mathrm{P})/\mathrm{Gl}(n, \mathbf{C})$.

The product group $\mathrm{Gl}(\mathrm{V}) \times \mathrm{Gl}(n, \mathbf{C})$ acts on the space T. We have a composition

$$\mathrm{T} \to \mathrm{Q}^{\mathrm{LF}(\xi)} \to \mathbf{M}^{\mathrm{LF}(\xi)}(\Lambda, \mathrm{P})$$

of categorical quotients by $\mathrm{Gl}(n, \mathbf{C})$ and Gl(V) respectively. The first expresses T as a principal bundle over $\mathrm{Q}^{\mathrm{LF}(\xi)}$, and it is preserved by the second group Gl(V). In this situation it follows that $\mathbf{M}^{\mathrm{LF}(\xi)}(\Lambda, \mathrm{P})$ is a categorical quotient of T by $\mathrm{Gl}(\mathrm{V}) \times \mathrm{Gl}(n)$. We also have a composition

$$\mathrm{T} \to \mathbf{R}(\Lambda, \xi, \mathrm{P}) \to \mathbf{M}'$$

of categorical quotients in the other order, and the first map is preserved by $\mathrm{Gl}(n, \mathbf{C})$ as can be seen by looking at the functors represented by T and $\mathbf{R}(\Lambda, \xi, \mathrm{P})$. Again, T is a principal bundle over $\mathbf{R}(\Lambda, \xi, \mathrm{P})$. Thus $\mathbf{M}'$ is a categorical quotient of T. Therefore there exists a unique isomorphism $\mathbf{M}^{\mathrm{LF}(\xi)}(\Lambda, \mathrm{P}) \cong \mathbf{M}'$ commuting with the maps from T. Thus the good quotient $\mathbf{R}(\Lambda, \xi, \mathrm{P})/\mathrm{Gl}(n, \mathbf{C})$ is naturally isomorphic to $\mathbf{M}^{\mathrm{LF}(\xi)}$.

Finally, we have to prove that the properly stable points for the action of $\mathrm{Sl}(n, \mathbf{C})$ are the p-stable Λ-modules. The stabilizer in $\mathrm{Gl}(n, \mathbf{C})$ of a point $(\mathscr{F}, \beta)$ is equal to the stabilizer in $\mathrm{Gl}(\mathrm{V}) \times \mathrm{Gl}(n)$ of the point $(\mathscr{F}, \alpha, \beta)$, and this in turn is equal to the stabilizer of $(\mathscr{F}, \alpha)$ in Gl(V); this stabilizer is naturally identified with the group of automorphisms of $\mathscr{F}$. Furthermore, in either of the two compositions above, an orbit of the second group is closed if and only if its preimage in T is a closed orbit for the product of the two groups. In the first case, an orbit of Gl(V) is closed if and only if the Λ-module $\mathscr{F}$ is a direct sum of p-stable ones; thus this criterion holds in the second case too. The closed orbits in $\mathbf{R}(\Lambda, \xi, \mathrm{P})$ correspond to the Λ-modules which are direct sums of p-stable Λ-modules. The group of automorphisms of determinant one of such a direct sum is finite if and only if the sum has exactly one p-stable component. Therefore a point $(\mathscr{F}, \beta) \in \mathbf{R}(\Lambda, \xi, \mathrm{P})$ has closed orbit and finite stabilizer in $\mathrm{Sl}(n, \mathbf{C})$, if and only if $\mathscr{F}$ is p-stable. This completes the proof of the theorem. □

Remark. — Since the space $\mathbf{R}(\Lambda, \xi, \mathrm{P})$ represents a functor, it is easy to form moduli spaces for p-semistable Λ-modules satisfying the condition $\mathrm{LF}(\xi_1, \ldots, \xi_k)$ that the $\mathrm{gr}(\mathscr{F}_s)$ are locally free at all $\xi_i(s)$, with frames at the points $\xi_1, \ldots, \xi_k$. Choose one of the points for defining $\mathbf{R}(\Lambda, \xi, \mathrm{P})$, and let $\mathbf{R}^{\mathrm{LF}(\xi_1, \ldots, \xi_k)}(\Lambda, \xi_1, \mathrm{P})$ denote the open subset representing the condition $\mathrm{LF}(\xi_1, \ldots, \xi_k)$. The scheme $\mathbf{R}(\Lambda, \xi_1, \ldots, \xi_k, P)$ of objects with frames at all the points is just a principal bundle over $\mathbf{R}^{\mathrm{LF}(\xi_1, \ldots, \xi_k)}(\Lambda, \xi_1, P)$, the bundle of sets of frames for the restrictions of the universal bundle to the various other sections.

We say that $\mathscr{F}$ *satisfies condition* $\mathrm{LF}(X)$ if $\mathscr{F}$ is locally free as a sheaf of $\mathcal{O}_X$-modules on all of X. This is also preserved by base change and, by the same argument as in 4.8.4, it is an open condition over the base. We obtain an open subset $\mathbf{R}^{\mathrm{LF}(X)}(\Lambda, \xi, P) \subset \mathbf{R}(\Lambda, \xi, P)$ parametrizing $(\mathscr{F}, \beta)$ where $\mathscr{F}$ satisfies $\mathrm{LF}(X)$ and β is a frame.

5. Analytic theory

Define the notion of *analytic sheaf of rings of differential operators* on a complex analytic space in exactly the same way as for the algebraic case (we avoid repeating the axioms here). Our first task is to show that an algebraic sheaf of rings of differential operators Λ on X over S gives rise to an analytic Λ^{an} on X^{an} over S^{an}. We include the more general situation of a base change in the following proposition.

Proposition **5.1.** — *Suppose* X *is quasiprojective over a noetherian algebraic base scheme* S. *Suppose* S′ *is a complex analytic space and* $S' \to S^{an}$ *is a morphism. Let* $X' = X^{an} \times_{S^{an}} S'$ *denote the analytic space obtained by base change. Suppose that* Λ *is a sheaf of rings of differential operators on* X *over* S. *Let* Λ_i^l *and* Λ_i^r *denote the pullbacks to* X′ *of the coherent analytic sheaves on* X^{an} *corresponding to the coherent sheaf* Λ_i *of left or right* $\mathcal{O}_X$*-modules. There is a natural isomorphism* $\Lambda_i^l \cong \Lambda_i^r$ *equal to the identity on algebraic sections. Put* Λ_i^1 *equal to this sheaf, and set*

$$\Lambda_i' = \lim_{\to, j} \operatorname{im}(\Lambda_i^1 \to \Lambda_j^1).$$

Then the union $\Lambda' = \bigcup_i \Lambda_i'$ *(with the given filtration) has a structure of analytic sheaf of rings of differential operators on* X′ *over* S′.

Proof. — First we treat the case $S' = S^{an}$. The sheaf $\mathrm{Gr}_1(\Lambda)$ acts on $\mathcal{O}_X$ by a derivation given by the symbol $\sigma : \mathrm{Gr}_1(\Lambda) \to \mathrm{Hom}(\Omega^1_{X/S}, \mathcal{O}_X)$. But $\Omega^1_{X^{an}/S^{an}}$ is the analytic sheaf associated to $\Omega^1_{X/S}$, so we obtain a derivation

$$\sigma' : \mathrm{Gr}_1(\Lambda)^{an} \to \mathrm{Hom}(\Omega^1_{X^{an}/S^{an}}, \mathcal{O}_{X^{an}}).$$

If $X = \mathrm{Spec}(B)$ is affine, and Λ corresponds to a ring L as explained in § 2, then the left and right analytic sheaves are given by the formulas (for $U \subset X^{an}$ open)

$$\Lambda_i^l(U) = \mathcal{O}_{X^{an}}(U) \otimes_B L_i$$

and

$$\Lambda_i^r(U) = L_i \otimes_B \mathcal{O}_{X^{an}}(U).$$

Using the derivation σ we obtain rules for passing elements of $\mathcal{O}_{X^{an}}(U)$ through expressions which are products of elements in L_1. These give isomorphisms

$$\mathcal{O}_{X^{an}}(U) \otimes_B L_i \cong L_i \otimes_B \mathcal{O}_{X^{an}}(U),$$

and hence $\Lambda_i^l \cong \Lambda_i^r$. If X is not affine, the isomorphisms thus obtained over the subsets in an affine covering glue together to give $\Lambda_i^l \cong \Lambda_i^r$. We obtain sheaves Λ_i^{an} which have left and right structures of coherent $\mathcal{O}_{X^{an}}$-modules. In this case, $\mathcal{O}_{X^{an}}$ is flat over $\mathcal{O}_X$, so the maps $\Lambda_i^{an} \to \Lambda_j^{an}$ are injective for $i \leqslant j$. Thus Λ^{an} is the union of the Λ_i^{an}. The left and right $\mathcal{O}_{X^{an}}$-module structures agree on the associated graded $\mathrm{Gr}(\Lambda^{an})$. One can verify all of the properties which correspond to 2.1.1-2.1.6.

