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COMPLEX IMMERSIONS AND QUILLEN METRICS

by JEAN-MICHEL BISMUT *and* GILLES LEBEAU

CONTENTS

INTRODUCTION

Let $i: Y \to X$ be an embedding of compact complex manifolds. Let η be a holomorphic vector bundle on Y and let

$$(0.1) \qquad (\xi, v): 0 \to \xi_m \underset{v}{\to} \xi_{m-1} \dots \underset{v}{\to} \xi_0 \to 0$$

be a holomorphic chain complex of vector bundles on X which, together with a restriction map, $r: \xi_{0\,|\,Y} \to \eta$, provides a resolution of the sheaf $i_* \mathcal{O}_Y(\eta)$.

For $0 \leqslant i \leqslant m$, let $\lambda(\xi_i)$ be the inverse of the determinant of the cohomology of ξ_i. Set $\lambda(\xi) = \bigotimes_{i=0}^{m} (\lambda(\xi_i))^{(-1)^i}$. Similarly, let $\lambda(\eta)$ be the inverse of the determinant of the cohomology of η. By Grothendieck-Knudsen-Mumford [KnM], the lines $\lambda(\xi)$

and $\lambda(\eta)$ are canonically isomorphic. Let σ be the nonzero element of the line $\lambda^{-1}(\eta)\otimes\lambda(\xi)$ which defines the canonical isomorphim.

Assume that TX, TY, $\xi_0, \ldots, \xi_m$, η are equipped with Hermitian metrics. Then by [Q2], [BGS3, Section 1d)], we can equip the lines $\lambda(\xi)$, $\lambda(\eta)$ with Hermitian metrics $\| \;\|_{\lambda(\xi)}$, $\| \;\|_{\lambda(\eta)}$, which are called *Quillen metrics*. The Quillen metric is the product of the standard L_2 metric coming from Hodge theory by the Ray-Singer analytic torsion of the Dolbeault complex [RS2]. The logarithm of the Ray-Singer analytic torsion is a linear combination of derivatives at zero of the zeta functions of the Hodge Laplacians acting on smooth forms of various degrees. The L_2 metric and the Ray-Singer analytic torsion have to be normalized. The normalizations which we use here are described in Section 1e) of this paper.

Let $\| \;\|_{\lambda^{-1}(\eta)\otimes\lambda(\xi)}$ be the corresponding Quillen metric on the line $\lambda^{-1}(\eta)\otimes\lambda(\xi)$. The purpose of this paper is to give a formula for $\operatorname{Log}(\|\sigma\|^2_{\lambda^{-1}(\eta)\otimes\lambda(\xi)})$ in terms of local secondary invariants of the holomorphic Hermitian bundles introduced above, provided that the metrics satisfy certain natural assumptions.

Our first assumption is that the metric g^{TX} on TX is Kähler and that the metric g^{TY} on TY is induced by the metric g^{TX}. Let N be the normal bundle to Y in X, and let g^N be the metric induced by g^{TX} on N. We assume in addition that assumption (A) in Bismut [B2, Definition 1.5] is verified, *i.e.* that the metrics $h^{\xi_0}, \ldots, h^{\xi_m}$ on $\xi_0, \ldots, \xi_m$ are in some sense compatible with the metrics g^N, g^η on N, η.

Let $\operatorname{Td}(TX, g^{TX})$ be the Todd form in Chern-Weil theory associated to the holomorphic Hermitian connection on (TX, g^{TX}). Other Chern-Weil forms will be denoted in a similar way. In particular $\operatorname{ch}(\xi, h^\xi)$ denotes the Chern-Weil representative of the Chern character of the **Z**-graded vector bundle ξ associated to the metrics $h^{\xi_0}, \ldots, h^{\xi_m}$.

Let $T(\xi, h^\xi)$ be the Bott-Chern current on X constructed in Bismut-Gillet-Soulé [BGS4], associated with the holomorphic chain complex (ξ, v). By [BGS4, Theorem 2.5], we know that if $\delta_{\{Y\}}$ is the current corresponding to integration over Y, then

$$\frac{\bar\partial\partial}{2i\pi} T(\xi, h^\xi) = \operatorname{Td}^{-1}(N, g^N)\operatorname{ch}(\eta, g^\eta)\,\delta_{\{Y\}} - \operatorname{ch}(\xi, h^\xi). \tag{0.2}$$

Let $\widetilde{\operatorname{Td}}(TY, TX|_Y, g^{TX|Y})$ be the Bott-Chern class associated with the exact sequence $0 \to TY \to TX|_Y \to N$ constructed in [BGS1, Section 1f)]. The class of forms $\widetilde{\operatorname{Td}}(TY, TX|_Y, g^{TX|Y})$ on Y is such that

$$\frac{\bar\partial\partial}{2i\pi} \widetilde{\operatorname{Td}}(TY, TX|_Y, g^{TX|Y}) = \operatorname{Td}(TX|_Y, g^{TX|Y}) - \operatorname{Td}(TY, g^{TY})\operatorname{Td}(N, g^N). \tag{0.3}$$

Finally let R(x) be the power series introduced by Gillet-Soulé [GS3], which is such that if $\zeta(s)$ is the Riemann zeta function, then

$$\mathrm{R}(x)=\sum_{\substack{n\geqslant 1\\ n\ \mathrm{odd}}}\left(\sum_{1}^{n}\frac{1}{j}+\frac{2\zeta'(-n)}{\zeta(-n)}\right)\zeta(-n)\frac{x^n}{n!}. \tag{0.4}$$

We identify R with the corresponding additive genus.

The main result of this paper (which is a consequence of Theorems 2.1 and 6.1) is as follows.

Theorem **0.1.** — *The following identity holds*

$$\begin{aligned}\mathrm{Log}(\|\sigma\|^2_{\lambda^{-1}(\eta)\otimes\lambda(\xi)})=&-\int_{\mathrm{X}}\mathrm{Td}(\mathrm{TX},g^{\mathrm{TX}})\,\mathrm{T}(\xi,h^{\xi})\\&+\int_{\mathrm{Y}}\mathrm{Td}^{-1}(\mathrm{N},g^{\mathrm{N}})\,\widetilde{\mathrm{Td}}(\mathrm{TY},\mathrm{TX}|_{\mathrm{Y}},g^{\mathrm{TX}|\mathrm{Y}})\,\mathrm{ch}(\eta,g^{\eta})\\&-\int_{\mathrm{X}}\mathrm{Td}(\mathrm{TX})\,\mathrm{R}(\mathrm{TX})\,\mathrm{ch}(\xi)+\int_{\mathrm{Y}}\mathrm{Td}(\mathrm{TY})\,\mathrm{R}(\mathrm{TY})\,\mathrm{ch}(\eta).\end{aligned} \tag{0.5}$$

That a formula like (0.5) holds is perhaps not too surprising. In fact if X, Y are themselves the fibres of a locally Kähler fibration over a complex manifold S (in the sense of Bismut-Gillet-Soulé [BGS3, Definition 1.25]), the curvature Theorem of [BGS1, Theorem 0.1] shows that the function on S defined by

$$\begin{aligned}f(s)=\mathrm{Log}(\|\sigma\|^2_{\lambda^{-1}(\eta)\otimes\lambda(\xi)})_s&+\int_{\mathrm{X}_s}\mathrm{Td}(\mathrm{TX},g^{\mathrm{TX}})\,\mathrm{T}(\xi,h^{\xi})\\&-\int_{\mathrm{Y}_s}\mathrm{Td}^{-1}(\mathrm{N},g^{\mathrm{N}})\,\widetilde{\mathrm{Td}}(\mathrm{TY},\mathrm{TX}|_{\mathrm{Y}},g^{\mathrm{TX}|\mathrm{Y}})\,\mathrm{ch}(\eta,g^{\eta})\end{aligned} \tag{0.6}$$

is pluriharmonic. Formula (0.5) says that $f(s)$ is, in fact, constant and equal to the topological quantity $-\int_{\mathrm{X}}\mathrm{Td}(\mathrm{TX})\,\mathrm{R}(\mathrm{TX})\,\mathrm{ch}(\xi)+\int_{\mathrm{Y}}\mathrm{Td}(\mathrm{TY})\,\mathrm{R}(\mathrm{TY})\,\mathrm{ch}(\eta)$. In this sense, our formula is a considerable refinement of the local Riemann-Roch-Grothendieck theorem on determinants of direct images proved in [BGS1, 2, 3].

The archetypical application of formula (0.5) is the case where X is a curve of genus $g\geqslant 1$, where Y is a point P in X, D the divisor associated to P, f the canonical section of D, μ a holomorphic line bundle on X, η the line μ_{P} and (ξ, v) the complex

$$0\to\mu\otimes[-\mathrm{D}]\underset{f}{\to}\mu\to 0.$$

If the various line bundles are equipped with Arakelov metrics [Ar], [F, p. 394], [La, p. 85], then formula (0.5) shows that $\text{Log}(\|\sigma\|^2_{\lambda^{-1}(\eta)\otimes\lambda(\xi)})=0$. This result was already proved before by [AlBMNV, Section 5D] using the pluriharmonicity of the function f in (0.6) on the moduli space of curves of genus > 2. If Y contains more than one point, Arakelov metrics do not verify assumption (A) of [B2]. Still the formulas of [AlBMNV]—which correspond to Arakelov adjunction formulas [La, Theorem IV 5.3]—also follow from our formula (0.5) by using classical anomaly formulas. Let us point out that the many difficulties we encounter in the analytic proof of formula (0.5) disappear when X is a curve, the proof being very easy in this case.

On the other hand, by a Grothendieck type of approach to an arithmetic version of a theorem of Riemann-Roch-Grothendieck, which would extend to arbitrary dimensions the Faltings-Riemann-Roch theorem for curves [F, Theorem 3], [La, Theorem V 3.4], Gillet and Soulé [GS3] conjectured that the additive genus R should play an important role. They did so by calculating the logarithm of the Ray-Singer analytic torsion [RS2] of the trivial line bundle on $\mathbf{P}^n$ equipped with the Fubini-Study metric, and by substracting off natural local quantities on $\mathbf{P}^n$. Recently, using our formula (0.5), they have finally proved the conjectured generalization of the Theorem of Faltings-Riemann-Roch [GS4].

To establish a formula like (0.5), one might be tempted to follow the now well-known strategy to the proof of the theorem of Riemann-Roch-Grothendieck, *i.e.* the deformation to the normal cone of Baum-Fulton-McPherson [BaFM], [BGS5, Section 4]. Such a strategy is very difficult to follow here, because it introduces singular fibres near which the behaviour of the Ray-Singer analytic torsion seems to be difficult to study.

Here, we choose a very different route to prove Theorem 0.1. Namely we obtain (0.5) directly by understanding in depth the Hodge theory of resolutions.

We now briefly describe the general strategy of the proof of Theorem 0.1, and also the techniques which we use in this paper.

1. Čech cohomology and Dolbeault cohomology

Let δ^X, δ^Y be the Čech coboundaries on X, Y. As is well known in the theory of determinants, we can construct the double complex $(\mathcal{O}_X(\xi), \delta^X+v)$ on X which has the following two properties:

- If $\tilde{\lambda}(\xi)$ is the determinant of its cohomology, then $\tilde{\lambda}(\xi)$ is canonically isomorphic to $\lambda(\xi)$.
- The restriction map $r: (\mathcal{O}_X(\xi), \delta^X+v) \to (\mathcal{O}_Y(\eta), \delta^Y)$ is a quasi-isomorphism of complexes. In fact, if we filter the double complex $(\mathcal{O}_X(\xi), \delta^X+v)$ by the map v, we

exactly obtain the complex $(\mathcal{O}_Y(\eta), \delta^Y)$. Let ρ be the section of $\lambda^{-1}(\eta) \otimes \tilde{\lambda}(\xi)$ which canonically identifies $\lambda(\eta)$ to $\tilde{\lambda}(\xi)$.

In Section 1, we construct the Dolbeault analogues $(E, \bar{\partial}^X + v)$ and $(F, \bar{\partial}^Y)$ of the complexes $(\mathcal{O}_X(\xi), \delta^X + v)$ and $(\mathcal{O}_Y(\eta), \delta^Y)$ and of the quasi-isomorphism r.

One essential idea of this paper is to make analytic sense of the degeneration of the complex $(E, \bar{\partial}^X + v)$ into the complex $(F, \bar{\partial}^Y)$. In fact we equip the line $\tilde{\lambda}(\xi)$ with a Quillen metric $\| \ \|_{\tilde{\lambda}(\xi)}$. We prove in Theorem 2.1 that

$$\operatorname{Log}(\| \sigma \|^2_{\lambda^{-1}(\eta) \otimes \lambda(\xi)}) = \operatorname{Log}(\| \rho \|^2_{\lambda^{-1}(\eta) \otimes \tilde{\lambda}(\xi)}).$$

Then, we must evaluate $\operatorname{Log}(\| \rho \|^2_{\lambda^{-1}(\eta) \otimes \tilde{\lambda}(\xi)})$.

2. A fundamental closed form

Let $\bar{\partial}^{X*}$, v^* be the adjoints of $\bar{\partial}^X$, v. Set $D^X = \bar{\partial}^X + \bar{\partial}^{X*}$, $V = v + v^*$. For $u > 0$, $T > 0$, set

$$B_{u,T} = u(D^X + TV). \tag{0.7}$$

Let N_V^X, N_H be the number operators which define the $\mathbf{Z}$-grading on $\Lambda(T^{*(0,1)}X)$, ξ respectively. Let $\beta_{u,T}$ be the one-form on $\mathbf{R}_+^* \times \mathbf{R}_+^*$

$$\beta_{u,T} = \frac{du}{u} \operatorname{Tr}_s[(N_V^X - N_H)\exp(-B_{u,T}^2)] - \frac{dT}{T} \operatorname{Tr}_s[N_H \exp(-B_{u,T}^2)]. \tag{0.8}$$

In (0.8), Tr_s is our notation of the supertrace. In Theorem 3.5, we prove that the form $\beta_{u,T}$ is closed. If Γ is a closed rectangle in $\mathbf{R}_+^* \times \mathbf{R}_+^*$, we get the basic identity

$$\int_\Gamma \beta = 0, \tag{0.9}$$

in Theorem 3.6.

As explained in Remark 3.7, Theorem 0.1 will be obtained by deforming Γ into the boundary of $\mathbf{R}_+^* \times \mathbf{R}_+^*$. In this process the contributions of each side of the rectangle diverge. Once divergences have been substracted off, one side of the rectangle ultimately calculates the logarithm of the Ray-Singer analytic torsion of $(E, \bar{\partial}^X + v)$, another the logarithm of the Ray-Singer analytic torsion of $(F, \bar{\partial}^Y)$, the third one the ratio of the L_2 metrics on $\lambda(\eta)$ and $\tilde{\lambda}(\xi)$, and the fourth one the right-hand side of formula (0.5).

Let us just say here that we devised this strategy by imitating the natural procedure one would follow if $(E, \bar{\partial}^X + v)$ and $(F, \bar{\partial}^Y)$ were finite-dimensional complexes.

3. Degeneration of a spectral sequence in Hodge Theory

Let $\bar{\partial}^{Y*}$ be the adjoint of $\bar{\partial}^Y$. Set $D^Y = \bar{\partial}^Y + \bar{\partial}^{Y*}$. We show in Sections 8 and 9 that in an adequate sense, the limit of the operators $D^X + TV$ (which act on E) as $T \to +\infty$ is equal to the operator D^Y (which acts on F). We now make this statement more precise. Let N_V^Y be the number operator which grades $\Lambda(T^{*(0,1)}Y) \otimes \eta$. A typical result, stated in Theorem 6.4, says that for $\alpha \geqslant \alpha_0 > 0$, $T \geqslant 1$,

$$|\operatorname{Tr}_s[(N_V^X - N_H)\exp(-\alpha(D^X+TV)^2)] - \operatorname{Tr}_s[N_V^Y \exp(-\alpha(D^Y)^2)]| \leqslant \frac{C}{\sqrt{T}}. \tag{0.10}$$

This is proved in Sections 8 and 9. The proof of (0.10) uses the fact that the metric g^{TX} is Kähler. Uniformity in $\alpha \geqslant \alpha_0$ is obtained by an adequate control of the lower part of the spectrum of $D^X + TV$, based on the quasi-isomorphism between the complexes $(E, \bar{\partial}^X + v)$ and $(F, \bar{\partial}^Y)$. This is one of the key steps of the proof where a purely algebraic fact is converted into relevant analytic information.

4. L_2 metrics and localization of harmonic forms

Roughly speaking, we show that, as $T \to +\infty$, the kernel of $D^X + TV$ is deformed into the kernel of D^Y. To calculate the ratio of the L_2 metrics on $\lambda(\eta)$ and $\tilde{\lambda}(\xi)$, we must in particular show that there are no spurious interactions between the connected components Y_j of Y. This is proved in Section 10, and is also an analytic consequence of the fact that r induces a quasi-isomorphism of complexes.

5. Local index Theory

In the whole paper, local index theory techniques play an important role. In particular we use the rescaling of the Clifford algebra introduced by Getzler [Ge] in his proof of the local Atiyah-Singer index Theorem.

Still, because the family of operators $(uD^X + TV)^2$ which we consider depends on two parameters $u > 0$, $T > 0$, we must adapt the rescaling of [Ge] to the variation of the two parameters. To illustrate this point, let us just say that if T is constant, as $u \to 0$, the local supertraces under consideration converge (by local index theory) to a smooth form on X. If $T \cong (1/u)$, by the results of Section 12, these local supertraces converge to a current concentrated on Y. A rather subtle problem which arises in our proof is to describe the convergence in the full region of variation of parameters $u \in]0, 1]$, $1 \leqslant T \leqslant (1/u)$. This analysis which is carried through in Section 11, is possible by a fine tuning of Getzler's rescaling.

The subtlety is best revealed by the fact that, as shown in [BGS4, Section 3], the current $T(\xi, h^{\xi})$ is not locally integrable near Y. The current $T(\xi, h^{\xi})$ on X can be defined as a principal part of a smooth current on $X\setminus Y$ whose integral near an ε-neighborhood on Y behaves like $\mathrm{Log}(\varepsilon)$ as $\varepsilon \to 0$, and the whole point is to show that our proof exactly produces this specific principal part.

6. Finite propagation speed and localization

By Theorem 6.4, for one given $u > 0$, as $T \to +\infty$, $\mathrm{Tr}_s[N_H \exp(-(uD^X+TV)^2)]$ has a limit. On the other hand, by the previous discussion and by the results in Section 12, for a given $T_0 \geqslant 1$, as $u \to 0$, $\mathrm{Tr}_s[N_H \exp(-(uD^X+(T_0/u)V)^2)]$ also has a limit. The question then arises to understand the behaviour of the quantity $\mathrm{Tr}_s[N_H \exp(-(uD^X+TV)^2)]$ in the range $u \in]0, 1]$, $T \geqslant (1/u)$. This is done in Section 13 of this paper, by using the fact that the operator $\cos(uD^X+TV)$ has finite propagation speed [CP, Chapter VII], [T, Chapter IV], which only depends on $u > 0$. This technique permits us to gently interpolate between the values $T=(1/u)$ and $T=+\infty$ of the parameter T. Finite propagation speed is one of the most important and significant of the techniques which are used in this paper.

7. Trivializations and functional analysis

Most of the analysis which is involved in this paper consists in writing the given operators as (2,2) or (3,3) matrices, which, as explained in the introduction to Section 12, have a preferred asymptotic structure as $T \to +\infty$. This preferred matrix structure is not invariant under conjugation. Therefore, the choice of trivializations of the considered vector bundles plays a key role in all the proofs. It turns out that the choice of the right trivialization depends heavily on the domain of variation of the parameters $u > 0$, $T > 0$. In many cases, the difficulty in the proofs lies in adjusting the trivialization to the domain of variation of the parameters, and also in delicately estimating the transition from one trivialization to another. In particular for $T \leqslant (1/u)$, the preferred coordinate system is a geodesic coordinate system centered at $x \in X$; for $T \cong (1/u)$, either a geodesic coordinate systems on X centered at $y_0 \in Y$ or a coordinate system of geodesics which are normal to Y; for $T \geqslant (1/u)$, a coordinate system of geodesics normal to Y.

Once the trivialization is chosen, most of the time, we need to estimate the resolvents of the considered rescaled operators, and in particular their regularizing properties, which should be uniform in the given domain of variation of parameters. To estimate the resolvents, we introduce an adequate family of norms depending on u, T and also on the grading of the considered vector spaces on which the considered operators act. In particular, because we deal with resolvents, Getzler's rescaling [Ge] imposes an analysis of its own, which can be avoided when only heat kernels are involved. Uniform coercivity estimates have to be proved on the operators considered. Their regularity properties are obtained by estimating uniformly iterated commutators with a class of test operators. These estimates have to be done carefully, since many estimates are borderline. Again, the choice of norms has to be delicately tuned to the domain of variation of parameters. Surprisingly enough, certain purely local problems, if seriously dealt with, are very difficult to solve. Solving them in detail partly explains the length of this paper.

8. Fighting the devil: heat equation and the logarithm

In the course of the proof of Theorem 0.1, one of the principal challenges is to study the behaviour, as $\varepsilon \to 0$, of the integral

$$\text{(0.11)} \qquad I_4^2(\varepsilon) = -\int_1^{+\infty} \left\{ \mathrm{Tr}_s[N_H \exp(-(\varepsilon D^X + \varepsilon TV)^2)] - \frac{1}{2} \dim N \chi(\eta) \right\} \frac{dT}{T}$$

(as the reader may guess, substracting off $(1/2) \dim N \chi(\eta)$ in the integrand (0.11) makes the integral converge). As shown in Theorem 6.14, the proof of which depends on Sections 7-13, once logarithmic divergences are substracted off, we ultimately produce the current $T(\xi, h^\xi)$ of Bismut-Gillet-Soulé [BGS4] and the analytic torsion forms $\mathbf{B}(TY, TX|_Y, g^{TX|Y})$ of Bismut [B3] associated with the exact sequence $0 \to TY \to TX|_Y \to N \to 0$.

9. Evaluation of the final formula: a geometric deformation

The analysis described above only involves a scaling on a two parameters family of operators, without modifying the geometry of the immersion $Y \to X$. The final step of our proof—which is the evaluation of the form $\mathbf{B}(TY, TX|_Y, g^{TX|Y})$ modulo ∂ and $\bar\partial$ coboundaries—was carried through in Bismut [B3], in particular by deforming the arbitrary short exact sequence of vector bundles $0 \to TY \to TX|_Y \to N \to 0$ to a split exact sequence (in which the vector bundle in the middle is the direct sum of the two others). The Bott-Chern class $\widetilde{\mathrm{Td}}(TY, TX|_Y, g^{TX|Y})$ and the additive genus $R(N)$ appear in (0.5) using the results of [B3]. The arguments in [B3] are the only ones which may resemble more classical arguments in the proof of the Theorem of Riemann-Roch-Grothendieck.

We now briefly describe our sources and our debts. First, this work is clearly connected with previous work by Bismut [B2] on the convergence of Chern character superconnection currents in the sense of Quillen [Q1] as the Quillen parameter tends to infinity, and to the subsequent construction by Bismut-Gillet-Soulé [BGS4] of the current $T(\xi, h^{\xi})$. Also Bismut-Gillet-Soulé [BGS5] gave a direct verification that the current $T(\xi, h^{\xi})$ verifies natural functorial properties compatible with a formula like (0.5). As we pointed out before, the genus $R(x)$ was conjectured to appear in a refined formula of Riemann-Roch-Grothendieck in Gillet-Soulé [GS3] and reobtained in Bismut [B3] as a piece of the analytic torsion forms of an exact sequence of holomorphic Hermitian vector bundles.

In [W], Witten gave a heat equation proof of the Morse inequalities, by a deformation of the Hodge de Rham complex which depends on the considered Morse function h. As a parameter t tends to infinity, the Witten Laplacian contains a potential $t^2 |dh|^2$ which makes the corresponding harmonic eigenforms localize on the critical points of h, together with other eigenforms associated with asymptotically zero eigenvalues, hence the Morse inequalities. In [W], Witten also suggested a proof of the Bott inequalities, in the case where the critical points of the function h form submanifolds. Morse and Bott inequalities were proved in [B5] using the Witten complex by a heat equation method, which involves a non-trivial deformation of the metric in the case of non isolated critical points. In [HeSj1], Helffer and Sjöstrand gave a rigorous analytic construction of the Thom-Smale-Witten complex of a Morse function with isolated critical points, by making mathematical sense of the instantons construction of Witten [W]. Also in [HeSj2], Helffer and Sjöstrand made a detailed analysis of the lower part of the spectrum and of the corresponding eigenforms of a Schrödinger operator with a potential tV as $t \to +\infty$, where the minima of V form submanifolds. In [HeSj3], they also proved the Bott inequalities using their previous results in [HeSj1].

It turns out that the analysis of the kernel of the operator $(D^X + TV)^2$ as $T \to +\infty$ is closely related to the analysis of Witten's Laplacian, $|dh|^2$ being replace by V^2. Any kind of "instanton" effect is here prevented by the existence of the quasi-isomorphism $r: (E, \bar{\partial}^X + v) \to (F, \bar{\partial}^Y)$.

Also Cheeger [Ch] and Müller [Mü] proved the equality between the Reidemeister torsion of a flat vector bundle and the corresponding Ray-Singer torsion [RS1]. This result is an equality between topological invariants. In [Ch] and [Mü], it is proved by studying the behaviour of the Reidemeister torsion and of the Ray-Singer torsion by surgery and by reducing the problem to a calculation on a sphere. More recently, Tangerman [Ta] announced a proof of the equality between the Reidemeister and analytic torsions by using the Witten complex associated with a Morse function with isolated critical points. Althouth the problem which is solved here is of a very different nature, there could be at least some analogy between the techniques announced in [Ta] and our own work.

We have tried to make the paper as self-contained as possible. It is organized as follows. In Section 1, we describe the geometric setting, we construct the Quillen metrics, and we introduce our fundamental assumptions on the metrics on the considered vector bundles. In Section 2, we prove that if $\tau \in \tilde{\lambda}^{-1}(\xi) \otimes \lambda(\xi)$ is the canonical section identifying $\lambda(\xi)$ with $\tilde{\lambda}(\xi)$, then $\|\tau\|_{\tilde{\lambda}^{-1}(\xi) \otimes \lambda(\xi)} = 1$. In Section 3, we construct our fundamental closed one-form β and the contours Γ in $\mathbf{R}_+^* \times \mathbf{R}_+^*$. In Section 4, we recall the definition of Quillen's superconnections [Q1], the results of [B2] on the convergence of superconnection Chern character currents of a resolution, and also the construction in [BGS4] of the current $T(\xi, h^\xi)$. In Section 5, we describe the results of [B3] on the construction of the analytic torsion forms associated to a short exact sequence of holomorphic Hermitian vector bundles.

In Section 6, we state seven intermediary results concerning the asymptotic behaviour of supertraces which involve the operator $\exp(-(u D^X + TV)^2)$. The proofs of six of these results is delayed to Sections 8-13. By pushing the contour Γ to the boundary of $\mathbf{R}_+^* \times \mathbf{R}_+^*$, we then derive Theorem 0.1 by a long but, in our opinion, quite interesting calculation. At the final stage, we use the results of Bismut [B3] on the construction and evaluation of the analytic torsion forms $\mathbf{B}(TY, TX|_Y, g^{TX|Y})$ together with the explicit formula by Bismut-Soulé [B3, Appendix 1] which relates the final evaluation of these forms to the genus R.

In Section 7, we recall the results of [B3] which concern the Hodge theory on a Hermitian vector space V of the Dolbeault complex associated with the resolution of the trivial sheaf concentrated at $\{0\}$ by the corresponding Koszul complex $\Lambda(V^*)$. The results of this Section are applied in Sections 9 and 13 to the fibres of the normal bundle N. We draw the attention of the reader on the Gaussian form β, which represents the canonical representative 1 of the cohomology of $\{0\}$ in the Hodge theory of the Dolbeault-Koszul complex. Part of the immense work of relating the Hodge theories on Y and X is done by such β's.

Sections 8-13 are devoted to the proof of six intermediary results stated in Section 6, which are needed in the proof of Theorem 0.1. In Sections 8 and 9, we study the family of operators $\exp(-\alpha(D^X + TV)^2)$ as T or/and α tend to $+\infty$. Section 10 describes the behaviour of the kernel of $D^X + TV$ as $T \to +\infty$. In Section 11, we establish uniform estimates on supertraces of operators involving $\exp(-(u D^X + TV)^2)$ in the range $u \in]0, 1]$, $1 \leqslant T \leqslant (1/u)$. If $u \to 0$, $T \cong (1/u)$, the behaviour of certain supertraces is studied in Section 12 and for $u \to 0$, $T \geqslant (1/u)$, the behaviour of such supertraces is described in Section 13. Sections 8-13 are accompanied with the description of the preferred coordinate systems, trivializations and functional analytic apparatus.

Finally, Section 14 contains a new direct proof of the result proved in [B3] concerning the asymptotic behaviour, as $T \to +\infty$, of certain differential forms associated with a short exact sequence of holomorphic Hermitian vector bundles. In [B3], the proof relied on a direct explicit computation of such forms by functional integration techniques. Here it is obtained by an adaptation of the functional analytic techniques which were developed in the previous Sections.

We now say a few words concerning our notation. If $\mathbf{A}$ is a $\mathbf{Z}_2$-graded algebra, and if A, B $\in \mathbf{A}$, we define the supercommutator [A, B] by the formula

$$[A, B] = AB - (-1)^{\deg A \deg B} BA. \tag{0.12}$$

If $E = E_+ \oplus E_-$ is a $\mathbf{Z}_2$-graded vector space, let $\tau = \pm 1$ on $E_\pm$ be the involution defining the grading. Then End(E) is a $\mathbf{Z}_2$-graded algebra, the even (resp. odd) elements commuting (resp. anticommuting) with τ. If $A \in \mathrm{End}(E)$, its supertrace $\mathrm{Tr}_s[A]$ is defined by

$$\mathrm{Tr}_s[A] = \mathrm{Tr}[\tau A]. \tag{0.13}$$

By [Q1] supertraces vanish on supercommutators. As in [B1], these notations will also be used in an infinite-dimensional context.

The results presented in this paper were announced in [BL].

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I - COMPLEX IMMERSIONS, DOLBEAULT RESOLUTIONS AND QUILLEN METRICS

In this Section, we introduce our basic setting, *i.e.*

- an immersion $i: Y \to X$ of compact complex manifolds.
- a holomorphic chain complex of vector bundles

$$(\xi, v): 0 \to \xi_m \underset{v}{\to} \ldots \underset{v}{\to} \xi_0 \to 0$$

on X, and a holomorphic restriction map $r: \xi_{0|Y} \to \eta$ such that we have the exact sequence of sheaves

$$0 \to \mathcal{O}_X(\xi_m) \underset{v}{\to} \mathcal{O}_X(\xi_{m-1}) \to \ldots \underset{v}{\to} \mathcal{O}_X(\xi_0) \underset{r}{\to} i_* \mathcal{O}_Y(\eta) \to 0.$$

For $0 \leqslant i \leqslant m$, let $\lambda(\xi_i)$ be the inverse of the determinant of the cohomology of the sheaf $\mathcal{O}_X(\xi_i)$. Set

$$\lambda(\xi) = \bigotimes_{i=0}^{m} (\lambda(\xi_i))^{(-1)^i}.$$

Let $\tilde{\lambda}(\xi)$ be the inverse of the determinant of the cohomology of the complex of sheaves $\mathcal{O}_X(\xi, v)$. Finally let $\lambda(\eta)$ be the inverse of the determinant of the cohomology of $\mathcal{O}_Y(\eta)$.

By Grothendieck-Knudsen-Mumford [KnM], the lines $\lambda(\xi)$, $\tilde{\lambda}(\xi)$ and $\lambda(\eta)$ are canonically isomorphic. More precisely $\lambda^{-1}(\eta) \otimes \tilde{\lambda}(\xi)$, $\tilde{\lambda}^{-1}(\xi) \otimes \lambda(\xi)$, and $\lambda^{-1}(\eta) \otimes \lambda(\xi)$ have canonical nonzero sections ρ, τ, σ and moreover $\sigma = \rho \otimes \tau$. In [KnM], these canonical isomorphisms are constructed in Čech cohomology. The first purpose of this Section is to give the corresponding construction by using the associated Dolbeault resolutions. In particular, in Theorem 1.7, we construct an explicit quasi-isomorphism between the double complex associated with the Dolbeault resolution of (ξ, v) on X and the Dolbeault resolution of η on Y, which corresponds to the tautological quasi-isomorphism in Čech cohomology.

The existence of the quasi-isomorphism of Dolbeault resolutions is the key algebraic input in the proof of our main result. From it, we will derive in Sections 9 and 10 results of a purely analytic nature, which are essential in the proof of Theorem 0.1.

We then give ourselves Hermitian metrics on TX, TY, $\xi_0, \ldots, \xi_m$, η. The second purpose of this Section is to construct the Quillen metrics on the lines $\lambda(\xi)$, $\tilde{\lambda}(\xi)$, $\lambda(\eta)$ in the sense of Quillen [Q2], Bismut-Gillet-Soulé [BGS3]. In particular, we adopt the normalization of the L_2 metric on Dolbeault resolutions suggested by Deligne [De].

The third purpose of this Section is to describe the compatibility assumption (A) between metrics on TX, $\xi_0, \ldots, \xi_m$ and η. This assumption was first introduced in [B2]. It will be satisfied in the whole paper.

This Section is organized as follows. In *a*), we introduce our main conventions concerning complex vector spaces, Hermitian metrics, the star operator acting on the associated exterior algebras. These conventions will be used in the whole paper. In *b*), we introduce the immersion $i\colon Y \to X$, the vector bundle η, and the complex (ξ, v). In *c*), we define the determinant lines $\lambda(\xi)$, $\tilde{\lambda}(\xi)$, $\lambda(\eta)$. In *d*), we describe the various canonical isomorphisms between determinant lines in Čech and Dolbeault cohomology. We also exhibit an explicit quasi-isomorphism between the Dolbeault resolutions of (ξ, v) and η. In *e*), we construct the Quillen metrics on the lines $\lambda(\xi)$, $\lambda(\eta)$, $\tilde{\lambda}(\xi)$, and we explain the main purpose of this paper, which is to calculate the Quillen norms of the sections ρ, τ, σ previously described. Finally in *f*), we introduce assumption (A) on the metrics on TX, $\xi_0, \ldots, \xi_m$, η.

a) Complex vector spaces and Hermitian products

To avoid any ambiguities, we now will explain the conventions concerning complex vector spaces and Hermitian products which will be used in the remainder of the paper.

Let $V_{\mathbf{R}}$ be a real even-dimensional vector space. Let $\mathbf{J}$ be a complex structure on $V_{\mathbf{R}}$, *i.e.* a linear map in $\mathrm{End}(V_{\mathbf{R}})$ such that $\mathbf{J}^2 = -1$. Set

$$\begin{aligned} V &= \{z \in V_{\mathbf{R}} \otimes_{\mathbf{R}} \mathbf{C};\ \mathbf{J} z = \sqrt{-1}\, z\}, \\ \bar{V} &= \{z \in V_{\mathbf{R}} \otimes_{\mathbf{R}} \mathbf{C};\ \mathbf{J} z = -\sqrt{-1}\, z\}. \end{aligned} \tag{1.1}$$

Here V will be called the complex vector space associated with $(V_{\mathbf{R}}, \mathbf{J})$. In some cases, we will also use the notation $V^{(1,0)}$, $V^{(0,1)}$ instead of V, $\bar{V}$. We have the identity

$$V_{\mathbf{R}} \otimes_{\mathbf{R}} \mathbf{C} = V \oplus \bar{V}.$$

There is a natural conjugation map $z \in V \mapsto \bar{z} \in \bar{V}$. Any $Z \in V_{\mathbf{R}} \otimes_{\mathbf{R}} \mathbf{C}$ can be written uniquely in the form

$$Z = z + z'; \qquad z \in V,\ z' \in \bar{V}.$$

Then $Z \in V_{\mathbf{R}}$ if and only if $Z = z + \bar{z}$, $z \in V$. If $z \in V$, z will represent $Z = z + \bar{z} \in V_{\mathbf{R}}$.

Let V^*, $\bar{V}^*$ be the vector spaces of $\mathbf{C}$-linear forms on V, $\bar{V}$ respectively.

Let $\langle\ ,\ \rangle$ be a $\mathbf{J}$-invariant scalar product on $V_{\mathbf{R}}$. We extend $\langle\ ,\ \rangle$ by $\mathbf{C}$-linearity to a bilinear symmetric form on $V_{\mathbf{R}} \otimes_{\mathbf{R}} \mathbf{C}$. The bilinear map $y, z \in V_{\mathbf{R}} \otimes_{\mathbf{R}} \mathbf{C} \mapsto \langle y, z \rangle$ vanishes when $y, z \in V$ or when $y, z \in \bar{V}$. The map $y, z \in V \to \langle y, \bar{z} \rangle$ is a Hermitian product on V.

If $Z \in V_{\mathbf{R}}$, $z \in V$, set

$$(1.2) \qquad \begin{aligned} |Z|^2 &= \langle Z, Z \rangle, \\ |z|^2 &= \langle z, \bar{z} \rangle. \end{aligned}$$

Clearly if $Z \in V_{\mathbf{R}}$ is such that $Z = z + \bar{z}$, with $z \in V$, then

$$(1.3) \qquad |Z|^2 = 2\,|z|^2.$$

Here the map $z \in V \mapsto Z = z + \bar{z} \in V_{\mathbf{R}}$ is not an isometry.

The Kähler form on $V_{\mathbf{R}}$ is the 2-form

$$(1.4) \qquad Z, Z' \in V_{\mathbf{R}} \mapsto \theta(Z, Z') = \langle Z, \mathbf{J} Z' \rangle.$$

Then θ extends by $\mathbf{C}$-linearity to a 2-form on $V_{\mathbf{R}} \otimes_{\mathbf{R}} \mathbf{C}$ of complex type (1,1).

The volume form dv_V of $V_{\mathbf{R}}$ is the form $(-\theta)^{\dim V}/(\dim V)!$, *i.e.* the usual volume form on $V_{\mathbf{R}}$ associated with the scalar product $\langle\ ,\ \rangle$ and the canonical orientation of $V_{\mathbf{R}}$.

The Hermitian product on V induces a Hermitian product on $\Lambda(\bar{V}^*)$ which we still denote $\langle\ ,\ \rangle$. Let $*$ be the Hodge operator, which maps $\Lambda^p(\bar{V}^*)$ into $\Lambda^{\dim V - p}(\bar{V}^*) \hat{\otimes} \Lambda^{\dim V}(V^*)$. To avoid any ambiguities in the normalization of $*$, we make the convention that if $\alpha, \beta \in \Lambda^p(\bar{V}^*)$,

$$(1.5) \qquad \alpha \wedge * \beta = \langle \alpha, \beta \rangle \frac{(-\theta)^{\dim V}}{(\dim V)!}.$$

b) Immersions and resolutions of vector bundles

Let X be a compact connected complex manifold of complex dimension l. Let $Y = \bigcup_1^d Y_j$ be a finite union of compact connected complex submanifolds of X, such that for $1 \leqslant j, j' \leqslant d$, $j \neq j'$, then $Y_j \cap Y_{j'} = \varnothing$. Let i be the immersion $Y \to X$. For $1 \leqslant j \leqslant d$, let l'_j be the complex dimension of Y_j.

Let η be a holomorphic vector bundle on the manifold Y. For $1 \leqslant j \leqslant d$, let η_j be the restriction of η to Y_j. In the sequel, we will often omit the subscript j in Y_j, η_j, l'_j.

Let

$$(\xi, v): 0 \to \xi_m \underset{v}{\to} \xi_{m-1} \dots \underset{v}{\to} \xi_0 \to 0 \tag{1.6}$$

be a holomorphic chain complex of vector bundles on X. In the sequel, we identify ξ with $\overset{m}{\underset{0}{\oplus}} \xi_k$. Then ξ is a $\mathbf{Z}$-graded vector bundle. Let r be a holomorphic restriction map $\xi_0|_Y \to \eta$.

For $0 \leqslant i \leqslant m$, let $\mathcal{O}_X(\xi_i)$ be the sheaf of holomorphic sections of ξ_i over X. Similarly let $\mathcal{O}_Y(\eta)$ be the sheaf of holomorphic sections of η over Y.

We assume that the complex (ξ, v) provides a projective resolution of the sheaf $i_* \mathcal{O}_Y(\eta)$, *i.e.* we have the exact sequence of sheaves

$$0 \to \mathcal{O}_X(\xi_m) \underset{v}{\to} \mathcal{O}_X(\xi_{m-1}) \to \dots \underset{v}{\to} \mathcal{O}_X(\xi_0) \underset{r}{\to} i_* \mathcal{O}_Y(\eta) \to 0. \tag{1.7}$$

c) The determinant fibres $\lambda(\xi)$, $\tilde{\lambda}(\xi)$ and $\lambda(\eta)$

Let δ^X, δ^{Y_j}, δ^Y be the Čech coboundary operators on X, Y_j $(1 \leqslant j \leqslant d)$, Y respectively. By definition the cohomology groups $H^*(X, \xi_i)$ $(0 \leqslant i \leqslant m)$, $H^*(Y_j, \eta_j)$ $(1 \leqslant j \leqslant d)$, $H^*(Y, \eta)$ are the cohomology groups of the complexes $(\mathcal{O}_X(\xi_i), \delta^X)$, $(\mathcal{O}_{Y_j}(\eta_j), \delta^{Y_j})$, $(\mathcal{O}_Y(\eta), \delta^Y)$. Of course $H^*(Y, \eta) = \overset{d}{\underset{j=1}{\oplus}} H^*(Y_j, \eta_j)$.

In the sequel, we will use the results of Grothendieck-Knudsen-Mumford [KnM] concerning determinants of the cohomology. However, we will disregard the questions of signs which appear in [KnM], since we have only to calculate norms of sections of determinant lines. These do not depend on signs. When we say that two determinant lines are canonically isomorphic, we will refer implicitly to [KnM] every time a sign has to be specified.

For $0 \leqslant i \leqslant m$, let $\lambda(\xi_i)$ be the inverse of the Grothendieck-Knudsen-Mumford determinant fibre associated with the sheaf $\mathcal{O}_X(\xi_i)$ [KnM, p. 46]. By definition

$$\lambda(\xi_i) = \overset{l}{\underset{p=0}{\otimes}} (\det H^p(X, \xi_i))^{(-1)^{p+1}}. \tag{1.8}$$

Similarly, for $1 \leqslant j \leqslant d$, $\lambda(\eta_j)$ denotes the inverse of the Grothendieck-Knudsen-Mumford determinant fibre associated with the sheaf $\mathcal{O}_{Y_j}(\eta_j)$. Then

$$\lambda(\eta_j) = \overset{l'_j}{\underset{p=0}{\otimes}} (\det H^p(Y_j, \eta_j))^{(-1)^{p+1}}. \tag{1.9}$$

Set

$$\begin{aligned}\lambda(\xi) &= \bigotimes_{i=0}^{m} (\lambda(\xi_i))^{(-1)^i},\\ \lambda(\eta) &= \bigotimes_{j=1}^{d} (\lambda(\eta_j)).\end{aligned} \tag{1.10}$$

Let N_δ be the operator acting on q-cochains on X by multiplication by q. Let N_H be the operator in $\operatorname{End}(\xi)$ acting on ξ_i by multiplication by i $(0 \leqslant i \leqslant m)$. We will grade the complex (ξ, v) by the operator $-N_H$, so that v increases the degree by one.

We now use sign conventions so that $\delta^X v + v\delta^X = 0$. We define the total **Z**-grading on the complex $(\mathcal{O}_X(\xi), \delta^X + v)$ by the operator $N_\delta - N_H$, so that the chain map $\delta^X + v$ increases the total degree by one. Set

$$\tilde{\lambda}(\xi) = \bigotimes_{p \in \mathbf{Z}} (\det H^p(\mathcal{O}_X(\xi), \delta^X + v))^{(-1)^{p+1}}. \tag{1.11}$$

We extend r to a map from $\mathcal{O}_X(\xi)$ into $i_* \mathcal{O}_Y(\eta)$, which vanishes on $\mathcal{O}_X(\xi_i)$ for $i \neq 0$ and coincides with the initial r for $i = 0$. Tautologically, the map $r: (\mathcal{O}_X(\xi), \delta^X + v) \to (\mathcal{O}_Y(\eta), \delta^Y)$ is a quasi-isomorphism of **Z**-graded complexes. Therefore

$$H^*(\mathcal{O}_X(\xi), \delta^X + v) \cong H^*(Y, \eta) = \bigoplus_{j=1}^{d} H^*(Y_j, \eta_j). \tag{1.12}$$

Definition **1.1.** — Let ρ be the canonical nonzero section of $\lambda^{-1}(\eta) \otimes \tilde{\lambda}(\xi)$ associated with the identification (1.12).

For $0 \leqslant i \leqslant m$, consider the exact sequence of complexes

$$0 \to (\bigoplus_{j \leqslant i-1} \mathcal{O}_X(\xi_j), \delta^X + v) \to (\bigoplus_{j \leqslant i} \mathcal{O}_X(\xi_j), \delta^X + v) \underset{\beta}{\to} (\mathcal{O}_X(\xi_i), \delta^X) \to 0. \tag{1.13}$$

In (1.13), we give the degree $-i$ to $\mathcal{O}_X(\xi_i)$, so that β is indeed a map of **Z**-graded complexes. From (1.13), we get a long exact sequence

$$\begin{aligned}\ldots \to H^p(\bigoplus_{j \leqslant i-1} \mathcal{O}_X(\xi_j), \delta^X + v) &\to H^p(\bigoplus_{j \leqslant i} \mathcal{O}_X(\xi_j), \delta^X + v)\\ &\to H^{p+i}(X, \xi_i) \to H^{p+1}(\bigoplus_{j \leqslant i-1} \mathcal{O}_X(\xi_j), \delta^X + v) \to \ldots\end{aligned} \tag{1.14}$$

From (1.14), by a standard construction [KnM, Lemma 2], [BGS1, Definition 1.1], we obtain a canonical isomorphism

$$\lambda(\xi_i) \cong \bigotimes_{p=-m}^{l} \left((\det H^p(\bigoplus_{j \leqslant i-1} \mathcal{O}_X(\xi_j), \delta^X + v))^{-1} \otimes (\det H^p(\bigoplus_{j \leqslant i} \mathcal{O}_X(\xi_j), \delta^X + v)) \right)^{(-1)^{p+i+1}}. \tag{1.15}$$

Using (1.10), we get a canonical isomorphism

$$\lambda(\xi) \cong \tilde{\lambda}(\xi). \tag{1.16}$$

Definition **1.2.** — Let τ be the nonzero section of the line $\tilde{\lambda}^{-1}(\xi) \otimes \lambda(\xi)$ associated with the canonical isomorphism (1.16).

In view of Definitions 1.1 and 1.2, we now set the following definition.

Definition **1.3.** — Let σ be the nonzero section of $\lambda^{-1}(\eta) \otimes \lambda(\xi)$ defined by

$$\sigma = \rho \otimes \tau. \tag{1.17}$$

Then σ is exactly the canonical nonzero section of the line $\lambda^{-1}(\eta) \otimes \lambda(\xi)$ constructed in Knudsen-Mumford [KnM, p. 46].

d) Canonical isomorphisms of determinant lines and Dolbeault resolutions

We now will construct the Dolbeault resolutions of the sheaves considered in Section 1c).

Let $\mathbf{J}$ be the complex structure of $T_{\mathbf{R}} X$. Set

$$T^{(1,0)}X = \{U \in T_{\mathbf{R}} X \otimes_{\mathbf{R}} \mathbf{C};\ \mathbf{J} U = \sqrt{-1}\, U\},$$
$$T^{(0,1)}X = \{U \in T_{\mathbf{R}} X \otimes_{\mathbf{R}} \mathbf{C};\ \mathbf{J} U = -\sqrt{-1}\, U\}.$$

Let $T^{*(1,0)}X$, $T^{*(0,1)}X$ be the complex duals to $T^{(1,0)}X$, $T^{(0,1)}X$ respectively. For simplicity, we will often write TX, T^*X instead of $T^{(1,0)}X$, $T^{*(1,0)}X$. We will use similar notation for the manifolds Y_j, Y.

Clearly

$$\Lambda(T^{*(0,1)}X) = \bigoplus_{p=0}^{l} \Lambda^p(T^{*(0,1)}X).$$

Let N_V^X be the operator defining the $\mathbf{Z}$-grading on $\Lambda(T^{*(0,1)}X)$. Then N_V^X acts on $\Lambda^p(T^{*(0,1)}X)$ by multiplication by p.

The spaces $\Lambda(T^{*(0,1)}X)$, ξ being $\mathbf{Z}$-graded inherit a corresponding $\mathbf{Z}_2$-grading. We can then form the $\mathbf{Z}_2$-graded tensor product $\Lambda(T^{*(0,1)}X) \hat{\otimes} \xi$. We define a

$\mathbf{Z}$-grading on $\Lambda(T^{*(0,1)}X) \hat{\otimes} \xi$ by the operator $N_V^X \otimes 1 - 1 \otimes N_H$, which we will often denote $N_V^X - N_H$.

Definition **1.4.** — For $0 \leqslant p \leqslant l$, $0 \leqslant i \leqslant m$, let E_i^p be the set of smooth sections of $\Lambda^p(T^{*(0,1)}X) \otimes \xi_i$ over the manifold X. Set

$$(1.18)\qquad \begin{aligned} &E_i^+ = \bigoplus_{p \text{ even}} E_i^p;\ E_i^- = \bigoplus_{p \text{ odd}} E_i^p;\ E_i = E_i^+ \oplus E_i^-,\\ &E_+ = \bigoplus_{p-i \text{ even}} E_i^p;\ E_- = \bigoplus_{p-i \text{ odd}} E_i^p;\ E = E_+ \oplus E_-. \end{aligned}$$

For $0 \leqslant i \leqslant m$, E_i is $\mathbf{Z}$-graded by N_V^X. The splitting $E_i = E_i^+ \oplus E_i^-$ describes the corresponding $\mathbf{Z}_2$-grading of E_i. Similarly, E can be identified with the set of smooth sections of $\Lambda(T^{*(0,1)}X) \hat{\otimes} \xi$ over X. The space E is naturally $\mathbf{Z}$-graded by the operator $N_V^X - N_H$. The splitting $E = E_+ \oplus E_-$ describes the corresponding $\mathbf{Z}_2$-grading of E.

Let $\bar{\partial}^X$ be the Dolbeault operator acting on E. If $(x^1, \ldots, x^l)$ is a holomorphic coordinate system on X, in a given local holomorphic trivialization of ξ, then

$$(1.19)\qquad \bar{\partial}^X = \sum_1^l d\bar{x}^i \wedge \frac{\partial}{\partial \bar{x}^i}.$$

The operator $\bar{\partial}^X$ acts on each E_i $(0 \leqslant i \leqslant m)$. For $0 \leqslant i \leqslant m$, we have the fundamental identity of $\mathbf{Z}$-graded vector spaces

$$(1.20)\qquad H^*(E_i, \bar{\partial}^X) \cong H^*(\mathcal{O}_X(\xi_i), \delta^X).$$

Equivalently, for $0 \leqslant i \leqslant m$,

$$(1.21)\qquad H^*(E_i, \bar{\partial}^X) \cong H^*(X, \xi_i).$$

The chain map v acts on ξ as an odd operator, *i.e.* it interchanges the even and odd parts of ξ. We extend v to an odd operator acting on $\Lambda(T^{*(0,1)}X) \hat{\otimes} \xi$, so that if $\alpha \in \Lambda^p(T^{*(0,1)}X)$, $f \in \xi$, then

$$v(\alpha \hat{\otimes} f) = (-1)^{\deg \alpha} \alpha \hat{\otimes} vf.$$

We have the obvious identities of operators acting on E

$$(1.22)\qquad (\bar{\partial}^X)^2 = 0;\ v^2 = 0;\ \bar{\partial}^X v + v \bar{\partial}^X = 0.$$

From (1.22), we deduce that

$$(1.23)\qquad (\bar{\partial}^X + v)^2 = 0.$$

We can then form the $\mathbf{Z}$-graded complex $(E, \bar{\partial}^X + v)$.

Proposition **1.5.** — *There is a canonical identification of* **Z**-*graded vector spaces*

$$\mathrm{H}^*(\mathrm{E}, \bar{\partial}^{\mathrm{X}}+v) \cong \mathrm{H}^*(\mathcal{O}_{\mathrm{X}}(\xi), \delta^{\mathrm{X}}+v). \tag{1.24}$$

Proof. — The proof of (1.24) is identical to the proof of the more classical result (1.20). In fact we form a triple complex with chain map $\delta^{\mathrm{X}}+\bar{\partial}^{\mathrm{X}}+v$ (and choice of signs such that this is indeed a coboundary map). By filtering this complex by the chain map $\bar{\partial}^{\mathrm{X}}$ and using the Poincaré lemma, we get the complex $(\mathcal{O}_{\mathrm{X}}(\xi), \delta^{\mathrm{X}}+v)$, by filtering the complex by the chain map δ^{X}, we get the complex $(\mathrm{E}, \bar{\partial}^{\mathrm{X}}+v)$. Our Proposition is proved. □

Let $\mathrm{N}_{\mathrm{V}}^{\mathrm{Y}}$ be the operator defining the **Z**-grading on

$$\Lambda(\mathrm{T}^{*(0,1)}\mathrm{Y}) = \bigoplus_{q=0}^{l'} \Lambda^q(\mathrm{T}^{*(0,1)}\mathrm{Y}).$$

This $\mathrm{N}_{\mathrm{V}}^{\mathrm{Y}}$ acts as the operator $\mathrm{N}_{\mathrm{V}}^{\mathrm{Y}}\otimes 1$ on $\Lambda(\mathrm{T}^{*(0,1)}\mathrm{Y})\otimes\eta$. It will be sometimes useful to assume that η has degree 0, so that $\Lambda(\mathrm{T}^{*(0,1)}\mathrm{Y})\otimes\eta=\Lambda(\mathrm{T}^{*(0,1)}\mathrm{Y})\hat{\otimes}\eta$.

Definition **1.6.** — For $1\leqslant j\leqslant d$, $0\leqslant q\leqslant l'_j$, let F_j^q be the set of smooth sections of $\Lambda^q(\mathrm{T}^{*(0,1)}\mathrm{Y}_j)\otimes\eta$ over the manifold Y_j. Set

$$\mathrm{F}_{j,+} = \bigoplus_{q \text{ even}} \mathrm{F}_j^q;\ \mathrm{F}_{j,-} = \bigoplus_{q \text{ odd}} \mathrm{F}_j^q;\ \mathrm{F}_j=\mathrm{F}_{j,+}\oplus\mathrm{F}_{j,-}; \tag{1.25}$$

$$\mathrm{F}_{\pm} = \bigoplus_{j=1}^{d} \mathrm{F}_{j,\pm};\ \mathrm{F}=\mathrm{F}_+\oplus\mathrm{F}_-.$$

For $q\in\mathbf{N}$, set

$$\mathrm{F}^q = \bigoplus_{j=1}^{d} \mathrm{F}_j^q. \tag{1.26}$$

Then

$$\mathrm{F} = \bigoplus_{q\in\mathbf{N}} \mathrm{F}^q. \tag{1.27}$$

The operator $\mathrm{N}_{\mathrm{V}}^{\mathrm{Y}}$ defines a **Z**-grading on F_j $(1\leqslant j\leqslant d)$, and on F.

For $1\leqslant j\leqslant d$, let $\bar{\partial}^{\mathrm{Y}_j}$ be the Dolbeault operator acting on F_j. Then we have the canonical identification of **Z**-graded vector spaces

$$\mathrm{H}^*(\mathrm{F}_j, \bar{\partial}^{\mathrm{Y}_j}) \cong \mathrm{H}^*(\mathrm{Y}_j, \eta_j). \tag{1.28}$$

We will use the notation

$$\bar{\partial}^{\mathrm{Y}} = \bigoplus_{1}^{d} \bar{\partial}^{\mathrm{Y}_j}; \tag{1.29}$$

Then $\bar{\partial}^{Y}$ acts on $F = \bigoplus_{j=1}^{d} F_j$. Also

$$(1.30) \qquad H^*(F, \bar{\partial}^Y) \cong \bigoplus_{j=1}^{d} H^*(Y_j, \eta_j).$$

Recall that r is the restriction map $\xi_0|_Y \to \eta$. We will extend r to a linear map from $(\Lambda(T^{*(0,1)}X) \hat{\otimes} \xi)|_Y$ into η. Namely if $\alpha \in \Lambda(T^{*(0,1)}X)|_Y$, $f \in \xi_k|_Y$ $(0 \leqslant k \leqslant m)$, set

$$(1.31) \qquad \begin{aligned} r(\alpha \hat{\otimes} f) = 0 \quad & \text{if} \quad k \neq 0, \\ i^*\alpha \hat{\otimes} rf \quad & \text{if} \quad k = 0. \end{aligned}$$

We now prove the following simple and essential result.

Theorem **1.7.** – *The map* $r: (E, \bar{\partial}^X + v) \to (F, \bar{\partial}^Y)$ *is a quasi-isomorphism of* **Z**-*graded complexes. It induces the canonical identification* (1.12) *of* $H^*(\mathcal{O}_X(\xi), \delta^X + v)$ *with* $H^*(Y, \eta) = \bigoplus_{j=1}^{d} H^*(Y_j, \eta_j)$. *In particular, the map r induces the canonical identification* ρ *of* $\lambda(\eta)$ *and* $\tilde{\lambda}(\xi)$ *described in Definition* 1.1.

Proof. – It is clear that

$$(1.32) \qquad r\bar{\partial}^X = \bar{\partial}^Y r.$$

Also

$$(1.33) \qquad rv = 0.$$

Therefore r is a map of chain complexes. We now briefly prove that r is a quasi-isomorphism which induces the canonical isomorphism of $H^*(\mathcal{O}_X(\xi), \delta^X + v)$ with $H^*(Y, \eta)$.

As in the proof of Proposition 1.5, we introduce the triple complex $(\tilde{E}, \delta^X + \bar{\partial}^X + v)$ and we also consider the complex $(\tilde{F}, \delta^Y + \bar{\partial}^Y)$. The map r sends $(\tilde{E}, \delta^X + \bar{\partial}^X + v)$ into $(\tilde{F}, \delta^Y + \bar{\partial}^Y)$. If we filter the complex $(\tilde{E}, \delta^X + \bar{\partial}^X + v)$ by the map $\bar{\partial}^X$, and the complex $(F, \delta^Y + \bar{\partial}^Y)$ by the map $\bar{\partial}^Y$, we find that r induces the tautological quasi-isomorphism $(\mathcal{O}_X(\xi), \delta^X + v) \to (\mathcal{O}_Y(\eta), \delta^Y)$. If we filter the complex $(\tilde{E}, \delta^X + \bar{\partial}^X + v)$ by the map δ^X and the complex $(\tilde{F}, \delta^Y + \bar{\partial}^Y)$ by the map δ^Y, we obtain the complexes $(E, \bar{\partial}^X + v)$ and $(F, \bar{\partial}^Y)$. The induced map r is exactly the one described in (1.31). Therefore the map $r: (E, \bar{\partial}^X + v) \to (F, \bar{\partial}^Y)$ is a quasi-isomorphism, which induces the canonical isomorphism (1.12).

Our Theorem is proved. □

For $0 \leqslant i \leqslant m$, consider the exact sequence of complexes

$$0 \to (\bigoplus_{j \leqslant i-1} E_j, \bar{\partial}^X + v) \to (\bigoplus_{\gamma \; j \leqslant i} E_j, \bar{\partial}^X + v) \to (E_i, \bar{\partial}^X) \to 0. \tag{1.34}$$

As before, in (1.34), we give the total degree $p-i$ to E_i^p, so that (1.34) is indeed an exact sequence of **Z**-graded complexes. From (1.34), we get the long exact sequence

$$\begin{aligned} \ldots \to H^p(\bigoplus_{j \leqslant i-1} E_j, \bar{\partial}^X + v) \to H^p(\bigoplus_{j \leqslant i} E_j, \bar{\partial}^X + v) \\ \to H^{p+i}(E_i, \bar{\partial}^X) \to H^{p+1}(\bigoplus_{j \leqslant i-1} E_j, \bar{\partial}^X + v) \to \ldots \end{aligned} \tag{1.35}$$

By Proposition 1.5, the vector spaces appearing in the exact sequences (1.14) and (1.35) are canonically isomorphic.

Proposition **1.8.** — *The exact sequences* (1.14) *and* (1.35) *are canonically isomorphic. In particular, the exact sequence* (1.35) *induces the canonical identification* τ *of* $\tilde{\lambda}(\xi)$ *and* $\lambda(\xi)$ *described in Definition* 1.2.

Proof. — By proceeding as in Proposition 1.5, we introduce the complexes $(\bigoplus_{i \leqslant j} \tilde{E}_i, \delta^X + \bar{\partial}^X + v)$ which fit into an exact sequence similar to (1.13) and (1.34). By filtering this triple complex with respect to $\bar{\partial}^X$ and δ^X, we obtain our Proposition. □

e) Quillen metrics on the lines $\lambda(\xi)$, $\lambda(\eta)$ and $\tilde{\lambda}(\xi)$

We assume that TX is equipped with a smooth Hermitian metric g^{TX}. Let $\langle \, , \rangle$ denote the corresponding scalar product on $T_{\mathbf{R}}X$. If **J** is the complex structure of $T_{\mathbf{R}}X$, the Kähler form ω^X on X is defined by

$$U, U' \in T_{\mathbf{R}}X \to \omega^X(U, U') = \langle U, \mathbf{J}U' \rangle. \tag{1.36}$$

As a complex submanifold of X, Y is also naturally equipped with a Hermitian metric g^{TY} whose Kähler form ω^Y is given by

$$\omega^Y = i^* \omega^X. \tag{1.37}$$

Let $h^{\xi_0}, \ldots, h^{\xi_m}$ be smooth Hermitian metrics on the vector bundles $\xi_0, \ldots, \xi_m$. Let h^ξ be the metric on $\xi = \bigoplus_{i=0}^{m} \xi_i$ which is the orthogonal sum of the metrics h^{ξ_i}. Let g^η be a smooth Hermitian metric on the vector bundle η.

We now briefly explain the construction of the Quillen metric on the determinant lines $\lambda(\xi_0), \ldots, \lambda(\xi_m)$ [Q2], [BGS3]. We use the conventions of Section 1a).

Let $*$ be the complex star operator associated to the given Hermitian metric on TX. For $0 \leqslant p \leqslant \dim X$, if $\alpha, \alpha' \in E_i^p$, set

$$\langle \alpha, \alpha' \rangle = \left(\frac{1}{2\pi}\right)^{\dim X} \int_X \langle \alpha \wedge * \alpha' \rangle_{\xi_i}. \tag{1.38}$$

Equivalently, the metric on TX induces a metric on $\Lambda(T^{*(0,1)}X)$. We equip $\Lambda(T^{*(0,1)}X) \otimes \xi_i$ with the obvious product metric. Let dv_X be the volume form on X associated with the metric g^{TX}. Then if $\alpha, \alpha' \in E_i^p$, we also have

$$\langle \alpha, \alpha' \rangle = \left(\frac{1}{2\pi}\right)^{\dim X} \int_X \langle \alpha, \alpha' \rangle_{\Lambda(T^{*(0,1)}X) \otimes \xi_i} dv_X. \tag{1.39}$$

The formulas (1.38), (1.39) define a Hermitian product on E_i^p. We equip $E_i = \bigoplus_{p=1}^{l} E_i^p$ with the orthogonal sum of the given Hermitian products on the E_i^p's.

Let $\bar\partial^{X*}$ be the formal adjoint of $\bar\partial^X$ with respect to the Hermitian product (1.38), (1.39).

For $0 \leqslant i \leqslant m$, $0 \leqslant p \leqslant \dim X$, let K_i^p be the finite-dimensional vector space

$$K_i^p = \{\alpha \in E_i^p;\ \bar\partial^X \alpha = 0;\ \bar\partial^{X*} \alpha = 0\}. \tag{1.40}$$

By Hodge theory, we know that K_i^p is canonically isomorphic to $H^p(X, \xi_i)$. As finite dimensional vector subspaces of the E_i^p's, the vector spaces $K_i^p \cong H^p(X, \xi_i)$ inherit the Hermitian product (1.38), (1.39). Using the identification of $\lambda(\xi_i)$ with $\bigotimes_{p=0}^{\dim X} (\det H^p(X, \xi_i))^{(-1)^{p+1}}$, we may equip $\lambda(\xi_i)$ with the obvious product metric which we denote $| \ |_{\lambda(\xi_i)}$.

Set $K_i = \bigoplus_{p=0}^{l} K_i^p$. Let $K_i^\perp$ be the space orthogonal to K_i in E_i. Let P_i, $P_i^\perp$ denote the orthogonal projection operators from E_i on K_i, $K_i^\perp$. The Laplacian $(\bar\partial^X + \bar\partial^{X*})^2$ acts on $K_i^\perp$ as an invertible operator, whose inverse is denoted $[(\bar\partial^X + \bar\partial^{X*})^2]^{-1}$.

Observe that the vector spaces E_i, K_i, $K_i^\perp$ are **Z**-graded by the operator N_V^X, and so they are $\mathbf{Z}_2$-graded. In particular if $A \in \mathrm{End}(E_i)$ is trace class, we can define its supertrace $\mathrm{Tr}_s^{E_i}[A]$.

Definition **1.9.** — For $0 \leqslant i \leqslant m$, $s \in \mathbf{C}$, $\mathrm{Re}(s) > \dim X$, set

$$\theta_{\xi_i}^X(s) = -\mathrm{Tr}_s^{E_i}[N_V^X[(\bar\partial^X + \bar\partial^{X*})^2]^{-s} P_i^\perp]. \tag{1.41}$$

By a result of Seeley [Se], the function $\theta_{\xi_i}^X(s)$ extends to a meromorphic function of $s \in \mathbf{C}$, which is holomorphic at $s = 0$.

Definition **1.10.** — For $0 \leqslant i \leqslant m$, the Quillen metric $\| \ \|_{\lambda(\xi_i)}$ on $\lambda(\xi_i)$ is given by

$$\| \ \|_{\lambda(\xi_i)} = \exp\left\{-\frac{1}{2}\frac{\partial\theta^X_{\xi_i}}{\partial s}(0)\right\} | \ |_{\lambda(\xi_i)}. \tag{1.42}$$

The factor $\exp\{-(\partial\theta^X_{\xi_i}/\partial s)(0)\}$ is the Ray-Singer analytic torsion [RS2] of the complex $(E_i, \bar\partial^X)$.

For $1 \leqslant i \leqslant m$, let $\| \ \|_{(\lambda(\xi_i))^{(-1)}}$ be the metric on the line $(\lambda(\xi_i))^{-1}$ which is dual to the metric $\| \ \|_{\lambda(\xi_i)}$ on $\lambda(\xi_i)$.

We then equip the line $\lambda(\xi) = \overset{m}{\underset{0}{\otimes}} (\lambda(\xi_i))^{(-1)^i}$ with the product metric

$$\| \ \|_{\lambda(\xi)} = \bigotimes_{i=0}^{m} \| \ \|_{(\lambda(\xi_i))^{(-1)^i}}. \tag{1.43}$$

We construct the Quillen metric $\| \ \|_{\lambda(\eta_j)}$ on the line $\lambda(\eta_j)$ $(1 \leqslant j \leqslant d)$ in a similar way. In particular if $1 \leqslant j \leqslant d$ and $\beta, \beta' \in F^q_j$, set

$$\langle \beta, \beta' \rangle = \left(\frac{1}{2\pi}\right)^{\dim Y_j} \int_{Y_j} \langle \beta \wedge * \beta' \rangle_{\eta_j}. \tag{1.44}$$

Equivalently if dv_{Y_j} is the volume element on Y_j, then

$$\langle \beta, \beta' \rangle = \left(\frac{1}{2\pi}\right)^{\dim Y_j} \int_{Y_j} \langle \beta, \beta' \rangle_{\Lambda(T^{*(0,1)} Y_j) \otimes \eta_j} dv_{Y_j}. \tag{1.45}$$

The definition of the Quillen metric $\| \ \|_{\lambda(\eta_j)}$ on the line $\lambda(\eta_j)$ then proceeds exactly as before. We equip the line $\lambda(\eta) = \bigotimes_{j=1}^{d} \lambda(\eta_j)$ with the Quillen metric $\| \ \|_{\lambda(\eta)}$ which is the product of the metrics $\| \ \|_{\lambda(\eta_j)}$.

We equip $F = \bigoplus_{j=1}^{d} F_j$ with the orthogonal sum of the Hermitian products (1.45). Let $\bar\partial^{Y_j *}$ be the formal adjoint of $\bar\partial^{Y_j}$ with respect to (1.45). Then $\bar\partial^{Y*} = \bigoplus_{j=1}^{d} \bar\partial^{Y_j *}$ is the formal adjoint of $\bar\partial^Y = \bigoplus_{j=1}^{d} \bar\partial^{Y_j}$.

Remark **1.11.** — Note a few differences with the conventions of Bismut-Gillet-Soulé [BGS3, Section 1d)]. The first is the factor $(1/2\pi)^{\dim X}$ or $(1/2\pi)^{\dim Y_j}$ in (1.38), (1.44) which did not appear in [BGS3]. This modification was suggested by

Deligne [De]. Also in [BGS3], $\theta^X_{\xi_i}(s)$ was instead

$$\bar{\theta}^X_{\xi_i}(s) = -\operatorname{Tr}_s^{E_i}[N^X_V[2(\bar{\partial}^X + \bar{\partial}^{X*})^2]^{-s} P_i^\perp]. \tag{1.46}$$

However one verifies easily that

$$\bar{\theta}^X_{\xi_i}(s) = -\operatorname{Tr}_s^{E_i}[N^X_V[(\bar{\partial}^X + 2\bar{\partial}^{X*})^2]^{-s} P_i^\perp]. \tag{1.47}$$

Now if the metric g^{TX} on TX is changed into $g^{TX}/2$, the operator $\bar{\partial}^{X*}$ is changed into $2\bar{\partial}^{X*}$. Therefore $\bar{\theta}^X_{\xi_i}(s)$ is exactly the function $\theta^X_{\xi_i}(s)$ in which the metric g^{TX} is replaced by $g^{TX}/2$. The effect of such of a change in our final result will be briefly considered in Remark 6.18.

We finally construct the Quillen metric on the line $\tilde{\lambda}(\xi)$, by imitating [BGS3, Section 2a)]. Namely, we equip $E = \oplus_{0 \leqslant i \leqslant m,\, 0 \leqslant p \leqslant l} E_i^p$ with the orthogonal sum of the Hermitian metrics (1.38), (1.39) on the E_i^p's.

Let v^* be the adjoint of v with respect to the Hermitian product h^ξ on ξ. Then $\bar{\partial}^{X*} + v^*$ is the formal adjoint of $\bar{\partial}^X + v$. Set

$$K = \{e \in E;\ (\bar{\partial}^X + v)e = 0;\ (\bar{\partial}^{X*} + v^*)e = 0\}. \tag{1.48}$$

By Hodge theory, we have a canonical identification of $\mathbf{Z}$-graded vector spaces $K \cong H^*(E, \bar{\partial}^X + v)$. The vector space K inherits a Hermitian product from the Hermitian product of E. Let $|\ \ |_{\tilde{\lambda}(\xi)}$ denote the corresponding metric on $\tilde{\lambda}(\xi)$.

Let $K^\perp$ be the vector space orthogonal to K in E. Then the operator $(\bar{\partial}^X + v + \bar{\partial}^{X*} + v^*)^2$ acts as an invertible operator on $K^\perp$, whose inverse is denoted $((\bar{\partial}^X + v + \bar{\partial}^{X*} + v^*)^2)^{-1}$. Let P, $P^\perp$ denote the orthogonal projection operators from E on K, $K^\perp$ respectively.

For $s \in \mathbf{C}$, $\operatorname{Re}(s) > \dim X$, set

$$\tilde{\theta}^X_\xi(s) = -\operatorname{Tr}_s^E[(N^X_V - N_H)((\bar{\partial}^X + v + \bar{\partial}^{X*} + v^*)^2)^{-s} P^\perp]. \tag{1.49}$$

Then $\tilde{\theta}^X_\xi(s)$ extends to a meromorphic function of $s \in \mathbf{C}$, which is holomorphic at $s = 0$.

The Quillen metric $\|\ \ \|_{\tilde{\lambda}(\xi)}$ on the line $\tilde{\lambda}(\xi)$ is defined by

$$\|\ \ \|_{\tilde{\lambda}(\xi)} = \exp\left\{-\frac{1}{2}\frac{\partial \tilde{\theta}^X_\xi}{\partial s}(0)\right\} |\ \ |_{\tilde{\lambda}(\xi)}. \tag{1.50}$$

The lines $\lambda(\xi)$, $\tilde{\lambda}(\xi)$, $\lambda(\eta)$ are now equipped with Quillen metrics $\|\ \ \|_{\lambda(\xi)}$, $\|\ \ \|_{\tilde{\lambda}(\xi)}$, $\|\ \ \|_{\lambda(\eta)}$. We equip the inverses or the tensor products of such lines with the inverses or the tensor products of the corresponding Quillen metrics.

Tautologically, by formula (1.17), we know that

$$\|\sigma\|_{\lambda^{-1}(\eta) \otimes \lambda(\xi)} = \|\rho\|_{\lambda^{-1}(\eta) \otimes \tilde{\lambda}(\xi)} \|\tau\|_{\tilde{\lambda}^{-1}(\xi) \otimes \lambda(\xi)}. \tag{1.51}$$

The main purpose of this paper is to calculate the norms which appear in (1.51).

This will be done under various assumptions on the metrics g^{TX}, $h^{\xi_0}, \dots, h^{\xi_m}$, g^{η} which are described in Section 1f).

f) Assumptions on the metrics on TX, ξ, η

Our first basic assumption is that the metric g^{TX} is Kähler, or equivalently that the Kähler form ω^X defined in (1.36) is closed. Therefore the metric g^{TY} on Y is Kähler, and the corresponding Kähler form ω^Y is closed.

Let N be the complex bundle normal to Y in X; N_j will denote the restriction of N to Y_j.

On Y, we have the exact sequence of holomorphic vector bundles

$$0 \to TY \to TX|_Y \to N \to 0. \tag{1.52}$$

We identify N with the bundle orthogonal to TY in $TX|_Y$. Therefore N is now equipped with a Hermitian metric g^N. Let P^{TY}, P^N denote the orthogonal projection operators from $TX|_Y$ on TY, N respectively.

Let g^{η} be a Hermitian metric on η.

We now describe the special choice of metrics $h^{\xi_0}, \dots, h^{\xi_m}$ on $\xi_0, \dots, \xi_m$ in [B2, Section 1].

Take $y_0 \in Y$. Let $x=(y, z)$, $y \in \mathbf{C}^{l'}$, $z \in \mathbf{C}^n$ be a holomorphic system of coordinates on an open neighborhood U of y_0 in X such that 0 represents y_0, and that

$$Y \cap U = \{x=(y, z) \in U,\ z=0\}. \tag{1.53}$$

Then $dz^1, \dots, dz^n$ span a trivial vector bundle $\tilde{N}^*$ on U, whose restriction to $Y \cap U$ is exactly N^*. Let $\tilde{N}$ be the dual of $\tilde{N}^*$. Clearly, $\tilde{N}$ extends $N|_{Y \cap U}$ to U. If U is small enough, the holomorphic vector bundle $\eta|_{Y \cap U}$ extends to a holomorphic vector bundle $\tilde{\eta}$ on U. We consider z as a section of $\tilde{N}$ on U which exactly vanishes on $Y \cap U$. Then the interior multiplication operator i_z acts naturally on the Koszul complex $\Lambda \tilde{N}^* \otimes \tilde{\eta}$.

By the local uniqueness of resolutions [Ser, Chapter IV, Appendix 1], [E, Theorem 8], we know that if U is small enough, there exists a **Z**-graded holomorphic acyclic chain complex (A, a) on U such that we have an identification of **Z**-graded holomorphic chain complexes

$$(\xi, v)|_U \cong (\Lambda \tilde{N}^* \otimes \tilde{\eta}, \sqrt{-1}\, i_z) \oplus (A, a). \tag{1.54}$$

Under the identification (1.54), the restriction map r is given by

$$(1.55)\qquad (f, g)\in(\tilde{\eta}\oplus A_0)\,|_{Y\cap U}\mapsto f\in\eta.$$

We now briefly describe the results of [B2, Section 1b)], which are simple consequences of the previous considerations.

For $y\in Y$, let $H_y(\xi, v)$ be the homology of the chain complex $(\xi, v)_y$. It is a **Z**-graded vector space. Then

• For $0\leqslant i\leqslant m$, the dimension of $H_{i,y}(\xi, v)$ is locally constant as y varies in Y. Therefore $H(\xi, v)$ is now a **Z**-graded holomorphic vector bundle on Y.

• For $y\in Y$, $u\in T_yX$, let $\partial_u v(y)$ be the derivative of the chain map v at y in the direction u calculated in any holomorphic trivialization of ξ near y. Then $\partial_u v(y)$ acts on $H_y(\xi, v)$ and decreases the total degree by one. The action of $\partial_u v(y)$ on $H_y(\xi, v)$ does not depend on the trivialization of (ξ, v) and only depends on the image $z\in N_y$ of $u\in T_yX$. So, from now on, we will write $\partial_z v(y)$ instead of $\partial_u v(y)$.

• For any $y\in Y$, $z\in N_y$, we have $(\partial_z v(y))^2=0$. Also, $\partial_z v(y)$ depends holomorphically on y, z.

• Let $\tilde{\pi}$ be the projection $N\to Y$. Then there is a canonical isomorphism of holomorphic **Z**-graded chain complexes on N

$$(1.56)\qquad (H(\xi, v), \partial_z v)\cong(\tilde{\pi}^*(\Lambda N^*\otimes\eta), \sqrt{-1}\, i_z).$$

Recall that $\xi_0, \ldots, \xi_m$ are equipped with Hermitian metrics $h^{\xi_0}, \ldots, h^{\xi_m}$. We then use the same notation as in Section 1e). By finite-dimensional Hodge theory, we know that for every $y\in Y$, there is a canonical isomorphism of **Z**-graded vector spaces

$$(1.57)\qquad H_y(\xi, v)\cong\{f\in\xi_y;\ v(y)f=0;\ v^*(y)f=0\}.$$

The identification (1.57) induces an identification of smooth vector bundles on Y. The bundle $H(\xi, v)$ will now be considered as a smooth vector subbundle of $\xi|_Y$. In particular $H(\xi, v)$ inherits a Hermitian metric $h^{H(\xi, v)}$ from the metric h^ξ on ξ.

Recall that N, η are already equipped with Hermitian metrics g^N, g^η. The bundle ΛN^* is naturally equipped with the metric induced by g^N. We equip $\Lambda N^*\otimes\eta$ with the tensor product of the metrics on ΛN^* and η.

Definition **1.12.** – We will say that the metrics $h^{\xi_0}, \ldots, h^{\xi_m}$ on $\xi_0, \ldots, \xi_m$ verify assumption (A) with respect to the metrics g^N, g^η if the identification of holomorphic **Z**-graded complexes (1.56) also identifies the metrics.

Proposition **1.13.** – *There exist Hermitian metrics* $h^{\xi_0}, \ldots, h^{\xi_m}$ *on* $\xi_0, \ldots, \xi_m$ *which verify assumption* (A) *with respect to the metrics* g^N, g^η *on* N, η.

Proof. – This result is proved in [B2, Proposition 1.6]. □

In the sequel, we suppose that the metrics $h^{\xi_0}, \ldots, h^{\xi_m}$ verify assumption (A) with respect to the metrics g^N, g^η on N, η.

II - EVALUATION OF THE QUILLEN NORM OF THE SECTION τ

Recall that τ is the canonical nonzero section of the line $\tilde{\lambda}^{-1}(\xi) \otimes \lambda(\xi)$, which was defined in Definition 1.2. The purpose of this Section is to calculate the Quillen norm of τ. Note that in this Section, we do not use the fact that $\mathcal{O}_X(\xi, v)$ is a resolution of $i_* \mathcal{O}_Y(\eta)$.

Theorem **2.1.** – *The following identity holds*

$$\|\tau\|_{\tilde{\lambda}^{-1}(\xi) \otimes \lambda(\xi)} = 1. \tag{2.1}$$

Proof. – For $0 \leqslant i \leqslant m$, we consider the double complex $(\bigoplus_{j \leqslant i} E_j, \bar{\partial}^X + v)$. This complex is again $\mathbf{Z}$-graded by $N_V^X - N_H$. By imitating the constructions of Sections 1c), 1d), 1e) (which correspond to the case $i=m$), we can construct the associated determinant fibre $\tilde{\lambda}_i(\xi)$, which we equip with the Quillen metric $\| \ \|_{\tilde{\lambda}_i(\xi)}$. Clearly $\tilde{\lambda}(\xi) = \tilde{\lambda}_m(\xi)$ and the Quillen metrics $\| \ \|_{\tilde{\lambda}(\xi)}$ and $\| \ \|_{\tilde{\lambda}_m(\xi)}$ coincide.

We now again consider the exact sequence of complexes (1.34)

$$0 \to (\bigoplus_{j \leqslant i-1} E_j, \bar{\partial}^X + v) \underset{\gamma}{\to} (\bigoplus_{j \leqslant i} E_j, \bar{\partial}^X + v) \underset{\gamma}{\to} (E_i, \bar{\partial}^X) \to 0. \tag{2.2}$$

Using (2.2) and the associated long exact sequence in cohomology, we get the identification of lines already described in (1.15)

$$(\lambda(\xi_i))^{(-1)^i} \cong (\tilde{\lambda}_{i-1}(\xi))^{-1} \otimes \tilde{\lambda}_i(\xi). \tag{2.3}$$

We will prove that for every $i = 0, \ldots, m$, the identification (2.3) is an isometry. Then (2.1) will trivially follow.

We fix i, $0 \leqslant i \leqslant m$. For $i=0$, there is nothing to prove, so me may assume that i is positive.