Suppose now that $S' \to S^{an}$ is a morphism of complex analytic spaces. We make the base change from Λ^{an} on X^{an} to Λ' on X', by the same construction as in Lemmas 2.5 and 2.6, but with Stein open sets playing the role of the affine open sets. The construction was based on two things: the notion of coherent sheaf associated to a module over the coordinate ring, which works also in the analytic case with a little bit of extra care (one must allow for sections over a subset those which are locally tensor products of elements of the module with functions in the coordinate ring of the subset); and the tensor product for obtaining the module which gives rise to the pullback sheaf on the product. In the analytic case, we use a completed tensor product for the Frechet topology. This means that one includes among the sections, limits of sequences of elements in the tensor product which converge uniformly on compact sets. The Frechet topologies on the coherent sheaves Λ_i considered as left and right $\mathcal{O}_{X^{an}}$-modules are the same: fix a finite set of sections which span the space of germs of sections around a compact subset, with respect to the left module structure; a convergent sequence of sections may be represented as a sequence of expressions in terms of the spanning set, with converging coefficients; when the expressions are changed into expressions with the coefficients on the right, some new terms are introduced depending on derivatives of the original coefficients; but the derivatives will also converge on a smaller compact subset, so the coefficients of the expressions on the right converge also. We obtain the isomorphisms $\Lambda_i^l \cong \Lambda_i^r$ from the formulas

$$B' \hat{\otimes}_B L_i \cong A' \hat{\otimes}_A L_i = L_i \hat{\otimes}_A A' \cong L_i \hat{\otimes}_B B',$$

where A, A', B and B' denote the rings of functions on Stein open subsets of S^{an}, S', X^{an}, and X' respectively, and L_i denote the modules corresponding to Λ_i^{an}. The middle equality depends on the fact that the left and right Frechet topologies on L_i are the same. Let $\Lambda_i' = \lim_{\to, j} \mathrm{im}(\Lambda_i^r \to \Lambda_j^r)$. The Λ_i' are locally equal to images of Λ_i^r, since the local rings of X' are noetherian. Hence the Λ_i' are coherent sheaves of $\mathcal{O}_{X'}$-modules for the left and right structures. The union $\Lambda' = \bigcup_i \Lambda_i'$ satisfies all of the necessary properties. □

Lemma **5.2.** — *Suppose $\mathcal{E}$ is an $\mathcal{O}_X$-coherent Λ-module on* X. *Suppose* $S' \to S^{an}$ *is a morphism of complex analytic spaces. Then the pullback $\mathcal{E}'$ of the associated analytic sheaf $\mathcal{E}^{an}$ to* X' *has a structure of Λ'-module.*

Proof. — The construction is again the same as in Lemma 2.7, with the remarks made in the previous proof taken into account. □

Analytic properties of Hilbert schemes

Suppose $X \to S$ is a projective morphism of algebraic schemes, with a relatively very ample $\mathcal{O}_X(1)$ chosen. Suppose that $\mathcal{W}$ is a coherent sheaf on X. Fix a polynomial P and let $H \to S$ denote the Hilbert scheme parametrizing quotients

$$\mathcal{W} \to \mathcal{E} \to 0$$

with Hilbert polynomial P (relative to S).

Proposition **5.3.** — *The associated complex analytic space* $H^{an} \to S^{an}$ *represents the functor which associates to each analytic space* $S' \to S^{an}$ *the set of coherent analytic quotients* $\mathcal{E}$ *of the pullback* $\mathcal{W}'$ *of* $\mathcal{W}^{an}$ *to* $X' = X^{an} \times_{S^{an}} S'$, *which are flat over* S' *and have relative Hilbert polynomial* P.

Proof. — This follows from Douady's theory. In fact, it is easier, since we begin with X and $\mathcal{W}$ algebraic, so we indicate the proof here. We may suppose that X is flat over S (replacing X by a projective space into which it embedds, if necessary). There is an integer n and a surjective morphism

$$\mathcal{O}_X(-n)^b \to \mathcal{W} \to 0.$$

Let $\mathcal{W}_H$ denote the pullback of $\mathcal{W}$ to $X \times_S H$. We obtain a surjection

$$\mathcal{O}_{X \times_S H}(-n)^b \to \mathcal{W}_H \to 0,$$

and composition with the universal quotient gives a surjection

$$\mathcal{O}_{X \times_S H}(-n)^b \to \mathcal{E}^{univ} \to 0$$

on $X \times_S H$. There is an integer m such that for any $s \in S$ and any quotient $\mathcal{E}_s$ of $\mathcal{W}_s$ with Hilbert polynomial P, we have $h^0(X_s, \mathcal{E}_s(m)) = P(m)$ and $H^i(X_s, \mathcal{E}_s(m)) = 0$ for $i > 0$. We may also assume that the same is true for $\mathcal{O}_X(m-n)$ (with its own Hilbert polynomial). For any quotient $\mathcal{E}_s$, let

$$0 \to \mathcal{K}_s \to \mathcal{O}_{X_s}(-n)^b \to \mathcal{E}_s \to 0$$

denote the kernel; we may suppose also that $H^i(X_s, \mathcal{K}_s(m)) = 0$ for $i > 0$. The base change theorems—those of Grauert for the complex analytic case—imply that if $S' \to S^{an}$ is a morphism of complex analytic spaces, and if $\mathcal{E}$ is any quotient of $\mathcal{W}'$ on $X' = X^{an} \times_{S^{an}} S'$ flat over S' with Hilbert polynomial P, then $H^0(X'/S', \mathcal{E}(m))$ and $H^0(X'/S', \mathcal{O}_{X'}(m-n))$ are locally free over S' and commute with further base change; that the higher direct images vanish; and that the map

$$H^0(X'/S', \mathcal{O}_{X'}(m-n)^b) \to H^0(X'/S', \mathcal{E}(m))$$

is surjective (and the same in the case of an algebraic base change). Finally, the universal quotient $\mathscr{W}_{\mathrm{H}} \to \mathscr{E}^{\mathrm{univ}}$ on $\mathrm{X} \times_{\mathrm{S}} \mathrm{H}$ gives rise to the surjection of locally free sheaves on H

$$\mathrm{H}^0(\mathrm{X} \times_{\mathrm{S}} \mathrm{H}/\mathrm{H}, \mathscr{O}_{\mathrm{X} \times_{\mathrm{S}} \mathrm{H}}(m-n)^b) \to \mathrm{H}^0(\mathrm{X} \times_{\mathrm{S}} \mathrm{H}/\mathrm{H}, \mathscr{E}^{\mathrm{univ}}(m)) \to 0,$$

hence to a map into the relative Grassmanian

$$\mathrm{H} \to \mathbf{Grass}_{\mathrm{S}}(\mathrm{H}^0(\mathrm{X}/\mathrm{S}, \mathscr{O}_{\mathrm{X}}(m-n)^b), \mathrm{P}(m)).$$

We may choose m so that this map is a closed embedding.

Now we complete the proof of the proposition. Suppose $\mathrm{S}' \to \mathrm{S}^{\mathrm{an}}$ is a morphism of complex analytic spaces. If $\mathrm{S}' \to \mathrm{H}^{\mathrm{an}}$ is a morphism, then the pullback of the universal quotient $(\mathscr{E}^{\mathrm{univ}})^{\mathrm{an}}$ over H^{an} gives an element of the specified set. Suppose, on the other hand, that we are given a coherent analytic quotient $\mathscr{W}' \to \mathscr{E} \to 0$ flat over S' with relative Hilbert polynomial P. We obtain a surjection of locally free sheaves on S',

$$\mathrm{H}^0(\mathrm{X}'/\mathrm{S}', \mathscr{O}_{\mathrm{X}'}(m-n)^b) \to \mathrm{H}^0(\mathrm{X}'/\mathrm{S}', \mathscr{E}(m)) \to 0.$$

The sheaf on the left is the pullback to S' of $\mathrm{H}^0(\mathrm{X}/\mathrm{S}, \mathscr{O}_{\mathrm{X}}(m-n)^b)$, so we obtain a holomorphic map

$$\varphi : \mathrm{S}' \to \mathbf{Grass}_{\mathrm{S}}(\mathrm{H}^0(\mathrm{X}/\mathrm{S}, \mathscr{O}_{\mathrm{X}}(m-n)^b), \mathrm{P}(m)).$$

On the other hand, we know that the proposition is true for base changes to the formal completions of S' at all points. This implies that, when restricted to the formal completions, the map φ has image in H. This implies that the functions in the ideal defining H restrict to zero on S', so we obtain a map $\mathrm{S}' \to \mathrm{H}$. It remains to be seen that these two constructions are inverses. If we start with a map $\mathrm{S}' \to \mathrm{H}$, take the corresponding quotient, and apply the above construction, we return to the same map. If we start with a quotient $\mathscr{E}$ on X', apply the above construction to obtain a map $\mathrm{S}' \to \mathrm{H}$, and take the pullback of the universal quotient, we obtain another quotient $\mathscr{E}^1$ of $\mathscr{W}'$. But restricted to all of the formal completions, these two quotients are the same. The condition for a section of $\mathscr{W}'$ to be contained in the kernel of one of the quotients can be tested in the formal completions; thus the kernels are the same, so the quotients are the same. □

Proposition **5.4.** — *Suppose* X *is projective over a base scheme* S, *and* Λ *is a sheaf of rings of differential operators on* X *over* S. *Fix a relatively very ample* $\mathscr{O}_{\mathrm{X}}(1)$ *and a polynomial* P. *Let* N *and* r *be the integers and* Q *be the* S*-scheme constructed Theorem* 3.8. *Then the associated complex analytic space* Q^{an} *over* S^{an} *represents the functor which, to each morphism of analytic spaces* $\mathrm{S}' \to \mathrm{S}^{\mathrm{an}}$, *associates the set of isomorphism classes of pairs* $(\mathscr{E}, \alpha)$, *where*: $\mathscr{E}$ *is a semistable* Λ'*-module on* $\mathrm{X}' = \mathrm{X} \times_{\mathrm{S}} \mathrm{S}'$, $\mathscr{O}_{\mathrm{X}'}$*-coherent and flat over* S', *with all fibers* $\mathscr{E}_s$ *being p-semistable, and with Hilbert polynomial* P; *and*

$$\alpha : (\mathscr{O}_{\mathrm{S}'})^{p(\mathrm{N})} \xrightarrow{\cong} \mathrm{H}^0(\mathrm{X}'/\mathrm{S}', \mathscr{E}(\mathrm{N})).$$