Consider the double complex S_i

$$
\begin{array}{ccccccccccc}
 & & 0 & & 0 & & 0 & & & & \\
 & & \uparrow & & \uparrow & & \uparrow & & & & \\
0 & \rightarrow & \xi_i & \underset{v}{\rightarrow} & 0 & \underset{v}{\rightarrow} & 0 & \underset{v}{\rightarrow} \dots \rightarrow & 0 & \rightarrow & 0 \\
 & & \uparrow b & & \uparrow b & & \uparrow b & & \uparrow b & & \\
0 & \rightarrow & \xi_i & \underset{v}{\rightarrow} & \xi_{i-1} & \underset{v}{\rightarrow} & \xi_{i-2} & \underset{v}{\rightarrow} \dots \underset{v}{\rightarrow} & \xi_0 & \rightarrow & 0 \\
 & & \uparrow b & & \uparrow b & & \uparrow b & & \uparrow b & & \\
0 & \rightarrow & 0 & \underset{v}{\rightarrow} & \xi_{i-1} & \underset{v}{\rightarrow} & \xi_{i-2} & \underset{v}{\rightarrow} \dots \underset{v}{\rightarrow} & \xi_0 & \rightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 0 & & 0 & &
\end{array}
\tag{2.4}
$$

In (2.4), the horizontal arrows v are either the original chain map v of the complex ξ or the zero map. The vertical arrows b are either the identity map or the zero map. Sign conventions can be chosen so that $bv+vb=0$. Therefore $v+b$ is a chain map on the complex S_i.

For $a\in\mathbf{C}$, $a'\in\mathbf{C}$, we can instead replace v, b by av, $a'b$ respectively. Let $(S_i, av+a'b)$ denote the corresponding chain complex.

The operator N_H (which defines the $\mathbf{Z}$-grading on ξ) still acts on the complex S_i in the obvious way. Let N'_H be the operator which takes the value $k-1$ on the kth nonzero row of S_i, the rows being numbered upwards. The complex $(S_i, av+a'b)$ is $\mathbf{Z}$-graded by the operator N'_H-N_H, and it inherits the corresponding $\mathbf{Z}_2$-grading.

Let Σ_i be the set of smooth sections of $\Lambda(T^{*(0,1)}X)\hat{\otimes} S_i$ over the manifold X. The $\mathbf{Z}$-grading of Σ_i will be defined by the operator $N_V^X+N'_H-N_H$. For a, $a'\in\mathbf{C}$, $\bar\partial^X+av+a'b$ is a chain map acting on Σ_i.

We now equip the complex S_i with the orthogonal sum of the metrics on the various ξ_k which appear in S_i. Also v^* still denotes the adjoint of v. Let b^* be the adjoint of b (which is here either the identity or the zero map). As in Section 1e), we equip Σ_i with the obvious Hermitian product. Then $\bar\partial^{X*}+\bar a v^*+\bar a' b^*$ is the formal adjoint of $\bar\partial^X+av+a'b$.

By proceeding as in Bismut-Gillet-Soulé [BGS3, Section 2], we can form an analytically defined holomorphic determinant line bundle μ_i on $\mathbf{C}^2$, whose fiber $\mu_{i,(a,a')}$ is canonically isomorphic to

$$\bigotimes_{q\in\mathbf{Z}} (\det H^q(\Sigma_i, \bar\partial^X+av+a'b))^{(-1)^{q+1}}.$$

For every $(a, a')\in\mathbf{C}^2$, we equip the fiber $\mu_{i,(a,a')}$ with the corresponding Quillen metric $\|\ \|_{\mu_{i,(a,a')}}$. By [BGS3, Theorem 1.6], the metrics $\|\ \|_{\mu_{i,(a,a')}}$ induce a smooth Hermitian metric $\|\ \|_{\mu_i}$ on the holomorphic line bundle μ_i.

Let ∇^{μ_i} be the holomorphic Hermitian connection on the line bundle $(\mu_i, \| \ \|_{\mu_i})$. By [BGS3, Proposition 2.1], the curvature of the connection ∇^{μ_i} vanishes. The holomorphic Hermitian bundle μ_i is then trivial on $\mathbf{C}^2$. We will identify the fibers $\mu_{i,(a,a')}$ with the fiber $\mu_{i,(1,0)}$ by parallel transport with respect to the connection ∇^{μ_i}. This identification preserves the metric.

Now in (2.4) the columns of the complex S_i are acyclic. Therefore if $a' \neq 0$, the complex $(S_i, av + a'b)$ is acyclic. So for $a' \neq 0$, the fibers $\mu_{i,(a,a')}$ are canonically trivial. More precisely, by [BGS3, Remark 1.10 and Section 2], on $\mathbf{C} \times \mathbf{C}^*$ the line μ_i has a canonical holomorphic nonzero section $T(\bar{\partial}^X + av + a'b)$. Since the curvature of ∇^{μ_i} vanishes, it follows that

$$(2.5) \qquad \bar{\partial}\partial \operatorname{Log}(\| T(\bar{\partial}^X + av + a'b) \|^2_{\mu_i}) = 0 \text{ on } \mathbf{C} \times \mathbf{C}^*.$$

Now for $\theta \in \mathbf{R}$

$$(2.6) \qquad e^{-i\theta N_H}(\bar{\partial}^X + av + a'b + \bar{\partial}^{X*} + \bar{a}v^* + \bar{a}'b^*)e^{i\theta N_H}$$
$$= \bar{\partial}^X + ae^{i\theta}v + a'b + \bar{\partial}^{X*} + \bar{a}e^{-i\theta}v^* + \bar{a}'b^*.$$

Also if $a' \in \mathbf{C}^*$, by definition [BGS3, Remark 1.10] (and taking into account Remark 1.11), we get

$$(2.7) \qquad \operatorname{Log}(\| T(\bar{\partial}^X + av + a'b) \|^2_{\mu_i})$$
$$= -\frac{\partial}{\partial s} \operatorname{Tr}_s[(N^X_V + N'_H - N_H)((\bar{\partial}^X + av + a'b + \bar{\partial}^{X*} + \bar{a}v^* + \bar{a}'b^*)^2)^{-s}](0).$$

From (2.6), (2.7), we deduce that $\| T(\bar{\partial}^X + av + a'b) \|^2_{\mu_i}$ only depends on a via $|a|$. A similar argument shows it depends on a' via $|a'|$.

For $a' \in \mathbf{C}^*$, the function $a \in \mathbf{C} \to \operatorname{Log}(\| T(\bar{\partial}^X + av + a'b) \|^2_{\mu_i})$ being smooth, radial and harmonic is constant. In particular

$$(2.8) \qquad \| T(\bar{\partial}^X + v + a'b) \|^2_{\mu_i} = \| T(\bar{\partial}^X + a'b) \|^2_{\mu_i}.$$

Now in (2.4), the columns are acyclic and split, in the sense that when b is non zero, it is the identity. Therefore the Bott-Chern classes associated with the columns of S_i (in the sense of [BGS1, Section 1f)] vanish identically. By a simple direct computation or by [BGS3, Theorem 2.4], we find that

$$(2.9) \qquad \| T(\bar{\partial}^X + b) \|^2_{\mu_i} = 1.$$

Let $\chi(\xi_i)$ be the Euler characteristic of ξ_i. Set

$$(2.10) \qquad d_i = (-1)^i(-\chi(\xi_i) - \chi(\xi_{i-1}) + \chi(\xi_{i-2}) - \ldots + (-1)^i \chi(\xi_0)).$$

By a direct computation or by [BGS3, Theorem 2.4] and by (2.9), we find that if $a' \in \mathbf{C}^*$

$$\left\| \frac{\mathrm{T}(\bar{\partial}^{\mathrm{X}} + a' b)}{a'^{d_i}} \right\|^2_{\mu_i} = 1. \tag{2.11}$$

From (2.8), (2.11), we see that if $a' \in \mathbf{C}^*$

$$\left\| \frac{\mathrm{T}(\bar{\partial}^{\mathrm{X}} + v + a' b)}{a'^{d_i}} \right\|^2_{\mu_i} = 1. \tag{2.12}$$

Recall that the fibres of μ_i are identified with $\mu_{i,(1,0)}$. Using (2.12), we find that the function

$$a' \in \mathbf{C}^* \to \frac{\mathrm{T}(\bar{\partial}^{\mathrm{X}} + v + a' b)}{a'^{d_i}} \in \mu_{i,(1,0)}$$

is constant. Now one verifies easily that

$$\mu_{i,(1,0)} = \tilde{\lambda}_{i-1}(\xi) \otimes \tilde{\lambda}_i^{-1}(\xi) \otimes (\lambda(\xi_i))^{(-1)^i}. \tag{2.13}$$

Also the identity (2.13) identifies the Quillen metrics.

Let τ_i be the nonzero section of $\mu_{i,(1,0)}$ which defines the canonical isomorphism (2.3). We claim that

$$\lim_{\substack{a' \in \mathbf{C}^* \\ a' \to 0}} \frac{\mathrm{T}(\bar{\partial}^{\mathrm{X}} + v + a' b)}{a'^{d_i}} = \tau_i. \tag{2.14}$$

Note that $(\mathrm{T}(\bar{\partial}^{\mathrm{X}} + v + a' b)/a'^{d_i}) \in \mu_{i,(1,a')}$ and so (2.14) can be verified locally near $0 \in \mathbf{C}$.

Now (2.14) is exactly the identity proved in [BGS3, eq. (2.23)] which was essential in proving [BGS3, Theorem 2.8]. The only difference with [BGS3] is that the chain map $\bar{\partial}^{\mathrm{X}}$ considered in [BGS3] is replaced by the chain map $\bar{\partial}^{\mathrm{X}} + v$ (in [BGS3], the map b was denoted v). The proof of [BGS3, Theorem 2.8] can otherwise be reproduced.

From (2.12), (2.14), we get

$$\| \tau_i \|^2_{\mu_{i,(1,0)}} = 1. \tag{2.15}$$

Since (2.13) is an identification of Hermitian lines, (2.1) follows from (2.15). Our Theorem is proved. □

Remark **2.2.** — An essentially equivalent proof of (2.1) is as follows. For $a \in \mathbf{C}$, we consider the double complex $(\mathrm{E}, \bar{\partial}^{\mathrm{X}} + av)$ and its Čech analogue

$\left(\overset{m}{\underset{0}{\oplus}} \mathcal{O}_X(\xi_i),\ \delta^X + av\right)$. Associated to this last complex, there is a determinant line bundle $\tilde{\lambda}(\xi)$ over **C** in the sense of Grothendieck-Knudsen-Mumford [KnM]. By [KnM], the line bundle $\tilde{\lambda}^{-1}(\xi) \otimes \lambda(\xi)$ has a nonzero holomorphic canonical section σ over **C**. By the obvious analogue of a result of Bismut-Gillet-Soulé [BGS3, Corollary 3.9], the Quillen metric is smooth on the Grothendieck-Knudsen-Mumford line bundle $\tilde{\lambda}(\xi)$. Since the curvature of the Quillen metric on $\tilde{\lambda}(\xi)$ vanishes, then

$$\bar{\partial}\partial \operatorname{Log}(\|\sigma\|^2) = 0. \tag{2.16}$$

On the other hand $\operatorname{Log}(\|\sigma\|^2)$ is a radial function. Then $\operatorname{Log}(\|\sigma\|^2)$ is constant. Trivially $\operatorname{Log}(\|\sigma\|^2)$ vanishes at $a=0$. Therefore $\operatorname{Log}(\|\sigma\|^2)$ is identically zero.

Our proof of Theorem 2.1 avoids any explicit consideration of the Grothendieck-Knudsen-Mumford line bundle.

III - TWO PARAMETERS SEMI-GROUPS AND CONTOUR INTEGRALS

The purpose of this Section is to construct a closed 1-form β on $\mathbf{R}_+^* \times \mathbf{R}_+^*$ associated to the two parameters semi-group $u > 0$, $\mathrm{T} > 0 \to \exp(-(u(\mathrm{D}^\mathrm{X}+\mathrm{TV}))^2)$, and a contour Γ in $\mathbf{R}_+^* \times \mathbf{R}_+^*$ depending on three parameters ε, A, T_0, so that $\int_\Gamma \beta = 0$. To prove Theorem 0.1, we will then push the contour Γ to the boundary of $\mathbf{R}_+^* \times \mathbf{R}_+^*$, and take the obvious limit in the previous identity.

The construction of the form β is directly related to results of [B2] concerning double transgression formulas for Quillen's superconnection forms, and their dependence on the considered metrics. In fact we interpret our result in terms of the scaling of the metrics g^{TX}, $h^{\xi_0}, \dots, h^{\xi_m}$ by the factors $1/u^2$, $1/\mathrm{T}^2, \dots, 1/\mathrm{T}^{2m}$ respectively. This scaling will also play a key role in Section 10.

This Section is organized as follows. In *a*), we describe the scaling of the metrics on TX, $\xi_0, \dots, \xi_m$. In b), we construct a basic closed 1-form, α, on $\mathbf{R}_+^* \times \mathbf{R}_+^*$. In c) we obtain our form β by a change of coordinates. In d), we describe the contour Γ in $\mathbf{R}_+^* \times \mathbf{R}_+^*$.

a) Scaling of metrics on X and ξ

Take $u > 0$, $\mathrm{T} > 0$. Suppose that the metrics g^{TX}, $h^{\xi_0}, \dots, h^{\xi_m}$ are replaced by the metrics g^{TX}/u^2, h^{ξ_0}, $h^{\xi_1}/\mathrm{T}^2, \dots, h^{\xi_m}/\mathrm{T}^{2m}$. Then the adjoints of the operators $\bar\partial^\mathrm{X}$, v become $u^2\,\bar\partial^{\mathrm{X}*}$, $\mathrm{T}^2 v^*$.

The basic idea of the paper is to study the deformation of various supertraces as u and T vary. However at a technical level, it is easier to scale $\bar\partial^\mathrm{X}$, $\bar\partial^{\mathrm{X}*}$ and v, v^* in the same way.

Our supertraces will be calculated on the $\mathbf{Z}_2$-graded vector space E.

Proposition **3.1.** — *For any* $u > 0$, $\mathrm{T} > 0$, *the following identities hold*

$$
\begin{aligned}
(3.1)\qquad & \mathrm{Tr}_s[\mathrm{N}_\mathrm{V}^\mathrm{X}\exp(-(\bar\partial^\mathrm{X}+v+u^2\,\bar\partial^{\mathrm{X}*}+\mathrm{T}^2 v)^2)] \\
&\quad = \mathrm{Tr}_s[\mathrm{N}_\mathrm{V}^\mathrm{X}\exp(-(u(\bar\partial^\mathrm{X}+\bar\partial^{\mathrm{X}*})+\mathrm{T}(v+v^*))^2)], \\
& \mathrm{Tr}_s[\mathrm{N}_\mathrm{H}\exp(-(\bar\partial^\mathrm{X}+v+u^2\,\bar\partial^{\mathrm{X}*}+\mathrm{T}^2 v^*)^2)] \\
&\quad = \mathrm{Tr}_s[\mathrm{N}_\mathrm{H}\exp(-(u(\bar\partial^\mathrm{X}+\bar\partial^{\mathrm{X}*})+\mathrm{T}(v+v^*))^2)].
\end{aligned}
$$

Proof. — Observe that

$$u^{N_V^X} T^{-N_H} (\bar\partial^X + v + u^2 \bar\partial^{X*} + T^2 v^*) T^{N_H} u^{-N_V^X} = u(\bar\partial^X + \bar\partial^{X*}) + T(v + v^*),$$
$$[N_V^X, N_H] = 0. \tag{3.2}$$

Using the fact that supertraces vanish on supercommutators [Q1], (3.1) follows. □

Remark **3.2.** — The key fact is that in the left-hand side of (3.1), the coboundary operator $\bar\partial^X + v$ does not change, only the metrics on TX and ξ are changing. In the right-hand side, the coboundary operator $\bar\partial^X + v$ is changed into $u\bar\partial^X + Tv$, and the metrics on TX, ξ do not vary. The correct geometric picture is the one given by the left-hand side.

b) A basic closed 1-form on $\mathbf{R}_+^* \times \mathbf{R}_+^*$.

Set

$$D^X = \bar\partial^X + \bar\partial^{X*},$$
$$V = v + v^*. \tag{3.3}$$

For $u > 0$, $T \geqslant 0$, set

$$A_{u,T} = uD^X + TV. \tag{3.4}$$

Then $A_{u,T}$ is an elliptic first order differential operator.

Theorem **3.3.** — *Let* $\alpha_{u,T}$ *be the smooth* 1-*form on* $\mathbf{R}_+^* \times \mathbf{R}_+^*$

$$\alpha_{u,T} = \frac{du}{u} \operatorname{Tr}_s[N_V^X \exp(-A_{u,T}^2)] - \frac{dT}{T} \operatorname{Tr}_s[N_H \exp(-A_{u,T}^2)]. \tag{3.5}$$

Then $\alpha_{u,T}$ *is a closed form.*

Proof. — For $u > 0$, $T \geqslant 0$, the operator $\exp(-A_{u,T}^2)$ is given by a smooth kernel on the manifold X. By proceding as in [B1, Proposition 2.8] *i.e.* by expressing the above supertraces as integrals on the manifold X of integrals of supertraces of heat kernels evaluated on the diagonal, one can easily justify the following manipulations of supertraces.

We have the identity

$$\frac{\partial}{\partial T} \operatorname{Tr}_s[N_V^X \exp(-A_{u,T}^2)] = \frac{\partial}{\partial b} \left\{ \operatorname{Tr}_s\left[N_V^X \exp\left(-A_{u,T}^2 - b\left[A_{u,T}, \frac{\partial A_{u,T}}{\partial T} \right] \right) \right] \right\}_{b=0}. \tag{3.6}$$

Now $\partial A_{u,T}/\partial T = V$. Since supertraces vanish on supercommutators, we rewrite (3.6) in the form

$$\frac{\partial}{\partial T}\operatorname{Tr}_s[N_V^X \exp(-A_{u,T}^2)] = -\frac{\partial}{\partial b}\{\operatorname{Tr}_s[[A_{u,T}, N_V^X]\exp(-A_{u,T}^2 - bV)]\}_{b=0}. \tag{3.7}$$

Clearly

$$[A_{u,T}, N_V^X] = -u\bar\partial^X + u\bar\partial^{X*}. \tag{3.8}$$

Using (3.6)-(3.8), we get

$$\begin{aligned}\frac{\partial}{\partial T}&\operatorname{Tr}_s[N_V^X \exp(-A_{u,T}^2)] \\ &= \frac{u\,\partial}{\partial b}\{\operatorname{Tr}_s[(\bar\partial^X - \bar\partial^{X*})\exp(-A_{u,T}^2 - b(v+v^*))]\}_{b=0}.\end{aligned} \tag{3.9}$$

Now

$$A_{u,T}^2 = [u\bar\partial^X + T\,v,\ u\bar\partial^{X*} + T\,v^*], \tag{3.10}$$

and so $A_{u,T}^2$ preserves the total degree in E. Also $\bar\partial^X$, v increase the total degree by one and $\bar\partial^{X*}$, v^* decrease the total degree by one. The degree counting argument of [BGS1, Proposition 1.8] gives

$$\begin{aligned}\frac{\partial}{\partial b}&\{\operatorname{Tr}_s[(\bar\partial^X - \bar\partial^{X*})\exp(-A_{u,T}^2 - b(v+v^*))]\}_{b=0} \\ &= \frac{\partial}{\partial b}\{\operatorname{Tr}_s[\bar\partial^X \exp(-A_{u,T}^2 - bv^*)] - \operatorname{Tr}_s[\bar\partial^{X*}\exp(-A_{u,T}^2 - bv)]\}_{b=0}.\end{aligned} \tag{3.11}$$

Using (3.9), (3.11), we obtain

$$\begin{aligned}\frac{\partial}{\partial T}&\left\{\frac{1}{u}\operatorname{Tr}_s[N_V^X \exp(-A_{u,T}^2)]\right\} \\ &= \frac{\partial}{\partial b}\{\operatorname{Tr}_s[\bar\partial^X \exp(-A_{u,T}^2 - bv^*)] - \operatorname{Tr}_s[\bar\partial^{X*}\exp(-A_{u,T}^2 - bv)]\}_{b=0}.\end{aligned} \tag{3.12}$$

By interchanging the roles of u and T, bearing in mind that the analogue of N_V^X is $-N_H$ and using again the degree counting argument of [BGS1, Proposition 1.8], we also get

$$\frac{\partial}{\partial u}\left\{\frac{1}{T}\operatorname{Tr}_s[N_H \exp(-A_{u,T}^2)]\right\} \tag{3.13}$$

$$= -\frac{\partial}{\partial b}\{\mathrm{Tr}_s[v\exp(-\mathrm{A}_{u,\mathrm{T}}^2 - b\,\bar{\partial}^{\mathrm{X}*})] - \mathrm{Tr}_s[v^*\exp(-\mathrm{A}_{u,\mathrm{T}}^2 - b\,\bar{\partial}^{\mathrm{X}})]\}_{b=0}.$$

Since supertraces vanish on supercommutators, we find that

$$\frac{\partial}{\partial b}\{\mathrm{Tr}_s[\bar{\partial}^{\mathrm{X}}\exp(-\mathrm{A}_{u,\mathrm{T}}^2 - bv^*)] - \mathrm{Tr}_s[\bar{\partial}^{\mathrm{X}*}\exp(-\mathrm{A}_{u,\mathrm{T}}^2 - bv)]\}_{b=0} \tag{3.14}$$

$$= \frac{\partial}{\partial b}\{\mathrm{Tr}_s[v\exp(-\mathrm{A}_{u,\mathrm{T}}^2 - b\,\bar{\partial}^{\mathrm{X}*})] - \mathrm{Tr}_s[v^*\exp(-\mathrm{A}_{u,\mathrm{T}}^2 - b\,\bar{\partial}^{\mathrm{X}})]\}_{b=0}.$$

From (3.12), (3.14), we deduce that the form $\alpha_{u,\mathrm{T}}$ is closed. □

Remark **3.4.** — Theorem 3.3 is in fact a consequence of a general result established in [B2, Theorem 2.2] on double transgression formulas for Quillen superconnection Chern character forms, which extend corresponding formulas of Bott and Chern [BoC, 3.28] for ordinary connections. Here we consider E as a vector bundle over a base S consisting of a point. In fact if h_{T}^{ξ} is the metric on ξ which is the direct sum of the metrics h^{ξ_0}, $h^{\xi_1}/\mathrm{T}^2, \ldots, h^{\xi_m}/\mathrm{T}^{2m}$, then

$$(h_{\mathrm{T}}^{\xi})^{-1}\frac{\partial h_{\mathrm{T}}^{\xi}}{\partial \mathrm{T}} = -\frac{2\,\mathrm{N_H}}{\mathrm{T}}. \tag{3.15}$$

Similarly if $g_u^{\Lambda \mathrm{T}^{*(0,1)}\mathrm{X}}$ denotes the metric induced by the metric g^{TX}/u^2 on $\Lambda(\mathrm{T}^{*(0,1)}\mathrm{X})$, then

$$g_u^{-1}\frac{\partial g_u^{\Lambda \mathrm{T}^{*(0,1)}\mathrm{X}}}{\partial u} = \frac{2\,\mathrm{N_V^X}}{u}. \tag{3.16}$$

By [B2, Theorem 2.2], since the base S is just a single point, we find that the form $\tilde{\alpha}_{u,\mathrm{T}}$ given by

$$\tilde{\alpha}_{u,\mathrm{T}} = \mathrm{Tr}_s\left[\left(\frac{du}{u}\mathrm{N_V^X} - \frac{d\mathrm{T}}{\mathrm{T}}\mathrm{N_H}\right)\exp(-(\bar{\partial}^{\mathrm{X}} + v + u^2\,\bar{\partial}^{\mathrm{X}*} + \mathrm{T}^2 v^*)^2)\right]$$

is closed. By Proposition 3.1, $\alpha_{u,\mathrm{T}} = \tilde{\alpha}_{u,\mathrm{T}}$, and so $\alpha_{u,\mathrm{T}}$ is closed.

c) A change of coordinates

For $u > 0$, $\mathrm{T} \geqslant 0$, set

$$\mathrm{B}_{u,\mathrm{T}} = u(\mathrm{D^X} + \mathrm{TV}). \tag{3.17}$$

Theorem **3.5.** — *Let* $\beta_{u,\mathrm{T}}$ *be the* 1-*form on* $\mathbf{R}_+^* \times \mathbf{R}_+^*$

(3.18) $$\beta_{u,\mathrm{T}}=\frac{du}{u}\mathrm{Tr}_s[(\mathrm{N}_{\mathrm{V}}^{\mathrm{X}}-\mathrm{N}_{\mathrm{H}})\exp(-\mathrm{B}_{u,\mathrm{T}}^2)]-\frac{d\mathrm{T}}{\mathrm{T}}\mathrm{Tr}_s[\mathrm{N}_{\mathrm{H}}\exp(-\mathrm{B}_{u,\mathrm{T}}^2)].$$

Then $\beta_{u,\mathrm{T}}$ *is a closed form.*

Proof. – In Theorem 3.3, we make the change of variables $u\rightarrow u$, $\mathrm{T}\rightarrow u\mathrm{T}$. Theorem 3.5 follows. □

d) A contour integral

We now fix constants ε, A, T_0 such that $0<\varepsilon<1\leqslant \mathrm{A}<+\infty$, $1\leqslant \mathrm{T}_0<+\infty$. Let $\Gamma=\Gamma_{\varepsilon,\mathrm{A},\mathrm{T}_0}$ be the oriented contour in $\mathbf{R}_+^*\times\mathbf{R}_+^*$

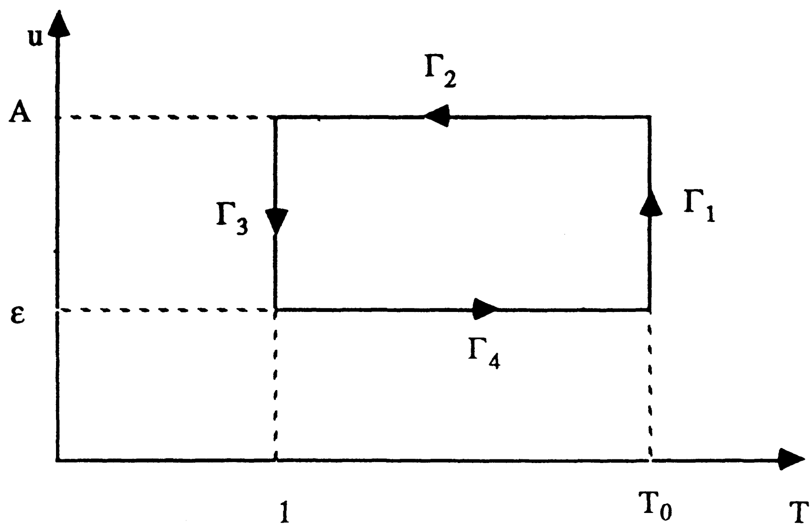


FIG. 1

As shown in Figure 1, the contour Γ is made of four oriented pieces:

$$\Gamma_1:\mathrm{T}=\mathrm{T}_0;\ \varepsilon\leqslant u\leqslant \mathrm{A},$$

$$\Gamma_2: 1\leqslant \mathrm{T}\leqslant \mathrm{T}_0;\ u=\mathrm{A},$$

$$\Gamma_3:\mathrm{T}=1;\ \varepsilon\leqslant u\leqslant \mathrm{A},$$

$$\Gamma_4: 1\leqslant \mathrm{T}\leqslant \mathrm{T}_0;\ u=\varepsilon.$$

The orientation of $\Gamma_1, \ldots, \Gamma_4$ is indicated in Figure 1.

For $1 \leqslant k \leqslant 4$, set

$$I_k^0 = \int_{\Gamma_k} \beta_{u,T}. \tag{3.19}$$

Theorem **3.6.** – *The following identity holds*

$$\sum_{k=1}^{4} I_k^0 = 0. \tag{3.20}$$

Proof. – Theorem 3.6 is a trivial consequence of Theorem 3.5. □

Remark **3.7.** – We now will make $A \to +\infty$, $T_0 \to +\infty$, $\varepsilon \to 0$ in this order in identity (3.20). Typically each term I_k^0 $(1 \leqslant k \leqslant 4)$ will diverge at one or several stages of this process. However because of the identity (3.20), the divergences will cancel, often for non trivial reasons. Once the divergences in each term will have been substracted off, we will ultimately obtain an identity which is exactly our main Theorem 6.1.

Roughly speaking, in this process,

- I_1^0 will calculate the Ray-Singer torsion of the complex $(F, \bar{\partial}^Y)$.
- I_2^0 will calculate the ratio of the metrics $| \ |^2_{\lambda(\xi)} / | \ |^2_{\lambda(\eta)}$.
- I_3^0 will calculate the Ray-Singer torsion of the complex $(E, \bar{\partial}^X + v)$.
- I_4^0 will produce highly non trivial local terms, which include the Bott-Chern current $T(\xi, h^\xi)$ of Bismut-Gillet-Soulé [BGS4] and the class $\mathbf{B}(TY, TX|_Y, g^{TX|_Y})$ of Bismut [B3].

Remark **3.8.** – Take $t_0 \in]0, 1[$. Replacing T_0 by t_0, we then obtain a contour $\Gamma_{\varepsilon, A, t_0}$. Again $\int_{\Gamma_{\varepsilon, A, t_0}} \beta = 0$. In principle by making $A \to +\infty$, $t_0 \to 0$, $\varepsilon \to 0$ in this order, we may reprove Theorem 2.1. However, to carry this out, one then has to deal analytically with spectral sequences which in general do not degenerate. The proof of Theorem 2.1 has exactly consisted in carefully avoiding this difficulty.

IV - A SINGULAR BOTT-CHERN CURRENT

This section has three purposes. We first briefly describe the superconnection formalism of Quillen [Q1]. We also recall the result in Bismut [B2] on the convergence of Quillen's Chern character currents associated with the Hermitian chain complex (ξ, v) as a parameter u tends to $+\infty$. Finally, we describe the construction by Bismut-Gillet-Soulé [BGS4] of a singular Bott-Chern current $T(\xi, h^{\xi})$. This current will appear in our final formula for $\operatorname{Log}(\|\sigma\|^2_{\lambda^{-1}(\eta)\otimes\lambda(\xi)})$ in Theorem 0.1.

This Section is organized as follows. In a), we introduce various polynomials in Chern-Weil theory which will appear in the remainder of the paper. In b), we describe Quillen's superconnections [Q1]. In c), we recall the results of [B2] on superconnection currents. Finally in d), we describe the construction in [BGS4] of the current $T(\xi, h^{\xi})$. This current is obtained by a non trivial extension in a geometric context of the formalism of the Ray-Singer analytic torsion [RS2].

The assumptions of Sections 1, 2, 3 will remain in force.

a) Characteristic classes in Chern-Weil theory

Recall that the Todd power series $\operatorname{Td}(x)$ is defined by

$$\operatorname{Td}(x)=\frac{x}{1-e^{-x}}. \tag{4.1}$$

Set

$$\begin{aligned} &\operatorname{Td}(x_1, \ldots, x_q)=\prod_1^q \operatorname{Td}(x_i) \\ &\operatorname{Td}'(x_1, \ldots, x_q)=\frac{\partial}{\partial b}[\operatorname{Td}(x_1+b, \ldots, x_q+b)]_{b=0}, \\ &(\operatorname{Td}^{-1})'(x_1, \ldots, x_q)=\frac{\partial}{\partial b}[\operatorname{Td}^{-1}(x_1+b, \ldots, x_q+b)]_{b=0}. \end{aligned} \tag{4.2}$$

The polynomials Td' and $(\operatorname{Td}^{-1})'$ play an important role in [BGS2, Section 2g)] and in [B2, Section 4].

Finally the Chern power series ch is defined by

$$\mathrm{ch}(x_1, \ldots, x_q) = \sum_1^q e^{x_i}. \tag{4.3}$$

We identify these power series the corresponding ad-invariant power series evaluated on (q, q) matrices.

Let B be a complex manifold.

Definition **4.1.** – Denote by P^B the set of smooth differential forms on B which are sums of forms of type (p, p). Denote by $P^{B,0}$ the set of forms $\omega \in P^B$ such that $\omega = \partial\alpha + \bar{\partial}\beta$, where α, β are smooth forms on B.

If E is a holomorphic vector bundle on B equipped with a Hermitian metric g^E, let ∇^E be the holomorphic Hermitian connection on (E, g^E), and let $(\nabla^E)^2$ be its curvature.

Let $q = \dim E$, and let $Q(x_1, \ldots, x_q)$ be a symmetric polynomial in $x_1, \ldots, x_q$. By Chern-Weil theory, the form $Q(-(\nabla^E)^2/2i\pi)$ is closed, and its cohomology class does not depend on the metric g^E. We will use the notation $Q(E, g^E)$ instead of $Q(-(\nabla^E)^2/2i\pi)$. Then $Q(E, g^E)$ lies in P^B. We will denote by $Q(E)$ the class of $Q(E, g^E)$ in $P^B/P^{B,0}$.

Let Φ be the homomorphism from $\Lambda^{\mathrm{even}}(T^*_{\mathbf{R}} B)$ into itself which to $\alpha \in \Lambda^{2p}(T^*_{\mathbf{R}} B)$ associates $(2\pi i)^{-p}\alpha$.

b) Quillen's superconnections

We make the same assumptions as in Sections 1, 2 and 3.

Definition **4.2.** – Let $\mathrm{ch}(\xi, h^\xi)$, $\mathrm{ch}'(\xi, h^\xi)$, be the differential forms on X

$$\begin{aligned} \mathrm{ch}(\xi, h^\xi) &= \sum_{i=0}^m (-1)^i \mathrm{ch}(\xi_i, h^{\xi_i}), \\ \mathrm{ch}'(\xi, h^\xi) &= \sum_{i=0}^m (-1)^i i\, \mathrm{ch}(\xi_i, h^{\xi_i}). \end{aligned} \tag{4.4}$$

We now briefly describe the superconnection formalism of Quillen [Q1]. Set

$$\xi_+ = \bigoplus_{i \text{ even}} \xi_i; \quad \xi_- = \bigoplus_{i \text{ odd}} \xi_i. \tag{4.5}$$

Then $\xi = \xi_+ \oplus \xi_-$ is a $\mathbf{Z}_2$-graded vector bundle. Let τ^ξ be the involution of ξ defining the $\mathbf{Z}_2$-grading, *i.e.* $\tau^\xi = \pm 1$ on $\xi_\pm$.

The bundle of algebras $\mathrm{End}(\xi)$ is $\mathbf{Z}_2$-graded, the even (resp. odd) elements of $\mathrm{End}(\xi)$ commuting (resp. anticommuting) with τ^ξ. The supertrace Tr_s is the linear form

$$A \in \mathrm{End}\,\xi \to \mathrm{Tr}_s[A] = \mathrm{Tr}[\tau^\xi A].$$

We then form the $\mathbf{Z}_2$-graded tensor product $\Lambda(T^*_{\mathbf{R}}X) \hat{\otimes} \mathrm{End}(\xi)$. We extend Tr_s to a linear map from $\Lambda(T^*_{\mathbf{R}}X) \hat{\otimes} \mathrm{End}(\xi)$ into $\Lambda(T^*_{\mathbf{R}}X)$ such that if $\omega \in \Lambda(T^*_{\mathbf{R}}X)$, $A \in \mathrm{End}\,\xi$, then

$$\mathrm{Tr}_s[\omega A] = \omega\, \mathrm{Tr}_s[A]. \tag{4.6}$$

By [Q1], the supertrace vanishes on supercommutators in $\Lambda(T^*_{\mathbf{R}}X) \hat{\otimes} \mathrm{End}(\xi)$.

For $0 \leqslant i \leqslant m$, let ∇^{ξ_i} be the holomorphic Hermitian connection on the vector bundle (ξ_i, h^{ξ_i}). Then the connection $\nabla^\xi = \bigoplus_{i=0}^{m} \nabla^{\xi_i}$ is the holomorphic Hermitian connection on (ξ, h^ξ).

Now $V = v + v^*$ is a self-adjoint section of $\mathrm{End}^{\mathrm{odd}}(\xi)$. For $u \geqslant 0$, set

$$C_u = \nabla^\xi + \sqrt{u}\, V. \tag{4.7}$$

Then C_u is a superconnection on the $\mathbf{Z}_2$-graded vector bundle $\xi = \xi_+ \oplus \xi_-$ in the sense of Quillen [Q1].

Our calculations will now be done in the graded algebra $\Lambda(T^*_{\mathbf{R}}X) \hat{\otimes} \mathrm{End}(\xi)$. Also, ∇^ξ will be considered as a first order odd differential operator, whose curvature $(\nabla^\xi)^2$ is the square of the connection ∇^ξ. Then C_u^2 is the curvature of the superconnection C_u. It is a smooth section of $(\Lambda(T^*_{\mathbf{R}}X) \hat{\otimes} \mathrm{End}(\xi))^{\mathrm{even}}$. By Quillen [Q1], we know that the forms $\Phi\, \mathrm{Tr}_s[\exp(-C_u^2)]$ are closed, and represent in cohomology the Chern character of the complex ξ.

Clearly

$$\begin{aligned} &\Phi\, \mathrm{Tr}_s[\exp(-C_0^2)] = \mathrm{ch}(\xi, h^\xi), \\ &\Phi\, \mathrm{Tr}_s[N_H \exp(-C_0^2)] = \mathrm{ch}'(\xi, h^\xi). \end{aligned} \tag{4.8}$$

Also by Bismut-Gillet-Soulé [BGS1, Theorem 1.9], the forms $\Phi\, \mathrm{Tr}_s[\exp(-C_u^2)]$ and the forms $\Phi\, \mathrm{Tr}_s[N_H \exp(-C_u^2)]$ lie in P^X.

c) Convergence of the superconnection currents of the complex (ξ, v)

Let $\mathscr{C}^1(X)$ be the set of differential forms on X which are continuous, with continuous first derivatives. Let $\|\ \|_{\mathscr{C}^1(X)}$ be a norm on $\mathscr{C}^1(X)$. We now recall results which are proved in [B2, Theorems 3.2 and 4.3].

Theorem **4.3.** — *There is a constant* $C > 0$ *such that for any* $\mu \in \mathscr{C}^1(X)$, *and any* $u \geq 1$

$$
\begin{aligned}
&\left| \int_X \mu \Phi \operatorname{Tr}_s[\exp(-C_u^2)] - \int_Y i^* \mu \operatorname{Td}^{-1}(N, g^N) \operatorname{ch}(\eta, g^\eta) \right| \\
&\quad \leq \frac{C}{\sqrt{u}} \| \mu \|_{\mathscr{C}^1(X)}, \\
&\left| \int_X \mu \Phi \operatorname{Tr}_s[N_H \exp(-C_u^2)] + \int_Y i^* \mu (\operatorname{Td}^{-1})'(N, g^N) \operatorname{ch}(\eta, g^\eta) \right| \\
&\quad \leq \frac{C}{\sqrt{u}} \| \mu \|_{\mathscr{C}^1(X)}.
\end{aligned} \tag{4.9}
$$

d) A Bott-Chern singular current

For the definition of the wave front set of a current, we refer to Hörmander [H, Chapter VIII].

Definition **4.4.** — Let P_Y^X denote the set of the currents on X which are sums of currents of type (p, p) whose wave front set is included in $N_{\mathbf{R}}^*$.

We now briefly describe the construction by Bismut-Gillet-Soulé [BGS4] of a singular Bott-Chern current. Let $\delta_{\{Y\}}$ be the current corresponding to integration over Y. If μ is a smooth form on X, by definition $\int_X \mu \delta_{\{Y\}} = \int_Y i^* \mu$.

Definition **4.5.** — For $s \in \mathbf{C}$, $0 < \operatorname{Re}(s) < (1/2)$, let $R(\xi, h^\xi)(s)$ be the current

$$
\begin{aligned}
R(\xi, h^\xi)(s) = \frac{1}{\Gamma(s)} \int_0^{+\infty} u^{s-1} \{ &\Phi \operatorname{Tr}_s[N_H \exp(-C_u^2)] \\
&+ (\operatorname{Td}^{-1})'(N, g^N) \operatorname{ch}(\eta, g^\eta) \delta_{\{Y\}} \} \, du.
\end{aligned} \tag{4.10}
$$

Clearly, by Theorem 4.3, the current $R(\xi, h^\xi)(s)$ is well-defined. Also one verifies easily that the map $s \to R(\xi, h^\xi)(s)$ extends to a map which is holomorphic near $s = 0$.

Definition **4.6.** — Let $T(\xi, h^\xi)$ be the current

$$
T(\xi, h^\xi) = \frac{\partial}{\partial s} R(\xi, h^\xi)(0). \tag{4.11}
$$

In [BGS4, Section 2a)], it was verified that $T(\xi, h^\xi)$ is given by the formula

(4.12)
$$\mathrm{T}(\xi, h^{\xi}) = \int_0^1 \Phi \operatorname{Tr}_s[\mathrm{N_H}(\exp(-\mathrm{C}_u^2) - \exp(-\mathrm{C}_0^2))] \frac{du}{u}$$
$$+ \int_1^{+\infty} \{\Phi \operatorname{Tr}_s[\mathrm{N_H} \exp(-\mathrm{C}_u^2)] + (\mathrm{Td}^{-1})'(\mathrm{N}, g^{\mathrm{N}}) \operatorname{ch}(\eta, g^{\eta}) \delta_{\{\mathrm{Y}\}}\} \frac{du}{u}$$
$$- \Gamma'(1) \{\operatorname{ch}'(\xi, h^{\xi}) + (\mathrm{Td}^{-1})'(\mathrm{N}, g^{\mathrm{N}}) \operatorname{ch}(\eta, g^{\eta}) \delta_{\{\mathrm{Y}\}}\}.$$

The following result is proved in [BGS4, Theorem 2.5].

Theorem **4.7.** – *The current* $\mathrm{T}(\xi, h^{\xi})$ *lies in* $\mathrm{P}_{\mathrm{Y}}^{\mathrm{X}}$. *Also the following equation of currents holds on* X

(4.13)
$$\frac{\bar{\partial}\partial}{2i\pi} \mathrm{T}(\xi, h^{\xi}) = \mathrm{Td}^{-1}(\mathrm{N}, g^{\mathrm{N}}) \operatorname{ch}(\eta, g^{\eta}) \delta_{\{\mathrm{Y}\}} - \operatorname{ch}(\xi, h^{\xi}).$$

Remark **4.8.** – In [BGS4] and in [BGS5, Section 2], the currents $\mathrm{T}(\xi, h^{\xi})$ were shown to have remarkable functorial properties, which are compatible with refinements of the Theorem of Riemann-Roch-Grothendieck. Our final formula for $\operatorname{Log}(\|\sigma\|^2_{\lambda^{-1}(\eta) \otimes \lambda(\xi)})$ shows directly that the currents $\mathrm{T}(\xi, h^{\xi})$ must verify some of the functorial properties established in [BGS5, Section 2].

V - THE GENERALIZED ANALYTIC TORSION FORMS OF A SHORT EXACT SEQUENCE

Let B be a complex manifold. Let $0 \to L \to M \to N \to 0$ be a short exact sequence of holomorphic vector bundles on B, and let g^M be a Hermitian metric on M. The purpose of this Section is to describe the results of Bismut [B3] which concern the construction of an associated differential form $\mathbf{B}(L, M, g^M) \in P^B$ and the evaluation of $\mathbf{B}(L, M, g^M)$ in $P^B/P^{B,0}$ in terms of a Bott-Chern class in the sense of [BGS1] and of an additive genus D evaluated on N. Also we recall the evaluation by Bismut-Soulé [B3, Theorem 1 of the Appendix] of the power series $D(x)$ in terms of the power series $R(x)$ of Gillet-Soulé [GS3].

This Section is organized as follows. In a), we recall various results on Clifford algebras. In b), we give a formula for the curvature $\mathscr{B}_u^2$ of the superconnection $\mathscr{B}_u$ considered in [B3]. We also introduce two essentially equivalent curvature operators $\mathscr{C}_u^2$ and $\mathscr{D}_u^2$. The operators $\mathscr{B}_u^2$, $\mathscr{C}_u^2$ and $\mathscr{D}_u^2$ will reappear in a rather extraordinary way in Sections 12 and 13. In c), and following [B3] we construct the generalized supertrace of $\exp(-\mathscr{B}_u^2)$, which is a smooth differential form on the manifold B. In d), we recall the results of [B3] which concern the behaviour as $u \to 0$ and $u \to +\infty$ of the generalized supertrace. In e), we recall the construction in [B3] of the form $\mathbf{B}(L, M, g^M)$ on B. The form $\mathbf{B}(L, M, g^M)$ is obtained by a non trivial extension of the Ray-Singer analytic torsion formalism [RS2]. In f), and following [B3], we evaluate $\mathbf{B}(L, M, g^M)$ in $P^B/P^{B,0}$. Finally in g), we recall an identity of Bismut-Soulé [B3].

This Section is self-contained. In the sequel, its results will be applied to the exact sequence $0 \to TY \to TX|_Y \to N \to 0$.

a) Clifford algebras and complex vector spaces

Let V be a finite dimensional complex Hermitian vector space of complex dimension k. Let $\bar{V}$ be the conjugate vector space. If $z \in V$, z represents $Z = z + \bar{z} \in V_{\mathbf{R}}$, so that

$$|Z|^2 = 2|z|^2. \tag{5.1}$$

Let $c(V_{\mathbf{R}})$ be the Clifford algebra of $V_{\mathbf{R}}$, *i.e.* the algebra generated over $\mathbf{C}$ by $U \in V_{\mathbf{R}}$ and the commutation relations $UU' + U'U = -2\langle U, U'\rangle$. Then $\Lambda(\bar{V}^*)$ and $\Lambda(V^*)$ are Clifford modules. Namely, if $X \in V$, $X' \in \bar{V}$, let $X^* \in \bar{V}^*$, $X'^* \in V^*$ be the corresponding elements by the Hermitian product on V. Set

$$\begin{aligned} &c(X) = \sqrt{2}\, X^* \wedge;\ c(X') = -\sqrt{2}\, i_{X'};\\ &\hat{c}(X) = -\sqrt{-2}\, i_X;\ \hat{c}(X') = -\sqrt{-2}\, X'^* \wedge. \end{aligned} \tag{5.2}$$

We extend the maps c, $\hat{c}$ into maps from $V_{\mathbf{R}} \otimes_{\mathbf{R}} \mathbf{C}$ by $\mathbf{C}$ linearity. Then for any $U \in V_{\mathbf{R}} \otimes_{\mathbf{R}} \mathbf{C}$, $c(U)$, $\hat{c}(U)$ act as odd operators on $\Lambda(\bar{V}^*)$, $\Lambda(V^*)$ respectively. If $U, U' \in V_{\mathbf{R}} \otimes_{\mathbf{R}} \mathbf{C}$,

$$\begin{aligned} &c(U)c(U') + c(U')c(U) = -2\langle U, U'\rangle,\\ &\hat{c}(U)\hat{c}(U') + \hat{c}(U')\hat{c}(U) = -2\langle U, U'\rangle. \end{aligned} \tag{5.3}$$

Also $c(U)$, $\hat{c}(U)$ act as odd operators on $\Lambda(V_{\mathbf{R}}^*) \otimes_{\mathbf{R}} \mathbf{C} = \Lambda(\bar{V}^*) \hat{\otimes} \Lambda(V^*)$. If $U, U' \in V_{\mathbf{R}} \otimes_{\mathbf{R}} \mathbf{C}$, we also have

$$c(U)\hat{c}(U') + \hat{c}(U')c(U) = 0. \tag{5.4}$$

b) Short exact sequences and superconnections in infinite dimensions

We now describe a construction by Bismut [B3] of a secondary invariant associated with a short exact sequence of holomorphic Hermitian vector bundles.

Let B be a compact complex manifold. Let

$$0 \to L \to M \to N \to 0 \tag{5.5}$$

be a short exact sequence of holomorphic vector bundles on B. Let $\mathbf{J}$ denote the complex structure on $M_{\mathbf{R}}$. Then $\mathbf{J}$ induces the complex structures of $L_{\mathbf{R}}$ and $N_{\mathbf{R}}$.

Let g^M be a Hermitian metric on M. Then g^M induces a Hermitian metric g^L on L. We identify N with the orthogonal bundle to L in M. Therefore N inherits a metric g^N. Let P^L, P^N denote the orthogonal projection operators from M on L, N respectively.

Let $e_1, \ldots, e_{2n}$ be an orthonormal base of $N_{\mathbf{R}}$. In the sequel, we use the notation of Section 5a), with $V = N$.

Definition **5.1.** — Let $S \in \mathrm{End}^{\mathrm{even}}(\Lambda(\bar{N}^*) \hat{\otimes} \Lambda(N^*))$ be given by the formula

$$S = \frac{\sqrt{-1}}{2} \sum_1^{2n} c(e_i)\, \hat{c}(J e_i). \tag{5.6}$$

Let ∇^L, ∇^M, ∇^N denote the holomorphic Hermitian connections on L, M, N respectively, and let R^L, R^M, R^N be their curvatures.

Classically [K, Propositions 6.4 and 6.5], we know that

$$\begin{aligned} &\nabla^L = P^L \nabla^M, \\ &\nabla^N = P^N \nabla^M. \end{aligned} \tag{5.7}$$

Let $\widehat{R^N}$ denote the natural action of R^N on $\Lambda(N^*)$. Then $\widehat{R^N}$ acts like $1 \otimes \widehat{R^N}$ on $\Lambda(N^*_{\mathbf{R}}) = \Lambda(\bar{N}^*) \hat{\otimes} \Lambda(N^*)$.

Let ${}^0\nabla^M = \nabla^L \oplus \nabla^N$ be the connection on M which is the direct sum of the connections ∇^L and ∇^N, Set

$$A = \nabla^M - {}^0\nabla^M. \tag{5.8}$$

Then A is a 1-form on B which takes its values in skew-adjoint elements of End(M) which interchange L and N.

Let $f_1, \ldots, f_{2k}$ be a base of $T_{\mathbf{R}} B$, let $f^1, \ldots, f^{2k}$ be the dual base of $T^*_{\mathbf{R}} B$.

Definition **5.2.** — If $Z \in M_{\mathbf{R}}$, set

$$\begin{aligned} &c(AP^L Z) = -\sum_1^{2k} f^j c(A(f_j) P^L Z), \\ &\hat{c}(J\,AP^L Z) = -\sum_1^{2k} f^j \hat{c}(J\,A(f_j) P^L Z). \end{aligned} \tag{5.9}$$

Let $\mathrm{Tr}[R^M]$ denote the (1,1) form on B which is the trace of R^M.

Definition **5.3.** — If $y \in B$, J_y denotes the set of smooth sections of $\Lambda_y(N^*_{\mathbf{R}}) = (\Lambda(\bar{N}^*) \hat{\otimes} \Lambda(N^*))_y$ over the fibre $M_{\mathbf{R}, y}$.

Since $\Lambda(N^*_{\mathbf{R}})$ is **Z**-graded, it is also $\mathbf{Z}_2$-graded. If $y \in B$, let $J_{+,y}$ (resp. $J_{-,y}$) be the set of smooth sections of $\Lambda^{\mathrm{even}}_y(N^*_{\mathbf{R}})$ (resp. $\Lambda^{\mathrm{odd}}_y(N^*_{\mathbf{R}})$) over the fibre $M_{\mathbf{R}, y}$. Clearly $J_y = J_{+,y} \oplus J_{-,y}$.

Moreover $J = J_+ \oplus J_-$ will be considered as an infinite dimensional $\mathbf{Z}_2$-graded vector bundle over B. By the same construction as in Section 4b), we define a $\mathbf{Z}_2$-grading on End(J). Our calculations will be done in the $\mathbf{Z}_2$-graded algebra $\Lambda(T^*_{\mathbf{R}} B) \hat{\otimes} \mathrm{End}(J)$.

We introduce the operators $\mathscr{B}^2_u$ considered in Bismut [B3, Theorem 3.10]. Let $e_1, \ldots, e_{2m}$ be an orthonormal base of $M_{\mathbf{R}}$.

Definition **5.4.** – For $u>0$, let $\mathscr{B}_u^2 \in (\Lambda(T_{\mathbb{R}}^* B) \widehat{\otimes} \operatorname{End}(J))^{\text{even}}$ be given by

$$\mathscr{B}_u^2 = -\frac{1}{2}\sum_1^{2m}\left(\nabla_{e_i} + \langle \frac{1}{2} R^M Z, e_i \rangle\right)^2 + \frac{u}{2}|P^N Z|^2 + \sqrt{u}\,S + \frac{\sqrt{-u}}{\sqrt{2}}\hat{c}(J\,AP^L Z) + \frac{1}{2}\operatorname{Tr}[R^M] + \widehat{R^N}. \tag{5.10}$$

Remark **5.5.** – In [B3], $\mathscr{B}_u^2$ is obtained as the curvature of a superconnection $\mathscr{B}_u$. The fact that $\mathscr{B}_u^2$ is the square of $\mathscr{B}_u$ plays a crucial in the proof in [B3, Theorem 3.12] of non trivial commutation rules for $\mathscr{B}_u^2$. Here this fact will not play any role.

Theorem **5.6.** – *For* $u>0$, *set*

$$\begin{aligned} \mathscr{C}_u^2 &= \exp\left(\frac{c(AP^L Z)}{\sqrt{2}}\right)\mathscr{B}_u^2 \exp\left(\frac{-c(AP^L Z)}{\sqrt{2}}\right), \\ \mathscr{D}_u^2 &= \exp\left(\frac{c(AP^L Z)}{\sqrt{2}} - \frac{\langle R^M P^N Z, P^L Z\rangle}{2}\right)\mathscr{B}_u^2 \exp\left(\frac{-c(AP^L Z)}{\sqrt{2}} + \frac{\langle R^M P^N Z, P^L Z\rangle}{2}\right). \end{aligned} \tag{5.11}$$

Then the following identities hold

$$\begin{aligned} \mathscr{C}_u^2 &= -\frac{1}{2}\sum_1^{2m}\left(\nabla_{e_i} + \frac{1}{2}\langle (R^M - P^L A^2 P^L) Z, e_i\rangle - \frac{c(AP^L e_i)}{\sqrt{2}}\right)^2 \\ &\quad + \frac{u|P^N Z|^2}{2} + \sqrt{u}\,S + \frac{1}{2}\operatorname{Tr}[R^M] + \widehat{R^N}, \\ \mathscr{D}_u^2 &= -\frac{1}{2}\sum_1^{2m}\left(\nabla_{e_i} + \frac{1}{2}\langle (R^M - P^L A^2 P^L) Z, e_i\rangle \right. \\ &\quad \left. + \frac{1}{2}\langle R^M P^N Z, P^L e_i\rangle - \frac{1}{2}\langle R^M P^L Z, P^N e_i\rangle - \frac{c(AP^L e_i)}{\sqrt{2}}\right)^2 \\ &\quad + \frac{u|P^N Z|^2}{2} + \sqrt{u}\,S + \frac{1}{2}\operatorname{Tr}[R^M] + \widehat{R^N}. \end{aligned} \tag{5.12}$$

Proof. – The first identity in (5.12) is proved in [B3, Theorem 4.12]. The second identity follows from the first one. □

Remark **5.7.** – Rather mysterious identities like (5.11), (5.12) will have a very clear geometric interpretation in Section 13i).

c) Generalized supertraces

Let dv_M, dv_N be the volume forms on the fibres of $M_{\mathbf{R}}$, $N_{\mathbf{R}}$ respectively. All the smooth kernels along the fibres of $M_{\mathbf{R}}$ will be calculated with respect to the form $dv_M(Z)/(2\pi)^{\dim M}$.

We denote by N_H the operator in $\operatorname{End}(\Lambda(N^*))$ which defines the $\mathbf{Z}$-grading of $\Lambda(N^*)$, *i.e.* N_H acts by multiplication by p on $\Lambda^p(N^*)$. Then N_H acts like $1 \otimes N_H$ on $\Lambda(\bar{N}^*) \hat{\otimes} \Lambda(N^*)$.

For $u > 0$, let $Q^y_u(Z, Z')$ $(Z, Z' \in M_{\mathbf{R}, y})$ denote the smooth kernel associated with the operator $\exp(-\mathscr{B}^{2, y}_u)$. The existence and uniqueness of $Q^y_u(Z, Z')$ are standard.

Observe that $Q^y_u(Z, Z') \in (\Lambda(T^*_{\mathbf{R}} B) \hat{\otimes} \operatorname{End}(\Lambda(\bar{N}^*) \hat{\otimes} \Lambda(N^*)))^{\text{even}}_y$. We now use the conventions of Quillen [Q1] described in Section 4b). In particular $\operatorname{Tr}_s[Q^y_u(Z, Z')]$ lies in $\Lambda^{\text{even}}(T^*_{\mathbf{R}} B)$.

By [B3, Theorem 4.1], we know that for $u > 0$, there exist $c > 0$, $C > 0$ such that if $y \in B$, $Z \in N_{\mathbf{R}, y}$, then

$$|Q^y_u(Z, Z)| \leqslant c \exp(-C|Z|^2). \tag{5.13}$$

Note that in (5.13), it is crucial that Z is restricted to vary in $N_{\mathbf{R}, y}$.

In view of (5.13) and following [B3, Definition 4.4], we now set the following definition.

Definition **5.8.** — For $u > 0$, set

$$\begin{aligned} \operatorname{Tr}_s[\exp(-\mathscr{B}^2_u)]_y &= \int_{N_{\mathbf{R}, y}} \operatorname{Tr}_s[Q^y_u(Z, Z)] \frac{dv_N(Z)}{(2\pi)^{\dim N}}, \\ \operatorname{Tr}_s[N_H \exp(-\mathscr{B}^2_u)]_y &= \int_{N_{\mathbf{R}, y}} \operatorname{Tr}_s[N_H Q^y_u(Z, Z)] \frac{dv_N(Z)}{(2\pi)^{\dim N}}. \end{aligned} \tag{5.14}$$

Note that $\operatorname{Tr}_s[\exp(-\mathscr{B}^2_u)]$ and $\operatorname{Tr}_s[N_H \exp(-\mathscr{B}^2_u)]$ are only generalized supertraces. In fact the operator $\exp(-\mathscr{B}^2_u)$ is in general not trace class.

Using (5.11), and the fact that supertraces vanish on supercommutators [Q1], it is clear that $\exp(-\mathscr{C}^2_u)$, $\exp(-\mathscr{D}^2_u)$, $N_H \exp(-\mathscr{C}^2_u)$, $N_H \exp(-\mathscr{D}^2_u)$ also have generalized supertraces $\operatorname{Tr}_s[\exp(-\mathscr{C}^2_u)]$, $\operatorname{Tr}_s[\exp(-\mathscr{D}^2_u)]$, $\operatorname{Tr}_s[N_H \exp(-\mathscr{C}^2_u)]$, $\operatorname{Tr}_s[N_H \exp(-\mathscr{D}^2_u)]$ and that

$$\begin{aligned} \operatorname{Tr}_s[\exp(-\mathscr{B}^2_u)] &= \operatorname{Tr}_s[\exp(-\mathscr{C}^2_u)] = \operatorname{Tr}_s[\exp(-\mathscr{D}^2_u)], \\ \operatorname{Tr}_s[N_H \exp(-\mathscr{B}^2_u)] &= \operatorname{Tr}_s[N_H \exp(-\mathscr{C}^2_u)] = \operatorname{Tr}_s[N_H \exp(-\mathscr{D}^2_u)]. \end{aligned} \tag{5.15}$$

d) Convergence of generalized supertraces

Recall that the map $\Phi: \Lambda^{\text{even}}(T^*_{\mathbf{R}} B) \to \Lambda^{\text{even}}(T^*_{\mathbf{R}} B)$ was defined in Section 4a).

If $u \in \mathbf{R}_+ \to \omega_u$ is a family of smooth forms on B, we will write that as $u \to 0$

$$\omega_u = \omega_0 + O(u) \tag{5.16}$$

if for any $k \in \mathbf{N}$, there exists $C_k > 0$ such that for $0 < u \leqslant 1$, the norm of $\omega_u - \omega_0$ in $\mathscr{C}^k(B)$ is dominated by $C_k u$. We use a similar notation when $u \to +\infty$.

We now recall several essential results of [B3].

Theorem **5.9.** — *For any $u > 0$, the forms* $\mathrm{Tr}_s[\exp(-\mathscr{B}_u^2)]$ *are closed, lie in* P^B, *and their cohomology class does not depend on $u > 0$. The forms* $\mathrm{Tr}_s[N_H \exp(-\mathscr{B}_u^2)]$ *lie in* P^B.

As $u \to 0$,

$$\begin{aligned} &\Phi \,\mathrm{Tr}_s[\exp(-\mathscr{B}_u^2)] = \mathrm{Td}^{-1}(N, g^N)\,\mathrm{Td}(M, g^M) + O(u), \\ &\Phi \,\mathrm{Tr}_s[N_H \exp(-\mathscr{B}_u^2)] = -(\mathrm{Td}^{-1})'(N, g^N)\,\mathrm{Td}(M, g^M) + O(u). \end{aligned} \tag{5.17}$$

As $u \to +\infty$,

$$\begin{aligned} &\Phi \,\mathrm{Tr}_s[\exp(-\mathscr{B}_u^2)] = \mathrm{Td}(L, g^L) + O\left(\frac{1}{\sqrt{u}}\right), \\ &\Phi \,\mathrm{Tr}_s[N_H \exp(-\mathscr{B}_u^2)] = \frac{\dim N}{2}\,\mathrm{Td}(L, g^L) + O\left(\frac{1}{\sqrt{u}}\right). \end{aligned} \tag{5.18}$$

Proof. — The results stated in Theorem are proved in Bismut [B3, Theorems 4.6, 4.8 and 7.7]. Note especially [B3, eq. (4.37)] which expresses $\mathrm{Tr}_s[N_H \exp(-\mathscr{B}_u^2)]$ in terms of other objects more commonly used in [B3]. □

Remark **5.10.** — In Section 14, we will give a new proof of the convergence results contained in (5.18).

Remark **5.11.** — It is of crucial importance that the same class $(\mathrm{Td}^{-1})'(N, g^N)$ appears in both formulas (4.9) and (5.17).

e) Generalized analytic torsion forms

We now reproduce the construction in Bismut [B3, Section 8] of generalized analytic torsion forms.

Definition **5.12.** — For $s \in \mathbf{C}$, $0 < \mathrm{Re}(s) < 1/2$, let $\mathbf{B}(s)$ be the form on B

$$\mathbf{B}(s)=\frac{1}{\Gamma(s)}\int_0^{+\infty} u^{s-1}\left\{\Phi \operatorname{Tr}_s[\mathrm{N_H}\exp(-\mathscr{B}_u^2)]-\frac{\dim \mathrm{N}}{2}\operatorname{Td}(\mathrm{L}, g^{\mathrm{L}})\right\} du. \tag{5.19}$$

One then easily verifies in [B3, Section 8a)] that $\mathbf{B}(s)$ extends to a function of s which is holomorphic near $s=0$.

Definition **5.13.** – Let $\mathbf{B}(\mathrm{L}, \mathrm{M}, g^{\mathrm{M}})$ be the form on B

$$\mathbf{B}(\mathrm{L}, \mathrm{M}, g^{\mathrm{M}})=\frac{\partial \mathbf{B}}{\partial s}(0). \tag{5.20}$$

By [B3, eq. (4.37), (8.2)], the following identity holds

$$\begin{aligned}\mathbf{B}(\mathrm{L}, \mathrm{M}, g^{\mathrm{M}})=&\int_0^1 \{\Phi \operatorname{Tr}_s[\mathrm{N_H}\exp(-\mathscr{B}_u^2)]+\operatorname{Td}(\mathrm{M}, g^{\mathrm{M}})(\operatorname{Td}^{-1})'(\mathrm{N}, g^{\mathrm{N}})\}\frac{du}{u}\\ &+\int_1^{+\infty}\left\{\Phi \operatorname{Tr}_s[\mathrm{N_H}\exp(-\mathscr{B}_u^2)]-\frac{\dim \mathrm{N}}{2}\operatorname{Td}(\mathrm{L}, g^{\mathrm{L}})\right\}\frac{du}{u}\\ &+\Gamma'(1)\left\{\operatorname{Td}(\mathrm{M}, g^{\mathrm{M}})(\operatorname{Td}^{-1})'(\mathrm{N}, g^{\mathrm{N}})+\frac{\dim \mathrm{N}}{2}\operatorname{Td}(\mathrm{L}, g^{\mathrm{L}})\right\}.\end{aligned} \tag{5.21}$$

The following result is proved [B3, Theorem 8.3].

Theorem **5.14.** – *The form* $\mathbf{B}(\mathrm{L}, \mathrm{M}, g^{\mathrm{M}})$ *lies in* $\mathrm{P^B}$. *Also*

$$\frac{\bar\partial\partial}{2i\pi}\mathbf{B}(\mathrm{L}, \mathrm{M}, g^{\mathrm{M}})=\operatorname{Td}(\mathrm{L}, g^{\mathrm{L}})-\frac{\operatorname{Td}(\mathrm{M}, g^{\mathrm{M}})}{\operatorname{Td}(\mathrm{N}, g^{\mathrm{N}})}. \tag{5.22}$$

f) Evaluation of the generalized analytic torsion forms

We now describe the main results of [B3] concerning the evaluation of the form $\mathbf{B}(\mathrm{L}, \mathrm{M}, g^{\mathrm{M}})$.

Recall that the Hirzebruch polynomial $\hat{\mathrm{A}}(x)$ is given by

$$\hat{\mathrm{A}}(x)=\frac{x/2}{\sinh(x/2)}. \tag{5.23}$$

Set

$$\hat{\mathrm{A}}(x_1, \ldots, x_q)=\prod_1^q \hat{\mathrm{A}}(x_i). \tag{5.24}$$

In particular

(5.25) $$\operatorname{Td}(x_1, \ldots, x_q) = \hat{A}(x_1, \ldots, x_q)\, e^{1/2 \sum_1^q x_i}.$$

For $u > 0$, $x \in \mathbf{C}$, set

(5.26) $$\varphi(u, x) = \frac{4}{u}\sinh\left(\frac{x+\sqrt{x^2+4u}}{4}\right)\sinh\left(\frac{-x+\sqrt{x^2+4u}}{4}\right).$$

Then as one easily verifies in [B3, eq. (6.3)],

(5.27) $$\varphi(0, x) = \hat{A}^{-1}(x)$$

Also by [B3, eq. (8.8)], for $x \in \mathbf{C}$, $|x| < 2\pi$, as $u \to +\infty$,

(5.28) $$\left(\frac{\partial \varphi}{\partial x}\Big/ \varphi\right)(u, x) = O\left(\frac{1}{\sqrt{u}}\right).$$

Definition **5.15.** — For $s \in \mathbf{C}$, $0 < \operatorname{Re}(s) < 1/2$, $x \in \mathbf{C}$, $|x| < 2\pi$, set

(5.29) $$C(s, x) = -\frac{1}{\Gamma(s)} \int_0^{+\infty} u^{s-1} \left(\frac{\partial \varphi}{\partial x}\Big/ \varphi\right)(u, x)\, du.$$

Then $C(s, x)$ extends to a holomorphic function of s near $s = 0$. Set

(5.30) $$D(x) = \frac{\partial C}{\partial s}(0, x).$$

The function $D(x)$ is holomorphic in $x \in \mathbf{C}$, $|x| < 2\pi$. We identify $D(x)$ with the additive genus

(5.31) $$D(x_1, \ldots, x_q) = \sum_1^q D(x_i).$$

Then $\operatorname{Td}(L)\, D(N)$ is a well defined element of $P^B/P^{B,0}$.

Let $\widetilde{\operatorname{Td}}(L, M, g^M)$ be the Bott-Chern class in $P^B/P^{B,0}$ associated to the exact sequence of holomorphic Hermitian vector bundles (5.5), which is constructed in [BGS1, Theorem 1.29] and is such that

(5.32) $$\frac{\bar{\partial}\partial}{2i\pi} \widetilde{\operatorname{Td}}(L, M, g^M) = \operatorname{Td}(M, g^M) - \operatorname{Td}(L, g^L)\operatorname{Td}(N, g^N).$$

The class $\widetilde{\mathrm{Td}}(\mathrm{L}, \mathrm{M}, g^{\mathrm{M}})$ is normalized by the fact that if the exact sequence (5.5) splits holomorphically (and here also metrically), then $\widetilde{\mathrm{Td}}(\mathrm{L}, \mathrm{M}, g^{\mathrm{M}})=0$ in $\mathrm{P}^{\mathrm{B}}/\mathrm{P}^{\mathrm{B},0}$.

The following result is proved in [B3, Theorem 8.5].

Theorem **5.16.** — *The following identity holds*

$$\mathbf{B}(\mathrm{L}, \mathrm{M}, g^{\mathrm{M}}) = -\mathrm{Td}^{-1}(\mathrm{N}, g^{\mathrm{N}})\widetilde{\mathrm{Td}}(\mathrm{L}, \mathrm{M}, g^{\mathrm{M}}) + \mathrm{Td}(\mathrm{L})\mathrm{D}(\mathrm{N}) \text{ in } \mathrm{P}^{\mathrm{B}}/\mathrm{P}^{\mathrm{B},0}. \tag{5.33}$$

g) Evaluation of the function D(x)

Let $\zeta(s)$ be the Riemann zeta function. We now state the result of Bismut and Soulé which is proved in [B3, Theorem 1 of the Appendix].

Theorem **5.17.** — *For* $x \in \mathbf{C}$, $|x| < 2\pi$, *then*

$$\mathrm{D}(x) = \sum_{\substack{n \geq 1 \\ n \text{ odd}}} \left(\Gamma'(1) + \sum_{1}^{n} \frac{1}{j} + \frac{2\zeta'(-n)}{\zeta(-n)}\right)\zeta(-n)\frac{x^n}{n!}. \tag{5.34}$$

We recall the definition of the formal power series $\mathrm{R}(x)$ introduced by Gillet and Soulé [GS3].

Definition **5.18.** — Let $\mathrm{R}(x)$ be the formal power series

$$\mathrm{R}(x) = \sum_{\substack{n \geq 1 \\ n \text{ odd}}} \left(\sum_{1}^{n} \frac{1}{j} + \frac{2\zeta'(-n)}{\zeta(-n)}\right)\zeta(-n)\frac{x^n}{n!}. \tag{5.35}$$

Proposition **5.19.** — *The following identity of formal power series holds*

$$\mathrm{D}(x) = \mathrm{R}(x) + \Gamma'(1)\frac{\hat{\mathrm{A}}'}{\hat{\mathrm{A}}}(x). \tag{5.36}$$

Proof. — This simple identity immediately follows from (5.34)-(5.35) and is proved in [B3, Remark 8.8]. □

In the sequel we will identify $\mathrm{R}(x)$ with the additive genus

$$\mathrm{R}(x_1, \ldots, x_q) = \sum_{1}^{q} \mathrm{R}(x_i). \tag{5.37}$$

Since the genus $\hat{A}$ is multiplicative, the genus $\hat{A}'/\hat{A}$ is additive. Therefore (5.36) can be rewritten as an identity of additive genera

$$D = R + \Gamma'(1)\frac{\hat{A}'}{\hat{A}}. \tag{5.38}$$

VI - THE QUILLEN NORM OF THE CANONICAL SECTION ρ

This Section is the heart of the paper. Its purpose is to calculate the Quillen norm of $\rho\in\lambda^{-1}(\eta)\otimes\tilde{\lambda}(\xi)$.

Recall that in Section 3, we constructed a closed differential 1-form β on $\mathbf{R}_+^*\times\mathbf{R}_+^*$, and a contour Γ depending on three parameters ε, A, T_0, such that $\int_\Gamma \beta=0$. In Theorem 3.6, we showed that this last relation is equivalent to the equation $\sum_{k=1}^4 I_k^0=0$, where the I_k^0's depend on ε, A, T_0. In this Section, we study each term I_k^0 separately, by making in succession $A\to+\infty$ (step α), $T_0\to+\infty$ (step β), $\varepsilon\to 0$ (step γ). At one or more of these three stages, divergences have to be subtracted off. For every $k\,(1\leqslant k\leqslant 4)$, the last stage (step δ) is to evaluate the final object I_k^3 in terms of the quantities introduced in Sections 1, 4 and 5. We finally obtain an identity $\sum_{k=1}^4 I_k^3=0$ which is equivalent to the formulas in Theorem 6.1 for $\operatorname{Log}(\|\rho\|^2_{\lambda^{-1}(\eta)\otimes\tilde{\lambda}(\xi)})$.

To calculate the asymptotics of the I_k^0's, we use all the notation and results of Section 1 and of Sections 3-5. We also state seven intermediary results. The proofs of six of these results are delayed to Sections 8-13. The purpose of these intermediary results is to calculate certain limits and also to establish some key estimates. These estimates are intimately related to the deepest aspects of the problem which is solved in this paper. Their main object is to handle simultaneously local cancellations in index theory and spectral theory in very degenerate situations. Also note that, as is best revealed in Sections 9-10, the quasi-isomorphism of Dolbeault complexes of Theorem 1.7 plays a key role in the proof of some of these intermediary results.

We hope that the behaviour of the term I_4^0 as $\varepsilon\to 0$ will attract the interest of the reader. Its devilish nature is best revealed by the fact that it produces altogether the current $T(\xi, h^\xi)$ of Section 4 and the form $\mathbf{B}(TY, TX|_Y, g^{TX|Y})$ of Section 5. It kept us busy for a long time.

Section 6 is organized as follows. In a), we state the main result of this paper, which consists of two equivalent formulas for $\mathrm{Log}(\|\rho\|^2_{\lambda^{-1}(\eta)\otimes\tilde{\lambda}(\xi)})$. In b), we again consider the rescaled metrics on TX, $\xi_0, \ldots, \xi_m$, which were already constructed in Section 3a). In c), we state seven key intermediary results. In d), we study the asymptotics of the I_k^0's, each of these terms being individually studied in four steps α)-δ). In e), we verify that the divergences which were obtained in steps α)-δ) effectively add up to zero, and we obtain the crucial identity $\sum_{k=1}^{4} \mathrm{I}_k^3 = 0$. In f), we obtain a first formula for $\mathrm{Log}(\|\rho\|^2_{\lambda^{-1}(\eta)\otimes\tilde{\lambda}(\xi)})$ in terms of the current $\mathrm{T}(\xi, h^\xi)$ and of the form $\mathbf{B}(\mathrm{TY}, \mathrm{TX}|_\mathrm{Y}, g^{\mathrm{TX}|\mathrm{Y}})$. Finally in g), we prove Theorem 6.1 by using the evaluation of $\mathbf{B}(\mathrm{TY}, \mathrm{TX}|_\mathrm{Y}, g^{\mathrm{TX}|\mathrm{Y}})$ of [B3] which was given in Section 5.

This Section is meant to encourage the reader to read the rest of the paper.

a) The main Theorem

Recall that the additive genus R of Gillet and Soulé [GS3] was defined in Definition 5.18.

We now state the main result of this paper, whose proof occupies Sections 6-13.

Theorem **6.1.** — *The following identities hold*

$$\begin{aligned}\mathrm{Log}(\|\rho\|^2_{\lambda^{-1}(\eta)\otimes\tilde{\lambda}(\xi)}) = &-\int_\mathrm{X} \mathrm{Td}(\mathrm{TX}, g^{\mathrm{TX}})\,\mathrm{T}(\xi, h^\xi)\\ &+\int_\mathrm{Y} \mathrm{Td}^{-1}(\mathrm{N}, g^\mathrm{N})\,\widetilde{\mathrm{Td}}(\mathrm{TY}, \mathrm{TX}|_\mathrm{Y}, g^{\mathrm{TX}|\mathrm{Y}})\,\mathrm{ch}(\eta, g^\eta)\\ &-\int_\mathrm{Y} \mathrm{Td}(\mathrm{TY})\,\mathrm{R}(\mathrm{N})\,\mathrm{ch}(\eta),\end{aligned} \tag{6.1}$$

$$\begin{aligned}\mathrm{Log}(\|\rho\|^2_{\lambda^{-1}(\eta)\otimes\tilde{\lambda}(\xi)}) = &-\int_\mathrm{X} \mathrm{Td}(\mathrm{TX}, g^{\mathrm{TX}})\,\mathrm{T}(\xi, h^\xi)\\ &+\int_\mathrm{Y} \mathrm{Td}^{-1}(\mathrm{N}, g^\mathrm{N})\,\widetilde{\mathrm{Td}}(\mathrm{TY}, \mathrm{TX}|_\mathrm{Y}, g^{\mathrm{TX}|\mathrm{Y}})\,\mathrm{ch}(\eta, g^\eta)\\ &-\int_\mathrm{X} \mathrm{Td}(\mathrm{TX})\,\mathrm{R}(\mathrm{TX})\,\mathrm{ch}(\xi) + \int_\mathrm{Y} \mathrm{Td}(\mathrm{TY})\,\mathrm{R}(\mathrm{TY})\,\mathrm{ch}(\eta).\end{aligned} \tag{6.1'}$$

b) A rescaled metric on E

Definition **6.2.** – For $T > 0$, we denote by $\langle\ ,\ \rangle_T$ the Hermitian product on E associated with the metrics g^{TX}, h^{ξ_0}, $h^{\xi_1}/T^2, \ldots, h^{\xi_m}/T^{2m}$ on TX, $\xi_0, \ldots, \xi_m$ respectively. Set

$$K_T = \{s \in E;\ (\bar\partial^X + v)s = 0;\ (\bar\partial^{X*} + T^2 v^*)s = 0\}. \tag{6.2}$$

Let P_T be the orthogonal projection operator from E on K_T with respect to the Hermitian product $\langle\ ,\ \rangle_T$.

In Section 1e), we saw that for any $T > 0$, there is a canonical isomorphism of **Z**-graded vector spaces

$$K_T \cong H^*(E, \bar\partial^X + v). \tag{6.3}$$

Let $|\ \ |_{\tilde\lambda(\xi), T}$ be the metric on the line $\tilde\lambda(\xi)$ inherited from the metric $\langle\ ,\ \rangle_T$ restricted to K_T. Clearly, with the notation of Section 1e), we have

$$K_1 = K;\ P_1 = P;$$
$$|\ \ |_{\tilde\lambda(\xi), 1} = |\ \ |_{\tilde\lambda(\xi)}.$$

For $T > 0$, set

$$\tilde K_T = \{s \in E;\ (D^X + TV)s = 0\}. \tag{6.4}$$

Let $\tilde P_T$ be the orthogonal projection operator from E on $\tilde K_T$ with respect to the Hermitian product $\langle\ ,\ \rangle = \langle\ ,\ \rangle_1$ on E.

By (3.2), we know that for $T > 0$

$$T^{-N_H}(\bar\partial^X + v + \bar\partial^{X*} + T^2 v^*)T^{N_H} = D^X + TV. \tag{6.5}$$

From (6.5), we deduce that

$$\tilde P_T = T^{-N_H} P_T T^{N_H}. \tag{6.6}$$

The map $s \in K_T \to T^{-N_H} s \in \tilde K_T$ is an isomorphism of **Z**-graded vector spaces. We thus find that as a **Z**-graded vector space, $\tilde K_T$ is also isomorphic to $H^*(E, \bar\partial^X + v)$.

Set

$$D^Y = \bar\partial^Y + \bar\partial^{Y*}. \tag{6.7}$$

For $1 \leqslant j \leqslant d$, let D^{Y_j} be the restriction of D^Y to Y_j.

Let Q be the orthogonal projection operator from F on $K' = \operatorname{Ker}(D^Y)$ with respect to the given Hermitian product on F.

c) Seven intermediary results

Recall that ω^X, ω^Y are the Kähler forms of X, Y. Since these forms are closed, they can be paired with characteristic classes of vector bundles on X, Y respectively.

For $0 \leqslant i \leqslant m$, $1 \leqslant j \leqslant d$, let $\chi(\xi_i)$, $\chi(\eta_j)$ be the Euler characteristics of ξ_i, η_j. By Theorems of Riemann-Roch-Hirzebruch [H] and Atiyah-Singer [AS], we know that

$$\begin{aligned} \chi(\xi_i) &= \int_X \operatorname{Td}(TX)\operatorname{ch}(\xi_i), \\ \chi(\eta_j) &= \int_{Y_j} \operatorname{Td}(TY_j)\operatorname{ch}(\eta_j). \end{aligned} \tag{6.8}$$

In the sequel, we will often use the notation

$$\begin{aligned} \int_Y \dim Y \operatorname{Td}(TY)\operatorname{ch}(\eta) &= \sum_{j=1}^{d} \dim Y_j \int_{Y_j} \operatorname{Td}(TY_j)\operatorname{ch}(\eta_j), \\ \dim N \chi(\eta) &= \sum_1^d \dim N_j \chi(\eta_j). \end{aligned} \tag{6.9}$$

We now state in Theorems 6.3 to 6.9 seven intermediary results which play an essential role in the proof of Theorem 6.1. The proofs of Theorems 6.4-6.9 are deferred to Sections 8-13.