Proof. — First we establish a GAGA principle: suppose T is an artinian scheme of finite type over **C**, and $\mathrm{T} \to \mathrm{S}$ is a morphism; then any $\Lambda_{\mathrm{T}}^{\mathrm{an}}$-module $\mathscr{E}_{\mathrm{T}}$ on the fiber

$X_T = X \times_S T$ is algebraic. Since X_T is projective over a complex artinian scheme, the coherent sheaf of $\mathcal{O}_{X_T^{an}}$-modules underlying $\mathcal{E}_T$ is algebraic, obtained from the coherent algebraic sheaf $\mathcal{E}_T^{alg}$. The sheaf $\Lambda_{1,s}^{an} \otimes_{\mathcal{O}_{X_T^{an}}} \mathcal{E}_T$ is equal to the analytic sheaf associated to $\Lambda_{1,s} \otimes_{\mathcal{O}_{X_T}} \mathcal{E}_T^{alg}$ (this isomorphism may be constructed using the methods of Lemma 2.4 and Proposition 5.1). The Λ^{an}-module structure of $\mathcal{E}_T$ is determined by a map

$$\Lambda_{1,s}^{an} \otimes_{\mathcal{O}_{X_T^{an}}} \mathcal{E}_T \to \mathcal{E}_T$$

of sheaves of $\mathcal{O}_{X_T^{an}}$-modules; this map is algebraic. The conditions satisfied by this map in order to give a structure of Λ_T^{an}-module are similarly satisfied by the corresponding algebraic map, so $\mathcal{E}_T^{alg}$ has a structure of Λ_T-module. Morphisms between analytic Λ_T^{an}-modules are also algebraic.

This GAGA principle implies that the p-semistable Λ_s^{an}-modules on fibers X_s over closed points form a bounded family, and that the integer N chosen for Theorem 3.8 will give the same properties here. Suppose $S' \to S^{an}$ is a morphism of complex analytic spaces and $\mathcal{E}$ is a Λ'-module on $X' = X^{an} \times_{S^{an}} S'$ which is $\mathcal{O}_{X'}$-coherent, flat over S', with relative Hilbert polynomial p, such that the fibers $\mathcal{E}_s$ over closed points $s \in S'$ are p-semistable. We obtain a map to the analytic space associated to the Hilbert scheme of quotients of $\Lambda_r \otimes_{\mathcal{O}_X} \mathcal{O}_X(-N)^{P(N)}$ with Hilbert polynomial P, by Proposition 5.3 and the same construction as before. When restricted to the formal completions at points of S', this map has image in the locally closed subscheme Q. Hence the map has image in the closure $\overline{Q}$. But all of the points map into the open set Q, we so obtain a map $S' \to Q$. As in Proposition 5.3, a map $S' \to Q$ gives rise to an element of the specified set, and these two constructions are inverses. Note that the coherent sheaf $\mathcal{E}$ appears as the quotient corresponding to the map from S' into the Hilbert scheme, so we obtain the isomorphism between the original $\mathcal{E}$ and the one given by the constructed map $S' \to Q$. □

Analytic properties of good quotients

We show that good quotients in the algebraic category give universal categorical quotients in the analytic category. In this section, the notation X plays a different role than in the rest of the paper.

Proposition **5.5.** — *Suppose* X *is a quasi-projective scheme over* **C**, *on which a semisimple group* G *acts linearizing an ample line bundle. Suppose that every point is semistable,* $X = X^{ss}$. *Let* $\varphi : X \to Y$ *be the good quotient given by* [Mu]. *Then* $\varphi^{an} : X^{an} \to Y^{an}$ *is a universal categorical quotient in the category of complex analytic spaces.*

Proof. — The proof is an analytic version of Mumford's original proof that his quotients were universal categorical quotients in the algebraic category. The only interesting point is that we seem to need some results from the theory of moment maps and symplectic quotients to obtain certain compact subsets.

First consider a special case. Assume that $X = \mathbf{A}^n$ is the affine n-space, and that G acts by a linear n-dimensional representation. Let R be the ring of G-invariants in the coordinate ring $\mathbf{C}[x_1, \ldots, x_n]$ of X. All points of X are semistable (with respect to the trivial line bundle), and the good quotient is $Y = \mathrm{Spec}(R)$. Let $\varphi : X \to Y$ denote the projection. Note that R is contained in the field $\mathbf{C}(x_1, \ldots, x_n)$, so Y is reduced and irreducible. The complex analytic space X^{an} is just $\mathbf{C}^n$. Suppose $U \subset Y^{an}$ is an open set which is a Stein space. Let $W = \varphi^{-1}(U)$. It is an open subset of $\mathbf{C}^n$, and in fact it is also Stein (it is equal to the intersection of the graph of φ, a closed subset, with the Stein space $\mathbf{C}^n \times U \subset \mathbf{C}^n \times Y^{an}$). We now prove the contention that if f is a G-invariant holomorphic function on W, then there is a holomorphic function g on U such that $f = \varphi^*(g)$.

Suppose $U_0 \subset U$ is a relatively compact open subset. Let C denote the compact closure of U_0. The first step is to show that there is a compact subset $D \subset W$ such that $\varphi(D) = C$. This relies on the theory of moment maps and symplectic quotients [GS] [KN] [Ki]. The standard embedding $X = \mathbf{A}^n \subset \mathbf{P}^n$ is compatible with a linear action of G on $\mathbf{P}^n$. This action linearizes the ample line bundle defined by the divisor at infinity $\mathbf{P}^n - X$, and with respect to this linearization, X is a subset of $(\mathbf{P}^n)^{ss}$ defined by the nonvanishing of a section of the line bundle. Hence Y is an affine open subset of the good quotient $(\mathbf{P}^n)^{ss}/G$, and X is the inverse image of Y. Let $K \subset G$ be a compact real form (a real form which is compact and meets every connected component of G). Let $\mathbf{k}$ denote the Lie algebra of K. There is a *moment map* for the action of K, which is a C^∞ function $\mu : \mathbf{P}^n \to \mathbf{k}^*$. The subset $\mu^{-1}(0) \subset \mathbf{P}^n$ is closed, and hence compact. It is K-invariant, so we can form the topological quotient $\mu^{-1}(0)/K$, which is again compact. The main facts we need are that $\mu^{-1}(0)$ is contained in the set of semistable points $(\mathbf{P}^n)^{ss}$, and that the map

$$\mu^{-1}(0)/K \to (\mathbf{P}^n)^{ss}/G$$

is a homeomorphism of usual topological spaces. See [Ki], p. 95, and the theorems thereafter. Now $\varphi^{-1}(C)$ is a closed subset of $(\mathbf{P}^n)^{ss}$, so the intersection $D = \varphi^{-1}(C) \cap \mu^{-1}(0)$ is compact. Note that $\varphi^{-1}(C)$ is contained in $W = \varphi^{-1}(U)$, so $D \subset W$. On the other hand, the map $\mu^{-1}(0) \to (\mathbf{P}^n)^{ss}/G$ is surjective by the above assertion, so $\varphi(D) = C$ as desired.

The D constructed above is K-invariant. By the Weierstrass approximation theorem, we may choose a sequence of polynomials $P_i(x_1, \ldots, x_n)$ such that

$$\sup_D | P_i - f | \to 0.$$

Let $Q_i(x) = \int_K P_i(kx)\, dk$ be the averages by the group K. Then Q_i are K-invariant but still

$$\sup_D | Q_i - f | \to 0.$$

Since $K \subset G$ is a compact real form, the polynomials Q_i are G-invariant, in other words they are elements of the ring R of algebraic regular functions on Y. They form a uniformly Cauchy sequence on the compact subset $C = \varphi(D)$. In particular, they are uniformly Cauchy on any compact subset of $U_0 \subset C$. Now the space of holomorphic functions on the reduced analytic space U_0 is complete with respect to uniform convergence on compact subsets, so the Q_i converge to a function g defined on U_0. Considered as functions on $D \subset W$, the Q_i converge to f. Now f and $h = \varphi^*(g)$ are two G-invariant holomorphic functions on $W_0 = \varphi^{-1}(U_0)$, which agree on the subset $W_0 \cap D$. This subset surjects onto U_0. Suppose $w \in W_0$, and let $u = \varphi(w)$. Let y be a point in $W_0 \cap D$ such that $\varphi(y) = u$. Then $f(y) = h(y)$. There is a unique closed G-orbit in $\varphi^{-1}(u)$ [Mu], contained in the closure of any other orbit. The functions f and h agree on the G-orbit of y, hence they agree on the closure of that orbit, so they agree on the unique closed orbit. On the other hand, f and h are constant on the closure of the G-orbit of w, which includes the closed orbit. Therefore $f(w) = h(w)$. Hence we have constructed a holomorphic function g on U_0 such that $f = \varphi^*(g)$ on W_0. This condition uniquely determines g, since φ is surjective. Finally, we may exhaust the original subset U by such relatively compact subsets U_0. The functions g defined on these subsets agree on overlaps, so they patch together to give a holomorphic function g on U with $f = \varphi^*(g)$. We have shown in this first case of $X = \mathbf{A}^n$ that Y^{an} is a categorical quotient of X^{an}, also universal for inclusions of open sets in Y^{an}.

Suppose that G acts linearly on affine space $\mathbf{A}^n$, and suppose that $X \subset \mathbf{A}^n$ is a closed G-invariant subscheme (not necessarily reduced), defined by a G-invariant ideal $I \subset \mathbf{C}[x_1, \ldots, x_n]$. Let $R = \mathbf{C}[x_1, \ldots, x_n]^G$ as before, with projection $\varphi : \mathbf{A}^n \to \mathrm{Spec}(R)$. The image $Y = \varphi(X) \subset \mathrm{Spec}(R)$ is a closed subscheme, defined by the ideal $J = I \cap R$. Since G is semisimple, we may choose a G-invariant complement to I, so

$$R/J = (\mathbf{C}[x_1, \ldots, x_n]/I)^G.$$

Thus Y is the good quotient X/G. We will show, under these circumstances, that $X^{an} \to Y^{an}$ is a universal categorical quotient in the category of complex analytic spaces. Suppose A is a complex analytic space with a morphism $a : A \to Y^{an}$. We may as well assume that A is a closed subspace of a Stein open set $V \subset \mathbf{C}^m$. Let $X' = \mathbf{A}^n \times \mathbf{A}^m$, with G acting trivially on the second factor. The quotient $Y' = X'/G$ is equal to $\mathrm{Spec}(R) \times \mathbf{A}^m$. Note that the first case treated above applies to $\varphi' : X' \to Y'$. Let $U = \mathrm{Spec}(R) \times V$, an open subset of $(Y')^{an} = \mathrm{Spec}(R)^{an} \times \mathbf{C}^m$. Note that A is now a closed analytic subspace of U (interpreting it as the graph of the map a). Furthermore,

$$X \times_Y A = (\varphi')^{-1}(A) \cap (X^{an} \times \mathbf{C}^m)$$

is a G-invariant closed analytic subspace of $W = (\varphi')^{-1}(U)$. Denote also the projection by $\varphi : X^{an} \times_{Y^{an}} A \to A$. Suppose Z is a complex analytic space, and $f : X \times_Y A \to Z$ is a G-invariant morphism. We have to show that f factors through A.

Set theoretically, f must factor through A. For if (x_1, u_1) and (x_2, u_2) are two points in $X^{an} \times_{Y^{an}} A$ which map to the same point in A, then $u_1 = u_2$, and $\varphi(x_1) = \varphi(x_2)$. Thus the closures of the orbits of x_1 and x_2 intersect in X^{an}. Hence the closures of the G-orbits of (x_1, u_1) and (x_2, u_2) intersect in $X^{an} \times_{Y^{an}} A$. Since the map f is G-invariant and the space Z is separated, this implies that $f(x_1, u_1) = f(x_2, u_2)$. Hence we obtain a function $g : A \to Z$ such that $f = g \circ \varphi$.

Recall from the earlier part of the proof that there is a subset $\mu^{-1}(0) \subset X'$ such that the map $\mu^{-1}(0) \to Y'$ is proper and surjective. Furthermore, if $y' \in Y'$ then $\mu^{-1}(0)$ meets the unique closed G-orbit in $(\varphi')^{-1}(y')$, cf. [GS] [KN] [Ki]. Let $E \subset X^{an} \times_{Y^{an}} A$ be the intersection of $\mu^{-1}(0)$ with $X^{an} \times_{Y^{an}} A$. It is proper over A. It also surjects onto A, because if $y' \in A$ then $X^{an} \times_{Y^{an}} A$ contains the closed orbit over y'. Suppose $S \subset Z$ is a closed subset. Then $f^{-1}(S) = \varphi^{-1}(g^{-1}(S))$ is a closed subset of $X^{an} \times_{Y^{an}} A$. Furthermore, $g^{-1}(S) = \varphi(f^{-1}(S) \cap E)$. But since E is proper over A, the image in A of a closed subset of E is closed. Thus $g^{-1}(S)$ is closed. This proves that the map g is continuous.

We have to show that g can be given a structure of morphism of complex analytic spaces. (This structure may not be determined by the function g we have defined so far, if A is not reduced.) Cover Z by open sets Z_α which have embeddings $Z_\alpha \subset \mathbf{C}^M$. Then $g^{-1}(Z_\alpha)$ are open subsets which cover A. In order to define the map $g : A \to Z$, we can localize, replacing A by a smaller open subset which maps into some Z_α. We may replace Z by Z_α, so as to reduce to the case where $Z \subset \mathbf{C}^M$. Keep the assumptions and notation established for A above.

Suppose $z : Z \to \mathbf{C}$ is one of the holomorphic coordinate functions from among those which induce the embedding $Z \subset \mathbf{C}^M$. We obtain a G-invariant holomorphic function $zf : X^{an} \times_{Y^{an}} A \to \mathbf{C}$, and we would like to show that there is a unique holomorphic function b on A such that $zf = \varphi^*(b)$.

For uniqueness, note that if b_1 and b_2 are two such functions which do not agree, then there is an artinian complex space $A' \subset A$ such that the restrictions b'_1 and b'_2 do not agree on A'. But an artinian complex space is the same as an artinian scheme, so $X^{an} \times_{Y^{an}} A'$ is algebraic over A'. Then $\varphi^*(b'_i)$ are algebraic functions whose associated analytic functions are equal to f. Hence $\varphi^*(b'_1)$ and $\varphi^*(b'_2)$ are equal as algebraic functions. By the universal categorical quotient property for $X \to Y$ in the algebraic category, this implies that $b'_1 = b'_2$, contradicting the possibility that b_1 and b_2 could be different. This shows that b is unique if it exists (which makes possible the patching arguments necessary to undo the previous reductions in the size of A).

Now $X^{an} \times_{Y^{an}} A$ is a closed analytic subspace of the Stein open subset $W \subset \mathbf{C}^{n+m}$. Thus the holomorphic function zf on $X^{an} \times_{Y^{an}} A$ is the restriction of a holomorphic function e on W. We may replace e by its average with respect to the action of the compact group K. This average still restricts to zf, since the latter is K-invariant. The new e is K-invariant, in other words the two different pullbacks of e to $G \times W$ agree on $K \times W$. Since K is a real form of G which meets every connected component of G, any connected component of $G \times W$ contains a totally real subset of half the real dimension where the two different pullbacks agree. Therefore the two pullbacks agree on $G \times W$, meaning that e is G-invariant.

Apply the result proved in the first case above, to the map $\varphi' : W \to U$. It says that there is a holomorphic function b' on U such that $(\varphi')^*(b') = e$. Let b be the restriction of b' to the subspace A. Then $\varphi^*(b)$ is equal to the restriction of e to $Z \times_Y A$, in other words $\varphi^*(b) = zf$. Applying this to all of the coordinate functions z which give the embedding $Z \subset \mathbf{C}^M$, we obtain a morphism $g : A \to \mathbf{C}^M$ such that $f = \varphi^*(g)$. In particular, the continuous function associated to g is equal to the continuous function defined above, so g maps A to Z set theoretically, at least. Suppose that z' is a function on an open subset of $\mathbf{C}^M$ which vanishes on the subspace Z. Then $z'g$ is a holomorphic function on an open subset of A such that $\varphi^*(z'g) = 0$. By the uniqueness proved above, this implies that $z'g = 0$. This shows that g maps the complex analytic space A to the complex analytic space Z. This completes the proof that $\varphi : X^{an} \to Y^{an}$ is a universal categorical quotient, in this second case where X is a closed subset of $\mathbf{A}^n$.

Finally, we will complete the proof of the proposition in general. Suppose that X is quasi-projective with G-action linearizing an ample line bundle, that all points of X are semistable, and that $\varphi : X \to Y = X/G$ is the good quotient of [Mu]. Let L denote the very ample line bundle on Y (its pull-back to X being the initial linearized line bundle). The question of whether φ is a universal categorical quotient in the analytic category is local on Y in the analytic topology, hence in particular it is local in the Zariski topology. Thus we may replace Y by a Zariski open subset. Choose a section s of some power of L on Y, such that the subset $s \neq 0$ is affine. Then replace Y by this subset. Now Y is affine and L is trivial. The map φ is affine, so X is affine too. The group G acts in a locally finite way on $H^0(X, \mathcal{O}_X)$, in other words that space is an increasing union of finite dimensional G-invariant subspaces. Hence we may choose a finite dimensional subspace $V \subset H^0(X, \mathcal{O}_X)$, invariant by G, which gives an embedding of X into the affine space V^*. The action of G on V^* is a linear representation, and the embedding $X \subset V^*$ is compatible with the action of G. This places us in the situation covered by the second case treated above, so $\varphi : X^{an} \to Y^{an}$ is a universal categorical quotient. This completes the proof of the proposition. □

The moduli spaces

Now we return to the following situation: S is a base scheme of finite type over $\mathbf{C}$, X is projective over S, and Λ is a sheaf of rings of differential operators on X over S. Fix a relatively very ample $\mathcal{O}_X(1)$ and a polynomial P, and let Q be the parametrizing scheme constructed in Theorem 3.8. The group $G = Sl(P(N), \mathbf{C})$ acts on Q. We have seen (in Corollary 4.6) that there is a G-linearized very ample invertible sheaf $\mathcal{L}$ on Q, and that $Q = Q^{ss}$ for this action. The good quotient is $\mathbf{M}(\Lambda, P) = Q/G$.