Theorem **6.3.** — *As $u \to 0$, then*

$$\begin{aligned} \operatorname{Tr}_s[(N_V^X - N_H)\exp(-u(D^X+V)^2)] &= \frac{1}{u}\int_X \frac{\omega^X}{2\pi}\operatorname{Td}(TX)\operatorname{ch}(\xi) \\ &+ \int_X (\dim X \operatorname{Td}(TX)\operatorname{ch}(\xi) - \operatorname{Td}'(TX)\operatorname{ch}(\xi) \\ &- \operatorname{Td}(TX)\operatorname{ch}'(\xi)) + O(u), \\ \operatorname{Tr}_s[N_V^Y \exp(-u(D^Y)^2)] &= \frac{1}{u}\int_Y \frac{\omega^Y}{2\pi}\operatorname{Td}(TY)\operatorname{ch}(\eta) \\ &+ \int_Y (\dim Y \operatorname{Td}(TY) - \operatorname{Td}'(TY))\operatorname{ch}(\eta) + O(u). \end{aligned} \tag{6.10}$$

Proof. — The metrics g^{TX}, g^{TY} being Kähler, Theorem 6.3 follows from Bismut-Gillet-Soulé [BGS2, Theorem 2.16]. Note that in [BGS2], the operators D^X, D^Y are replaced by the operators $\sqrt{2}D^X$, $\sqrt{2}D^Y$. This explains why the expressions $\omega^X/4\pi$, $\omega^Y/4\pi$ in [BGS2] are here changed into $\omega^X/2\pi$, $\omega^Y/2\pi$. □

Theorem **6.4.** — *For any* $\alpha_0 > 0$, *there exists* $C > 0$ *such that for* $\alpha \geqslant \alpha_0$, $T \geqslant 1$,

$$\left| \mathrm{Tr}_s[\mathrm{N_H} \exp(-\alpha(\mathrm{D^X}+\mathrm{TV})^2)] - \frac{1}{2}\dim \mathrm{N}\,\chi(\eta)\right| \leqslant \frac{\mathrm{C}}{\sqrt{\mathrm{T}}}, \tag{6.11}$$

$$|\mathrm{Tr}_s[(\mathrm{N_V^X}-\mathrm{N_H})\exp(-\alpha(\mathrm{D^X}+\mathrm{TV})^2)] - \mathrm{Tr}_s[\mathrm{N_V^Y}\exp(-\alpha(\mathrm{D^Y})^2)]| \leqslant \frac{\mathrm{C}}{\sqrt{\mathrm{T}}}.$$

Theorem **6.5.** — *There exist* $c > 0$, $C > 0$ *such that for* $\alpha \geqslant 1$, $T \geqslant 1$,

$$|\mathrm{Tr}_s[(\mathrm{N_V^X}-\mathrm{N_H})\exp(-\alpha(\mathrm{D^X}+\mathrm{TV})^2)] - \mathrm{Tr}_s[(\mathrm{N_V^X}-\mathrm{N_H})\tilde{\mathrm{P}}_\mathrm{T}]| \leqslant c\exp(-\mathrm{C}\alpha) \tag{6.12}$$

Theorem **6.6.** — *There exist* $C > 0$, $\gamma \in]0, 1]$ *such that for* $u \in]0, 1]$, $0 \leqslant T \leqslant 1/u$, *then*

$$\begin{aligned} &|\mathrm{Tr}_s[\mathrm{N_H}\exp(-(u\mathrm{D^X}+\mathrm{TV})^2)] \\ &\quad - \int_X \mathrm{Td}(\mathrm{TX}, g^{\mathrm{TX}})\,\Phi\,\mathrm{Tr}_s[\mathrm{N_H}\exp(-\mathrm{C}_{\mathrm{T}^2}^2)]| \leqslant \mathrm{C}(u(1+\mathrm{T}))^\gamma. \end{aligned} \tag{6.13}$$

There exists a constant $C' > 0$ *such that for* $u \in]0, 1]$, $0 \leqslant T \leqslant 1$

$$|\mathrm{Tr}_s[\mathrm{N_H}\exp(-(u\mathrm{D^X}+\mathrm{TV})^2)] - \mathrm{Tr}_s[\mathrm{N_H}\exp(-(u\mathrm{D^X})^2)]| \leqslant \mathrm{C'T}. \tag{6.14}$$

For $1 \leqslant j \leqslant d$, consider the exact sequence of holomorphic Hermitian vector bundles

$$0 \to \mathrm{TY}_j \to \mathrm{TX}|_{\mathrm{Y}_j} \to \mathrm{N}_j \to 0. \tag{6.15}$$

We will now use the notation of Section 5, which we will apply to the exact sequence (6.15). In particular for $1 \leqslant j \leqslant d$, $u > 0$, we construct the operator $\mathscr{B}_{j,u}^2$ as in Definition 5.4. Most of the time, and following the conventions in (6.8), the subscript j will be omitted.

Theorem **6.7.** — *For any* $T > 0$, *the following identity holds*

$$\begin{aligned} &\lim_{u \to 0} \mathrm{Tr}_s\left[\mathrm{N_H}\exp\left(-\left(u\mathrm{D^X}+\frac{\mathrm{T}}{u}\mathrm{V}\right)^2\right)\right] \\ &\quad = \int_Y \Phi\,\mathrm{Tr}_s[\mathrm{N_H}\exp(-\mathscr{B}_{\mathrm{T}^2}^2)]\,\mathrm{ch}(\eta, g^\eta). \end{aligned} \tag{6.16}$$

Theorem **6.8.** — *There exist* $C > 0$, $\delta \in]0, 1]$ *such that for* $u \in]0, 1]$, $T \geqslant 1$

$$\left|\mathrm{Tr}_s\left[\mathrm{N_H}\exp\left(-\left(u\mathrm{D^X}+\frac{\mathrm{T}}{u}\mathrm{V}\right)^2\right)\right] - \frac{1}{2}\dim\mathrm{N}\,\chi(\eta)\right| \leqslant \frac{\mathrm{C}}{\mathrm{T}^\delta}. \tag{6.17}$$

Recall that the metric $|\ \ |_{\tilde{\lambda}(\xi),\mathrm{T}}$ on the line $\tilde{\lambda}(\xi)$ was defined in Section 6b).

Theorem **6.9.** – *As* $\mathrm{T} \to +\infty$

$$\mathrm{Log}\left(\frac{|\ \ |^2_{\tilde{\lambda}(\xi),\mathrm{T}}}{|\ \ |^2_{\tilde{\lambda}(\xi)}}\right) = \dim \mathrm{N}\,\chi(\eta)\,\mathrm{Log}(\mathrm{T}) - \mathrm{Log}(|\rho|^2_{\lambda^{-1}(\eta)\otimes\tilde{\lambda}(\xi)}) + \varepsilon\left(\frac{1}{\mathrm{T}}\right). \tag{6.18}$$

Remark **6.10.** – Theorems 6.4, 6.5, 6.6, 6.7 and 6.8 are related to each other and to results of Sections 4 and 5. In fact by Theorem 6.8, and also by Theorems 4.3 and 6.6, we know that for $\mathrm{T} > 0$, as $u \to 0$, $\mathrm{Tr}_s[\mathrm{N_H} \exp(-(u\mathrm{D}^{\mathrm{X}} + (\mathrm{T}/u)\mathrm{V})^2)]$ remains uniformly bounded. Of course this also follows from Theorem 6.7. Similarly, using Theorems 4.3, 6.6 and 6.7, we find that if $0 < \mathrm{T} \leqslant 1$

$$\left|\int_{\mathrm{Y}} \Phi\,\mathrm{Tr}_s[\mathrm{N_H}\exp(-\mathscr{B}^2_{\mathrm{T}^2})]\,\mathrm{ch}(\eta, g^\eta) + \int_{\mathrm{Y}} i^*(\mathrm{Td}(\mathrm{TX}, g^{\mathrm{TX}}))\,\mathrm{ch}(\eta, g^\eta)(\mathrm{Td}^{-1})'(\mathrm{N}, g^{\mathrm{N}})\right| \leqslant \mathrm{CT}^\gamma, \tag{6.19}$$

which also follows from Theorem 5.9. Finally, by Theorems 6.7 and 6.8, for $\mathrm{T} \geqslant 1$

$$\left|\int_{\mathrm{Y}} \Phi\,\mathrm{Tr}_s[\mathrm{N_H}\exp(-\mathscr{B}^2_{\mathrm{T}^2})]\,\mathrm{ch}(\eta, g^\eta) - \frac{1}{2}\dim \mathrm{N}\,\chi(\eta)\right| \leqslant \frac{\mathrm{C}}{\mathrm{T}^\delta} \tag{6.20}$$

Equation (6.20) is also a consequence of Theorem 5.9 and of (6.8).

Sections 8 and 9 are devoted to the proofs of Theorems 6.4 and 6.5, Section 10 to the proof of Theorem 6.9, Section 11 to the proof of Theorem 6.6, Section 12 to the proof of Theorem 6.7 and Section 13 to the proof of Theorem 6.8.

d) The asymptotics of the I^0_k's

We now use the notation of Section 3d). Recall that by Theorem 3.6

$$\sum_{k=1}^{4} \mathrm{I}^0_k = 0. \tag{6.21}$$

We will study each term $\mathrm{I}^0_1, \ldots, \mathrm{I}^0_4$ separately, by making in succession $\mathrm{A} \to +\infty$, $\mathrm{T}_0 \to +\infty$, $\varepsilon \to 0$.

1) *The term* I_1^0

Clearly

$$I_1^0 = \int_\varepsilon^A \mathrm{Tr}_s[(N_V^X - N_H)\exp(-u^2(D^X + T_0 V)^2)]\frac{du}{u}. \tag{6.22}$$

α) $A \to +\infty$

One easily verifies that as $A \to +\infty$

$$\begin{aligned} & I_1^0 - \mathrm{Tr}_s[(N_V^X - N_H)\tilde{P}_{T_0}]\,\mathrm{Log}(A) \to \\ & I_1^1 = \int_\varepsilon^1 \mathrm{Tr}_s[(N_V^X - N_H)\exp(-u^2(D^X + T_0 V)^2)]\frac{du}{u} \\ & \quad + \int_1^{+\infty} \{\mathrm{Tr}_s[(N_V^X - N_H)\exp(-u^2(D^X + T_0 V)^2)] \\ & \quad - \mathrm{Tr}_s[(N_V^X - N_H)\tilde{P}_{T_0}]\}\frac{du}{u}. \end{aligned} \tag{6.23}$$

β) $T_0 \to +\infty$

By Theorem 6.4, as $T_0 \to +\infty$

$$\begin{aligned} & \int_\varepsilon^1 \mathrm{Tr}_s[(N_V^X - N_H)\exp(-u^2(D^X + T_0 V)^2)]\frac{du}{u} \to \\ & \int_\varepsilon^1 \mathrm{Tr}_s[N_V^Y \exp(-u^2(D^Y)^2)]\frac{du}{u}. \end{aligned} \tag{6.24}$$

By (6.6), we find that for any $T \geqslant 1$

$$\mathrm{Tr}_s[(N_V^X - N_H)\tilde{P}_T] = \mathrm{Tr}_s[(N_V^X - N_H)P_T]. \tag{6.25}$$

Since we have the isomorphisms of $\mathbf{Z}$-graded vector bundles $K_T \cong H^*(E, \bar{\partial}^X + v)$, then

$$\mathrm{Tr}_s[(N_V^X - N_H)P_T] = \sum_{p \in \mathbf{Z}} (-1)^p p \dim H^p(E, \bar{\partial}^X + v). \tag{6.26}$$

Also by Theorem 1.7, the $\mathbf{Z}$-graded complexes $(E, \bar{\partial}^X + v)$ and $(F, \bar{\partial}^Y)$ are quasi-isomorphic. Therefore for $p \in \mathbf{Z}$

$$\dim H^p(E, \bar{\partial}^X + v) = \dim H^p(Y, \eta). \tag{6.27}$$

Recall that Q is the orthogonal projection operator from F on $K' = \mathrm{Ker}(D^Y)$. By Hodge Theory

$$\mathrm{Tr}_s[N_V^Y Q] = \sum_{p \in \mathbf{Z}} (-1)^p p \dim H^p(Y, \eta). \tag{6.28}$$

From (6.26)-(6.28), we deduce that for $T \geqslant 1$

$$\mathrm{Tr}_s[(N_V^X - N_H)\tilde{P}_T] = \mathrm{Tr}_s[N_V^Y Q]. \tag{6.29}$$

By Theorem 6.5, we find that there exist $c > 0$, $C > 0$ such that for $u \geqslant 1$, $T_0 \geqslant 1$

$$\begin{aligned} &|\mathrm{Tr}_s[(N_V^X - N_H)\exp(-u^2(D^X + T_0 V)^2)] \\ &\quad - \mathrm{Tr}_s[(N_V^Y - N_H)\tilde{P}_{T_0}]| \leqslant c \exp(-C u^2). \end{aligned} \tag{6.30}$$

Using Theorem 6.4, (6.29), (6.30) and the dominated convergence Theorem, we find that as $T_0 \to +\infty$

$$\begin{aligned} &\int_1^{+\infty} \{\mathrm{Tr}_s[(N_V^X - N_H)\exp(-u^2(D^X + T_0 V)^2)] - \mathrm{Tr}_s[(N_V^X - N_H)\tilde{P}_{T_0}]\} \frac{du}{u} \\ &\quad \to \int_1^{+\infty} \{\mathrm{Tr}_s[N_V^Y \exp(-u^2(D^Y)^2)] - \mathrm{Tr}_s[N_V^Y Q]\} \frac{\mathrm{du}}{\mathrm{u}}. \end{aligned} \tag{6.31}$$

From (6.23), (6.24), (6.31), we see that as $T_0 \to +\infty$

$$\begin{aligned} I_1^1 \to I_1^2 = &\int_\varepsilon^1 \mathrm{Tr}_s[N_V^Y \exp(-u^2(D^Y)^2)] \frac{du}{u} \\ &+ \int_1^{+\infty} \{\mathrm{Tr}_s[N_V^Y \exp(-u^2(D^Y)^2)] - \mathrm{Tr}_s[N_V^Y Q]\} \frac{du}{u}. \end{aligned} \tag{6.32}$$

γ) <u>$\varepsilon \to 0$</u>

Using Theorem 6.3, we find easily that as $\varepsilon \to 0$

$$\begin{aligned} I_1^2 &+ \frac{1}{2}\int_Y \frac{\omega^Y}{2\pi} \mathrm{Td}(TY)\,\mathrm{ch}(\eta)\left(1 - \frac{1}{\varepsilon^2}\right) \\ &+ \int_Y (\dim Y\, \mathrm{Td}(TY) - \mathrm{Td}'(TY))\,\mathrm{ch}(\eta)\,\mathrm{Log}(\varepsilon) \\ &\to I_1^3 = \int_0^1 \Big\{\mathrm{Tr}_s[N_V^Y \exp(-u^2(D^Y)^2)] - \frac{1}{u^2}\int_Y \frac{\omega^Y}{2\pi}\mathrm{Td}(TY)\,\mathrm{ch}(\eta) \\ &\quad - \int_Y (\dim Y\, \mathrm{Td}(TY) - \mathrm{Td}'(TY))\,\mathrm{ch}(\eta)\Big\} \frac{du}{u} \end{aligned} \tag{6.33}$$

$$+\int_1^{+\infty}\{\mathrm{Tr}_s[\mathrm{N}_\mathrm{V}^\mathrm{Y}\exp(-u^2(\mathrm{D}^\mathrm{Y})^2)]-\mathrm{Tr}_s[\mathrm{N}_\mathrm{V}^\mathrm{Y}\mathrm{Q}]\}\frac{du}{u}.$$

δ) Evaluation of I_1^3

Set $\mathrm{Q}^\perp=\mathrm{I}-\mathrm{Q}$. The analogue of formula (1.41) for $\theta_\eta^\mathrm{Y}(s)$ is

(6.34) $$\theta_\eta^\mathrm{Y}(s)=-\mathrm{Tr}_s[\mathrm{N}_\mathrm{V}^\mathrm{Y}[(\bar\partial^\mathrm{Y}+\bar\partial^{\mathrm{Y}*})^2]^{-s}\mathrm{Q}^\perp].$$

Theorem **6.11.** – *The following identity holds*

(6.35) $$\mathrm{I}_1^3=-\frac{1}{2}\Big\{\frac{\partial\theta_\eta^\mathrm{Y}}{\partial s}(0)-\int_\mathrm{Y}\frac{\omega^\mathrm{Y}}{2\pi}\mathrm{Td}(\mathrm{TY})\,\mathrm{ch}(\eta)$$
$$-\Gamma'(1)\Big(\int_\mathrm{Y}(\dim\mathrm{Y}\,\mathrm{Td}(\mathrm{TY})-\mathrm{Td}'(\mathrm{TY}))\,\mathrm{ch}(\eta)-\mathrm{Tr}_s[\mathrm{N}_\mathrm{V}^\mathrm{Y}\mathrm{Q}]\Big)\Big\}.$$

Proof. – For $s\in\mathbf{C}$, $\mathrm{Re}(s)>\sup_{1\leqslant j\leqslant d}(\dim\mathrm{Y}_j)$, then

(6.36) $$\theta_\eta^\mathrm{Y}(s)=-\frac{1}{\Gamma(s)}\int_0^{+\infty}u^{s-1}\{\mathrm{Tr}_s[\mathrm{N}_\mathrm{V}^\mathrm{Y}\exp(-u(\mathrm{D}^\mathrm{Y})^2)]-\mathrm{Tr}_s[\mathrm{N}_\mathrm{V}^\mathrm{Y}\mathrm{Q}]\}\,du.$$

So (6.35) is now a trivial consequence of (6.33), Theorem 6.3 and (6.36). □

2) *The term* I_2^0

The term I_2^0 is given by

(6.37) $$\mathrm{I}_2^0=\int_1^{\mathrm{T}_0}\mathrm{Tr}_s[\mathrm{N}_\mathrm{H}\exp(-\mathrm{A}^2(\mathrm{D}^\mathrm{X}+\mathrm{TV})^2)]\frac{d\mathrm{T}}{\mathrm{T}}.$$

α) $\mathrm{A}\to+\infty$

Clearly

(6.38) $$\mathrm{I}_2^0\to\mathrm{I}_2^1=\int_1^{\mathrm{T}_0}\mathrm{Tr}_s[\mathrm{N}_\mathrm{H}\tilde{\mathrm{P}}_\mathrm{T}]\frac{d\mathrm{T}}{\mathrm{T}}.$$

β) $\mathrm{T}_0\to+\infty$

By making $\alpha\to+\infty$ in Theorem 6.4, we find that for $\mathrm{T}\geqslant1$

(6.39) $$\Big|\mathrm{Tr}_s[\mathrm{N}_\mathrm{H}\tilde{\mathrm{P}}_\mathrm{T}]-\frac{1}{2}\dim\mathrm{N}\,\chi(\eta)\Big|\leqslant\frac{\mathrm{C}}{\sqrt{\mathrm{T}}}.$$

From (6.38), (6.39), we deduce that as $\mathrm{T}_0\to+\infty$

(6.40)
$$I_2^1 - \frac{1}{2}\dim N\,\chi(\eta)\,\mathrm{Log}(T_0) \to$$

$$I_2^2 = \int_1^{+\infty} (\mathrm{Tr}_s[N_H \tilde{P}_T] - \frac{1}{2}\dim N\,\chi(\eta))\frac{dT}{T}.$$

γ) $\underline{\varepsilon \to 0}$

I_2^2 remains constant and equal to I_2^3.

δ) Evaluation of I_2^3

Let $|\ |_{\lambda^{-1}(\eta)\otimes\tilde{\lambda}(\xi)}$ be the metric on the line $\lambda^{-1}(\eta)\otimes\tilde{\lambda}(\xi)$ which is the tensor product of the metrics $|\ |_{\lambda^{-1}(\eta)}$ and $|\ |_{\tilde{\lambda}(\xi)}$.

Theorem **6.12.** — *The following identity holds*

(6.41)
$$I_2^3 = -\frac{1}{2}\mathrm{Log}(|\rho|^2_{\lambda^{-1}(\eta)\otimes\tilde{\lambda}(\xi)}).$$

Proof. — For $T_0 \geqslant 1$, set

(6.42)
$$I_{2,T_0}^3 = \int_1^{T_0} \mathrm{Tr}_s[N_H \tilde{P}_T]\frac{dT}{T} - \frac{1}{2}\dim N\,\chi(\eta)\,\mathrm{Log}(T_0).$$

Clearly as $T_0 \to +\infty$

(6.43)
$$I_{2,T_0}^3 \to I_2^3.$$

Using (6.6), we find that

(6.44)
$$I_{2,T_0}^3 = \int_1^{T_0} \mathrm{Tr}_s[N_H P_T]\frac{dT}{T} - \frac{1}{2}\dim N\,\chi(\eta)\,\mathrm{Log}(T_0).$$

By Hodge Theory, the map $s\in K_1 \to P_T s \in K_T$ is the canonical isomorphism of K_1 with K_T (where these two finite dimensional **Z**-graded vector spaces are themselves identified with $H^*(E, \bar{\partial}^X + v)$). In particular, if $s\in K_1$, $1 < T < T'$, then

(6.45)
$$P_{T'}s = P_{T'}P_T s.$$

Using (6.45), we find that if $s\in K_1$, $s'\in K_1$, we get

(6.46)
$$\frac{\partial}{\partial T}\langle P_T s, P_T s'\rangle_T = \left\langle \frac{\partial P_T}{\partial T} P_T s, P_T s' \right\rangle_T$$
$$+ \left\langle P_T s, \frac{\partial P_T}{\partial T} P_T s' \right\rangle_T - \frac{2}{T}\langle N_H P_T s, P_T s'\rangle_T.$$

Since $P_T^2 = P_T$, then

$$\frac{\partial P_T}{\partial T} P_T + P_T \frac{\partial P_T}{\partial T} = \frac{\partial P_T}{\partial T}. \tag{6.47}$$

From (6.47), we find that $(\partial P_T/\partial T)$ maps K_T into its orthogonal $K_T^\perp$ with respect to the Hermitian product $\langle\ ,\ \rangle_T$. We thus rewrite (6.46) in the form

$$\frac{\partial}{\partial T} \langle P_T s, P_T s' \rangle_T = -\frac{2}{T} \langle N_H P_T s, P_T s' \rangle_T. \tag{6.48}$$

Using (6.48), we deduce easily that

$$\frac{\partial}{\partial T} \operatorname{Log}\left(\frac{|\ |^2_{\tilde{\lambda}(\xi), T}}{|\ |^2_{\tilde{\lambda}(\xi)}}\right) = \frac{2}{T} \operatorname{Tr}_s [N_H P_T]. \tag{6.49}$$

From (6.44), (6.49), we get

$$I^3_{2, T_0} = \frac{1}{2} \operatorname{Log}\left(\frac{|\ |^2_{\tilde{\lambda}(\xi), T_0}}{|\ |^2_{\tilde{\lambda}(\xi)}}\right) - \frac{1}{2} \dim N\, \chi(\eta) \operatorname{Log}(T_0). \tag{6.50}$$

Using Theorem 6.9 and (6.43), (6.50), we get (6.41). Our Theorem is proved. □

3) *The term* I_3^0

We have the identity

$$I_3^0 = -\int_\varepsilon^A \operatorname{Tr}_s [(N_V^X - N_H) \exp(-u^2 (D^X + V)^2)] \frac{du}{u}. \tag{6.51}$$

α) <u>A → +∞</u>

Clearly, as A → +∞

$$\begin{aligned} & I_3^0 + \operatorname{Tr}_s [(N_V^X - N_H) P] \operatorname{Log}(A) \to \\ & I_3^1 = -\int_\varepsilon^1 \operatorname{Tr}_s [(N_V^X - N_H) \exp(-u^2 (D^X + V)^2)] \frac{du}{u} \\ & \quad - \int_1^{+\infty} \{\operatorname{Tr}_s [(N_V^X - N_H) \exp(-u^2 (D^X + V)^2] \\ & \quad - \operatorname{Tr}_s [(N_V^X - N_H) P]\} \frac{du}{u}. \end{aligned} \tag{6.52}$$

β) <u>$T_0 \to +\infty$</u>

The term I_3^1 remains constant and equal to I_3^2.

γ) $\varepsilon \to 0$

By Theorem 6.3, we find that as $\varepsilon \to 0$, then

$$(6.53)\qquad I_3^2 + \frac{1}{2}\int_X \frac{\omega^X}{2\pi}\operatorname{Td}(TX)\operatorname{ch}(\xi)\left(\frac{1}{\varepsilon^2}-1\right)$$
$$-\int_X (\dim X \operatorname{Td}(TX)\operatorname{ch}(\xi) - \operatorname{Td}'(TX)\operatorname{ch}(\xi)$$
$$-\operatorname{Td}(TX)\operatorname{ch}'(\xi))\operatorname{Log}(\varepsilon) \to$$
$$I_3^3 = -\int_0^1 \Big\{\operatorname{Tr}_s[(N_V^X - N_H)\exp(-u^2(D^X+V)^2)]$$
$$-\frac{1}{u^2}\int_X \frac{\omega^X}{2\pi}\operatorname{Td}(TX)\operatorname{ch}(\xi) - \int_X (\dim X \operatorname{Td}(TX)\operatorname{ch}(\xi)$$
$$-\operatorname{Td}'(TX)\operatorname{ch}(\xi) - \operatorname{Td}(TX)\operatorname{ch}'(\xi))\Big\}\frac{du}{u}$$
$$-\int_1^{+\infty}\{\operatorname{Tr}_s[(N_V^X - N_H)\exp(-u^2(D^X+V)^2)] - \operatorname{Tr}_s[(N_V^X - N_H)P]\}\frac{du}{u}.$$

δ) Evaluation of I_3^3

Recall that the function $\tilde{\theta}_\xi^X(s)$ was defined in equation (1.49).

Theorem **6.13.** — *The following identity holds*

$$(6.54)\qquad I_3^3 = \frac{1}{2}\Big\{\frac{\partial\tilde{\theta}_\xi^X}{\partial s}(0) - \int_X \frac{\omega^X}{2\pi}\operatorname{Td}(TX)\operatorname{ch}(\xi)$$
$$-\Gamma'(1)\Big(\Big(\int_X (\dim X \operatorname{Td}(TX)\operatorname{ch}(\xi) - \operatorname{Td}'(TX)\operatorname{ch}(\xi)$$
$$-\operatorname{Td}(TX)\operatorname{ch}'(\xi)) - \operatorname{Tr}_s[(N_V^X - N_H)P]\Big)\Big\}.$$

Proof. — By using Theorem 6.3 and the analogue of (6.36) for $\tilde{\theta}_\xi^X(s)$, we obtain (6.54). □.

4) *The term* I_4^0

We have the identity

$$(6.55)\qquad I_4^0 = -\int_1^{T_0}\operatorname{Tr}_s[N_H\exp(-\varepsilon^2(D^X+TV)^2)]\frac{dT}{T}.$$

α) $\underline{A \to +\infty}$

I_4^0 remains constant and equal to I_4^1.

β) $\underline{T_0 \to +\infty}$

Using Theorem 6.4, we find that as $T_0 \to +\infty$

$$I_4^1 + \frac{1}{2} \dim N \chi(\eta) \operatorname{Log}(T_0) \to \tag{6.56}$$

$$I_4^2 = -\int_1^{+\infty} \left\{ \operatorname{Tr}_s [N_H \exp(-(\varepsilon D^X + \varepsilon TV)^2)] - \frac{1}{2} \dim N \chi(\eta) \right\} \frac{dT}{T}.$$

γ) $\underline{\varepsilon \to 0}$

We are finally reaching the problem at its heart. Set

$$J_1^0 = -\int_\varepsilon^1 \operatorname{Tr}_s [N_H \exp(-(\varepsilon D^X + TV)^2)] \frac{dT}{T}, \tag{6.57}$$

$$J_2^0 = -\int_\varepsilon^1 \operatorname{Tr}_s \left[N_H \exp\left(-\left(\varepsilon D^X + \frac{T}{\varepsilon} V \right)^2 \right) \right] \frac{dT}{T},$$

$$J_3^0 = -\int_1^{+\infty} \left\{ \operatorname{Tr}_s \left[N_H \exp\left(-\left(\varepsilon D^X + \frac{T}{\varepsilon} V \right)^2 \right) \right] - \frac{1}{2} \dim N \chi(\eta) \right\} \frac{dT}{T}.$$

Clearly

$$I_4^2 = J_1^0 + J_2^0 + J_3^0 - \dim N \chi(\eta) \operatorname{Log}(\varepsilon). \tag{6.58}$$

1. *The term* J_1^0

We have the identity

$$\begin{aligned} J_1^0 = &-\int_\varepsilon^1 \{ \operatorname{Tr}_s [N_H \exp(-(\varepsilon D^X + TV)^2)] - \operatorname{Tr}_s [N_H \exp(-(\varepsilon D^X)^2)] \} \frac{dT}{T} \\ &+ \operatorname{Tr}_s [N_H \exp(-(\varepsilon D^X)^2)] \operatorname{Log}(\varepsilon). \end{aligned} \tag{6.59}$$

Let D_i^X be the restriction of D^X to E_i. The McKean-Singer formula for $\chi(\xi_i)$ [MKS] asserts that for any $\varepsilon > 0$

$$\chi(\xi_i) = \operatorname{Tr}_s [\exp(-(\varepsilon D_i^X)^2)]. \tag{6.60}$$

Therefore

$$\operatorname{Tr}_s [N_H \exp(-(\varepsilon D^X)^2)] = \sum_0^m (-1)^i i \chi(\xi_i). \tag{6.61}$$

Using (4.4) and (6.8), we get

$$\operatorname{Tr}_s[N_H \exp(-(\varepsilon D^X)^2)] = \int_X \operatorname{Td}(TX)\,\operatorname{ch}'(\xi). \tag{6.62}$$

We now use again Theorem 6.6, which guarantees that the integrand in the integral in the right-hand side of (6.59) has a limit as $\varepsilon \to 0$, and also that the dominated convergence theorem can be used in the integral. Combining this with (6.62), we find that as $\varepsilon \to 0$

$$\begin{aligned} & J_1^0 - \int_0^1 \operatorname{Td}(TX)\,\operatorname{ch}'(\xi)\,\operatorname{Log}(\varepsilon) \to \\ & J_1^1 = -\int_0^1 \left\{ \int_X \operatorname{Td}(TX, g^{TX})\,\Phi\,\operatorname{Tr}_s[N_H(\exp(-C_{T^2}^2) - \exp(-C_0^2))] \right\} \frac{dT}{T}. \end{aligned} \tag{6.63}$$

2. *The term* J_2^0

We here make the crucial step of writing J_2^0 in the form

$$\begin{aligned} J_2^0 = & -\int_\varepsilon^1 \left\{ \operatorname{Tr}_s\left[N_H \exp\left(-\left(\varepsilon D^X + \frac{T}{\varepsilon} V \right)^2 \right) \right] \right. \\ & \left. - \int_X \operatorname{Td}(TX, g^{TX})\,\Phi\,\operatorname{Tr}_s[N_H \exp(-C_{(T/\varepsilon)^2}^2)] \right\} \frac{dT}{T} \\ & - \int_1^{1/\varepsilon} \left\{ \int_X \operatorname{Td}(TX, g^{TX})\,\Phi\,\operatorname{Tr}_s[N_H \exp(-C_{T^2}^2)] \right\} \frac{dT}{T}. \end{aligned} \tag{6.64}$$

By Theorem 6.6, there exist $C > 0$, $\gamma \in]0, 1]$ such that for $0 < \varepsilon \leqslant T \leqslant 1$

$$\begin{aligned} & \left| \operatorname{Tr}_s\left[N_H \exp\left(-\left(\varepsilon D^X + \frac{T}{\varepsilon} V \right)^2 \right) \right] \right. \\ & \left. - \int_X \operatorname{Td}(T^X, g^{TX})\,\Phi\,\operatorname{Tr}_s[N_H \exp(-C_{(T/\varepsilon)^2}^2)] \right| \\ & \leqslant C(\varepsilon + T)^\gamma \\ & \leqslant C(2T)^\gamma. \end{aligned} \tag{6.65}$$

We now combine Theorems 4.3, 5.9, 6.7 and the inequality (6.65), which guarantees that we can use the dominated convergence Theorem in the first integral in the right-hand side of (6.64). We thus find that as $\varepsilon \to 0$

$$J_2^0 + \int_Y i^*(\operatorname{Td}(TX))(\operatorname{Td}^{-1})'(N)\,\operatorname{ch}(\eta)\,\operatorname{Log}(\varepsilon) \to \tag{6.66}$$

$$J_2^1 = -\int_Y \int_0^1 \{\Phi \operatorname{Tr}_s[N_H \exp(-\mathscr{B}_{T^2}^2)]$$

$$+ i^*(\operatorname{Td}(TX, g^{TX}))(\operatorname{Td}^{-1})'(N, g^N)\} \frac{dT}{T} \operatorname{ch}(\eta, g^\eta)$$

$$-\int_1^{+\infty} \left\{ \int_X \operatorname{Td}(TX, g^{TX}) \Phi \operatorname{Tr}_s[N_H \exp(-C_{T^2}^2)] \right.$$

$$\left. + \int_Y i^*(\operatorname{Td}(TX, g^{TX}))(\operatorname{Td}^{-1})'(N, g^N) \operatorname{ch}(\eta, g^\eta) \right\} \frac{dT}{T}.$$

Of course Theorems 4.3 and 5.9 guarantee that the integrals in the right-hand side of (6.66) make sense.

3. *The term* J_3^0

Using Theorems 6.7, 6.8 and the dominated convergence Theorem, we find that as $\varepsilon \to 0$

$$(6.67) \qquad J_3^0 \to J_3^1 = -\int_1^{+\infty} \left\{ \left(\int_Y \Phi \operatorname{Tr}_s[N_H \exp(-\mathscr{B}_{T^2}^2)] \operatorname{ch}(\eta, g^\eta) \right) - \frac{1}{2} \dim N \chi(\eta) \right\} \frac{dT}{T}.$$

By Theorem 5.5 and by the index formula (6.8), we can rewrite J_3^1 in the form

$$(6.68) \qquad J_3^1 = -\int_Y \left(\int_1^{+\infty} \left\{ \Phi \operatorname{Tr}_s[N_H \exp(-\mathscr{B}_{T^2}^2)] - \frac{1}{2} \dim N \operatorname{Td}(TY, g^{TY}) \right\} \frac{dT}{T} \right) \operatorname{ch}(\eta, g^\eta).$$

4. *The asympotics of* I_4^2

By using (6.58), (6.63), (6.66), (6.68), we find that as $\varepsilon \to 0$

$$(6.69) \qquad I_4^2 + \left\{ \dim N \chi(\eta) - \int_X \operatorname{Td}(TX) \operatorname{ch}'(\xi) + \int_Y i^*(\operatorname{Td}(TX))(\operatorname{Td}^{-1})'(N) \operatorname{ch}(\eta) \right\} \operatorname{Log}(\varepsilon) \to$$

$$
\begin{aligned}
I_4^3 = & -\int_0^1 \left\{\int_X \operatorname{Td}(TX, g^{TX})\, \Phi \operatorname{Tr}_s [N_H (\exp(-C_{T^2}^2) - \exp(-C_0^2))]\right\} \frac{dT}{T} \\
& -\int_1^{+\infty} \left\{\int_X \operatorname{Td}(T^X, g^{TX})\, \Phi \operatorname{Tr}_s [N_H \exp(-C_{T^2}^2)] \right. \\
& \left. + \int_Y i^* (\operatorname{Td}(TX, g^{TX}))\, (\operatorname{Td}^{-1})' (N, g^N) \operatorname{ch}(\eta, g^\eta)\right\} \frac{dT}{T} \\
& -\int_Y \int_0^1 \{\Phi \operatorname{Tr}_s [N_H \exp(-\mathscr{B}_{T^2}^2)] \\
& + i^* (\operatorname{Td}(TX, g^{TX}))\, (\operatorname{Td}^{-1})' (N, g^N)\} \frac{dT}{T} \operatorname{ch}(\eta, g^\eta) \\
& -\int_Y \int_1^{+\infty} \left\{\Phi \operatorname{Tr}_s [N_H \exp(-\mathscr{B}_{T^2}^2)] \right. \\
& \left. -\frac{1}{2} \dim N \operatorname{Td}(TY, g^{TY})\right\} \frac{dT}{T} \operatorname{ch}(\eta, g^\eta).
\end{aligned}
$$

δ) Evaluation of I_4^3

We now use the notation of Sections 4 and 5.

Theorem **6.14.** – *The following identity holds*

$$
\begin{aligned}
I_4^3 = & -\frac{1}{2} \left\{\int_X \operatorname{Td}(TX, g^{TX})\, T(\xi, h^\xi) \right. \\
& + \int_Y \mathbf{B}(TY, TX|_Y, g^{TX|Y}) \operatorname{ch}(\eta, g^\eta) \\
& \left. + \Gamma'(1) \left(\int_X \operatorname{Td}(TX) \operatorname{ch}'(\xi) - \frac{1}{2} \dim N\, \chi(\eta) \right) \right\}.
\end{aligned}
\tag{6.70}
$$

Proof. – By using formulas (4.12) for $T(\xi, h^\xi)$, (5.21) for $\mathbf{B}(TY, TX|_Y, g^{TX|Y})$ and (6.69) for I_4^3 and also formula (6.8), we get (6.70). □

e) Matching the divergences

We now obtain the key identity which leads to the proof of Theorem 6.1.

Theorem **6.15.** — *The following identity holds*

$$\sum_{k=1}^{4} I_k^3 = 0. \tag{6.71}$$

Proof. — Recall that $\sum_{k=1}^{4} I_k^0 = 0$. The sum of the diverging terms which have been added at the three steps $A \to +\infty$, $T_0 \to +\infty$, $\varepsilon \to 0$ is then tautologically zero. The identity (6.71) then follows.

We will here directly verify that these diverging terms add up to zero. We will thus check that our previous calculations are correct. Also we will establish identities which will be useful when proving Theorem 6.16.

α) $\underline{A \to +\infty}$

By formulas (6.23), (6.52), which concern I_1^0, I_3^0, we must analyse the diverging term

$$(-\operatorname{Tr}_s[(N_V^X - N_H)\tilde{P}_{T_0}] + \operatorname{Tr}_s[(N_V^X - N_H)P])\operatorname{Log}(A). \tag{6.72}$$

Recall that $P = P_1$. Using (6.25), (6.29), we find that (6.72) is exactly zero. Therefore

$$\sum_{k=1}^{4} I_k^1 = 0. \tag{6.73}$$

β) $\underline{T_0 \to +\infty}$

By formulas (6.40), (6.56), which refer to I_2^1, I_4^1, we get the diverging terms

$$\left(-\frac{1}{2}\dim N\chi(\eta) + \frac{1}{2}\dim N\chi(\eta)\right)\operatorname{Log}(T_0) = 0. \tag{6.74}$$

From (6.73), (6.74) we find that

$$\sum_{k=1}^{4} I_k^2 = 0. \tag{6.75}$$

γ) $\underline{\varepsilon \to 0}$

We get a first sort of terms in formulas (6.33) and (6.53) concerning I_1^2, I_3^2,

$$\frac{1}{2}\left\{\int_Y \frac{\omega^Y}{2\pi}\operatorname{Td}(TY)\operatorname{ch}(\eta) - \int_X \frac{\omega^X}{2\pi}\operatorname{Td}(TX)\operatorname{ch}(\xi)\right\}\left(1 - \frac{1}{\varepsilon^2}\right). \tag{6.76}$$

Now the form ω^X is closed. Since $i^* \omega^X = \omega^Y$, using (4.13), it is clear that

$$\int_X \frac{\omega^X}{2\pi} \operatorname{Td}(TX) \operatorname{ch}(\xi) = \int_Y \frac{\omega^Y}{2\pi} \operatorname{Td}(TY) \operatorname{ch}(\eta). \tag{6.77}$$

So (6.76) is exactly zero.

Also by equations (6.33), (6.53), (6.69) which concern I_1^2, I_3^2, I_4^2, we get the diverging terms

$$\begin{aligned}\Big\{ \int_Y (\dim Y \operatorname{Td}(TY) - \operatorname{Td}'(TY)) \operatorname{ch}(\eta) - \int_X (\dim X \operatorname{Td}(TX) \operatorname{ch}(\xi) \\ - \operatorname{Td}'(TX) \operatorname{ch}(\xi) - \operatorname{Td}(TX) \operatorname{ch}'(\xi)) + \dim N \chi(\eta) \\ - \int_X \operatorname{Td}(TX) \operatorname{ch}'(\xi) + \int_Y i^*(\operatorname{Td}(TX)) (\operatorname{Td}^{-1})'(N) \operatorname{ch}(\eta) \Big\} \operatorname{Log}(\varepsilon).\end{aligned} \tag{6.78}$$

For $1 \leqslant j \leqslant d$, we have the identity

$$\dim Y_j + \dim N_j = \dim X. \tag{6.79}$$

Using formulas (6.8), (6.9) we find that

$$\begin{aligned}\int_Y \dim Y \operatorname{Td}(TY) \operatorname{ch}(\eta) + \dim N \chi(\eta) \\ = \sum_1^d (\dim Y_j + \dim N_j) \chi(\eta_j) \\ = \dim X \sum_1^d \chi(\eta_j) = \dim X \chi(\eta).\end{aligned} \tag{6.80}$$

Also $\chi(\xi) = \chi(\eta)$. Using (6.8) and (6.80), we find that

$$\int_Y \dim Y \operatorname{Td}(TY) \operatorname{ch}(\eta) - \int_X \dim X \operatorname{Td}(TX) \operatorname{ch}(\xi) + \dim N \chi(\eta) = 0. \tag{6.81}$$

By (4.13), we get

$$\int_X \operatorname{Td}'(TX) \operatorname{ch}(\xi) = \int_Y i^*(\operatorname{Td}'(TX)) \operatorname{Td}^{-1}(N) \operatorname{ch}(\eta). \tag{6.82}$$

We have the identities in $H^*(Y)$

$$\begin{aligned}\operatorname{Td}(TX|_Y) = \operatorname{Td}(TY) \operatorname{Td}(N), \\ \operatorname{Td}'(TX|_Y) = \operatorname{Td}'(TY) \operatorname{Td}(N) + \operatorname{Td}(TY) \operatorname{Td}'(N),\end{aligned} \tag{6.83}$$

$$(\mathrm{Td}^{-1})'(\mathrm{N}) = -\frac{\mathrm{Td}'(\mathrm{N})}{\mathrm{Td}^2(\mathrm{N})}.$$

From (6.82), (6.83), we deduce that

$$\int_{\mathrm{X}} \mathrm{Td}'(\mathrm{TX})\,\mathrm{ch}(\xi) + \int_{\mathrm{Y}} \{i^*(\mathrm{Td}(\mathrm{TX}))(\mathrm{Td}^{-1})'(\mathrm{N}) - \mathrm{Td}'(\mathrm{TY})\}\,\mathrm{ch}(\eta) = 0. \tag{6.84}$$

By (6.81), (6.84), (6.78) is exactly zero. We thus obtain (6.71). □

f) A formula for $\mathrm{Log}(\|\rho\|^2_{\lambda^{-1}(\eta)\otimes\tilde{\lambda}(\xi)})$

We now obtain an explicit expression for $\mathrm{Log}(\|\rho\|^2_{\lambda^{-1}(\eta)\otimes\tilde{\lambda}(\xi)})$.

Recall that $\hat{\mathrm{A}}(x) = (x/2)/\sinh(x/2)$. As in Section 5g), we identify the function $\hat{\mathrm{A}}'/\hat{\mathrm{A}}(x)$ with the corresponding additive genus.

Theorem **6.16.** — *The following identity holds*

$$\begin{aligned}\mathrm{Log}(\|\rho\|^2_{\lambda^{-1}(\eta)\otimes\tilde{\lambda}(\xi)}) = &-\int_{\mathrm{X}} \mathrm{Td}(\mathrm{TX}, g^{\mathrm{TX}})\,\mathrm{T}(\xi, h^{\xi}) \\ &-\int_{\mathrm{Y}} \mathbf{B}(\mathrm{TY}, \mathrm{TX}|_{\mathrm{Y}}, g^{\mathrm{TX}|\mathrm{Y}})\,\mathrm{ch}(\eta, g^{\eta}) \\ &+\Gamma'(1)\int_{\mathrm{Y}} \mathrm{Td}(\mathrm{TY})\frac{\hat{\mathrm{A}}'}{\hat{\mathrm{A}}}(\mathrm{N})\,\mathrm{ch}(\eta).\end{aligned} \tag{6.85}$$

Proof. — By Theorem 6.15, we know $\sum_{k=1}^{4} \mathrm{I}_k^3 = 0$. Using Theorems 6.11, 6.12, 6.13, 6.14, we find that

$$\begin{aligned}-\mathrm{Log}(|\rho|^2_{\lambda^{-1}(\eta)\otimes\tilde{\lambda}(\xi)}) &- \frac{\partial\theta^{\mathrm{Y}}_{\eta}}{\partial s}(0) + \frac{\partial\tilde{\theta}^{\mathrm{X}}_{\xi}}{\partial s}(0) - \int_{\mathrm{X}} \mathrm{Td}(\mathrm{TX}, g^{\mathrm{TX}})\,\mathrm{T}(\xi, h^{\xi}) \\ &-\int_{\mathrm{Y}} \mathbf{B}(\mathrm{TY}, \mathrm{TX}|_{\mathrm{Y}}, g^{\mathrm{TX}|\mathrm{Y}})\,\mathrm{ch}(\eta, g^{\eta}) \\ &+\int_{\mathrm{Y}} \frac{\omega^{\mathrm{Y}}}{2\pi}\mathrm{Td}(\mathrm{TY})\,\mathrm{ch}(\eta) - \int_{\mathrm{X}} \frac{\omega^{\mathrm{X}}}{2\pi}\mathrm{Td}(\mathrm{TX})\,\mathrm{ch}(\xi) \\ &+\Gamma'(1)\Big\{\int_{\mathrm{Y}} (\dim \mathrm{Y}\,\mathrm{Td}(\mathrm{TY}) - \mathrm{Td}'(\mathrm{TY}))\,\mathrm{ch}(\eta)\end{aligned} \tag{6.86}$$

$$-\operatorname{Tr}_s[N_V^Y Q]+\int_X(-\dim X\,\operatorname{Td}(TX)\operatorname{ch}(\xi)+\operatorname{Td}'(TX)\operatorname{ch}(\xi)$$
$$+\operatorname{Td}(TX)\operatorname{ch}'(\xi))+\operatorname{Tr}_s[(N_V^X-N_H)P]$$
$$-\int_X \operatorname{Td}(TX)\operatorname{ch}'(\xi)+\frac{1}{2}\dim N\,\chi(\eta)\Big\}=0.$$

By (1.42), (1.50),

$$\operatorname{Log}(|\rho|^2_{\lambda^{-1}(\eta)\otimes\tilde\lambda(\xi)})+\frac{\partial\theta_\eta^Y}{\partial s}(0)-\frac{\partial\tilde\theta_\xi^X}{\partial s}(0)=\operatorname{Log}(\|\rho\|^2_{\lambda^{-1}(\eta)\otimes\tilde\lambda(\xi)}). \tag{6.87}$$

We now use identities (6.25), (6.29), (6.77), (6.81), (6.82), (6.83), (6.86), (6.87), and we get

$$\operatorname{Log}(\|\rho\|^2_{\lambda^{-1}(\eta)\otimes\tilde\lambda(\xi)})=-\int_X \operatorname{Td}(X,g^{TX})\,T(\xi,h^\xi)$$
$$-\int_Y \mathbf{B}(TY,TX|_Y,g^{TX|Y})$$
$$+\Gamma'(1)\Big\{\int_Y \operatorname{Td}(TY)\frac{\operatorname{Td}'}{\operatorname{Td}}(N)\operatorname{ch}(\eta)-\frac{1}{2}\dim N\,\chi(\eta)\Big\}. \tag{6.88}$$

Since the Todd genus is multiplicative, the genus Td′/Td is additive. Moreover $\operatorname{Td}(x)=\hat{A}(x)e^{x/2}$, and so

$$\frac{\operatorname{Td}'}{\operatorname{Td}}(x)=\frac{\hat{A}'}{\hat{A}}(x)+\frac{1}{2}. \tag{6.89}$$

Therefore

$$\frac{\operatorname{Td}'}{\operatorname{Td}}(N)=\frac{\hat{A}'}{\hat{A}}(N)+\frac{1}{2}\dim N. \tag{6.90}$$

Using (6.8), (6.90), we find that

$$\int_Y \operatorname{Td}(TY)\frac{\operatorname{Td}'}{\operatorname{Td}}(N)\operatorname{ch}(\eta)-\frac{1}{2}\dim N\,\chi(\eta)$$
$$=\int_Y \operatorname{Td}(TY)\frac{\hat{A}'}{\hat{A}}(N)\operatorname{ch}(\eta). \tag{6.91}$$

From (6.88), (6.91), we get (6.85). □

We now will use the results of Bismut [B3] to give a more explicit formula for $\operatorname{Log}(\|\rho\|^2_{\lambda^{-1}(\eta)\otimes\tilde\lambda(\xi)})$.

Recall that $\widetilde{\mathrm{Td}}(\mathrm{TY}, \mathrm{TX}|_{\mathrm{Y}}, g^{\mathrm{TX}|_{\mathrm{Y}}})$ is the Bott-Chern class in $\mathrm{P}^{\mathrm{Y}}/\mathrm{P}^{\mathrm{Y},0}$ associated to the Todd genus Td and the exact sequence of holomorphic Hermitian vector bundles $0 \to \mathrm{TY} \to \mathrm{TX}|_{\mathrm{Y}} \to \mathrm{N} \to 0$. Also the additive genus D was defined in Section 5g).

Theorem **6.17.** — *The following identity holds*

$$\operatorname{Log}(\| \rho \|^2_{\lambda^{-1}(\eta) \otimes \tilde{\lambda}(\xi)}) = - \int_{\mathrm{X}} \mathrm{Td}(\mathrm{X}, g^{\mathrm{TX}})\, \mathrm{T}(\xi, h^{\xi}) + \int_{\mathrm{Y}} \mathrm{Td}^{-1}(\mathrm{N}, g^{\mathrm{N}})\, \widetilde{\mathrm{Td}}(\mathrm{TY}, \mathrm{TX}|_{\mathrm{Y}}, g^{\mathrm{TX}|_{\mathrm{Y}}})\, \mathrm{ch}(\eta, g^{\eta}) - \int_{\mathrm{Y}} \mathrm{Td}(\mathrm{TY}) \left(\mathrm{D} - \Gamma'(1) \frac{\hat{\mathrm{A}}'}{\hat{\mathrm{A}}} \right)(\mathrm{N})\, \mathrm{ch}(\eta). \tag{6.92}$$

Proof. — The form $\mathrm{ch}(\eta, g^{\eta})$ lies in P^{Y} and is ∂ and $\bar{\partial}$ closed. Therefore it can be paired with elements of $\mathrm{P}^{\mathrm{Y}}/\mathrm{P}^{\mathrm{Y},0}$. Using Theorem 5.16, we find that

$$\int_{\mathrm{Y}} \mathbf{B}(\mathrm{TY}, \mathrm{TX}|_{\mathrm{Y}}, g^{\mathrm{TX}|_{\mathrm{Y}}})\, \mathrm{ch}(\eta, g^{\eta}) = - \int_{\mathrm{Y}} \mathrm{Td}^{-1}(\mathrm{N}, g^{\mathrm{N}})\, \widetilde{\mathrm{Td}}(\mathrm{TY}, \mathrm{TX}|_{\mathrm{Y}}, g^{\mathrm{TX}|_{\mathrm{Y}}})\, \mathrm{ch}(\eta, g^{\eta}) + \int_{\mathrm{Y}} \mathrm{Td}(\mathrm{TY})\, \mathrm{ch}(\eta)\, \mathrm{D}(\mathrm{N}). \tag{6.93}$$

Equation (6.92) follows from (6.85) and (6.93). □

g) Proof of Theorem 6.1

The identity (6.1) immediately follows from Proposition 5.19 and Theorem 6.17. Also we have the identity in $\mathrm{H}^*(\mathrm{Y})$

$$\mathrm{Td}(\mathrm{TY}) = \frac{i^* \mathrm{Td}(\mathrm{TX})}{\mathrm{Td}(\mathrm{N})}, \qquad \mathrm{R}(\mathrm{N}) = i^* \mathrm{R}(\mathrm{TX}) - \mathrm{R}(\mathrm{TY}). \tag{6.94}$$

Using Theorem 4.7 and (6.94), we get

$$\int_{\mathrm{Y}} \mathrm{Td}(\mathrm{TY})\, \mathrm{R}(\mathrm{N})\, \mathrm{ch}(\eta) \tag{6.95}$$

$$= \int_X \mathrm{Td}(\mathrm{TX})\,\mathrm{R}(\mathrm{TX})\,\mathrm{ch}(\xi) - \int_Y \mathrm{Td}(\mathrm{TY})\,\mathrm{R}(\mathrm{TY})\,\mathrm{ch}(\eta).$$

So (6.1′) follows from (6.1) and from (6.95). The proof of Theorem 6.1 is completed. □

Remark **6.18.** – Take $a > 0$. Assume that in formulas (1.41), (1.49), and in their analogues on Y, the operators $\bar\partial^X$, $\bar\partial^Y$ are replaced by the operators $\sqrt{a}\,\bar\partial^X$, $\sqrt{a}\,\bar\partial^Y$. Let $\| \ \|_{\tilde\lambda(\xi),\,a}$, $\| \ \|_{\lambda(\eta),\,a}$, $\| \ \|_{\lambda^{-1}(\eta)\otimes\tilde\lambda(\xi),\,a}$ be the corresponding Quillen metrics on the lines $\tilde\lambda(\xi)$, $\lambda(\eta)$, $\lambda^{-1}(\eta)\otimes\tilde\lambda(\xi)$. By redoing the calculations in (6.22), (6.95), we find that

$$\mathrm{Log}(\|\rho\|^2_{\lambda^{-1}(\eta)\otimes\tilde\lambda(\xi),\,a}) = \mathrm{Log}(\|\rho\|^2_{\lambda^{-1}(\eta)\otimes\tilde\lambda(\xi)}) + \left(\int_Y \mathrm{Td}(\mathrm{TY})\frac{\mathrm{Td}'}{\mathrm{Td}}(\mathrm{N})\,\mathrm{ch}(\eta) - \dim \mathrm{N}\,\chi(\eta)\right)\mathrm{Log}(a), \tag{6.96}$$

$$\mathrm{Log}(\|\rho\|^2_{\lambda^{-1}(\eta)\otimes\tilde\lambda(\xi),\,a}) = \mathrm{Log}(\|\rho\|^2_{\lambda^{-1}(\eta)\otimes\tilde\lambda(\xi)}) + \left(\int_X \mathrm{Td}'(\mathrm{TX})\,\mathrm{ch}(\xi) - \int_Y \mathrm{Td}'(\mathrm{TY})\,\mathrm{ch}(\eta) - \dim \mathrm{X}\,\chi(\xi) + \dim(\mathrm{Y})\,\chi(\eta)\right)\mathrm{Log}(a) \tag{6.96'}$$

Equations (6.96), (6.96′) can also be obtained as a consequence of the general anomaly formula of [BGS3, Theorem 1.23].

VII - THE HODGE THEORY OF THE RESOLUTION OF A POINT

As pointed out in the Introduction, the Hodge theory of the resolution of the point $\{0\}$ in a complex Hermitian vector space plays an essential, if modest, role in our whole paper.

The purpose of this Section is in fact to recall the elementary results established in [B3] concerning the Dolbeault resolution of the Koszul complex on a complex Hermitian vector space V. In the next Sections, the results of this Section will be applied to the fibres of the normal bundle N to Y in X.

In a), we describe the Koszul complex of V, and in b), we discuss the Hodge theory of the associated Dolbeault double complex.

We use the notation of Sections 1a) and 5a). This Section is otherwise self-contained.

a) The resolution of a point in a complex vector space

We use the notation of Sections 1a) and 5a). In particular $V_{\mathbf{R}}$ is a real even dimensional vector space. Also **J** is a complex structure on $V_{\mathbf{R}}$, V is the associated complex vector space. Let n be the complex dimension of V. Recall that if $z \in V$, z represents $Z = z + \bar{z} \in V_{\mathbf{R}}$.

Let i be the embedding $\{0\} \to V$.

In the sequel, $\Lambda(V^*)$ will be considered as a **Z**-graded vector bundle on V. If $z \in V$, let i_z be the interior multiplication operator by z. Then i_z acts on $\Lambda(V^*)$. We now consider the holomorphic chain complex on V

$$(\Lambda V^*, \sqrt{-1}\, i_z) : 0 \to \Lambda^n(V^*) \xrightarrow[\sqrt{-1}\, i_z]{} \Lambda^{n-1}(V^*) \xrightarrow[\sqrt{-1}\, i_z]{} \dots \xrightarrow[\sqrt{-1}\, i_z]{} \Lambda^0(V^*) = \mathbf{C} \to 0. \tag{7.1}$$

Let r be the restriction map: $\alpha \in \Lambda^0(V^*)|_{\{0\}} \to \alpha \in \mathbf{C}$. Then by [GrH, p. 688], we have the exact sequence of sheaves

$$0 \to \mathcal{O}_V(\Lambda^n(V^*)) \xrightarrow[\sqrt{-1}\, i_z]{} \mathcal{O}_V(\Lambda^{n-1}(V^*)) \dots \xrightarrow[\sqrt{-1}\, i_z]{} \mathcal{O}_V \xrightarrow[r]{} i_* \mathbf{C} \to 0, \tag{7.2}$$

i.e. the complex $(\Lambda V^*, \sqrt{-1}\, i_z)$ resolves the sheaf $i_* \mathbf{C}$.

We can then apply the results of Sections 1b), c), d) in this situation. Let N_V, N_H be the operators which define the $\mathbf{Z}$-grading on $\Lambda(\bar{V}^*)$, $\Lambda(V^*)$ respectively. We define the $\mathbf{Z}$-grading on $\Lambda(\bar{V}^*) \hat{\otimes} \Lambda(V^*)$ by the operator $N_V - N_H$.

Let Γ be the set of smooth sections of $\Lambda(\bar{V}^*) \hat{\otimes} \Lambda(V^*)$ over V. Then Γ is also $\mathbf{Z}$-graded by the operator $N_V - N_H$. Let $\bar{\partial}$ be the standard Dolbeault operator acting on Γ. If $z \in V$, the operator i_z acts naturally on Γ, with the convention that if $Z \in V_{\mathbf{R}}$ is written in the form $Z = z + \bar{z}$, $z \in V$, then at $Z \in V_{\mathbf{R}}$, $i_z \alpha$ is exactly $i_z(\alpha(Z))$. Both operators $\bar{\partial}$ and $\sqrt{-1}\, i_z$ are odd and increase the total degree in Γ by one. Also $\bar{\partial} + \sqrt{-1}\, i_z$ is a chain map *i.e.*

$$(\bar{\partial} + \sqrt{-1}\, i_z)^2 = 0. \tag{7.3}$$

As in (1.31), we extend r to a linear map from Γ into $\mathbf{C}$. Namely, if α is a smooth section of $\Lambda^p(\bar{V}^*) \hat{\otimes} \Lambda^q(V^*)$, if $p+q > 0$, $r\alpha = 0$, and if $p = q = 0$, $r\alpha = \alpha(0) \in \mathbf{C}$.

The Dolbeault complex of $\{0\}$ is simply $\mathbf{C} \xrightarrow[\bar{\partial}^{\{0\}}]{} 0$. By [B3, Proposition 1.1], which is a special case of Theorem 1.7, the map $r: (\Gamma, \bar{\partial} + \sqrt{-1}\, i_z) \to (\mathbf{C}, \bar{\partial}^{\{0\}})$ is a quasi-isomorphism of $\mathbf{Z}$-graded complexes. Therefore the cohomology of the complex $(\Gamma, \bar{\partial} + \sqrt{-1}\, i_z)$ is concentrated in degree zero, and is one dimensional.

b) The Hodge theory of the complex $(\Gamma, \bar{\partial} + \sqrt{-1}\, i_z)$

We now recall the results of [B3, Section 1] which concern the Hodge theory of the complex $(\Gamma, \bar{\partial} + \sqrt{-1}\, i_z)$.

As in Section 1a), we assume that V is equipped with a Hermitian product. Let $dv_V(Z)$ be the volume form on $V_{\mathbf{R}}$. Let Γ^0 be the set of the square integrable sections of $\Lambda(\bar{V}^*) \hat{\otimes} \Lambda(V^*)$ over $V_{\mathbf{R}}$. We equip Γ^0 with the Hermitian product

$$\alpha, \beta \to \langle \alpha, \beta \rangle = \left(\frac{1}{2\pi}\right)^{\dim V} \int_{V_{\mathbf{R}}} \langle \alpha, \beta \rangle_{\Lambda(\bar{V}^*) \hat{\otimes} \Lambda(V^*)}\, dv_V. \tag{7.4}$$

The adjoint of the operator i_z is the operator $i_z^* = \bar{z} \wedge$. Let $\bar{\partial}^*$ be the formal adjoint of $\bar{\partial}$ with respect to the Hermitian product (7.4). Then $\bar{\partial}^* - \sqrt{-1}\, i_z^*$ is the formal adjoint of $\bar{\partial} + \sqrt{-1}\, i_z$. Also

$$(\bar{\partial}^* - \sqrt{-1}\, i_z^*)^2 = 0. \tag{7.5}$$

Recall that θ is the Kähler form on $V_{\mathbf{R}}$, so that if $X, Y \in V_{\mathbf{R}}$

$$\theta(X, Y) = \langle X, \mathbf{J} Y \rangle. \tag{7.6}$$

Then θ is a (1, 1) form on V or equivalently an element of $\Lambda^1(\bar{V}^*) \hat{\otimes} \Lambda^1(V^*)$, whose total degree is zero.

Let L be the operator

$$\alpha \in \Lambda(\bar{V}^*) \hat{\otimes} \Lambda(V^*) \to L\alpha = \theta \wedge \alpha \in \Lambda(\bar{V}^*) \hat{\otimes} \Lambda(V^*). \tag{7.7}$$

Let Λ be the adjoint of L.

Definition **7.1.** — S *denotes the operator in* $\mathrm{End}^{\mathrm{even}}(\Lambda(\bar{V}^*) \hat{\otimes} \Lambda(V^*))$

$$S = -(L + \Lambda) \tag{7.8}$$

Then S is a self-adjoint operator. It obviously acts on Γ and Γ^0. Let $e_1, \ldots, e_{2n}$ be an orthonormal base of $V_{\mathbf{R}}$. The Laplacian Δ of $V_{\mathbf{R}}$ is given by

$$\Delta = \sum_1^{2n} (\nabla_{e_i})^2$$

The operator Δ also acts on Γ.

The following result is proved in [B3, Proposition 1.4].

Proposition **7.2.** — *The following identities hold*

$$\begin{aligned} &(\bar{\partial} + \sqrt{-1}\, i_z + \bar{\partial}^* - \sqrt{-1}\, i_z^*)^2 = -\frac{\Delta}{2} + \frac{|Z|^2}{2} + S, \\ &S = \frac{\sqrt{-1}}{2} \sum_1^{2n} c(e_i)\, \hat{c}(\mathbf{J} e_i). \end{aligned} \tag{7.9}$$

Let $C_0^\infty(V_{\mathbf{R}})$ be the set of real smooth functions of $V_{\mathbf{R}}$ which have compact support. Let $\mathscr{L}$ be the differential operator

$$\mathscr{L} = \frac{1}{2}(-\Delta + |Z|^2 - 2n). \tag{7.10}$$

Then $\mathscr{L}$ is the harmonic oscillator on $V_{\mathbf{R}}$. By [GlJ, Theorem 1.5.1], we know that $\mathscr{L}$ is essentially self-adjoint on $C_0^\infty(V_{\mathbf{R}})$, that its closure is nonnegative and has compact resolvent. The spectrum of $\mathscr{L}$ is the set of nonnegative integers **N** and the kernel of $\mathscr{L}$ is one-dimensional and is spanned by the function $\exp(-(|Z|^2/2))$.

Let Γ_0 be the set of C^∞ sections of $\Lambda(\bar{V}^*) \hat{\otimes} \Lambda(V^*)$ with compact support. Clearly

$$\Gamma_0 = C_0^\infty(V_{\mathbf{R}}) \otimes (\Lambda(\bar{V}^*) \hat{\otimes} \Lambda(V^*)). \tag{7.11}$$

Also $\mathscr{L}$ acts as the operator $\mathscr{L} \otimes 1$ on Γ_0. By Proposition 7.2, we have the identity

$$(\bar{\partial} + \sqrt{-1}\, i_z + \bar{\partial}^* - \sqrt{-1}\, i_z^*)^2 = \mathscr{L} + S + n. \tag{7.12}$$

Therefore the operator $(\bar\partial+\sqrt{-1}\,i_z+\bar\partial^*-\sqrt{-1}\,i_z^*)^2$ is essentially self-adjoint on Γ_0, its closure is nonegative and has compact resolvent.

By definition, the form $\exp(\theta)$ is given by

$$\exp(\theta)=1+\frac{\theta}{1!}+\ldots+\frac{\theta^{\dim V}}{(\dim V)!}. \tag{7.13}$$

Since θ has total degree zero, $\exp(\theta)$ also has total degree zero.

Proposition **7.3.** — *The lowest eigenvalue of the self-adjoint operator* $S\in\operatorname{End}(\Lambda(\bar V^*)\hat\otimes\Lambda(V^*))$ *is equal to* $-n$, *and the corresponding eigenspace is spanned by the form* $\exp(\theta)$. *Also*

$$|\exp(\theta)|^2_{\Lambda(\bar V^*)\hat\otimes\Lambda(V^*)}=2^{\dim V}. \tag{7.14}$$

Proof. — Proposition 7.3 follows from [B3, Proposition 1.5], and [B3, eq. (1.25) and (1.31)]. □

Theorem **7.4.** — *Let* $\beta\in\Gamma^0$ *be given by*

$$\beta=\exp\left(\theta-\frac{|Z|^2}{2}\right). \tag{7.15}$$

Then β *has total degree zero, and also*

$$\left(\frac{1}{2\pi}\right)^{\dim V}\int_{V_{\mathbf{R}}}\langle\beta,\beta\rangle_{\Lambda(\bar V^*)\hat\otimes\Lambda(V^*)}\,dv_V=1. \tag{7.16}$$

Moreover β *spans the one-dimensional kernel of the operator*

$$(\bar\partial+\sqrt{-1}\,i_z+\bar\partial^*-\sqrt{-1}\,i_z^*)^2.$$

Finally

$$\begin{gathered}(\bar\partial+\sqrt{-1}\,i_z)\,\beta=0,\\ (\bar\partial^*-\sqrt{-1}\,i_z^*)\,\beta=0,\\ \left\langle\left(N_H-\frac{n}{2}\right)\beta,\beta\right\rangle=\left\langle\left(N_V-\frac{n}{2}\right)\beta,\beta\right\rangle=0.\end{gathered} \tag{7.17}$$

Proof. — Theorem 7.4 is proved in [B3, Theorem 1.6]. The fact that β spans the kernel of $(\bar\partial+\sqrt{-1}\,i_z+\bar\partial^*-\sqrt{-1}\,i_z^*)^2$ can also be derived from the considerations which follow Proposition 7.2 and from Proposition 7.3. Moreover (7.15) is a consequence of (7.14) and of the trivial

$$\int_{V_{\mathbf{R}}}\exp(-|Z|)^2\,dv_V=\pi^{\dim V}. \quad \square \tag{7.18}$$

Remark **7.5.** — Since β has total degree zero, then

$$(N_V - N_H)\,\beta = 0. \tag{7.19}$$

So (7.19) is compatible with the last identity in (7.17).

Remark **7.6.** — As pointed in [B3, Section 1d)], the L_2 cohomology of the complex $(\Gamma^0, \bar\partial + \sqrt{-1}\, i_z)$ is concentrated in degree zero and $H^0(\Gamma^0, \bar\partial + \sqrt{-1}\, i_z)$ is spanned by β. Also observe that

$$r\beta = 1. \tag{7.20}$$

Since $(\bar\partial + \sqrt{-1}\, i_z)\,\beta = 0$, β is a representative of the element in $H^0(\Gamma, \bar\partial + \sqrt{-1}\, i_z)$ which correspond to $1 \in \mathbf{C}$ via the quasi-isomorphism $r : (\Gamma, \bar\partial + \sqrt{-1}\, i_z) \to (\mathbf{C}, \bar\partial^{\{0\}})$. The form β is in fact the unique harmonic representative in Γ^0 of the corresponding cohomology class.

Observe that with the notation of Section 5a)

$$\sqrt{-1}\,(i_z - i_z^*) = \frac{\sqrt{-1}}{\sqrt{2}}\, \hat{c}(\mathbf{J}\,Z). \tag{7.21}$$

Let D be the operator

$$D = \bar\partial + \bar\partial^*. \tag{7.22}$$

We then have the identity

$$\bar\partial + \sqrt{-1}\, i_z + \bar\partial^* - \sqrt{-1}\, i_z^* = D + \frac{\sqrt{-1}}{\sqrt{2}}\, \hat{c}(\mathbf{J}\,Z). \tag{7.23}$$

VIII - A TAYLOR EXPANSION OF THE OPERATOR $D^X + TV$ NEAR Y

The purpose of the next two sections is to prove Theorems 6.4 and 6.5. In fact we state in Theorems 8.2 and 8.3 two very general results, from which Theorems 6.4 and 6.5 immediately follow.

Let L be a smooth section of $\text{End}^{\text{even}}(\Lambda(T^{*(0,1)}X) \hat{\otimes} \xi)$. Theorem 8.2 states that for $\alpha_0 > 0$, $\alpha \geqslant \alpha_0$, as $T \to +\infty$, $\text{Tr}_s[L \exp(-\alpha(D^X + TV)^2)]$ converges uniformly to a limit which can expressed in terms of the operator $D^Y = \bar{\partial}^Y + \bar{\partial}^{Y*}$ on Y. Theorem 8.3 is concerned with the determination of a uniform rate of convergence in $T \geqslant 1$ of $\text{Tr}_s[L \exp(-\alpha(D^X + TV)^2)]$ as $\alpha \to +\infty$. This last problem will be extensively dealt with in Section 9.

The proof of Theorem 8.2 is also delayed to Section 9. Still it relies heavily on the constructions of this Section, which consist of:

- The identification of a tubular neighborhood of Y in X with an open set in the total space of the normal bundle N to Y in X, by using geodesic coordinates normal to Y.
- The construction of an orthogonal splitting $\xi = \xi^+ \oplus \xi^-$ of ξ near Y, which preserves the **Z**-grading of ξ, which is stable under the action of V, and is such that $\xi^-|_Y = H(\xi, v)$. The construction of this splitting is taken from [B2].
- A trivialization of $\Lambda(T^{*(0,1)}X) \hat{\otimes} \xi$ near Y along geodesics normal to Y.
- The determination of the asymptotic expansion of the operator $D^X + TV$ in the considered trivialization when $T \to +\infty$ under the change of variables in the normal bundle N, $Z \to Z/\sqrt{T}$.
- The explicitation of remarkable algebraic properties of the asymptotic expansion of the operator $D^X + TV$, in which the harmonic oscillator considered in Section 7 and its kernel, which is spanned by β, play a key role. These algebraic considerations will ultimately explain why the limit as $T \to +\infty$ of $\text{Tr}_s[L \exp(-\alpha(D^X + TV)^2)]$ can

be evaluated in terms of the operator D^Y. Here the Kähler condition on the metric g^{TX} plays an essential role.

This Section is organized as follows. In a), we give our main assumptions and notation. In b), we state the two main results of Sections 8 and 9, and we derive Theorems 6.4 and 6.5 from them. In c), we express the operators D^X and D^Y as standard Dirac operators in the sense of Atiyah-Bott-Patodi [ABoP]. In d), we construct the holomorphic Hermitian connections on the vector bundles of the exact sequence $0 \to TY \to TX|_Y \to N \to 0$. In e), we construct the global normal geodesic coordinate system on X near Y. In f), we describe the splitting $\xi = \xi^+ \oplus \xi^-$ of ξ near Y. In g), we construct a trivialization of $\Lambda(T^{*(0,1)}X) \hat{\otimes} \xi$ near Y. In h), we obtain the Taylor expansion of the rescaled Dirac operator $D^X + TV$ near Y in the essential Theorem 8.18. Finally in i), we give remarkable algebraic properties of this Taylor expansion in Theorem 8.21.

The results of this Section will be used in Sections 9, 10 and 13. The splitting $\xi = \xi^+ \oplus \xi^-$ will reappear until Section 13.

a) Assumptions and notation

We make the same assumptions and we use the same notation as in Sections 1-6. When vector bundles have been canonically identified, we now consider them as being equal.

Recall that $H(\xi, v)$ is the **Z**-graded holomorphic vector bundle on Y, which is the homology of the complex $(\xi, v)|_Y$. To make our notation simpler, we will write H instead of $H(\xi, v)$. By equation (1.57)

$$H = \{f \in \xi|_Y;\ vf = 0;\ v^* f = 0\}. \tag{8.1}$$

Then H inherits its Hermitian metric h^H from the metric h^ξ on ξ.

Also by equation (1.56), we know that

$$H = \Lambda N^* \otimes \eta. \tag{8.2}$$

By assumption (A) in Section 1f), (8.2) is an identification of holomorphic **Z**-graded Hermitian vector bundles on Y.

Recall that N is identified with the orthogonal bundle to TY in $TX|_Y$. We thus have the identification of C^∞ vector bundles on Y

$$TX|_Y = TY \oplus N. \tag{8.3}$$

From (8.3), we deduce the identification of C^∞ vector bundles

$$\Lambda(T^{*(0,1)}X)|_Y = \Lambda(T^{*(0,1)}Y) \hat{\otimes} \Lambda(\bar{N}^*). \tag{8.4}$$

Using (8.2), (8.4), we find that $\Lambda(\bar{N}^*)\hat{\otimes}\Lambda(N^*)\otimes\eta$ is now a subvector bundle of $(\Lambda(T^{*(0,1)}X)\otimes\xi)|_Y$.

For $y\in Y$, let θ_y denote the Kähler form of the fiber $N_{\mathbf{R},y}$. If $\mathbf{J}$ is the complex structure of $N_{\mathbf{R}}$, if $Z, Z'\in N_{\mathbf{R},y}$, then

$$\theta(Z, Z')=\langle Z, \mathbf{J}Z'\rangle.$$

Also $\theta\in\Lambda^1(\bar{N}^*)\hat{\otimes}\Lambda^1(N^*)$. Similarly $\exp(\theta)\in\Lambda(\bar{N}^*)\hat{\otimes}\Lambda(N^*)$.

Definition **8.1.** — Let φ denote the linear map

$$\varphi: a\in\Lambda(T^{*(0,1)}Y)\otimes\eta\to\frac{a\exp(\theta)}{2^{(\dim N)/2}}\in\Lambda(T^{*(0,1)}X)|_Y\hat{\otimes}H. \tag{8.5}$$

Using (7.14), we find that φ is norm preserving.

Let q be the orthogonal projection from $(\Lambda(T^{*(0,1)}X)\hat{\otimes}\xi)|_Y$ on the image of φ.

b) The main Theorems

Let L be a smooth section of $\mathrm{End}^{\mathrm{even}}(\Lambda(T^{*(0,1)}X)\hat{\otimes}\xi)$ over X. Let $L|_Y$ be the restriction of L to Y.

Set

$$L^{\theta}=\varphi^{-1}qL|_Y q\varphi. \tag{8.6}$$

Then L^{θ} is a smooth section of $\mathrm{End}^{\mathrm{even}}(\Lambda(T^{*(0,1)}Y)\otimes\eta)$.

We now state the two essential results of this section.

Theorem **8.2.** — *For any* $\alpha_0>0$, *there exists* $C>0$ *such that for* $\alpha\geqslant\alpha_0$, $T\geqslant 1$

$$|\mathrm{Tr}[L\exp(-\alpha(D^X+TV)^2)]-\mathrm{Tr}[L^{\theta}\exp(-\alpha(D^Y)^2)]|\leqslant\frac{C}{\sqrt{T}}. \tag{8.7}$$

Theorem **8.3.** — *There exist constants* $c>0$, $C>0$ *such that for any* $\alpha\geqslant 1$, $T\geqslant 1$

$$|\mathrm{Tr}[L\exp(-\alpha(D^X+TV)^2)]-\mathrm{Tr}[L\tilde{P}_T]|\leqslant c\exp(-C\alpha). \tag{8.8}$$

Before we proceed, let us show how to deduce Theorems 6.4 and 6.5 from Theorems 8.2 and 8.3.

Proof of Theorem **6.4.** — The proof of Theorem 6.4 relies on the following simple result.

Proposition **8.4.** — *The following identities hold*

$$N_H^\theta = \frac{1}{2}\dim N, \qquad (8.9)$$
$$(N_V^X - N_H)^\theta = N_V^Y.$$

Proof. — If $a \in \Lambda(T^{*(0,1)}Y) \otimes \eta$, since η is identified with H_0, we get

$$\frac{N_H(a\exp(\theta))}{2^{(\dim N)/2}} = \frac{a\, N_H \exp(\theta)}{2^{(\dim N)/2}}. \qquad (8.10)$$

Now by the last equation in (7.17), or by a direct computation, we find that

$$\left\langle \frac{N_H \exp(\theta)}{2^{(\dim N)/2}}, \frac{\exp(\theta)}{2^{(\dim N)/2}} \right\rangle = \frac{1}{2}\dim N. \qquad (8.11)$$

From (8.11), we get the first line of (8.9). Similarly

$$(N_V^X - N_H)(a\exp(\theta)) = (N_V^Y a)\exp(\theta) + a(N_V^X - N_H)\exp(\theta). \qquad (8.12)$$

As pointed out after equation (7.13), $\exp(\theta)$ is of total degree zero in $\Lambda(\bar{N}^*) \hat{\otimes} \Lambda(N^*)$, *i.e.*

$$(N_V^X - N_H)\exp(\theta) = 0. \qquad (8.13)$$

The second line of (8.9) follows from (8.12), (8.13). □

Theorem 6.4 follows from Theorem 8.2 and from Proposition 8.4. In fact, in Theorem 8.2, since θ is even, q commutes with the involution defining the $\mathbf{Z}_2$-grading on $\Lambda(T^{*(0,1)}X) \hat{\otimes} \xi$. Therefore we can replace the traces Tr by supertraces Tr_s. The second line of (6.11) is now obvious. Also by the McKean-Singer formula [MKS], for $1 \leqslant j \leqslant d$, $\alpha > 0$,

$$\chi(\eta_j) = \mathrm{Tr}_s[\exp(-\alpha(D^{Y_j})^2)]. \qquad (8.14)$$

The first line of (6.11) trivially follows from Theorem 8.2, Proposition 8.4 and equation (8.14). □

Proof of Theorem **6.5.** — Let τ be the involution defining the $\mathbf{Z}_2$-grading on $\Lambda(T^{*(0,1)}X) \hat{\otimes} \xi$. Theorem 6.5 follows from Theorem 8.3, with $L = \tau(N_V^X - N_H)$.

c) The Dirac operators D^X and D^Y

Let ∇^{TX} (resp. ∇^{TY}) be the holomorphic Hermitian connection on TX (resp. TY). Since the metric g^{TX} (resp. g^{TY}) is Kähler, ∇^{TX} (resp. ∇^{TY}) induces on $T_{\mathbf{R}}X$ (resp. $T_{\mathbf{R}}Y$)

the corresponding Levi-Civita connection. Then ∇^{TX} (resp. ∇^{TY}) induces a unitary connection on $\Lambda(T^{*(0,1)}X)$ (resp. $\Lambda(T^{*(0,1)}Y)$) which we still note ∇^{TX} (resp. ∇^{TY}).

Let ∇^X (resp. ∇^Y) be the unitary connection on $\Lambda(T^{*(0,1)}X)\otimes\xi$ (resp. $\Lambda(T^{*(0,1)}Y)\otimes\eta$)

(8.15) $$\nabla^X=\nabla^{TX}\hat{\otimes}1+1\hat{\otimes}\nabla^\xi$$

(resp.

(8.15′) $$\nabla^Y=\nabla^{TY}\otimes 1+1\otimes\nabla^\eta).$$

We now use the notation of Section 5a). If $U\in T_{\mathbf{R}}X$, $c(U)$ acts as an odd operator on $\Lambda(T^{*(0,1)}X)$. Also $c(U)$ acts as $c(U)\hat{\otimes}1$ on $\Lambda(T^{*(0,1)}X)\hat{\otimes}\xi$.

Proposition **8.5.** – *Let* $e_1, \ldots, e_{2l}$ (resp. $e'_1, \ldots, e'_{2l'}$) *be an orthonormal base of* $T_{\mathbf{R}}X$ (resp. $T_{\mathbf{R}}Y$). *Then the following identity holds*

(8.16) $$D^X=\sum_1^{2l}\frac{c(e_i)}{\sqrt{2}}\nabla^X_{e_i}$$

(*resp.*

(8.16′) $$D^Y=\sum_1^{2l'}\frac{c(e'_i)}{\sqrt{2}}\nabla^Y_{e'_i}).$$

Proof. – Since the metric g^{TX} (resp. g^{TY}) is Kähler, (8.16) (resp. (8.16′)) follows from [Hi, p. 13].

d) The canonical exact sequence on Y

We now consider the exact sequence of holomorphic Hermitian vector bundles

$$0\to TY\to TX|_Y\to N\to 0.$$

Recall that N is identified with the orthogonal bundle to TY in $TX|_Y$. Let P^{TY}, P^N be the orthogonal projection operators from $TX|_Y$ on TY, N respectively.

The restriction of ∇^{TX} to $TX|_Y$ is exactly the holomorphic Hermitian connection $\nabla^{TX|_Y}$ on $TX|_Y$. Let ∇^N be the holomorphic Hermitian connection on N.

Proposition **8.6.** – *The following identities hold*

(8.17) $$\nabla^{TY}=P^{TY}\nabla^{TX|_Y},\qquad \nabla^N=P^N\nabla^{TX|_Y}.$$

Proof. – Proposition 8.6 is proved in [K, Propositions 6.4 and 6.6]. □

We now use the notation of Section 5b).

Definition **8.7.** – Let ${}^0\nabla^{TX|_Y} = \nabla^{TY} \oplus \nabla^N$ be the connection on $TX|_Y$ which is the direct sum of the connections ∇^{TY} and ∇^N. Set

$$A = \nabla^{TX|_Y} - {}^0\nabla^{TX|_Y}. \tag{8.18}$$

Then A is a 1-form on Y which takes values in skew-adjoint endomorphisms of $TX|_Y$ which exchange TY and N. Actually A defines the second fundamental form of Y.

We now recall the definition of the mean curvature vector [KN, p. 34].

Definition **8.8.** – If $e_1, \ldots, e_{2l'}$ is an orthonormal base of $T_{\mathbf{R}}Y$, set

$$\nu = \frac{1}{2l'} \sum_1^{2l'} A(e_i) e_i. \tag{8.19}$$

Then ν is a section of $N_{\mathbf{R}}$. Of course if $l'=0$, ν is by definition set equal to 0.

e) A coordinate system on X near Y

If $y \in Y$, $Z \in N_{\mathbf{R}, y}$, let $t \in \mathbf{R} \to x_t = \exp_y^X(tZ) \in X$ be the geodesic in X which is such that $x_0 = y$, $dx/dt|_{t=0} = Z$.