Corollary **5.6.** — *The associated complex analytic space* $\mathbf{M}^{an}(\Lambda, P)$ *is a universal categorical quotient of* Q^{an} *by the action of* G *in the category of complex analytic spaces.*

Proof. — This follows from Proposition 5.5, noting that if S is not quasiprojective then at least it can be covered by quasiprojective open sets; the quotients obtained from the previous proposition then glue together to give a quotient over S. □

We record here the universal property of $\mathbf{M}^{an}(\Lambda, P)$ implied by this corollary. Let $\mathbf{M}^{an, \natural}(\Lambda, P)$ denote the functor which, to each morphism $S' \to S^{an}$ of complex analytic spaces associates the set $\mathbf{M}^{an, \natural}(\Lambda, P)$ (S') of isomorphism classes of p-semistable Λ'-modules $\mathscr{E}$ on X', $\mathscr{O}_{X'}$-coherent and flat over S', with relative Hilbert polynomial P. Suppose $Z \to U \to S^{an}$ are morphisms of complex analytic spaces, and

$$\mathbf{M}^{an, \natural}(\Lambda, P) \times_{S^{an}} U \to Z$$

is a natural transformation of functors of complex analytic spaces over U. Then there exists a unique factorization through a morphism of complex analytic spaces

$$\mathbf{M}^{an}(\Lambda, P) \times_{S^{an}} U \to Z.$$

This follows from the same arguments used in the proof of Theorem 1.21.

The representation spaces

We have a similar result for the representation spaces $\mathbf{R}(\Lambda, \xi, P)$ constructed at the end of § 4.

Lemma **5.7.** — *Suppose that the fibers* X_s *are irreducible and* $\xi : S \to X$ *is a section. The space* $\mathbf{R}^{an}(\Lambda, \xi, P)$ *represents the functor which to each analytic space* $S' \to S$ *associates the set of pairs* $(\mathscr{E}, \beta)$ *where* $\mathscr{E}$ *is a p-semistable* Λ^{an}*-module on* X' *over* S', *with Hilbert polynomial* P, *satisfying condition* LF(ξ), *and* $\beta : \xi^*(\mathscr{E}) \cong \mathscr{O}^n_{S'}$.

Proof. — This follows by applying Proposition 5.3-5.5 to the construction of the representation space given in the previous section. □

Standardized sequences of diffeomorphisms

Suppose that $X \to S$ is a smooth projective morphism. This does not imply that X itself is smooth (we cannot avoid considering nonsmooth, and even nonreduced base schemes S). To treat convergence questions in the relative case, we must investigate the relationship between the various different fibers X_s.

Suppose $s(i) \in S$ is a sequence of points converging to $t \in S$. We construct a *standardized sequence of diffeomorphisms* $\Psi_i : X^{\infty}_{s(i)} \cong X^{\infty}_t$, defined for $i \gg 0$. Here X^{∞}_s denotes the C^{∞} manifold underlying X_s. Choose a sufficiently small open neighborhood $U \subset S^{an}$ of t and a finite open covering $f^{-1}(U) =: X_U = \bigcup Z_\alpha$ together with holomorphic trivializations $(\psi_\alpha, f) : Z_\alpha \cong V_\alpha \times U$ over U, where $V_\alpha \subset X_t$ are open discs and $f : X_U$ is the projection.

Choose a sufficiently small neighborhood N of the diagonal

$$X_t \subset N \subset \prod_\alpha X_t$$

where the product is taken over the same index set as the covering. Let $T \subset \prod_\alpha [0, 1]$ denote the set of points r with $\Sigma_\alpha r_\alpha = 1$. We can choose a C^∞ function

$$\mu : N \times T \to X_t$$

such that $\mu(x, r) = x$ if x is in the diagonal X_t, and such that if $r_\alpha = 0$ then $\mu(x, r.)$ is independent of the α-th coordinate of x. In order to choose μ, note first of all that N can be given a structure of fiberwise convex open neighborhood of the zero section in a vector bundle over the diagonal X_t. Let $m : [0, 1] \times N \to N$ denote the scalar multiplication for this vector bundle structure. We may proceed by induction and assume that $\mu_{\partial T} : N \times \partial T \to X_t$ is already chosen. Choose a C^∞ function $u : T \to [0, 1]$ so that $u(r) = 1$ for $r \in \partial T$ and $u(r) = 0$ for r in a disc in the middle of T. Choose a function $\tau : T \to \partial T$ which is C^∞ except on the interior of the disc where u vanishes, and such that $\tau|_{\partial T}$ is the identity. Then put $\mu(x, r) = \mu_{\partial T}(m(u(r), x), \tau(r))$.

Choose a partition of unity $1 = \Sigma_\alpha u_\alpha$ with u_α compactly supported in Z_α, in the following " C^∞ " way. Fix an embedding $X \subset \mathbf{P}^N \times S$; and choose open sets Z_α^{big} of $\mathbf{P}^N \times U$ restricting to Z_α in X_U. Then let $\{ u_\alpha^{\mathrm{big}}, u_0 \}$ be a partition of unity for $\mathbf{P}^N \times U$ for the covering consisting of the Z_α^{big} and the complement of X_U. Put $u_\alpha = u_\alpha^{\mathrm{big}}|_X$.

Our trivializations have first projections $\psi_\alpha : Z_\alpha \to X_t$. For $x \in X_U$ set

$$\psi(x) = (\ldots, \psi_\alpha(x), \ldots) \in \prod_\alpha X_t;$$

in cases where $x \notin Z_\alpha$, choose a point at random for the value of $\psi_\alpha(x)$. Similarly, set $u(x) = (\ldots, u_\alpha(x), \ldots) \in T$. Then put

$$\Psi(x) = \mu(\psi(x), u(x)).$$

From the above properties of μ, there is a smaller open neighborhood $t \in U' \subset U$ such that for any $s \in U'$, $\Psi|_{X_s}$ is a C^∞ diffeomorphism from X_s to X_t. Put $\Psi_i = \Psi|_{X_{s(i)}}$: this is our sequence of *standardized diffeomorphisms*, defined for $i \gg 0$. A different choice of trivializations, partition of unity, etc. yields a different sequence of diffeomorphisms Ψ_i'. These two sequences are related in the following way: for any $k \geqslant 0$, $\Psi_i' \Psi_i^{-1} \to 1_{X_t}$ in the C^k norm for diffeomorphisms of X_t, as $i \to \infty$.

We say that a sequence of diffeomorphisms $\Psi_i' : X_{s(i)}^\infty \cong X_t^\infty$ is a *standardized sequence* if for one or for any collection of trivializations, partitions of unity, and μ as above, the resulting standardized sequence Ψ_i gives $\Psi_i' \Psi_i^{-1} \to 1_{X_t}$ in any C^k norm as $i \to \infty$.

Suppose $\mathscr{V}$ is an algebraic vector bundle over X. Let V_s denote the associated C^∞ vector bundles on the fibers X_s. Fix a standardized sequence of diffeomorphisms Ψ_i, according to our original construction. Then we can construct, in a similar way, a *standardized sequence of bundle isomorphisms* $\zeta_i : \Psi_{i,*}(V_{s(i)}) \cong V_t$. Use a lifting of the μ constructed above to a bundle map (still a projection) from the direct sum $\bigoplus \mathrm{pr}_\alpha^*(V_t)$ over $N \times T$, to the bundle V_t over the diagonal. Again, say that ζ_i' is a standardized sequence of bundle isomorphisms over any standardized sequence Ψ'_i if, for some standardized sequence ζ_i over Ψ_i constructed as described, the $\zeta'_i\,\zeta_i^{-1} \to 1$ in any C^k norm for diffeomorphisms of the total space of V_t.

Bundles in the split almost polynomial case

Suppose that X is smooth and projective over S, with connected fibers. Suppose that $\xi : S \to X$ is a section. Suppose (Λ, ζ) is a split almost polynomial sheaf of rings of differential operators on X over S (see § 2). Let $H = \mathrm{Gr}_1(\Lambda)^*$ and $K = \wedge^2 H$, with the splitting $\zeta : H^* \to \Lambda_1$, the derivation $\delta : \mathcal{O}_X \to H$ corresponding to the symbol of Λ, and the morphism of sheaves $\gamma : H \to K \otimes_{\mathcal{O}_X} \Lambda_1$ described at the end of § 2.

Fix a point $s \in S$. Let $\mathscr{C}_s^\infty$ denote the sheaf of C^∞ functions on X_s, and let H_s^∞ and K_s^∞ denote the C^∞ vector bundles (or locally free sheaves over $\mathscr{C}_s^\infty$) corresponding to H_s and K_s. Note that $H_s^\infty = H_s \otimes_{\mathcal{O}_{X_s}} \mathscr{C}_s^\infty$ and similarly for K. We can extend the derivation δ_s to a derivation

$$\delta_s^\infty : \mathscr{C}_s^\infty \to H_s^\infty$$

by noting that δ_s corresponds to a morphism $\Omega^1_{X_s} \to H_s$, which gives $\mathscr{A}^{1,0}(X_s) \to H_s^\infty$. The derivation δ_s^∞ is the composition of the usual ∂ with this morphism.

Fix a polynomial P which corresponds to the relative Hilbert polynomial of a locally free sheaf of rank n on X. Let $\mathbf{R}^{\mathrm{LF}(X)}(\Lambda, \xi, P)$ denote the scheme parametrizing pairs $(\mathscr{E}, \beta)$ where $\mathscr{E}$ is a p-semistable Λ-module with Hilbert polynomial P, satisfying condition LF(X), and $\beta : \xi^*(\mathscr{E}) \cong \mathcal{O}_S^n$ is a frame (see the end of § 4).

Suppose $(\mathscr{E}, \beta)$ corresponds to a point $y \in \mathbf{R}^{\mathrm{LF}(X_s)}(\Lambda_s, \xi(s), P)$. Let E denote the C^∞ vector bundle on X_s corresponding to the locally free sheaf $\mathscr{E}$, and let $\bar\partial : E \to A^{1,0}(E)$ denote the operator defining the holomorphic structure of $\mathscr{E}$. Thus $\mathscr{E}^{\mathrm{an}}$ is the sheaf of sections e of E with $\bar\partial(e) = 0$. The frame β gives $\beta : E_{\xi(s)} \cong \mathbf{C}^n$. The Λ-module structure of $\mathscr{E}$ is determined by an operator

$$\varphi_\varepsilon : \mathscr{E} \to H \otimes_{\mathcal{O}_X} \mathscr{E}$$

satisfying the Leibniz rule $\varphi_\varepsilon(ae) = a\varphi_\varepsilon(e) + \delta_s(a)\, e$, and $\varphi'_\varepsilon\,\gamma\varphi_\varepsilon = 0$. By enforcing the same Leibniz rule using the derivation δ_s^∞ we obtain an operator

$$\varphi : E \to H^\infty \otimes_{\mathscr{C}_s^\infty} E.$$

This satisfies the equation

$$\bar\partial\varphi + \varphi\bar\partial = 0,$$

an equation of operators from E to $A^{0,1}(H^\infty \otimes_{\mathscr{C}_s^\infty} E)$. Given any such operator, we can form an operator

$$\varphi'\,\gamma : H^\infty \otimes_{\mathscr{C}_s^\infty} E \to K^\infty \otimes_{\mathscr{C}_s^\infty} E.$$

If φ comes from a Λ-module structure, then $\varphi'\,\gamma\varphi = 0$. We have the following converse.

Lemma **5.8.** — *Suppose* E *is a* C^∞ *vector bundle on* X_s *with a holomorphic structure given by the operator* $\overline{\partial}$, *and an operator*

$$\varphi : E \to H^\infty \otimes_{\mathscr{C}^\infty_s} E.$$

Suppose that $\overline{\partial}$ *is integrable,* $\overline{\partial}\varphi + \varphi\overline{\partial} = 0$, *and* $\varphi' \gamma\varphi = 0$. *Then there exists a unique* Λ-*module* $\mathscr{E}$ *on* X_s, *together with isomorphism* $\mathscr{E} \otimes_{\mathcal{O}_{X_s}} \mathscr{C}^\infty_s \cong E$, *such that the operators* $\overline{\partial}$ *and* φ *are those given by* $\mathscr{E}$.

Proof. — From the integrability of $\overline{\partial}$ we get a holomorphic vector bundle $\mathscr{E}^{an}$. The second equation gives an operator φ^{an}_ε on sheaves of holomorphic sections. Serre's GAGA theorems give an algebraic object $(\mathscr{E}, \varphi_\varepsilon)$, and the equation $\varphi' \gamma\varphi$ implies $\varphi'_\varepsilon \gamma\varphi_\varepsilon$. This gives a Λ-module structure for $\mathscr{E}$ (Lemma 2.13). □

A convergence statement for Hilbert schemes

We would like to use this analytic description of Λ-modules to study the question of convergence of a sequence of points in $\mathbf{R}^{LF(X)}(\Lambda, \xi, P)$. Before getting to the theorem, we need the following statement about convergence in Hilbert schemes.

Suppose $X \to S$ is smooth. Suppose $\mathscr{W}$ is a locally free sheaf over X; let W denote the corresponding C^∞ bundle. Let $\mathbf{Hilb}^{lf(X)}(\mathscr{W}, P)$ denote the Hilbert scheme over S of quotients of $\mathscr{W}$ which are locally free with Hilbert polynomial P. Suppose $s(i)$ is a sequence of points converging to t in S. Choose a standardized sequence of diffeomorphisms $\Psi_i : X_{s(i)} \cong T$ and a standardized sequence of bundle isomorphisms $\zeta_i : \Psi_{i,*}(W_{s(i)}) \cong W_t$.

We may assume that there exists an open set $U_a \subset X_t$ meeting each connected component of X_t, such that for $u \in U_a$ there exists a section $\xi_u : S' \to X$ defined on a usual open neighborhood S' of t (which also contains $s(i)$), such that $\xi_u(t) = u$ and $\Psi_i(\xi_u(s(i))) = u$ for all i. We may suppose that S' is part of an étale neighborhood and that the sections are defined algebraically. (The open set U_a consists of the points where only one element of the partition of unity used to construct the Ψ_i is nonzero.)

Lemma **5.9.** — *In the above situation, suppose* $y(i) \in \mathbf{Hilb}^{lf(X)}(\mathscr{W}_{s(i)}, P)$ *and* $z \in \mathbf{Hilb}^{lf(X)}(\mathscr{W}_t, P)$. *These correspond to quotients which we denote in terms of* C^∞ *bundles as* $a_i : W_{s(i)} \to E_i \to 0$ *and* $a : W_t \to E \to 0$. *Assume that there exists a sequence of bundle isomorphisms* $\eta_i : \Psi_{i,*}(E_i) \cong E$ *such that, with* $b_i : W_t \to E \to 0$ *denoting the transported morphisms* $b_i = \eta_i \Psi_{i,*}(a_i) \zeta_i^{-1}$, *we have* $b_i \to a$ *in* L^1 *norm. Then the points* $y(i)$ *approach the point* z *in* $\mathbf{Hilb}^{lf(X)}(\mathscr{W}, P)$.

Proof. — It suffices to show that there is a subsequence $\{ i' \}$ such that $y(i') \to z$. To see this, note that in order to show that $y(i) \to z$ it suffices to show that any subsequence has a subsequence which approaches z.

If $b_i \to a$ in L^1 norm then there is a subset $\Sigma \subset X_t$ of full measure and a subsequence $\{ i' \}$ such that $b_{i'}(u) \to a(u)$ for $u \in \Sigma$ ([Ru], Exercise 18, p. 76-77). Replace our sequence now by this subsequence.

The intersection $\Sigma \cap U$ is a set of positive measure. There exist points $u_1, \ldots, u_k \in \Sigma$ such that, denoting ξ_{u_j} by ξ_j, the morphism

$$\mathbf{Hilb}^{\mathrm{lf(X)}}(\mathscr{W}, P) \to \prod_{j=1}^{k} \mathbf{Grass}(\xi_j^* \mathscr{W}, n)$$

is a locally closed embedding. Although this is well known, we briefly note the proof. Choose an embedding

$$\psi_m : \mathbf{Hilb}(\mathscr{W}, P) \hookrightarrow \mathbf{Grass}(H^0(X/S, \mathscr{W}(m)), P(m)).$$

Denote with a script $\mathscr{E}$ the algebraic quotients corresponding to points of the Hilbert scheme. The set of quotients $\mathscr{E}$ which occur is bounded. We may choose the points $u_j \in \Sigma \cap U$ such that there are sufficiently many sections ξ_j in a general enough position so that

$$H^0(X_s, \mathscr{W}(m)) \hookrightarrow \bigoplus_{j=1}^{k} \mathscr{W}(m)_{\xi_j(s)}$$

and for any quotient $\mathscr{E}$ on a fiber X_s over $s \in S'$, we have

$$H^0(X_s, E(m)) \hookrightarrow \bigoplus_{j=1}^{k} \mathscr{E}(m)_{\xi_j(s)}.$$

Let $T = \prod_{j=1}^{k} \mathbf{Grass}(\xi_j^*(\mathscr{W}(m)), n)$. Denote a point in T by $t = (t_1, \ldots, t_k)$ where $t_j : \xi_j^*(\mathscr{W}(m)) \to V_j$ is a quotient of rank n. Let $T_1 \subset T$ be the locally closed subset whose S''-valued points (for $S'' \to S'$) are those t such that composition

$$F_t : H^0(X''/S'', \mathscr{W}(m)) \longrightarrow \bigoplus_{j=1}^{k} \xi_j^*(\mathscr{W}(m)) \xrightarrow{(t_1, \ldots, t_k)} \bigoplus_{j=1}^{k} V_j$$

has an image which is a locally free sheaf or rank $P(m)$ over S''. This gives a natural map $F : T_1 \to \mathbf{Grass}(H^0(X/S, \mathscr{W}(m)), P(m))\ (S'')$. If $c : \mathscr{W} \to \mathscr{E} \to 0$ is an S''-valued point in $\mathbf{Hilb}^{\mathrm{lf(X)}}(\mathscr{W}, P)$ then we obtain a point $t \in T(S'')$ and a diagram

$$\begin{tikzcd} H^0(X''/S'', \mathscr{W}(m)) \arrow[r] \arrow[d] & H^0(X''/S'', \mathscr{E}(m)) \arrow[d] \\ \bigoplus_{j=1}^{k} \xi_j^*(\mathscr{W}(m)) \arrow[r] & \bigoplus_{j=1}^{k} V_j. \end{tikzcd}$$

The vertical arrows are injective, so the vertical map on the right gives an isomorphism between $H^0(X''/S'', \mathscr{E}(m))$ and the image of the map F_t defined above. But the top row is the point $\psi_m(c) \in \mathbf{Grass}(H^0(X/S, \mathscr{W}(m)), P(m))\ (S'')$. Therefore the diagram

$$\begin{tikzcd} \mathbf{Hilb}^{\mathrm{lf(X)}}(\mathscr{W}, P) \arrow[r] \arrow[dr] & T_1 \arrow[d] \\ & \mathbf{Grass}(H^0(X/S, \mathscr{W}(m)), P(m)) \end{tikzcd}$$

commutes. Since the diagonal map ψ_m is a locally closed embedding, the map $\mathbf{Hilb}^{\mathrm{lf(X)}}(\mathscr{W}, P) \to T_1$ is a locally closed embedding.