For $0 < \varepsilon < +\infty$, set

$$B_\varepsilon = \{Z \in N_{\mathbf{R}};\ |Z| < \varepsilon\}. \tag{8.20}$$

Since X and Y are compact, there exists $\varepsilon_0 > 0$ such that for $0 < \varepsilon < \varepsilon_0$, the map $(y, Z) \in N_{\mathbf{R}} \to \exp_y^X(Z) \in X$ is a diffeomorphism from B_ε on a tubular neighborhood $\mathcal{U}_\varepsilon$ of Y in X. From now on, we will identify B_ε with $\mathcal{U}_\varepsilon$. Also we will use the notation $x = (y, Z)$ instead of $x = \exp_y^X(Z)$. Finally we identify $y \in Y$ with $(y, 0) \in N_{\mathbf{R}}$.

Let dv_N denote the volume element of the fibers $N_{\mathbf{R}}$. Then $dv_Y(y)\, dv_{N_y}(Z)$ is a natural volume element on the total space of $N_{\mathbf{R}}$.

Let $k(y, Z)$ be the smooth positive function defined on B_{ε_0} by the equation

$$dv_X(y, Z) = k(y, Z)\, dv_Y(y)\, dv_{N_y}(Z). \tag{8.21}$$

The function k has a positive lower bound on $\mathcal{U}_{\varepsilon_0/2}$. Also if $y \in Y$

$$k(y) = 1. \tag{8.22}$$

Then $\partial k/\partial Z(y)$ lies in $N^*_{\mathbf{R}, y}$. We identify $\partial k/\partial Z(y)$ with an element of $N_{\mathbf{R}, y}$ by the metric.

Proposition **8.9.** – *The following identity holds*

$$\frac{\partial k}{\partial Z}(y) = -2(\dim Y)\,\nu. \tag{8.23}$$

Proof. – Let R^{TX} be the curvature of the connection ∇^{TX}. Take $y \in Y$, $Z \in N_{\mathbf{R}, y}$. Let D/Dt be the covariant differentiation operator along the geodesic $t \to (y, tZ)$ with respect to the connection ∇^{TX}. If J is a Jacobi field associated with the geodesic $t \to (y, tZ)$, then

$$\frac{D^2 J}{Dt^2} + R^{TX}(J, Z)Z = 0.$$

To calculate the Jacobian of the map $(x, Z) \to \exp_y^X(Z)$, we must consider two kinds of initial conditions:

- The initial condition

$$J_0 = 0, \ \frac{DJ_0}{Dt} \in N_{\mathbf{R}, y}.$$

This corresponds to infinitesimal displacements of the geodesic $t \to (y, tZ)$ where only Z varies.

- The initial condition

$$J_0 \in (T_{\mathbf{R}} Y)_y,$$
$$\frac{DJ_0}{Dt} = A(J_0) Z \in (T_{\mathbf{R}} Y)_y.$$

This corresponds to infinitesimal displacements of the geodesic $t \to (y, tZ)$, where $y \in Y$ moves in the direction J_0, and $Z \in N_{\mathbf{R}}$ is parallel with respect to the connection ∇^N.

We then easily deduce that if $e_1, \ldots, e_{2l'}$ is an orthonormal base of $(T_{\mathbf{R}} Y)_y$, for any $Z \in N_{\mathbf{R}, y}$

$$\left\langle \frac{\partial k}{\partial Z}(y), Z \right\rangle = \sum_1^{2l'} \langle A(e_i) Z, e_i \rangle. \tag{8.24}$$

Equivalently

$$\left\langle \frac{\partial k}{\partial Z}(y), Z \right\rangle = -\sum_1^{2l'} \langle A(e_i) e_i, Z \rangle. \tag{8.25}$$

So (8.23) follows from (8.25). □

f) A splitting of ξ near Y

Let $H^{\perp}$ be the orthogonal bundle to H in $\xi|_Y$. We now recall the construction in Bismut [B2, Section 3f)] of a splitting $\xi=\xi^{+}\oplus\xi^{-}$ near Y which extends the splitting $\xi|_Y=H\oplus H^{\perp}$.

By (8.1), we have the identity

$$H=\{f\in\xi|_Y;\ V^2 f=0\}. \tag{8.26}$$

For $y\in Y$, let $\mu(y)$ be the smallest nonzero eigenvalue of the self-adjoint nonnegative operator $V^2(y)$. Since H is a smooth vector bundle on Y, the function $y\in Y\to\mu(y)\in\mathbf{R}_+^*$ is continuous. Since Y is compact, the function μ has a positive lower bound $2b$ on Y.

We may and we will assume that $\varepsilon_0>0$ is small enough so that if $x\in\mathcal{U}_{\varepsilon_0}$, b is not an eigenvalue of $V^2(x)$.

Definition **8.10.** – For $0\leqslant k\leqslant m$, $x\in\mathcal{U}_{\varepsilon_0}$, $\xi_{k,x}^{-}$ (*resp.* $\xi_{k,x}^{+}$) denotes the direct sum of the eigenspaces of the restriction of $V^2(x)$ to $\xi_{k,x}$ corresponding to eigenvalues which are smaller (resp. larger) than b.

For $0\leqslant k\leqslant m$, the $\xi_{k,x}^{\pm}$'s are the fibres of smooth vector subbundles $\xi_k^{\pm}$ of ξ_k over $\mathcal{U}_{\varepsilon_0}$. Clearly on $\mathcal{U}_{\varepsilon_0}$, for $0\leqslant k\leqslant m$,

$$\xi_k=\xi_k^{+}\oplus\xi_k^{-}. \tag{8.27}$$

Set

$$\xi^{\pm}=\bigoplus_{k=0}^{m}\xi_k^{\pm};\ \xi_+^{\pm}=\bigoplus_{k\text{ even}}\xi_k^{\pm};\ \xi_-^{\pm}=\bigoplus_{k\text{ odd}}\xi_k^{\pm}. \tag{8.28}$$

In (8.27), (8.28), the various splittings are orthogonal. We equip ξ^{+}, ξ^{-} with the metrics $h^{\xi^{+}}$, $h^{\xi^{-}}$ induced by the metric h^{ξ}.

Then v, v^* and V preserve ξ^{+}, ξ^{-}. Let V^{+}, V^{-} be the restrictions of V to ξ^{+}, ξ^{-}. We will often write V in matrix form with respect to the splitting $\xi=\xi^{+}\oplus\xi^{-}$

$$V=\begin{bmatrix}V^{+} & 0\\ 0 & V^{-}\end{bmatrix}. \tag{8.29}$$

Clearly by (8.26), we have the identity of **Z**-graded vector bundles over Y

$$\xi^{-}|_Y=H. \tag{8.30}$$

From (8.2), (8.30), we get

$$\xi^{-}|_Y=\Lambda N^*\otimes\eta. \tag{8.31}$$

The equalities in (8.30), (8.31) also identify the metrics. From (8.30), we find $\xi_-|_Y$ is a **Z**-graded holomorphic Hermitian vector bundle on Y.

Let $P^{\xi^\pm}$ be the orthogonal projection operator from ξ on $\xi^\pm$. By (8.30), the restriction $P^{\xi^-}|_Y$ of P^{ξ^-} to Y is the orthogonal projection operator P^H from $\xi|_Y$ on H.

Let $\tilde{\nabla}^{\xi^\pm}$ be the Hermitian connection on $\xi^\pm$

$$\tilde{\nabla}^{\xi^\pm} = P^{\xi^\pm} \nabla^\xi. \tag{8.32}$$

Proposition **8.11.** – *The connection* $i^* \tilde{\nabla}^{\xi^-}$ *on* $\xi^-|_Y = H$ *is exactly the holomorphic Hermitian connection* ∇^H *on* H.

Proof. – This result is proved in [B2, Proposition 1.8]. □

Definition **8.12.** – Let $\tilde{\nabla}^\xi = \tilde{\nabla}^{\xi^+} \oplus \tilde{\nabla}^{\xi^-}$ be the connection on $\xi|_{\mathcal{U}_{\varepsilon_0}} = \xi^+ \oplus \xi^-$ which is the direct sum of the connections $\tilde{\nabla}^{\xi^+}$ and $\tilde{\nabla}^{\xi^-}$. Set

$$B = \nabla^\xi - \tilde{\nabla}^\xi. \tag{8.33}$$

Then B is a 1-form on $\mathcal{U}_{\varepsilon_0}$ which takes values in endomorphisms of ξ which interchange ξ^+ and ξ^-.

By Section 5a), if $Z \in N_{\mathbf{R}}$, $\hat{c}(Z)$ acts on $\Lambda(N^*)$. We assume that $\hat{c}(Z)$ acts like $\hat{c}(Z) \otimes 1$ on $\Lambda(N^*) \otimes \eta$.

Proposition **8.13.** – *If* $y \in Y$, $Z \in N_{\mathbf{R}, y}$, *the following identity holds*

$$\tilde{\nabla}_Z^{\xi^-} V^-(y) = \frac{\sqrt{-1}}{\sqrt{2}} \hat{c}(JZ). \tag{8.34}$$

Proof. – This result is proved in [B2, Section 1c) and Section 3j)]. We reproduce the proof for the sake of completeness.

Let ∇'^ξ be an arbitrary connection on ξ near Y which preserves the **Z**-grading of ξ. Let Γ be the connection form of ∇'^ξ in a given holomorphic trivialization of the complex (ξ, v) near $y \in Y$. If $U \in (T_{\mathbf{R}} X)_y$, then

$$\nabla_U'^\xi v(y) = \partial_U v(y) + [\Gamma_y(U), v(y)]. \tag{8.35}$$

Recall that $\partial_U v(y)$ acts on $H_y(\xi, v) = H_y$, and that this action only depends on the image Z of U in $N_{\mathbf{R}, y}$. From (8.35), we deduce that $\nabla_U'^\xi v(y)$ acts on H_y in the same way as $\partial_U v(y)$. In particular, if $Z \in N_{\mathbf{R}, y}$, we have the identity of operators acting on H_y

$$\nabla_Z'^\xi v(y) = \partial_Z v(y). \tag{8.36}$$

Now v and the connection $\tilde{\nabla}^{\xi}$ preserve ξ^-, and also $\xi^-|_Y = H$. From (8.35), (8.36), we deduce that

$$\tilde{\nabla}^{\xi^-}_Z v(y) = \partial_Z v(y). \tag{8.37}$$

If $z \in N_y$, if $Z = z + \bar{z}$, using (1.56), we rewrite (8.37) in the form

$$\tilde{\nabla}^{\xi^-}_Z v(y) = \sqrt{-1}\, i_z. \tag{8.38}$$

By (5.2), (8.38) is equivalent to

$$\tilde{\nabla}^{\xi^-}_Z v(y) = -\frac{\hat{c}(z)}{\sqrt{2}}. \tag{8.39}$$

Let f, g be smooth sections of ξ^- near $y_0 \in Y$. Since v, v^* act on ξ^-, and since the connection $\tilde{\nabla}^{\xi^-}$ is unitary, we find that

$$\langle \tilde{\nabla}^{\xi^-}_Z v(y) f, g \rangle = \langle f, \tilde{\nabla}^{\xi^-}_Z v^*(y) g \rangle. \tag{8.40}$$

Equivalently

$$(\tilde{\nabla}^{\xi^-}_Z v(y))^* = \tilde{\nabla}^{\xi^-}_Z v^*(y). \tag{8.41}$$

From (8.39), (8.41), we get

$$\tilde{\nabla}^{\xi^-}_Z v^*(y) = \frac{\hat{c}(\bar{z})}{\sqrt{2}}. \tag{8.42}$$

Using (8.39), (8.42), we find that

$$\tilde{\nabla}^{\xi^-}_Z V^-(y) = \frac{1}{\sqrt{2}}(-\hat{c}(z) + \hat{c}(\bar{z})), \tag{8.43}$$

which is equivalent to (8.34). □

We now state a simple but essential result.

Proposition **8.14.** — *There exists a constant* $C > 0$ *such that for any* $x = (y, Z) \in \mathcal{U}_{\varepsilon_0/2}$, $f \in \xi_x$, *then*

$$|V(x) f|^2 \geq C |Z|^2 |f|^2. \tag{8.44}$$

Proof. — This result is proved in [B2, Proposition 3.3]. We will here deduce Proposition 8.14 from Proposition 8.13. Recall that ξ^+ and ξ^- are orthogonal in ξ. Also the complex (ξ, v) is acyclic on $X \setminus Y$. It is then clear that there is a constant $C > 0$ such that if $f \in \xi^+$, (8.44) holds.

Take $y \in \mathrm{Y}$, $\mathrm{Z} \in \mathrm{N}_{\mathbf{R}, y}$. We identify $\xi^-_{(y, \mathrm{Z})}$ with ξ^-_y by parallel transport with respect to the connection $\tilde{\nabla}^{\xi^-}$ along the geodesic $t \to (y, t\mathrm{Z})$. This identification clearly preserves the metric. Then $\mathrm{V}^-(y, \mathrm{Z})$ acts on $\xi^-_y = (\Lambda \mathrm{N}^* \otimes \eta)_y$. Since V^- vanishes on Y, by Taylor's formula, we get

$$\text{(8.45)} \qquad \mathrm{V}^-(y, \mathrm{Z}) = \tilde{\nabla}^{\xi^-}_{\mathrm{Z}} \mathrm{V}(y) + O(|\mathrm{Z}|^2).$$

From (8.45), we deduce that if $f \in \xi^-_{y0}$, then

$$\text{(8.46)} \qquad |\mathrm{V}^-(y, \mathrm{Z}) f|^2 \geqslant \frac{1}{2} |\tilde{\nabla}^{\xi^-}_{\mathrm{Z}} \mathrm{V}(y) f|^2 - O(|\mathrm{Z}|^4) |f|^2.$$

We now use Proposition 8.13, and also the fact that

$$\text{(8.47)} \qquad (\hat{c}(\mathbf{J}\mathrm{Z}))^2 = -|\mathrm{Z}|^2.$$

From (8.46), (8.47) we then get

$$\text{(8.48)} \qquad |\mathrm{V}^-(y, \mathrm{Z}) f|^2 \geqslant \left(\frac{1}{4} |\mathrm{Z}|^2 - O(|\mathrm{Z}|^4)\right) |f|^2.$$

If $|\mathrm{Z}|$ is small enough, we deduce (8.44) from (8.48). Now since (ξ, v) is acyclic on $\mathrm{X} \setminus \mathrm{Y}$, V^- is invertible on $\mathcal{U}_{\varepsilon 0/2} \setminus \mathrm{Y}$. We thus obtain (8.44) for arbitrary $(y, \mathrm{Z}) \in \mathcal{U}_{\varepsilon 0/2}$. □

g) A trivialization of $\Lambda(\mathrm{T}^{*(0,1)}\mathrm{X}) \hat{\otimes} \xi$ along geodesics normal to Y

Take $x = (y, \mathrm{Z}) \in \mathcal{U}_{\varepsilon 0}$. We will identify ξ_x with ξ_y by parallel transport with respect to the connection $\tilde{\nabla}^{\xi}$ along the geodesic $t \to (y, t\mathrm{Z})$. Under this identification, $\xi^{\pm}_x$ is identified to $\xi^{\pm}_y$, and the identification preserves the metric and the **Z**-grading of ξ, $\xi^{\pm}$. Also if $x = (y, \mathrm{Z}) \in \mathcal{U}_{\varepsilon 0}$, $\mathrm{V}(x)$, $\mathrm{V}^+(x)$, $\mathrm{V}^-(x)$ act as self-adjoint operators on ξ_y, ξ^+_y, ξ^-_y respectively.

If $x = (y, \mathrm{Z}) \in \mathcal{U}_{\varepsilon 0}$, we identify $(\Lambda(\mathrm{T}^{*(0,1)}\mathrm{X})_x$ with $\Lambda(\mathrm{T}^{*(0,1)}\mathrm{X})_y$ by parallel transport with respect to the connection ∇^{TX} along the geodesic $t \to (y, t\mathrm{Z})$. This identification still preserves the metric and the **Z**-grading.

If $x = (y, \mathrm{Z}) \in \mathcal{U}_{\varepsilon 0}$, $(\Lambda(\mathrm{T}^{*(0,1)}\mathrm{X}) \hat{\otimes} \xi)_x$ is identified with $(\Lambda(\mathrm{T}^{*(0,1)}\mathrm{X}) \hat{\otimes} \xi)_y$, and this identification preserves the metric and the **Z**-gradings associated to the operators $\mathrm{N}^{\mathrm{X}}_{\mathrm{V}}$ and N_{H}.

h) A Taylor expansion of the operator $\mathrm{D}^{\mathrm{X}} + \mathrm{TV}$ near Y

Recall that $\tilde{\pi}$ is the canonical projection $\mathrm{N} \to \mathrm{Y}$.

Definition **8.15.** — Take $\alpha > 0$. Let $\mathbf{E}(\alpha)$ (resp. $\mathbf{E}$) be the set of smooth sections of $\tilde{\pi}^*((\Lambda(T^{*(0,1)}X) \hat{\otimes} \xi)|_Y)$ on B_α (resp. on the total space of $N_{\mathbf{R}}$).

If $f, g \in \mathbf{E}$ have compact support, set

$$\langle f, g\rangle = \left(\frac{1}{2\pi}\right)^{\dim X} \int_Y \left(\int_{N_{\mathbf{R}, y}} \langle f, g\rangle (y, Z)\, dv_N(Z)\right) dv_Y(y). \tag{8.49}$$

By using the construction of Section 8e), if $f \in \mathbf{E}$ has compact support in B_{ε_0}, we may and we will identify f with an element of E which has compact support in $\mathcal{U}_{\varepsilon_0}$. Still observe that in (8.21), k is in general not identically equal to 1, so this identification is not unitary with respect to the Hermitian products (1.38) and (8.49).

The holomorphic Hermitian connection ∇^N on N induces a splitting $TN = N \oplus T^H N$ of the tangent space to N, where $T^H N$ is the horizontal part of TN with respect to the connection ∇^N. If $U \in T_{\mathbf{R}} Y$, let U^H denote the horizontal lift of U in $T^H_{\mathbf{R}} N$, so that $U^H \in T^H_{\mathbf{R}} N$, $\tilde{\pi}_* U^H = U$.

Recall that the connection ${}^0\nabla^{TX|_Y}$ on $TX|_Y$ was defined in Definition 8.7. Then ${}^0\nabla^{TX|_Y}$ induces a connection on $\Lambda(T^{*(0,1)}X)|_Y$, which we still denote ${}^0\nabla^{TX|_Y}$. Let $\tilde{\nabla}^{\xi|_Y}$ be the restriction of the connection $\tilde{\nabla}^{\xi}$ to $\xi|_Y$. Let ${}^0\tilde{\nabla}^Y$ be the connection on $(\Lambda(T^{*(0,1)}X) \hat{\otimes} \xi)|_Y$

$${}^0\tilde{\nabla}^Y = {}^0\nabla^{TX|_Y} \hat{\otimes} 1 + 1 \hat{\otimes} \tilde{\nabla}^{\xi|_Y}. \tag{8.50}$$

The connection ${}^0\tilde{\nabla}^Y$ lifts to a connection on $\tilde{\pi}^*((\Lambda(T^{*(0,1)}X) \hat{\otimes} \xi)|_Y)$, which we still denote ${}^0\tilde{\nabla}^Y$.

Recall that $B = \nabla^{\xi} - \tilde{\nabla}^{\xi}$. If $y \in Y$, $U \in (T_{\mathbf{R}} X)_y$, $B_y(U)$ acts on ξ_y and preserves the $\mathbf{Z}$-grading of ξ_y. Therefore $B_y(U) \in \mathrm{End}^{\mathrm{even}}(\xi_y)$. Then $c(U) B_y(U)$ acts on $(\Lambda(T^{*(0,1)}X) \hat{\otimes} \xi)_y$. In what follows, we omit the subscript y in B_y.

Let $e_1, \ldots, e_{2l'}$ be an orthonormal base of $T_{\mathbf{R}} Y$, let $e_{2l'+1}, \ldots, e_{2l}$ be an orthonormal base of $N_{\mathbf{R}}$.

Definition **8.16.** — Let M, D^H, D^N be the operators acting on $\mathbf{E}$

$$\begin{aligned} M &= \sum_{i=1}^{2l} \frac{c(e_i)}{\sqrt{2}} B(e_i), \\ D^H &= \sum_{i=1}^{2l'} \frac{c(e_i)}{\sqrt{2}}\, {}^0\tilde{\nabla}^Y_{e_i^H}, \\ D^N &= \sum_{i=2l'+1}^{2l} \frac{c(e_i)}{\sqrt{2}}\, {}^0\tilde{\nabla}^Y_{e_i}. \end{aligned} \tag{8.51}$$

Clearly, D^N acts along the fibres of $N_{\mathbf{R}}$. Let $\bar{\partial}^N$ be the $\bar{\partial}$ operator along the fibres of $N_{\mathbf{R}}$, and let $\bar{\partial}^{N*}$ be its formal adjoint with respect to the Hermitian product (8.49).

Then one easily sees that

$$D^N = \bar{\partial}^N + \bar{\partial}^{N*}. \tag{8.52}$$

Also one verifies that M, D^H, D^N are self-adjoint with respect to the Hermitian product (8.49). It is crucial for D^H to be self-adjoint that the connection ${}^0\nabla^{TX|_Y}$ preserves the splitting $TX|_Y = TY \oplus N$.

Using the identification of $(\Lambda(T^{*(0,1)}X) \hat{\otimes} \xi)_{(y,Z)}$ with $(\Lambda(T^{*(0,1)}X) \hat{\otimes} \xi)_y$, we can now consider the connection ∇^X as a unitary connection on $\tilde{\pi}^*((\Lambda(T^{*(0,1)}X) \hat{\otimes} \xi)|_Y)$ over B_{ε_0}. Similarly D^X acts on $\mathbf{E}(\varepsilon_0)$.

The operator D^X is in general not self-adjoint with respect to the Hermitian product (8.49). However in view of (8.21), $k^{1/2} D^X k^{-1/2}$ acts as a self-adjoint operator on $\mathbf{E}(\varepsilon_0)$ with respect to the Hermitian product (8.49).

For $T \geqslant 1$, let Q_T be a first order differential operator acting on $\mathbf{E}(\varepsilon_0\sqrt{T})$. Then Q_T can be written in the form

$$Q_T = \sum_1^{2l'} a_i(T, y, Z)\, {}^0\tilde{\nabla}^Y_{e_i^H} + \sum_{2l'+1}^{2l} b_i(T, y, Z)\, {}^0\tilde{\nabla}^Y_{e_i} + c(T, y, Z), \tag{8.53}$$

where $a_i(T, y, Z)$, $b_i(T, y, Z)$, $c(T, y, Z)$ are endomorphisms of

$$\tilde{\pi}^*((\Lambda(T^{*(0,1)}X) \hat{\otimes} \xi)|_Y)$$

which depend smoothly on (y, Z).

Assume there exist constants $C > 0$, $p \in \mathbf{N}$ such that for any $T \geqslant 1$, $(y, Z) \in B_{\varepsilon_0\sqrt{T}}$, then

$$\begin{aligned} &|a_i(T, y, Z)| \leqslant C|Z|;\ 1 \leqslant i \leqslant 2l', \\ &|b_i(T, y, Z)| \leqslant C|Z|^2;\ 2l'+1 \leqslant i \leqslant 2l, \\ &|c(T, y, Z)| \leqslant C(|Z| + |Z|^p). \end{aligned} \tag{8.54}$$

We will then use the notation

$$Q_T = O(|Z|^2 \partial^N + |Z|\partial^H + |Z| + |Z|^p). \tag{8.55}$$

In (8.55), ∂^H and ∂^N represent horizontal and vertical differentiation operators respectively.

Definition **8.17.** — Let $T > 0$. If $f \in \mathbf{E}(\varepsilon_0)$, let $F_T f \in \mathbf{E}(\varepsilon_0\sqrt{T})$ be given by

$$(F_T f)(y, Z) = f\left(y, \frac{Z}{\sqrt{T}}\right);\ (y, Z) \in B_{\varepsilon_0\sqrt{T}}. \tag{8.56}$$

Take $y \in Y$, $Z \in N_{\mathbf{R}, y}$. Let $\tilde{D}/Dt$ denote the covariant differentiation operator along $t \to (y, tZ)$ with respect to the connection $\tilde{\nabla}^{\xi}$. In the sequel, we use the notation

$$\tilde{\nabla}^{\xi}_{Z} \tilde{\nabla}^{\xi}_{Z} V(y) = \left(\frac{\tilde{D}^2}{Dt^2} V(y, tZ) \right)_{t=0}.$$

So $\tilde{\nabla}^{\xi}_{Z} \tilde{\nabla}^{\xi}_{Z} V(y)$ is a quadratic function of $Z \in N_{\mathbf{R}, y}$.

The first essential result of this Section is as follows.

Theorem **8.18.** — *As* $T \to +\infty$, *then*

$$(8.57) \qquad \begin{aligned} F_T k^{1/2} D^X k^{-1/2} F_T^{-1} &= \sqrt{T} D^N + D^H + M \\ &\quad + \frac{1}{\sqrt{T}} O(|Z|^2 \partial^N + |Z| \partial^H + |Z|), \\ F_T V(y, Z) F_T^{-1} &= V^+(y) + \frac{1}{\sqrt{T}} \tilde{\nabla}^{\xi}_{Z} V(y) \\ &\quad + \frac{1}{2T} \tilde{\nabla}^{\xi}_{Z} \tilde{\nabla}^{\xi}_{Z} V(y) + \frac{1}{T^{3/2}} O(|Z|^3). \end{aligned}$$

In particular as $T \to +\infty$,

$$(8.58) \qquad \begin{aligned} F_T k^{1/2} (D^X + TV) k^{-1/2} F_T^{-1} &= TV^+(y) + \sqrt{T}(D^N + \tilde{\nabla}^{\xi}_{Z} V(y)) \\ + D^H + M + \frac{1}{2} \tilde{\nabla}^{\xi}_{Z} \tilde{\nabla}^{\xi}_{Z} V(y) &+ \frac{1}{\sqrt{T}} O(|Z|^2 \partial^N + |Z| \partial^H + |Z| + |Z|^3). \end{aligned}$$

Proof. — Since TX, ξ are identified with $\tilde{\pi}^*(TX|_Y)$, $\tilde{\pi}^*(\xi|_Y)$ over $\mathcal{U}_{\varepsilon_0}$, we can consider ∇^{TX}, ∇^{ξ} as unitary connections on $\tilde{\pi}^*(TX|_Y)$, $\tilde{\pi}^*(\xi|_Y)$ over $\mathcal{U}_{\varepsilon_0}$. Set

$$(8.59) \qquad \begin{aligned} \Gamma &= \nabla^{TX} - {}^0\nabla^{TX|_Y}, \\ \tilde{\Gamma} &= \nabla^{\xi} - \tilde{\nabla}^{\xi|_Y}. \end{aligned}$$

Now TX, ξ have been trivialized along the geodesics $t \to (y, tZ)$ by parallel transport with respect to the connections ∇^{TX}, $\tilde{\nabla}^{\xi}$. We then find that with the notation in (8.18), (8.33), if $y \in Y$

$$(8.60) \qquad \begin{aligned} \Gamma_y &= \tilde{\pi}^* A_y, \\ \tilde{\Gamma}_y &= B_y. \end{aligned}$$

We denote by Γ^{Λ} the action of Γ on $\tilde{\pi}^*(\Lambda(T^{*(0,1)}X)|_Y)$.

If $(y, Z) \in \mathcal{U}_{\varepsilon_0}$, $U \in (T_{\mathbf{R}}X)_y$, let $\tau U \in (T_{\mathbf{R}}X)_{(y, Z)}$ be the parallel transport of U with respect to the connection ∇^{TX} along the geodesic $t \to (y, tZ)$. Take $e_1, \ldots, e_{2l}$ as in

Definition 8.16. Using Proposition 8.5, we find that if $h \in \mathrm{E}(\varepsilon_0)$, then

$$\mathrm{D}^{\mathrm{X}} h(y, \mathrm{Z}) = \sum_1^{2l} \frac{c(e_i)}{\sqrt{2}} \nabla^{\mathrm{X}}_{\tau e_i} h(y, \mathrm{Z}). \tag{8.61}$$

Using (8.59), (8.61), we get

$$\mathrm{D}^{\mathrm{X}} = \sum_1^{2l} c(e_i)\, {}^0\tilde{\nabla}^{\mathrm{Y}}_{\tau e_i} + \sum_1^{2l} \frac{c(e_i)}{\sqrt{2}} \Gamma^{\Lambda}(\tau e_i) + \sum_1^{2l} \frac{c(e_i)}{\sqrt{2}} \tilde{\Gamma}(\tau e_i). \tag{8.62}$$

Let $y = (y^1, \ldots, y^{l'})$ be holomorphic system of coordinates on a neighborhood $\mathscr{V}$ of y_0 in Y. We assume that $\mathrm{N}_{\mathbf{R}}$ is trivialized over $\mathscr{V}$ so that

$$\tilde{\pi}^{-1}(\mathscr{V}) = \mathscr{V} \times \mathbf{R}^{2n}.$$

Set $\mathscr{W} = \mathrm{B}_{\varepsilon_0} \cap \tilde{\pi}^{-1}(\mathscr{V})$. The map $(y, \mathrm{Z}) \in \mathscr{W} \to \exp_y(\mathrm{Z}) \in \mathrm{X}$ identifies $\mathscr{W}$ with an open neighborhood of y in X, on which $\mathrm{T}_{\mathbf{R}}\mathrm{X}$ splits into

$$\mathrm{T}_{\mathbf{R}}\mathrm{X} = \mathbf{R}^{2l'} \oplus \mathbf{R}^{2n}. \tag{8.63}$$

Of course $\mathbf{R}^{2l'}$, $\mathbf{R}^{2n}$ are integrable subbundles of $\mathrm{T}_{\mathbf{R}}\mathrm{X}$ over $\mathscr{W}$. Also on $\mathscr{V}$, the splitting (8.63) coincides with the splitting

$$\mathrm{T}_{\mathbf{R}}\mathrm{X}|_{\mathrm{Y}} = \mathrm{T}_{\mathbf{R}}\mathrm{Y} \oplus \mathrm{N}_{\mathbf{R}}.$$

Let p_1, p_2 be the projection operators from $\mathrm{T}_{\mathbf{R}}\mathrm{X}$ on $\mathbf{R}^{2l'}$, $\mathbf{R}^{2n}$ respectively. Using (8.62), we find that

$$\begin{aligned} \mathrm{F}_{\mathrm{T}} k^{1/2} \mathrm{D}^{\mathrm{X}} k^{-1/2} \mathrm{F}_{\mathrm{T}}^{-1} = {} & \sqrt{\mathrm{T}} \sum_1^{2l} \frac{c(e_i)}{\sqrt{2}}\, {}^0\tilde{\nabla}^{\mathrm{Y}}_{p_2 \tau e_i(y, (\mathrm{Z}/\sqrt{\mathrm{T}}))} \\ & + \sum_1^{2l} \frac{c(e_i)}{\sqrt{2}}\, {}^0\tilde{\nabla}^{\mathrm{Y}}_{p_1 \tau e_i(y, (\mathrm{Z}/\sqrt{\mathrm{T}}))} + \sum_1^{2l} \frac{c(e_i)}{\sqrt{2}} \Gamma^{\Lambda}_{(y, (\mathrm{Z}/\sqrt{\mathrm{T}}))}(\tau e_i) \\ & + \sum_1^{2l} \frac{c(e_i)}{\sqrt{2}} \tilde{\Gamma}_{(y, (\mathrm{Z}/\sqrt{\mathrm{T}}))}(\tau e_i) - \frac{1}{2k(y, (\mathrm{Z}/\sqrt{\mathrm{T}}))} \sum_1^{2l} \frac{c(e_i)}{\sqrt{2}} (\nabla_{\tau e_i} k)\left(y, \frac{\mathrm{Z}}{\sqrt{\mathrm{T}}}\right). \end{aligned} \tag{8.64}$$

Recall that $e_1, \ldots, e_{2l'} \in \mathrm{T}_{\mathbf{R}}\mathrm{Y}$, $e_{2l'+1}, \ldots, e_{2l} \in \mathrm{N}_{\mathbf{R}}$. From (8.64), we find that as $\mathrm{T} \to +\infty$,

$$\mathrm{F}_T k^{1/2} \mathrm{D}^{\mathrm{X}} k^{-1/2} \mathrm{F}_{\mathrm{T}}^{-1} = \sqrt{\mathrm{T}} \sum_{2l'+1}^{2l} \frac{c(e_i)}{\sqrt{2}}\, {}^0\tilde{\nabla}^{\mathrm{Y}}_{e_i} \tag{8.65}$$

$$+\sum_1^{2l'} \frac{c(e_i)}{\sqrt{2}} {}^0\tilde{\nabla}^{\mathrm{Y}}_{e_i} + \sum_1^{2l} \frac{c(e_i)}{\sqrt{2}} {}^0\tilde{\nabla}^{\mathrm{Y}}_{(\partial/\partial t)(p_2 \tau e_i)(y, t\mathrm{Z})_{t=0}}$$

$$+\mathrm{M} + \sum_1^{2l} \frac{c(e_i)}{\sqrt{2}} \Gamma^{\wedge}_y(e_i) - \frac{1}{2} \sum_{2l'+1}^{2l} \frac{c(e_i)}{\sqrt{2}} \nabla_{e_i} k(y)$$

$$+\frac{1}{\sqrt{\mathrm{T}}} O(|\mathrm{Z}|^2 \partial^{\mathrm{N}} + |\mathrm{Z}| \partial^{\mathrm{H}} + |\mathrm{Z}|).$$

By (8.60), $\Gamma_y = \tilde{\pi}^* \mathrm{A}_y$ and so

$$\sum_1^{2l} \frac{c(e_i)}{\sqrt{2}} \Gamma^{\wedge}_y(e_i) = \sum_1^{2l'} \frac{c(e_i)}{\sqrt{2}} \mathrm{A}^{\wedge}_y(e_i). \tag{8.66}$$

Also one easily verifies that if $\mathrm{U} \in (\mathrm{T}_{\mathbf{R}} \mathrm{Y})_y$, then

$$\mathrm{A}^{\wedge}_y(\mathrm{U}) = \frac{1}{4} \sum_{1 \leq j, k \leq 2l} \langle \mathrm{A}_y(\mathrm{U}) e_j, e_k \rangle c(e_j) c(e_k) + \frac{1}{2} \mathrm{Tr}[\mathrm{A}_y(\mathrm{U})]. \tag{8.67}$$

Now for $\mathrm{U} \in \mathrm{T}_{\mathbf{R}} \mathrm{Y}$, $\mathrm{A}_y(\mathrm{U})$ exchanges $\mathrm{T}_y \mathrm{Y}$ and N_y. In particular $\mathrm{Tr}[\mathrm{A}_y(\mathrm{U})] = 0$, and so

$$\mathrm{A}^{\wedge}_y(\mathrm{U}) = \frac{1}{2} \sum_{\substack{1 \leq j \leq 2l' \\ 2l'+1 \leq k \leq 2l}} \langle \mathrm{A}_y(\mathrm{U}) e_j, e_k \rangle c(e_j) c(e_k). \tag{8.68}$$

From (8.66), (8.68), we deduce that

$$\sum_1^{2l} \frac{c(e_i)}{\sqrt{2}} \Gamma^{\wedge}_y(e_i) = -\frac{1}{2\sqrt{2}} \left\langle \sum_1^{2l'} \mathrm{A}_y(e_i) e_i, e_k \right\rangle \mathrm{c}(\mathrm{e}_k) \tag{8.69}$$

$$+\frac{1}{4\sqrt{2}} \sum_{\substack{1 \leq i, j \leq 2l' \\ 2l'+1 \leq k \leq 2l}} \langle \mathrm{A}_y(e_i) e_j - \mathrm{A}_y(e_j) e_i, e_k \rangle c(e_i) c(e_j) c(e_k).$$

Since the connection ∇^{TX} is torsion free, if $\mathrm{U}, \mathrm{V} \in (\mathrm{T}_{\mathbf{R}} \mathrm{Y})_y$

$$\mathrm{A}_y(\mathrm{U}) \mathrm{V} - \mathrm{A}_y(\mathrm{V}) \mathrm{U} = 0. \tag{8.70}$$

So from (8.19), (8.69), (8.70), we get

$$\sum_1^{2l} \frac{c(e_i)}{\sqrt{2}} \Gamma^{\wedge}_y(e_i) = -\frac{\dim \mathrm{Y}}{\sqrt{2}} c(\mathrm{v}_y). \tag{8.71}$$

Also by Proposition 8.9, we know that

$$-\frac{1}{2}\sum_{2l'+1}^{2l}\frac{c(e_i)}{\sqrt{2}}\nabla_{e_i}k(y)=\frac{\dim \mathrm{Y}}{\sqrt{2}}c(\mathrm{v}_y). \tag{8.72}$$

So from (8.65), (8.71), (8.72), we find that as $\mathrm{T}\to+\infty$

$$\begin{aligned}\mathrm{F_T}k^{1/2}\mathrm{D^X}k^{-1/2}\mathrm{F_T^{-1}}=\sqrt{\mathrm{T}}\sum_{2l'+1}^{2l}\frac{c(e_i)}{\sqrt{2}}\,{}^0\tilde{\nabla}^{\mathrm{Y}}_{e_i}\\+\sum_1^{2l'}\frac{c(e_i)}{\sqrt{2}}\,{}^0\tilde{\nabla}^{\mathrm{Y}}_{e_i}+\sum_1^{2l}\frac{c(e_i)}{\sqrt{2}}\tilde{\nabla}^{\mathrm{Y}}_{(\partial/\partial t)(p_2\tau e_i)(y,\,t\mathrm{Z})_{t=0}}\\+\mathrm{M}+\frac{1}{\sqrt{\mathrm{T}}}O(|\mathrm{Z}|^2\partial^{\mathrm{N}}+|\mathrm{Z}|\partial^{\mathrm{H}}+|\mathrm{Z}|).\end{aligned} \tag{8.73}$$

Let C be the Christoffel symbols of the connection ∇^{TX} over $\mathrm{T_{\mathbf{R}}X}$ in the trivialization (8.63) of $\mathrm{T_{\mathbf{R}}X}$. Since τe_i is the parallel transport of e_i along $t\to(y,t\mathrm{Z})$, then

$$\frac{\partial}{\partial t}(\tau e_i)(y,t\mathrm{Z})=-\mathrm{C}_{(y,t\mathrm{Z})}(\mathrm{Z})\tau e_i(y,t\mathrm{Z}). \tag{8.74}$$

Since ∇^{TX} is torsion free, if $\mathrm{U},\mathrm{V}\in\mathrm{T_{\mathbf{R}}X}\simeq\mathbf{R}^{2l}$, then $\mathrm{C(U)V=C(V)U}$. We can then rewrite (8.74) in the form

$$\frac{\partial}{\partial t}\tau e_i(y,t\mathrm{Z})=-\mathrm{C}_{(y,t\mathrm{Z})}(\tau e_i)(y,t\mathrm{Z})\mathrm{Z}. \tag{8.75}$$

Since the curve $t\to(y,t\mathrm{Z})$ is a geodesic, then

$$\mathrm{C}_{(y,t\mathrm{Z})}(\mathrm{Z})\mathrm{Z}=0. \tag{8.76}$$

In particular for any $y\in\mathscr{V}$, $\mathrm{Z}\in\mathrm{N}_{\mathbf{R},y}\simeq\mathbf{R}^{2n}$, then

$$\mathrm{C}_y(\mathrm{Z})\mathrm{Z}=0. \tag{8.77}$$

Since the connection ∇^{TX} is torsion free, we deduce from (8.77) that if $\mathrm{Z},\mathrm{Z}'\in\mathrm{N}_{\mathbf{R},y}$

$$\mathrm{C}_y(\mathrm{Z})\mathrm{Z}'=0. \tag{8.78}$$

Using (8.75), (8.78), we find that for $1\leqslant i\leqslant 2l$

$$\frac{\partial}{\partial t}(\tau e_i)(y,t\mathrm{Z})_{t=0}=-\mathrm{C}_y(p_1e_i)\mathrm{Z}, \tag{8.79}$$

and so

$$\frac{\partial}{\partial t} p_2 \tau e_i(y, tZ)_{t=0} = -p_2 C_y(p_1 e_i) Z. \tag{8.80}$$

Using (8.51), (8.73), (8.80), we see that as $T \to +\infty$,

$$\begin{aligned} F_T k^{1/2} D^X k^{-1/2} F_T^{-1} = \sqrt{T} D^N + \sum_1^{2l'} \frac{c(e_i)}{\sqrt{2}} {}^0\tilde{\nabla}^Y_{e_i - p_2 C_y(e_i) Z} \\ + M + \frac{1}{\sqrt{T}} O(|Z|^2 \partial^N + |Z| \partial^H + |Z|). \end{aligned} \tag{8.81}$$

If $Z \in \mathbf{R}^{2n}$ is considered as a constant vector field on $\mathbf{R}^{2l'} \oplus \mathbf{R}^{2n}$, for $1 \leqslant i \leqslant 2l'$, we have the identity

$$\nabla^{TX|_Y}_{e_i} Z = C(e_i) Z. \tag{8.82}$$

By Proposition 8.6, $\nabla^N = P^N \nabla^{TX|_Y}$. So from (8.82), we get

$$\nabla^N_{e_i} Z = p_2 C(e_i) Z. \tag{8.83}$$

Using (8.83), we find that for $1 \leqslant i \leqslant 2l'$

$$e_i^H(Z) = e_i - p_2 C(e_i) Z. \tag{8.84}$$

From (8.81), (8.84), we get the first line of (8.57).

Since $V^-(y) = 0$, the second line in (8.57) is obvious by Taylor expansion.

Our Theorem is proved. □

i) The projection of the operator $D^H + M + (1/2) \tilde{\nabla}^\xi_Z \tilde{\nabla}^\xi_Z V(y)$.

Definition **8.19.** — Let $\mathbf{E}^\pm$ be the set of smooth sections of $\tilde{\pi}^*((\Lambda(T^{*(0,1)}X) \hat{\otimes} \xi^\pm)|_Y)$ on the total space of $N_{\mathbf{R}}$.

Then $\mathbf{E}$ splits into

$$\mathbf{E} = \mathbf{E}^+ \oplus \mathbf{E}^-. \tag{8.85}$$

The operators D^H and D^N preserve $\mathbf{E}^+$ and $\mathbf{E}^-$. Let $D^{H,\pm}$, $D^{N,\pm}$ be the restrictions of D^H, D^N to $\mathbf{E}^\pm$.

Let $\mathbf{E}^0$, $\mathbf{E}^{\pm,0}$ be the Hilbert spaces of square integrable sections of $\tilde{\pi}^*((\Lambda(T^{*(0,1)}X) \hat{\otimes} \xi)|_Y)$, $\tilde{\pi}^*((\Lambda(T^{*(0,1)}X) \hat{\otimes} \xi^\pm)|_Y)$. We equip $\mathbf{E}^0$, $\mathbf{E}^{\pm,0}$ with the

Hermitian product (8.49). Then $\mathbf{E}^0$ splits orthogonally into

(8.86) $$\mathbf{E}^0 = \mathbf{E}^{+,0} \oplus \mathbf{E}^{-,0}.$$

Let F^0 be the Hilbert space of square integrable sections of $\Lambda(T^{*(0,1)}Y) \otimes \eta$ over Y. We equip F^0 with the Hermitian product constructed in (1.44), (1.45).

Using (8.4), (8.31), we have the identity

(8.87) $$(\Lambda(T^{*(0,1)}X) \hat{\otimes} \xi^-)|_Y = \Lambda(T^{*(0,1)}Y) \hat{\otimes} \Lambda(\bar{N}^*) \hat{\otimes} \Lambda(N^*) \otimes \eta.$$

We now use the notation of Sections 7 and 8 a). In particular, if $y \in Y$, θ_y denote the Kähler form of the fibre $N_{\mathbf{R},y}$. If $y \in Y$, $Z \in N_{\mathbf{R},y}$, set

(8.88) $$\beta_y = \exp\left(\theta_y - \frac{|Z|^2}{2}\right).$$

Definition **8.20.** – Let ψ be the linear map

(8.89) $$\psi : \sigma \in F^0 \to \sigma\beta \in \mathbf{E}^0.$$

Let $\mathbf{E}'^{,0}$ be the image of F^0 by ψ in $\mathbf{E}^0$. Then $\mathbf{E}'^{,0} \subset \mathbf{E}^{-,0}$.

By Theorem 7.4, it is clear that ψ is an isometry. Using the notation of Definition 8.1, we find that if $\sigma \in F^0$

(8.90) $$\psi\sigma = 2^{\dim N/2} \exp\left(\frac{-|Z|^2}{2}\right) \varphi\sigma.$$

Let p be the orthogonal projection operator from $\mathbf{E}^0$ on $\mathbf{E}'^{,0}$. Recall that q is the orthogonal projection operator from $(\Lambda(T^{*(0,1)}X) \hat{\otimes} \xi)|_Y$ on $\Lambda(T^{*(0,1)}Y) \hat{\otimes} \{\exp(\beta)\} \otimes \eta$. One then finds easily that if $s \in \mathbf{E}^0$,

(8.91) $$ps(y, Z) = \frac{1}{\pi^{\dim N}} \exp\left(\frac{-|Z|^2}{2}\right) q \int_{N_{\mathbf{R},y}} \exp\left(\frac{-|Z'|^2}{2}\right) s(y, Z')\, dv_N(Z').$$

Recall that P^{ξ^-} is the orthogonal projection operator from ξ on ξ^-. The operator $P^{\xi^-}|_Y$ acts like $1 \hat{\otimes} P^{\xi^-}|_Y$ on $(\Lambda(T^{*(0,1)}X) \hat{\otimes} \xi)|_Y$.

Clearly $(\Lambda(T^{*(0,1)}X) \hat{\otimes} \xi^-)|_Y$ is the kernel of $V^+|_Y$. Similarly by using Theorem 7.4, equation (7.23), and Proposition 8.13, we find that $\mathbf{E}'^{,0}$ is exactly the kernel of the operator $D^{N,-} + \tilde{\nabla}_Z^{\xi^-} V^-(y)$. In view of Theorem 8.18, this hierarchy of kernels will be of **utmost importance** in the whole paper.

The second essential result, which is complementary to Theorem 8.18, is as follows.

Theorem **8.21.** — *The following identities of operators in* $\mathrm{End}((\Lambda(T^{*(0,1)}X)\hat{\otimes}\xi)|_Y)$ *hold*

$$P^{\xi^-|_Y} M P^{\xi^-|_Y} = 0, \qquad q\tilde{\nabla}^{\xi}_Z \tilde{\nabla}^{\xi}_Z V(y)\, q = 0. \tag{8.92}$$

The following identity of operators in End(F) *holds*

$$\psi^{-1} p\, D^H p\, \psi = D^Y. \tag{8.93}$$

In particular

$$\psi^{-1} p \left(D^H + M + \frac{1}{2}\tilde{\nabla}^{\xi}_Z \tilde{\nabla}^{\xi}_Z V(y)\right) p\, \psi = D^Y. \tag{8.94}$$

Proof. — Recall that for any $U \in (T_{\mathbf{R}}X)_y$, $B_y(U)$ exchanges ξ_y^+ and ξ_y^-. The first line in (8.92) follows from formula (8.51) for M. Clearly

$$\tilde{\nabla}^{\xi}_Z \tilde{\nabla}^{\xi}_Z V(y) = \tilde{\nabla}^{\xi}_Z \tilde{\nabla}^{\xi}_Z v(y) + \tilde{\nabla}^{\xi}_Z \tilde{\nabla}^{\xi}_Z v^*(y). \tag{8.95}$$

Recall that the connection $\tilde{\nabla}^{\xi}$ preserves the **Z**-grading of ξ. Therefore $\tilde{\nabla}^{\xi^-}_Z \tilde{\nabla}^{\xi^-}_Z V^-(y)$ is the sum of two operators acting on $\xi^-|_Y = \Lambda N^* \otimes \eta$, one of which increases the degree in $\Lambda N^* \otimes \eta$ by one, the other decreases this degree by one. Also $\exp(\theta)$ is of total degree zero in $\Lambda(\bar{N}^*) \hat{\otimes} \Lambda(N^*)$. We then find that for any $Z \in N_{\mathbf{R}}$,

$$q\tilde{\nabla}^{\xi}_Z \tilde{\nabla}^{\xi}_Z V(y)\, q = 0. \tag{8.96}$$

The second identity in (8.92) follows.

By Proposition 8.11, the connection $i^*\tilde{\nabla}^{\xi^-}$ on $\xi^-|_Y = \Lambda N^* \otimes \eta$ is exactly the holomorphic Hermitian connection of $\Lambda N^* \otimes \eta$. Also ∇^N preserves the norm $|Z|^2$ of $Z \in N_{\mathbf{R}}$. Finally the Kähler form $\theta \in \Lambda(\bar{N}^*) \hat{\otimes} \Lambda(N^*)$ is parallel with respect to the connection induced by ∇^N. Therefore if $U \in T_{\mathbf{R}} Y$, if $\sigma \in F$, then

$${}^0\tilde{\nabla}^Y_U(\sigma\beta) = (\nabla^{TY}_U \sigma)\,\beta. \tag{8.97}$$

From (8.51), (8.97), we get

$$D^H(\sigma\beta) = (D^Y \sigma)\,\beta. \tag{8.98}$$

So (8.93) follows. Then (8.94) is a trivial consequence of (8.92), (8.93). The proof of Theorem 8.21 is completed. □

Remark **8.22.** — A result closely related to (8.93) is proved in [B3, Theorem 2.7].

IX - THE ASYMPTOTICS FOR LARGE α, T OF SUPERTRACES INVOLVING THE OPERATOR $\exp(-\alpha(D^X+TV)^2)$

The purpose of this section is to prove Theorems 8.2 and 8.3. Set $A_T = D^X + TV$. Then if C is an adequately chosen contour in $\mathbf{C}$, we have the identity

$$\exp(-\alpha A_T^2) = \frac{1}{2i\pi}\int_C \exp(-\alpha\lambda^2)(\lambda - A_T)^{-1}\, d\lambda.$$

We will derive the needed informations on $\exp(-\alpha A_T^2)$ from the behaviour as $T \to +\infty$ of the resolvent $(\lambda - A_T)^{-1}$.

To study this resolvent, we roughly proceed as follows. Let β be the form constructed in Theorem 7.4 associated to the fibres of the normal bundle N. Let E^0 be the Hilbert space of square integrable sections of $\Lambda(T^{*(0,1)}X)\,\hat{\otimes}\,\xi$ over X. We then construct an orthogonal splitting $E^0 = E_T^0 \oplus E_T^{0,\perp}$ of E^0. Here E_T^0 is essentially the image of F^0 by multiplication by a rescaled truncated version of β, which concentrates on Y as $T \to +\infty$. We write A_T in matrix form

$$A_T = \begin{bmatrix} A_{T,1} & A_{T,2} \\ A_{T,3} & A_{T,4} \end{bmatrix}$$

with respect to this splitting, and we calculate the resolvent $(\lambda - A_T)^{-1}$ using the fact that if $\lambda \in C$, as $T \to +\infty$, $(\lambda - A_{T,4})^{-1}$ is invertible. With adequate estimates on the matrix elements of $(\lambda - A_T)^{-1}$, we thus obtain Theorem 8.2 for bounded α. Obtaining Theorem 8.2 for unbounded α requires a precise information on the kernel of A_T. This information is of a purely algebraic nature, and is given to us by the quasi-isomorphism $r: (E, \bar{\partial}^X + v) \to (F, \bar{\partial}^Y)$ of Theorem 1.7. We thus get Theorem 8.2 in full generality. Theorem 8.3 also follows from the same sort of arguments.

Of course the results of Section 8, and in particular Proposition 8.13, Theorems 8.18 and 8.21 are constantly used in the whole Section. Of critical importance is the fact that, as shown in the proof of Theorem 9.11, the supercommutator

$[D^X, V]$ is an operator of order zero, or equivalently that the principal symbols of D^X and V anticommute.

This section is organized as follows. In a), we construct the splitting $E^0 = E_T^0 \oplus E_T^{0,\perp}$. In b), we express $A_T = D^X + TV$ as a (2,2) matrix, and we establish various estimates on the matrix components $A_{T,j}$ $(1 \leqslant j \leqslant 4)$. In particular, we prove in Theorem 9.8 that as $T \to +\infty$, $A_{T,1}$ "converges" adequately to D^Y and we establish in Theorem 9.14 a key coercitivity estimate on $A_{T,4}^2$. In c), we establish various estimates on the resolvent of $A_{T,4}$, and in d), we estimate the resolvent of D^Y. In e), we calculate the resolvent of A_T by using the coercitivity of $A_{T,4}^2$. We prove in Theorem 9.23 that the resolvent of A_T converges in the adequate Schatten class to the resolvent of D^Y. We thus obtain in Theorem 9.24 a resolvent version of Theorem 8.2. In f), using Theorem 1.7, we show that there is a gap in the spectrum of A_T^2 at 0 which is uniform as $T \to +\infty$. In g), we prove Theorem 8.2 in full generality. Finally in h), we prove Theorem 8.3.

a) An orthogonal splitting of the Hilbert space $\mathbf{E}^0$

For $\mu \geqslant 0$, let E^μ (resp. $\mathbf{E}^\mu$, resp. F^μ) be the set of sections of $\Lambda(T^{*(0,1)}X) \hat{\otimes} \xi$ over X (resp. of $\tilde{\pi}^*((\Lambda(T^{*(0,1)}X) \hat{\otimes} \xi)|_Y)$ on the total space of $N_{\mathbf{R}}$, resp. of $\Lambda(T^{*(0,1)}Y) \otimes \eta$ over Y) which lie in the μ^{th} Sobolev space. Let $\| \ \|_{E^\mu}$ (resp. $\| \ \|_{\mathbf{E}^\mu}$, resp. $\| \ \|_{F^\mu}$) be a Sobolev norm on E^μ (resp. $\mathbf{E}^\mu$, resp. F^μ). We will always assume that the norm $\| \ \|_{E^0}$ (resp. $\| \ \|_{\mathbf{E}^0}$, resp. $\| \ \|_{F^0}$) is the norm associated with the Hermitian product (1.38) (resp. (8.49), resp. (1.44)).

Recall that $\varepsilon_0 > 0$ was defined in Section 8 e). We now take $\varepsilon \in]0, \varepsilon_0/2]$. In the sequel the constants in our estimates will depend on ε. In Theorem 9.11, we will have to choose ε small enough so that the corresponding estimates hold; ε can otherwise be assumed to be fixed.

Let γ be a smooth function on $\mathbf{R}$ with values in [0, 1] such that

$$\gamma(a) = 1 \quad \text{for} \quad a \leqslant \frac{1}{2}, \qquad = 0 \quad \text{for} \quad a \geqslant 1. \tag{9.1}$$

If $Z \in N_{\mathbf{R}}$, set

$$\rho(Z) = \gamma\left(\frac{|Z|}{\varepsilon}\right). \tag{9.2}$$

For $T > 0$, $y \in Y$, set

$$\alpha_T(y)=\int_{N_{\mathbf{R},y}} \exp(-T|Z|^2)\,\rho^2(Z)\,\frac{dv_N(Z)}{(2\pi)^{\dim N}}. \tag{9.3}$$

Clearly for $1 \leqslant j \leqslant d$, α_T takes the constant value $\alpha_{T,j}$ on Y_j. We now will write α_T instead of $\alpha_T(y)$. Since for $|Z| \leqslant \varepsilon/2$, $\rho(Z)=1$, there exist $c>0$, $C>0$ such that for $T \geqslant 1$

$$\frac{c}{T^{\dim N}} \leqslant \alpha_T \leqslant \frac{C}{T^{\dim N}}. \tag{9.4}$$

Definition **9.1.** — For $\mu \geqslant 0$, $T>0$, let I_T be the linear map

$$\sigma \in F^\mu \to I_T\sigma(y,Z)=(\alpha_T 2^{\dim N})^{-1/2}\,\rho(Z)\exp\left(\theta-\frac{T|Z|^2}{2}\right)\sigma(y)\in \mathbf{E}^\mu. \tag{9.5}$$

For $\mu \geqslant 0$, $T>0$, let $\mathbf{E}^\mu_T$ be the image of F^μ in E^μ by I_T. Let $\mathbf{E}^{0,\perp}_T$ be the orthogonal space to $\mathbf{E}^0_T$ in $\mathbf{E}^0$, let p_T, $p_T^\perp$ be the orthogonal projection operators from $\mathbf{E}^0$ on $\mathbf{E}^0_T$, $\mathbf{E}^{0,\perp}_T$ respectively.

Then I_T maps F^0 onto $\mathbf{E}^0_T$ isometrically. Recall that the operator q was defined in Section 8 a).

Proposition **9.2.** — *If* $s\in \mathbf{E}^0$, *if* $y\in Y$, $Z\in N_{\mathbf{R},y}$ *then*

$$(p_T s)(y,Z)=\frac{1}{\alpha_T}\rho(Z)\exp\left(\frac{-T|Z|^2}{2}\right) q\int_{N_{\mathbf{R},y}} \rho(Z')\exp\left(\frac{-T|Z'|^2}{2}\right)s(y,Z')\,\frac{dv_N(Z')}{(2\pi)^{\dim N}}. \tag{9.6}$$

Proof. — The proof is elementary and is left to the reader. □

Proposition **9.3.** — *There exists* $C>0$ *such that if* $T\geqslant 1$, $\sigma\in F^1$

$$\|I_T\sigma\|_{\mathbf{E}^1} \leqslant C(\|\sigma\|_{F^1}+\sqrt{T}\,\|\sigma\|_{F^0}). \tag{9.7}$$

There exists $C>0$ *such that for any* $T\geqslant 1$, *any* $s\in\mathbf{E}^1$, *then*

$$\|p_T s\|_{\mathbf{E}^1} \leqslant C(\|s\|_{\mathbf{E}^1}+\sqrt{T}\,\|s\|_{\mathbf{E}^0}). \tag{9.8}$$

Given $\gamma>0$, *there exists* $C'>0$ *such that for* $T\geqslant 1$, *for* $s\in\mathbf{E}^0$, *then*

$$\|p_T|Z|^\gamma s\|_{\mathbf{E}^0} \leqslant \frac{C'}{T^{\gamma/2}}\|s\|_{\mathbf{E}^0}. \tag{9.9}$$

Proof.— (9.7), (9.8), (9.9) follow from (9.4)-(9.6). □

Recall that on $B_{\varepsilon_0} \simeq \mathcal{U}_{\varepsilon_0}$, $\Lambda(T^{*(0,1)}X) \hat{\otimes} \xi$ is identified with $\tilde{\pi}^*((\Lambda(T^{*(0,1)}X) \hat{\otimes} \xi|_Y)$. Therefore if $\sigma \in F^\mu$, we can also consider $k^{-1/2} I_T \sigma$ as an element of E^μ.

Definition **9.4.** — For $\mu \geqslant 0$, $T > 0$, let J_T be the linear map

$$\sigma \in F^\mu \to J_T \sigma = k^{-1/2} I_T \sigma \in E^\mu. \tag{9.10}$$

Let E^μ_T be the image of F^μ in E^μ by J_T. Let $E^{0,\perp}_T$ be the orthogonal space to E^0_T in E^0.

For $\mu \geqslant 0$, set

$$E^{\mu,\perp}_T = E^\mu \cap E^{0,\perp}_T. \tag{9.11}$$

Then E^0 splits orthogonally into

$$E^0 = E^0_T \oplus E^{0,\perp}_T. \tag{9.12}$$

Let $\bar{p}_T$, $\bar{p}^\perp_T$ be the orthogonal projection operators from E^0 on E^0_T, $E^{0,\perp}_T$.

The map $s \in \mathbf{E}^0_T \to k^{-1/2} s \in E^0_T$ identifies the Hilbert spaces $\mathbf{E}^0_T$ and E^0_T.

Using formula (9.6), one easily verifies that if $s \in E^0$, $k^{-1/2} p_T k^{1/2} s$ is a well-defined element of $\mathbf{E}^0_T$.

Proposition **9.5.** — *The following identity holds*

$$\bar{p}_T = k^{-1/2} p_T k^{1/2}. \tag{9.13}$$

Proof. — The proof of Proposition 9.5 is trivial and is left to the reader. □

b) The operator $D^X + TV$ as a (2,2) matrix

For $T > 0$, set

$$A_T = D^X + TV. \tag{9.14}$$

Definition **9.6.** — Let R_T be the first order differential operator acting on $\mathbf{E}(\varepsilon_0)$

$$\begin{aligned} R_T = {} & k^{1/2} A_T k^{-1/2} - TV^+(y) - D^N - T\tilde{\nabla}^\xi_Z V(y) \\ & - D^H - M - \frac{1}{2} T \tilde{\nabla}^\xi_Z \tilde{\nabla}^\xi_Z V(y). \end{aligned} \tag{9.15}$$

We now use a notation similar to the notation in (8.55).

Proposition **9.7.** — *As* $T \to +\infty$

$$R_T = O(|Z|^2 \partial^N + |Z| \partial^H + |Z| + T|Z|^3). \tag{9.16}$$

Proof. — Equation (9.16) immediately follows from Theorem 8.18. □

Set

$$A_{T,1} = \bar{p}_T A_T \bar{p}_T,\ A_{T,2} = \bar{p}_T A_T \bar{p}_T^{\perp},\qquad (9.17)$$
$$A_{T,3} = \bar{p}_T^{\perp} A_T \bar{p}_T,\ A_{T,4} = \bar{p}_T^{\perp} A_T \bar{p}_T^{\perp}.$$

We then write the operator A_T in matrix form

$$A_T = \begin{bmatrix} A_{T,1} & A_{T,2} \\ A_{T,3} & A_{T,4} \end{bmatrix}. \qquad (9.18)$$

We will now establish various estimates on the $A_{T,j}$'s as $T \to +\infty$.

1. The operator $A_{T,1}$

If $T \in \mathbf{R}^+ \to B_T$ is a family of first order differential operators with smooth coefficients acting on F, we will write that as $T \to +\infty$

$$B_T = O\left(\frac{1}{\sqrt{T}}\right) \qquad (9.19)$$

if there exists $C > 0$ such that for $T \geqslant 1$, the sup of the norms of the coefficients are dominated by $C/\sqrt{T}$.

By (9.5), (9.10), one sees easily that for any $T > 0$, $J_T^{-1} A_{T,1} J_T$ is a first order formally self-adjoint differential operator acting on F.

Theorem **9.8.** — *As* $T \to \infty$

$$J_T^{-1} A_{T,1} J_T = D^Y + O\left(\frac{1}{\sqrt{T}}\right). \qquad (9.20)$$

Proof. — Using Proposition 9.5, we find that

$$J_T^{-1} A_{T,1} J_T = I_T^{-1} p_T (k^{1/2} A_T k^{-1/2}) p_T I_T. \qquad (9.21)$$

Since $V^+(y)$ maps ξ_y into ξ_y^+, $p_T V^+(y) = 0$. From (9.15), (9.21), we get

$$J_T^{-1} A_{T,1} J_T = I_T^{-1} p_T \Big(D^N + T\, \tilde{\nabla}_Z^{\xi} V(y) + D^H + M + \frac{1}{2} \tilde{\nabla}_Z^{\xi} \tilde{\nabla}_Z^{\xi} V(y) + R_T \Big) p_T I_T. \qquad (9.22)$$

By Theorem 7.4, equation (7.23), and Proposition 8.13, we know that

$$(D^N + T\tilde{\nabla}^\xi_Z V(y)) \exp\left(\theta - \frac{T|Z|^2}{2}\right) = 0. \tag{9.23}$$

So from (8.51), (9.5) and (9.23), we find that if $\sigma \in F$

$$\begin{aligned}(D^N + T\tilde{\nabla}^\xi_Z V(y))\, p_T I_T \sigma \\ = (\alpha_T 2^{\dim N})^{-1/2} \frac{c}{\sqrt{2}}\left(\frac{\partial \rho}{\partial Z}(Z)\right) \exp\left(\theta - \frac{T|Z|^2}{2}\right)\sigma.\end{aligned} \tag{9.24}$$

Now if $U \in N_{\mathbf{R}}$, by (5.2), $c(U)$ is the sum of two operators, one of which increases the total degree in $\Lambda(\bar{N}^*)$ by one, the other decreases the total degree in $\Lambda(\bar{N}^*)$ by one. Since $\exp(\theta)$ is of total degree zero in $\Lambda(\bar{N}^*) \hat{\otimes} \Lambda(N^*)$, we find that

$$qc(U)\exp(\theta) = 0. \tag{9.25}$$

From (9.6), (9.24), (9.25), we deduce that

$$I_T^{-1} p_T (D^N + T\tilde{\nabla}^\xi_Z V(y))\, p_T I_T = 0. \tag{9.26}$$

Recall that $\rho(Z)$ only depends on $|Z|$. Since the connection ∇^N preserves the metric of N, if $U \in T_{\mathbf{R}} Y$, then

$$\nabla_{U^H} \rho = 0. \tag{9.27}$$

Using the same arguments as in the proof of Theorem 8.21 and also (9.27), we find that

$$I_T^{-1} p_T D^H p_T I_T = D^Y. \tag{9.28}$$

Also by Theorem 8.21, we get

$$I_T^{-1} p_T \left(M + \frac{1}{2}\tilde{\nabla}^\xi_Z \tilde{\nabla}^\xi_Z V(y)\right) p_T I_T I_T = 0. \tag{9.29}$$

Finally using Proposition 9.7 and (9.9), we find easily that as $T \to +\infty$

$$I_T^{-1} p_T R_T p_T I_T = O\left(\frac{1}{\sqrt{T}}\right). \tag{9.30}$$

Theorem 9.8 follows from (9.22), (9.26)-(9.30). □

2. The operators $A_{T,2}$, $A_{T,3}$

We first establish several technical results.

Proposition **9.9.** – *For any* $T>0$, *the following identity of operators acting on* $\mathbf{E}^1$ *holds*

$$[D^H, p_T]=0. \tag{9.31}$$

There exists $C>0$ *such that for any* $T\geqslant 1$, *any* $s\in\mathbf{E}^1$, *then*

$$\| p_T(D^N+T\tilde{\nabla}^{\xi}_Z V(y))s\|_{\mathbf{E}^0}\leqslant\frac{C}{\sqrt{T}}\|s\|_{\mathbf{E}^0}. \tag{9.32}$$

There exists $C'>0$ *such that for any* $T\geqslant 1$, *any* $s\in\mathbf{E}^1$ *with support in* $B_{3\varepsilon_0/4}$, *then*

$$\| p_T R_T s\|_{\mathbf{E}^0}\leqslant\frac{C}{\sqrt{T}}\|s\|_{\mathbf{E}^1}. \tag{9.33}$$

Proof. – Since

$$^0\tilde{\nabla}^Y\theta=0,$$

it is clear that

$$^0\tilde{\nabla}^Y q=0. \tag{9.34}$$

Also if $U\in T_{\mathbf{R}}Y$

$$[c(U), q]=0. \tag{9.35}$$

We now use the fact that ∇^N preserves the norm in N and more specifically equation (9.27). We then obtain equation (9.31).

By Proposition 9.2, if $s\in\mathbf{E}^1$, then

$$p_T(D^N+T\tilde{\nabla}^{\xi}_Z V(y))s(y,Z)=\frac{1}{\alpha_T}\rho(Z)\exp\left(\frac{-T|Z|^2}{2}\right)$$
$$q\int_{N_{\mathbf{R},y}}\rho(Z')\exp\left(\frac{-T|Z'|^2}{2}\right)(D^N+T\tilde{\nabla}^{\xi}_{Z'}V(y))s(y,Z')\frac{dv_N(Z')}{(2\pi)^{\dim N}}. \tag{9.36}$$

We use (9.23), (9.24), and the fact that the operator $D^N+T\tilde{\nabla}^{\xi}_Z V(y)$ is fibrewise self-adjoint. Integrating by parts in (9.36), we get

$$(9.37)\qquad \begin{aligned} &p_{\mathrm{T}}(\mathrm{D}^{\mathrm{N}}+\mathrm{T}\,\tilde{\nabla}^{\xi}_{\mathrm{Z}}\mathrm{V}(y))\,s(y,\mathrm{Z}) \\ &\quad= -\frac{1}{\alpha_{\mathrm{T}}}\rho(\mathrm{Z})\exp\left(\frac{-\mathrm{T}|\mathrm{Z}|^2}{2}\right)q\int_{\mathrm{N}_{\mathbf{R},y}}\exp\left(\frac{-\mathrm{T}|\mathrm{Z}'|^2}{2}\right) \\ &\qquad\qquad \frac{1}{\sqrt{2}}\,c\left(\frac{\partial\rho}{\partial\mathrm{Z}}(\mathrm{Z}')\right)s(y,\mathrm{Z}')\,\frac{dv_{\mathrm{N}}(\mathrm{Z}')}{(2\pi)^{\dim\mathrm{N}}}. \end{aligned}$$

Now $\partial\rho/\partial\mathrm{Z}(\mathrm{Z})$ vanishes for $|\mathrm{Z}|$ small enough. From (9.37), we get (9.32).

Finally using Propositions 9.3 and 9.7, we get (9.33). □

Theorem **9.10.** – *There exists* $\mathrm{C}>0$ *such that for any* $\mathrm{T}\geqslant 1$, *any* $s\in\mathrm{E}^{1,\perp}_{\mathrm{T}}$, $s'\in\mathrm{E}^{1}_{\mathrm{T}}$, *then*

$$(9.38)\qquad \begin{aligned} \|\mathrm{A}_{\mathrm{T},2}\,s\|_{\mathrm{E}^0} &\leqslant \mathrm{C}\left(\frac{\|s\|_{\mathrm{E}^1}}{\sqrt{\mathrm{T}}}+\|s\|_{\mathrm{E}^0}\right), \\ \|\mathrm{A}_{\mathrm{T},3}\,s'\|_{\mathrm{E}^0} &\leqslant \mathrm{C}\left(\frac{\|s'\|_{\mathrm{E}^1}}{\sqrt{\mathrm{T}}}+\|s'\|_{\mathrm{E}^0}\right). \end{aligned}$$

Proof. – Let δ be a smooth function on $\mathbf{R}$ with values in [0, 1] such that

$$\begin{aligned} \delta(a)&=1 \quad \text{for } a\leqslant\frac{1}{2}, \\ &=0 \quad \text{for } a\geqslant\frac{3}{4}. \end{aligned}$$

Set

$$\psi(\mathrm{Z})=\delta\left(\frac{|\mathrm{Z}|}{\varepsilon_0}\right).$$

We will consider ψ as a function defined on X, which vanishes on $\mathrm{X}\setminus\mathscr{U}_{3\varepsilon_0/4}$. Also since $\varepsilon\leqslant\varepsilon_0/2$, ψ is equal to 1 on the support of ρ.

Take $s\in\mathrm{E}^{1,\perp}_{\mathrm{T}}$. Set

$$(9.39)\qquad \bar{s}=\psi s.$$

Since $\bar{p}_{\mathrm{T}}s=0$, using Propositions 9.2 and 9.5, we find that $\bar{p}_{\mathrm{T}}\bar{s}=0$, *i.e.* $\bar{s}\in\mathrm{E}^{1,\perp}_{\mathrm{T}}$. Also $\mathrm{A}_{\mathrm{T}}s=\mathrm{A}_{\mathrm{T}}\bar{s}$ on the support of ρ, and so by Propositions 9.2 and 9.5, $\bar{p}_{\mathrm{T}}\mathrm{A}_{\mathrm{T}}s=\bar{p}_{\mathrm{T}}\mathrm{A}_{\mathrm{T}}\bar{s}$, *i.e.*

(9.40) $$A_{T,2}s = A_{T,2}\bar{s}.$$

Clearly since $\operatorname{Im}[V^+(y)] \in \xi_y^+$, then

(9.41) $$qV^+(y)\bar{s}(y, Z) = 0.$$

Using Proposition 9.5 and Definition 9.6, we find that

(9.42) $$A_{T,2}s(y, Z) = k^{-1/2}(y, Z)p_T\left\{D^N + T\tilde{\nabla}_Z^\xi V(y) + D^H + M + \frac{1}{2}T\tilde{\nabla}_Z^\xi\tilde{\nabla}_Z^\xi V(y) + R_T\right\}k^{1/2}\bar{s}(y, Z).$$

Since $\bar{p}_T\bar{s} = 0$, by Proposition 9.5, $p_T k^{1/2}\bar{s} = 0$. By Proposition 9.9, $[D^H, p_T] = 0$, and so

(9.43) $$p_T D^H k^{1/2}\bar{s} = 0.$$

Using Propositions 9.3, 9.9, (9.42) and (9.43), we get the first inequality in (9.38).

Take now $s' \in E_T^\perp$. Then $s' \in \xi^-$, and so $V^+(y)s' = 0$. Using Proposition 9.5 and Definition 9.6, we find that

(9.44) $$A_{T,3}s' = k^{-1/2}(y, Z)p_T^\perp\left\{D^N + T\tilde{\nabla}_Z^\xi V(y) + D^H + M + \frac{1}{2}T\tilde{\nabla}_Z^\xi\tilde{\nabla}_Z^\xi V(y) + R_T\right\}k^{1/2}s'(y, Z).$$

By taking adjoints in equation (9.32), we know that if $s \in \mathbf{E}^0$, then

(9.45) $$\|(D^N + T\tilde{\nabla}_Z^\xi V(y))p_T s\|_{\mathbf{E}^0} \leqslant \frac{C}{\sqrt{T}}\|s\|_{\mathbf{E}^0}.$$

Also since $\bar{p}_T s' = s'$, using Proposition 9.5, we find that $p_T k^{1/2}s' = k^{1/2}s'$. From (9.45), we then deduce that

(9.46) $$\|k^{-1/2}p_T^\perp(D^N + T\tilde{\nabla}_Z^\xi V(y))k^{1/2}s'\|_{\mathbf{E}^0} \leqslant \frac{C}{\sqrt{T}}\|s'\|_{\mathbf{E}^0}.$$

Also by Proposition 9.9, $[p_T, D^H] = 0$. Since $p_T k^{1/2}s' = k^{1/2}s'$, we get

(9.47) $$p_T^\perp D^H k^{1/2}s' = 0.$$

Using (9.44)-(9.47) and Propositions 9.3 and 9.9, we get the second inequality in (9.38). □

3. The operator $A_{T,4}$

Recall that the vector spaces E^0_T, $E^{0,\perp}_T$ implicitly depend on $\varepsilon \in]0, (\varepsilon_0/2)]$.

Theorem **9.11.** — *There exist* $\varepsilon \in]0, (\varepsilon_0/4)]$, $C > 0$, $b > 0$ *such that for any* $T \geqslant 1$, *any* $s \in E^{1,\perp}_T$, *then*

$$\| A_T s \|^2_{E^0} \geqslant C (\| s \|^2_{E^1} + (T - b) \| s \|^2_{E^0}). \tag{9.48}$$

Proof. — The proof of Theorem 9.11 will consist of three main steps:

- In a first step, we show that for $\varepsilon \in]0, (\varepsilon_0/4)]$ small enough, if the support of $s \in E^{1,\perp}_T$ is included in $\mathscr{U}_{2\varepsilon}$, then (9.48) holds.
- $\varepsilon \in]0, (\varepsilon_0/4)]$ being now fixed, we show that if $s \in E^1$ vanishes on $\mathscr{U}_{\varepsilon/2}$, (9.48) still holds.
- Using partition of unity, we finally prove (9.48) in full generality.

Already observe that if $U \in T_{\mathbf{R}} X$, $c(U)$ acts as $c(U) \hat{\otimes} 1$ on $\Lambda (T^{*(0,1)} X) \hat{\otimes} \xi$, and V acts like $1 \hat{\otimes} V$ on $\Lambda (T^{*(0,1)} X) \hat{\otimes} \xi$. Since $c(U)$ and V are odd operators, then

$$[c(U), V] = 0. \tag{9.49}$$

Let $e_1, \ldots, e_{2l}$ be an orthonormal base of $T_{\mathbf{R}} X$. Using Proposition 8.5 and (9.49), we find that

$$[D^X, V] = \sum_1^{2l} \frac{c(e_i)}{\sqrt{2}} \nabla^{\xi}_{e_i} V(x). \tag{9.50}$$

Therefore $[D^X, V]$ is a differential operator of order zero. This fact will play a key role in the proof of Theorem 9.11.

Step n° 1 : The case where s is supported near Y.

The key step in the proof of Theorem 9.11 is as follows.

Proposition **9.12.** — *There exist* $\varepsilon \in]0, (\varepsilon_0/4)]$, $C > 0$, $b > 0$ *such that for any* $T \geqslant 1$, $s \in E^{1,\perp}_T$ *whose support is included in* $\mathscr{U}_{2\varepsilon}$, *then*

$$\| A_T s \|^2_{E^0} \geqslant C (\| s \|^2_{E^1} + (T - b) \| s \|^2_{E^0}). \tag{9.51}$$

Proof. — We temporarily fix $\varepsilon \in]0, (\varepsilon_0/4)]$. In the following estimates, the constants which cannot be chosen independently of ε will be marked with a subscript 0 like $C_0, C'_0, \ldots$

Take $s \in \mathbf{E}_{\mathrm{T}}^{1,\perp}$ whose support is included in $\mathscr{U}_{2\varepsilon}$. Since $\varepsilon \leqslant \varepsilon_0/4$, we can define $s' \in \mathbf{E}^1$ by the formula

$$s' = k^{1/2} s. \tag{9.52}$$

By Proposition 9.5, since $\bar{p}_{\mathrm{T}} s = 0$, then

$$p_{\mathrm{T}} s' = 0. \tag{9.53}$$

Also

$$\| \mathrm{A}_{\mathrm{T}} s \|_{\mathbf{E}^0}^2 = \| k^{1/2} \mathrm{A}_{\mathrm{T}} k^{-1/2} s' \|_{\mathbf{E}^0}^2. \tag{9.54}$$

By (9.15), we get

$$(k^{1/2} \mathrm{A}_{\mathrm{T}} k^{-1/2}) s' = \mathrm{T}\mathrm{V}^{+}(y) s' + (\mathrm{D}^{\mathrm{N}} + \mathrm{T} \tilde{\nabla}_{\mathrm{Z}}^{\xi} \mathrm{V}(y)) s' + \left(\mathrm{D}^{\mathrm{H}} + \mathrm{M} + \frac{\mathrm{T}}{2} \tilde{\nabla}_{\mathrm{Z}}^{\xi} \tilde{\nabla}_{\mathrm{Z}}^{\xi} \mathrm{V}(y)\right) s' + \mathrm{R}_{\mathrm{T}} s'. \tag{9.55}$$

Let L_{T} be the first order differential operator

$$\mathrm{L}_{\mathrm{T}} = \mathrm{T}\mathrm{V}^{+}(y) + \mathrm{D}^{\mathrm{N}} + \mathrm{T} \tilde{\nabla}_{\mathrm{Z}}^{\xi} \mathrm{V}(y) + \mathrm{D}^{\mathrm{H}}. \tag{9.56}$$

Then by (9.55), we get

$$\| \mathrm{A}_{\mathrm{T}} s \|_{\mathbf{E}^0}^2 \geqslant \frac{1}{2} \| \mathrm{L}_{\mathrm{T}} s' \|_{\mathbf{E}^0}^2 - \left\| \left(\mathrm{M} + \frac{\mathrm{T}}{2} \tilde{\nabla}_{\mathrm{Z}}^{\xi} \tilde{\nabla}_{\mathrm{Z}}^{\xi} \mathrm{V}(y) + \mathrm{R}_{\mathrm{T}}\right) s' \right\|_{\mathbf{E}^0}^2. \tag{9.57}$$

We now will estimate the various terms in the right-hand side of (9.57).

Recall that by (8.86), $\mathbf{E}^0$ splits orthogonally into

$$\mathbf{E}^0 = \mathbf{E}^{+,0} \oplus \mathbf{E}^{-,0}. \tag{9.58}$$

Also L_{T} acts as an unbounded formally self-adjoint operator on $\mathbf{E}^0$, which preserves $\mathbf{E}^{+,0}$ and $\mathbf{E}^{-,0}$. We now write s' in the form

$$s' = s'^{+} + s'^{-};\ s'^{\pm} \in \mathbf{E}^{\pm,0}. \tag{9.59}$$

By (9.53), $p_{\mathrm{T}} s' = 0$, and so using (9.59), we get

$$p_{\mathrm{T}} s'^{\pm} = 0. \tag{9.60}$$

1) An estimate on $\| \mathrm{L}_{\mathrm{T}} s'^{+} \|_{\mathbf{E}^0}^2$.

Clearly

$$\mathrm{L}_{\mathrm{T}}^2 = (\mathrm{D}^{\mathrm{N}} + \mathrm{D}^{\mathrm{H}})^2 + \mathrm{T}[\mathrm{D}^{\mathrm{N}} + \mathrm{D}^{\mathrm{H}}, \mathrm{V}^{+}(y) + \tilde{\nabla}_{\mathrm{Z}}^{\xi} \mathrm{V}(y)] + \mathrm{T}^2 (\mathrm{V}^{+}(y) + \tilde{\nabla}_{\mathrm{Z}}^{\xi} \mathrm{V}(y))^2. \tag{9.61}$$

There exists $c > 0$ such that for any $y \in Y, f \in \xi_y^+$

$$(9.62) \qquad |V^+(y) f|^2 \geqslant c |f|^2.$$

From (9.62), we deduce

$$(9.63) \qquad \|(V^+(y) + \tilde{\nabla}_Z^\xi V(y)) s'^+\|_{E^0}^2 \geqslant \left(\frac{c}{2} - c' \varepsilon^2\right) \|s'^+\|_{E^0}^2.$$

For the same reason as in (9.49)-(9.50), the operator $[D^N + D^H, V^+(y) + \tilde{\nabla}_Z^\xi V(y)]$ is of order zero. Therefore

$$(9.64) \qquad |\langle [D^N + D^H, V^+(y) + \tilde{\nabla}_Z^\xi V(y)] s'^+, s'^+ \rangle_{E^0}| \leqslant C \|s'^+\|_{E^0}^2.$$

Finally, since the operator $D^N + D^H$ is elliptic of order one on B_{ε_0}, there exist $C' > 0$, $C'' > 0$ such that for any $s \in E_T^{1, \perp}$ whose support is included in $\mathcal{U}_{2\varepsilon}$

$$(9.65) \qquad \|(D^N + D^H) s'^+\|_{E^0}^2 \geqslant C' \|s'^+\|_{E^1}^2 - C'' \|s'^+\|_{E^0}^2.$$

From (9.61)-(9.65), we get

$$(9.66) \qquad \|L_T s'^+\|_{E^0}^2 \geqslant C' \|s'^+\|_{E^1}^2 + \left(T^2 \left(\frac{c}{2} - \frac{c' \varepsilon^2}{2}\right) - CT - C''\right) \|s'^+\|_{E^0}^2.$$

2) An estimate on $\|L_T s'^-\|_{E^0}^2$.

Recall that $D^{N,-}$, $D^{H,-}$ are the restrictions of D^N, D^H to E^-. Similarly let L_T^- be the restriction of L_T to E^-.

By Proposition 8.13, we know that

$$(9.67) \qquad \tilde{\nabla}_Z^\xi V^-(y) = \frac{\sqrt{-1}}{\sqrt{2}} \hat{c}(JZ).$$

In view of (9.67), it is clear that

$$(9.68) \qquad [D^{H,-}, \tilde{\nabla}_Z^{\xi^-} V^-(y)] = 0.$$

Similarly, since the connection ∇^N is holomorphic and unitary

$$(9.69) \qquad [D^{H,-}, D^{N,-}] = 0.$$

Using (9.67)-(9.69), we get

$$(9.70) \qquad (L_T^-)^2 = (D^{H,-})^2 + \left(D^{N,-} + \frac{T\sqrt{-1}}{\sqrt{2}} \hat{c}(JZ)\right)^2.$$

From (9.70), we find that

$$(9.71) \qquad \| L_T^- s'^- \|^2_{\mathbf{E}^0} = \| D^{H,-} s'^- \|^2_{\mathbf{E}^0} + \left\| \left(D^{N,-} + \frac{T\sqrt{-1}}{\sqrt{2}} \hat{c}(JZ) \right) s'^- \right\|^2_{\mathbf{E}^0}.$$

Let $\mathbf{E}'^{\,0}_T$ be the image of F^0 in $\mathbf{E}^0$ by the linear map

$$\sigma \in F^0 \to \sigma \exp\left(\theta - \frac{T|Z|^2}{2} \right) \in \mathbf{E}^0.$$

Let p'_T be the orthogonal projection operator from $\mathbf{E}^0$ on $\mathbf{E}'^{\,0}_T$. By (8.91), we find that

$$(9.72) \qquad p'_T s'^- (y, Z) = \left(\frac{T}{\pi} \right)^{\dim N} \exp\left(\frac{-T|Z|^2}{2} \right) q \int_{N_{\mathbf{R},y}} \exp\left(\frac{-T|Z'|^2}{2} \right) s'^- (y, Z')\, dv_N(Z').$$

Observe that

$$(9.73) \qquad F_T \left(D^{N,-} + \frac{T\sqrt{-1}}{\sqrt{2}} \hat{c}(JZ) \right) F_T^{-1} = \sqrt{T} \left(D^{N,-} + \frac{\sqrt{-1}}{\sqrt{2}} \hat{c}(JZ) \right).$$

By Theorem 7.4 and by equations (7.23) and (9.73), we find that

$$(9.74) \qquad \left(D^{N,-} + \frac{\sqrt{-1}}{\sqrt{2}} T\hat{c}(JZ) \right) s'^- = \left(D^{N,-} + \frac{\sqrt{-1}}{\sqrt{2}} T\hat{c}(JZ) \right) (s'^- - p'_T s'^-).$$

For $y \in Y$, let $\mathbf{F}_y^{-,0}$ be the set of square integrable sections of

$$\tilde{\pi}^* ((\Lambda(T^{*(0,1)}Y) \hat{\otimes} \Lambda(\bar{N}^*) \otimes \xi^-)|_y)$$

over the fibre $N_{\mathbf{R},y}$. Recall that $\xi_y^- = (\Lambda(N^*) \otimes \eta)_y$. By Theorem 7.4 and by (7.23), the kernel of the operator $(D^{N,-} + (\sqrt{-1}/\sqrt{2})\hat{c}(JZ))_y^2$ acting as an unbounded operator on $\mathbf{F}_y^{-,0}$ is the image of $\Lambda(T^{*(0,1)}Y)_y \otimes \eta_y$ by the map ψ considered in Definition 8.20. Also the spectrum of $(D^{N,-} + (\sqrt{-1}/\sqrt{2})\hat{c}(JZ))_y^2$ is discrete and does not depend on $y \in Y$.

Using (9.73), (9.74), we find there exists $C''' > 0$ such that for $s \in E_T^{1,\perp}$ whose support is included in $\mathcal{U}_{2\varepsilon}$, then

$$(9.75) \qquad \left\| \left(D^{N,-} + \frac{T\sqrt{-1}}{\sqrt{2}} \hat{c}(JZ) \right) s'^- \right\|^2_{\mathbf{E}^0} \geqslant TC''' (\| s'^- - p'_T s'^- \|^2_{\mathbf{E}^0}).$$

Equivalently

$$\left\|\left(\mathrm{D}^{\mathrm{N},-}+\frac{\mathrm{T}\sqrt{-1}}{\sqrt{2}}\,\hat{c}(\mathrm{J\,Z})\right)s'^{-}\right\|^2_{\mathbf{E}^0} \geq \mathrm{TC}'''(\|s'^{-}\|^2_{\mathbf{E}^0}-\|p'_{\mathrm{T}}s'^{-}\|^2_{\mathbf{E}^0}). \tag{9.76}$$

Let $\Delta^{\mathrm{N}_{\mathbf{R}}}$ be the flat Laplacian along the fibres of $\mathrm{N}_{\mathbf{R}}$. Using Proposition 7.2 and (9.73), we also find there exists $\kappa > 0$ such that for any $\alpha \in]0, 1]$

$$\left\|\left(\mathrm{D}^{\mathrm{N},-}+\frac{\mathrm{T}\sqrt{-1}}{\sqrt{2}}\,\hat{c}(\mathrm{J\,Z})\right)s'^{-}\right\|^2_{\mathbf{E}^0} \geq \frac{\alpha}{2}\langle -\Delta^{\mathrm{N}_{\mathbf{R}}}s'^{-}, s'^{-}\rangle_{\mathbf{E}^0} + \frac{\alpha}{2}\mathrm{T}^2\,\|\,|\mathrm{Z}|\,s'^{-}\|^2_{\mathbf{E}^0}-\alpha\kappa\mathrm{T}\,\|s'^{-}\|^2_{\mathbf{E}^0}. \tag{9.77}$$

We now fix $\alpha \in]0, 1]$ such that $\mathrm{C}'''-\alpha\kappa \geq (\mathrm{C}'''/2)$. Using (9.76), (9.77), we find that

$$\left\|\left(\mathrm{D}^{\mathrm{N},-}+\frac{\mathrm{T}\sqrt{-1}}{\sqrt{2}}\,\hat{c}(\mathrm{J\,Z})\right)s'^{-}\right\|^2_{\mathbf{E}^0} \geq \frac{1}{4}\alpha\langle -\Delta^{\mathrm{N}_{\mathbf{R}}}s'^{-}, s'^{-}\rangle_{\mathbf{E}^0} + \frac{\mathrm{TC}'''}{4}\|s'^{-}\|^2_{\mathbf{E}^0}+\frac{\alpha}{4}\mathrm{T}^2\,\|\,|\mathrm{Z}|\,s'^{-}\|^2_{\mathbf{E}^0}-\frac{\mathrm{TC}'''}{2}\|p'_{\mathrm{T}}s'^{-}\|^2_{\mathbf{E}^0}. \tag{9.78}$$

The support of s' is included in $\mathrm{B}_{\varepsilon_0/2}$. By using elliptic estimates, there exists $d' > 0$, $d'' > 0$ such that

$$\frac{1}{4}\alpha\langle -\Delta^{\mathrm{N}_{\mathbf{R}}}s'^{-}, s'^{-}\rangle_{\mathbf{E}^0}+\|\mathrm{D}^{\mathrm{H},-}s'^{-}\|^2_{\mathbf{E}^0} \geq d'\,\|s'^{-}\|^2_{\mathbf{E}^1}-d''\,\|s'^{-}\|^2_{\mathbf{E}^0}. \tag{9.79}$$

By (9.60), $p_{\mathrm{T}}s'^{-}=0$. Using (9.6), (9.72), we get

$$p'_{\mathrm{T}}s'^{-}(y, \mathrm{Z})=\left(\frac{\mathrm{T}}{\pi}\right)^{\dim \mathrm{N}}\exp\left(\frac{-\mathrm{T}|\mathrm{Z}|^2}{2}\right) q\int_{\mathrm{N}_{\mathbf{R}},y}\exp\left(\frac{-\mathrm{T}|\mathrm{Z}'|^2}{2}\right)(1-\rho)(\mathrm{Z}')\,s'^{-}(y, \mathrm{Z}')\,dv_{\mathrm{N}}(\mathrm{Z}'). \tag{9.80}$$

The function $1-\rho(\mathrm{Z})$ vanishes for $|\mathrm{Z}| \leq (\varepsilon/2)$. So from (9.80), we get

$$\|p'_{\mathrm{T}}s'^{-}\|^2_{\mathbf{E}^0} \leq \frac{\mathrm{C}_0}{\sqrt{\mathrm{T}}}\|s'^{-}\|_{\mathbf{E}^0}. \tag{9.81}$$

Using (9.71), (9.78), (9.79), we finally obtain

$$\|L_T^- s'^-\|^2_{E^0} \geqslant d'\|s'^-\|^2_{E^1} + \left(\frac{TC'''}{4} - \frac{\sqrt{T}\,C_0 C'''}{2} - d''\right)\|s'^-\|^2_{E^0} + \frac{\alpha T^2}{4}\||Z|s'^-\|^2_{E^0}. \tag{9.82}$$

3) An estimate for $\|(M+(1/2)\tilde{\nabla}^\xi_E\tilde{\nabla}^\xi_Z V(y)+R_T)s'\|^2_{E^0}$.

In what follows, the constants c, c' (which do not depend on $\varepsilon > 0$) may vary from line to line. Clearly

$$\|M s'\|^2_{E^0} \leqslant c\|s'\|^2_{E^0}. \tag{9.83}$$

Also s', $s'^{\pm}$ vanish on $X \setminus B_{2\varepsilon}$. Therefore

$$\begin{aligned} &\|T\tilde{\nabla}^\xi_Z\tilde{\nabla}^\xi_Z V(y)s'^+\|^2_{E^0} \leqslant c\varepsilon^4 T^2\|s'^+\|^2_{E^0},\\ &\|T\tilde{\nabla}^\xi_Z\tilde{\nabla}^\xi_Z V(y)s'^-\|^2_{E^0} \leqslant c\varepsilon^2 T^2\||Z|s'^-\|^2_{E^0}. \end{aligned} \tag{9.84}$$

By Proposition 9.7, we also get

$$\|R_T s'\|^2_{E^0} \leqslant c(\varepsilon^2\|s'\|^2_{E^1} + \varepsilon^6 T^2\|s'^+\|^2_{E^0} + \varepsilon^4 T^2\||Z|s'^-\|^2_{E^0}). \tag{9.85}$$

So from (9.83)-(9.85), we obtain

$$\begin{aligned} &\left\|\left(M + \frac{1}{2}\tilde{\nabla}^\xi_Z\tilde{\nabla}^\xi_Z V(y) + R_T\right)s'\right\|^2_{E^0}\\ &\qquad \leqslant c(\varepsilon^2\|s'\|^2_{E^1} + \|s'\|^2_{E^0} + \varepsilon^4 T^2\|s'^+\|^2_{E^0} + \varepsilon^2 T^2\||Z|s'^-\|^2_{E^0}). \end{aligned} \tag{9.86}$$

4) An estimate for $\|A_T s\|^2_{E^0}$.

Using (9.57), (9.66), (9.82), (9.86), we get

$$\begin{aligned} \|A_T s\|^2_{E^0} \geqslant{} & \frac{1}{2}C'\|s'^+\|^2_{E^1} + \frac{1}{2}d'\|s'^-\|^2_{E^1} - 2c\varepsilon^2\|s'\|^2_{E^1}\\ &+ \left(T^2\left(\frac{c}{4} - \frac{c'\varepsilon^2}{4} - c\varepsilon^4\right) - \frac{CT}{2} - \frac{C''}{2} - c\right)\|s'^+\|^2_{E^0}\\ &+ \left(\frac{TC'''}{8} - \frac{\sqrt{T}\,C_0 C'''}{4} - \frac{d''}{2} - c\right)\|s'^-\|^2_{E^0}\\ &+ T^2\left(\frac{\alpha}{8} - \varepsilon^2 c\right)\||Z|s'^-\|^2_{E^0}. \end{aligned} \tag{9.87}$$

From (9.87), it is clear that for $\varepsilon > 0$ small enough, (9.51) holds. Proposition 9.12 is proved. □

Step n° 2: The case where s vanishes near Y.

We now fix $\varepsilon \in]0, (\varepsilon_0/4)]$ as in Proposition 9.12.

Proposition **9.13.** – *There exist* $C > 0$, $b > 0$, *such that for any* $T \geqslant 1$, *any* $s \in E^1$ *which vanishes on* $\mathcal{U}_\varepsilon$, *then*

$$\| A_T s \|_{E^0}^2 \geqslant C(\| s \|_{E^1}^2 + (T-b) \| s \|_{E^0}^2). \tag{9.88}$$

Proof. – Clearly

$$A_T^2 = (D^X)^2 + T[D^X, V] + T^2 V^2. \tag{9.89}$$

Also V is invertible on $X \diagdown \mathcal{U}_\varepsilon$. Therefore there exists $C > 0$ such that if $s \in E^1$ vanishes on $\mathcal{U}_\varepsilon$, then

$$\| V s \|_{E^0}^2 \geqslant C \| s \|_{E^0}^2. \tag{9.90}$$

Also by (9.50), $[D^X, V]$ is an operator of order zero, and so

$$| \langle D^X, V] s, s \rangle_{E^0} | \leqslant C' \| s \|_{E^0}^2. \tag{9.91}$$

Finally since D^X is an elliptic first order differential operator, there exists $C'' > 0$, $C''' > 0$ such that

$$\| D^X s \|_{E^0}^2 \geqslant C'' \| s \|_{E^1}^2 - C''' \| s \|_{E^0}^2. \tag{9.92}$$

From (9.89)-(9.92), we get

$$\| A_T s \|_{E^0}^2 \geqslant C'' \| s \|_{E^1}^2 + (CT^2 - C'T - C''') \| s \|_{E^0}^2. \tag{9.93}$$

Equation (9.88) follows. □

Step n° 3: Proof of Theorem 9.11.

We choose $\varepsilon \in]0, (\varepsilon_0/4)]$ as in Proposition 9.12. Let $a \to \gamma(a)$ be the function considered in (9.1). We consider the function $Z \in N_{\mathbf{R}} \to \gamma(|Z|/2\varepsilon)$ as a smooth function on X which vanishes on $X \diagdown \mathcal{U}_{2\varepsilon}$. Set

$$\begin{aligned} \tau_1 &= \frac{\gamma}{(\gamma^2 + (1-\gamma)^2)^{1/2}} \left(\frac{|Z|}{2\varepsilon} \right) \\ \tau_2 &= \frac{1-\gamma}{(\gamma^2 + (1-\gamma)^2)^{1/2}} \left(\frac{|Z|}{2\varepsilon} \right). \end{aligned} \tag{9.94}$$

Then τ_1, τ_2 are smooth functions on X such that $\tau_1^2+\tau_2^2=1$. Also on $\mathscr{U}_\varepsilon$, τ_1 is equal to 1 and τ_2 vanishes.

Take $s\in E_T^{1,\perp}$. For $j=1, 2$, set

$$s_j=\tau_j s. \tag{9.95}$$

Since $\bar{p}_T s=0$, using (9.6), (9.13), it is clear that

$$\bar{p}_T s_1=0. \tag{9.96}$$

Since $\tau_1^2+\tau_2^2=1$, we find that

$$\|A_T s\|_{E^0}^2=\sum_{j=1}^{2}\|A_T s_j\|_{E^0}^2+\sum_{j=1}^{2}\langle[\tau_j, A_T^2]s, s_j\rangle_{E^0}. \tag{9.97}$$

Also by (9.50), $[D^X, V]$ and V^2 are differential operators of order zero. From (9.89), we get

$$[\tau_j, A_T^2]=[\tau_j, (D^X)^2]. \tag{9.98}$$

Also $[\tau_j, (D^X)^2]$ is a differential operator of order one. We thus find that for any $\eta>0$, there exists $C_\eta>0$ such that for any s taken as before

$$\left|\sum_{j=1}^{2}\langle[\tau_j, A_T^2]s, s_j\rangle\right|\leqslant\eta\|s\|_{E^1}^2+C_\eta\|s\|_{E^0}^2. \tag{9.99}$$

From (9.97), (9.99), we get

$$\|A_T s\|_{E^0}^2\geqslant\sum_{j=1}^{2}\|A_T s_j\|_{E^0}^2-\eta\|s\|_{E^1}^2-C_\eta\|s\|_{E^0}^2. \tag{9.100}$$

We now use (9.100) and Propositions 9.12 and 9.13. We obtain

$$\|A_T s\|_{E^0}^2\geqslant C\sum_{j=1}^{2}\|s_j\|_{E^1}^2-\eta\|s\|_{E^1}^2+C(T-b)\sum_{j=1}^{2}\|s_j\|_{E^0}^2-C_\eta\|s\|_{E^0}^2. \tag{9.101}$$

Since $\tau_1^2+\tau_2^2=1$, it is clear that

$$\begin{gathered}\sum_{j=1}^{2}\|s_j\|_{E^0}^2=\|s\|_{E^0}^2\\ \sum_{j=1}^{2}\|s_j\|_{E^1}^2\geqslant\frac{1}{2}\|s\|_{E^1}^2-C'\|s\|_{E^0}^2.\end{gathered} \tag{9.102}$$

From (9.101), (9.102), we get

$$(9.103)\qquad \|A_T s\|^2_{E^0} \geqslant \left(\frac{C}{2} - \eta\right) \|s\|^2_{E^1} + (CT - Cb - C_\eta - CC') \|s\|^2_{E^0}.$$

By taking $\eta \in]0, (C/4)]$, we obtain (9.48). Theorem 9.11 is proved. □

From Theorem 9.11, we obtain the following important estimate.

Theorem **9.14.** – *There exist* $T_0 > 0$, $c > 0$ *such that for any* $T \geqslant T_0$,, $s \in E_T^{1,\perp}$, *then*

$$(9.104)\qquad \|A_{T,4} s\|_{E^0} \geqslant c(\|s\|_{E^1} + \sqrt{T}\|s\|_{E^0}).$$

Proof. – Clearly

$$A_{T,4} s = A_T s - A_{T,2} s.$$

Then (9.104) follows from Theorems 9.10 and 9.11. □

c) Uniform estimates on the resolvent of $A_{T,4}$.

We now fix $\varepsilon \in]0, (\varepsilon_0/4)]$ **once and for all** as in Theorem 9.11. Also c denotes the positive constant which was determined in Theorem 9.14.

We still write the operators acting on E^0 in matrix form with respect to the splitting

$$E^0 = E_T^0 \oplus E_T^{0,\perp}.$$

Definition **9.15.** – Let A'_T be the operator

$$(9.105)\qquad A'_T = \begin{bmatrix} A_{T,1} & 0 \\ 0 & A_{T,4} \end{bmatrix}.$$

Proposition **9.16.** – *There exist* $T_0 \geqslant 1$, $C > 0$ *such that:*

- *For any* $T \geqslant T_0$, *the operator* A'_T *is self-adjoint with domain* E^1, *and the operator* $A_{T,4}$ *is one to one from* $E_T^{1,\perp}$ *into* $E_T^{0,\perp}$.
- *For any* $T \geqslant T_0$, $\lambda \in \mathbf{C}$, $|\lambda| \leqslant c/2\sqrt{T}$, $s \in E_T^{0,\perp}$, *then*

$$(9.106)\qquad \begin{aligned} &\|(\lambda - A_{T,4})^{-1} s\|_{E_T^{0,\perp}} \leqslant \frac{C}{\sqrt{T}} \|s\|_{E_T^{0,\perp}}, \\ &\|(\lambda - A_{T,4})^{-1} s\|_{E_T^{1,\perp}} \leqslant C \|s\|_{E_T^{0,\perp}}, \end{aligned}$$

Proof. – By (9.50), $[D^X, V]$ is an operator of order zero. Since D^X is elliptic of order 1, using (9.89), we find there exists $C > 0$ such that for any $T \geqslant 1$, and any $s \in E^1$,

$$(9.107)\qquad \|s\|_{E^1} \leqslant C(\|A_T s\|_{E^0} + \sqrt{T}\|s\|_{E^0}).$$

Also by using the estimate (9.8) in Proposition 9.3, Proposition 9.5 and Theorem 9.10, we find that if $s \in E^1$

$$\text{(9.108)} \qquad \|(A_T - A'_T)s\|_{E^0} \leqslant C\left(\frac{\|s\|_{E^1}}{\sqrt{T}} + \|s\|_{E^0}\right).$$

From (9.107), (9.108), we get

$$\text{(9.109)} \qquad \|(A_T - A'_T)s\|_{E^0} \leqslant C'\left(\frac{\|A_T s\|_{E^0}}{\sqrt{T}} + \|s\|_{E^0}\right).$$

For $T \geqslant 1$ large enough, $C'/\sqrt{T}$ is strictly smaller than 1. Also for any $T \geqslant 1$, A_T is a self-adjoint operator with domain E^1. By the Kato-Rellich theorem [ReSi, Theorem X.12], we deduce that for $T \geqslant 1$ large enough, the operator A'_T is self-adjoint with domain E^1. In particular for $T \geqslant 1$ large enough, $A_{T,4}$ is self-adjoint with domain $E_T^{1,\perp}$. From Theorem 9.14, we see that for T large enough, $A_{T,4}$ is one to one from $E_T^{1,\perp}$ into $E_T^{0,\perp}$.

The first line in (9.106) follows from Theorem 9.14. Also by Theorem 9.14, if $s \in E_T^{0,\perp}$, $|\lambda| \leqslant c/2\sqrt{T}$, then

$$\|(\lambda - A_{T,4})^{-1} s\|_{E_T^{1,\perp}} \leqslant \frac{1}{c}\left\|\left(\frac{A_{T,4}}{\lambda - A_{T,4}}\right)s\right\|_{E_T^{0,\perp}} \leqslant C\|s\|_{E_T^{0,\perp}}.$$

Proposition 9.16 is proved. □

Definition **9.17.** – If H, H′ are separable Hilbert space, if $1 \leqslant p < +\infty$, set

$$\mathscr{L}_p(H, H') = \{A \in \mathscr{L}(H, H');\ \operatorname{Tr}[(A^* A)^{p/2}] < +\infty\}.$$

If $A \in \mathscr{L}_p(H, H')$, set

$$\|A\|_p = \{\operatorname{Tr}[(A^* A)^{p/2}]\}^{1/p}.$$

Then by [ReSi, Th. IX, p. 42], $\|\ \|_p$ is a norm on $\mathscr{L}_p(H, H')$. Similarly, if $A \in \mathscr{L}(H, H')$, let $\|A\|_\infty$ be the usual norm of A.

In the sequel, the norms $\|\ \|_p$, $\|\ \|_\infty$ will always be calculated with respect to the Sobolev spaces of order zero like E_T^0, $E_T^{0,\perp}$, F^0.

Recall that T_0 has been determined in Proposition 9.16.

Proposition **9.18.** – *If* $p \geqslant 2\dim X+1$, *there exists* $C>0$ *such that for* $T \geqslant T_0$, $\lambda \in \mathbf{C}$, $|\lambda| \leqslant c/2\sqrt{T}$, *then*

$$\begin{aligned} &\|(\lambda - A_{T,4})^{-1}\|_\infty \leqslant \frac{C}{\sqrt{T}}, \\ &\|(\lambda - A_{T,4})^{-1}\|_p \leqslant C, \\ &\|A_{T,2}(\lambda - A_{T,4})^{-1}\|_\infty \leqslant \frac{C}{\sqrt{T}}. \end{aligned} \tag{9.110}$$

Proof. – The first line of (9.110) was proved in Proposition 9.16. Also

$$\|(\lambda - A_{T,4})^{-1}\|_p \leqslant \|(D^X + \sqrt{-1})^{-1}\|_p \|(D^X + \sqrt{-1})(\lambda - A_{T,4})^{-1}\|_\infty. \tag{9.111}$$

Since D^X is elliptic of order one, when $p \geqslant 2\dim X+1$, $\|(D^X+\sqrt{-1})^{-1}\|_p < +\infty$. Also by Proposition 9.16, for $T \geqslant T_0$

$$\|(D^X + \sqrt{-1})(\lambda - A_{T,4})^{-1}\|_\infty \leqslant C.$$

The second line in (9.110) follows. Using (9.38), (9.106), we get the third line in (9.110). □

d) Estimates on the resolvent of D^Y.

Recall that the linear isometry $J_T: F^0 \to E_T^0$ was defined in Definition 9.4.

Let $\mathrm{Sp}(D^Y)$ be the spectrum of D^Y. Also $c_2 \in]0, 1]$ is a constant fixed once and for all such that

$$\mathrm{Sp}(D^Y) \cap \{\lambda \in \mathbf{R};\ |\lambda| \leqslant 2c_2\} \subset \{0\}. \tag{9.112}$$

Here $c>0$ is the constant determined in Theorem 9.14. Take $c_1 \in]0, (c/2)]$. The precise value of c_1 will be fixed in Theorem 9.21.

For $T \geqslant 1$, set

$$U_T = \left\{\lambda \in \mathbf{C};\ |\lambda| \leqslant c_1\sqrt{T};\ \inf_{\mu \in \mathrm{Sp}(D^Y)} |\lambda - \mu| \geqslant \frac{c_2}{4}\right\}. \tag{9.113}$$

Proposition **9.19.** – *There exists a constant* $C>0$ *such that for* $T \geqslant 1$, $\lambda \in U_T$, *then*

$$\|A_T^3(\lambda - J_T D^Y J_T^{-1})^{-1}\|_\infty \leqslant C. \tag{9.114}$$

Proof. — Clearly if $\lambda \in U_T$,

$$\|(\lambda - D^Y)^{-1}\|_\infty \leqslant C. \tag{9.115}$$

Since J_T is an isometry from F^0 into E^0_T, we get

$$\|(\lambda - J_T D^Y J_T^{-1})^{-1}\|_\infty \leqslant C. \tag{9.116}$$

By the resolvent equation, we find that

$$(\lambda - D^Y)^{-1} = (\sqrt{-1} - D^Y)^{-1} + (\sqrt{-1} - \lambda)(\lambda - D^Y)^{-1}(\sqrt{-1} - D^Y)^{-1}. \tag{9.117}$$

From (9.117), we see that if $\lambda \in U_T$, $\sigma \in F^0$,

$$\|(\lambda - D^Y)^{-1}\sigma\|_{F^1} \leqslant C'(1 + |\lambda|)\|\sigma\|_{F^0}. \tag{9.118}$$

Using (9.118) and Proposition 9.3, we find that if $\lambda \in U_T$, $s \in E^0_T$,

$$\|(\lambda - J_T D^Y J_T^{-1})^{-1} s\|_{E^1_T} \leqslant C''(1 + \sqrt{T})\|s\|_{E^0_T}. \tag{9.119}$$

From Theorem 9.10 and from (9.116), (9.119), we get (9.114). □

e) Estimates on the resolvent of A_T.

By Proposition 9.16, for $T \geqslant T_0$, $\lambda \in U_T$, the operator $\lambda - A_{T,4}$ is an invertible operator from $E^{1,\perp}_T$ into $E^{0,\perp}_T$.

Definition **9.20.** — For $T \geqslant T_0$, $\lambda \in U_T$, let $M_T(\lambda)$ be the linear map from E^1_T into E^0_T

$$M_T(\lambda) = \lambda - A_{T,1} - A_{T,2}(\lambda - A_{T,4})^{-1} A_{T,3}. \tag{9.120}$$

If $s \in E^0$, set

$$s_1 = \bar{p}_T s; \; s_2 = \bar{p}^\perp_T s. \tag{9.121}$$

Of course $s = s_1 + s_2$.

Take then $T \geqslant T_0$, $\lambda \in U_T$, $s \in E^1$, $s' \in E^0$. Consider the equation

$$(\lambda - A_T) s = s'. \tag{9.122}$$

By (9.18), (9.121), it is clear that (9.122) is equivalent to

$$\begin{aligned} M_T(\lambda) s_1 &= s'_1 + A_{T,2}(\lambda - A_{T,4})^{-1} s'_2, \\ s_2 &= (\lambda - A_{T,4})^{-1}(s'_2 + A_{T,3} s_1). \end{aligned} \tag{9.123}$$

From (9.123), we deduce that to estimate $(\lambda - A_T)^{-1}$, we need first to estimate $M_T^{-1}(\lambda)$.

Theorem **9.21.** — *If* $c_1 \in]0, (c/2)]$ *is small enough, there exists* $T_0 \geqslant 1$ *such that if* $T \geqslant T_0$, $\lambda \in U_T$, *then* $M_T(\lambda)$ *is invertible and moreover for any integer* $p \geqslant 2\dim X + 1$, *there exist* $C > 0$ *such that for* $T \geqslant T_0$, $\lambda \in U_T$

$$\begin{aligned} &\| M_T^{-1}(\lambda) \|_\infty \leqslant C, \\ &\| A_{T,3} M_T^{-1}(\lambda) \|_\infty \leqslant C, \\ &\| M_T^{-1}(\lambda) \|_p \leqslant C(1 + |\lambda|), \\ &\| J_T^{-1} (M_T^{-1}(\lambda))^p J_T - (\lambda - D^Y)^{-p} \|_1 \leqslant \frac{C}{\sqrt{T}} (1 + |\lambda|)^{p+1}. \end{aligned} \tag{9.124}$$

Proof. — Set

$$C_T = J_T^{-1} A_{T,1} J_T - D^Y. \tag{9.125}$$

For $\lambda \in U_T$, set

$$\begin{aligned} m_T(\lambda) = 1 - J_T C_T (\lambda - D^Y)^{-1} J_T^{-1} \\ - A_{T,2} (\lambda - A_{T,4})^{-1} A_{T,3} (\lambda - J_T D^Y J_T^{-1})^{-1}. \end{aligned} \tag{9.126}$$

Clearly

$$M_T(\lambda) = m_T(\lambda)(\lambda - J_T D^Y J_T^{-1}). \tag{9.127}$$

Now by Theorem 9.8 and inequality (9.118), we find that

$$\| C_T (\lambda - D^Y)^{-1} \|_\infty \leqslant \frac{C}{\sqrt{T}} (1 + |\lambda|). \tag{9.128}$$

Also, by Propositions 9.18 and 9.19, we get

$$\| A_{T,2} (\lambda - A_{T,4})^{-1} A_{T,3} (\lambda - J_T D^Y J_T^{-1})^{-1} \|_\infty \leqslant \frac{C'}{\sqrt{T}}. \tag{9.129}$$

From (9.126)-(9.129), it is clear that if $1/\sqrt{T}$ and $|\lambda|/\sqrt{T}$ are small enough, the operator $m_T(\lambda)$ is invertible, and moreover for $T \geqslant 1$

$$\| m_T^{-1}(\lambda) - 1 \|_\infty \leqslant \frac{C}{\sqrt{T}} (1 + |\lambda|). \tag{9.130}$$

In particular

$$\| m_T^{-1}(\lambda) \|_\infty \leqslant C'. \tag{9.131}$$

Under such conditions, by (9.127), we get

$$M_T^{-1}(\lambda) = (\lambda - J_T D^Y J_T)^{-1} m_T^{-1}(\lambda). \tag{9.132}$$

From (9.115), (9.131), we obtain the first inequality in (9.124). The second inequality in (9.124) follows from (9.114), (9.131) and (9.132). From (9.131), (9.132), we also get

$$\| M_T^{-1}(\lambda) \|_p \leqslant C' \|(\lambda - D^Y)^{-1}\|_p. \tag{9.133}$$

Using the identity (9.117), we find that if $\lambda \in U_T$

$$\|(\lambda - D^Y)^{-1}\|_p \leqslant C''(1 + |\lambda|). \tag{9.134}$$

The third inequality in (9.124) follows from (9.133), (9.134). Finally using (9.130)-(9.134), we get the last inequality in (9.124). □

From now on, $c_1 \in]0, (c/2)]$ is taken as in Theorem 9.21.

If $B \in \mathscr{L}(E^0)$, for any $T \geqslant 1$, we write B as a matrix with respect to the splitting $E^0 = E_T^0 \oplus E_T^{0,\perp}$ in the form

$$B = \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix}.$$

Definition **9.22.** – If $B \in \mathscr{L}(E^0)$, $C \in \mathscr{L}(F^0)$, set

$$d(B, C) = \sum_{j=2}^{4} \|B_j\|_1 + \|J_T^{-1} B_1 J_T - C\|_1. \tag{9.135}$$

Clearly if $B \in \mathscr{L}_1(E^0)$, $C \in \mathscr{L}_1(F^0)$

$$|\operatorname{Tr}(B) - \operatorname{Tr}(C)| \leqslant d(B, C). \tag{9.136}$$

Theorem **9.23.** – *There exists* $T_0 \geqslant 1$ *such that for any* $T \geqslant T_0$, $\lambda \in U_T$, $\lambda - A_T$ *is invertible. For any integer* $p \geqslant 2 \dim X + 2$, *there exists* $C > 0$ *such that if* $T \geqslant T_0$, $\lambda \in U_T$, *then*

$$d((\lambda - A_T)^{-p}, (\lambda - D^Y)^{-p}) \leqslant \frac{C}{\sqrt{T}}(1 + |\lambda|)^{p+1}. \tag{9.137}$$

Proof. – Set

$$B_T = (\lambda - A_T)^{-1}. \tag{9.138}$$

In view of (9.123), we find that

$$\begin{aligned} &B_{T,1}=M_T^{-1}(\lambda),\\ &B_{T,2}=M_T^{-1}(\lambda)A_{T,2}(\lambda-A_{T,4})^{-1},\\ &B_{T,3}=(\lambda-A_{T,4})^{-1}A_{T,3}M_T^{-1}(\lambda),\\ &B_{T,4}=(\lambda-A_{T,4})^{-1}(1+A_{T,3}B_{T,2}). \end{aligned} \tag{9.139}$$

If $\lambda\in U_T$, then $|\lambda|\leqslant c_1\sqrt{T}$. Using Proposition 9.18 and Theorem 9.21, we find that if $p\geqslant 2\dim X+2$, $T\geqslant T_0$, $\lambda\in U_T$, for $2\leqslant j\leqslant 4$, then

$$\|B_{T,j}\|_{p-1}\leqslant C;\ \|B_{T,j}\|_\infty\leqslant\frac{C}{\sqrt{T}}. \tag{9.140}$$

From Theorem 9.21 and from (9.140), we deduce that if $j_1,\ldots,j_p\in\{1,2,3,4\}$, if one of the j_k's is not equal to 1, then

$$\|B_{T,j_1}\ldots B_{T,j_p}\|_1\leqslant\frac{C}{\sqrt{T}}(1+|\lambda|)^{p-1}. \tag{9.141}$$

Theorem 9.23 follows from the fourth inequality in (9.124) and from (9.141). □

Let L be a smooth section of $\mathrm{End}^{\mathrm{even}}(\Lambda(T^{*(0,1)}X)\widehat{\otimes}\xi)$.We define L^θ as in (8.6).

We now obtain the essential technical result of this section.

Theorem **9.24.** – *For any integer* $p\geqslant 2\dim X+2$, *there exists a constant* $C>0$ *such that for any* $T\geqslant T_0$, *any* $\lambda\in U_T$,

$$|\mathrm{Tr}[L(\lambda-A_T)^{-p}]-\mathrm{Tr}[L^\theta(\lambda-D^Y)^{-p}]|\leqslant\frac{C}{\sqrt{T}}(1+|\lambda|)^{p+1}. \tag{9.142}$$

Proof. – Clearly

$$\begin{aligned} |\mathrm{Tr}[L(\lambda-A_T)^{-p}]-\mathrm{Tr}[L^\theta(\lambda-D^Y)^{-p}]|&\leqslant C\sum_{j=2}^{4}\|(\lambda-A_T)^{-p}\|_1\\ &+|\mathrm{Tr}[\bar{p}_T L\bar{p}_T(\lambda-A_T)_1^{-p}]-\mathrm{Tr}[L^\theta(\lambda-D^Y)^{-p}]|. \end{aligned} \tag{9.143}$$

By Theorem 9.23, for $2\leqslant j\leqslant 4$, if $T\geqslant T_0$, $\lambda\in U_T$

$$\|(\lambda-A_T)_j^{-p}\|_1\leqslant\frac{C}{\sqrt{T}}(1+|\lambda|)^{p+1}. \tag{9.144}$$

Also since J_T is an isometry of Hilbert spaces, we get

$$\begin{aligned} &|\mathrm{Tr}[\bar{p}_T L\bar{p}_T(\lambda-A_T)_1^{-p}]-\mathrm{Tr}[L^\theta(\lambda-D^Y)^{-p}]|\\ &\leqslant\|J_T^{-1}\bar{p}_T L\bar{p}_T J_T-L^\theta\|_\infty\|(\lambda-A_T)_1^{-p}\|_1 \end{aligned} \tag{9.145}$$

$$+C\|J_T^{-1}(\lambda-A_T)_1^{-p}J_T-(\lambda-D^Y)^{-p}\|_1.$$

Over $\mathscr{U}_{\varepsilon_0}$, we have identified $(\Lambda(T^{*(0,1)}X)\hat{\otimes}\xi)_{(y,Z)}$ with $(\Lambda(T^{*(0,1)}X)\hat{\otimes}\xi)_y$. Therefore for $|Z|<\varepsilon_0$, $L(y,Z)$ now acts on $(\Lambda(T^{*(0,1)}X)\hat{\otimes}\xi)_y$. Using the notation of Section 8 a), and also Propositions 9.2 and 9.5, we get

$$(J_T^{-1}\bar{p}_T L\bar{p}_T J_T)(y)=\frac{1}{\alpha_T}\int_{N_{\mathbf{R},y}}\rho^2(Z')\exp(-T|Z'|^2)\,\varphi^{-1}q\,L(y,Z')\,q\,\varphi\frac{dv_N(Z')}{(2\pi)^{\dim N}}. \tag{9.146}$$

Using (9.145), (9.146), and the fact that ρ and L are smooth, we find there exists $C>0$ such that for $T\geqslant 1$, $y\in Y$

$$|(J_T^{-1}\bar{p}_T L\bar{p}_T J_T-L^\theta)(y)|\leqslant\frac{C}{\sqrt{T}}. \tag{9.147}$$

By Theorem 9.23, we know that

$$\|(\lambda-A_T)_1^{-p}\|_1\leqslant\|(\lambda-D_Y)^{-p}\|_1+\frac{C}{\sqrt{T}}(1+|\lambda|)^{p+1}. \tag{9.148}$$

On the other hand, by (9.134), if $\lambda\in U_T$

$$\|(\lambda-D_Y)^{-p}\|_1\leqslant C(1+|\lambda|)^p. \tag{9.149}$$

Using again Theorem 9.23 and (9.145)-(9.149), we obtain

$$|\mathrm{Tr}[\bar{p}_T L\bar{p}_T(\lambda-A_T)_1^{-p}]-\mathrm{Tr}[L^\theta(\lambda-D^Y)^{-p}]|\leqslant\frac{C'}{\sqrt{T}}(1+|\lambda|)^{p+1}. \tag{9.150}$$

From (9.143), (9.144), (9.150), we get (9.142). □

f) The spectrum of A_T.

We now will obtain a crucial information on the spectrum of the operator A_T.

Recall that the constant $c_2\in]0,1]$ was fixed once and for all in Section 9d). Let $\mathrm{Sp}(A_T)$ be the spectrum of A_T.

Theorem **9.25.** – *There exists* $T_0\geqslant 1$, *such that for* $T\geqslant T_0$

$$\mathrm{Sp}(A_T)\cap\{\lambda\in\mathbf{R};\,|\lambda|\leqslant c_2\}\subset\{0\}. \tag{9.151}$$

Proof. — Let γ be the circle in $\mathbf{C}$ of center 0 and radius c_2. From (9.112), (9.113), we find that for T large enough, $\gamma \subset U_T$. Using Theorem 9.23, we see that for T large enough

$$\gamma \cap \mathrm{Sp}(A_T) = \varnothing. \tag{9.152}$$

Let $\tilde{K}_T^{c_2}$ be the direct sum of the eigenspaces of A_T associated to eigenvalues λ of A_T such that $|\lambda| < c_2$. For T large enough, set

$$\tilde{P}_T^{c_2} = \frac{1}{2\pi i}\int_\gamma (\lambda - A_T)^{-1}\, d\lambda. \tag{9.153}$$

Then $\tilde{P}_T^{c_2}$ is exactly the orthogonal projection operator from E^0 on $\tilde{K}_T^{c_2}$. Integrating by parts in (9.153), for any $p \in \mathbf{N}$, we get

$$\tilde{P}_T^{c_2} = \frac{1}{2\pi i}\int_\gamma \lambda^{p-1}(\lambda - A_T)^{-p}\, d\lambda. \tag{9.154}$$

Using Theorem 9.23, we find that if Q is the orthogonal projection operator from F^0 on $K' = \mathrm{Ker}(D^Y)$, then for T large enough,

$$d(\tilde{P}_T^{c_2}, Q) \leqslant \frac{C}{\sqrt{T}}. \tag{9.155}$$

From (9.155), we deduce that for T large enough

$$\dim \tilde{K}_T^{c_2} = \dim K'. \tag{9.156}$$

Recall that the finite dimensional vector subspace K_T of E was defined in (6.2). By Hodge theory

$$\dim K_T = \dim H^*(E, \bar{\partial}^X + v). \tag{9.157}$$

Also by Theorem 1.7, the complexes $(E, \bar{\partial}^X + v)$ and $(F, \bar{\partial}^Y)$ are quasi-isomorphic. Therefore

$$\dim H^*(E, \bar{\partial}^X + v) = \dim H^*(Y, \eta). \tag{9.158}$$

Using again Hodge theory, we find that

$$\dim H^*(Y, \eta) = \dim K'. \tag{9.159}$$

Recall that $\tilde{K}_T = \mathrm{Ker}(A_T)$. By equation (6.6), we find that for $T > 0$

$$\dim \tilde{K}_T = \dim K_T. \tag{9.160}$$

By (9.157)-(9.160), we deduce that for any $T > 0$

(9.161) $$\dim \tilde{K}_T = \dim K'.$$

Now clearly

(9.162) $$\dim \tilde{K}_T^{c_2} \geqslant \dim \tilde{K}_T.$$

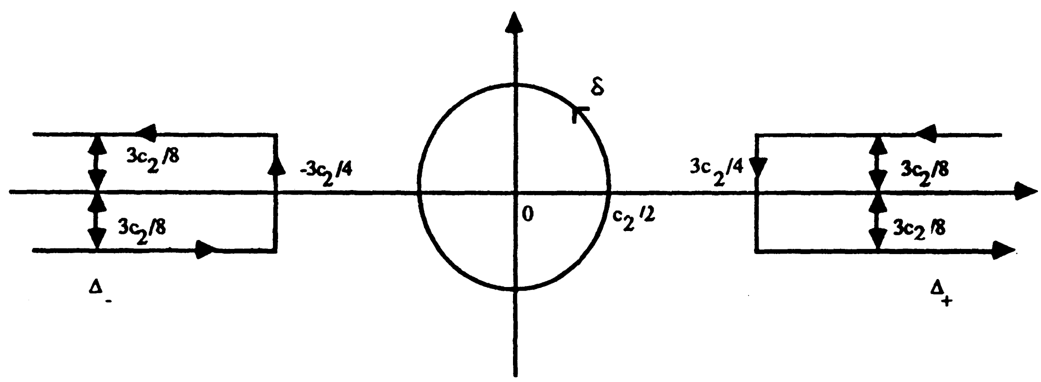
In view of (9.156), (9.161), (9.162), we see that for T large enough

(9.163) $$\tilde{K}_T^{c_2} = \tilde{K}_T.$$

Our Theorem is proved. □

Remark **9.26.** – Theorem 9.25 plays a very important role in the whole paper. The key equality (9.161) follows from sheaf theoretic considerations, and does not have a direct analytic proof.

g) Proof of Theorem 8.2.



Let $\Delta = \Delta_+ \cup \Delta_-$ be the oriented contour in **C** indicated above. Similarly let δ be the circle in **C** of center 0 and radius $c_2/2$.

By Theorem 9.25, for T large enough, the eigenvalues of the operator A_T lie in the interior of the domain limited by the contour $\Delta \cup \delta$, and inside δ, the only possible eigenvalue is 0. Therefore, for T large enough and $\alpha > 0$

(9.164) $$\exp(-\alpha A_T^2) = \frac{1}{2\pi i}\int_\Delta \exp(-\alpha\lambda^2)(\lambda - A_T)^{-1}\, d\lambda + \frac{1}{2\pi i}\int_\delta (\lambda - A_T)^{-1}\, d\lambda.$$

Of course, a similar formula holds when A_T is replaced by D^Y.

Take $p \in \mathbf{N}$. Let f_p be the unique holomorphic function defined on $\mathbf{C} \setminus \sqrt{-1}\,\mathbf{R}$ with values in $\mathbf{C}$ which has the following two properties.

- As $\lambda \to \pm\infty$, $f_p(\lambda) \to 0$.
- The following equation holds

$$\frac{f_p^{(p-1)}(\lambda)}{(p-1)!} = \exp(-\lambda^2). \tag{9.165}$$

Clearly

$$\frac{1}{2\pi i}\int_\Delta \exp(-\alpha\lambda^2)(\lambda - A_T)^{-1}\,d\lambda = \frac{1}{2\pi i}\int_\Delta \frac{f_p(\sqrt{\alpha}\lambda)}{(\sqrt{\alpha})^{p-1}}(\lambda - A_T)^{-p}\,d\lambda. \tag{9.166}$$

Let L be a smooth section of $\mathrm{End}^{\mathrm{even}}(\Lambda(T^{*(0,1)}X) \hat{\otimes} \xi)$, let L^θ be defined as in (8.6).

Assume that $p \geqslant 2\dim X + 2$. By Theorem 9.24, we find that

$$\begin{aligned} &\left| \mathrm{Tr}\left[\frac{1}{2\pi i}\int_{\Delta \cap U_T} \frac{f_p(\sqrt{\alpha}\lambda)}{(\sqrt{\alpha})^{p-1}} L(\lambda - A_T)^{-p}\right]\right. \\ &\quad \left. - \mathrm{Tr}\left[\frac{1}{2\pi i}\int_{\Delta \cap U_T} \frac{f_p(\sqrt{\alpha}\lambda)}{(\sqrt{\alpha})^{p-1}} L^\theta(\lambda - D^Y)^{-p}\right]\right| \\ &\quad \leqslant \frac{C}{\sqrt{T}}\int_\Delta \frac{|f_p(\sqrt{\alpha}\lambda)|}{(\sqrt{\alpha})^{p-1}}(1+|\lambda|)^{p+1}\,d\lambda. \end{aligned} \tag{9.167}$$

Now there exists $c' \in]0,1[$ such that if $\lambda \in \Delta$, then $|\mathrm{Im}\,\lambda| \leqslant c'|\mathrm{Re}\,\lambda|$. Also there exist $C' > 0$, $C'' > 0$ such that if $\lambda \in \mathbf{C}$, $|\mathrm{Im}\,\lambda| \leqslant c'|\mathrm{Re}\,\lambda|$,

$$|f_p(\lambda)| \leqslant C' \exp(-C''|\lambda|^2). \tag{9.168}$$

Take $\alpha_0 > 0$. From (9.167), we deduce that there exist $c_0 > 0$, $C_0 > 0$, such that if $\alpha \geqslant \alpha_0$,

$$\int_\Delta \frac{|f_p(\sqrt{\alpha}\lambda)|}{(\sqrt{\alpha})^{p-1}}(1+|\lambda|)^{p+1}\,d\lambda \leqslant C_0\exp(-c_0\alpha). \tag{9.169}$$

Also for T large enough, if $\lambda \in \Delta \cap {}^cU_T$, then $|\lambda| \geqslant c_1\sqrt{T}$, $|\mathrm{Im}\,\lambda| = 3c_2/8$. Using the resolvent identity as in (9.117) and also Theorem 9.23, we find that for T large enough, and $\lambda \in \Delta \cap {}^cU_T$,

$$\|(\lambda - A_T)^{-p}\|_1 \leqslant C(1+|\lambda|)^p. \tag{9.170}$$

From (9.170), it is clear that there exist $c_0' > 0$, $C_0' > 0$ such that for T large enough and $\alpha \geqslant \alpha_0$

$$\left|\mathrm{Tr}\left[\frac{1}{2\pi i}\int_{\Delta\cap {}^c\mathrm{U_T}} \frac{f_p(\sqrt{\alpha}\lambda)}{(\sqrt{\alpha})^{p-1}} \mathrm{L}(\lambda-\mathrm{A_T})^{-p}\,d\lambda\right]\right| \leqslant \mathrm{C}_0' \exp(-c_0'\alpha \mathrm{T}). \tag{9.171}$$

Of course an inequality similar to (9.171) holds when $\mathrm{A_T}$ is replaced by $\mathrm{D^Y}$.

By proceeding as in (9.153), (9.154), we also find that

$$\frac{1}{2\pi i}\int_\delta (\lambda-\mathrm{A_T})^{-1}\,d\lambda = \frac{1}{2\pi i}\int_\delta \lambda^{p-1}(\lambda-\mathrm{A_T})^{-p}\,d\lambda. \tag{9.172}$$

From Theorem 9.24, we then deduce that

$$\begin{aligned}\Bigg|\mathrm{Tr}&\left[\frac{1}{2\pi i}\int_\delta \lambda^{p-1}\mathrm{L}(\lambda-\mathrm{A_T})^{-p}\,d\lambda\right]\\ &-\mathrm{Tr}\left[\frac{1}{2\pi i}\int_\delta \lambda^{p-1}\mathrm{L}^\theta(\lambda-\mathrm{D^Y})^{-p}\,d\lambda\right]\Bigg| \leqslant \frac{\mathrm{C}}{\sqrt{\mathrm{T}}}.\end{aligned} \tag{9.173}$$

From (9.164), (9.167), (9.169), (9.173), we find that for T large enough

$$|\mathrm{Tr}[\mathrm{L}\exp(-\alpha \mathrm{A_T^2})] - \mathrm{Tr}[\mathrm{L}^\theta \exp(-\alpha(\mathrm{D^Y})^2)]| \leqslant \frac{\mathrm{C}}{\sqrt{\mathrm{T}}} \tag{9.174}$$

which is exactly Theorem 8.2.

h) Proof of Theorem 8.3.

We use the notation of Section 9g). By (9.151), (9.164), (9.166), for T large enough,

$$\mathrm{Tr}[\mathrm{L}\exp(-\alpha \mathrm{A_T^2})] - \mathrm{Tr}[\mathrm{L}\tilde{\mathrm{P}}_\mathrm{T}] = \mathrm{Tr}\left[\frac{1}{2\pi i}\int_\Delta \frac{f_p(\sqrt{\alpha}\lambda)}{(\sqrt{\alpha})^{p-1}}\mathrm{L}(\lambda-\mathrm{A_T})^{-p}\,d\lambda\right]. \tag{9.175}$$

Then using (9.134), (9.137), (9.170), we find that there exist $c > 0$, $\mathrm{C} > 0$ such that for $\alpha \geqslant 1$

$$|\mathrm{Tr}[\mathrm{L}\exp(-\alpha \mathrm{A_T^2})] - \mathrm{Tr}[\mathrm{L}\tilde{\mathrm{P}}_\mathrm{T}]| \leqslant c\exp(-\mathrm{C}\alpha). \tag{9.176}$$

The proof of Theorem 8.3 is completed. □

X - THE L_2 METRICS ON THE LINES $\tilde{\lambda}(\xi)$ AND $\lambda(\eta)$

The purpose of this Section is to prove Theorem 6.9, *i.e.* to show that as $T \to +\infty$, $-\mathrm{Log}(\|\rho\|^2_{\lambda^{-1}(\eta)\otimes\tilde{\lambda}(\xi)})$ is the constant term in the asymptotic expansion of $\mathrm{Log}(\|\ \|^2_{\tilde{\lambda}(\xi),T}/\|\ \|^2_{\tilde{\lambda}(\xi)})$.

Observe that the quasi-isomorphism $r: (E, \bar{\partial}^X + v) \to (F, \bar{\partial}^Y)$ of Theorem 1.7 was already used in Section 9f) to obtain a critical information on the dimension of the kernel of A_T. In this Section, the full strength of Theorem 1.7 will be needed, *i.e.* the fact that the canonical section $\rho \in \lambda^{-1}(\eta) \otimes \tilde{\lambda}(\xi)$ is precisely constructed through the quasi-isomorphism r.

In the case where Y has only one connected component, Theorem 6.9 is a rather easy consequence of Theorems 9.23 and 9.25. When Y has more than one connected component, the problem is complicated by the fact that one has to show that as $T \to +\infty$, the harmonic forms on Y_j $(1 \leqslant j \leqslant d)$ lift "approximately" to harmonic forms on X with respect to the Hermitian product $\langle\ ,\ \rangle_T$ which are $O(T^{-\infty})$ on compact subsets of $X \setminus Y_j$. This property is the asymptotic version of the mutual orthogonality of the F_j's in F.

This Section is organized as follows. In a), we lift harmonic forms in F to elements of $\mathrm{Ker}(A_T)$. In b), we use the results of a) to lift harmonic forms in F to harmonic forms in $(E, \bar{\partial}^X + v)$ with respect to the Hermitian product $\langle\ ,\ \rangle_T$. In c), we calculate the asymptotics as $T \to +\infty$ of the Hermitian product on $H^*(E, \bar{\partial}^X + v)$ induced by the metric $\langle\ ,\ \rangle_T$ on E via Hodge theory. Finally in d), we prove Theorem 6.9.

We here use the notation of Sections 1, 6, 8 and 9.

a) The lift of harmonic forms on Y to the kernel of A_T

We take $\varepsilon_0 > 0$ as in Sections 8e), f). For $1 \leqslant j \leqslant d$, set

$$B_{j,\varepsilon_0/2} = \left\{ Z \in N_{\mathbf{R},j};\ |Z| < \frac{\varepsilon_0}{2} \right\}. \tag{10.1}$$

As in Section 8e), we identify $B_{j,\varepsilon_0/2}$ with a tubular neighborhood $\mathcal{U}_{j,\varepsilon_0/2}$ of Y_j in X. Also since $\mathcal{U}_{\varepsilon_0}$ is a tubular neighborhood of $Y = \bigcup_1^d Y_j$ in X, we deduce that if $j \neq j'$, then

$$\bar{\mathscr{U}}_{j,\varepsilon_0/2} \cap \bar{\mathscr{U}}_{j',\varepsilon_0/2} = \varnothing. \tag{10.2}$$

Recall that Q is the orthogonal projection operator from F^0 on $\mathrm{Ker}(D^Y)$. We fix $\varepsilon \in]0, (\varepsilon_0/4)]$ as in Proposition 9.12. The linear map $J_T: F^0 \to E^0_T$ which depends on ε, was defined in Definition 9.4.

Theorem **10.1.** — *For any* $q \in \mathbf{N}$, *there exists* $C_q > 0$ *such that for any* j, $1 \leqslant j \leqslant d$, *for any* $\sigma \in \mathrm{Ker}(D^{Y_j})$, *for any* $T \geqslant 1$

$$\sup_{x \in X \setminus \mathscr{U}_{j,\varepsilon_0/2}} |\tilde{P}_T J_T \sigma|(x) \leqslant \frac{C_q}{T^q} \|\sigma\|_{F^0}. \tag{10.3}$$

There exists $C > 0$ *such that for any* $\sigma \in \mathrm{Ker}(D^Y)$, *any* $T \geqslant 1$

$$\| Q r \tilde{P}_T (\alpha_T 2^{\dim N})^{1/2} J_T \sigma - \sigma \|_{F^0} \leqslant \frac{C}{\sqrt{T}} \|\sigma\|_{F^0}. \tag{10.4}$$

Proof. — The proof of Theorem 10.1 will be divided into the obvious two parts:

A) *Proof of* (10.3).

Let $E^0(X \setminus \mathscr{U}_{j,\varepsilon_0/2})$ be the Hilbert space of sections of $\Lambda(T^{*(0,1)}X) \hat{\otimes} \xi$ over $X \setminus \mathscr{U}_{j,\varepsilon_0/2}$ which are square integrable. We equip $E^0(X \setminus \mathscr{U}_{j,\varepsilon_0/2})$ with the Hermitian product induced by the Hermitian product (1.38), (1.39) on E^0.

We will first prove that as $T \to +\infty$

$$\|\tilde{P}_T J_T \sigma\|_{E^0(X \setminus \mathscr{U}_{j,\varepsilon_0/2})} = O(T^{-\infty}). \tag{10.5}$$

Recall that the constant $c_2 \in]0, 1]$ was determined in (9.112). Let δ be the circle of center 0 and radius $c_2/2$ in $\mathbf{C}$. By Theorem 9.25, we know that for T large enough

$$\tilde{P}_T J_T \sigma = \frac{1}{2\pi i} \int_\delta (\lambda - A_T)^{-1} J_T \sigma \, d\lambda. \tag{10.6}$$

To prove (10.5), we only need to show that uniformly in $\lambda \in \delta$, as $T \to +\infty$

$$\|(\lambda - A_T)^{-1} J_T \sigma\|_{E^0(X \setminus \mathscr{U}_{j,\varepsilon_0/2})} = O(T^{-\infty}). \tag{10.7}$$

Now by Theorem 9.25, for T large enough, if $\lambda \in \delta$, $\|(\lambda - A_T)^{-1}\|_\infty$ is uniformly bounded. Therefore to prove (10.7), we only need to show that for any $\lambda \in \delta$, $m \in \mathbf{N}$, there exists $s_m(\lambda, T) \in E^1$ such that

$$\begin{aligned} & s_m(\lambda, T) = 0 \quad \text{on } X \setminus \mathscr{U}_{j,\varepsilon_0/2}, \\ & \|(\lambda - A_T) s_m(\lambda, T) - J_T \sigma\|_{E^0} = O(T^{-m/2}). \end{aligned} \tag{10.8}$$

We now use the notation of Section 8h). By proceeding as in the proof of Theorem 8.18, we find that there exist first order differential operators $\mathcal{O}_1, \ldots, \mathcal{O}_j, \ldots$ acting on $\mathbf{E}$ such that for any $m \in \mathbf{N}$

$$\text{(10.9)} \qquad \mathrm{F_T} k^{1/2} \mathrm{A_T} k^{-1/2} \mathrm{F_T^{-1}} = \mathrm{TV}^+(y) + \sqrt{\mathrm{T}}\,(\mathrm{D^N} + \tilde{\nabla}^{\xi}_{\mathrm{Z}} \mathrm{V}(y)) + \mathrm{D^H} + \mathrm{M} + \frac{1}{2} \tilde{\nabla}^{\xi}_{\mathrm{Z}} \tilde{\nabla}^{\xi}_{\mathrm{Z}} \mathrm{V}(y) + \sum_{p=1}^{m-1} \mathrm{T}^{-p/2} \mathcal{O}_p + O(\mathrm{T}^{-m/2}).$$

Moreover the coefficients of the operator $\mathcal{O}_j$ are polynomials in Z. Finally there exists $m' \in \mathbf{N}$ such that for any $k' \in \mathbf{N}$, $\mathrm{T} \geqslant 1$, the derivatives of order $\leqslant k'$ of the coefficients of the operator $O(\mathrm{T}^{-m/2})$ are dominated by $\mathrm{CT}^{-m/2}(1 + |\mathrm{Z}|)^{m'}$.

Let $f(\lambda, \mathrm{T})$ be a formal power series in $\mathbf{E}^0$

$$\text{(10.10)} \qquad f(\lambda, \mathrm{T}) = \sum_{k=0}^{+\infty} \mathrm{T}^{-k/2} f_k(\lambda);\; f_k(\lambda) \in \mathbf{E}^0.$$

Recall that ψ is the linear map $\sigma \in \mathrm{F}^0 \to \sigma\beta \in \mathbf{E}^0$. Take $\sigma \in \mathrm{Ker}(\mathrm{D}^{\mathrm{Y}_j})$. Tautologically, σ vanishes on $\mathrm{Y}_{j'}$ $(j' \neq j)$. The equations on $\mathbf{E}^0$ which we now consider will in effect be solved in $\mathbf{E}^0_j$ (which is the Hilbert space $\mathbf{E}^0$ attached to the single manifold Y_j). For $\lambda \in \delta$, consider the equation of formal power series

$$\text{(10.11)} \qquad \Big(-\mathrm{TV}^+(y) - \sqrt{\mathrm{T}}\,(\mathrm{D^N} + \tilde{\nabla}^{\xi}_{\mathrm{Z}} \mathrm{V}(y)) + \lambda - \mathrm{D^H} - \mathrm{M} - \frac{1}{2} \tilde{\nabla}^{\xi}_{\mathrm{Z}} \tilde{\nabla}^{\xi}_{\mathrm{Z}} \mathrm{V}(y) - \sum_{p=1}^{+\infty} \mathrm{T}^{-p/2} \mathcal{O}_p \Big) f(\lambda, \mathrm{T}) = \psi\sigma.$$

Recall that by Definition 8.19, $\mathbf{E}^0$ splits into

$$\text{(10.12)} \qquad \mathbf{E}^0 = \mathbf{E}^{+,0} \oplus \mathbf{E}^{-,0}.$$

Also by definition, $\mathbf{E}'^{,0}$ is the image of ψ in $\mathbf{E}^{-,0}$. Let $\mathbf{E}'^{,0,\perp}$ be the orthogonal space to $\mathbf{E}'^{,0}$ in $\mathbf{E}^0$. Let $\mathbf{E}'^{,0,\perp,-}$ be the orthogonal space to $\mathbf{E}'^{,0}$ in $\mathbf{E}^{-,0}$. Then $\mathbf{E}^0$ splits orthogonally into

$$\text{(10.13)} \qquad \mathbf{E}^0 = \mathbf{E}^{+,0} \oplus \mathbf{E}'^{,0} \oplus \mathbf{E}'^{,0,\perp,-}.$$

Clearly $\mathbf{E}^{-,0}$ is the kernel of $\mathrm{V}^+(y)$. Similarly by Theorem 7.4 and by (7.23), $\mathbf{E}'^{,0}$ is the kernel of $\mathrm{D}^{\mathrm{N},-} + \tilde{\nabla}^{\xi}_{\mathrm{Z}} \mathrm{V}^-(y)$.

We now decompose $f(\lambda, \mathrm{T})$ and $f_k(\lambda)$ according the splitting (10.13) of $\mathbf{E}^0$

$$\text{(10.14)} \qquad \begin{aligned} f(\lambda, \mathrm{T}) &= f^+(\lambda, \mathrm{T}) + \psi g(\lambda, \mathrm{T}) + f^{\perp,-}(\lambda, \mathrm{T}), \\ f_k(\lambda) &= f_k^+(\lambda) + \psi g_k(\lambda) + f_k^{\perp,-}(\lambda). \end{aligned}$$

Also we use Theorem 8.21, which asserts in particular that M maps $\mathbf{E}^-$ into $\mathbf{E}^+$ and that $1/2\,\tilde{\nabla}^\xi_Z \tilde{\nabla}^\xi_Z \mathrm{V}(y)$ maps $\mathbf{E}'^{,0}$ into $\mathbf{E}'^{,0,\perp,-}$. Using (8.94) and identifying the powers of $\sqrt{\mathrm{T}}$ in (10.11), we find that

$$
(10.15)\quad
\begin{aligned}
&\bullet\ f_0^+(\lambda)=f_1^+(\lambda)=0;\ f_0^{\perp,-}(\lambda)=0;\ (\lambda-\mathrm{D}^{\mathrm{Y}})\,g_0(\lambda)=\sigma;\\
&\bullet\ \text{for } k\geqslant 2,\ \mathrm{V}^+(y)\,f_k^+(\lambda) \text{ depends linearly on}\\
&\qquad \{f_m^+(\lambda)\}_{m\leqslant k-1},\ \{f_m^{\perp,-}(\lambda)\}_{m\leqslant k-2},\ \{g_m(\lambda)\}_{m\leqslant k-2};\\
&\bullet\ \text{for } k\geqslant 0,\ (\mathrm{D}^{\mathrm{N}}+\tilde{\nabla}^\xi_Z \mathrm{V}^-(y))\,f_k^{\perp,-}(\lambda) \text{ depends linearly on}\\
&\qquad \{f_m^+(\lambda)\}_{m\leqslant k-1},\ \{f_m^{\perp,-}(\lambda)\}_{m\leqslant k-1},\ \{g_m(\lambda)\}_{m\leqslant k-1};\\
&\bullet\ \text{for } k\geqslant 1,\ (\lambda-\mathrm{D}^{\mathrm{Y}})\,g_k(\lambda) \text{ depends linearly on}\\
&\qquad \{f_m^+(\lambda,\mathrm{T})\}_{m\leqslant k},\ \{f_m^{\perp,-}(\lambda,\mathrm{T})\}_{m\leqslant k},\ \{g_m(\lambda)\}_{m\leqslant k-1}.
\end{aligned}
$$

Let $\Delta^{\mathrm{N}_{\mathbf{R}}}$ be the Laplacian in the fibres of $\mathrm{N}_{\mathbf{R}}$. Let $e_1, \ldots, e_{2n}$ be an orthonormal base of $\mathrm{N}_{\mathbf{R}}$. Set

$$\mathrm{S}=\frac{\sqrt{-1}}{2}\sum_1^{2n} c(e_i)\,\hat{c}(\mathbf{J}\,e_i).$$

By Proposition 7.2, equation (7.23) and Proposition 8.13, we find that

$$
(10.16)\qquad (\mathrm{D}^{\mathrm{N},-}+\tilde{\nabla}^\xi_Z \mathrm{V}^-(y))^2=-\frac{\Delta^{\mathrm{N}_{\mathbf{R}}}}{2}+\frac{|\mathrm{Z}|^2}{2}+\mathrm{S}.
$$

Let $\mathscr{L}$ be the harmonic oscillator on $\mathbf{R}^{2n}$

$$\mathscr{L}=\frac{1}{2}(-\Delta+|\mathrm{Z}|^2-2n).$$

Let $\mathscr{L}^{-1}$ denote the inverse of $\mathscr{L}$ acting on the orthogonal space $\mathrm{L}_2^\perp$ to the kernel of $\mathscr{L}$, $\{\exp-(|\mathrm{Z}|^2/2)\}$, in $\mathrm{L}_2(\mathbf{R}^{2n})$. We extend $\mathscr{L}^{-1}$ by the zero map on $\{\exp-(|\mathrm{Z}|^2/2)\}$. By [ReSi, Theorem V-13], $\mathscr{L}^{-1}$ acts on the Schwartz space $\mathrm{S}(\mathbf{R}^{2n})$. Similarly, for any $a>0$, $(\mathscr{L}+a)^{-1}$ acts on the Schwartz space $\mathrm{S}(\mathbf{R}^{2n})$.

Let $\mathrm{U}_1, \ldots, \mathrm{U}_k \ldots$ and $\mathrm{V}_1, \ldots, \mathrm{V}_{k'} \ldots$ be arbitrary smooth sections of $\mathrm{T}_{\mathbf{R}}\mathrm{Y}$ and $\mathrm{N}_{\mathbf{R}}$ which span $(\mathrm{T}_{\mathbf{R}}\mathrm{Y})_y$ and $\mathrm{N}_{\mathbf{R},y}$ at every $y\in\mathrm{Y}$. Let $\mathrm{S}(\mathrm{N}_{\mathbf{R}})$ be the vector space of smooth sections s of $\tilde{\pi}^*((\Lambda(\mathrm{T}^{*(0,1)}\mathrm{X})\,\hat{\otimes}\,\xi)|_{\mathrm{Y}})$ over $\mathrm{N}_{\mathbf{R}}$ such that, if L is any differential operator on $\mathrm{N}_{\mathbf{R}}$ which is a product of a finite number of the operators ${}^0\tilde{\nabla}^{\mathrm{Y}}_{\mathrm{U}_k^{\mathrm{H}}}$ or ${}^0\tilde{\nabla}^{\mathrm{Y}}_{\mathrm{V}_{k'}}$, for any $q\in\mathrm{N}$, $|\mathrm{Z}|^q\mathrm{L}\,s(y,\mathrm{Z})$ remains bounded. Let $(\mathrm{D}^{\mathrm{N},-}+(\sqrt{-1}/\sqrt{2})\,\tilde{\nabla}^{\xi^-}_Z \mathrm{V}^-(y))^{-1}$ be the inverse of $\mathrm{D}^{\mathrm{N},-}+(\sqrt{-1}/\sqrt{2})\,\tilde{\nabla}^{\xi^-}_Z \mathrm{V}^-(y)$ acting on $\mathbf{E}'^{,0,\perp,-}$. We extend this operator to a linear map acting on $\mathbf{E}^{-,0}$ which is zero on $\mathbf{E}'^{,0}$. Clearly

$$\left(\mathrm{D}^{\mathrm{N},-}+\frac{\sqrt{-1}}{\sqrt{2}}\tilde{\nabla}^{\xi^-}_{\mathrm{Z}}\mathrm{V}^-(y)\right)^{-1}=\left[\left(\mathrm{D}^{\mathrm{N},-}+\frac{\sqrt{-1}}{\sqrt{2}}\tilde{\nabla}^{\xi^-}_{\mathrm{Z}}\mathrm{V}^-(y)\right)^2\right]^{-1}\left(\mathrm{D}^{\mathrm{N},-}+\frac{\sqrt{-1}}{\sqrt{2}}\tilde{\nabla}^{\xi^-}_{\mathrm{Z}}\mathrm{V}^-(y)\right). \tag{10.17}$$

Using equation (10.16) and the previous considerations on the harmonic oscillator, we find that the operator $(\mathrm{D}^{\mathrm{N},-}+(\sqrt{-1}/\sqrt{2})\tilde{\nabla}^{\xi^-}_{\mathrm{Z}}\mathrm{V}^-(y))^{-1}$ acts on the Schwartz space $\mathrm{S}(\mathrm{N}_{\mathbf{R}})$.

If $\lambda\in\delta$, we know that $\lambda\notin\mathrm{Sp}(\mathrm{D}^{\mathrm{Y}})$. We then deduce from the previous considerations that equation (10.11) determines a unique power series $f(\lambda,\mathrm{T})\in\mathrm{E}^0$. Moreover for every $n\geqslant 0$, $f_n^+(\lambda)$, $f_n^{\perp,-}(\lambda)$ lie in $\mathrm{S}(\mathrm{N}_{\mathbf{R}})$, and $g_n(\lambda)$ lies in F.

Set

$$s_m(\lambda,\mathrm{T})=\frac{\rho(\mathrm{Z})k^{-1/2}(y,\mathrm{Z})}{(\alpha_{\mathrm{T}}2^{\dim\mathrm{N}})^{1/2}}\mathrm{F}_{\mathrm{T}}^{-1}\left(\sum_{k=0}^{m+1}f_k(\lambda)\mathrm{T}^{-k/2}\right)(y,\mathrm{Z}). \tag{10.18}$$

We consider $s_m(\lambda,\mathrm{T})$ as an element of E which vanishes on $\mathrm{X}\diagdown\mathcal{U}_{j,\varepsilon}$. Then if $e_1,\ldots,e_{2l}$ is an orthonormal base of $\mathrm{T}_{\mathbf{R}}\mathrm{X}$, using Proposition 8.5, we get

$$(\lambda-\mathrm{A}_{\mathrm{T}})s_m(\lambda,\mathrm{T})=\frac{\rho(\mathrm{Z})}{(\alpha_{\mathrm{T}}2^{\dim\mathrm{N}})^{1/2}}(\lambda-\mathrm{A}_{\mathrm{T}})k^{-1/2}\mathrm{F}_{\mathrm{T}}^{-1}\left(\sum_{k=0}^{m+1}f_k(\lambda)\mathrm{T}^{-k/2}\right)-\sum_{1}^{2l}\frac{c(e_i)(\nabla_{e_i}\rho(\mathrm{Z}))k^{-1/2}}{\sqrt{2}(\alpha_{\mathrm{T}}2^{\dim\mathrm{N}})^{1/2}}\mathrm{F}_{\mathrm{T}}^{-1}\left(\sum_{k=0}^{m+1}f_k(\lambda)\mathrm{T}^{-k/2}\right). \tag{10.19}$$

Since ρ is equal to 1 on $\mathcal{U}_{j,\varepsilon/2}$, $\nabla_{e_i}\rho$ is equal to 0 on $\mathcal{U}_{j,\varepsilon/2}$. Using (10.9), (10.11), (10.19) and the fact that the $f_k(\lambda)$'s lie in $\mathrm{S}(\mathrm{N}_{\mathbf{R}})$, we easily deduce that (10.8) holds. We have thus proved (10.5).

Since $\mathrm{A}_{\mathrm{T}}=\mathrm{D}^{\mathrm{X}}+\mathrm{TV}$, and $\mathrm{A}_{\mathrm{T}}\tilde{\mathrm{P}}_{\mathrm{T}}\mathrm{J}_{\mathrm{T}}\sigma=0$, we get

$$\mathrm{D}^{\mathrm{X}}\tilde{\mathrm{P}}_{\mathrm{T}}\mathrm{J}_{\mathrm{T}}\sigma=-\mathrm{TV}\tilde{\mathrm{P}}_{\mathrm{T}}\mathrm{J}_{\mathrm{T}}\sigma, \tag{10.20}$$

and so $\|\mathrm{D}^{\mathrm{X}}\tilde{\mathrm{P}}_{\mathrm{T}}\mathrm{J}_{\mathrm{T}}\sigma\|_{\mathrm{E}^0(\mathrm{X}\diagdown\mathcal{U}_{j,\varepsilon})}=O(\mathrm{T}^{-\infty})$. Similarly $\mathrm{A}_{\mathrm{T}}^2\tilde{\mathrm{P}}_{\mathrm{T}}\mathrm{J}_{\mathrm{T}}\sigma=0$, and so using (9.89), and the fact that by (9.50), $[\mathrm{D}^{\mathrm{X}},\mathrm{V}]$ is an operator of order zero, we get $\|(\mathrm{D}^{\mathrm{X}})^2\tilde{\mathrm{P}}_{\mathrm{T}}\mathrm{J}_{\mathrm{T}}\sigma\|_{\mathrm{E}^0(\mathrm{X}\diagdown\mathcal{U}_{j,\varepsilon})}=O(\mathrm{T}^{-\infty})$. Also $\mathrm{A}_{\mathrm{T}}^3\tilde{\mathrm{P}}\mathrm{J}_{\mathrm{T}}\sigma=0$. Now $\mathrm{A}_{\mathrm{T}}^3$ is the sum of $(\mathrm{D}_{\mathrm{X}})^3$ and of an operator of order two with polynomial coefficients in T. Since $(\mathrm{D}^{\mathrm{X}})^2$ is an elliptic operator of order two, we deduce from the previous estimates that $\|(\mathrm{D}^{\mathrm{X}})^3\tilde{\mathrm{P}}_{\mathrm{T}}\mathrm{J}_{\mathrm{T}}\sigma\|_{\mathrm{E}^0(\mathrm{X}\diagdown\mathcal{U}_{j,\varepsilon+\varepsilon_0/16})}=O(\mathrm{T}^{-\infty})$. By iterating this procedure, we find that for any $k\in\mathbf{N}$

$$\|(\mathrm{D}^{\mathrm{X}})^k\tilde{\mathrm{P}}_{\mathrm{T}}\mathrm{J}_{\mathrm{T}}\sigma\|_{\mathrm{E}^0(\mathrm{X}\diagdown\mathcal{U}_{j,\varepsilon+\varepsilon_0/8})}=O(\mathrm{T}^{-\infty}). \tag{10.21}$$

Since D^X is an elliptic operator, and since $\varepsilon \leqslant (\varepsilon_0/4)$, we immediately deduce (10.3) from (10.21).

B) *Proof of* (10.4).

Take $\lambda \in \delta$ and $\sigma \in \mathrm{Ker}(D^Y)$. We again consider equation (10.11). In view of (10.15) and of the fact that $D^Y \sigma = 0$, we get in particular

$$g_0(\lambda) = \frac{\sigma}{\lambda}. \tag{10.22}$$

For $m \in \mathbf{N}$, set

$$R_m(\lambda, T) = F_T k^{1/2} (\lambda - A_T) k^{-1/2} \rho F_T^{-1} \left(\sum_0^{m+1} f_k(\lambda) T^{-k/2} \right) - (F_T \rho) \psi \sigma. \tag{10.23}$$

Recall that $\mathrm{Ker}(D^Y) \subset F$. Since $\mathrm{Ker}(D^Y)$ is finite dimensional, all the norms on $\mathrm{Ker}(D^Y)$ are equivalent. By (10.9), (10.11) and by the considerations after (10.19), we find that $R_m(\lambda, T) \in S(N_{\mathbf{R}})$ and that if q is an arbitrary semi-norm on $S(N_{\mathbf{R}})$, there exists $C > 0$ such that for any $T \geqslant 1$

$$q(R_m(\lambda, T)) \leqslant \frac{C}{T^{m/2}} \| \sigma \|_{F^0}. \tag{10.24}$$

Set

$$s'_m(\lambda, T) = k^{-1/2} \rho F_T^{-1} \left(\sum_0^{m+1} f_k(\lambda) T^{-k/2} \right). \tag{10.25}$$

Then by (10.23), (10.25), we find that

$$\begin{aligned} s'_m(\lambda, T) - (\lambda - A_T)^{-1} (\alpha_T 2^{\dim N})^{1/2} J_T \sigma \\ = (\lambda - A_T)^{-1} k^{-1/2} F_T^{-1} R_m(\lambda, T). \end{aligned} \tag{10.26}$$

Using (10.6), (10.26), we get

$$\begin{aligned} \frac{1}{2\pi i} \int_\delta s'_m(\lambda, T)\, d\lambda - \tilde{P}_T (\alpha_T 2^{\dim N})^{1/2} J_T \sigma \\ = \frac{1}{2\pi i} \int_\delta (\lambda - A_T)^{-1} k^{-1/2} F_T^{-1} R_m(\lambda, T)\, d\lambda. \end{aligned} \tag{10.27}$$

From (7.20) and from the fact that ρ and k are equal to 1 on Y, we see that

$$r k^{-1/2} \rho F_T^{-1} (\psi g_0(\lambda)) = g_0(\lambda). \tag{10.28}$$

By (10.15), $f_0(\lambda)=\psi g_0(\lambda)$. So from (10.22), (10.28), we find that

$$r\frac{1}{2\pi i}\int_\delta k^{-1/2}\rho F_T^{-1}(f_0(\lambda))\,d\lambda=\sigma. \tag{10.29}$$

So for a given $m\in\mathbf{N}$, by (10.25), (10.29), we obtain

$$\left\|\frac{Qr}{2\pi i}\int_\delta s'_m(\lambda, T)\,d\lambda-\sigma\right\|_{F^0}\leqslant\frac{C}{\sqrt{T}}\|\sigma\|_{F^0}. \tag{10.30}$$

Using (10.27), (10.30), we see that to prove (10.4), we only need to show that if $m\in\mathbf{N}$ is large enough, for $\lambda\in\delta$, $T\geqslant 1$

$$\|r(\lambda-A_T)^{-1}k^{-1/2}F_T^{-1}R_m(\lambda, T)\|_{F^0}\leqslant\frac{C'}{\sqrt{T}}\|\sigma\|_{F^0}. \tag{10.31}$$

Let μ be an integer. From (10.24), we find easily that

$$\|F_T^{-1}R_m(\lambda, T)\|_{E^\mu}\leqslant CT^{1/2\,(\mu-\inf_{1\leqslant j\leqslant d}(\dim N_j)-m)}\|\sigma\|_{E^0}. \tag{10.32}$$

Also

$$(\bar\lambda-A_T)(\lambda-A_T)=(D^X-\bar\lambda)(D^X-\lambda)+T(-(\lambda+\bar\lambda)V+[D^X, V])+T^2V^2. \tag{10.33}$$

Since by (9.50), $[D^X, V]$ is an operator of order zero, and since D^X is an elliptic operator of order one, using (10.33), we find that there exists $C>0$, such that for $T\geqslant 1$, $\lambda\in\delta$, $s\in E^1$,

$$\|s\|_{E^1}^2\leqslant C(\|(\lambda-A_T)s\|_{E^0}^2+T\|s\|_{E^0}^2). \tag{10.34}$$

In particular, if O is a differential operator of order $p-1$ acting on E with scalar principal symbol, then for $T\geqslant 1$, $\lambda\in\delta$, $s\in E$,

$$\|Os\|_{E^1}^2\leqslant C(\|(\lambda-A_T)Os\|_{E^0}^2+T\|Os\|_{E^0}^2). \tag{10.35}$$

Now

$$[A_T, O]=[D^X, O]+T[V, O],$$

and $[D^X, O]$ and $[V, O]$ are differential operators of order $p-1$ and $p-2$ respectively. Using (10.35), we find there exists $C'>0$ such that for $T\geqslant 1$, $\lambda\in\delta$, $s\in E$,

$$\|s\|_{E^p}^2\leqslant C'(\|(\lambda-A_T)s\|_{E^{p-1}}^2+T^2\|s\|_{E^{p-1}}^2). \tag{10.36}$$

Using (10.36) and recursion on p, we deduce that for any $p\in\mathbf{N}$, there exists $C_p>0$ such that for any $T\geqslant 1$, $\lambda\in\delta$, $s\in E$,

$$\|s\|^2_{E^p}\leqslant C_p T^{2p}(\|(\lambda-A_T)s\|^2_{E^{p-1}}+\|s\|^2_{E^0}). \tag{10.37}$$

By Theorem 9.25, for T large enough, if $\lambda\in\delta$, $\|(\lambda-A_T)^{-1}\|_\infty$ is uniformly bounded. From (10.37), we find that for T large enough, $\lambda\in\delta$, $s\in E$,

$$\|(\lambda-A_T)^{-1}s\|^2_{E^p}\leqslant C_p T^{2p}\|s\|^2_{E^{p-1}}. \tag{10.38}$$

Take $p\in\mathbf{N}$, $p>2\dim X$. Using (10.32), (10.38), we see that for T large enough,

$$\begin{aligned}\|(\lambda-A_T)^{-1}k^{-1/2}F_T^{-1}R_m(\lambda,T)\|_{E^p}\\ \leqslant C_p T^{p+1/2(p-1-\inf_{1\leqslant j\leqslant d}(\dim N_j)-m)}\|\sigma\|^2_{F^0}.\end{aligned} \tag{10.39}$$

By taking m large enough, we get from (10.39)

$$\|(\lambda-A_T)^{-1}k^{-1/2}F_T^{-1}R_m(\lambda,T)\|_{E^p}\leqslant\frac{C_p}{\sqrt{T}}\|\sigma\|_{F^0}. \tag{10.40}$$

On the other hand, for $p>2\dim X$, E^p embeds continuously in the set of continuous sections of $\Lambda(T^{*(0,1)}X)\hat{\otimes}\xi$ over X. We thus deduce (10.31) from (10.40).