We complete the proof of the lemma. Let $A_{ij} : \mathcal{W}(m)_{\xi_j(s(i))} \to \mathcal{E}_i(m)_{\xi_j(s(i))}$ denote the points in $\mathbf{Grass}(\xi_j^* \mathcal{W}(m), n)_{s(i)}$ corresponding to the quotients a_i over the points $\xi_j(s(i))$, and let $A'_j : \mathcal{W}(m)_{\xi_j(t)} \to \mathcal{E}(m)_{\xi_j(t)}$ denote the points in $\mathbf{Grass}(\xi_j^* \mathcal{W}(m), n)_t$ corresponding to the quotient a. The fact that $b_i(u_j) \to a(u_j)$ implies that $A_{ij} \to A'_j$. The collection $(A_{i1}, \ldots, A_{ik})$ (resp. $A'_1, \ldots, A'_k$) is the image in T of $y(i) \in \mathbf{Hilb}^{\mathrm{lf(X)}}(\mathcal{W}, P)$ (resp. z). Thus $y(i) \to z$. □

Suppose that we are in the situation of the previous lemma. Since a universal family of quotients exists, and $y(i) \to z$, we can choose a standardized sequence of bundle isomorphisms $\eta_i^{\mathrm{std}} : \Psi_{i,*}(E_i) \cong E$. Suppose that there is a subbundle $\mathcal{V} \subset \mathcal{W}$ which is locally a direct summand. Then the associated C^∞ bundles V_s are direct summands of W_s and we can choose the standardized sequence of isomorphisms ζ_i such that $\zeta_i(\Psi_{i,*} V_{s(i)}) \cong V_t$.

Lemma **5.10.** — *Under these circumstances, suppose that the morphism* $a : V_t \to E$ *is surjective. Suppose that the morphisms* $b_i|_{V_t}$ *approach* a *in* C^0 *norm. Then the automorphisms* $\eta_i^{\mathrm{std}} \eta_i^{-1}$ *approach the identity in* C^0 *norm.* □

Proof. — Choose sections $v_1, \ldots, v_n$ of $\mathcal{V}$ over an open set in X, so that $av_i|_{X_t}$ are a frame for E. Then $\eta_i^{\mathrm{std}}(a_i v_i|_{X_{s(i)}}) \to av_i|_{X_t}$ and $\zeta_i(v_i|_{X_{s(i)}}) \to v_i|_{X_t}$ in C^0 norm. We have

$$\eta_i(a_i v_i|_{X_{s(i)}}) = b_i(\zeta_i(v_i|_{X_{s(i)}})),$$

so the hypothesis implies that $\eta_i(a_i v_i|_{X_{s(i)}}) \to av_i|_{X_t}$ in C^0 norm. Comparing these, $\eta_i^{\mathrm{std}} \eta_i^{-i} \to 1$ in C^0 norm. □

Suppose $\xi : S \to X$ is a section, and let $\mathbf{Hilb}^{\mathrm{lf(X)}}(\mathcal{W}, \xi, P)$ denote the frame bundle over $\mathbf{Hilb}^{\mathrm{lf(X)}}(\mathcal{W}, P)$, parametrizing pairs $(\mathcal{E}, \beta)$ where E is a quotient of $\mathcal{W}$ and $\beta : \xi^*(\mathcal{E}) \cong \mathcal{O}^n$ is a frame along the section.

Corollary **5.11.** — *Suppose*

$$y(i) \in \mathbf{Hilb}^{\mathrm{lf(X)}}(\mathcal{W}_{s(i)}, \xi(s(i)), P) \quad \textit{and} \quad z \in \mathbf{Hilb}^{\mathrm{lf(X)}}(\mathcal{W}_t, \xi(t), P),$$

corresponding respectively to pairs (E_i, β_i) *and* (E, β). *Suppose that there exists a subbundle* $\mathcal{V} \subset \mathcal{W}$ *which is a local direct summand, that the* ζ_i *are chosen compatibly, and that* $a : V_t \to E$ *is surjective. Suppose that there exist bundle isomorphisms* $\eta_i : \Psi_{i,*}(E_i) \cong E$ *such that* $b_i = \eta_i \Psi_{i,*}(a_i) \zeta_i^{-1}$ *approach* a *in* L^1 *norm, and that* $b_i|_{V_t} \to a|_{V_t}$ *in* C_0 *norm. Suppose that* $\beta_i \eta_i^{-1} \to \beta$. *Then the points* $y(i)$ *approach* z *in* $\mathbf{Hilb}^{\mathrm{lf(X)}}(\mathcal{W}, \xi, P)$.

Proof. — By Lemma 5.9, the points in $\mathbf{Hilb}^{\mathrm{lf(X)}}(\mathcal{W}, P)$ converge. By Lemma 5.10 the η_i are comparable to a standardized sequence of bundle isomorphisms η_i^{std}, so $\beta_i(\eta_i^{\mathrm{std}})^{-1} \to \beta$. This implies that the points in the frame bundle converge. □

Convergence in $\mathbf{R}^{\mathrm{LF(X)}}(\Lambda, \xi, P)$

Return to the case of a split almost polynomial Λ on X smooth over S. We give some notation to make the statement of the next theorem simpler. Suppose $y(i)$ is a sequence of points and z is another point in $\mathbf{R}^{\mathrm{LF(X)}}(\Lambda, \xi, \mathrm{P})$. Let $s(i)$ and t be their images in S. Let $(\mathscr{E}_i, \beta_{\mathrm{E}_i})$ and $(\mathscr{F}, \beta)$ denote the Λ-modules with frame over $\mathrm{X}_{s(i)}$ or X_t, corresponding to the points $y(i)$ and z. Let E_i and F denote the underlying C^∞ bundles, with operators $\overline{\partial}_{\mathrm{E}_i}$ and φ_{E_i} on E_i, or $\overline{\partial}$ and φ on F. Suppose that a standardized sequence of diffeomorphisms $\Psi_i : \mathrm{X}^\infty_{s(i)} \cong \mathrm{X}^\infty_t$ is fixed. Suppose we have chosen a sequence of bundle isomorphisms $\eta_i : \Psi_{i,*}(\mathrm{E}_i) \cong \mathrm{F}$. Then by transport, we obtain a sequence of operators

$$\varphi_i \stackrel{\mathrm{def}}{=} \eta_{i,*}\, \Psi_{i,*}(\varphi_{\mathrm{E}_i})$$

and

$$\overline{\partial}_i \stackrel{\mathrm{def}}{=} \eta_{i,*}\, \Psi_{i,*}(\overline{\partial}_{\mathrm{E}_i})$$

on the bundle F. Let $x = \xi(t)$ and $x(i) = \xi(s(i))$ in X_t. We obtain a sequence of frames $\beta_i : \mathrm{F}_{x(i)} \cong \mathbf{C}^n$ by composing the β_{E_i} with η_i^{-1}.

Fix $q \gg 0$, large compared with the dimension of the fibers X_s. Recall that L^q_k denotes the Banach space of functions whose first k derivatives are in L^q. If V is a C^∞ bundle, let $\mathrm{L}^q_k(\mathrm{V})$ denote the Banach space of sections of V whose first k derivatives are in L^q (we supress the subscript k when it is equal to 0). The norm is the sum of the L^q norms of the derivatives in question.

Theorem **5.12.** — *Suppose $y(i)$ is a sequence of points in $\mathbf{R}^{\mathrm{LF(X)}}(\Lambda, \xi, \mathrm{P})$ lying over $s(i) \in \mathrm{S}$, and z is a point lying over $t \in \mathrm{S}$. Suppose that $s(i)$ converge to t in the analytic topology of* S. *Fix a standardized sequence of diffeomorphisms $\Psi_i : \mathrm{X}_{s(i)} \cong \mathrm{X}_t$. Let* T *denote the complexified tangent bundle of X_t. Let $(\mathrm{E}_i, \overline{\partial}_{\mathrm{E}_i}, \varphi_{\mathrm{E}_i}, \beta_{\mathrm{E}_i})$ be the objects corresponding to $y(i)$, and let $(\mathrm{F}, \overline{\partial}, \varphi, \beta)$ be the object corresponding to z. Then the points $y(i)$ converge to z in the analytic topology of $\mathbf{R}^{\mathrm{LF(X)}}(\Lambda, \xi, \mathrm{P})$ if and only if there exists a sequence of bundle isomorphisms $\eta_i : \Psi_{i,*}(\mathrm{E}_i) \cong \mathrm{F}$ such that, with the above notations, the following convergence statements hold: the operators φ_i converge to φ in the operator norm for operators from $\mathrm{L}^q_1(\mathrm{F})$ to $\mathrm{L}^q(\mathrm{F} \otimes \mathrm{T})$; the operators $\overline{\partial}_i$ converge to $\overline{\partial}$ in the operator norm for operators from $\mathrm{L}^q_1(\mathrm{F})$ to $\mathrm{L}^q(\mathrm{F} \otimes \mathrm{T})$; and the frames $\beta_i : \mathrm{F}_{x(i)} \cong \mathbf{C}^n$ converge to $\beta : \mathrm{F}_x \cong \mathbf{C}^n$ (note that the condition that Ψ_i are standardized implies $x(i) \to x$).*

Proof. — If the points $y(i)$ converge to z then we can choose η_i to be a standardized sequence of bundle isomorphisms, which will have the required convergence properties.