Our Theorem is proved. □

b) The lift of harmonic forms in F to harmonic forms on X for the metric $\langle\ ,\ \rangle_T$ on E

We now use the notation of Section 6b). In particular for $T\geqslant 0$, the Hermitian product $\langle\ ,\ \rangle_T$ on E was defined in Definition 6.2, The finite dimensional vector subspace K_T of E was introduced in equation (6.2), P_T denotes the orthogonal projection operator from E on K_T with respect to the Hermitian product $\langle\ ,\ \rangle_T$. By definition $K_1=K$, $P_1=P$. Recall that by (6.3), for any $T>0$, K_T is canonically identified with $H^*(E,\bar\partial^X+v)$. As we saw after equation (6.44), the linear map $P_T: K=K_1\to K_T$ provides the canonical identification of K with K_T.

Identity (6.6) says that

$$P_T=T^{N_H}\tilde{P}_T T^{-N_H}. \tag{10.41}$$

Definition **10.2.** — For $T>0$, let B_T be the linear map

$$\sigma\in F\to(B_T\sigma)(y,Z)=k^{-1/2}(y,Z)\,\rho(Z)\exp\left(T\theta-\frac{T|Z|^2}{2}\right)\sigma(y)\in E. \tag{10.42}$$

Theorem **10.3.** — *For* $T > 0$, *let* C_T *be the linear map*

$$\sigma \in \mathrm{Ker}(D^Y) \to C_T \sigma = Q r P_T B_T \sigma \in \mathrm{Ker}(D^Y). \tag{10.43}$$

There exists $c > 0$ *such that for any* $T \geqslant 1$

$$\| C_T - 1 \| \leqslant \frac{c}{\sqrt{T}}. \tag{10.44}$$

For any $q \in \mathbf{N}$, *there exists* $C_q > 0$ *such that if* $1 \leqslant j \leqslant d$, *if* $T \geqslant 1$ *and if* $\sigma \in \mathrm{Ker}(D^{Y_j})$, *then*

$$\sup_{y \in Y \setminus Y_j} | C_T \sigma(y) | \leqslant \frac{C_q}{T^q} \| \sigma \|_{F^0}. \tag{10.45}$$

There exists $T_0 \geqslant 1$ *such that for* $T \geqslant T_0$, C_T *is invertible. Then for* $T \geqslant T_0$, *if* $s \in K$,

$$P_T s = P_T B_T C_T^{-1} Q r s. \tag{10.46}$$

For any $q \in \mathbf{N}$, *there exists* $C'_q > 0$ *such that if* $1 \leqslant j \leqslant d$, *if* $T \geqslant T_0$ *and if* $\sigma \in \mathrm{Ker}(D^{Y_j})$, *then*

$$\sup_{y \in Y \setminus Y_j} | C_T^{-1} \sigma(y) | \leqslant \frac{C'_q}{T^q} \| \sigma \|_{F^0}. \tag{10.47}$$

Proof. — Clearly

$$\begin{aligned} r T^{N_H} &= r, \\ T^{-N_H} \exp(T\theta) &= \exp(\theta). \end{aligned} \tag{10.48}$$

Using (10.41), (10.48), we find that if $\sigma \in \mathrm{Ker}(D^Y)$

$$C_T \sigma = Q r \widetilde{P}_T (\alpha_T 2^{\dim N})^{1/2} J_T \sigma. \tag{10.49}$$

So (10.44) is equivalent to (10.4), (10.45) follows from (10.3) and (10.49).

By (10.44), for T large enough, C_T is invertible. By definition, we know that for T large enough, if $\sigma \in \mathrm{Ker}(D^Y)$, then

$$Q r P_T B_T C_T^{-1} \sigma = \sigma. \tag{10.50}$$

On the other hand, for $T > 0$, the linear map $s \in K \to P_T s \in K_T$ provides the canonical identification of K with K_T. In particular $P_T s - s$ is a $\bar{\partial}^X + v$ coboundary. Using Theorem 1.7, we find that $r(P_T s - s)$ is a $\bar{\partial}^Y$ coboundary. Therefore

$$Q r P_T s = Q r s. \tag{10.51}$$

Also, since the map $r: (\mathrm{E}, \bar{\partial}^{\mathrm{X}}+v) \to (\mathrm{F}, \bar{\partial}^{\mathrm{Y}})$ is a quasi-isomorphism, for $s \in \mathrm{K}$, there is a unique $s' \in \mathrm{K}_{\mathrm{T}}$ such that

(10.52) $$\mathrm{Q}rs' = \mathrm{Q}rs.$$

From (10.51), (10.52), we get

(10.53) $$s' = \mathrm{P}_{\mathrm{T}} s.$$

Using (10.50), we know that for T large enough, for $s \in \mathrm{K}$

(10.54) $$\mathrm{Q}r\mathrm{P}_{\mathrm{T}}\mathrm{B}_{\mathrm{T}}\mathrm{C}_{\mathrm{T}}^{-1}\mathrm{Q}rs = \mathrm{Q}rs.$$

From (10.52)-(10.54), we get (10.46).

The space $\mathrm{Ker}(\mathrm{D}^{\mathrm{Y}})$ splits into $\mathrm{Ker}(\mathrm{D}^{\mathrm{Y}}) \cong \bigoplus_{j=1}^{d} \mathrm{Ker}(\mathrm{D}^{\mathrm{Y}_j})$. Let D_{T}, E_{T} be the diagonal and non diagonal parts of the operator C_{T} with respect to this splitting of $\mathrm{Ker}(\mathrm{D}^{\mathrm{Y}})$. Using (10.45), we know that as $\mathrm{T} \to +\infty$

(10.55) $$\|\mathrm{E}_{\mathrm{T}}\| = O(\mathrm{T}^{-\infty}).$$

Also by (10.44), for T large enough, D_{T} is invertible, and moreover

(10.56) $$\mathrm{C}_{\mathrm{T}}^{-1} = \mathrm{D}_{\mathrm{T}}^{-1}(1+\mathrm{E}_{\mathrm{T}}\mathrm{D}_{\mathrm{T}}^{-1})^{-1}.$$

Using (10.44), (10.55), (10.56), we find that if E'_{T} is the non diagonal part of $\mathrm{C}_{\mathrm{T}}^{-1}$, then as $\mathrm{T} \to +\infty$

(10.57) $$\|\mathrm{E}'_{\mathrm{T}}\| = O(\mathrm{T}^{-\infty}).$$

Since norms on finite dimensional vector spaces are equivalent, we immediately deduce (10.47) from (10.57).

The proof of Theorem 10.3 is completed. □

Recall that by Theorem 1.7

(10.58) $$\mathrm{H}^*(\mathrm{E}, \bar{\partial}^{\mathrm{X}}+v) \cong \bigoplus_{1}^{d} \mathrm{H}^*(\mathrm{Y}_j, \eta_j).$$

Definition **10.4.** – For $1 \leqslant j \leqslant d$, let $\mathrm{H}_j^*(\mathrm{E}, \bar{\partial}^{\mathrm{X}}+v)$ be the vector subspace of $\mathrm{H}^*(\mathrm{E}, \bar{\partial}^{\mathrm{X}}+v)$ corresponding to $\mathrm{H}(\mathrm{Y}_j, \eta_j)$ under the canonical isomorphism (10.58).

Definition **10.5.** – For $1 \leqslant j \leqslant d$, let $\mathrm{K}(j)$ be the vector subspace of K corresponding to $\mathrm{H}_j^*(\mathrm{E}, \bar{\partial}^{\mathrm{X}}+v)$ via the canonical isomorphism $\mathrm{K} \cong \mathrm{H}^*(\mathrm{E}, \bar{\partial}^{\mathrm{X}}+v)$.

If $1 \leqslant j \leqslant d$, $s \in \mathrm{E}$, let $r_j s \in \mathrm{F}_j$ be the restriction of rs to Y_j. Similarly, for $1 \leqslant j \leqslant d$, let Q_j be the orthogonal projection operator from F^0 on $\mathrm{K}'_j = \mathrm{Ker}(\mathrm{D}^{\mathrm{Y}_j})$.

Proposition **10.6.** — *For* $1 \leqslant j \leqslant d$, *the following identity holds*

$$\mathrm{K}(j)=\{s \in \mathrm{K};\ \mathrm{Q}_{j'} r_{j'} s=0 \text{ for } j' \neq j\}. \tag{10.59}$$

Proof. — Using Theorem 1.7, it is clear that if $s \in \mathrm{K}$, $rs \in \mathrm{F}$ is $\bar{\partial}^{\mathrm{Y}}$ closed. Then $s \in \mathrm{K}(j)$ if and only if for $j' \neq j$, $r_{j'} s \in \mathrm{F}_{j'}$ is $\bar{\partial}^{\mathrm{Y}_j}$ exact, *i.e.* if $\mathrm{Q}_{j'} r_{j'} s=0$. □

Theorem **10.7.** — *For any* $q \in \mathbf{N}$, *there exists a constant* C_q *such that for any* j, $1 \leqslant j \leqslant d$, *any* $s \in \mathrm{K}(j)$, *for* $\mathrm{T} \geqslant 1$

$$\sup_{x \in \mathrm{X} \setminus \mathscr{U}_{j, \varepsilon_0/2}} |\mathrm{P}_{\mathrm{T}} s|(x) \leqslant \frac{\mathrm{C}_q}{\mathrm{T}^q} \|s\|_{\mathrm{E}^0}. \tag{10.60}$$

Proof. — By Theorem 10.3, for T large enough, if $s \in \mathrm{K}(j)$

$$\mathrm{P}_{\mathrm{T}} s=\mathrm{P}_{\mathrm{T}} \mathrm{B}_{\mathrm{T}} \mathrm{C}_{\mathrm{T}}^{-1} \mathrm{Q} rs. \tag{10.61}$$

By Proposition 10.6, $\mathrm{Q}(rs)$ vanishes except on Y_j. Therefore using (10.42), (10.61), we find that

$$\mathrm{P}_{\mathrm{T}} s=\sum_{j'=1}^{d} \mathrm{T}^{\mathrm{N}_{\mathrm{H}}} \tilde{\mathrm{P}}_{\mathrm{T}}(\alpha_{\mathrm{T}, j'} 2^{\dim \mathrm{N}_{j'}})^{1/2} \mathrm{J}_{\mathrm{T}} \mathrm{Q}_{j'} \mathrm{C}_{\mathrm{T}}^{-1} \mathrm{Q}_j r_j s. \tag{10.62}$$

For T large enough, $\|\mathrm{C}_{\mathrm{T}}^{-1}\|$ is bounded. Using Theorem 10.1, we find that as $\mathrm{T} \to +\infty$

$$\sup_{x \in \mathrm{X} \setminus \mathscr{U}_{j, \varepsilon_0/2}} |(\mathrm{T}^{\mathrm{N}_{\mathrm{H}}} \tilde{\mathrm{P}}_{\mathrm{T}}(\alpha_{\mathrm{T}, j} 2^{\dim \mathrm{N}_j})^{1/2} \mathrm{J}_{\mathrm{T}} \mathrm{Q}_j \mathrm{C}_{\mathrm{T}}^{-1} \mathrm{Q}_j r_j s)(x)|=O(\mathrm{T}^{-\infty}) \|s\|_{\mathrm{E}^0}. \tag{10.63}$$

On the other hand, by (10.47) in Theorem 10.3, we know that if $j' \neq j$, as $\mathrm{T} \to +\infty$

$$\sup_{x \in \mathrm{Y}_{j'}} |(\mathrm{C}_{\mathrm{T}}^{-1} \mathrm{Q}_j r_j s)(x)|=O(\mathrm{T}^{-\infty}) \|s\|_{\mathrm{E}^0}. \tag{10.64}$$

Using the fact that J_{T} is an isometry, we see from (10.64) that for $j' \neq j$

$$\|\tilde{\mathrm{P}}_{\mathrm{T}}(\alpha_{\mathrm{T}, j'} 2^{\dim \mathrm{N}_{j'}})^{1/2} \mathrm{J}_{\mathrm{T}} \mathrm{Q}_{j'} \mathrm{C}_{\mathrm{T}}^{-1} \mathrm{Q}_j r_j s\|_{\mathrm{E}^0}=O(\mathrm{T}^{-\infty}) \|s\|_{\mathrm{E}^0}. \tag{10.65}$$

By proceeding as in (10.20), (10.21) we deduce from (10.65) that if $j' \neq j$

$$\sup_{x \in \mathrm{X}} |(\tilde{\mathrm{P}}_{\mathrm{T}}(\alpha_{\mathrm{T}, j'} 2^{\dim \mathrm{N}_{j'}})^{1/2} \mathrm{J}_{\mathrm{T}} \mathrm{Q}_{j'} \mathrm{C}_{\mathrm{T}}^{-1} \mathrm{Q}_j r_j s)(x)|=O(\mathrm{T}^{-\infty}) \|s\|_{\mathrm{E}^0}. \tag{10.66}$$

From (10.63), (10.66), we get (10.60). Our Theorem is proved. □

Remark **10.8.** — Theorem 10.7 is a very important result. It asserts that as $\mathrm{T} \to +\infty$, the harmonic representatives of elements of $\mathrm{H}_j^*(\mathrm{E}, \bar{\partial}^{\mathrm{X}}+v)$ localize near Y_j.

c) The asymptotics of the Hermitian product induced by $\langle\ ,\ \rangle_T$ on $H^*(E, \bar\partial^X + v)$

We now prove the main result of this Section.

Theorem **10.9.** — *For* $1 \leqslant j, j' \leqslant d$, *take* $s \in K(j)$, $s' \in K(j')$. *Then if* $j \neq j'$, *as* $T \to +\infty$, *for any* $m \in \mathbf{N}$

$$\langle P_T s, P_T s' \rangle_T = O(T^{-m}). \tag{10.67}$$

If $j=j'$, *as* $T \to +\infty$

$$\langle P_T s, P_T s' \rangle_T = T^{-\dim N_j}\left(\langle Q_j r_j s, Q_j r_j s' \rangle + O\left(\frac{1}{\sqrt{T}}\right)\right). \tag{10.68}$$

Proof. — By definition, if $s \in K(j)$, $s' \in K(j')$, for any $x \in X$

$$\begin{aligned}\langle P_T s, P_T s' \rangle_T(x) &= \langle T^{-N_H} P_T s, T^{-N_H} P_T s' \rangle (x) \\ &= \langle \tilde{P}_T T^{-N_H} s, T^{-N_H} P_T s' \rangle (x) = \langle T^{-N_H} P_T s, \tilde{P}_T T^{-N_H} s' \rangle (x).\end{aligned} \tag{10.69}$$

Also recall that $\|\tilde{P}_T\| \leqslant 1$. If $j \neq j'$, we now use Theorem 10.7 and (10.69) and we obtain (10.67).

More generally, if $s, s' \in K$, by (10.41), (10.42), (10.46), (10.69), we get

$$\langle P_T s, P_T s' \rangle_T = \langle \tilde{P}_T (\alpha_T 2^{\dim N})^{1/2} J_T C_T^{-1} Q rs, \tilde{P}_T (\alpha_T 2^{\dim N})^{1/2} J_T C_T^{-1} Q rs' \rangle. \tag{10.70}$$

If $s, s' \in K(j)$, by Proposition 10.6, $Q(rs)$ vanishes on $\bigcup_{j' \neq j} Y_{j'}$. Also by (9.155), (9.163), we know that

$$d(\tilde{P}_T, Q) \leqslant \frac{C}{\sqrt{T}}. \tag{10.71}$$

Recall that J_T is an isometry from F^0 on its image E_T^0. From (10.44), (10.47), (10.70), (10.71), we deduce that as $T \to +\infty$

$$\begin{aligned}\langle P_T s, P_T s' \rangle_T = \alpha_{T,j} 2^{\dim N_j}\Big(&\langle Q_j (r_j s), Q_j (r_j s') \rangle \\ &+ O\left(\frac{1}{\sqrt{T}}\right)\Big) + O(T^{-\infty}).\end{aligned} \tag{10.72}$$

By formula (9.3), it is clear that as $T \to +\infty$

$$\alpha_{T,j} 2^{\dim N_j} = \frac{1}{T^{\dim N_j}}(1 + O(T^{-\infty})). \tag{10.73}$$

From (10.72), (10.73), we get (10.68). Our Theorem is proved. □

d) Proof of Theorem 6.9

The vector spaces K and K_T are **Z**-graded by the operator $N_V^X - N_H$. Let

(10.74)
$$K = \bigoplus_{p=0}^{\dim X} K^p,$$
$$K_T = \bigoplus_{p=0}^{\dim X} K_T^p,$$

be the corresponding splittings of K and K_T. Recall that $P_T: K \to K_T$ provides the canonical identification of K and K_T and preserves the **Z**-grading.

Let $| \ |_{\det(K^p)}$, $| \ |_{\det(K_T^p)}$ be the metrics on the lines $\det(K^p)$, $\det(K_T^p)$ induced by the Hermitian products $\langle \ , \ \rangle$, $\langle \ , \ \rangle_T$ on E. Let $| \ |_{\det(K^p), T}$ be the metric on the line $\det(K^p)$ which is the pull back of the metric $| \ |_{\det(K_T^p)}$ by the canonical isomorphism $P_T: \det(K^p) \to \det(K_T^p)$.

Recall that $K' = \operatorname{Ker}(D^Y)$. The vector space K′ is **Z**-graded by the operator N_V^Y, and so

(10.75)
$$K' = \bigoplus_{p \geq 0} K'^p.$$

Let $| \ |_{\det(K'^p)}$ be the metric on the line $\det(K'^p)$ induced by the metric (1.44), (1.45) on F.

By Theorem 1.7, the map $s \in K \to Q r s \in K'$ provides the canonical isomorphism of K and K′. This isomorphism also preserves the **Z**-grading. Let $| \ |_{\det(K^p), \infty}$ be the metric on the line $\det(K^p)$ which is the pull-back of the metric $| \ |_{\det(K'^p)}$ under the canonical isomorphism $\det(K^p) \cong \det(K'^p)$.

Of course K^p, K_T^p, K'^p split into

(10.76)
$$K^p = \bigoplus_{j=1}^{d} K^p(j),$$
$$K_T^p = \bigoplus_{j=1}^{d} K_T^p(j),$$
$$K'^p = \bigoplus_{j=1}^{d} K'^p(j).$$

Also the various canonical isomorphisms described before preserve the splittings (10.76).

Let now $s^p_{1,1}, \ldots, s^p_{1,i_1}$ be a base of $K^p(1)$, let $s^p_{2,1}, \ldots, s^p_{2,i_2}$ be a base of $K^p(2), \ldots$ Let $\sigma^p \in \det(K^p)$ be given by

$$(10.77) \qquad \sigma^p = \bigwedge_{j=1}^{d} \bigwedge_{k=1}^{i_j} s^p_{j,k}.$$

We choose the $s_{j,k}$'s so that

$$(10.78) \qquad |\sigma^p|_{\det(K^p)} = 1.$$

Then by definition, for any $T > 0$

$$(10.79) \qquad \begin{aligned} \frac{|\ |^2_{\det(K^p),T}}{|\ |^2_{\det(K^p)}} &= \det(\langle P_T s^p_{j,k}, P_T s^p_{j',k'} \rangle), \\ \frac{|\ |^2_{\det(K^p),\infty}}{|\ |^2_{\det(K^p)}} &= \prod_{j=1}^{d} \det(\langle Q_j r_j s^p_{j,k}, Q_j r_j s^p_{j,k'} \rangle). \end{aligned}$$

Now by Theorem 10.9, it is clear that as $T \to +\infty$,

$$(10.80) \qquad \det(\langle P_T s^p_{j,k}, P_T s^p_{j',k'} \rangle) = T^{-\sum_{j=1}^{d} \dim N_j \dim H^p(Y_j, \eta_j)} \left(\prod_{j=1}^{d} \det(\langle Q_j r_j s^p_{j,k}, Q_j r_j s^p_{j,k'} \rangle) + O\left(\frac{1}{\sqrt{T}}\right) \right).$$

From (10.79), (10.80), we deduce that with the notation of Theorem 6.9,

$$(10.81) \qquad \frac{|\ |^2_{\lambda(\xi),T}}{|\ |^2_{\lambda(\xi)}} = T^{\sum_1^d \dim N_j \chi(\eta_j)} \left(|\rho^{-1}|^2_{\lambda^{-1}(\xi) \otimes \lambda(\eta)} + O\left(\frac{1}{\sqrt{T}}\right) \right).$$

Theorem 6.9 follows. □

XI - THE ANALYSIS OF THE TWO PARAMETERS SEMI-GROUP $\exp(-(u D^X + TV)^2)$ IN THE RANGE $u \in]0, 1]$, $T \in [0, (1/u)]$

a) Rescaling the Clifford algebra: Getzler's trick.
b) Lichnerowicz's formula.
c) The limit as $u \to 0$ of $\mathrm{Tr}_s[N_H \exp(-(u D^X + TV)^2)]$.
d) Localization of the problem.
e) A rescaling of the normal coordinate Z_0.
f) A local coordinate system near Y and a trivialization of $\Lambda(T^{*(0,1)}X) \otimes \xi$.
g) The Taylor expansion of the operator $(D^X)^2$.
h) Replacing the manifold X by $(T_{\mathbf{R}} X)_{y_0}$.
i) Rescaling of the variable Z and of the Clifford variables.
j) The matrix structure of the operator $L^{3, Z_0/T}_{u, T}$.
k) A family of Sobolev spaces with weights.
l) Estimates on the resolvent of $L^{3, Z_0/T}_{u, T}$.
m) Regularizing properties of the resolvent of $L^{3, Z_0/T}_{u, T}$.
n) Uniform estimates on the kernel $P^{3, Z_0/T}_{u, T}$.
o) Estimates on $(\lambda - L^{3, Z_0/T}_{u, T})^{-1} - (\lambda - L^{3, Z_0/T}_{0, T})^{-1}$.
p) Proof of Theorem 11.13.

The purpose of this Section is to prove Theorem 6.6. The main point of Theorem 6.6 is to show the existence of $C > 0$, $\gamma \in]0, 1]$ such that if $u \in]0, 1]$, $0 \leqslant T \leqslant (1/u)$, then

$$\left| \mathrm{Tr}_s[N_H \exp(-(u D^X + TV)^2)] - \int_X \mathrm{Td}(TX, g^{TX}) \Phi \mathrm{Tr}_s[N_H \exp(-C_{T^2}^2)] \right| \leqslant C(u(1+T))^\gamma. \tag{11.1}$$

When T remains uniformly bounded, this estimate immediately follows from local cancellations in index theory. On the other hand, as pointed out in Remark 6.10, the estimate (11.1) implicitly reflects the results of [B2] stated in Theorem 4.3 on the convergence as $T \to +\infty$ of the currents $\Phi \mathrm{Tr}_s[N_H \exp(-C_{T^2}^2)]$.

The proof of (11.1) relies on three main ideas:

- The first simple idea is that the estimate (11.1) is local on X and that difficulties may only occur near Y.
- The second main idea is to combine the rescaling techniques of Getzler [Ge] with the splitting $\xi = \xi^+ \oplus \xi^-$ of ξ near Y which was constructed in [B2, Section 1] and in Section 8f), and was already used in the proofs of Theorems 6.4 and 6.5.

In [Ge], Getzler introduced a rescaling of the Clifford algebra of a vector space in order to prove the local index Theorem of Atiyah-Singer. One of the main ideas of the proof of (11.1) is to introduce a two parameters rescaling of the Clifford algebra of $T_{\mathbf{R}}X$. When T remains bounded, this rescaling essentially coincides with Getzler's [Ge]. However when T gets very large (*i.e.* of the order of $1/u$), the Clifford variables coming from the normal bundle $N_{\mathbf{R}}$ do not get rescaled at all, as in fact will be the case in Section 12. This fine tuning of Getzler's rescaling permits us to obtain the estimate (11.1) in the range $u \in]0, 1]$, $T \in [0, (1/u)]$.

The splitting $\xi = \xi^+ \oplus \xi^-$ of ξ near Y also plays an important role in the proof of the estimate (11.1). In fact, as suggested by the results of [B2] and of Sections 8 and 9, for a given $u \in]0, 1]$, as $T \to +\infty$, ξ^+ tends to be eliminated. The difficulty is then to obtain some sort of uniform control for $u \in]0, 1]$, $1 \leqslant T \leqslant (1/u)$.

- Contrary to the proofs of the local Atiyah-Singer index Theorem, functional analytic techniques play a prominent role here. In fact, to handle together the difficulties coming from the splitting $\xi = \xi^+ \oplus \xi^-$ and from a concentration phenomenon near Y as $T \to +\infty$, we construct the heat kernels through the resolvent of the operators obtained by rescaling from the operators $(uD^X + TV)^2$. Uniform estimates on the resolvents in $u \in]0, 1]$, $T \in [0, (1/u)]$ are obtained by using Sobolev spaces of sections on $(T_{\mathbf{R}}X)_{y_0}$ $(y_0 \in Y)$ of $\Lambda(T_{\mathbf{R}}^*X \hat{\otimes} \xi)_{y_0}$, with weights which explicitly depend on the **Z**-grading.

This Section is organized as follows. In a), we recall Getzler's rescaling technique of the Clifford algebra [Ge]. In b) we establish Lichnerowicz's formula for $(D^X)^2$. In c), we calculate the limit as $u \to 0$ of $\mathrm{Tr}_s[N_H \exp(-(uD^X + TV)^2)]$ and we obtain the second easy half of Theorem 6.6. In d), we show that the proof of the estimate (11.1) can be localized near Y. In e) and f), we construct a coordinate system and a trivialization of $\Lambda(T^{*(0,1)}X) \hat{\otimes} \xi$ near Y. In g), and following [Ge], we obtain a Taylor expansion of the operator $(D^X)^2$. In h), we reduce the proof of (11.1) to an equivalent problem on $\mathbf{C}^l$. In i), we perform Getzler's rescaling on the operator $(uD^X + TV)^2$ and in j), we describe certain key algebraic features of the new rescaled operator $L_{u,T}^{3, Z_0/T}$ with respect to the splitting $\xi = \xi^+ \oplus \xi^-$. In k), we introduce graded Sobolev spaces with weights. In l) and m), we prove uniform estimates on the resolvent $L_{u,T}^{3, Z_0/T}$. Special attention has to be devoted to the fact that $L_{u,T}^{3, Z_0/T}$ is no longer a self-adjoint operator. In n), we show that the rescaled heat kernels $P_{u,T}^{3, Z_0/T}$ decay rapidly in directions normal to Y, and this uniformly in $u \in]0, 1]$. In o), we estimate the operator $L_{u,T}^{3, Z_0/T} - L_{0,T}^{3, Z_0/T}$ and the difference of the resolvents of $L_{u,T}^{3, Z_0/T}$ and $L_{0,T}^{3, Z_0/T}$ in a purely operator theoretic sense. Finally in p), we prove the estimate (11.1).

We here use the notation of Sections 1, 4, 6, 8 and 9.

a) Rescaling the Clifford algebra: Getzler's trick

We here use the notation of Section 5a). In particular V denotes a complex Hermitian vector space of complex dimension k.

Recall that $\Lambda(\bar{V}^*)$ is a $c(V_{\mathbf{R}})$ Clifford module. Moreover $\Lambda(\bar{V}^*)=\bigoplus_{p=0}^{k}\Lambda^p(\bar{V}^*)$ is a **Z**-graded vector space, and so it inherits a corresponding $\mathbf{Z}_2$-grading. If $e\in c(V_{\mathbf{R}})$, let $\mathrm{Tr}_s[e]$ be the supertrace of e acting on $\Lambda(\bar{V}^*)$.

Let $e_1, \ldots, e_{2k}$ be an orthonormal oriented base of $V_{\mathbf{R}}$. Then by [ABo, p. 484], $c(e_1)\ldots c(e_{2k})$ is the only monomial in the $c(e_i)$'s $(1\leqslant i\leqslant 2k)$ whose supertrace is nonzero and moreover

$$\mathrm{Tr}_s[c(e_1)\ldots c(e_{2k})]=(-2i)^k. \tag{11.2}$$

We now briefly describe Getzler's trick in local index theory [Ge]. If $X\in V_{\mathbf{R}}$, let $X^*\in V_{\mathbf{R}}^*$ correspond to X by the scalar product of $V_{\mathbf{R}}$. Then the operators $X^*\wedge$ and i_X both act on $\Lambda(V_{\mathbf{R}}^*)$.

For $a>0$, $X\in V_{\mathbf{R}}$, let $c^a(X)\in\mathrm{End}(\Lambda(V_{\mathbf{R}}^*))$ be given by

$$c^a(X)=\frac{1}{a}X^*\wedge -a i_X. \tag{11.3}$$

Clearly if X, $Y\in V_{\mathbf{R}}$, then

$$c^a(X)c^a(Y)+c^a(Y)c^a(X)=-2\langle X, Y\rangle.$$

Therefore the map $c(X)\in c(V_{\mathbf{R}})\to c^a(X)\in\mathrm{End}(\Lambda(V_{\mathbf{R}}^*))$ induces an injective algebra homomorphism $\psi_a\colon c(V_{\mathbf{R}})\to\mathrm{End}(\Lambda(V_{\mathbf{R}}^*))$. Observe that if N is the number operator which defines the **Z**-grading of $\Lambda(V_{\mathbf{R}}^*)$, then

$$\psi_a=a^{-N}\psi_1 a^N.$$

Now for $1\leqslant i_1<\ldots<i_p\leqslant 2k$, $1\leqslant j_1<\ldots<j_q\leqslant 2k$, the operators

$$e^{i_1}\wedge\ldots e^{i_p}\wedge i_{e_{j_1}}\ldots i_{e_{j_q}}$$

are linearly independent in $\mathrm{End}(\Lambda(V_{\mathbf{R}}^*))$. Due to (11.3), if $e\in c(V_{\mathbf{R}})$, $\psi^a(e)$ is a linear combination of such operators.

Definition **11.1.** — For $e\in c(V_{\mathbf{R}})$, let $\{\psi^a(e)\}^{\max}\in\mathbf{C}$ be the coefficient of the operator $e^1\wedge\ldots\wedge e^{2k}$ in the expansion of $\psi^a(e)$.

Proposition **11.2.** — *If* $e\in c(V_{\mathbf{R}})$, *for any* $a>0$, *then*

$$\mathrm{Tr}_s[e]=(-2i)^k a^{2k}\{\psi^a(e)\}^{\max}. \tag{11.4}$$

Proof. — Clearly

$$[\psi^a(c(e_1)\ldots c(e_{2k}))]^{\max}=a^{-2k}.$$

Also the supertraces of other monomials in $c(e_1), \ldots, c(e_{2k})$ than $c(e_1)\ldots c(e_{2k})$ vanish. Using (11.2), Proposition 11.2 follows. □

b) Lichnerowicz's formula

Let $e_1(x), \ldots, e_{2l}(x)$ be a locally defined smooth section of the bundle of orthonormal frames in $T_{\mathbf{R}}X$.

We now recall the definition of the horizontal (or Bochner) Laplacian.

Definition **11.3.** – Let Δ^X be the second order differential operator acting on E

$$\Delta^X = \sum_1^{2l} (\nabla^X_{e_i})^2 - \nabla^X_{\sum_1^{2l} \nabla^{TX}_{e_i} e_i}. \tag{11.5}$$

Let K be the scalar curvature of the Riemannian manifold X.

Let $(\nabla^{TX})^2$, $(\nabla^{\xi})^2$ be the curvatures of the connections ∇^{TX}, ∇^{ξ} on TX, ξ, let $\mathrm{Tr}[(\nabla^{TX})^2]$ denote the trace of $(\nabla^{TX})^2$, considered as a 2-form on X with values in End (TX).

Proposition **11.4.** – *Let* $e_1, \ldots, e_{2l}$ *be an orthonormal base of* $T_{\mathbf{R}}X$. *Then the following identity holds*

$$(D^X)^2 = -\frac{\Delta^X}{2} + \frac{K}{8} + \frac{1}{4} \sum_{1 \leqslant i, j \leqslant 2l} c(e_i)\, c(e_j) \left[(\nabla^{\xi})^2 + \frac{1}{2}\mathrm{Tr}[(\nabla^{TX})^2]\right](e_i, e_j). \tag{11.6}$$

Proof. – By [Hi, p. 13], since the metric g^{TX} is Kähler, the operator $\sqrt{2}\, D^X = \sqrt{2}\,(\bar\partial^X + \bar\partial^{X*})$ is a standard Dirac operator of Lichnerowicz's type. Therefore the formula of [B6, Proposition 1.2] is applicable to the operator $2(D^X)^2$. □

Proposition **11.5.** – *Let* $e_1, \ldots, e_{2l}$ *be an orthonormal base of* $T_{\mathbf{R}}X$. *Then for any* $u > 0$, $T \geqslant 0$

$$(u D^X + TV)^2 = -\frac{u^2 \Delta^X}{2} + \frac{u^2 K}{8} + \frac{u^2}{4} \sum_{i \leqslant i, j \leqslant 2l} c(e_i)\, c(e_j) \left[(\nabla^{\xi})^2 + \frac{1}{2}\mathrm{Tr}[(\nabla^{TX})^2]\right](e_i, e_j) + uT \sum_1^{2l} \frac{c(e_i)}{\sqrt{2}} \nabla^{\xi}_{e_i} V + T^2 V^2. \tag{11.7}$$

Proof. – Clearly

$$(u D^X + TV)^2 = u^2 (D^X)^2 + uT\,[D^X, V] + T^2 V^2.$$

We now use formula (9.50) for $[D^X, V]$ and Proposition 11.4 to get (11.7). □

Definition **11.6.** — For $u>0$, $T\geq 0$, let $P_{u,T}(x,x')$ $(x,x'\in X)$ be the smooth kernel associated with the operator $\exp(-(uD^X+TV)^2)$, calculated with respect to the volume element $dv_X/(2\pi)^{\dim X}$.

If $h\in E$, for $u>0$, $T\geq 0$, $x\in X$, we then have

$$\exp(-(uD^X+TV)^2)h(x)=\int_X P_{u,T}(x,x')h(x')\frac{dv_X(x')}{(2\pi)^{\dim X}}. \tag{11.8}$$

For $x\in X$, $P_{u,T}(x,x)$ lies in $\operatorname{End}_x^{\text{even}}(\Lambda(T^{*(0,1)}X)\hat{\otimes}\xi)$.

c) The limit as $u\to 0$ of $\operatorname{Tr}_s[N_H\exp(-uD^X+TV)^2)]$

Recall that if $s\in\mathbf{R}_+$, C_s is the superconnection on ξ

$$C_s=\nabla^\xi+\sqrt{s}V.$$

Proposition **11.7.** — *Let* $T_0\in[0,+\infty[$. *There exists* $C>0$ *such that for any* $u\in]0,1]$, $T\in[0,T_0]$, *then*

$$\left|\operatorname{Tr}_s[N_H\exp(-(uD^X+TV)^2)] - \int_X \operatorname{Td}(TX,g^{TX})\Phi\operatorname{Tr}_s[N_H\exp(-C_{T^2}^2)]\right|\leq Cu, \tag{11.9}$$

$$|\operatorname{Tr}_s[N_H\exp(-(uD^X+TV)^2)]-\operatorname{Tr}_s[N_H\exp(-(uD^X)^2)]|\leq CT.$$

Proof. — Clearly

$$\operatorname{Tr}_s[N_H\exp(-(uD^X+TV)^2)]=\int_X \operatorname{Tr}_s[N_H P_{u,T}(x,x)]\frac{dv_X(x)}{(2\pi)^{\dim X}}. \tag{11.10}$$

In view of formula (11.7) for $(uD^X+TV)^2$, as in [BGS2, Theorem 2.26, eq. (2.127)], we may use local index theory techniques to show that for any $T\geq 0$, any $x\in X$, as $u\to 0$

$$\operatorname{Tr}_s[N_H P_{u,T}(x,x)]\frac{dv_X(x)}{(2\pi)^{\dim X}} \to \{\operatorname{Td}(TX,g^{TX})\Phi\operatorname{Tr}_s[N_H\exp(-C_{T^2}^2)]\}_x^{\max}. \tag{11.11}$$

Take $T_0\geq 0$. The arguments in [BGS2, proof of Theorem 2.26] easily show that there exists $C>0$ such that for $u\in]0,1]$, $T\in[0,T_0]$, $x\in X$,

$$\left| \operatorname{Tr}_s[\mathrm{N_H P}_{u,\mathrm{T}}(x, x)] \frac{dv_{\mathrm{X}}(x)}{(2\pi)^{\dim \mathrm{X}}} - \{\mathrm{Td}(\mathrm{TX}, g^{\mathrm{TX}})\,\Phi\operatorname{Tr}_s[\mathrm{N_H}\exp(-\mathrm{C}^2_{\mathrm{T}^2})]\}^{\max} \right| \leqslant \mathrm{C}u. \tag{11.12}$$

The first inequality in (11.9) follows from (11.10), (11.12). Also by proceeding as in equations (3.6)-(3.9), we find that

$$\frac{1}{\mathrm{T}} \frac{\partial}{\partial \mathrm{T}} \operatorname{Tr}_s[\mathrm{N_H}\exp(-(u\mathrm{D^X}+\mathrm{TV})^2)] = -\frac{\partial}{\partial b}\{\operatorname{Tr}_s[\mathrm{V}\exp(-(u\mathrm{D^X}+\mathrm{TV})^2+b(v-v^*))]\}_{b=0}. \tag{11.13}$$

Now the same arguments as before easily show that for $\mathrm{T} \leqslant \mathrm{T}_0$, as $u \to 0$, the right-hand side of (11.13) remains uniformly bounded. We then see that, for $u \in]0, 1]$, $\mathrm{T} \in [0, \mathrm{T}_0]$,

$$|\operatorname{Tr}_s[\mathrm{N_H}\exp(-(u\mathrm{D^X}+\mathrm{TV})^2)] - [\mathrm{N_H}\exp(-(u\mathrm{D^X})^2)]| \leqslant \mathrm{CT}^2. \tag{11.14}$$

The second inequality in (11.9) follows. □

Remark **11.8.** – The second inequality in (11.9) is exactly inequality (6.14), which is part of Theorem 6.6. To complete the proof of Theorem 6.6, we must prove inequality (6.13), which is much sharper than the first inequality in (11.9), since T is allowed to vary in the interval $[0, (1/u)]$.

d) Localization of the problem

Let $a > 0$ be the injectivity radius of X. For $b \in]0, (a/2)]$, $x \in \mathrm{X}$, let $\mathrm{B^X}(x, b)$ be the open ball of center x and radius b. Let $\partial\mathrm{B^X}(x, b)$ be the smooth boundary of $\mathrm{B^X}(x, b)$.

We now fix $b \in]0, (a/2)]$.

Definition **11.9.** – For $x_0 \in \mathrm{X}$, let $\mathrm{P}^{x_0}_{u,\mathrm{T}}(x', x'')$ $(x', x'' \in \mathrm{B^X}(x_0, b))$ be the smooth heat kernel associated to the operator $\exp(-(u\mathrm{D^X}+\mathrm{TV})^2)$ with Dirichlet conditions on $\partial\mathrm{B^X}(x_0, b)$.

Proposition **11.10.** – *There exist* $c > 0$, $\mathrm{C} > 0$ *such that for any* $x_0 \in \mathrm{X}$, $u \in]0, 1]$, $\mathrm{T} \in [0, (1/u)]$, $x \in \mathrm{B^X}(x_0, b/2)$, *then*

$$\|(\mathrm{P}_{u,\mathrm{T}} - \mathrm{P}^{x_0}_{u,\mathrm{T}})(x, x)\| \leqslant c\exp\left(-\frac{\mathrm{C}}{u^2}\right). \tag{11.15}$$

Proof. — Clearly

$$(11.16)\qquad (u D^x + TV)^2 = u^2 (D^X)^2 + u T [D^X, V] + T^2 V^2.$$

As we saw in (9.50), the operator $[D^X, V]$ is of order zero. The proof will then consist in using the fact that for $T \leqslant (1/u)$, the operator $uT[D^X, V]$ remains uniformly bounded, and also the fact that the operator V^2 is non negative. Let Δ be the Laplace-Beltrami operator on X. For $t > 0$, let $p_t(x', x'')(x', x'' \in X)$ be the scalar heat kernel on X associated with the operator $\exp(t\Delta/2)$, calculated with respect to the volume element $dv_X(x'')/(2\pi)^{\dim X}$.

Take $x_0 \in X$. For $u > 0$, $x \in B^X(x_0, b/2)$, let Q_x^u be the probability law on $\mathscr{C}([0, 1]; X)$ of the Brownian bridge $t \in [0, 1] \to x_t^u \in X$ associated with the metric g^{TX}/u^2, such that $x_0^u = x_1^u = x$. For the definition and the properties of the Brownian bridge, we refer to [B4, Chapter II].

For $0 \leqslant t \leqslant 1$, let τ_0^t be the parallel transport operator from $(\Lambda(T^{*(0,1)}X) \hat{\otimes} \xi)_{x_t^u}$ into $\Lambda(T^{*(0,1)}X) \hat{\otimes} \xi)_x$ along the path x^u with respect to the unitary connection ∇^X. The fact that the operator τ_0^t is well-defined and depends continuously on $t \in [0, 1]$ follows from Dynkin [Dy], Itô [I]. Set $\tau_t^0 = (\tau_0^t)^{-1}$. Let $e_1, \ldots, e_{2l}$ be an orthonormal base of $(T_{\mathbf{R}}X)_x$. For $0 \leqslant t \leqslant 1$, let $M_t \in \mathrm{End}_x(\Lambda(T^{*(0,1)}X) \hat{\otimes} \xi)$ be given by

$$(11.17)\qquad \begin{aligned} M_t = &-\frac{u^2}{4} c(e_i) c(e_j) \tau_0^t \left((\nabla^\xi)^2 + \frac{1}{2} \mathrm{Tr}[(\nabla^{TX})^2] \right)_{x_t^u} (\tau_t^0 e_i, \tau_t^0 e_j) \tau_t^0 \\ &- \tau_0^t (u T [D^X, V] + T^2 V^2)(x_t^u) \tau_t^0. \end{aligned}$$

Consider the differential equation

$$(11.18)\qquad \begin{aligned} &\frac{dH_t}{dt} = H_t M_t, \\ &H_0 = I_{(\Lambda(T^{*(0,1)}X) \hat{\otimes} \xi)_x}. \end{aligned}$$

In (11.18), H_t lies in $\mathrm{End}_x^{\mathrm{even}}(\Lambda(T^{*(0,1)}X) \hat{\otimes} \xi)$. Finally, let S be the stopping time

$$(11.19)\qquad S = \inf\{t > 0;\ x_t^u \in \partial B^X(x_0, b)\}.$$

Let $E^{Q_x^u}$ be the expectation operator with respect to the probability measure Q_x^u. By using Lichnerowicz's formula of Proposition 11.5, and also Itô's formula as in [B5, Theorem 2.5], we find easily that

$$(11.20)\qquad (P_{u,T} - P_{u,T}^{x_0})(x, x) = p_{u^2}(x, x)\, E^{Q_x^u}\left[1_{S \leqslant 1} \exp\left(-\frac{u^2}{8} \int_0^1 K(x_t^u)\, dt \right) H_1 \tau_0^1 \right].$$

Let H_t^* be the adjoint of H_t. Since M_t is self-adjoint, we find that

$$\frac{dH_t^*}{dt} = M_t H_t^*, \qquad (11.21)$$
$$H_0^* = I_{(\Lambda(T^{*(0,1)}X) \hat{\otimes} \xi)_x}.$$

In particular, if $h \in (\Lambda(T^{*(0,1)}X) \hat{\otimes} \xi)_x$, we deduce from (11.21) that

$$\frac{d}{dt} |H_t^* h|^2 = 2 \langle M_t H_t^* h, H_t^* h \rangle. \qquad (11.22)$$

Now the operator $\tau_0^t V^2(x_u^t) \tau_t^0$ is self-adjoint and nonnegative. Using (11.22) and Gronwall's lemma, we see that there exists $C > 0$ such that for $u \in]0, 1]$, $u T \leqslant 1$, $t \leqslant 1$

$$|H_t^* h| \leqslant C |h|. \qquad (11.23)$$

From (11.23), we get

$$\| H_t \| \leqslant C. \qquad (11.24)$$

Using (11.20), (11.24), we obtain

$$\| (P_{u,T} - P_{u,T}^{x_0})(x, x) \| \leqslant C p_{u^2}(x, x) Q_x^u (S \leqslant 1). \qquad (11.25)$$

Let $p_t^{x_0}(x', x'')$ $(x', x'' \in B^X(x_0, b))$ be the scalar heat kernel associated with the operator $\exp(t\Delta/2)$ and Dirichlet boundary conditions on $\partial B^X(x_0, b)$. Using Itô's formula again, we get

$$(p_{u^2} - p_{u^2}^{x_0})(x, x) = p_{u^2}(x, x) Q_x^u (S \leqslant 1). \qquad (11.26)$$

Therefore, by (11.25), (11.26), for $u \in]0, 1]$, $u T \leqslant 1$

$$\| (P_{u,T} - P_{u,T}^{x_0})(x, x) \| \leqslant C (p_{u^2} - p_{u^2}^{x_0})(x, x). \qquad (11.27)$$

Now classically, we know there exist $c > 0$, $C > 0$ such that for any $x \in B^X(x_0, b/2)$, $u \in]0, 1]$

$$(p_{u^2} - p_{u^2}^{x_0})(x, x) \leqslant c \exp\left(- \frac{C}{u^2} \right). \qquad (11.28)$$

Then (11.15) follows from (11.27), (11.28). □

Remark **11.11.** — Recall that we want to show that there exist $C > 0$, $\gamma \in]0, 1]$ such that for $u \in]0, 1]$, $T \in [0, (1/u)]$

$$\left| \int_X \left\{ \mathrm{Tr}_s [N_H P_{u,T}(x, x)] \frac{dv_X(x)}{(2\pi)^{\dim X}} - \mathrm{Td}(TX, g^{TX}) \Phi \mathrm{Tr}_s [N_H \exp(-C_{T^2}^2)] \right\} \right| \leqslant C(u(1+T))^{\gamma}. \tag{11.29}$$

By Proposition 11.7, it is clear that only large values of T may cause trouble. Also Proposition 11.10 shows that inequality (11.29) can in fact be proved locally on X.

e) A rescaling of the normal coordinate Z_0

We use the notation of Section 8e). We fix $\varepsilon \in]0, \inf(\varepsilon_0/2, a/2)]$. Then $\mathcal{U}_\varepsilon$ is a tubular neighborhood of Y which is identified with the open set B_ε in $N_{\mathbf{R}}$ defined in (8.20).

Definition **11.12.** — Let $\beta_T(x)$ be the smooth function of $T \geqslant 0$, $x \in X$ such that

$$\beta_T(x) \frac{dv_X(x)}{(2\pi)^{\dim X}} = \{\mathrm{Td}(TX, g^{TX}) \Phi \mathrm{Tr}_s [N_H \exp(-C_{T^2}^2)]\}_x^{\max}. \tag{11.30}$$

The key result of this Section is as follows.

Theorem **11.13.** — *There exists* $\gamma \in]0, 1]$ *such that for any* $p \in \mathbf{N}$, *there is* $C_p > 0$ *such that if* $u \in]0, 1]$, $T \in [1, (1/u)]$, $y_0 \in Y$, $Z_0 \in N_{\mathbf{R}, y_0}$, $|Z_0| \leqslant (\varepsilon/2) T$, *then*

$$\frac{1}{T^{2\dim N}} \left| \mathrm{Tr}_s \left[N_H P_{u,T}\left(\left(y_0, \frac{Z_0}{T}\right), \left(y_0, \frac{Z_0}{T}\right)\right)\right] - \beta_T\left(y_0, \frac{Z_0}{T}\right) \right| \leqslant C_p((1+|Z_0|))^{-p}(u(1+T))^{\gamma}. \tag{11.31}$$

Proof. — The proof of Theorem 11.13 is given in the next subsections. □

Remark **11.14.** — Let us briefly show how to derive (11.29) from Theorem 11.13. Clearly

$$\begin{aligned} &\int_{\mathcal{U}_{\varepsilon/2}} |[\mathrm{Tr}_s [N_H P_{u,T}(x, x)] - \beta_T(x)]| \frac{dv_X(x)}{(2\pi)^{\dim X}} \\ &= \left(\frac{1}{2\pi}\right)^{\dim X} \int_Y \left[\frac{1}{T^{2\dim N}} \int_{|Z_0| \leqslant (\varepsilon T/2)} \left| \mathrm{Tr}_s \left[N_H P_{u,T}\left(\left(y_0, \frac{Z_0}{T}\right), \left(y_0, \frac{Z_0}{T}\right)\right)\right] \right.\right. \\ &\qquad \left.\left. - \beta_T\left(y_0, \frac{Z_0}{T}\right) \right| k\left(y_0, \frac{Z_0}{T}\right) dv_N(Z_0) \right] dv_Y(y_0). \end{aligned} \tag{11.32}$$

From (11.31), (11.32), we deduce that

$$\int_{\mathscr{U}_{\varepsilon/2}} |[\mathrm{Tr}_s[\mathrm{N_H}\,\mathrm{P}_{u,\mathrm{T}}(x, x)] - \beta_{\mathrm{T}}(x)]| \frac{dv_{\mathrm{X}}(x)}{(2\pi)^{\dim \mathrm{X}}} \leqslant \mathrm{C}(u(1+\mathrm{T}))^{\gamma}. \tag{11.33}$$

On the other hand, we can cover $\mathrm{X}\setminus\mathscr{U}_{\varepsilon/2}$ by a finite number of open balls $\mathrm{B^X}(x, c)$ (with $0 < c \leqslant (a/2)$) such that $\mathrm{B^X}(x, c) \cap \mathscr{U}_{\varepsilon/4} = 0$. On each of these $\mathrm{B^X}(x, c)$, we can use (11.31) with $\mathrm{Y} = \varnothing$. We thus find that

$$\int_{\mathrm{B^X}(x, c)} |[\mathrm{Tr}_s[\mathrm{N_H}\,\mathrm{P}_{u,\mathrm{T}}(x, x)] - \beta_{\mathrm{T}}(x)]| \frac{dv_{\mathrm{X}}(x)}{(2\pi)^{\dim \mathrm{X}}} \leqslant \mathrm{C}(u(1+\mathrm{T}))^{\gamma}. \tag{11.34}$$

From (11.33), (11.34), we obtain (11.29) for $u \in]0, 1]$, $\mathrm{T} \in [1, 1/u]$.

The purpose of the subsections which follow is to prove Theorem 11.13.

f) A local coordinate system near Y and a trivialization of $\Lambda(\mathrm{T}^{*(0,1)}\mathrm{X}) \hat{\otimes} \xi$

Recall that a is the injectivity radius of X and that $\varepsilon \in]0, \inf((\varepsilon_0/2), (a/2))]$.

Take $y_0 \in \mathrm{Y}$. If $\mathrm{Z} \in (\mathrm{T_{\mathbf{R}}X})_{y_0}$, $t \in \mathbf{R} \to x_t = \exp^{\mathrm{X}}_{y_0}(t\mathrm{Z})$ denotes the geodesic in X such that $x_0 = y_0$, $dx/dt|_{t=0} = \mathrm{Z}$. If $|\mathrm{Z}| < \varepsilon$, we identify $\mathrm{Z} \in (\mathrm{T_{\mathbf{R}}X})_{y_0}$ with $\exp^{\mathrm{X}}_{y_0}(\mathrm{Z}) \in \mathrm{X}$. Let $\mathrm{B}^{\mathrm{TX}}_{y_0}(0, \varepsilon)$ be the ball in $(\mathrm{T_{\mathbf{R}}X})_{y_0}$ of center 0 and radius ε. The ball $\mathrm{B}^{\mathrm{TX}}_{y_0}(0, \varepsilon)$ in $(\mathrm{T_{\mathbf{R}}X})_{y_0}$ is then identified with the ball $\mathrm{B^X}(y_0, \varepsilon)$ in X.

Let $dv_{\mathrm{TX}}(\mathrm{Z})$ be the volume element in $(\mathrm{T_{\mathbf{R}}X})_{y_0}$. Let $k'(\mathrm{Z})$ be the positive smooth function on $\mathrm{B}^{\mathrm{TX}}_{y_0}(0, \varepsilon)$ such that

$$dv_{\mathrm{X}}(\mathrm{Z}) = k'(\mathrm{Z})\, dv_{\mathrm{TX}}(\mathrm{Z}). \tag{11.35}$$

Then $k'(0) = 1$.

We now fix $\mathrm{Z}_0 \in \mathrm{N}_{\mathbf{R}, y_0}$, $|\mathrm{Z}_0| \leqslant (\varepsilon/2)$. Take $\mathrm{Z} \in (\mathrm{T_{\mathbf{R}}X})_{y_0}$, $|\mathrm{Z}| \leqslant (\varepsilon/2)$. The curve $t \in [0, 1] \to \mathrm{Z}_0 + t\mathrm{Z}$ lies in $\mathrm{B}^{\mathrm{TX}}_{y_0}(0, \varepsilon)$. We identify $\mathrm{TX}_{\mathrm{Z}_0+\mathrm{Z}}$, $\Lambda(\mathrm{T}^{*(0,1)}\mathrm{X})_{\mathrm{Z}_0+\mathrm{Z}}$ with $\mathrm{TX}_{\mathrm{Z}_0}$, $\Lambda(\mathrm{T}^{*(0,1)}\mathrm{X})_{\mathrm{Z}_0}$ (resp. $\xi_{\mathrm{Z}_0+\mathrm{Z}}$ with ξ_{Z_0}) by parallel transport with respect to the connection ∇^{TX} (resp. $\tilde{\nabla}^{\xi}$) along the line $t \in [0, 1] \to \mathrm{Z}_0 + t\mathrm{Z}$.

When $\mathrm{Z}_0 \in \mathrm{N}_{\mathbf{R}, y_0}$, $|\mathrm{Z}_0| \leqslant (\varepsilon/2)$ is itself allowed to vary, we will identify $\mathrm{TX}_{\mathrm{Z}_0}$, $\Lambda(\mathrm{T}^{*(0,1)}\mathrm{X})_{\mathrm{Z}_0}$ (resp. ξ_{Z_0}) with TX_{y_0}, $\Lambda(\mathrm{T}^{*(0,1)}\mathrm{X})_{y_0}$ (resp. ξ_{y_0}) by parallel transport with respect to the connection ∇^{TX} (resp. $\tilde{\nabla}^{\xi}$) along $t \in [0, 1] \to t\mathrm{Z}_0$. Therefore we identify $(\Lambda(\mathrm{T}^{*(0,1)}\mathrm{X}) \hat{\otimes} \xi)_{\mathrm{Z}_0+\mathrm{Z}}$ with $(\Lambda(\mathrm{T}^{*(0,1)}\mathrm{X}) \hat{\otimes} \xi)_{y_0}$ by parallel transport with respect to the connection $\nabla^{\mathrm{TX}} \otimes 1 + 1 \otimes \tilde{\nabla}^{\xi}$ along the broken curve

$$t \in [0, 1] \to 2t\mathrm{Z}_0, \qquad 0 \leqslant t \leqslant \frac{1}{2};$$

$$Z_0+(2t-1)Z, \qquad \frac{1}{2}\leqslant t\leqslant 1.$$

In this case the identification depends explicitly on Z_0 and Z, and not only on Z_0+Z.

g) The Taylor expansion of the operator $(D^X)^2$

Recall that B was defined in Definition 8.12 by

$$B=\nabla^{\xi}-\tilde{\nabla}^{\xi}. \tag{11.36}$$

We fix $y_0\in Y_0$, $Z_0\in N_{\mathbf{R},y_0}$, $|Z_0|\leqslant(\varepsilon/2)$. We then use the trivialization of TX, $\Lambda(T^{*(0,1)}X)$, ξ considered in Section 11f), which depends on Z_0.

If $|Z|\leqslant(\varepsilon/2)$, let Γ_Z^{TX,Z_0}, Γ_Z^{ξ,Z_0} be the connection forms of the connections ∇^{TX}, ∇^{ξ} on TX, ξ evaluated at Z_0+Z. It is clear that

$$\begin{aligned}&\Gamma_0^{TX,Z_0}=0,\\&\Gamma_0^{\xi,Z_0}=B_{Z_0}.\end{aligned} \tag{11.37}$$

Also by [ABoP, Proposition 3.7], we know that

$$\Gamma_Z^{TX,Z_0}=\frac{1}{2}(\nabla^{TX})^2_{Z_0}(Z,\,.)+O(|Z|^2). \tag{11.38}$$

In the sequel, we consider D^X, V as differential operators acting on the space of sections of $(\Lambda(T^{*(0,1)}X)\hat{\otimes}\xi)_{Z_0}$ which depend smoothly on $Z\in(T_{\mathbf{R}}X)_{y_0}$, $|Z|\leqslant(\varepsilon/2)$.

If $U\in(T_{\mathbf{R}}X)_{Z_0+Z}$, let ∇_U be the standard differentiation operator in the direction U acting on smooth sections of $(\Lambda(T^{*(0,1)}X)\hat{\otimes}\xi)_{Z_0}$. Let $e_1,\ldots,e_{2l}$ be an orthonormal base of $(T_{\mathbf{R}}X)_{Z_0}$. For $1\leqslant i\leqslant 2l$, let $\tau e_i^{Z_0}(Z)$ be the parallel transport of e_i with respect to the connection ∇^{TX} along the curve $t\in[0,1]\to Z_0+tZ$. We will often use the notation $\tau e_i^{Z_0}$ instead of $\tau e_i^{Z_0}(Z)$.

From (8.16), we find that in the considered trivialization of $\Lambda(T^{*(0,1)}X)\hat{\otimes}\xi$

$$\begin{aligned}D^X=\sum_1^{2l}\frac{c(e_i)}{\sqrt{2}}\Big(&\nabla_{\tau e_i^{Z_0}(Z)}+\frac{1}{4}\sum_{1\leqslant j,j'\leqslant 2l}\langle\Gamma_Z^{TX,Z_0}(\tau e_i^{Z_0}(Z))\,e_j,e_{j'}\rangle c(e_j)c(e_{j'})\\&+\frac{1}{2}\operatorname{Tr}[\Gamma_Z^{TX,Z_0}(\tau e_i^{Z_0}(Z))]+\Gamma_Z^{\xi,Z_0}(\tau e_i^{Z_0}(Z)\Big).\end{aligned} \tag{11.39}$$

Let Op be the set of scalar differential operators on $B_{y_0}^{TX}(0,\varepsilon/2)$ with smooth coefficients. By (11.39), it is clear that

$$D^X\in(c(T_{\mathbf{R}}X)\hat{\otimes}\operatorname{End}\xi)_{Z_0}\otimes\mathrm{Op}.$$

Also V acts at Z as $V(Z_0+Z)\in(\mathrm{End}\,\xi)_{Z_0}$. Using (9.50), (11.39), we thus find that

$$(D^X)^2,\ [D^X, V],\ V^2\in(c(T_{\mathbf{R}}X)\,\hat{\otimes}\,\mathrm{End}\,\xi)_{Z_0}\otimes \mathrm{Op}. \tag{11.40}$$

For $p\in\mathbf{N}$, $O(|Z|^p)$ will denote an expression in $\mathrm{End}(TX)_{Z_0}$ or $(\mathrm{End}\,\xi)_{Z_0}$ which is such that for $k\in\mathbf{N}$, $k\leqslant p$, its derivatives are $O(|Z|^{p-k})$ as $|Z|\to 0$. Note that $O(|Z|^p)$ will never contain any Clifford variable.

We now rewrite Lichnerowicz's formula of Proposition 11.4 in a form close to the one considered by Getzler [Ge].

Proposition **11.15.** – *The following identity holds*

$$\begin{aligned}(D^X)^2=&-\frac{1}{2}\sum_i\Big(\nabla_{e_i+O(|Z|)}+\frac{1}{8}\sum_{j\neq j'}\langle((\nabla^{TX})^2_{Z_0}(Z, e_i)\\ &+O(|Z|^2))e_j, e_{j'}\rangle c(e_j)c(e_{j'})+O(1)\Big)^2+O(1)\\ &+\frac{1}{4}\sum_{j\neq j'}c(e_j)c(e_{j'})\Big(\Big((\nabla^{\xi})^2_{Z_0}+\frac{1}{2}\mathrm{Tr}\,[(\nabla^{TX})^2_{Z_0}]\Big)(e_j, e_{j'})+O(|Z|)\Big)\\ &+\nabla_{O(|Z|)}+\sum_{j\neq j'}\langle O(|Z|^2)e_j, e_{j'}\rangle c(e_j)c(e_{j'}).\end{aligned}\tag{11.41}$$

Proof. – We use formula (11.5) for Δ^X with e_i replaced by $\tau e_i^{Z_0}(Z)$ $(1\leqslant i\leqslant 2l)$, Proposition 11.4 and also (11.36), (11.37). Then (11.41) immediately follows. □

Remark **11.16.** – A minor difference with [Ge] is that we have trivialized the vector bundle ξ using the connection $\tilde{\nabla}^{\xi}$, while a direct application of [Ge] would require us to use the connection ∇^{ξ}.

Of course, since the metric g^{TX} is Kähler, we know that

$$\langle((\nabla^{TX})^2_{Z_0}(Z, e_i))e_j, e_{j'}\rangle=\langle((\nabla^{TX})^2_{Z_0}(e_j, e_{j'}))Z, e_i\rangle. \tag{11.42}$$

h) Replacing the manifold X by $(T_{\mathbf{R}}X)_{y_0}$

Take $y_0\in Y$. For $Z_0\in N_{\mathbf{R},y_0}$, $|Z_0|\leqslant(\varepsilon/2)$, it will now be very useful to identify $(\Lambda(T^{*(0,1)}X)\,\hat{\otimes}\,\xi)_{Z_0}$ with $(\Lambda(T^{*(0,1)}X)\,\hat{\otimes}\,\xi)_{y_0}$ as indicated in Section 11f).

Definition **11.17.** – Let $\mathbf{H}_{y_0}$ be the set of smooth sections of $(\Lambda(T^{*(0,1)}X)\,\hat{\otimes}\,\xi)_{y_0}$ on $(T_{\mathbf{R}}X)_{y_0}$.

Let Δ be the ordinary flat Laplacian on $(T_{\mathbf{R}}X)_{y_0}$. Then Δ acts naturally on $\mathbf{H}_{y_0}$.

Let $\gamma(a)$ be the smooth function of $a \in \mathbf{R}$ considered in Section 9a), If $Z \in (T_{\mathbf{R}} X)_{y_0}$, set

$$\rho(Z) = \gamma\left(\frac{2|Z|}{\varepsilon}\right). \tag{11.43}$$

Then

$$\begin{aligned} \rho(Z) &= 1 \quad \text{if } |Z| \leqslant \frac{\varepsilon}{4}, \\ &= 0 \quad \text{if } |Z| \geqslant \frac{\varepsilon}{2}. \end{aligned} \tag{11.44}$$

We now fix $Z_0 \in N_{\mathbf{R}, y_0}$, $|Z_0| \leqslant (\varepsilon/2)$. As indicated in Section 11f), the trivialization under consideration of $\Lambda(T^{*(0,1)}X) \hat{\otimes} \xi$ depends explicitly on Z_0. Therefore the action of D^X also depends on Z_0.

Definition **11.18.** — For $u > 0$, $T \geqslant 0$, let $L_{u,T}^{1,Z_0}$, M_u^{1,Z_0} be the operators acting on $\mathbf{H}_{y_0}$

$$\begin{aligned} L_{u,T}^{1,Z_0} &= (1-\rho^2(Z))\left(-\frac{u^2 \Delta}{2} + T^2 P^{\xi_{y_0}^+}\right) + \rho^2(Z)(u D^X + TV(Z_0 + Z))^2, \\ M_u^{1,Z_0} &= -u^2(1-\rho^2(Z))\frac{\Delta}{2} + \rho^2(Z)(u D^X)^2. \end{aligned} \tag{11.45}$$

Then $L_{u,T}^{1,Z_0}$ is a second order elliptic operator with smooth coefficients. Let $P_{u,T}^{1,Z_0}(Z, Z')$ $(Z, Z' \in (T_{\mathbf{R}} X)_{y_0})$ be the smooth kernel associated with the operator $\exp(-L_{u,T}^{1,Z_0})$ calculated with respect to the volume $k'(Z_0)(dv_{TX}(Z')/(2\pi)^{\dim X})$. The same argument as in the proof of Proposition 11.10 shows there exist $c > 0$, $C > 0$ such that if $u \in]0, 1]$, $T \in [0, (1/u)]$, $y_0 \in Y$, $Z_0 \in N_{\mathbf{R}, y_0}$, $|Z_0| \leqslant (\varepsilon/2)$, then

$$|P_{u,T}((y_0, Z_0), (y_0, Z_0)) - P_{u,T}^{1,Z_0}(0, 0)| \leqslant c \exp\left(-\frac{C}{u^2}\right). \tag{11.46}$$

In the next subsections, we will prove that there exists $\gamma \in]0, 1]$, such that for any $p \in \mathbf{N}$, there is $C_p^1 > 0$ such that if $u \in]0, 1]$, $T \in [1, (1/u)]$, $y_0 \in Y$, $Z_0 \in N_{\mathbf{R}, y_0}$, $|Z_0| \leqslant (\varepsilon T/2)$, then

$$\begin{aligned} &\frac{1}{T^{2\dim N}} \left| \operatorname{Tr}_s[N_H P_{u,T}^{1, Z_0/T}(0, 0)] - \beta_T\left(y_0, \frac{Z_0}{T}\right) \right| \\ &\qquad \leqslant C_p^1 (1 + |Z_0|)^{-p} (u(1+T))^{\gamma}. \end{aligned} \tag{11.47}$$

If u, T, y_0, Z_0 are taken as before, then

$$\exp\left(-\frac{C}{u^2}\right) \leqslant \exp\left(-\frac{C}{2u^2}-\frac{2C|Z_0|^2}{\varepsilon^2}\right). \tag{11.48}$$

Theorem 11.13 then follows from (11.46)-(11.48).

We now concentrate on the proof of (11.47).

i) Rescaling of the variable Z and of the Clifford variables

Take $y_0 \in Y$, $Z_0 \in N_{\mathbf{R}, y_0}$, $|Z_0| \leqslant (\varepsilon/2)$. As explained in Section 11f), we identify TX_{Z_0}, $(\Lambda(T^{*(0,1)}X)\hat{\otimes}\xi)_{Z_0}$ with TX_{y_0}, $(\Lambda(T^{*(0,1)}X)\hat{\otimes}\xi)_{y_0}$ by parallel transport with respect to the connections ∇^{TX}, $\nabla^{TX}\otimes 1 + 1 \otimes \tilde{\nabla}^{\xi}$ along the curve $t \in [0, 1] \to tZ_0$. These identifications play a crucial role in the sequel.

For $u > 0$, set F_u be the linear map

$$h \in \mathbf{H}_{y_0} \to F_u h \in \mathbf{H}_{y_0};\ F_u\, h(Z) = h\left(\frac{Z}{u}\right). \tag{11.49}$$

For $u \geqslant 0$, $T > 0$, set

$$\begin{aligned} L^{2, Z_0}_{u, T} &= F_u^{-1} L^{1, Z_0}_{u, T} F_u,\\ M^{2, Z_0}_u &= F_u^{-1} M^{1, Z_0}_u F_u. \end{aligned} \tag{11.50}$$

From (11.40), (11.45), we find that

$$L^{2, Z_0}_{u, T},\ M^{2, Z_0}_u \in (c(T_{\mathbf{R}}X)\hat{\otimes}\operatorname{End}\xi)_{y_0} \otimes \mathrm{Op}. \tag{11.51}$$

Let $e_1, \ldots, e_{2l'}$ be an orthonormal oriented base of $(T_{\mathbf{R}}Y)_{y_0}$, let $e_{2l'+1}, \ldots, e_{2l}$ be an orthonormal oriented base of $N_{\mathbf{R}, y_0}$, $e^1, \ldots, e^{2l'}$ and $e^{2l'+1}, \ldots, e^{2l}$ denote the corresponding dual bases of $(T^*_{\mathbf{R}}Y)_{y_0}$ and $N^*_{\mathbf{R}, y_0}$. Then $e_1, \ldots, e_{2l}$ and $e^1, \ldots, e^{2l}$ are orthonormal oriented bases of $(T_{\mathbf{R}}X)_{y_0}$ and $(T^*_{\mathbf{R}}X)_{y_0}$ respectively.

We now will use the rescaling technique of Getzler [Ge] outlined in Section 11a), which we will adapt to our special needs.

Definition **11.19.** — For $u > 0$, $T > 0$, set

$$\begin{aligned} c_{u,T}(e_j) &= \frac{\sqrt{2}\, e^j}{u} \wedge - \frac{u}{\sqrt{2}} i_{e_j},\ 1 \leqslant j \leqslant 2l';\\ c_{u,T}(e_j) &= \frac{\sqrt{2}\, e^j}{uT} \wedge - \frac{uT}{\sqrt{2}} i_{e_j},\ 2l'+1 \leqslant j \leqslant 2l. \end{aligned} \tag{11.52}$$

The operators $c_{u,\mathrm{T}}(e_j)$ act naturally on $(\Lambda(\mathrm{T}_{\mathbf{R}}^* \mathrm{X}) \hat{\otimes} \xi)_{y_0}$.

Definition **11.20.** — For $u>0$, $\mathrm{T}>0$, let $\mathrm{L}_{u,\mathrm{T}}^{3,\mathrm{Z}_0}$, $\mathrm{M}_{u,\mathrm{T}}^{3,\mathrm{Z}_0} \in \mathrm{End}(\Lambda(\mathrm{T}_{\mathbf{R}}^* \mathrm{X}) \hat{\otimes} \xi)_{y_0} \otimes \mathrm{Op}$ be the operators obtained from $\mathrm{L}_{u,\mathrm{T}}^{2,\mathrm{Z}_0}$, $\mathrm{M}_u^{2,\mathrm{Z}_0}$ by replacing the Clifford variables $c(e_j)$ by the operators $c_{u,\mathrm{T}}(e_j)$ considered in Definition 11.19.

Let $\mathrm{P}_{u,\mathrm{T}}^{3,\mathrm{Z}_0}(\mathrm{Z}, \mathrm{Z}')$ be the smooth kernel associated with the operator $\exp(-\mathrm{L}_{u,\mathrm{T}}^{3,\mathrm{Z}_0})$ which is calculated with respect to the volume element $k'(\mathrm{Z}_0)\,(dv_{\mathrm{TX}}(\mathrm{Z}')/(2\pi)^{\dim \mathrm{X}})$. Then $\mathrm{P}_{u,\mathrm{T}}^{3,\mathrm{Z}_0}(0, 0)$ can be expanded in the form

$$\mathrm{P}_{u,\mathrm{T}}^{3,\mathrm{Z}_0}(0,0) = \sum_{\substack{1 \leqslant i_1 < \ldots < i_p \leqslant 2l \\ 1 \leqslant j_1 < \ldots < j_q \leqslant 2l}} e^{i_1} \wedge \ldots e^{i_p} \wedge i_{e_{j_1}} \ldots i_{e_{j_q}} \hat{\otimes} \mathrm{Q}_{i_1 \ldots i_p}^{j_1 \ldots j_q}, \qquad \mathrm{Q}_{i_1 \ldots i_p}^{j_1 \ldots j_q} \in \mathrm{End}(\xi)_{y_0}. \tag{11.53}$$

Set

$$[\mathrm{P}_{u,\mathrm{T}}^{3,\mathrm{Z}_0}(0,0)]^{\max} = \mathrm{Q}_{1 \ldots 2l} \in (\mathrm{End}\,\xi)_{y_0}. \tag{11.54}$$

Equivalently $[\mathrm{P}_{u,\mathrm{T}}^{3,\mathrm{Z}_0}(0,0)]^{\max}$ is the operator in $(\mathrm{End}\,\xi)_{y_0}$ which appears after $e^1 \wedge \ldots \wedge e^{2l}$ in the expansion (11.53).

Proposition **11.21.** — *The following identity holds*

$$\frac{1}{\mathrm{T}^{2\dim \mathrm{N}}} \mathrm{Tr}_s[\mathrm{N_H} \mathrm{P}_{u,\mathrm{T}}^{1,\mathrm{Z}_0}(0,0)] = (-i)^{\dim \mathrm{X}}\, \mathrm{Tr}_s[\mathrm{N_H}[\mathrm{P}_{u,\mathrm{T}}^{3,\mathrm{Z}_0}(0,0)]^{\max}]. \tag{11.55}$$

Proof. — Equation (11.55) is a trivial consequence of Proposition 11.2. □

Remark **11.22.** — If T remains in a compact set in $\mathbf{R}_+^*$, by making $u \to 0$ in (11.55), and by using Proposition 11.15, we would essentially reproduce the proof by Getzler of the local index Theorem [Ge]. The critical fact is that here T varies in the interval $[0, (1/u)]$. In particular in (11.52), for $\mathrm{T}=(1/u)$, the Clifford variables $c(e_i)\,(2l'+1 \leqslant i \leqslant 2l)$ are not rescaled at all. The two parameters rescaling of Definition 11.19 will permit us to interpolate between the values 1 and $(1/u)$ of T, the value 1 corresponding to the situation considered in [Ge] and the value $\mathrm{T}=(1/u)$ to the problem which is solved in Section 12.

We now briefly imitate the procedure in Getzler [Ge]. By using Proposition 11.15 and the fact that $\rho(0)=1$, we find that $\mathrm{L}_{u,\mathrm{T}}^{3,\mathrm{Z}_0}$ can be extended by continuity at $u=0$. More precisely, we have the formula

$$\mathrm{L}_{0,\mathrm{T}}^{3,\mathrm{Z}_0} = -\frac{1}{2} \sum_{i=1}^{2l} \Bigg(\nabla_{e_i} + \frac{1}{4} \sum_{1 \leqslant j, j' \leqslant 2l'} \langle (\nabla^{\mathrm{TX}})^2_{\mathrm{Z}_0}(\mathrm{Z}, e_i) e_j, e_{j'} \rangle e^j \wedge e^{j'} \wedge + \frac{1}{4\mathrm{T}^2} \sum_{2l'+1 \leqslant j, j' \leqslant 2l} \langle (\nabla^{\mathrm{TX}})^2_{\mathrm{Z}_0}(\mathrm{Z}, e_i) e_j, e_{j'} \rangle e^j \wedge e^{j'} \wedge \tag{11.56}$$

$$+\frac{1}{2T}\sum_{\substack{1\leqslant j\leqslant 2l'\\ 2l'+1\leqslant j'\leqslant 2l}} \langle (\nabla^{TX})^2_{Z_0}(Z, e_i)\, e_j, e_{j'}\rangle\, e^j\wedge e^{j'}\wedge\Big)^2$$

$$+\frac{1}{2}\sum_{1\leqslant j,j'\leqslant 2l'} e^j\wedge e^{j'}\wedge\left((\nabla^{\xi})^2_{Z_0}+\frac{1}{2}\operatorname{Tr}[(\nabla^{TX})^2_{Z_0}]\right)(e_j, e_{j'})$$

$$+\frac{1}{2T^2}\sum_{2l'+1\leqslant j,j'\leqslant 2l} e^j\wedge e^{j'}\wedge\left((\nabla^{\xi})^2_{Z_0}+\frac{1}{2}\operatorname{Tr}[(\nabla^{TX})^2_{Z_0}]\right)(e_j, e_{j'})$$

$$+\frac{1}{T}\sum_{\substack{1\leqslant j\leqslant 2l'\\ 2l'+1\leqslant j\leqslant 2l}} e^j\wedge e^{j'}\wedge\left((\nabla^{\xi})^2_{Z_0}+\frac{1}{2}\operatorname{Tr}[(\nabla^{TX})^2_{Z_0}]\right)(e_j, e_{j'})$$

$$+T\sum_{1\leqslant j\leqslant 2l'} e^j\wedge(\nabla^{\xi}_{e_j}V)(Z_0)$$

$$+\sum_{2l'+1\leqslant j\leqslant 2l} e^j\wedge(\nabla^{\xi}_{e_j}V)(Z_0)+T^2V^2(Z_0).$$

Let $P^{3,Z_0}_{0,T}(Z, Z')\,(Z, Z'\in(T_{\mathbf{R}}X)_{y_0})$ be the smooth heat kernel associated with the operator $\exp(-L^{3,Z_0}_{0,T})$ calculated with respect to the volume element $k'(Z_0)(dv_{TX}(Z')/(2\pi)^{\dim X})$. Recall that the function β_T on X was defined in Definition 11.12. The standard local index Theorem in the form proved in [Ge] asserts that

$$(11.57)\qquad (-i)^{\dim X}\operatorname{Tr}_s[N_H[P^{3,Z_0}_{0,T}(0,0)]^{\max}]=\frac{\beta_T(y_0, Z_0)}{T^{2\dim N}}.$$

Using (11.55), (11.57), we see that to establish (11.47)—*i.e.* to prove Theorem 11.13—we only need to show that there exists $\gamma\in]0,1]$ such that for $p\in\mathbf{N}$, we can find $C_p>0$ for which when $u\in]0,1]$, $T\in[0,(1/u)]$, $y_0\in Y$, $Z_0\in N_{\mathbf{R},y_0}$, $|Z_0|\leqslant(\varepsilon T/2)$, then

$$(11.58)\qquad |(P^{3,Z_0/T}_{u,T}-P^{3,Z_0/T}_{0,T})(0,0)|\leqslant C_p(1+|Z_0|)^{-p}(u(1+T))^{\gamma}.$$

j) The matrix structure of the operator $L^{3,Z_0/T}_{u,T}$

We still use the notation of Section 11i). In the expressions which follow, all the terms containing wedge products or interior multiplication operators will be explicitly written. Terms like $O(1)$, $O(|Z|)$ will never contain such terms.

By Propositions 11.5 and 11.15, we find that if $|Z_0|\leqslant(\varepsilon T/2)$, then

(11.59)
$$
\begin{aligned}
\mathrm{M}_{u,\mathrm{T}}^{3,\mathrm{Z}_0/\mathrm{T}} = & -(1-\rho^2(u\mathrm{Z}))\frac{\Delta}{2} + \rho^2(u\mathrm{Z})\Bigg\{-\frac{1}{2}\sum_{i=1}^{2l}\Bigg(\nabla_{\tau e_i^{\mathrm{Z}_0/\mathrm{T}}(u\mathrm{Z})} \\
& + \frac{1}{4}\sum_{1\leqslant j,j'\leqslant 2l'}\Big\langle (\nabla^{\mathrm{TX}})^2_{\mathrm{Z}_0/\mathrm{T}}(\mathrm{Z},e_i) + \frac{1}{u}O(|u\mathrm{Z}|^2)e_j, e_{j'}\Big\rangle \\
& \Big(e^j\wedge -\frac{u^2}{2}i_{e_j}\Big)\Big(e^{j'}\wedge -\frac{u^2}{2}i_{e_{j'}}\Big) \\
& + \frac{1}{4}\sum_{2l'+1\leqslant j,j'\leqslant 2l}\Big\langle\Big((\nabla^{\mathrm{TX}})^2_{\mathrm{Z}_0/\mathrm{T}}(\mathrm{Z},e_i) + \frac{1}{u}O(|u\mathrm{Z}|^2)\Big)e_j, e_{j'}\Big\rangle \\
& \Big(\frac{e^j}{\mathrm{T}}\wedge -\frac{u^2\mathrm{T}}{2}i_{e_j}\Big)\Big(\frac{e^{j'}}{\mathrm{T}}\wedge -\frac{u^2\mathrm{T}}{2}i_{e_{j'}}\Big) \\
& + \frac{1}{2}\sum_{\substack{1\leqslant j\leqslant 2l'\\ 2l'+1\leqslant j'\leqslant 2l}}\Big\langle\Big((\nabla^{\mathrm{TX}})^2_{\mathrm{Z}_0/\mathrm{T}}(\mathrm{Z},e_i) + \frac{1}{u}O(|u\mathrm{Z}|^2)\Big)e_j, e_{j'}\Big\rangle \\
& \Big(e^j\wedge -\frac{u^2}{2}i_{e_j}\Big)\Big(\frac{e^{j'}}{\mathrm{T}}\wedge -\frac{u^2\mathrm{T}}{2}i_{e_{j'}}\Big) + uO(1)\Bigg)^2 + u^2O(1) \\
& + \frac{1}{2}\sum_{1\leqslant j,j'\leqslant 2l'}\Big((\nabla^{\xi})^2_{\mathrm{Z}_0/\mathrm{T}} + \frac{1}{2}\mathrm{Tr}\,[(\nabla^{\mathrm{TX}})^2_{\mathrm{Z}_0/\mathrm{T}}] + O(|u\mathrm{Z}|)\Big)(e_j,e_{j'}) \\
& \Big(e^j\wedge -\frac{u^2}{2}i_{e_j}\Big)\Big(e^{j'}\wedge -\frac{u^2}{2}i_{e_{j'}}\Big) \\
& + \frac{1}{2}\sum_{2l'+1\leqslant j,j'\leqslant 2l}\Big((\nabla^{\xi})^2_{\mathrm{Z}_0/\mathrm{T}} + \frac{1}{2}\mathrm{Tr}\,[(\nabla^{\mathrm{TX}})^2_{\mathrm{Z}_0/\mathrm{T}}] + O(|u\mathrm{Z}|)\Big)(e_j,e_{j'}) \\
& \Big(\frac{e^j}{\mathrm{T}}\wedge -\frac{u^2\mathrm{T}}{2}i_{e_j}\Big)\Big(\frac{e^{j'}}{\mathrm{T}}\wedge -\frac{u^2\mathrm{T}}{2}i_{e_{j'}}\Big) \\
& + \sum_{\substack{1\leqslant j\leqslant 2l'\\ 2l'+1\leqslant j'\leqslant 2l}}\Big((\nabla^{\xi})^2_{\mathrm{Z}_0/\mathrm{T}} + \frac{1}{2}\mathrm{Tr}\,[(\nabla^{\mathrm{TX}})^2_{\mathrm{Z}_0/\mathrm{T}}] + O(|u\mathrm{Z}|)\Big)(e_j,e_{j'}) \\
& \Big(e^j\wedge -\frac{u^2}{2}i_{e_j}\Big)\Big(\frac{e^{j'}}{\mathrm{T}}\wedge -\frac{u^2\mathrm{T}}{2}i_{e_{j'}}\Big) + u\nabla_{O(u|\mathrm{Z}|)}\Bigg\}.
\end{aligned}
$$

Also

(11.60)
$$\mathrm{L}_{u,\mathrm{T}}^{3,\mathrm{Z}_0/\mathrm{T}} = \mathrm{M}_{u,\mathrm{T}}^{3,\mathrm{Z}_0/\mathrm{T}} + \mathrm{T}\,\rho^2(u\mathrm{Z})\sum_{j=1}^{2l'}\Big(e^j\wedge -\frac{u^2}{2}i_{e_j}\Big)\nabla^{\xi}_{\tau e_j^{\mathrm{Z}_0/\mathrm{T}}(u\mathrm{Z})}\mathrm{V}\Big(\frac{\mathrm{Z}_0}{\mathrm{T}} + u\mathrm{Z}\Big)$$

$$+\rho^2(uZ)\sum_{j=2l'+1}^{2l}\left(e^j\wedge-\frac{u^2T^2}{2}i_{e_j}\right)\nabla^{\xi}_{\tau e_j^{Z_0/T}(uZ)}V\left(\frac{Z_0}{T}+uZ\right)$$

$$+T^2\rho^2(uZ)V^2\left(\frac{Z_0}{T}+uZ\right)+T^2(1-\rho^2(uZ))P^{\xi^+_{y_0}}.$$

It will be very useful to write some operators of order zero which appear in (11.60) in matrix form with respect to the splitting of $\xi=\xi^-\oplus\xi^+$.

Observe that since our trivialization of ξ preserves the splitting $\xi=\xi^-\oplus\xi^+$, we can write the operator $T^2V^2((Z_0/T)+uZ)$ in the form

$$T^2V^2\left(\frac{Z_0}{T}+uZ\right)=T^2\begin{bmatrix}(V^-)^2\left(\frac{Z_0}{T}+uZ\right) & 0\\ 0 & (V^+)^2\left(\frac{Z_0}{T}+uZ\right)\end{bmatrix}. \tag{11.61}$$

The matrix form of the operator

$$T\sum_{j=1}^{2l'}\left(e^j\wedge-\frac{u^2}{2}i_{e_j}\right)\nabla^{\xi}_{\tau e_j^{Z_0/T}(uZ)}V\left(\frac{Z_0}{T}+uZ\right) \tag{11.62}$$

will be described in a subtler way. By definition

$$\left[\nabla^{\xi}_{\tau e_j^{Z_0/T}(uZ)}V\left(\frac{Z_0}{T}+uZ\right)\right]_{\substack{Z_0=0\\ Z=0}}=\nabla^{\xi}_{e_j}V(y_0). \tag{11.63}$$

By [B2, Proposition 3.5], we know that if $U\in(T_{\mathbf{R}}Y)_{y_0}$, $\nabla^{\xi}_U V(y_0)$ maps $\xi^-_{y_0}$ into $\xi^+_{y_0}$. The argument is as follows. If f is smooth section of $\xi^-|_Y$, then $Vf=0$. Therefore

$$\nabla^{\xi}_U Vf+V\nabla^{\xi}_U f=0. \tag{11.64}$$

Now by definition $\mathrm{Im}(V|_Y)=\xi^+|_Y$. From (11.64), we deduce that $\nabla^{\xi}_U V(y_0)$ indeed maps $\xi^-_{y_0}$ into $\xi^+_{y_0}$.

For $1\leqslant j\leqslant 2l'$, we now write $\nabla^{\xi}_{\tau e_j^{Z_0/T}(uZ)}V((Z_0/T)+uZ)$ in matrix form

$$\nabla^{\xi}_{\tau e_j^{Z_0/T}(uZ)}V\left(\frac{Z_0}{T}+uZ\right)=\begin{bmatrix}E_{j,1} & E_{j,2}\\ E_{j,3} & E_{j,4}\end{bmatrix}. \tag{11.65}$$

We just saw that, for $1\leqslant j\leqslant 2l'$, for $Z_0=0$ and $Z=0$, $E_{j,1}$ vanishes. Therefore, for $1\leqslant j\leqslant 2l'$,

$$E_{j,1}=O\left(\frac{|Z_0|}{T}+u|Z|\right). \tag{11.66}$$

To prove (11.58), we will express $\exp(-L_{u,T}^{3,Z_0/T})$ as an integral over a contour Γ in **C** of the form

$$\exp(-L_{u,T}^{3,Z_0/T}) = \frac{1}{2\pi i}\int_\Gamma \exp(-\lambda)(\lambda - L_{u,T}^{3,Z_0/T})^{-1}\, d\lambda. \tag{11.67}$$

Using the matrix structure of $L_{u,T}^{3,Z_0/T}$, it will then be relatively easy to "dominate" $(\lambda - L_{u,T}^{3,Z_0/T})^{-1} - (\lambda - L_{0,T}^{3,Z_0/T})^{-1}$ for $\lambda \in \Gamma$, and to obtain inequality (11.58).

k) A family of Sobolev spaces with weights

In formula (11.59), we see that operators like

$$\rho(uZ)\left\langle (\nabla^{TX})^2_{Z_0/T}(Z, e_i)e_j, e_{j'}\right\rangle \left(e^j \wedge -\frac{u^2}{2} i_{e_j}\right)\left(e^{j'} \wedge -\frac{u^2}{2} i_{e_{j'}}\right)$$

do appear. These operators are not uniformly bounded for the usual L_2 Hermitian product. In this Section, we introduce a new family of Hermitian products such that these operators will remain uniformly bounded. This fact will play a key role in the sequel.

We equip $\Lambda(T_{\mathbf{R}}^* X)_{y_0}$ with the metric induced by the metric g^{TX}. We denote by $|\ |$ the corresponding norm. Clearly

$$\Lambda(T_{\mathbf{R}}^* X)_{y_0} = \Lambda(T_{\mathbf{R}}^* Y)_{y_0} \hat{\otimes} \Lambda(N_{\mathbf{R}}^*)_{y_0}. \tag{11.68}$$

Set $n = \dim N$. For $0 \leqslant p \leqslant 2l'$, $0 \leqslant q \leqslant 2n$, set

$$\Lambda^{(p,q)}(T_{\mathbf{R}}^* X)_{y_0} = \Lambda^p(T_{\mathbf{R}}^* Y)_{y_0} \hat{\otimes} \Lambda^q(N_{\mathbf{R}}^*)_{y_0}. \tag{11.69}$$

The various $\Lambda^{(p,q)}(T_{\mathbf{R}}^* X)_{y_0}$ are mutually orthogonal in $\Lambda(T_{\mathbf{R}}^* X)_{y_0}$.

Let $\mathbf{I}_{y_0}$ be the set of smooth sections of $\Lambda(T_{\mathbf{R}}^* X)_{y_0} \hat{\otimes} \xi_{y_0}$ over $(T_{\mathbf{R}} X)_{y_0}$, let $\mathbf{I}_{(p,q),y_0}$ be the set of smooth sections of $\Lambda^{(p,q)}(T_{\mathbf{R}}^* X)_{y_0} \hat{\otimes} \xi_{y_0}$ over $(T_{\mathbf{R}} X)_{y_0}$. Let $\mathbf{I}_{y_0}^0$ be the set of square integrable sections of $(\Lambda(T_{\mathbf{R}}^* X) \hat{\otimes} \xi)_{y_0}$ over $(T_{\mathbf{R}} X)_{y_0}$, let $\mathbf{I}_{(p,q),y_0}^0$ be the set of square integrable sections of $(\Lambda^{(p,q)}(T_{\mathbf{R}}^* X) \hat{\otimes} \xi)_{y_0}$ over $(T_{\mathbf{R}} X)_{y_0}$.

Definition **11.23.** – For $u \in]0, 1]$, $T \in [1, (1/u)]$, $y_0 \in Y$, $Z_0 \in N_{\mathbf{R},y_0}$, $|Z_0| \leqslant (\varepsilon T/2)$, $s \in \mathbf{I}_{(p,q),y_0}^0$, set

$$|s|_{u,T,Z_0,0}^2 = \int_{(T_{\mathbf{R}} X)_{y_0}} |s|^2 \left(1 + (|Z| + |Z_0|)\,\rho\left(\frac{uZ}{2}\right)\right)^{2(2l'-p)} \left(1 + \frac{|Z|}{T}\rho\left(\frac{uZ}{2}\right)\right)^{2(2n-q)} dv_{TX}(Z). \tag{11.70}$$

Notice that since the function ρ has compact support, for any $u>0$, $T>0$, the norm $|\ |_{u,T,Z_0,0}$ is equivalent to the usual L_2 norm of $I^0_{y_0}$.

Then (11.70) induces a Hermitian product $\langle\ ,\ \rangle_{u,T,Z_0,0}$ on $I^0_{(p,q),y_0}$. We equip $I^0_{y_0}=\oplus I^0_{(p,q),y_0}$ with the direct sum of the Hermitian products (11.70).

We will say that a family of operators acting on $I^0_{y_0}$, which depend on $u\in]0,1]$, $T\in[1,(1/u)]$, $y_0\in Y$, $|Z_0|\in N_{\mathbf{R},y_0}$, $|Z_0|\leqslant(\varepsilon T/2)$, is uniformly bounded if their norms calculated with respect to the norms $|\ |_{u,T,Z_0,0}$ are uniformly bounded.

The Hilbert norms $|\ |_{u,T,Z_0,0}$ have been taylor-made for the Proposition which follows to be true.

Proposition **11.24.** – *The following families of operators acting on* $I^0_{y_0}$ *and depending on* $u\in]0,1]$, $T\in[1,(1/u)]$, $y_0\in Y$, $Z_0\in N_{\mathbf{R},y_0}$, $|Z_0|\leqslant(\varepsilon T/2)$ *are uniformly bounded*

$$
(11.71)\qquad
\begin{aligned}
&1_{u|Z|\leqslant(\varepsilon/2)}\left(e^i\wedge-\frac{u^2}{2}i_{e_i}\right),\quad 1\leqslant i\leqslant 2l';\\
&1_{u|Z|\leqslant(\varepsilon/2)}|Z|\left(e^i\wedge-\frac{u^2}{2}i_{e_i}\right),\quad 1\leqslant i\leqslant 2l';\\
&1_{u|Z|\leqslant(\varepsilon/2)}|Z_0|\left(e^i\wedge-\frac{u^2}{2}i_{e_i}\right),\quad 1\leqslant i\leqslant 2l';\\
&1_{u|Z|\leqslant(\varepsilon/2)}\left(e^i\wedge-\frac{u^2T^2}{2}i_{e_i}\right),\quad 2l'+1\leqslant i\leqslant 2l;\\
&1_{u|Z|\leqslant(\varepsilon/2)}|Z|\left(\frac{e^i}{T}\wedge-\frac{u^2T}{2}i_{e_i}\right),\quad 2l'+1\leqslant i\leqslant 2l.
\end{aligned}
$$

Proof. – If $|Z|\leqslant(\varepsilon/2u)$, then $\rho(uZ/2)=1$. The Proposition follows from the fact that if $u\in]0,1]$, $T\in[1,(1/u)]$, $|Z_0|\leqslant(\varepsilon T/2)$, $|Z|\leqslant(\varepsilon/2u)$, then

$$
(11.72)\qquad
\begin{aligned}
&\frac{1}{1+|Z|+|Z_0|}\leqslant 1;\quad \frac{|Z|}{1+|Z|+|Z_0|}\leqslant 1;\quad \frac{|Z_0|}{1+|Z|+|Z_0|}\leqslant 1;\\
&u^2(1+|Z|+|Z_0|)\leqslant Cu;\quad u^2|Z|(1+|Z|+|Z_0|)\leqslant C;\\
&u^2|Z_0|(1+|Z|+|Z_0|)\leqslant C;\\
&\frac{1}{1+\dfrac{|Z|}{T}}\leqslant 1;\quad \frac{\dfrac{|Z|}{T}}{1+\dfrac{|Z|}{T}}\leqslant 1;\\
&u^2T^2\left(1+\frac{|Z|}{T}\right)\leqslant C;\quad u^2T|Z|\left(1+\frac{|Z|}{T}\right)\leqslant C.\quad\square
\end{aligned}
$$

For $\mu \in \mathbf{R}$, let $\mathbf{I}^{\mu}_{y_0}$, $\mathbf{I}^{\pm,\mu}_{y_0}$ be the set of sections of $(\Lambda(T^*_{\mathbf{R}}X)\hat{\otimes}\xi)_{y_0}$, $(\Lambda(T^*_{\mathbf{R}}X)\hat{\otimes}\xi^{\pm})_{y_0}$ over $(T_{\mathbf{R}}X)_{y_0}$ which lie in the μ^{th} Sobolev space. If $s \in \mathbf{I}^{\pm,\mu}_{y_0}$, we write s in the form

$$s = s^+ + s^-; \qquad s^{\pm} \in \mathbf{I}^{\pm,\mu}_{y_0}.$$

Definition **11.25.** — If $u \in]0, 1]$, $T \in [1, (1/u)]$, $y_0 \in Y$, $Z_0 \in N_{\mathbf{R}, y_0}$, $|Z_0| \leqslant (\varepsilon T/2)$, $s \in \mathbf{I}^1_{y_0}$, set

$$(11.73) \qquad \begin{aligned} |s|^2_{u,T,Z_0,1} = {} & |s|^2_{u,T,Z_0,0} + T^2 \, |s^+|^2_{u,T,Z_0,0} \\ & + T^2 \left| \rho(uZ) V^- \left(\frac{Z_0}{T} + uZ\right) s^- \right|^2_{u,T,Z_0,0} + \sum_{i=1}^{2l} |\nabla_{e_i} s|^2_{u,T,Z_0,0}. \end{aligned}$$

Then (11.73) defines a Hilbert norm on $\mathbf{I}^1_{y_0}$. Also $(\mathbf{I}^1_{y_0}, |\ |_{u,T,Z_0,1})$ is continuously embedded in $(\mathbf{I}^0_{y_0}, |\ |_{u,T,Z_0,0})$. We identify $\mathbf{I}^0_{y_0}$ with its antidual by the Hermitian product $\langle\ ,\ \rangle_{u,T,Z_0,0}$. The Hermitian product $\langle\ ,\ \rangle_{u,T,Z_0,0}$ permits us to identify $\mathbf{I}^{-1}_{y_0}$ with the antidual of $\mathbf{I}^1_{y_0}$. Let $|\ |_{u,T,Z_0,-1}$ be the norm on $\mathbf{I}^{-1}_{y_0}$ associated with the norm $|\ |_{u,T,Z_0,1}$ on $\mathbf{I}^1_{y_0}$. We then have the family of continuous dense embeddings with norms smaller than one

$$\mathbf{I}^1_{y_0} \to \mathbf{I}^0_{y_0} \to \mathbf{I}^{-1}_{y_0}.$$

Theorem **11.26.** — *There exist constants* $C_1 > 0$, $C_2 > 0$, $C_3 > 0$, $C_4 > 0$ *such that if* $u \in]0, 1]$, $T \in [1, (1/u)]$, $y_0 \in Y$, $Z_0 \in N_{\mathbf{R}, y_0}$, $|Z_0| \leqslant (\varepsilon T/2)$, *for any* $s, s' \in \mathbf{I}_{y_0}$ *with compact support, then*

$$(11.74) \qquad \begin{aligned} & \operatorname{Re} \langle L^{3, Z_0/T}_{u,T} s, s \rangle_{u,T,Z_0,0} \geqslant C_1 \, |s|^2_{u,T,Z_0,1} - C_2 \, |s|^2_{u,T,Z_0,0}, \\ & |\operatorname{Im} \langle L^{3, Z_0/T}_{u,T} s, s \rangle_{u,T,Z_0,0}| \leqslant C_3 \, |s|_{u,T,Z_0,1} \, |s|_{u,T,Z_0,0}, \\ & |\langle L^{3, Z_0/T}_{u,T} s, s' \rangle_{u,T,Z_0,0}| \leqslant C_4 \, |s|_{u,T,Z_0,1} \, |s'|_{u,T,Z_0,1}. \end{aligned}$$

Proof. — Since ρ has compact support, there exists $C > 0$ such that under the stated conditions on u, T, y_0, Z_0, if ∇ denotes the gradient in the variable Z, then

$$(11.75) \qquad \begin{aligned} & \left| \nabla \left(1 + (|Z| + |Z_0|) \, \rho\left(\frac{uZ}{2}\right) \right) \right| \leqslant C, \\ & \left| \nabla \left(1 + \frac{|Z|}{T} \, \rho\left(\frac{uZ}{2}\right) \right) \right| \leqslant C. \end{aligned}$$

Observe that if $|Z| \leqslant (\varepsilon/2)$, the vectors $\tau e^{Z_0/T}_1(Z), \ldots, \tau e^{Z_0/T}_{2l}(Z)$ span $(T_{\mathbf{R}}X)_{Z_0/T+Z}$. By using (11.59), Proposition 11.24, and (11.75), we find that there exist $C>0$, $C'>0$, $C''>0$ such that if $s, s' \in \mathbf{I}_{y_0}$ have compact support, then

$$
(11.76)\quad
\begin{aligned}
&\operatorname{Re}\langle M^{3, Z_0/T}_{u,T} s, s\rangle_{u,T,Z_0,0} \geqslant C\,|\nabla s|^2_{u,T,Z_0,0} - C'\,|s|^2_{u,T,Z_0,0},\\
&|\operatorname{Im}\langle M^{3, Z_0/T}_{u,T} s, s\rangle_{u,T,Z_0,0}| \leqslant C''\,(|s|_{u,T,Z_0,0} + |\nabla s|_{u,T,Z_0,0})\,|s|_{u,T,Z_0,0},\\
&|\langle M^{3, Z_0/T}_{u,T} s, s'\rangle_{u,T,Z_0,0}| \leqslant C''\,(|s|_{u,T,Z_0,0} + |\nabla s|_{u,T,Z_0,0})\\
&\qquad\qquad (|s'|_{u,T,Z_0,0} + |\nabla s'|_{u,T,Z_0,0}).
\end{aligned}
$$

Also by (11.65), (11.66) and Proposition 11.24, we get

$$
(11.77)\quad
\begin{aligned}
&\Big|\Big\langle \rho^2(uZ)\,T \sum_{j=1}^{2l'}\Big(e^j\wedge -\frac{u^2}{2} i_{e_j}\Big)\nabla^{\xi}_{\tau e_j Z_0/T(uZ)} V\Big(\frac{Z_0}{T}+uZ\Big) s^+, s'^+\Big\rangle_{u,T,Z_0,0}\Big| \leqslant CT\,|s^+|_{u,T,Z_0,0}\,|s'^+|_{u,T,Z_0,0},\\
&\Big|\Big\langle \rho^2(uZ)\,T \sum_{j=1}^{2l'}\Big(e^j\wedge -\frac{u^2}{2} i_{e_j}\Big)\nabla^{\xi}_{\tau e_j Z_0/T(uZ)} V\Big(\frac{Z_0}{T}+uZ\Big) s^\pm, s'^\mp\Big\rangle_{u,T,Z_0,0}\Big|\\
&\qquad \leqslant CT\,(|s^+|_{u,T,Z_0,0}\,|s'^-|_{u,T,Z_0,0} + |s^-|_{u,T,Z_0,0}\,|s'^+|_{u,T,Z_0,0}),\\
&\Big|\Big\langle \rho^2(uZ)\,T \sum_{j=1}^{2l'}\Big(e^j\wedge -\frac{u^2}{2} i_{e_j}\Big)\nabla^{\xi}_{\tau e_j Z_0/T(uZ)} V\Big(\frac{Z_0}{T}+uZ\Big) s^-, s'^-\Big\rangle_{u,T,Z_0,0}\Big| \leqslant C\,|s|_{u,T,Z_0,0}\,|s'|_{u,T,Z_0,0},\\
&\Big|\Big\langle \rho^2(uZ) \sum_{j=2l'+1}\Big(e^j\wedge -\frac{u^2T^2}{2} i_{e_j}\Big)\nabla^{\xi}_{\tau e_j Z_0/T(uZ)} V\Big(\frac{Z_0}{T}+uZ\Big) s, s'\Big\rangle_{u,T,Z_0,0}\Big| \leqslant C\,|s|_{u,T,Z_0,0}\,|s'|_{u,T,Z_0,0}.
\end{aligned}
$$

Using the last two lines in (11.76) and (11.77), we get the last two lines in (11.74). Moreover for any $\eta > 0$,

$$
(11.78)\quad T \leqslant \frac{1}{2}\Big(\eta T^2 + \frac{1}{\eta}\Big).
$$

Using the first inequality in (11.76), (11.77) with $s=s'$ and (11.78) with $\eta>0$ small enough, we obtain the first inequality in (11.74). Our Theorem is proved. □

1) Estimates on the resolvent of $L^{3, Z_0/T}_{u,T}$

In the sequel, we consider the operators $L^{3, Z_0/T}_{u,T}$ as unbounded operators acting on $\mathbf{I}^0_{y_0}$ with domain $\mathbf{I}^2_{y_0}$.

If $A \in \mathscr{L}(\mathbf{I}^0_{y_0})$ (resp. $\mathscr{L}(\mathbf{I}^{-1}_{y_0}, \mathbf{I}^1_{y_0})$), we note $\| A \|^{0,0}_{u,T,Z_0}$ (resp. $\| A \|^{-1,1}_{u,T,Z_0}$) the norm of A with respect to the norm $|\ |_{u,T,Z_0,0}$ (resp. the norms $|\ |_{u,T,Z_0,-1}$ and $|\ |_{u,T,Z_0,1}$).

Theorem **11.27.** – *There exist* $C>0$, $A>0$, $\delta>0$ *such that if*

$$U=\{\lambda \in \mathbf{C};\ \operatorname{Re}(\lambda) \leqslant \delta \operatorname{Im}^2(\lambda) - A\}, \tag{11.79}$$

if $u \in]0,1]$, $T \in [1,(1/u)]$, $y_0 \in Y$, $Z_0 \in N_{\mathbf{R},y_0}$, $|Z_0| \leqslant (\varepsilon T/2)$, $\lambda \in U$, *the resolvent* $(\lambda - L^{3, Z_0/T}_{u,T})^{-1}$ *exists, extends to a continuous linear operator from* $\mathbf{I}^{-1}_{y_0}$ *into* $\mathbf{I}^1_{y_0}$, *and moreover*

$$\begin{aligned} &\| (\lambda - L^{3, Z_0/T}_{u,T})^{-1} \|^{0,0}_{u,T,Z_0} \leqslant C \\ &\| (\lambda - L^{3, Z_0/T}_{u,T})^{-1} \|^{-1,1}_{u,T,Z_0} \leqslant C(1+|\lambda|)^2. \end{aligned} \tag{11.80}$$

Proof. – By the first inequality in (11.74), we find if $\lambda \in \mathbf{R}$, $\lambda \leqslant -C_2$, if $s \in \mathbf{I}_{y_0}$ has compact support, then

$$\operatorname{Re} \langle (L^{3, Z_0/T}_{u,T} - \lambda) s, s \rangle_{u,T,Z_0,0} \geqslant C_1 |s|^2_{u,T,Z_0,0}. \tag{11.81}$$

From (11.81), we get

$$|s|_{u,T,Z_0,0} \leqslant C_1^{-1} |(L^{3, Z_0/T}_{u,T} - \lambda) s|_{u,T,Z_0,0}. \tag{11.82}$$

Since $L^{3, Z_0/T}_{u,T} - \lambda$ is an elliptic operator of order two which coincides with $-(\Delta/2) - \lambda$ for $|Z|$ large enough, if $|\ |_{\mathbf{I}^2_{y_0}}$ is a Sobolev norm on $\mathbf{I}^2_{y_0}$, there exists $C'>0$ (which depends on u, T, Z_0, λ) such that

$$|s|_{\mathbf{I}^2_{y_0}} \leqslant C' (|(\lambda - L^{3, Z_0/T}_{u,T}) s|_{u,T,Z_0,0} + |s|_{u,T,Z_0,0}). \tag{11.83}$$

From (11.82), (11.83), we find that if $\lambda \in \mathbf{R}$, $\lambda \leqslant -C_2$, the resolvent $(\lambda - L^{3, Z_0/T}_{u,T})^{-1}$ exists.

Take now $\lambda = a + ib \in \mathbf{C}$, a, b, $\in \mathbf{R}$. If $s \in \mathbf{I}^2_{y_0}$ has compact support, then

$$\begin{aligned} |\langle (L^{3, Z_0/T}_{u,T} - \lambda) s, s \rangle_{u,T,Z_0,0}| \geqslant \sup \{ &\operatorname{Re} \langle L^{3, Z_0/T}_{u,T} s, s \rangle_{u,T,Z_0,0} \\ &- a |s|^2_{u,T,Z_0,0}, |\operatorname{Im} \langle L^{3, Z_0/T}_{u,T} s, s \rangle_{u,T,Z_0,0} - b |s|^2_{u,T,Z_0,0}| \}. \end{aligned} \tag{11.84}$$

Using (11.74) and (11.84), we get

(11.85) $$|\langle (\lambda - L_{u,T}^{3,Z_0/T}) s, s \rangle_{u,T,Z_0,0}| \geqslant \sup\{C_1 |s|_{u,T,Z_0,1}^2 - (C_2+a)|s|_{u,T,Z_0,0}^2, \ -C_3 |s|_{u,T,Z_0,1} |s|_{u,T,Z_0,0} + |b|\, |s|_{u,T,Z_0,0}^2\}.$$

Set

(11.86) $$C(\lambda) = \inf_{t \in \mathbf{R}, t \geqslant 1} \sup\{C_1 t^2 - (C_2+a), \ -C_3 t + |b|\}.$$

Note that $|s|_{u,T,Z_0,0} \leqslant |s|_{u,T,Z_0,1}$. From (11.85), (11.86), we deduce that

(11.87) $$|(\lambda - L_{u,T}^{3,Z_0/T}) s|_{u,T,Z_0,0} \geqslant C(\lambda) |s|_{u,T,Z_0,0}.$$

It is easy to verify that if $A > 0$ is large enough, and if $\delta > 0$ is small enough, if U is defined by (11.79), then

(11.88) $$C_0 = \inf_{\lambda \in U} C(\lambda) > 0.$$

We now fix $A > 0$, $\delta > 0$ such that (11.88) holds. From (11.87), (11.88), we deduce

(11.89) $$|(\lambda - L_{u,T}^{3,Z_0/T}) s|_{u,T,Z_0,0} \geqslant C_0 |s|_{u,T,Z_0,0}.$$

Using (11.89), we find that if $\lambda \in U$, if the resolvent $(\lambda - L_{u,T}^{3,Z_0/T})^{-1}$ exists, then

(11.90) $$\|(\lambda - L_{u,T}^{3,Z_0/T})^{-1} s\|_{u,T,Z_0}^{0,0} \leqslant C_0^{-1}.$$

From (11.90), we see that if $\lambda' \in \mathbf{C}$, $|\lambda' - \lambda| \leqslant (C_0/2)$, then the resolvent $(\lambda' - L_{u,T}^{3,Z_0/T})^{-1}$ still exists. Now we saw before that if $\lambda \in \mathbf{R}$, $\lambda \leqslant -C_2$, the resolvent $(\lambda - L_{u,T}^{3,Z_0/T})^{-1}$ exists. In particular there is at least one $\lambda \in U$ where the resolvent $(\lambda - L_{u,T}^{3,Z_0/T})^{-1}$ exists. It is now clear that the resolvent $(\lambda - L_{u,T}^{3,Z_0/T})^{-1}$ exists for every $\lambda \in U$, and that (11.90) holds. We have thus proved the first inequality in (11.80).

Using the third inequality in (11.74), it is clear that $L_{u,T}^{3,Z_0/T}$ can be extended to a continuous linear map from $\mathbf{I}_{y_0}^1$ into $\mathbf{I}_{y_0}^{-1}$. Moreover using the first inequality in (11.74), we find that if $\lambda_0 \in \mathbf{R}$, $\lambda_0 \leqslant -C_2$, if $s \in \mathbf{I}_{y_0}$ has compact support, then

(11.91) $$|s|_{u,T,Z_0,1} \leqslant C_1^{-1} \|(\lambda_0 - L_{u,T}^{3,Z_0/T}) s\|_{u,T,Z_0,-1}.$$

From the first inequality in (11.74) and from (11.91), we find that if $\lambda_0 \in \mathbf{R}$, $\lambda_0 \leqslant -C_2$, $\lambda_0 - L_{u,T}^{3,Z_0/T}$ is a one to one linear map from $\mathbf{I}_{y_0}^1$ into $\mathbf{I}_{y_0}^{-1}$ and that

(11.92) $$\|(\lambda_0 - L_{u,T}^{3,Z_0/T})^{-1}\|_{u,T,Z_0}^{-1,1} \leqslant C_1^{-1}.$$

Note that (11.91), (11.92) are in fact a consequence of Lax-Milgram's lemma.

Take now $\lambda_0 \in \mathbf{R}$, $\lambda_0 \leqslant -C_2$. If $\lambda \in U$, then

$$(11.93) \qquad (\lambda - L_{u,T}^{3,Z_0/T})^{-1} = (\lambda_0 - L_{u,T}^{3,Z_0/T})^{-1} + (\lambda_0 - \lambda)(\lambda - L_{u,T}^{3,Z_0/T})^{-1}(\lambda_0 - L_{u,T}^{3,Z_0/T})^{-1}.$$

From (11.90), (11.92), (11.93), we deduce that $(\lambda - L_{u,T}^{3,Z_0/T})^{-1}$ extends to a continuous linear map from $I_{y_0}^{-1}$ into $I_{y_0}^0$. Also if $\|(\lambda - L_{u,T}^{3,Z_0/T})^{-1}\|_{u,T,Z_0}^{-1,0}$ denotes the norm of $(\lambda - L_{u,T}^{3,Z_0/T})^{-1}$ when $I_{y_0}^{-1}$ and $I_{y_0}^0$ are respectively equipped with the norms $|\ |_{u,T,Z_0,-1}$, $|\ |_{u,T,Z_0,0}$, we find that

$$(11.94) \qquad \|(\lambda - L_{u,T}^{3,Z_0/T})^{-1}\|_{u,T,Z_0}^{-1,0} \leqslant C_1^{-1} + C_0^{-1} C_1^{-1} |\lambda - \lambda_0|.$$

Moreover

$$(11.95) \qquad (\lambda - L_{u,T}^{3,Z_0/T})^{-1} = (\lambda_0 - L_{u,T}^{3,Z_0/T})^{-1} + (\lambda_0 - \lambda)(\lambda_0 - L_{u,T}^{3,Z_0/T})^{-1}(\lambda - L_{u,T}^{3,Z_0/T})^{-1}.$$

From (11.92), (11.95), we deduce that $(\lambda - L_{u,T}^{3,Z_0/T})^{-1}$ extends to a continuous linear map from $I_{y_0}^{-1}$ into $I_{y_0}^1$ and that

$$(11.96) \qquad \|(\lambda - L_{u,T}^{3,Z_0/T})^{-1}\|_{u,T,Z_0}^{-1,1} \leqslant C_1^{-1} + C_1^{-1} |\lambda - \lambda_0| \, \|(\lambda - L_{u,T}^{3,Z_0/T})^{-1}\|_{u,T,Z_0}^{-1,0}.$$

Using (11.94), (11.96), we get the second inequality in (11.80). The proof of Theorem 11.27 is thus completed. □

m) Regularizing properties of the resolvent of $L_{u,T}^{3,Z_0/T}$

Since Y is a compact manifold, there exists a finite family of smooth functions $f_1, \ldots, f_r$ defined on X with values in $[0, 1]$ which have the following properties.

- $Y = \bigcap_{j=1}^{r} \{x \in X; f_j(x) = 0\}$
- On Y, $df_1, \ldots, df_r$ span $N_{\mathbf{R}}^*$.

Clearly if $|Z_0| \leqslant (\varepsilon T/2)$, the functions $Z \in (T_{\mathbf{R}} X)_{y_0} \to \rho(uZ) f_j((Z_0/T) + uZ)$ $(1 \leqslant j \leqslant r)$ are well defined.

Take now $u \in]0, 1]$, $T \in [1, (1/u)]$, $y_0 \in Y$, $Z_0 \in N_{\mathbf{R}, y_0}$, $|Z_0| \leqslant (\varepsilon T/2)$.

Definition **11.28.** – Let $\mathcal{Q}_{u,T,Z_0}$ be the family of operators acting on I_{y_0}

$$(11.97) \qquad \mathcal{Q}_{u,T,Z_0} = \left\{ \nabla_{e_i}, \ 1 \leqslant i \leqslant 2l; \ T \rho(uZ) f_j\left(\frac{Z_0}{T} + uZ\right), \ 1 \leqslant j \leqslant r \right\}.$$

For $k \in \mathbf{N}$, let $\mathscr{Q}^k_{u,\mathrm{T},\mathrm{Z}_0}$ be the family of operators Q acting on $\mathbf{I}_{y_0}$ which can be written in the form

$$\mathrm{Q}=\mathrm{Q}_1 \dots \mathrm{Q}_k, \quad \mathrm{Q}_i \in \mathscr{Q}_{u,\mathrm{T},\mathrm{Z}_0} (1 \leqslant i \leqslant k). \tag{11.98}$$

If $k \in \mathbf{N}$, we equip the Sobolev space $\mathbf{I}^k_{y_0}$ with the Hilbert norm $\| \quad \|_{u,\mathrm{T},\mathrm{Z}_0,k}$ such that if $s \in \mathbf{I}^k_{y_0}$

$$\| s \|^2_{u,\mathrm{T},\mathrm{Z}_0,k} = \sum_{l=0}^{k} \sum_{\mathrm{Q} \in \mathscr{Q}^l_{u,\mathrm{T},\mathrm{Z}_0}} | \mathrm{Q} s |^2_{u,\mathrm{T},\mathrm{Z}_0,0}. \tag{11.99}$$

Proposition **11.29.** — *Take* $k \in \mathbf{N}$. *There exists* $\mathrm{C}_k > 0$ *such that, for any* $u \in]0, 1]$, $\mathrm{T} \in [1, (1/u)]$, $y_0 \in \mathrm{Y}$, $\mathrm{Z}_0 \in \mathrm{N}_{\mathbf{R}, y_0}$, $|\mathrm{Z}_0| \leqslant (\varepsilon \mathrm{T}/2)$, *if* $\mathrm{Q}_1, \dots, \mathrm{Q}_k \in \mathscr{Q}$, *if* $s, s' \in \mathbf{I}_{y_0}$ *have compact support, then*

$$\begin{aligned} &| \langle [\mathrm{Q}_1, [\mathrm{Q}_2, \dots [\mathrm{Q}_k, \mathrm{L}^{3,\mathrm{Z}_0/\mathrm{T}}_{u,\mathrm{T}}]] \dots] s, s' \rangle_{u,\mathrm{T},\mathrm{Z}_0,0} | \\ &\qquad \leqslant \mathrm{C}_k | s |_{u,\mathrm{T},\mathrm{Z}_0,1} | s' |_{u,\mathrm{T},\mathrm{Z}_0,1}. \end{aligned} \tag{11.100}$$

Proof. — We first prove (11.100) for $k=1$. We thus have to show if $\mathrm{Q} \in \mathscr{Q}_{u,\mathrm{T},\mathrm{Z}_0}$

$$| \langle [\mathrm{Q}, \mathrm{L}^{3,\mathrm{Z}_0/\mathrm{T}}_{u,\mathrm{T}}] s, s' \rangle_{u,\mathrm{T},\mathrm{Z}_0,0} | \leqslant \mathrm{C} | s |_{u,\mathrm{T},\mathrm{Z}_0,1} | s' |_{u,\mathrm{T},\mathrm{Z}_0,1}. \tag{11.101}$$

Suppose that $\mathrm{Q}=\nabla_{e_i} (1 \leqslant i \leqslant 2l)$. We use formula (11.60) for $\mathrm{L}^{3,\mathrm{Z}_0/\mathrm{T}}_{u,\mathrm{T}}$. By Proposition 11.24, the term $[\mathrm{Q}, \mathrm{M}^{3,\mathrm{Z}_0/\mathrm{T}}_{u,\mathrm{T}}]$ can be easily dealt with. Also

$$\begin{aligned} &\left[\nabla_{e_i}, \rho^2 (u\mathrm{Z}) \mathrm{T} \sum_{j=1}^{2l'} \left(e^j \wedge -u^2 \frac{i_{e_j}}{2} \right) \nabla^{\xi}_{\tau e_j^{\mathrm{Z}_0/\mathrm{T}} (u\mathrm{Z})} \mathrm{V} \left(\frac{\mathrm{Z}_0}{\mathrm{T}} + u\mathrm{Z} \right) \right] \\ &\qquad = u\mathrm{T} \left\{ \nabla_{e_i} \left(\rho^2 (.) \sum_{j=1}^{2l'} \left(e^j \wedge -u^2 \frac{i_{e_j}}{2} \right) \nabla^{\xi}_{\tau e_j^{\mathrm{Z}_0/\mathrm{T}}} \mathrm{V} \left(\frac{\mathrm{Z}_0}{\mathrm{T}} + . \right) \right) \right\} (u\mathrm{Z}). \end{aligned} \tag{11.102}$$

Since $u\mathrm{T} \leqslant 1$, using Proposition 11.24, we can also control the contribution of (11.102) to $| \langle [\mathrm{Q}, \mathrm{L}^{3,\mathrm{Z}_0/\mathrm{T}}_{u,\mathrm{T}}] s, s' \rangle_{u,\mathrm{T},\mathrm{Z}_0,0} |$. Similarly

$$\begin{aligned} &\left[\nabla_{e_i}, \mathrm{T}^2 \rho^2 (u\mathrm{Z}) \mathrm{V}^2 \left(\frac{\mathrm{Z}_0}{\mathrm{T}} + u\mathrm{Z} \right) \right] \\ &\quad = \rho (u\mathrm{Z}) u\mathrm{T} \left\{ 2\mathrm{T} (\nabla_{e_i} \rho) (u\mathrm{Z}) \mathrm{V}^2 \left(\frac{\mathrm{Z}_0}{\mathrm{T}} + u\mathrm{Z} \right) \right. \\ &\qquad \left. + \mathrm{T} \rho (u\mathrm{Z}) \left[\mathrm{V} \left(\frac{\mathrm{Z}_0}{\mathrm{T}} + u\mathrm{Z} \right), (\nabla_{e_i} \mathrm{V}) \left(\frac{\mathrm{Z}_0}{\mathrm{T}} + u\mathrm{Z} \right) \right] \right\}. \end{aligned} \tag{11.103}$$

Since $u\mathrm{T} \leqslant 1$, from (11.103), we deduce that

(11.104)
$$\left|\left\langle\left[\nabla_{e_i}, \mathrm{T}^2 \rho^2(u, \mathrm{Z}) \mathrm{V}^2\left(\frac{\mathrm{Z}_0}{\mathrm{T}}+u\mathrm{Z}\right)\right] s(\mathrm{Z}), s'(\mathrm{Z})\right\rangle\right| \leqslant \mathrm{CT}\left(\left|\rho(u\mathrm{Z})\mathrm{V}\left(\frac{\mathrm{Z}_0}{\mathrm{T}}+u\mathrm{Z}\right)s(\mathrm{Z})\right| |s'(\mathrm{Z})| + |s(\mathrm{Z})| \left|\rho(u\mathrm{Z})\mathrm{V}\left(\frac{\mathrm{Z}_0}{\mathrm{T}}+u\mathrm{Z}\right)s'(\mathrm{Z})\right|\right).$$

From (11.102)-(11.104), we easily obtain (11.101) when $\mathrm{Q}=\nabla_{e_i}$ $(1 \leqslant i \leqslant m)$. Also for $1 \leqslant j \leqslant r$, if $\mathrm{Q}=\mathrm{T}\rho(u\mathrm{Z}) f_j((\mathrm{Z}_0/\mathrm{T})+u\mathrm{Z})$, then

(11.105)
$$[\mathrm{Q}, \mathrm{L}_{u,\mathrm{T}}^{3,\mathrm{Z}_0/\mathrm{T}}]=\mathrm{T}\left[\rho(u\mathrm{Z}) f_j\left(\frac{\mathrm{Z}_0}{\mathrm{T}}+u\mathrm{Z}\right), \mathrm{M}_{u,\mathrm{T}}^{3,\mathrm{Z}_0/\mathrm{T}}\right].$$

Clearly, for $1 \leqslant i \leqslant 2l$

(11.106)
$$\nabla_{e_i}\mathrm{T}\left(\rho(u\mathrm{Z}) f_j\left(\frac{\mathrm{Z}_0}{\mathrm{T}}+u\mathrm{Z}\right)\right)=u\mathrm{T}\left((\nabla_{e_i}\rho)(u\mathrm{Z}) f_j\left(\frac{\mathrm{Z}_0}{\mathrm{T}}+u\mathrm{Z}\right) + \rho(u\mathrm{Z})(\nabla_{e_i} f_j)\left(\frac{\mathrm{Z}_0}{\mathrm{T}}+u\mathrm{Z}\right)\right).$$

Since $u\mathrm{T} \leqslant 1$, using Proposition 11.24, we also obtain (11.101) when $\mathrm{Q}=\mathrm{T}\rho(u\mathrm{Z}) f_j((\mathrm{Z}_0/\mathrm{T})+u\mathrm{Z})$ $(1 \leqslant j \leqslant r)$. We have thus proved (11.100) when $k=1$.

We now briefly indicate the principle of the proof of (11.100) when $k=2$. Take $\mathrm{Q}_1=\nabla_{e_i}$, $\mathrm{Q}_2=\nabla_{e_{i'}}$, $1 \leqslant i, i' \leqslant 2l$. The only difficulty comes from the commutator $[\nabla_{e_{i'}}, [\nabla_{e_i}, \mathrm{T}^2\rho^2(u\mathrm{Z})\mathrm{V}^2((\mathrm{Z}_0/\mathrm{T})+u\mathrm{Z})]]$. However two successive derivations in the variable Z introduce a factor u^2, and $u^2\mathrm{T}^2 \leqslant 1$. We thus obtain (11.100) in this case. The other cases left when $k=2$ are trivial.

Finally when $k \geqslant 3$, one easily verifies that (11.100) follows from Proposition 11.24. □

If $\mathrm{A} \in \mathscr{L}(\mathbf{I}_{y_0}^k, \mathbf{I}_{y_0}^{k'})$, we denote by $|||\mathrm{A}|||_{u,\mathrm{T},\mathrm{Z}_0}^{k,k'}$ the norm of A with respect to the norms $\| \quad \|_{u,\mathrm{T},\mathrm{Z}_0,k}$ and $\| \quad \|_{u,\mathrm{T},\mathrm{Z}_0,k'}$ on $\mathbf{I}_{y_0}^k$ and $\mathbf{I}_{y_0}^{k'}$.

We now fix $\mathrm{A} > 0$, $\delta > 0$, as in Theorem 11.27. Also U denotes the subset of $\mathbf{C}$ defined in (11.79).

Theorem **11.30.** – *For any* $k \in \mathbf{N}$, *there exist* $m_k \in \mathbf{N}$, $\mathrm{C}_k > 0$ *such that if* $u \in]0, 1]$, $\mathrm{T} \in [1, (1/u)]$, $y_0 \in \mathrm{Y}$, $\mathrm{Z}_0 \in \mathrm{N}_{\mathbf{R}, y_0}$, $|\mathrm{Z}_0| \leqslant (\varepsilon\mathrm{T}/2)$, $\lambda \in \mathrm{U}$, *the resolvent* $(\lambda-\mathrm{L}_{u,\mathrm{T}}^{3,\mathrm{Z}_0/\mathrm{T}})^{-1}$ *maps* $\mathbf{I}_{y_0}^k$ *into* $\mathbf{I}_{y_0}^{k+1}$ *and moreover*

$$(11.107) \qquad |||(\lambda - L_{u,T}^{3,Z_0/T})^{-1}|||_{u,T,Z_0}^{k,k+1} \leqslant C_k(1+|\lambda|)^{m_k}.$$

Proof. — We first prove (11.107) when $k=0$. Since for $1 \leqslant j \leqslant r$, f_j vanishes on Y, there exists $C>0$ such that if $y_0 \in Y_0$, $Z_0 \in N_{\mathbf{R},y_0}$, $|Z_0| \leqslant \varepsilon$, then

$$(11.108) \qquad |f_j(y_0, Z_0)| \leqslant C|Z_0|.$$

Using Proposition 8.14 and (11.108), we find that for any $x \in \mathscr{U}_\varepsilon$, $s \in \xi_x$, then

$$(11.109) \qquad |f_j(x)s| \leqslant C'|V(x)s|.$$

From (11.109), we deduce that

$$(11.110) \qquad \|s\|_{u,T,Z_0,1} \leqslant C''|s|_{u,T,Z_0,1}.$$

We then obtain (11.107) for $k=0$ from Theorem 11.27 and from (11.110).

More generally, if $Q_1, \ldots, Q_{k+1} \in \mathscr{Q}_{u,T,Z_0}$, we can express

$$Q_1 \ldots Q_{k+1}(\lambda - L_{u,T}^{3,Z_0/T})^{-1}$$

as a linear combination of operators of the type

$$(11.111) \qquad Q_1[Q_2,[Q_3, \ldots (\lambda - L_{u,T}^{3,Z_0/T})^{-1}]]Q_{k'+1} \ldots Q_{k+1}, \; k' \leqslant k.$$

Let $\mathscr{R}_{u,T,Z_0}$ be the family of operators

$$(11.112) \qquad \mathscr{R}_{u,T,Z_0} = \{[Q_{i_1},[Q_{i_2}, \ldots [Q_{i_p},(L_{u,T}^{3,Z_0/T})]]]\}$$

Clearly, any commutator $[Q_{i_1},[Q_{i_2}, \ldots [Q_{i_p},(\lambda - L_{u,T}^{3,Z_0/T})^{-1}]]]$ is a linear combination of operators of the form

$$(11.113) \qquad \begin{gathered}(\lambda - L_{u,T}^{3,Z_0/T})^{-1} R_1 (\lambda - L_{u,T}^{3,Z_0/T})^{-1} R_2 \ldots R_{k'}(\lambda - L_{u,T}^{3,Z_0/T})^{-1};\\ R_1, \ldots, R_{k'} \in \mathscr{R}_{u,T,Z_0}.\end{gathered}$$

By Proposition 11.29, the operators $R_j \in \mathscr{R}_{u,T,Z_0}$ are uniformly bounded in $\mathscr{L}((\mathbf{I}_{y_0}^1, |\ |_{u,T,Z_0,1}), (\mathbf{I}_{y_0}^{-1}, |\ |_{u,T,Z_0,-1}))$. By Theorem 11.27, we thus find that there exist $C>0$, $m \in \mathbf{N}$, such that the norm in $\mathscr{L}((\mathbf{I}_{y_0}^0, |\ |_{u,T,Z_0,0}), (\mathbf{I}_{y_0}^1, |\ |_{u,T,Z_0,1}))$ of the operators (11.113) is dominated by $C(1+|\lambda|)^m$.

As we saw before the operators Q ($Q \in \mathscr{Q}_{u,T,Z_0}$) are bounded in $\mathscr{L}((\mathbf{I}_{y_0}^1, |\ |_{u,T,Z_0,1}), (\mathbf{I}_{y_0}^0, |\ |_{u,T,Z_0,0}))$. Our Theorem follows. □

n) Uniform estimates on the kernel $P^{3, Z_0/T}_{u, T}$

Theorem **11.31.** – *For any* $m \in \mathbf{N}$, $m' \in \mathbf{N}$, *there exists* $C > 0$ *such that for* $u \in]0, 1]$, $T \in [1, (1/u)]$, $y_0 \in Y$, $Z_0 \in N_{\mathbf{R}, y_0}$, $|Z_0| \leq (\varepsilon T/2)$, *then*

$$(1+|Z_0|)^m \sup_{\substack{|\alpha| \leq m', |\alpha'| \leq m' \\ |Z| \leq 1, |Z'| \leq 1}} \left| \frac{\partial^{|\alpha|+|\alpha'|}}{\partial Z^\alpha \partial Z'^{\alpha'}} P^{3, Z_0/T}_{u, T}(Z, Z') \right| \leq C. \tag{11.114}$$

Proof. – Let Γ be the contour in $\mathbf{C}$

$$\Gamma = \{\lambda \in \mathbf{C};\ \mathrm{Re}(\lambda) = \delta\, \mathrm{Im}^2(\lambda) - A\}. \tag{11.115}$$

Using Theorem 11.27, we find that

$$\exp\left(\frac{-L^{3, Z_0/T}_{u, T}}{2}\right) = \frac{1}{2\pi i} \int_\Gamma \exp\left(\frac{-\lambda}{2}\right) (\lambda - L^{3, Z_0/T}_{u, T})^{-1} d\lambda. \tag{11.116}$$

Equivalently, for any $k \in \mathbf{N}$

$$\exp\left(\frac{-L^{3, Z_0/T}_{u, T}}{2}\right) = \frac{(-1)^{k-1}(k-1)!\, 2^{k-1}}{2\pi i} \int_\Gamma \exp\left(\frac{-\lambda}{2}\right) (\lambda - L^{3, Z_0/T}_{u, T})^{-k} d\lambda. \tag{11.117}$$

Using Theorem 11.30 and (11.117), we find that for any $k \in \mathbf{N}$

$$\left|\left|\left| \exp\left(-\frac{L^{3, Z_0/T}_{u, T}}{2}\right) \right|\right|\right|^{0, k}_{u, T, Z_0} \leq C_k. \tag{11.118}$$

Let $A_{u, T, Z_0/T}$ be the operator

$$A_{u, T, Z_0/T} = \sum_1^r \left| T \rho(uZ) f_j \left(\frac{Z_0}{T} + uZ\right) \right|^2. \tag{11.119}$$

From (11.118), we deduce that for any $k, k', k'' \in \mathbf{N}$

$$\left|\left|\left| \Delta^k A^{k'}_{u, T, Z_0/T} \Delta^{k''} \exp\left(-\frac{L^{3, Z_0/T}_{u, T}}{2}\right) \right|\right|\right|^{0, 0}_{u, T, Z_0} \leq C. \tag{11.120}$$

We claim that for any $k''' \in \mathbf{N}$

$$\left|\left|\left| \exp\left(-\frac{L^{3, Z_0/T}_{u, T}}{2}\right) \Delta^{k'''} \right|\right|\right|^{0, 0}_{u, T, Z_0} \leq C. \tag{11.121}$$

In fact let $L_{u,T}^{3,Z_0/T*}$ be the formal adjoint of $L_{u,T}^{3,Z_0/T}$ with respect to the usual (unweighted) Hermitian product on $\mathbf{I}_{y_0}^0$

$$s, s' \in \mathbf{I}_{y_0}^0 \to \langle s, s' \rangle = \int_{(T_{\mathbf{R}} X)_{y_0}} \langle s, s' \rangle \, dv_{TX}(Z) \tag{11.122}$$

Then $L_{u,T}^{3,Z_0/T}$ has essentially the same structure as the operator $L_{u,T}^{3,Z_0/T}$, except that the operators $e^i \wedge$, i_{e_i} are changed into i_{e_i}, $e^i \wedge$ respectively. If $s \in \mathbf{I}_{(p,q),y_0}^0$, we now set

$$|s|_{u,T,Z_0,0}^{\prime 2} = \int_{(T_{\mathbf{R}} X)} |s|^2 \left(1 + (|Z| + |Z_0|)\, \rho\left(\frac{uZ}{2}\right)\right)^{2(p-2l')} \left(1 + \frac{|Z|}{T} \rho\left(\frac{uZ}{2}\right)\right)^{2(q-2n)} dv_{TX}(Z). \tag{11.123}$$

The obvious analogue of Proposition 11.24 still holds. Moreover under the stated conditions on u, T, y_0, Z_0, the analogue of (11.75) is now

$$\begin{aligned} &\left|\nabla\left(1 + (|Z| + |Z_0|)\, \rho\left(\frac{uZ}{2}\right)\right)^{-1}\right| \leqslant C\left(1 + (|Z| + |Z_0|)\, \rho\left(\frac{uZ}{2}\right)\right)^{-1}, \\ &\left|\nabla\left(1 + \frac{|Z|}{T} \rho\left(\frac{uZ}{2}\right)\right)^{-1}\right| \leqslant \frac{C}{1 + \frac{|Z|}{T} \rho\left(\frac{uZ}{2}\right)}. \end{aligned} \tag{11.124}$$

The analysis of the operator $\exp(-L_{u,T}^{3,Z_0/T*}/2)$ proceeds exactly as before with respect to the new Hilbert norm. By taking adjoints again with respect to the ordinary (unweighted) Hermitian product on $\mathbf{I}_{y_0}^0$, we thus obtain (11.121).

From (11.120), (11.121), we find that for any $k, k', k'', k''' \in \mathbf{N}$

$$|||\Delta^k A_{u,T,Z_0/T}^{k'} \Delta^{k''} \exp(-L^{3,Z_0/T}) \Delta^{k'''}|||_{u,T,Z_0}^{0,0} \leqslant C''. \tag{11.125}$$

Let $J_{y_0}^0$ be the set of square integrable sections of $(\Lambda(T_{\mathbf{R}}^* X) \hat{\otimes} \xi)_{y_0}$ over $\{Z \in (T_{\mathbf{R}}^* X)_{y_0};\ |Z| \leqslant 3/2\}$. We equip $J_{y_0}^0$ with the Hermitian product

$$s, s' \in J_{y_0}^0 \to \langle s, s' \rangle = \int_{|Z| \leqslant 3/2} \langle s, s' \rangle \, dv_{TX}(Z). \tag{11.126}$$

Let $|\ |$ denote the associated norm $J_{y_0}^0$. If $A \in \mathscr{L}(J_{y_0}^0)$, let $\|A\|_\infty$ be the norm of A with respect to the norm $|\ |$.

Clearly $J^0_{y_0}$ embeds in $\mathbf{I}^0_{y_0}$. Moreover if $s \in J^0_{y_0}$, from Definition 11.23, we deduce that

$$(11.127)\qquad \begin{aligned} &|s| \leqslant |s|_{u, T, Z_0, 0},\\ &|s|_{u, T, Z_0, 0} \leqslant C(1+|Z_0|)^{2l'}|s|. \end{aligned}$$

We now consider the operators $\Delta^k A^{k'}_{u, T, Z_0/T} \Delta^{k''} \exp(-L^{3, Z_0/T}_{u, T}) \Delta^{k'''}$ as acting on $J^0_{y_0}$. From (11.125), (11.127), we find that for any k, k', k'', $k''' \in \mathbf{N}$, then

$$(11.128)\qquad \| \Delta^k A^{k'}_{u, T, Z_0/T} \Delta^{k''} \exp(-L^{3, Z_0/T}_{u, T}) \Delta^{k'''} \|_\infty \leqslant C''(1+|Z_0|)^{2l'}.$$

By (11.128) and by Sobolev's inequalities, we deduce that for any k', k'', $k''' \in \mathbf{N}$

$$(11.129)\qquad \sup_{\substack{|Z| \leqslant 5/4\\ |Z'| \leqslant 5/4}} |A^{k'}_{u, T, Z_0/T} \Delta^{k''}_Z \Delta^{k'''}_{Z'} P^{3, Z_0/T}_{u, T}(Z, Z')| \leqslant C(1+|Z_0|)^{2l'}.$$

If $x \in X$, let $d(x, Y)$ be the Riemannian distance from x to Y. Since on $Y = \bigcap_{j=1}^{r} \{x \in X, f_j(x) = 0\}$, $df_1, \ldots, df_r$ span $N^*_{\mathbf{R}}$, there exists $C' > 0$ such that for any $x \in X$

$$(11.130)\qquad \sum_1^r f_j^2(x) \geqslant C' d^2(x, Y).$$

Clearly, by (11.44), if $u \in]0, (\varepsilon/5)]$, $|Z| \leqslant 5/4$, $\rho(uZ) = 1$. From (11.129), (11.130), we deduce that if $u \in]0, (\varepsilon/5)]$, $T \in [1, (1/u)]$, $|Z_0| \leqslant (\varepsilon T/2)$

$$(11.131)\qquad \sup_{\substack{|Z| \leqslant 5/4\\ |Z'| \leqslant 5/4}} \left| \left(T d\left(\frac{Z_0}{T} + uZ, Y\right)\right)^{2k'} \Delta^{k''}_Z \Delta^{k'''}_{Z'} P^{3, Z_0/T}_{u, T}(Z, Z') \right| \leqslant C(1+|Z_0|)^{2l'}.$$

If $T \leqslant 1/u$, $|Z| \leqslant 5/4$, then $uT|Z| \leqslant 5/4$. Since (11.131) is valid also for $k' = 0$, we deduce from (11.130) that for any k', k'', $k''' \in \mathbf{N}$, if $u \in]0, 1]$, $T \in [1, (1/u)]$, $|Z_0| \leqslant (\varepsilon T/2)$

$$(11.132)\qquad \sup_{\substack{|Z| \leqslant 5/4\\ |Z'| \leqslant 5/4}} \left| \left(T d\left(\frac{Z_0}{T}, Y\right)\right)^{2k'} \Delta^{k''}_Z \Delta^{k'''}_{Z'} P^{3, Z_0/T}_{u, T}(Z, Z') \right| \leqslant C(1+|Z_0|)^{2l'}.$$

Now by definition $d((Z_0/T), Y) = |Z_0|/T$. So (11.132) can be written in the form

$$(11.133)\qquad \sup_{\substack{|Z| \leqslant 5/4\\ |Z'| \leqslant 5/4}} \left| |Z_0|^{2k'} \Delta^{k''}_Z \Delta^{k'''}_{Z'} P^{3, Z_0/T}_{u, T}(Z, Z') \right| \leqslant C(1+|Z_0|)^{2l'}.$$

Using again Sobolev's inequalities, we deduce (11.114) from (11.133). Our Theorem is proved. □

o) Estimates on $(\lambda - \mathbf{L}_{u,\mathrm{T}}^{3,\mathrm{Z}_0/\mathrm{T}})^{-1} - (\lambda - \mathbf{L}_{0,\mathrm{T}}^{3,\mathrm{Z}_0/\mathrm{T}})$

Let $s \in \mathbf{I}_{(p,q),y_0}$ with compact support. Then as $u \to 0$, $|s|_{u,\mathrm{T},\mathrm{Z}_0,0}$ has a limit $|s|_{0,\mathrm{T},\mathrm{Z}_0,0}$ such that

$$|s|_{0,\mathrm{T},\mathrm{Z}_0,0}^2 = \int_{(\mathrm{T}_{\mathbf{R}}\mathrm{X})_{y_0}} |s|^2 (1+|\mathrm{Z}|+|\mathrm{Z}_0|)^{2(2l'-p)} \left(1+\frac{|\mathrm{Z}|}{\mathrm{T}}\right)^{2(2n-q)} dv_{\mathrm{TX}}(\mathrm{Z}). \tag{11.134}$$

Let $\mathbf{I}'^{0}_{(p,q),y_0}$ be the Hilbert space which is the closure of the set of the $s \in \mathbf{I}_{(p,q),y_0}$ with compact support with respect to the norm $|\ \ |_{0,\mathrm{T},\mathrm{Z}_0,0}$, and let $\mathbf{I}'^{0}_{y_0}$ be the orthogonal sum of the $\mathbf{I}'^{0}_{(p,q),y_0}$'s. Note that in general, $\mathbf{I}'^{0}_{y_0}$ is strictly included in $\mathbf{I}^{0}_{y_0}$.

Let $e_1, \ldots, e_{2l}$ be an orthornormal base of $(\mathrm{T}_{\mathbf{R}}\mathrm{X})_{y_0}$.

Definition **11.32.** – Let $\mathbf{I}'^{1}_{y_0}$ be the set of $s \in \mathbf{I}'^{0}_{y_0}$ such that for $1 \leqslant i \leqslant 2l$, $\nabla_{e_i} s \in \mathbf{I}'^{0}_{y_0}$. If $s \in \mathbf{I}'^{1}_{y_0}$, set

$$|s|_{0,\mathrm{T},\mathrm{Z}_0,1}^2 = |s|_{0,\mathrm{T},\mathrm{Z}_0,0}^2 + \mathrm{T}^2 |s^+|_{0,\mathrm{T},\mathrm{Z}_0,0}^2 + \mathrm{T}^2 \left| \mathrm{V}^-\left(\frac{\mathrm{Z}_0}{\mathrm{T}}\right) s^- \right|_{0,\mathrm{T},\mathrm{Z}_0,0}^2 + \sum_{i=1}^{2l} |\nabla_{e_i} s|_{0,\mathrm{T},\mathrm{Z}_0,0}^2. \tag{11.135}$$

Again if $s \in \mathbf{I}_{y_0}$ has compact support, as $u \to 0$, $|s|_{u,\mathrm{T},\mathrm{Z}_0,1} \to |s|_{0,\mathrm{T},\mathrm{Z}_0,1}$.

Let $\mathbf{I}'^{-1}_{y_0}$ be the antidual of $\mathbf{I}'^{1}_{y_0}$, and let $|\ \ |_{0,\mathrm{T},\mathrm{Z}_0,-1}$ be the norm on $\mathbf{I}'^{-1}_{y_0}$ corresponding to $|\ \ |_{0,\mathrm{T},\mathrm{Z}_0,1}$. Identifying $\mathbf{I}'^{0}_{y_0}$ with its antidual by the Hermitian product associated to $|\ \ |_{0,\mathrm{T},\mathrm{Z}_0,0}$, we have the continuous embeddings with norm smaller than one $\mathbf{I}'^{1}_{y_0} \to \mathbf{I}'^{0}_{y_0} \to \mathbf{I}'^{-1}_{y_0}$.

If $\alpha = (\alpha_1, \ldots, \alpha_{2l})$ is a multiindex, set $\mathrm{Z}^\alpha = \mathrm{Z}^{\alpha_1} \ldots \mathrm{Z}^{\alpha_{2l}}$.

Definition **11.33.** – If $k = -1, 0, 1$, $k' \in \mathbf{N}$, let $\mathbf{I}'^{(k,k')}_{y_0}$ be the set of $s \in \mathbf{I}'^{k}_{y_0}$ such that if $|\alpha| \leqslant k'$, $\mathrm{Z}^\alpha s \in \mathbf{I}'^{k}_{y_0}$. If $s \in \mathbf{I}'^{(k,k')}_{y_0}$, set

$$|s|_{0,\mathrm{T},\mathrm{Z}_0,(k,k')}^2 = \sum_{|\alpha| \leqslant k'} |\mathrm{Z}^\alpha s|_{0,\mathrm{T},\mathrm{Z}_0,k}^2. \tag{11.136}$$

Clearly $\mathbf{I}'^{(k,0)}_{y_0} = \mathbf{I}'^{k}_{y_0}$ and $|\ \ |_{0,\mathrm{T},\mathrm{Z}_0,(k,0)} = |\ \ |_{0,\mathrm{T},\mathrm{Z}_0,k}$.

Proposition **11.34.** — *For* $k \in \mathbf{N}$, *there exist* $C > 0$, $k' \in \mathbf{N}$ *such that if* $T \in [1, +\infty[$, $y_0 \in Y$, $Z_0 \in N_{\mathbf{R}, y_0}$, $|Z_0| \leqslant (\varepsilon T/2)$, $\lambda \in U$, *if* $s \in \mathbf{I}_{y_0}$ *has compact support, then*

$$(11.137) \qquad |(\lambda - L_{0,T}^{3, Z_0/T})^{-1} s|_{0,T,Z_0,(1,k)} \leqslant C(1+|\lambda|)^{k'} |s|_{0,T,Z_0,(0,k)}.$$

Proof. — Using Theorem 11.27 and proceeding as in the proof of Theorem 11.30, we see that to prove Proposition 11.34, we only need to show that if Σ is an operator of the type

$$(11.138) \qquad \Sigma = [Z^{i_1}, [Z^{i_2}, \ldots [Z^{i_p}, L_{0,T}^{3, Z_0/T}]]],$$

then

$$(11.139) \qquad \|\Sigma\|_{0,T,Z_0}^{1,0} \leqslant C.$$

Now clearly

$$(11.140) \qquad \Sigma = [Z^{i_1}, [Z^{i_2}, \ldots [Z^{i_p}, M_{0,T}^{3, Z_0/T}]]].$$

Then (11.139) follows from formula (11.59) for $M_{u,T}^{3, Z_0/T}$. □

Theorem **11.35.** — *There exists* $C > 0$ *such that for any* $u \in]0, 1]$, $T \in [1, (1/u)]$, $y_0 \in Y$, $Z_0 \in N_{\mathbf{R}, y_0}$, $|Z_0| \leqslant (\varepsilon T/2)$, *if* $s \in \mathbf{I}_{y_0}$ *has compact support, then*

$$(11.141) \qquad |(L_{u,T}^{3, Z_0/T} - L_{0,T}^{3, Z_0/T}) s|_{u,T,Z_0,-1} \leqslant C u T (1+|Z_0|) |s|_{0,T,Z_0,(1,4)}.$$

Proof. — Take $s, s' \in \mathbf{I}_{y_0}$ with compact support. Using (11.75), we find that

$$(11.142) \qquad |\langle (1-\rho^2(uZ)) \Delta s, s' \rangle_{u,T,Z_0,0}| \leqslant C u |s|_{0,T,Z_0,(1,1)} |s'|_{u,T,Z_0,1},$$

$$\left| \left\langle \left(\rho^2(uZ) \left(\sum_1^{2l} \nabla^2_{\tau e_i^{Z_0/T}(uZ)} \right) - \sum_1^{2l} \nabla^2_{e_i} \right) s, s' \right\rangle_{u,T,Z_0,0} \right| \leqslant C u |s|_{0,T,Z_0,(1,1)} |s'|_{u,T,Z_0,1}.$$

Observe that

$$(11.143) \qquad |\ |_{u,T,Z_0,0} \leqslant |\ |_{0,T,Z_0,0}.$$

From Proposition 11.24 and from (11.143), we find that for $1 \leqslant i \leqslant 2l'$

$$(11.144) \qquad |(\rho(uZ)-1) e^i \wedge s|_{u,T,Z_0,0} \leqslant C u |s|_{0,T,Z_0,0}.$$

Also for $T \in [1, (1/u)]$, $|Z_0| \leqslant (\varepsilon T/2)$, we get

$$(11.145) \qquad u^2 \left(1 + (|Z| + |Z_0|) \rho \left(\frac{uZ}{2} \right) \right) \leqslant C u.$$

From (11.143), (11.145), we deduce that for $1 \leqslant i \leqslant 2l'$

$$|\rho(uZ)u^2 i_{e_i} s|_{u,T,Z_0,0} \leqslant Cu|s|_{0,T,Z_0,0}. \tag{11.146}$$

So using (11.144), (11.146), we get for $1 \leqslant i \leqslant 2l'$

$$\left|\left(\rho(uZ)\left(e^i \wedge - \frac{u^2}{2} i_{e_i}\right) - e^i \wedge\right) s\right|_{u,T,Z_0,0} \leqslant Cu|s|_{0,T,Z_0,0}. \tag{11.147}$$

Similarly, by using again Proposition 11.24, we obtain for $2l'+1 \leqslant i \leqslant 2l$

$$\left|\left(\rho(uZ)\left(\frac{e^i \wedge}{T} - \frac{u^2 T}{2} i_{e_i}\right) - \frac{e^i \wedge}{T}\right) s\right|_{u,T,Z_0,0} \leqslant Cu|s|_{0,T,Z_0,0}. \tag{11.148}$$

From (11.59), (11.142), (11.147), (11.148), we finally obtain

$$|\langle (M_{u,T}^{3,Z_0/T} - M_{0,T}^{3,Z_0/T}) s, s' \rangle_{u,T,Z_0,0}| \leqslant Cu|s|_{0,T,Z_0,(1,4)} |s'|_{u,T,Z_0,1}. \tag{11.149}$$

By (11.149), we get

$$|(M_{u,T}^{3,Z_0/T} - M_{0,T}^{3,Z_0/T}) s|_{u,T,Z_0,-1} \leqslant Cu|s|_{0,T,Z_0,(1,4)}. \tag{11.150}$$

Using (11.147), (11.148) we find that

$$\begin{aligned} &\left|\left(T\rho^2(uZ)\sum_{j=1}^{2l'}\left(e^j \wedge - \frac{u^2}{2} i_{e_j}\right)(\nabla^{\xi}_{\tau e_j^{Z_0/T}(uZ)} V)\left(\frac{Z_0}{T} + uZ\right)\right.\right. \\ &\qquad \left.\left. - T\sum_{j=1}^{2l'} e^j \wedge (\nabla^{\xi}_{e_j} V)\left(\frac{Z_0}{T}\right)\right) s\right|_{u,T,Z_0,0} \leqslant CuT|s|_{0,T,Z_0,(0,1)}, \\ &\left|\left(\rho^2(uZ)\sum_{j=2l'+1}^{2l}\left(e^j \wedge - \frac{u^2 T^2}{2} i_{e_j}\right)(\nabla^{\xi}_{\tau e_j^{Z_0/T}(uZ)} V)\left(\frac{Z_0}{T} + uZ\right)\right.\right. \\ &\qquad \left.\left. - \sum_{j=2l'+1}^{2l} e^j \wedge (\nabla^{\xi}_{e_j} V)\left(\frac{Z_0}{T}\right)\right) s\right|_{u,T,Z_0,0} \leqslant CuT|s|_{0,T,Z_0,(0,1)}. \end{aligned} \tag{11.151}$$

Also for any $Z \in (T_{\mathbf{R}} X)_{y_0}$,

$$\begin{aligned} &\left|\left(T^2 \rho^2(uZ)(V^+)^2\left(\frac{Z_0}{T} + uZ\right) - T^2(V^+)^2\left(\frac{Z_0}{T}\right)\right) s^+(Z)\right| \\ &\qquad \leqslant CuT^2|Z|\,|s^+(Z)|. \end{aligned} \tag{11.152}$$

From (11.152), we get

$$(11.153)\qquad \left|\left(T^2\rho^2(uZ)(V^+)^2\left(\frac{Z_0}{T}+uZ\right)-T^2(V^+)^2\left(\frac{Z_0}{T}\right)\right)s^+\right|_{u,T,Z_0,0} \leqslant CuT\,|s|_{0,T,Z_0,(1,1)}.$$

Recall that V^- vanishes on Y. Therefore

$$(11.154)\qquad V^-\left(\frac{Z_0}{T}+uZ\right)=O\left(\frac{|Z_0|}{T}+u|Z|\right).$$

From (11.154), we find that for $T \leqslant (1/u)$, for $Z \in (T_{\mathbf{R}}X)_{y_0}$,

$$(11.155)\qquad \left|\left(T^2\rho^2(uZ)(V^-)^2\left(\frac{Z_0}{T}+uZ\right)-T^2(V^-)^2\left(\frac{Z_0}{T}\right)\right)s^-(Z)\right| \leqslant CuT(|Z|^2+|Z|\,|Z_0|)\,|s^-(Z)|.$$

From (11.155), we get

$$(11.156)\qquad \left|\left(T^2\rho^2(uZ)(V^-)^2\left(\frac{Z_0}{T}+uZ\right)-T^2(V^-)^2\left(\frac{Z_0}{T}\right)\right)s^-\right|_{u,T,Z_0,0} \leqslant CuT(1+|Z_0|)\,|s|_{0,T,Z_0,(0,2)}.$$

From (11.150), (11.151), (11.153), (11.156), we get (11.141). □

Theorem **11.36.** — *There exist* $C>0$, $k\in\mathbf{N}$ *such that for* $u\in\,]0,1]$, $T\in[1,(1/u)]$, $y_0\in Y$, $Z_0\in N_{\mathbf{R},y_0}$, $|Z_0|\leqslant(\varepsilon T/2)$, $\lambda\in U$, *if* $s\in \mathbf{I}^0_{y_0}$, *then*

$$(11.157)\qquad |((\lambda-L^{3,Z_0/T}_{u,T})^{-1}-(\lambda-L^{3,Z_0/T}_{0,T})^{-1})s|_{0,T,Z_0,0} \leqslant CuT(1+|Z_0|)(1+|\lambda|)^k\,|s|_{0,T,Z_0,(0,4)}.$$

Proof. — We use the formula

$$(11.158)\qquad (\lambda-L^{3,Z_0/T}_{u,T})^{-1}-(\lambda-L^{3,Z_0/T}_{0,T})^{-1} =(\lambda-L^{3,Z_0/T}_{u,T})^{-1}(L^{3,Z_0/T}_{u,T}-L^{3,Z_0/T}_{0,T})(\lambda-L^{3,Z_0/T}_{0,T})^{-1}.$$

By Theorem 11.27, Proposition 11.34 and Theorem 11.35, we get (11.156). □

p) Proof of Theorem 11.13.

Let Γ be the contour (11.115). By Theorem 11.27 and by its analogue for $L^{3,Z_0/T}_{0,T}$,

(11.159)
$$\exp(-L_{u,T}^{3,Z_0/T}) - \exp(-L_{0,T}^{3,Z_0/T}) = \frac{1}{2\pi i}\int_\Gamma \exp(-\lambda)((\lambda - L_{u,T}^{3,Z_0/T})^{-1} - (\lambda - L_{0,T}^{3,Z_0/T})^{-1})\,d\lambda.$$

Let $J_{y_0}^0$ be the set of square integrable sections of $(\Lambda(T_{\mathbf{R}}^*X)\hat{\otimes}\xi)_{y_0}$ over $\{Z\in(T_{\mathbf{R}}^*X)_{y_0};\ |Z|\leqslant 3/2\}$ which we equip with the Hermitian product (11.126). If $A\in\mathscr{L}(J_{y_0}^0)$, we denote by $\|A\|_\infty$ the associated norm of A. Using Theorem 11.36 and proceeding as in (11.125)-(11.128), we find that

(11.160)
$$\|\exp(-L_{u,T}^{3,Z_0/T}) - \exp(-L_{0,T}^{3,Z_0/T})\|_\infty \leqslant C u T (1+|Z_0|)^{2l'+1}.$$

By Theorem 11.31 and (11.160), Theorem 11.31 is also true for $u=0$.

Let now φ be a smooth function defined on $(T_{\mathbf{R}}X)_{y_0}$ with values in $\mathbf{R}$, with support in $\{Z\in(T_{\mathbf{R}}X)_{y_0};\ |Z|\leqslant 1\}$ and such that $\int_{(T_{\mathbf{R}}X)_{y_0}} \varphi(Z)\,dv_{TX}(Z)=1$. Take $\beta\in]0,1]$. Using Theorem 11.31, it is clear that for $U, U'\in(\Lambda(T_{\mathbf{R}}^*X)\hat{\otimes}\xi)_{y_0}$

(11.161)
$$\Big|\langle(P_{u,T}^{3,Z_0/T} - P_{0,T}^{3,Z_0/T})(0,0)U,U'\rangle - \int_{(T_{\mathbf{R}}X)_{y_0}\times(T_{\mathbf{R}}X)_{y_0}} \langle(P_{u,T}^{3,Z_0/T} - P_{0,T}^{3,Z_0/T})(Z,Z')U,U'\rangle \frac{1}{\beta^{2l}}\varphi\left(\frac{Z}{\beta}\right)\frac{1}{\beta^{2l}}\varphi\left(\frac{Z'}{\beta}\right)dv_{TX}(Z)\,dv_{TX}(Z')\Big| \leqslant C\beta.$$

On the other hand, by (11.160), we get

(11.162)
$$\Big|\int_{(T_{\mathbf{R}}X)_{y_0}\times(T_{\mathbf{R}}X)_{y_0}} \langle(P_{u,T}^{3,Z_0/T} - P_{0,T}^{3,Z_0/T})(Z,Z')U,U'\rangle \frac{1}{\beta^{2l}}\varphi\left(\frac{Z}{\beta}\right)\frac{1}{\beta^{2l}}\varphi\left(\frac{Z'}{\beta}\right)dv_{TX}(Z)\,dv_{TX}(Z')\Big| \leqslant \frac{CuT}{\beta^{2l}}(1+|Z_0|)^{2l'+1}.$$

By choosing $\beta=(uT)^{1/(2l+1)}$, we obtain from (11.161), (11.162)

(11.163)
$$|\langle(P_{u,T}^{3,Z_0/T} - P_{0,T}^{3,Z_0/T})(0,0)U,U'\rangle| \leqslant C'(uT)^{1/(2l+1)}(1+|Z_0|)^{2l'+1}.$$

By (11.163), we find that

(11.164)
$$|(P_{u,T}^{3,Z_0/T} - P_{0,T}^{3,Z_0/T})(0,0)| \leqslant C'(uT)^{1/(2l+1)}(1+|Z_0|)^{2l'+1}.$$

Take now $m\in\mathbf{N}$. By Theorem 11.31 and by (11.164), we obtain for $u\in]0,1]$, $T\in[1,(1/u)]$,

$$\left|\left(\mathrm{P}_{u,\mathrm{T}}^{3,\mathrm{Z}_0/\mathrm{T}} - \mathrm{P}_{0,\mathrm{T}}^{3,\mathrm{Z}_0/\mathrm{T}}\right)(0,0)\right| \leqslant \frac{\mathrm{C}''}{(1+|\mathrm{Z}_0|)^{m/2}} (u\mathrm{T})^{1/(2(2l+1))} (1+|\mathrm{Z}_0|)^{l'+1/2}. \tag{11.165}$$

By taking m large enough in (11.165), we obtain (11.58) for $u \in]0, 1]$, $\mathrm{T} \in [1, (1/u)]$.

The proof of Theorem 11.13 is completed. □

XII - THE ANALYSIS OF THE KERNEL OF $\exp(-(u\mathrm{D}^{\mathrm{X}}+(\mathrm{T}/u)\mathrm{V})^2)$ FOR T POSITIVE AS u TENDS TO ZERO

The purpose of this Section is to prove Theorem 6.7, *i.e.* to show that for any $\mathrm{T}>0$

(12.1)
$$\lim_{u\to 0} \mathrm{Tr}_s\left[\mathrm{N_H}\exp\left(-\left(u\mathrm{D}^{\mathrm{X}}+\frac{\mathrm{T}}{u}\mathrm{V}\right)^2\right)\right] = \int_{\mathrm{Y}} \Phi\, \mathrm{Tr}_s\,[\mathrm{N_H}\exp(-\mathscr{B}^2_{\mathrm{T}^2})]\,\mathrm{ch}(\eta, g^{\eta}).$$

In particular, the operator $\mathscr{B}^2_{\mathrm{T}^2}$ associated with the exact sequence $0 \to \mathrm{TY} \to \mathrm{TX}|_{\mathrm{Y}} \to \mathrm{N} \to 0$ appears here for the first time and is produced by a non-trivial asymptotic analysis near Y.

Note that the range of variation of the parameters $(u, (\mathrm{T}/u))$ ($u \in]0, 1]$, T fixed) is included in the range considered in Section 11. The splitting $\xi=\xi^- \oplus \xi^+$ plays again an important role, not only, as in Section 11, for functional analytic reasons, but also for computational purposes. As in Section 11, this splitting defines a corresponding splitting of the vector space of smooth sections of $\Lambda(\mathrm{T}^{*(0,1)}\mathrm{X}) \hat{\otimes} \xi$, and the analysis is in fact done on a splitted infinite dimensional vector space.

In Section 13, a more elaborated splitting of infinite dimensional vector spaces will be considered. Still the computational ideas being essentially the same, we briefly describe them in a finite dimensional context. Let $\mathrm{E}=\mathrm{E}^- \oplus \mathrm{E}^+$ be a finite dimensional vector space, and let M be an element of End(E) which we write in matrix form as

(12.2)
$$\mathrm{M}=\begin{bmatrix} \mathrm{M}_1 & \mathrm{M}_2 \\ \mathrm{M}_3 & \mathrm{M}_4 \end{bmatrix}.$$

Then under natural assumptions on $\lambda \in \mathbf{C}$, $\lambda - M$ is invertible and moreover

$$(12.3)\quad (\lambda-M)^{-1}=\begin{bmatrix}(\lambda-M_1-M_2(\lambda-M_4)^{-1}M_3)^{-1} & (\lambda-M_1-M_2(\lambda-M_4)^{-1}M_3)^{-1}M_2(\lambda-M_4)^{-1}\\ (\lambda-M_4)^{-1}M_3(\lambda-M_1-M_2(\lambda-M_4)^{-1}M_3)^{-1} & (\lambda-M_4)^{-1}+(\lambda-M_4)^{-1}M_3(\lambda-M_1-M_2(\lambda-M_4)^{-1}M_3)^{-1}M_2(\lambda-M_4)^{-1}\end{bmatrix}.$$

Assume now that $M = M_T$ and that as $T \to +\infty$

$$(12.4)\quad \begin{aligned} &M_{1,T}=M_1+O\left(\frac{1}{\sqrt{T}}\right); \qquad M_{2,T}=\sqrt{T}\,M_2+O(1);\\ &M_{3,T}=\sqrt{T}\,M_3+O(1); \qquad M_{4,T}=TM_4+O\left(\frac{1}{\sqrt{T}}\right).\end{aligned}$$

From (12.3), (12.4), we deduce, under natural assumptions, that as $T \to +\infty$

$$(12.5)\quad (\lambda-M_T)^{-1}\to\begin{bmatrix}(\lambda-M_1+M_2M_4^{-1}M_3)^{-1} & 0\\ 0 & 0\end{bmatrix}.$$

By using a contour integral formula, we can prove, under certain circumstances, that if P^{E^-} is the projection operator $E \to E^-$, then as $T \to +\infty$

$$(12.6)\quad \exp(-M_T)\to P^{E^-}\exp(-(M_1-M_2M_4^{-1}M_3))P^{E^-}.$$

Formula (12.6) was used in an infinite dimensional context by Berline-Vergne [BeV] in their proof of the local families index Theorem of Bismut [B1].