Suppose we are given a sequence of bundle isomorphisms η_i as in the statement of the theorem. Fix a number ℓ in the same way that N was fixed in the construction of Theorem 3.8. Let M denote the C^∞ line bundle on X_t corresponding to the holomorphic bundle $\mathcal{O}_{\mathrm{X}_t}(\ell)$. Let $\overline{\partial}_{\mathrm{M}}$ denote the operator giving the holomorphic structure. We can choose a sequence of standardized bundle isomorphisms $\Psi_{i,*}(\mathcal{O}_{\mathrm{X}_{s(i)}}(\ell)) \cong \mathrm{M}$ converging to the identity in any norm. Let $\overline{\partial}_{\mathrm{M},i}$ denote the operators on M obtained by transporting the holomorphic structures from $\mathcal{O}_{\mathrm{X}_{s(i)}}(\ell)$. These converge to $\overline{\partial}_{\mathrm{M}}$ in any operator norm. Note that the operators $\overline{\partial}_i$ and $\overline{\partial}_{\mathrm{M},i}$ do not have type (0, 1) on X_t, as the diffeomorphisms Ψ_i do not necessarily respect the holomorphic structure. These operators have values in the full complexified tangent bundle T.

Setting $\bar{\partial}_{F\otimes M, i} = \bar{\partial}_i \otimes 1 + 1 \otimes \bar{\partial}_{M, i}$ we obtain a sequence of operators

$$\bar{\partial}_{F\otimes M, i} : L^q_1(F \otimes M) \to L^q(F \otimes M \otimes T),$$

which by hypothesis converge to $\bar{\partial}_{F\otimes M} \stackrel{\text{def}}{=} \bar{\partial} \otimes 1 + 1 \otimes \bar{\partial}_M$ in the operator norm for operators from L^q_1 to L^q.

Choose C^∞ metrics for X_t and the holomorphic bundles $(F, \bar{\partial})$ and $(M, \bar{\partial}_M)$. The kernel $K = \ker(\bar{\partial}_{F\otimes M})$ is finite-dimensional, so we may choose a continuous projection $\varpi : L^q_1(F \otimes M) \to K$. Let $j : W \hookrightarrow L^q(F \otimes M \otimes T)$ denote the space of vectors orthogonal to the image of $\bar{\partial}_{F\otimes M}$ with respect to the L^2 inner product given by the chosen metrics. Harmonic theory implies that W is a closed subspace, and that the map

$$f \stackrel{\text{def}}{=} \bar{\partial}_{F\otimes M} + \varpi + j : L^q_1(F \otimes M) \oplus W \to L^q(F \otimes M \otimes T) \oplus K$$

is an isomorphism. Since $\bar{\partial}_{F\otimes M, i} \to \bar{\partial}_{F\otimes M}$ in operator norm, the maps

$$f_i \stackrel{\text{def}}{=} \bar{\partial}_{F\otimes M, i} + \varpi + j$$

are isomorphisms for $i \gg 0$, and the inverses converge $f_i^{-1} \to f^{-1}$ in operator norm. The image $f_i^{-1}(K)$ is equal to the kernel of

$$\bar{\partial}_{F\otimes M, i} + j : L^q_1(F \otimes M) \oplus W \to L^q(F \otimes M \otimes T),$$

but this kernel contains the kernel of $\bar{\partial}_{F\otimes M, i}$. By the hypothesis on ℓ, the kernel of $\bar{\partial}_{F\otimes M, i}$ has the same dimension as K, therefore

$$f_i^{-1}(K) = \ker(\bar{\partial}_{F\otimes M, i}) \subset L^q_1(F \otimes M).$$

Rename the resulting maps as $g_i : K \to L^q_1(F \otimes M)$. These converge to the original inclusion $g : K \to L^q_1(F \otimes M)$. Fix an isomorphism $K \cong \mathbf{C}^k$. The maps g_i, when transported back using Ψ_i^{-1} and η_i^{-1}, give frames $\gamma_i : H^0(X_{s(i)}, E_i(\ell)) \cong \mathbf{C}^k$; and the map g corresponds to a frame $\gamma : H^0(X_t, F(\ell)) \cong \mathbf{C}^k$. We obtain a sequence of points (E_i, γ_i, β_i) in the parametrizing scheme $T^{LF(X)}$, the frame bundle over $Q^{LF(X)}$ (see the discussion preceeding Theorem 4.10). We claim that these points converge to the point (F, γ, β). This statement will imply the theorem.

Let $\mathscr{W} = \Lambda_1 \otimes_{\mathcal{O}_X} \mathcal{O}_X(\ell)^{P(\ell)}$. Recall that $Q^{LF(X)}$ embeds as a locally closed subset in the Hilbert scheme $\mathbf{Hilb}^{lf(X)}(\mathscr{W}, P)$ of quotients of $\mathscr{W}$ (note that the condition LF(X), that $\mathrm{gr}(\mathscr{E})$ is locally free, is stronger than the condition lf(X) that $\mathscr{E}$ is locally free). The frame bundle $T^{LF(X)}$ embeds as a locally closed subset in the parameter scheme $\mathbf{Hilb}^{lf(X)}(\mathscr{W}, \xi, P)$ for locally free quotients of $\mathscr{W}$ provided with a frame along the section ξ. Keep the notation of Lemma 5.9. Choose a standardized sequence of isomorphisms $\zeta_i : \Psi_{i,*}(W_{s(i)}) \cong W_t$. Let $a_i : W_{s(i)} \to E_i \to 0$ be the quotients corresponding to the Λ-modules E_i with the frames γ_i constructed above. Transport to X_t using the isomorphisms ζ_i and η_i: we obtain quotients

$$a_i' : W_t \to F \to 0.$$

The frames β_i also give frames $\beta_i' : F_{x(i)} \cong \mathbf{C}^n$, which converge to the frame $\beta : F_x \cong \mathbf{C}^n$. This takes care of the part of the hypotheses of Corollary 5.11 concerning the frame. In order to apply Corollary 5.11 we must show that the maps a_i' converge, in terms of the fixed bundle structure of F, to the map $a : W_t \to F \to 0$ corresponding to the Λ-module F with its frame γ.

Locally we can choose a frame $(\nu, \lambda_1, \ldots, \lambda_r)$ for Λ_1 (considered as left $\mathcal{O}_X$-module), such that ν corresponds to the identity in $\Lambda_0 \cong \mathcal{O}_X$, and the λ_i give a frame for H. This collection of sections is then also a frame for Λ_1 considered as a right $\mathcal{O}_X$-module. Let $b_1, \ldots, b_{P(\ell)}$ denote a frame for $\mathcal{O}_X(-N)^{P(\ell)}$ consisting of the standard frame for $\mathbf{C}^{P(\ell)}$ times a nonvanishing section of $\mathcal{O}_X(-N)$. Then

$$\{ \nu \otimes b_i, \lambda_j \otimes b_i \}$$

s a frame for $\mathscr{W}$. Evaluation at any $s \in S$ gives a frame $\{ \nu \otimes b_k(s), \lambda_j \otimes b_k(s) \}$ for W_s. When transported to W_t by our standardized sequence of isomorphisms, the frames for $W_{s(i)}$ converge to the frame for W_t. Let $\mathscr{V} = \Lambda_0 \otimes_{\mathcal{O}_X} \mathcal{O}_X(-\ell)^{P(\ell)}$, and note that $a : V_t \to F$ is surjective. The elements $\nu \otimes b_i(s)$ provide a frame for V_s. It suffices to prove that the sequence of sections

$$\eta_i\, a_i(\nu \otimes b_k(s(i))) \qquad \text{or} \qquad \eta_i\, a_i(\lambda_j \otimes b_k(s(i)))$$

converge to $a(\nu \otimes b_k(t))$, in C^0, or $a(\lambda_j \otimes b_k(t))$, in L^1, respectively. The image $a_s(\nu \otimes b_k(s))$ is the section $c_k(s)$ of E_s corresponding to the k-th element of the frame chosen for $H^0(X_s, E_s(N))$ via the local trivialization of $\mathcal{O}_X(-N)$. The $\eta_i\, c_k(s(i))$ converge to $c_k(t)$ in L_1^q and hence in C^0. On the other hand,

$$a_s(\lambda_j \otimes b_k(s)) = \varphi_s(c_k(s))\, (\lambda_j(s))$$

where $\varphi_s : E_s \to E_s \otimes_{\mathcal{O}_{X_s}} H$ is the operator giving the Λ-module structure of E_s. We obtain

$$\eta_i\, a_i(\lambda_j \otimes b_k(s(i))) = \eta_{i,*}(\varphi_i)\, (\eta_i\, c_k(s))\, (\Psi_{i,*}\, \lambda_j(s)).$$

The operators $\eta_{i,*}(\varphi_i)$ converge to φ in the operator norm for operators from L_1^q to L^q by hypothesis, the $\Psi_{i,*}\, \lambda_j(s)$ converge to $\lambda_j(t)$ in any norm, and the $\eta_i(c_k(s(i)))$ converge to $c_k(t)$ in L_1^q, so

$$\eta_i\, a_i(\lambda_j \otimes b_k(s(i))) \to a(\lambda_j \otimes b_k(s))$$

in L^q and hence in L^1. By Corollary 5.11, (E_i, γ_i, β_i) converge to (E, γ, β). $\square$

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