In this Section and also in Section 13, infinite-dimensional and still local versions of (12.6) will appear, the difficulty being of course to make sense of the various asymptotic formulas in (12.4). Understanding the algebra underlying (12.5) is a useful guide to the computations in this Section and in Section 13.

This Section is organized as follows. In a), we introduce our assumptions and notation. In b), we show that the problem under consideration in Theorem 6.7 can be localized near Y. In c), we construct a coordinate system near $y_0 \in Y$ and also a trivialization of $\Lambda(T^{*(0,1)}X)\hat{\otimes}\xi$. In d), we replace the manifold X by $(T_{\mathbf{R}}X)_{y_0}$. In e), we rescale the coordinate Z in $(T_{\mathbf{R}}X)_{y_0}$ and also the Clifford variables. In f), we calculate the asymptotics as $u \to 0$ of the operator $L^{3,y_0}_{u,T/u}$ obtained from $(uD^X+(T/u)V)^2$ by such a rescaling. The building blocks of the operator $\mathscr{B}^2_{T^2}$ appear in this process. In g), we obtain uniform estimates on the rescaled heat kernels. In h), we prove the convergence as $u \to 0$ of the resolvent of the rescaled operator $L^{3,y_0}_{u,T/u}$ to the resolvent of $\mathscr{B}^{2,y_0}_{T^2}+(\nabla^{\eta})^2_{y_0}$ in the sense of distributions. In i), we prove Theorem 6.7. Finally in j), we show how to simplify the functional analytic constructions of Section 11 in the specific situation considered in this Section.

a) Assumptions and notation

Consider the exact sequence of holomorphic Hermitian vector bundles on Y

$$0 \rightarrow \mathrm{TY} \rightarrow \mathrm{TX}|_{\mathrm{Y}} \rightarrow \mathrm{N} \rightarrow 0. \tag{12.7}$$

We use the notation of Section 5 applied to this exact sequence. In particular for $u > 0$, $\mathscr{B}_u^2$ denotes the operator constructed in Definition 5.4, and $\mathrm{Q}_u^y(\mathrm{Z}, \mathrm{Z}')$ ($y \in \mathrm{Y}$, $\mathrm{Z}, \mathrm{Z}' \in (\mathrm{T}_{\mathbf{R}}\mathrm{X})_y$) denotes the heat kernel associated with the operator $\exp(-\mathscr{B}_u^{2,y})$ calculated with respect to the volume element of $(\mathrm{T}_{\mathbf{R}}\mathrm{X})_y$, $(dv_{\mathrm{TX}}/(2\pi)^{\dim \mathrm{X}})$. We otherwise use the notation of Section 11.

Clearly for $u > 0$, $\mathrm{T} > 0$

$$\mathrm{Tr}_s\left[\mathrm{N_H} \exp\left(-\left(u\mathrm{D^X} + \frac{\mathrm{T}}{u}\mathrm{V}\right)^2\right)\right] = \int_{\mathrm{X}} \mathrm{Tr}_s[\mathrm{N_H}\mathrm{P}_{u,\mathrm{T}/u}(x, x)] \frac{dv_{\mathrm{X}}(x)}{(2\pi)^{\dim \mathrm{X}}}. \tag{12.8}$$

Note that $\mathrm{T} \in]0, +\infty[$ will be fixed in the whole section.

b) The problem is localizable near Y

We now will show that the proof of Theorem 6.7 can be localized in a neighborhood of any given $y_0 \in \mathrm{Y}$.

Proposition **12.1.** – *Take* $\alpha > 0$. *There exist* $c > 0$, $\mathrm{C} > 0$ *such that for any* $x \in \mathrm{X}$, *with* $d(x, \mathrm{Y}) \geqslant \alpha$, *and any* $u \in]0, 1]$, *then*

$$|\mathrm{P}_{u,\mathrm{T}/u}(x, x)| \leqslant c \exp\left(-\frac{\mathrm{C}}{u^2}\right). \tag{12.9}$$

Proof. – Let a be be the injectivity radius of X. Take $b = \inf((a/2), (\alpha/2))$. Let $\mathrm{P}^x_{u,\mathrm{T}/u}(x', x'')$ be the heat kernel associated with the operator $\exp(-(u\mathrm{D^X} + (\mathrm{T}/u)\mathrm{V})^2)$ and the Dirichlet boundary conditions on $\partial \mathrm{B^X}(x, b)$. Then by Proposition 11.10, there exist $c > 0$, $\mathrm{C} > 0$ such that if $u \in]0, 1]$, $\mathrm{T} \in]0, 1]$

$$|(\mathrm{P}_{u,\mathrm{T}/u} - \mathrm{P}^x_{u,\mathrm{T}/u})(x, x)| \leqslant c \exp\left(-\frac{\mathrm{C}}{u^2}\right). \tag{12.10}$$

Observe that the condition $\mathrm{T} \leqslant 1$ can easily be lifted by a simple scaling argument.

We now use the notation of the proof of Proposition 11.10, with $x_0 = x$, and T replaced by T/u.

By using Lichnerowicz's formula of Proposition 11.5, and also Itô's formula, we find that if S is defined by (11.19) (with $x_0 = x$), then

$$(12.11)\qquad P^x_{u,T/u}(x,x)=p_{u^2}(x,x)\,E^{Q^u_x}\left[1_{S>1}\exp\left(\frac{-u^2}{8}\int_0^1 K(x^u_t)\,dt\right)H_1\,\tau^1_0\right].$$

For $0 \leqslant t \leqslant 1$, M_t is still given by (11.17) (with T replaced by T/u), and H_t is the solution of the differential equation (11.18). Since V is invertible on $X \setminus Y$, there exists $C > 0$ such that for any $x \in X$ with $d(x, Y) \geqslant \alpha$, and any $x' \in B(x, b)$, if $f \in \xi_{x'}$,

$$(12.12)\qquad |V(x')f|^2 \geqslant C\,|f|^2.$$

Using now equation (11.18) and Gronwall's lemma, we find that if $h \in (\Lambda(T^{*(0,1)}X) \hat{\otimes} \xi)_x$, on $(S > 1)$ (so that x^u_t remains in $B^X(x, b)$ for $t \in [0, 1]$), then

$$(12.13)\qquad |H^*_1 h|^2 \leqslant \exp\left(-\frac{CT^2}{u^2}+C'T\right).$$

Classically, for $u \in]0, 1]$

$$(12.14)\qquad p_{u^2}(x,x) \leqslant \frac{C''}{u^{2\dim X}}.$$

Also recall that the linear map τ^1_0 is unitary. From (12.11)-(12.14), we obtain

$$(12.15)\qquad |P^x_{u,T}(x,x)| \leqslant \frac{C'''}{u^{2\dim X}}\exp\left(-\frac{CT^2}{u^2}+C'T\right).$$

Since T is positive, from (12.10), (12.15), we get (12.9). □

We now fix $\varepsilon > 0$ as in Section 11 e). The tubular neighborhood $\mathcal{U}_\varepsilon$ of Y in X was defined in Section 8 e). The main result of this Section is as follows.

Theorem **12.2.** — *For any* $T \in]0, +\infty[$

$$(12.16)\qquad \lim_{u\to 0}\int_{\mathcal{U}_{\varepsilon/8}} \mathrm{Tr}_s[N_H P_{u,T/u}(x,x)]\frac{dv_X(x)}{(2\pi)^{\dim X}} = \int_Y \Phi\,\mathrm{Tr}_s[N_H\exp(-\mathcal{B}^2_{T^2})]\,\mathrm{ch}(\eta, g^\eta).$$

Proof. — The whole section is devoted to the proof of Theorem 12.2. □

Remark **12.3.** — Clearly

$$(12.17)\qquad \mathrm{Tr}_s\left[N_H\exp\left(-\left(uD^X+\frac{T}{u}V\right)^2\right)\right]=\int_X \mathrm{Tr}_s[N_H P_{u,T/u}(x,x)]\frac{dv_X(x)}{(2\pi)^{\dim X}}.$$

Theorem 6.7 is then a trivial consequence of Proposition 12.1, Theorem 12.2 and of (12.17).

As explained in Section 5c), for any $T > 0$, $y_0 \in Y$, $Z_0 \in N_{\mathbf{R}, y_0}$, $\mathrm{Tr}_s[Q_{T^2}^{y_0}(Z_0, Z_0)]$ lies in $\Lambda(T_{\mathbf{R}}^* Y)_{y_0}$. Therefore $\mathrm{Tr}_s[Q_{T^2}^{y_0}(Z_0, Z_0)]\,\mathrm{ch}(\eta, g^\eta)_{y_0}$ also lies in $\Lambda(T_{\mathbf{R}}^* Y)_{y_0}$.

The main technical result of this Section is as follows.

Theorem **12.4.** — *For any* $T > 0$, $y_0 \in Y$, $Z_0 \in N_{\mathbf{R}, y_0}$, *then*

$$\lim_{u \to 0} \frac{1}{(2\pi)^{\dim X}} u^{2 \dim N} \mathrm{Tr}_s[N_H P_{u, T/u}((y_0, u Z_0), (y_0, u Z_0))] = \left[\frac{1}{(2\pi)^{\dim N}} \Phi \mathrm{Tr}_s[Q_{T^2}^{y_0}(Z_0, Z_0)]\,\mathrm{ch}(\eta, g^\eta)_{y_0} \right]^{\max}. \tag{12.18}$$

For any $T > 0$, $p \in \mathbf{N}$, *there exists* $C > 0$ *such that for any* $u \in]0, 1]$, $y_0 \in Y$, $Z_0 \in N_{\mathbf{R}, y_0}$, $|Z_0| \leqslant \varepsilon/8u$, *then*

$$u^{2 \dim N} |\mathrm{Tr}_s[N_H P_{u, T/u}((y_0, u Z_0), (y_0, u Z_0))]| \leqslant C(1 + |Z_0|)^{-p}. \tag{12.19}$$

Proof. — The proof of Theorem 12.4 is delayed to Sections 12c) − 12i). □

Remark **12.5.** — Using (8.21), we find that

$$\int_{\mathscr{U}_{\varepsilon/8}} \mathrm{Tr}_s[N_H P_{u, T/u}(x, x)] \frac{dv_X(x)}{(2\pi)^{\dim X}} = \left(\frac{1}{2\pi}\right)^{\dim X} \int_Y \left\{ \int_{|Z_0| \leqslant \varepsilon/8u} u^{2 \dim N} \mathrm{Tr}_s[N_H P_{u, T/u}((y_0, u Z_0), (y_0, u Z_0))] k(y_0, u Z_0)\, dv_N(Z_0) \right\} dv_Y(y_0). \tag{12.20}$$

Using Theorem 12.4 and dominated convergence, we find that for any $T > 0$, as $u \to 0$

$$\int_{\mathscr{U}_{\varepsilon/8}} \mathrm{Tr}_s[N_H P_{u, T/u}(x, x)] \frac{dv_X(x)}{(2\pi)^{\dim X}} \to \int_Y \left\{ \int_N \Phi \mathrm{Tr}_s[N_H Q_{T^2}^{y_0}(Z_0, Z_0)] \frac{dv_N(Z_0)}{(2\pi)^{\dim N}} \right\} \mathrm{ch}(\eta, g^\eta)_{y_0}. \tag{12.21}$$

Using Definition 5.8, we can rewrite (12.21) in the form

$$\int_{\mathscr{U}_{\varepsilon/8}} \mathrm{Tr}_s[N_H P_{u, T/u}(x, x)] \frac{dv_X(x)}{(2\pi)^{\dim X}} \to \int_Y \Phi \mathrm{Tr}_s[N_H \exp(-\mathscr{B}_{T^2}^2)]\,\mathrm{ch}(\eta, g^\eta), \tag{12.22}$$

which is exactly Theorem 12.2.

We now concentrate on the proof of Theorem 12.4.

c) A local coordinate system near $y_0 \in Y$ and a trivialization of $\Lambda(T^{*(0,1)}X) \hat{\otimes} \xi$.

We here use the notation of Section 11 f). Take $y_0 \in Y$. If $Z \in (T_{\mathbf{R}}X)_{y_0}$, $t \in \mathbf{R} \to x_t = \exp^X_{y_0}(tZ) \in X$ still denotes the geodesic in X such that $x_0 = y_0$, $dx/dt|_{t=0} = Z$. If $|Z| < \varepsilon$, we identify $Z \in (T_{\mathbf{R}}X)_{y_0}$ with $\exp^X_{y_0}(Z) \in X$. The ball $B^{TX}(0, \varepsilon)$ in $(T_{\mathbf{R}}X)_{y_0}$ is identified with the ball $B^X(y_0, \varepsilon)$ in X. The function $k'(Z)$ on $B^{TX}(0, \varepsilon)$ is still defined by equation (11.35). Recall that $k'(0) = 1$.

If $|Z| < \varepsilon$, we identify TX_Z, $\Lambda(T^{*(0,1)}X)_Z$ with TX_{y_0}, $\Lambda(T^{*(0,1)}X)_{y_0}$ (resp. ξ_Z with ξ_{y_0}) by parallel transport with respect to the connection ∇^{TX} (resp. $\tilde{\nabla}^{\xi}$) along the curve $t \in [0, 1] \to tZ$.

With respect to Section 11 f), the main difference is that we do not need any more the intermediary $Z_0 \in N_{\mathbf{R}, y_0}$, which is now identically zero.

For $|Z| < \varepsilon/2$, let Γ^{TX}_Z, Γ^{ξ}_Z be the connection forms of the connections ∇^{TX}, ∇^{ξ} in the considered trivializations of TX, ξ. Using (11.37), (11.38), we get

$$\Gamma^{\xi}_0 = B_{y_0},$$
$$(12.23) \qquad \Gamma^{TX}_Z = \frac{1}{2}(\nabla^{TX})^2_{y_0}(Z, .) + O(|Z|^2).$$

If $|Z| \leqslant \varepsilon$, $U \in (T_{\mathbf{R}}X)_Z$, ∇_U denotes the standard differentiation operator in the direction U acting on smooth sections of $(\Lambda(T^{*(0,1)}X) \hat{\otimes} \xi)_{y_0}$ over $(T_{\mathbf{R}}X)_{y_0}$.

d) Replacing the manifold X by $(T_{\mathbf{R}}X)_{y_0}$

We still define the vector space $\mathbf{H}_{y_0}$ as in Definition 11.17. Also the function $\rho(Z)$ is given by formula (11.43).

Let Δ be the ordinary flat Laplacian on $(T_{\mathbf{R}}X)_{y_0}$.

As in Section 11, both operators D^X and V now act on smooth sections of $(\Lambda(T^{*(0,1)}X) \hat{\otimes} \xi)_{y_0}$ over $\{Z \in (T_{\mathbf{R}}X)_{y_0}; |Z| < \varepsilon\}$.

Definition **12.6.** — For $u > 0$, $T > 0$, $y_0 \in Y$, let $L^{1,y_0}_{u,T/u}$ be the operator acting on $\mathbf{H}_{y_0}$

$$\begin{aligned} L^{1,y_0}_{u,T/u} &= -u^2(1-\rho^2(Z))\frac{\Delta}{2} + \rho^2(Z)\left(u D^X + \frac{T}{u} V(Z)\right)^2 \\ &\quad + \frac{T^2}{u^2}(1-\rho^2(Z)) P^{\xi^+}_{y_0}, \\ M^{1,y_0}_u &= -u^2(1-\rho^2(Z))\frac{\Delta}{2} + \rho^2(Z) u^2 (D^X)^2. \end{aligned} \tag{12.24}$$

Let $P^{1,y_0}_{u,T/u}(Z, Z')$ $(Z, Z' \in (T_{\mathbf{R}}X)_{y_0})$ be the smooth kernel associated with the operator $\exp(-L^{1,y_0}_{u,T/u})$, which is calculated with respect to the measure $dv_{TX}(Z')/(2\pi)^{\dim X}$.

Using the notation of Definition 11.18, we find that $L^{1,y_0}_{u,T/u} = L^{1,0}_{u,T/u}$.

The same arguments as in the proof of Proposition 11.10 show that given $T > 0$, there exist $c > 0$, $C > 0$ such that for $u \in]0, 1]$, $y_0 \in Y$, $Z_0 \in N_{\mathbf{R},y_0}$, $|Z_0| \leqslant \varepsilon/8$, then

$$|P_{u,T/u}(Z_0, Z_0)\, k'(Z_0) - P^1_{u,T/u}(Z_0, Z_0)| \leqslant c \exp\left(\frac{-C}{u^2}\right). \tag{12.25}$$

From (12.25) it is clear that to prove Theorem 12.4, we only need to show that for any $T > 0$, $Z_0 \in N_{\mathbf{R},y_0}$

$$\begin{aligned} &\lim_{u \to 0} \left(\frac{1}{2\pi}\right)^{\dim X} u^{2\dim N}\, \mathrm{Tr}_s[N_H P^{1,y_0}_{u,T/u}(u Z_0, u Z_0)] \\ &= \left(\frac{1}{2\pi}\right)^{\dim N} [\Phi\, \mathrm{Tr}_s[N_H Q^{y_0}_{T^2}(Z_0, Z_0)]\, \mathrm{ch}(\eta, g^\eta)_{y_0}]^{\max}, \end{aligned} \tag{12.26}$$

and that given $T > 0$, $p \in \mathbf{N}$, there exists $C > 0$ such that for $u \in]0, 1]$, $y_0 \in Y$, $Z_0 \in N_{\mathbf{R},y_0}$, $|Z_0| \leqslant \varepsilon/8u$, then

$$u^{2\dim N} |\mathrm{Tr}_s[N_H P^{1,y_0}_{u,T/u}(u Z_0, u Z_0)]| \leqslant C(1 + |Z_0|)^{-p}. \tag{12.27}$$

e) Rescaling of the variable Z and of the horizontal Clifford variables

For $u > 0$, let F_u be the linear map acting on $\mathbf{H}_{y_0}$ defined in (11.49). For $u > 0$, $T > 0$, set

$$\begin{aligned} L^{2,y_0}_{u,T/u} &= F_u^{-1} L^{1,y_0}_{u,T/u} F_u, \\ M^{2,y_0}_u &= F_u^{-1} M^{1,y_0}_u F_u. \end{aligned} \tag{12.28}$$

As in (11.51), we find that

$$L^{2,y_0}_{u,T/u},\ M^{2,y_0}_u \in (c(T_{\mathbf{R}}X)\,\hat{\otimes}\,\xi)_{y_0} \otimes \mathrm{Op}.$$

Let $e_1, \ldots, e_{2l'}$ be an orthonormal oriented base of $(T_{\mathbf{R}}Y)_{y_0}$, let $e_{2l'+1}, \ldots, e_{2l}$ be an orthonormal oriented base of $(N_{\mathbf{R}})_{y_0}$. Let $e^1, \ldots, e^{2l'}$ and $e^{2l'+1}, \ldots, e^{2l}$ be the corresponding dual bases of $(T^*_{\mathbf{R}}Y)_{y_0}$ and $(N^*_{\mathbf{R}})_{y_0}$.

Definition **12.7.** — Let $\mathbf{K}_{y_0}$, $\mathbf{K}^{\pm}_{y_0}$ be the sets of smooth sections of $(\Lambda(T^*_{\mathbf{R}}Y)\,\hat{\otimes}\,\Lambda(\bar{N}^*)\,\hat{\otimes}\,\xi)_{y_0}$, $(\Lambda(T^*_{\mathbf{R}}Y)\,\hat{\otimes}\,\Lambda(\bar{N}^*)\,\hat{\otimes}\,\xi^{\pm})_{y_0}$ over $(T_{\mathbf{R}}X)_{y_0}$.

Then $\mathbf{K}_{y_0} = \mathbf{K}^+_{y_0} \oplus \mathbf{K}^-_{y_0}$. For $1 \leqslant i \leqslant 2l'$, the operators $e^i \wedge$, i_{e_i} act on $\Lambda(T^*_{\mathbf{R}}Y)_{y_0}$ as odd operators. Therefore they also act as odd operators on $\mathbf{K}_{y_0}$, $\mathbf{K}^{\pm}_{y_0}$.

Similarly, by Section 5 a), $\Lambda(\bar{N}^*)$ is a $c(N_{\mathbf{R}})$-Clifford module. Therefore for $2l'+1 \leqslant i \leqslant 2l$, the operators $c(e_i)$ act as odd operators on $\mathbf{K}_{y_0}$, $\mathbf{K}^{\pm}_{y_0}$.

Definition **12.8.** — For $u > 0$, $1 \leqslant i \leqslant 2l'$, set

$$c_u(e_i) = \sqrt{2}\,\frac{e^i \wedge}{u} - \frac{u}{\sqrt{2}}\, i_{e_i}. \tag{12.29}$$

For $u > 0$, $T > 0$, let $L^{3,y_0}_{u,T/u}$, $M^{3,y_0}_u \in \mathrm{End}(\mathbf{K}_{y_0})$ be the operators obtained from $L^{1,y_0}_{u,T/u}$, M^{2,y_0}_u by replacing the Clifford variables $c(e_i)$ by $c_u(e_i)$ for $1 \leqslant i \leqslant 2l'$ while leaving unchanged the operators $c(e_j)\,(2l'+1 \leqslant j \leqslant 2l)$.

Let $P^{3,y_0}_{u,T/u}(Z, Z')$ $(Z, Z' \in (T_{\mathbf{R}}X)_{y_0})$ be the smooth kernel associated with the operator $\exp(-L^{3,y_0}_{u,T/u})$, which is calculated with respect to the measure $dv_{TX}(Z')/(2\pi)^{\dim X}$. Then $P^{3,y_0}_{u,T/u}(Z, Z')$ can be expanded in the form

$$P^{3,y_0}_{u,T/u}(Z, Z') = \sum_{\substack{1 \leqslant i_1 < \ldots < i_p \leqslant 2l' \\ 1 \leqslant j_1 < \ldots < j_q \leqslant 2l'}} e^{i_1} \wedge \ldots \wedge e^{i_p} \wedge i_{e_{j_1}} \ldots i_{e_{j_q}} \tag{12.30}$$
$$\hat{\otimes}\, Q^{j_1 \ldots j_q}_{i_1 \ldots i_p}(Z, Z'),\quad Q^{j_1 \ldots j_q}_{i_1 \ldots i_p}(Z, Z') \in \mathrm{End}(\Lambda(\bar{N}^*)_{y_0}\,\hat{\otimes}\,\xi_{y_0}).$$

Set

$$[P^{3,y_0}_{u,T/u}(Z, Z')]^{\max} = Q_{1 \ldots 2l'}(Z, Z') \in \mathrm{End}(\Lambda(\bar{N}^*)\,\hat{\otimes}\,\xi)_{y_0}. \tag{12.31}$$

Equivalently, $[P^{1,y_0}_{u,T/u}(Z, Z')]^{\max}$ is the operator which factors $e^1 \wedge \ldots \wedge e^{2l'}$ in (12.30).

Proposition **12.9.** — *For any $u > 0$, $T > 0$, $y_0 \in Y$, $Z_0 \in N_{\mathbf{R},y_0}$, the following identity holds*

$$u^{2\dim N}\,\mathrm{Tr}_s[N_H P^{1,y_0}_{u,T/u}(uZ_0, uZ_0)] = (-i)^{\dim Y}\,\mathrm{Tr}_s[N_H [P^{3,y_0}_{u,T/u}(Z_0, Z_0)]^{\max}]. \tag{12.32}$$

Proof. – Clearly

$$\text{(12.33)} \qquad P^{1,y_0}_{u,T/u}(u Z_0, u Z_0) = u^{-2\dim X} P^{2,y_0}_{u,T/u}(Z_0, Z_0).$$

Then (12.32) is a trivial consequence of (12.33) and of Proposition 11.2. □

f) The asymptotics of the operator $L^{3,y_0}_{u,T/u}$ as $u \to 0$.

If $\beta \in \Lambda(T^*_{\mathbf{R}} Y)_{y_0}$, β acts on $\Lambda(T^*_{\mathbf{R}} Y)_{y_0}$ by the map

$$\alpha \in \Lambda(T^*_{\mathbf{R}} Y)_{y_0} \to \beta \wedge \alpha \in \Lambda(T^*_{\mathbf{R}} Y)_{y_0}.$$

Clearly $i^*(\nabla^{TX})^2$, $i^*(\nabla^{\xi})^2$, which are the restrictions to Y of the curvatures $(\nabla^{TX})^2$, $(\nabla^{\xi})^2$ of ∇^{TX}, ∇^{ξ}, lie in $\Lambda^2(T^*_{\mathbf{R}} Y) \hat{\otimes} \operatorname{End}(TX|_Y)$, $\Lambda^2(T^*_{\mathbf{R}} Y) \hat{\otimes} \operatorname{End}(\xi|_Y)$. Therefore $i^*(\nabla^{TX})^2_{y_0}$, $i^*(\nabla^{\xi})^2_{y_0}$, $i^* 1/2 \operatorname{Tr}[(\nabla^{TX})^2_{y_0}]$ act on $\mathbf{K}_{y_0}$.

For $1 \leqslant i \leqslant 2l$, set $\tau e_i = \tau e_i^{Z_0}$ (here $Z_0 = 0$). Using (12.24), we get

$$\text{(12.34)} \qquad L^{3,y_0}_{u,T/u} = M^{3,y_0}_u + \rho^2(uZ)\left\{\frac{T}{u}\sum_1^{2l'}\left(e^i \wedge -\frac{u^2}{2} i_{e_i}\right)(\nabla^{\xi}_{\tau e_i} V)(uZ)\right.$$
$$\left. + T\sum_{2l'+1}^{2l} \frac{c(e_i)}{\sqrt{2}}(\nabla^{\xi}_{\tau e_i} V)(uZ) + \frac{T^2}{u^2} V^2(uZ)\right\} + \frac{T^2}{u^2}(1-\rho^2(uZ)) P^{\xi^+}_{y_0}.$$

Observe that in (12.34), the divergent factor T/u has been forced into $L^{3,y_0}_{u,T/u}$ by the Clifford rescaling. It is exactly at this stage that the Todd form $\operatorname{Td}(TX, g^{TX})$ and the Chern character form $\operatorname{ch}(\xi, h^{\xi})$ will in fact interact with each other.

We will now describe the asymptotics as $u \to 0$ of the various terms in (12.34).

We first write the asymptotics as $u \to 0$ of M^{3,y_0}_u. In the sequel, all the operators $e^i \wedge$, $i_{e_i} (1 \leqslant i \leqslant 2l')$ will be explicitly written. Expressions like $O(|Z|)$, $O(|Z|^2)$... will never contain such terms in implicit form.

Using Proposition 11.4, (12.23) and proceeding as in (11.59), we find easily that

$$\text{(12.35)} \qquad M^{3,y_0}_u = -(1-\rho^2(uZ))\frac{\Delta}{2} - \rho^2(uZ)\left\{\frac{1}{2}\sum_1^{2l}\left(\nabla_{\tau e_i(uZ)}\right.\right.$$
$$+\frac{1}{4}\sum_{1 \leqslant j, j' \leqslant 2l'}\left\langle ((\nabla^{TX})^2_{y_0}(Z, e_i) + \frac{1}{u} O(|uZ|^2)) e_j, e_{j'}\right\rangle$$
$$\left(e^j \wedge -\frac{u^2}{2} i_{e_j}\right)\left(e^{j'} \wedge -\frac{u^2}{2} i_{e_{j'}}\right) + \frac{1}{4}\sum_{\substack{1 \leqslant j \leqslant 2l' \\ 2l'+1 \leqslant j' \leqslant 2l}} \langle O(u|Z|) e_j, e_{j'}\rangle$$

$$\left(\sqrt{2}\, e^j \wedge -\frac{u^2}{\sqrt{2}} i_{e_j}\right) c(e_{j'}) + u\,O(1)\Big)^2 + u^2\,O(1)$$

$$+\frac{1}{2}\sum_{1\leqslant j, j'\leqslant 2l'}\left((\nabla^{\xi})^2_{y_0} + \frac{1}{2}\operatorname{Tr}[(\nabla^{TX})^2_{y_0}] + O(|uZ|)\right)$$

$$(e_j, e_{j'})\left(e^j \wedge -\frac{u^2}{2} i_{e_j}\right)\left(e^{j'} \wedge -\frac{u^2}{2} i_{e_{j'}}\right) + \frac{u}{2}\sum_{\substack{1\leqslant j\leqslant 2l'\\ 2l'+1\leqslant j'\leqslant 2l}} O(1)(e_j, e_{j'})$$

$$\left(\sqrt{2}\, e^j \wedge -\frac{u^2}{\sqrt{2}} i_{e_j}\right) c(e_{j'}) + u\nabla_{O(u|Z|)}\Big\}.$$

We will write that as $u \to 0$, M_u^{3, y_0} converges to the differential operator M_0^{3, y_0} if the smooth coefficients of M_u^{3, y_0} converge to the corresponding coefficients of M_0^{3, y_0} together with their derivatives, uniformly over compact sets of $(T_{\mathbf{R}} X)_{y_0}$.

From (12.35), we deduce the following result.

Theorem **12.10.** – *Let* M_0^{3, y_0} *be the operator*

$$M_0^{3, y_0} = -\frac{1}{2}\sum_{1}^{2l}\left(\nabla_{e_i} + \frac{1}{2}\langle i^*(\nabla^{TX})^2_{y_0} Z, e_i\rangle\right)^2 + i^*\left((\nabla^{\xi})^2_{y_0} + \frac{1}{2}\operatorname{Tr}[(\nabla^{TX})^2_{y_0}]\right). \tag{12.36}$$

Then as $u \to 0$, $M_u^{3, y_0} \to M_0^{3, y_0}$.

Proof. – From (12.35), we find that as $u \to 0$

$$M_u^{3, y_0} \to -\frac{1}{2}\sum_{1}^{2l}\left(\nabla_{e_i} + \frac{1}{4}\sum_{1\leqslant j, j'\leqslant 2l'}\langle(\nabla^{TX})^2_{y_0}(Z, e_i)e_j, e_{j'}\rangle e^j \wedge e^{j'} \wedge\right)^2 + \frac{1}{2}\sum_{1\leqslant j, j'\leqslant 2l'}\left((\nabla^{\xi})^2_{y_0} + \frac{1}{2}\operatorname{Tr}[(\nabla^{TX})^2_{y_0}]\right)(e_j, e_{j'})\, e^j \wedge e^{j'} \wedge. \tag{12.37}$$

Since the metric g^{TX} is Kähler, we know that

$$\langle(\nabla^{TX})^2_{y_0}(Z, e_i)e_j, e_{j'}\rangle = \langle(\nabla^{TX})^2_{y_0}(e_j, e_{j'})Z, e_i\rangle. \tag{12.38}$$

Then (12.36) follows from (12.37), (12.38). □

Recall that the 1-form A on Y with values in $\operatorname{End}(TX|_Y)$ was defined in Definition 8.7. Note that A is exactly the object considered in (5.8) associated with the exact sequence $0 \to TY \to TX|_Y \to N \to 0$.

We now use the notation of Section 5 a).

Definition **12.11.** – Let $S \in \mathrm{End}(\Lambda(\bar{N}^*) \hat{\otimes} \Lambda(N^*))$ be given by

$$S = \frac{\sqrt{-1}}{2} \sum_{2l'+1}^{2l} c(e_i)\hat{c}(J e_i). \tag{12.39}$$

Clearly S extends to an even operator acting on

$$(\Lambda(T_{\mathbf{R}}^* Y) \hat{\otimes} \Lambda(\bar{N}^*) \hat{\otimes} \Lambda(N^*) \otimes \eta)_{y_0}.$$

Also recall that by (8.31), $\xi^- |_Y = \Lambda(N^*) \otimes \eta$. Therefore S acts on $(\Lambda(T_{\mathbf{R}}^* Y) \hat{\otimes} \Lambda(\bar{N}^*) \otimes \xi^-)_{y_0}$.

With the notation of Definition 5.2, if $Z \in (T_{\mathbf{R}} X)_{y_0}$, $\hat{c}(J\,AP^{TY} Z)$ is a 1-form on $(T_{\mathbf{R}} Y)_{y_0}$ taking values in odd endomorphisms of $\Lambda(N^*)_{y_0}$. Therefore $\hat{c}(J\,AP^{TY} Z)$ acts as an even linear map on

$$(\Lambda(T_{\mathbf{R}}^* Y) \hat{\otimes} \Lambda(\bar{N}^*) \hat{\otimes} \Lambda(N^*) \otimes \eta)_{y_0} = (\Lambda(T_{\mathbf{R}}^* Y) \hat{\otimes} \Lambda(\bar{N}^*))_{y_0} \hat{\otimes} \xi_{y_0}^-.$$

In the sequel, $i^* \nabla^\xi V$ denotes the 1-form on Y taking values in $\mathrm{End}^{\mathrm{odd}}(\xi|_Y)$

$$i^* \nabla^\xi V = \sum_1^{2l'} e^i \wedge \nabla_{e_i}^\xi V. \tag{12.40}$$

Also if $\tilde{D}/Dt$ denotes the covariant differentiation operator with respect to the connection $\tilde{\nabla}^\xi$ along the curve $t \in \mathbf{R} \to tZ$, set for $1 \leqslant i \leqslant 2l'$

$$(\tilde{\nabla}_Z^\xi \nabla_{\tau e_i}^\xi V)(y_0) = \frac{\tilde{D}}{Dt} [\nabla_{\tau e_i}^\xi V(tZ)]|_{t=0}. \tag{12.41}$$

In the expressions which follow, Taylor expansions of matrix valued operators will be taken in a naive sense. In particular $O(|Z|)$, $O(|Z|^2)$, $O(u^2)$ may contain operators like $e^i \wedge$ or i_{e_i} in implicit form.

We now prove one of the essential geometric results of this Section.

Theorem **12.12.** – *For any* $y_0 \in Y$, $Z \in (T_{\mathbf{R}} X)_{y_0}$, *as* $u \to 0$

$$\frac{1}{u} \sum_1^{2l'} \left(e^i \wedge - \frac{u^2}{2} i_{e_i}\right) \nabla_{\tau e_i}^\xi V(uZ) = \frac{1}{u} i^* \nabla^\xi V(y_0) + \sum_1^{2l'} e^i \wedge (\tilde{\nabla}_Z^\xi \nabla_{\tau e_i}^\xi V)(y_0) + \frac{1}{u} O(|uZ|^2 + u^2), \tag{12.42}$$

$$\sum_{2l'+1}^{2l} \frac{c(e_i)}{\sqrt{2}} (\nabla_{\tau e_i}^\xi V)(uZ) = \sum_{2l'+1}^{2l} \frac{c(e_i)}{\sqrt{2}} (\nabla_{e_i}^\xi V)(y_0) + O(|uZ|),$$

$$\frac{1}{u^2}(\mathrm{V}^+)^2(u\mathrm{Z})=\frac{1}{u^2}(\mathrm{V}^+)^2(y_0)+\frac{1}{u^2}O(|u\mathrm{Z}|),$$

$$\frac{1}{u^2}(\mathrm{V}^-)^2(u\mathrm{Z})=\left(\tilde{\nabla}^{\xi}_{\mathrm{Z}}\mathrm{V}^-(y_0)+\frac{1}{u}O(|u\mathrm{Z}|^2)\right)^2.$$

Moreover the following identities hold

(12.43)
$$\mathrm{P}^{\xi^-}i^*\nabla^{\xi}\mathrm{V}\,\mathrm{P}^{\xi^-}=0,$$

$$\mathrm{P}^{\xi^-}\left(\sum_{i=1}^{2l'}e^i\wedge(\tilde{\nabla}^{\xi}_{\mathrm{Z}}\nabla^{\xi}_{\tau e_i}\mathrm{V})(y_0)\right)\mathrm{P}^{\xi^-}=\mathrm{P}^{\xi^-}\frac{\sqrt{-1}}{\sqrt{2}}\hat{c}(\mathrm{J\,AP^{TY}Z})\mathrm{P}^{\xi^-},$$

$$\mathrm{P}^{\xi^-}\left(\sum_{i=2l'+1}^{2l}\frac{c(e_i)}{\sqrt{2}}(\nabla^{\xi}_{e_i}\mathrm{V})(y_0)\right)\mathrm{P}^{\xi^-}=\mathrm{P}^{\xi^-}\mathrm{S}_{y_0}\mathrm{P}^{\xi^-},$$

$$(\tilde{\nabla}^{\xi}_{\mathrm{Z}}\mathrm{V}^-(y_0))^2=\frac{|\mathrm{P^N Z}|^2}{2}\mathrm{P}^{\xi^-},$$

$$i^*(\tilde{\nabla}^{\xi^-})^2=i^*(\mathrm{P}^{\xi^-}(\nabla^{\xi})^2\mathrm{P}^{\xi^-}-\mathrm{P}^{\xi^-}(\nabla^{\xi}\mathrm{V})\mathrm{P}^{\xi^+}[(\mathrm{V}^+)^2]^{-1}\mathrm{P}^{\xi^+}(\nabla^{\xi}\mathrm{V})\mathrm{P}^{\xi^-}).$$

Proof. — Since ξ_{Z} is identified with ξ_{y_0} by parallel transport along the curve $t\to t\mathrm{Z}$ with respect to the connection $\tilde{\nabla}^{\xi}$. and since $\mathrm{V}^-(y_0)=0$, (12.42) follows by Taylor expansion.

As we saw after (11.64), if $\mathrm{U}\in(\mathrm{T_R Y})_{y_0}$, $\nabla^{\xi}_{\mathrm{U}}\mathrm{V}(y_0)$ maps $\xi^-_{y_0}$ into $\xi^+_{y_0}$. The first identity in (12.43) follows.

Recall that $\mathrm{B}=\nabla^{\xi}-\tilde{\nabla}^{\xi}$. Also B takes values in endomorphisms of ξ which exchange ξ^+ and ξ^-. If $\mathrm{Z}\in\mathrm{T_R X}|_{\mathscr{U}_\varepsilon}$, we then find that

(12.44)
$$\nabla^{\xi}_{\mathrm{Z}}\mathrm{V}=\tilde{\nabla}^{\xi}_{\mathrm{Z}}\mathrm{V}+[\mathrm{B}(\mathrm{Z}),\mathrm{V}].$$

Now V preserves ξ^+ and ξ^-. Therefore

(12.45)
$$\mathrm{P}^{\xi^-}\nabla^{\xi}_{\mathrm{Z}}\mathrm{V}\,\mathrm{P}^{\xi^-}=\mathrm{P}^{\xi^-}\tilde{\nabla}^{\xi}_{\mathrm{Z}}\mathrm{V}\,\mathrm{P}^{\xi^-}.$$

Note that (12.45) is valid on $\mathscr{U}_\varepsilon$ and not only on Y. Using (12.45) and Proposition 8.13, we get the third identity in (12.43).

Take $i\in\{1,\ldots,2l'\}$. From (12.45), we find that

(12.46)
$$\mathrm{P}^{\xi^-}\nabla^{\xi}_{\tau e_i}\mathrm{V}\,\mathrm{P}^{\xi^-}=\mathrm{P}^{\xi^-}\tilde{\nabla}^{\xi}_{\tau e_i}\mathrm{V}\,\mathrm{P}^{\xi^-},$$

and so

(12.47)
$$\mathrm{P}^{\xi^-}\tilde{\nabla}^{\xi}_{\mathrm{Z}}\nabla^{\xi}_{\tau e_i}\mathrm{V}\,\mathrm{P}^{\xi^-}=\mathrm{P}^{\xi^-}\tilde{\nabla}^{\xi}_{\mathrm{Z}}\tilde{\nabla}^{\xi}_{\tau e_i}\mathrm{V}\,\mathrm{P}^{\xi^-}.$$

Moreover

$$\tilde{\nabla}^{\xi}_{Z} \tilde{\nabla}^{\xi}_{\tau e_i} V = \tilde{\nabla}^{\xi}_{\tau e_i} \tilde{\nabla}^{\xi}_{Z} V - \tilde{\nabla}^{\xi}_{[\tau e_i, Z]} V + [(\tilde{\nabla}^{\xi})^2 (Z, \tau e_i), V]. \tag{12.48}$$

Using Proposition 8.13, (12.47), (12.48), we then obtain

$$\begin{aligned} &P^{\xi^-} \tilde{\nabla}^{\xi}_{Z} \nabla^{\xi}_{\tau e_i} V(y_0) P^{\xi^-} \\ &= P^{\xi^-} \left(\tilde{\nabla}^{\xi}_{e_i} \frac{\sqrt{-1}}{\sqrt{2}} \hat{c}(J P^N Z) - \frac{\sqrt{-1}}{\sqrt{2}} \hat{c}(J P^N [\tau e_i, Z]) \right) P^{\xi^-} (y_0). \end{aligned} \tag{12.49}$$

Since we have the identification of holomorphic Hermitian vector bundles $\xi^- |_Y = \Lambda N^* \otimes \eta$, we deduce from (12.49) that

$$\begin{aligned} &P^{\xi^-} \tilde{\nabla}^{\xi}_{Z} \nabla^{\xi}_{\tau e_i} V(y_0) P^{\xi^-} \\ &= P^{\xi^-} \frac{\sqrt{-1}}{\sqrt{2}} \hat{c}(J(\nabla^N_{e_i} P^N Z - P^N [\tau e_i, Z])) P^{\xi^-} (y_0). \end{aligned} \tag{12.50}$$

Now by definition, since $e_i \in (T_{\mathbf{R}} Y)_{y_0}$

$$P^N \nabla^{TX}_{e_i} Z = \nabla^N_{e_i} P^N Z + A(e_i) P^{TY} Z. \tag{12.51}$$

Therefore

$$\nabla^N_{e_i} P^N Z - P^N [\tau e_i, Z] = -A(e_i) P^{TY} Z + P^N (\nabla^{TX}_{e_i} Z - [\tau e_i, Z]). \tag{12.52}$$

Since the connection ∇^{TX} is torsion free, we find that

$$\nabla^{TX}_{e_i} Z - [\tau e_i, Z] = \nabla^{TX}_{Z} \tau e_i. \tag{12.53}$$

From (12.50)-(12.53), we get

$$\begin{aligned} &P^{\xi^-} (\tilde{\nabla}^{\xi}_{Z} \nabla^{\xi}_{\tau e_i} V)(y_0) P^{\xi^-} \\ &= P^{\xi^-} \frac{\sqrt{-1}}{\sqrt{2}} \hat{c}(-J A(e_i) P^{TY} Z + J P^N \nabla^{TX}_{Z} \tau e_i) P^{\xi^-} (y_0). \end{aligned} \tag{12.54}$$

Now, by construction

$$(\nabla^{TX}_{Z} \tau e_i)(y_0) = 0. \tag{12.55}$$

From (12.54), (12.55), we obtain the second identity in (12.43).

By Proposition 8.13, we get

$$\tilde{\nabla}^{\xi}_{Z} V^- (y_0) = \frac{\sqrt{-1}}{\sqrt{2}} \hat{c}(J P^N Z). \tag{12.56}$$

The fourth identity in (12.43) follows from (12.56). The fifth identity in (12.43) follows from [BeV, Lemma 1.17] or [B2, Proposition 3.5]. The proof of Theorem 12.12 is completed. □

Remark **12.13.** – In view of Theorems 12.10 and 12.12, we see that the main actors which appear in the operator $\mathscr{B}^2_{T^2}$ are already on stage. Via Proposition 12.9, the proof of (12.26) will now consist in explaining why, as $u \to 0$, $\xi|_Y$ is replaced by $\xi^-|_Y = \Lambda N^* \hat{\otimes} \eta$. This is in fact an infinite dimensional version of the results of [B2].

g) Uniform estimates on $P^{3, y_0}_{u, T/u}$.

Theorem **12.14.** – *For any* $T > 0$, $m \in \mathbf{N}$, *there exists* $C > 0$ *such that if* $u \in]0, 1]$, $y_0 \in Y$, *then*

$$\sup_{\substack{Z_0 \in N_{\mathbf{R}, y_0} \\ |Z_0| \leqslant \varepsilon/8u}} (1 + |Z_0|)^m \, |P^{3, y_0}_{u, T/u}(Z_0, Z_0)| \leqslant C. \tag{12.57}$$

For any $T > 0$, $M > 0$, $m' \in \mathbf{N}$, *there exists* $C' > 0$ *such that if* $u \in]0, 1]$, $y_0 \in Y$, *then*

$$\sup_{\substack{Z, Z' \in N_{\mathbf{R}, y_0} \\ |Z|, |Z'| \leqslant M \\ |\alpha|, |\alpha'| \leqslant m'}} \left| \frac{\partial^{|\alpha| + |\alpha'|}}{\partial Z^\alpha \, \partial Z'^{\alpha'}} P^{3, y_0}_{u, T/u}(Z, Z') \right| \leqslant C'. \tag{12.58}$$

Proof. – We will use the notation of Section 11. In fact recall that $L^{1, y_0}_{u, T/u} = L^{1, 0}_{u, T/u}$. Therefore $L^{3, 0}_{u, T/u}$ can be obtained from $L^{3, y_0}_{u, T/u}$ by replacing, for $2l' + 1 \leqslant i \leqslant 2l$, the Clifford variables $c(e_i)$ by the operators $(\sqrt{2}/T) e^i \wedge -(T i_{e_i}/\sqrt{2})$, and this is harmless for our estimates. Also to simplify our notation and to make our references to Section 11 easier, we will assume that $T = 1$.

By inequality (11.125) calculated with $Z_0 = 0$, $T = 1/u$, we find that for any k, k', k'', $k''' \in \mathbf{N}$

$$|||\Delta^k A^{k'}_{u, 1/u, 0} \, \Delta^{k''} \exp(-L^{3, 0}_{u, 1/u}) \, \Delta^{k'''}|||^{0, 0}_{u, 1/u, 0} \leqslant C. \tag{12.59}$$

Let Γ_{y_0} be the lattice of elements of $(T_{\mathbf{R}} X)_{y_0}$ which have integer coordinates with respect to the base $e_1, \ldots, e_{2l}$. If $a \in \Gamma_{y_0}$, let $J^{0, a}_{y_0}$ be the set of square integrable sections of $(\Lambda(T^*_{\mathbf{R}} X) \hat{\otimes} \xi)_{y_0}$ over $\{Z \in (T_{\mathbf{R}} X)_{y_0}; |Z - a| \leqslant 3/2\}$. We equip $J^{0, a}_{y_0}$ with the obvious analogue of the Hermitian product (11.126). Let $|\ |$ denote the corresponding norm. Then $J^{0, a}_{y_0}$ embeds into $I^0_{y_0}$. Moreover, from (11.70), we deduce that if $s \in J^{0, a}_{y_0}$

$$\begin{aligned} |s| &\leqslant |s|_{u, 1/u, 0, 0} \\ |s|_{u, 1/u, 0, 0} &\leqslant C(1 + |a|)^{2l'} |s|. \end{aligned} \tag{12.60}$$

If $B \in \mathscr{L}(J_{y_0}^{0,a})$, let $\|B\|_\infty$ be the norm of B with respect to the norm $|\ |$ on $J_{y_0}^{0,a}$. From (12.59), (12.60), we find that

$$\|\Delta^k A_{u,1/u,0}^{k'} \Delta^{k''} \exp(-L_{u,1/u}^{3,0}) \Delta^{k'''}\|_\infty \leqslant C'(1+|a|)^{2l'}. \tag{12.61}$$

Using (12.61) with $k=k'=0$ and Sobolev's inequalities, we get (12.58). From (12.61), we also deduce that for any $k' \in \mathbf{N}$

$$\sup_{\substack{|Z-a| \leqslant 1 \\ |Z'-a| \leqslant 1}} |A_{u,1/u,0}^{k'}(Z) P_{u,1/u}^{3,y_0}(Z,Z')| \leqslant C(1+|a|)^{2l'}. \tag{12.62}$$

Using (11.130), (12.62) and the fact that if $|Z| \leqslant \varepsilon/8u$, $\rho(uZ)=1$, we get

$$\sup_{\substack{|Z-a| \leqslant 1,\ |Z| \leqslant \varepsilon/8u \\ |Z'-a| \leqslant 1}} \left| \left(\frac{1}{u} d(uZ, Y)\right)^{2k'} P_{u,1/u}^{3,y_0}(Z,Z') \right| \leqslant C(1+|a|)^{2l'}. \tag{12.63}$$

If $Z \in N_{\mathbf{R},y_0}$, $d(uZ,Y)/u = |Z|$. We thus deduce from (12.63) that

$$\sup_{\substack{Z \in N_{\mathbf{R},y_0} \\ |Z-a| \leqslant 1,\ |Z| \leqslant \varepsilon/8u}} |Z|^{2k'} |P_{u,1/u}^{3,y_0}(Z,Z)| \leqslant C(1+|a|)^{2l'}. \tag{12.64}$$

By taking $a \in N_{\mathbf{R},y_0} \cap \Gamma_{y_0}$ and $k' \in \mathbf{N}$ large enough in (12.64), we get (12.57). Our Theorem is proved, when $T=1$. The extension of our results to the case where T is any number in $]0, +\infty[$ is easy by a trivial scaling argument. □

h) Convergence of the resolvent in distribution sense

We still use the notation of Section 11.

If $s \in \mathbf{I}_{(p,q),y_0}$ has compact support, it is clear that if $T \in]0, +\infty[$, as $u \to 0$, the function $u \to |s|_{u,T/u,0,0}$ has a limit $|s|_{0,0}$ given by

$$|s|_{0,0}^2 = \int_{(T_{\mathbf{R}}X)_{y_0}} |s|^2 (1+|Z|)^{2(2l'-p)} dv_{TX}(Z). \tag{12.65}$$

Let $\mathbf{I}_{y_0}'^0$ denotes the Hilbert space associated with the norm $|\ |_{0,0}$. Then $\mathbf{I}_{y_0}'^0$ still splits into $\mathbf{I}_{y_0}'^0 = \mathbf{I}_{y_0}'^{+,0} \oplus \mathbf{I}_{y_0}'^{-,0}$.

Definition **12.15.** – Take $y_0 \in Y$. If $s \in \mathbf{I}_{y_0}^-$ has compact support, set

$$|s|_{0,1}^{-,2} = |s|_{0,0}^2 + \frac{T^2}{2} \|P^N Z|s|_{0,0}^2 + \sum_{i=1}^{2l} |\nabla_{e_i} s|_{0,0}^2. \tag{12.66}$$

Observe that if $Z \in (T_{\mathbf{R}} X)_{y_0}$, by Theorem 12.12, we find that as $u \to 0$

$$\left(\frac{V^-(uZ)}{u}\right)^2 \to \frac{1}{2} |P^N Z|^2 P^{\xi_{y_0}^-}. \tag{12.67}$$

From (12.67), we see that if $s \in \mathbf{I}_{y_0}^-$ has compact support, then as $u \to 0$, $|s|_{u, T/u, 0, 1} \to |s|_{0, 1}^-$.

Let $\mathbf{I}_{y_0}'^{-, 1}$ denote the Hilbert space associated with the norm $|\ |_{0, 1}^-$. Let $|\ |_{0, 0}^-$ be the restriction of the norm $|\ |_{0, 0}$ to $\mathbf{I}_{y_0}'^{-, 0}$. Then $(\mathbf{I}_{y_0}'^{-, 1}, |\ |_{0, 1}^-)$ embeds continuously as a dense subspace of $(\mathbf{I}_{y_0}'^{-, 0}, |\ |_{0, 0}^-)$ and the norm of the embedding is smaller than one. We identify $(\mathbf{I}_{y_0}'^{-, 0}, |\ |_{0, 0}^-)$ with its antidual by the Hermitian product associated with $|\ |_{0, 0}^-$. Then if $(\mathbf{I}_{y_0}'^{-, -1}, |\ |_{0, -1}^-)$ is the Hilbert space which is antidual to $(\mathbf{I}_{y_0}'^{-, 1}, |\ |_{0, 1}^-)$, we have the continuous dense embeddings with norms smaller than one

$$(\mathbf{I}_{y_0}'^{-, 1}, |\ |_{0, 1}^-) \to (\mathbf{I}_{y_0}'^{-, 0}, |\ |_{0, 0}^-) \to (\mathbf{I}_{y_0}'^{-, -1}, |\ |_{0, -1}^-). \tag{12.68}$$

Also for $u \in]0, 1]$, $(\mathbf{I}_{y_0}'^{-, 0}, |\ |_{0, 0}^-)$ embeds continuously in $(\mathbf{I}_{y_0}^0, |\ |_{u, T/u, 0, 0})$ and the norm of the embedding is smaller than a constant $C(T)$. This essentially follows from the fact that

$$\frac{u|Z|}{T} \rho\left(\frac{uZ}{2}\right) \leqslant \frac{C}{T}. \tag{12.69}$$

If $B \in \mathscr{L}(\mathbf{I}_{y_0}'^{-, -1}, \mathbf{I}_{y_0}'^{-, 1})$, let $\|B\|_0^{-1, 1}$ be the norm of B with respect to the norms $|\ |_{0, -1}^-$ and $|\ |_{0, 1}^-$.

Now in the operator $\mathscr{B}_{T^2}^{2, y_0}$, we replace, for $2l'+1 \leqslant j \leqslant 2l$, the Clifford variables $c(e_j)$ by the operators $\sqrt{2}(e^j/T) \wedge -(T/\sqrt{2}) i_{e_j}$. For $1 \leqslant j \leqslant 2l'$, the operators $e^j \wedge$ act $\mathbf{I}_{y_0}^-$. Recall that $\xi_{y_0}^- = (\Lambda N^* \hat{\otimes} \eta)_{y_0}$. We thus obtain an operator acting on $\mathbf{I}_{y_0}^-$, which we still note $\mathscr{B}_{T^2}^{2, y_0}$. Also the operator $(\nabla^\eta)_{y_0}^2$ acts on $\mathbf{I}_{y_0}^-$.

Similarly, the operator $L_{u, T/u}^{3, 0}$ is obtained from the operator $L_{u, T/u}^{3, y_0}$ by replacing, for $2l'+1 \leqslant j \leqslant 2l$, the Clifford variables $c(e_j)$ by the operators $\sqrt{2}(e^j/T) \wedge -(T/\sqrt{2}) i_{e_j}$. We will use the notation $L_{u, T/u}^{3, y_0}$ instead of $L_{u, T/u}^{3, 0}$. The operator $L_{u, T/u}^{3, y_0}$ now acts on $\mathbf{I}_{y_0}$.

For $A > 0$, $\delta > 0$, set

$$U = \{\lambda \in \mathbf{C},\ \mathrm{Re}(\lambda) \leqslant \delta \,\mathrm{Im}^2(\lambda) - A\}. \tag{12.70}$$

By Theorem 11.27, we know that if A is large enough and if δ is small enough, there exists $C > 0$ such that for any $u \in]0, 1]$, if $\lambda \in U$, the resolvent $(\lambda - L_{u, T/u}^{3, y_0})^{-1}$ exists, extends to a continuous linear map from $\mathbf{I}_{y_0}^{-1}$ into $\mathbf{I}_{y_0}^1$, and that moreover

$$\|(\lambda - L_{u, T/u}^{3, y_0})^{-1}\|_{u, T/u, 0}^{-1, 1} \leqslant C(1 + |\lambda|)^2. \tag{12.71}$$

By proceeding as in the proof of Theorem 11.27, we may as well assume that if $\lambda \in \mathrm{U}$, the resolvent $(\lambda - \mathscr{B}^{2, y_0}_{\mathrm{T}^2} - (\nabla^{\eta})^2_{y_0})^{-1}$ exists, extends to a continuous linear map from $\mathbf{I}'^{-,-1}_{y_0}$ into $\mathbf{I}'^{-,1}_{y_0}$, and moreover

$$\| (\lambda - \mathscr{B}^{2, y_0}_{\mathrm{T}^2} - (\nabla^{\eta})^2_{y_0})^{-1} \|_0^{-1,1} \leqslant \mathrm{C}(1 + |\lambda|)^2. \tag{12.72}$$

From (12.71), (12.72), it is clear that if $\lambda \in \mathrm{U}$, $(\lambda - \mathrm{L}^{3, y_0}_{u, \mathrm{T}/u})^{-1}$ and $(\lambda - \mathscr{B}^{2, y_0}_{\mathrm{T}^2} - (\nabla^{\eta})^2_{y_0})^{-1}$ define matrix valued distributions on $(\mathrm{T}_{\mathbf{R}} \mathrm{X})_{y_0} \times (\mathrm{T}_{\mathbf{R}} \mathrm{X})_{y_0}$.

The main technical result of this Section is as follows.

Theorem **12.16.** – *For any* $\mathrm{T} > 0$, $y_0 \in \mathrm{Y}$, $\lambda \in \mathrm{U}$, *as* $u \to 0$

$$(\lambda - \mathrm{L}^{3, y_0}_{u, \mathrm{T}/u})^{-1} \to \mathrm{P}^{\xi^-_{y_0}} (\lambda - \mathscr{B}^{2, y_0}_{\mathrm{T}^2} - (\nabla^{\eta})^2_{y_0})^{-1} \mathrm{P}^{\xi^-_{y_0}} \tag{12.73}$$

in the sense of distributions on $(\mathrm{T}_{\mathbf{R}} \mathrm{X})_{y_0} \times (\mathrm{T}_{\mathbf{R}} \mathrm{X})_{y_0}$.

Proof. – Set

$$\begin{aligned} \mathrm{L}_{u,1} &= \mathrm{P}^{\xi^-_{y_0}} \mathrm{L}^{3, y_0}_{u, \mathrm{T}/u} \mathrm{P}^{\xi^-_{y_0}}, \quad \mathrm{L}_{u,2} = \mathrm{P}^{\xi^-_{y_0}} \mathrm{L}^{3, y_0}_{u, \mathrm{T}/u} \mathrm{P}^{\xi^+_{y_0}}, \\ \mathrm{L}_{u,3} &= \mathrm{P}^{\xi^+_{y_0}} \mathrm{L}^{3, y_0}_{u, \mathrm{T}/u} \mathrm{P}^{\xi^-_{y_0}}, \quad \mathrm{L}_{u,4} = \mathrm{P}^{\xi^+_{y_0}} \mathrm{L}^{3, y_0}_{u, \mathrm{T}/u} \mathrm{P}^{\xi^+_{y_0}}. \end{aligned} \tag{12.74}$$

We then write the operator $\mathrm{L}^{3, y_0}_{u, \mathrm{T}/u}$ in matrix form with respect to the splitting $\mathbf{I}_{y_0} = \mathbf{I}^-_{y_0} \oplus \mathbf{I}^+_{y_0}$, so that

$$\mathrm{L}^{3, y_0}_{u, \mathrm{T}/u} = \begin{bmatrix} \mathrm{L}_{u,1} & \mathrm{L}_{u,2} \\ \mathrm{L}_{u,3} & \mathrm{L}_{u,4} \end{bmatrix}. \tag{12.75}$$

If $\mathrm{C} \in \mathscr{L}(\mathbf{I}^{+,-1}_{y_0}, \mathbf{I}^{+,1}_{y_0})$, we denote by $\|\mathrm{C}\|^{-1,1}_{u, \mathrm{T}/u, 0}$ the norm of C with respect to the norms induced by $|\ |_{u, \mathrm{T}/u, 0, -1}$, $|\ |_{u, \mathrm{T}, 0, 1}$ on $\mathbf{I}^{+,-1}_{y_0}$, $\mathbf{I}^{+,1}_{y_0}$.

Using Theorem 11.26 (where s, s' are now restricted to vary in $\mathbf{I}^+_{y_0}$ and still have compact support), and by proceeding as in the proof of Theorem 11.27, we find that if $\lambda \in \mathrm{U}$, the resolvent $(\lambda - \mathrm{L}_{u,4})^{-1}$ exists, extends to a continuous linear map from $\mathbf{I}^{+,-1}_{y_0}$ into $\mathbf{I}^{+,1}_{y_0}$ and moreover

$$\| (\lambda - \mathrm{L}_{u,4})^{-1} \|^{-1,1}_{u, \mathrm{T}/u, 0} \leqslant \mathrm{C}(1 + |\lambda|)^2. \tag{12.76}$$

If $s \in \mathbf{I}_{y_0}$, we still write $s = s^+ + s^-$, with $s^{\pm} \in \mathbf{I}^{\pm}_{y_0}$. We now fix $\lambda \in \mathrm{U}$. The constants $\mathrm{C} > 0$ which appear in the sequel may depend on $|\lambda|$ and T.

To prove our Theorem, we only need to show that if $s' \in \mathbf{I}_{y_0}$ has compact support, then as $u \to 0$

$$(\lambda - \mathrm{L}^{3, y_0}_{u, \mathrm{T}/u})^{-1} s' \to (\lambda - \mathscr{B}^{2, y_0}_{\mathrm{T}^2} - (\nabla^{\eta})^2_{y_0})^{-1} s'^- \text{ in the sense of distributions.} \tag{12.77}$$

Set

$$s=(\lambda-L^{3,y_0}_{u,T/u})^{-1}s'. \tag{12.78}$$

Since $|s'|_{u,T/u,0,-1}\leqslant |s'|_{u,T/u,0,0}$, we find that as $u\to 0$, $|s'|_{u,T/u,0,-1}$ remains bounded. From (12.71), (12.78), we deduce that $|s|_{u,T/u,0,1}$ stays bounded. Moreover

$$|s|_{u,T/u,0,1}\geqslant \frac{T}{u}|s^+|_{u,T/u,0,0}. \tag{12.79}$$

Therefore, as $u\to 0$, $|s^+|_{u,T/u,0,0}\to 0$, and so

$$s^+\to 0 \text{ in the sense of distributions.} \tag{12.80}$$

Using (12.75), (12.78), we get

$$\begin{aligned}&(\lambda-L_{u,1})s^- - L_{u,2}s^+=s'^-,\\ &-L_{u,3}s^-+(\lambda-L_{u,4})s^+=s'^+.\end{aligned} \tag{12.81}$$

From (12.81), we deduce that

$$\begin{aligned}&s^+=(\lambda-L_{u,4})^{-1}(s'^+ + L_{u,3}s^-),\\ &(\lambda-L_{u,1}-L_{u,2}(\lambda-L_{u,4})^{-1}L_{u,3})s^-=s'^- + L_{u,2}(\lambda-L_{u,4})^{-1}s'^+.\end{aligned} \tag{12.82}$$

Set

$$E_u=\lambda-L_{u,1}-L_{u,2}(\lambda-L_{u,4})^{-1}L_{u,3}. \tag{12.83}$$

From (12.71), (12.76), (12.82), we find that E_u is one to one from $I^{-,1}_{y_0}$ into $I^{-,-1}_{y_0}$, and that if $\|E_u^{-1}\|^{-1,1}_{u,T/u,0}$ denotes the norm of E_u^{-1} with respect to the norms induced by $|\ |_{u,T/u,0,-1}$ and $|\ |_{u,T/u,0,1}$ on $I^{-,-1}_{y_0}$, $I^{-,1}_{y_0}$, then

$$\|E_u^{-1}\|^{-1,1}_{u,T/u,0}\leqslant C. \tag{12.84}$$

From (12.82), (12.83), we find that

$$s^-=E_u^{-1}L_{u,2}(\lambda-L_{u,4})^{-1}s'^+ + E_u^{-1}s'^-. \tag{12.85}$$

We now study the two terms which appear in the right-hand side of (12.85).

1. *The term* $E_u^- L_{u,2}(\lambda-L_{u,4})^- s'^+$.

Using (12.84), we get

$$|E_u^{-1}L_{u,2}(\lambda-L_{u,4})^{-1}s'^+|_{u,T/u,0,0}\leqslant C|L_{u,2}(\lambda-L_{u,4})^{-1}s'^+|_{u,T/u,0,-1}. \tag{12.86}$$

Observe that $V^2(uZ)$ preserves $\xi^+_{y_0}$ and $\xi^-_{y_0}$ and that the principal symbol of $L^{3,y_0}_{u,T/u}$ is scalar, so that $L_{u,2}$ is a first order differential operator. Using the previous considerations, formulas (11.59), (11.60), (12.34) for $L^{3, y_0}_{u, T/u}$, M^{3, y_0}_u, Proposition 11.24 and proceeding as in the proof of Theorem 11.26, we find that if $s'' \in \mathbf{I}^{+, 0}_{y_0}$, then

$$(12.87) \qquad | L_{u,2} s'' |_{u, T/u, 0, -1} \leqslant \frac{C}{u} | s'' |_{u, T/u, 0, 0}.$$

From (12.76), (12.86), (12.87), we obtain

$$(12.88) \qquad \begin{aligned} &| E_u^{-1} L_{u,2} (\lambda - L_{u,4})^{-1} s'^{+} |_{u, T/u, 0, 0} \\ &\leqslant \frac{C}{u} | (\lambda - L_{u,4})^{-1} s'^{+} |_{u, T/u, 0, 0} \leqslant C | (\lambda - L_{u,4})^{-1} s'^{+} |_{u, T/u, 0, 1} \\ &\leqslant C' | s'^{+} |_{u, T/u, 0, -1} \leqslant C'' u | s'^{+} |_{u, T/u, 0, 0}. \end{aligned}$$

By (12.88), we find that as $u \to 0$

$$(12.89) \qquad E_u^{-1} L_{u,2} (\lambda - L_{u,4})^{-1} s'^{+} \to 0 \text{ in the sense of distributions.}$$

2. *The term* $E_u^{-1} s'^{-}$.

In view of (12.80), (12.85), (12.89), we find that to prove (12.77), we only need to show that as $u \to 0$

$$(12.90) \qquad E_u^{-1} s'^{-} \to (\lambda - \mathscr{B}^{2, y_0}_{T^2} - (\nabla^{\eta})^2_{y_0})^{-1} s'^{-} \text{ in the sense of distributions.}$$

Clearly

$$(12.91) \qquad \begin{aligned} E_u^{-1} - (\lambda - \mathscr{B}^{2, y_0}_{T^2} - (\nabla^{\eta})^2_{y_0})^{-1} = E_u^{-1} (L_{u,1} + L_{u,2} (\lambda - L_{u,4})^{-1} \\ L_{u,3} - \mathscr{B}^{2, y_0}_{T^2} - (\nabla^{\eta})^2_{y_0}) (\lambda - \mathscr{B}^{2, y_0}_{T^2} - (\nabla^{\eta})^2_{y_0})^{-1}. \end{aligned}$$

Set

$$(12.92) \qquad \sigma = (\lambda - \mathscr{B}^{2, y_0}_{T^2} - (\nabla^{\eta})^2_{y_0})^{-1} s'^{-}.$$

Since s'^{-} has compact support, using the notation of Section 12h), we see that for any multiindex α, $Z^{\alpha} s'^{-} \in \mathbf{I}'^{-, 0}_{y_0}$. As in the proof of Proposition 11.34, one verifies that the commutators $[Z^{\alpha_1}, \ldots, [Z^{\alpha_p}, \mathscr{B}^{2, y_0}_{T^2} + (\nabla^{\eta})^2_{y_0}] \ldots]$ are bounded operators from $(\mathbf{I}'^{-, 1}_{y_0}, | \ |^{-}_{0,1})$ into $(\mathbf{I}'^{-, -1}_{y_0}, | \ |^{-}_{0,-1})$. By proceeding as in the proof of Theorem 11.30, we find that for any multiindex α, $Z^{\alpha} \sigma \in \mathbf{I}'^{-, 1}_{y_0}$.

In view of (12.84), (12.91), we see that to prove (12.90), we only need to show that as $u \to 0$

$$(12.93) \qquad | (L_{u,1} + L_{u,2} (\lambda - L_{u,4})^{-1} L_{u,3} - \mathscr{B}^{2, y_0}_{T^2} - (\nabla^{\eta})^2_{y_0}) \sigma |_{u, T/u, 0, -1} \to 0.$$

We identify the operators M_u^{3, y_0} and $M_{u, T/u}^{3, 0}$, *i.e.* for $2l'+1 \leqslant j \leqslant 2l$, we replace in M_u^{3, y_0} the Clifford variables $c(e_j)$ by the operators $(\sqrt{2}\, e^j/T) \wedge - (T/\sqrt{2})\, i_{e_j}$. Clearly,

$$(12.94) \qquad L_{u,1} = P^{\xi_{y_0}^-} M_u^{3, y_0} P^{\xi_{y_0}^-} + \rho^2(uZ) \left\{ \frac{T}{u} \sum_1^{2l'} \left(e^i \wedge - \frac{u^2}{2} i_{e_i} \right) \right.$$

$$P^{\xi_{y_0}^-} (\nabla^{\xi}_{\tau e_i} V)(uZ) P^{\xi_{y_0}^-} + T \sum_{2l'+1}^{2l} \left(\frac{e^i}{T} \wedge - \frac{T}{2} i_{e_i} \right)$$

$$\left. P^{\xi_{y_0}^-} (\nabla^{\xi}_{\tau e_i} V)(uZ) P^{\xi_{y_0}^-} + \frac{T^2}{u^2} (V^-)^2 (uZ) \right\}.$$

Recall that for any α, $Z^\alpha \sigma \in I_{y_0}'^{-,1}$. By proceeding as in the proof of Theorem 11.26, using (12.94), and also Theorems 12.10, 12.12 and in particular the fundamental identities (12.43), we find that as $u \to 0$

$$(12.95) \qquad \left| \left(L_{u,1} - P^{\xi_{y_0}^-} M_0^{3, y_0} P^{\xi_{y_0}^-} - \frac{T\sqrt{-1}}{\sqrt{2}} \hat{c}(J\, A P^{TY} Z) \right. \right.$$

$$\left. \left. - T S_{y_0} - T^2 \frac{|P^N Z|^2}{2} \right) \sigma \right|_{u, T/u, 0, -1} \to 0.$$

We now calculate the asymptotics of $L_{u,2} (\lambda - L_{u,4})^{-1} L_{u,3}\, \sigma$. Set

$$(12.96) \qquad L'_{u,2} = P^{\xi_{y_0}^-} \left(M_u^{3, y_0} + \rho^2(uZ) T \sum_{2l'+1}^{2l} \left(\frac{e^i}{T} \wedge - \frac{T}{2} i_{e_i} \right) (\nabla^{\xi}_{\tau e_i} V)(uZ) \right) P^{\xi_{y_0}^+},$$

$$L''_{u,2} = \rho^2(uZ) T \sum_1^{2l'} \left(e^i \wedge - \frac{u^2}{2} i_{e_i} \right) P^{\xi_{y_0}^-} (\nabla^{\xi}_{\tau e_i} V)(uZ) P^{\xi_{y_0}^+},$$

$$L'_{u,3} = P^{\xi_{y_0}^+} \left(M_u^{3, y_0} + \rho^2(uZ) T \sum_{2l'+1}^{2l} \left(\frac{e^i}{T} \wedge - \frac{T}{2} i_{e_i} \right) (\nabla^{\xi}_{\tau e_i} V)(uZ) \right) P^{\xi_{y_0}^-},$$

$$L''_{u,3} = \rho^2(uZ) T \sum_1^{2l'} \left(e^i \wedge - \frac{u^2}{2} i_{e_i} \right) P^{\xi_{y_0}^+} (\nabla^{\xi}_{\tau e_i} V)(uZ) P^{\xi_{y_0}^-}.$$

Then

$$(12.97) \qquad \begin{aligned} L_{u,2} &= L'_{u,2} + \frac{1}{u} L''_{u,2}, \\ L_{u,3} &= L'_{u,3} + \frac{1}{u} L''_{u,3}. \end{aligned}$$

Also $L'_{u,2}$, $L'_{u,3}$ are first order differential operators, and $L''_{u,2}$, $L''_{u,3}$ are operators of order zero.

Observe that if $\tau \in I^{+}_{y_0}$, $\tau' \in I^{-}_{y_0}$ have compact support, then

$$
\begin{aligned}
&|\tau|_{u,T/u,0,0} \leqslant Cu|\tau|_{u,T/u,0,1},\\
&|\tau|_{u,T/u,0,-1} \leqslant Cu|\tau|_{u,T/u,0,0},\\
&|\tau'|_{u,T/u,0,0} \leqslant |\tau'|_{u,T/u,0,1},\\
&|\tau'|_{u,T/u,0,-1} \leqslant |\tau'|_{u,T/u,0,0}.
\end{aligned}
\tag{12.98}
$$

In the inequalities which follow, the positive constants $C > 0$ may vary from line to line.

Using (12.76), (12.98) and proceeding as in the proof of Theorem 11.26, we find that

$$
\begin{aligned}
|L'_{u,2}(\lambda - L_{u,4})^{-1} L'_{u,3}\sigma|_{u,T/u,0,-1} &\leqslant C|(\lambda - L_{u,4})^{-1} L'_{u,3}\sigma|_{u,T/u,0,0}\\
&\leqslant Cu|(\lambda - L_{u,4})^{-1} L'_{u,3}\sigma|_{u,T/u,0,1}\\
&\leqslant Cu|L'_{u,3}\sigma|_{u,T/u,0,-1} \leqslant Cu|\sigma|_{u,T/u,0,0}.
\end{aligned}
\tag{12.99}
$$

Similarly using in particular (11.72) to handle the operators $u^2 i_{e_i}$ $(1 \leqslant i \leqslant 2l')$, we get

$$
\begin{aligned}
&\left|L'_{u,2}(\lambda - L_{u,4})^{-1}\frac{L''_{u,3}}{u}\sigma\right|_{u,T/u,0,-1} \leqslant C|L''_{u,3}\sigma|_{u,T/u,0,-1}\\
&\quad\leqslant Cu|L''_{u,3}\sigma|_{u,T/u,0,0} \leqslant Cu|\sigma|_{u,T/u,0,0},\\
&\left|\frac{L''_{u,2}}{u}(\lambda - L_{u,4})^{-1} L'_{u,3}\sigma\right|_{u,T/u,0,-1} \leqslant \frac{C}{u}|(\lambda - L_{u,4})^{-1} L'_{u,3}\sigma|_{u,T/u,0,0}\\
&\quad\leqslant C|(\lambda - L_{u,4})^{-1} L'_{u,3}\sigma|_{u,T/u,0,1} \leqslant C|L'_{u,3}\sigma|_{u,T/u,0,-1}\\
&\quad\leqslant Cu|L'_{u,3}\sigma|_{u,T/u,0,0} \leqslant Cu|\sigma|_{u,T/u,0,1},\\
&\left|\frac{L''_{u,2}}{u}(\lambda - L_{u,4})^{-1}\frac{(L''_{u,3} - L''_{0,3})}{u}\sigma\right|_{u,T/u,0,-1}\\
&\quad\leqslant \frac{C}{u^2}|(\lambda - L_{u,4})^{-1}(L''_{u,3} - L''_{0,3})\sigma|_{u,T/u,0,0}\\
&\quad\leqslant \frac{C}{u}|(\lambda - L_{u,4})^{-1}(L''_{u,3} - L''_{0,3})\sigma|_{u,T/u,0,1}\\
&\quad\leqslant \frac{C}{u}|(L''_{u,3} - L''_{0,3})\sigma|_{u,T/u,0,-1}\\
&\quad\leqslant C|(L''_{u,3} - L''_{0,3})\sigma|_{u,T/u,0,0} \leqslant Cu|(1 + |Z|)\sigma|_{u,T/u,0,0}.
\end{aligned}
\tag{12.100}
$$

One easily verifies that as $u \to 0$, the right-hand sides of the inequalities (12.99), (12.100) tend to zero, in particular by using (12.67), which implies that as $u \to 0$, $|\sigma|_{u, T/u, 0, 1}$ has a finite limit. Also

$$(12.101) \qquad \left| \frac{(L''_{u,2} - L''_{0,2})}{u} (\lambda - L_{u,4})^{-1} \frac{L''_{0,3}}{u} \sigma \right|_{u, T/u, 0, -1} \leqslant \frac{C}{u} |(1+|Z|)(\lambda - L_{u,4})^{-1} L''_{0,3} \sigma|_{u, T/u, 0, 0}.$$

By studying the commutators $[Z^i, L_{u,4}]$ as in the proof of Proposition 11.34, we find that for $1 \leqslant i \leqslant 2l$

$$(12.102) \qquad |Z^i (\lambda - L_{u,4})^{-1} L''_{0,3} \sigma|_{u, T/u, 0, 1} \leqslant C(|L''_{0,3} \sigma|_{u, T/u, 0, -1} + |Z^i L''_{0,3} \sigma|_{u, T/u, 0, -1}).$$

So from (12.96), (12.101), (12.102), we obtain

$$(12.103) \qquad \left| \frac{(L''_{u,2} - L''_{0,2})}{u} (\lambda - L_{u,4})^{-1} \frac{L''_{0,3}}{u} \sigma \right|_{u, T/u, 0, -1} \leqslant C u (|\sigma|_{u, T/u, 0, 0} + ||Z| \sigma|_{u, T/u, 0, 0}).$$

The right-hand side of (12.103) also tends to zero as $u \to 0$.

We now calculate the asymptotics as $u \to 0$ of $\frac{L''_{0,2}}{u} (\lambda - L_{u,4})^{-1} \frac{L''_{0,3}}{u} \sigma$. Set

$$(12.104) \qquad L'_{u,4} = P^{\xi^+_{y_0}} M_u^{3, y_0} P^{\xi^+_{y_0}},$$

$$L''_{u,4} = \rho^2 (uZ) \left\{ T \left(\sum_1^{2l'} \left(e^i \wedge - \frac{u^2}{2} i_{e_i} \right) P^{\xi^+_{y_0}} (\nabla^{\xi}_{\tau e_i} V)(uZ) P^{\xi^+_{y_0}} \right) + Tu \sum_{2l'+1}^{2l} \left(\frac{e^i}{T} \wedge - \frac{T}{2} i_{e_i} \right) P^{\xi^+_{y_0}} (\nabla^{\xi}_{\tau e_i} V)(uZ) P^{\xi^+_{y_0}} \right\},$$

$$L'''_{u,4} = T^2 ((1 - \rho^2 (uZ)) P^{\xi^+_{y_0}} + \rho^2 (uZ)(V^+)^2 (uZ)).$$

Then

$$(12.105) \qquad L_{u,4} = L'_{u,4} + \frac{L''_{u,4}}{u} + \frac{L'''_{u,4}}{u^2}.$$

Here $L'''_{u,4}$ is a matrix-valued positive operator acting on $\xi^+_{y_0}$. Therefore if $\lambda \in U$, $u \in]0, 1]$, $(\lambda u^2 - L'''_{u,4})^{-1}$ exists and is uniformly bounded. Also

$$\frac{1}{u^2}\left((\lambda - L_{u,4})^{-1} - \left(\lambda - \frac{L'''_{u,4}}{u^2}\right)^{-1}\right) = (\lambda - L_{u,4})^{-1}\left(L'_{u,4} + \frac{L''_{u,4}}{u}\right)(\lambda u^2 - L'''_{u,4})^{-1}. \tag{12.106}$$

Then by using (12.76), (12.98), we get

$$\begin{aligned} &\left| L''_{0,2}(\lambda - L_{u,4})^{-1}\frac{L''_{u,4}}{u}(\lambda u^2 - L'''_{u,4})^{-1} L''_{0,3}\,\sigma\right|_{u,T/u,0,-1} \\ &\leqslant C\left|(\lambda - L_{u,4})^{-1}\frac{L''_{u,4}}{u}(\lambda u^2 - L'''_{u,4})^{-1} L''_{0,3}\,\sigma\right|_{u,T/u,0,0} \\ &\leqslant C\,|L''_{u,4}(\lambda u^2 - L'''_{u,4})^{-1} L''_{0,3}\,\sigma|_{u,T/u,0,-1} \leqslant C u\,|\sigma|_{u,T/u,0,0}. \end{aligned} \tag{12.107}$$

Moreover

$$\begin{aligned} &|L''_{0,2}(\lambda - L_{u,4})^{-1} L'_{u,4}(\lambda u^2 - L'''_{u,4})^{-1} L''_{0,3}\,\sigma|_{u,T/u,0,-1} \\ &\leqslant C u\,|L'_{u,4}(\lambda u^2 - L'''_{u,4})^{-1} L''_{0,3}\,\sigma|_{u,T/u,0,-1}. \end{aligned} \tag{12.108}$$

If $\tau \in \mathbf{I}^1_{y_0}$, set

$$|\tau|^{\sim 2}_{u,T/u,0,1} = |\tau|^2_{u,T/u,0,0} + \sum_{i=1}^{2l} |\nabla_{e_i}\tau|^2_{u,T/u,0,0}. \tag{12.109}$$

Then for $u > 0$, $|\ |^{\sim}_{u,T/u,0,1}$ is a norm on $\mathbf{I}^1_{y_0}$. Let $|\ |^{\sim}_{u,T/u,0,-1}$ be the corresponding norm on $\mathbf{I}^{-1}_{y_0}$. If $\tau \in \mathbf{I}^1_{y_0}$, $|\tau|^{\sim}_{u,T/u,0,1} \leqslant |\tau|_{u,T/u,0,1}$. Therefore if $\tau' \in \mathbf{I}^{-1}_{y_0}$, $|\tau'|_{u,T/u,0,-1} \leqslant |\tau'|^{\sim}_{u,T/u,0,-1}$. From (12.108), we thus get

$$\begin{aligned} &|L''_{0,2}(\lambda - L_{u,4})^{-1} L'_{u,4}(\lambda u^2 - L'''_{u,4})^{-1} L''_{0,3}\,\sigma|_{u,T/u,0,-1} \\ &\leqslant C u\,|L'_{u,4}(\lambda u^2 - L'''_{u,4})^{-1} L''_{0,3}\,\sigma|^{\sim}_{u,T/u,0,-1}. \end{aligned} \tag{12.110}$$

Then $L'_{u,4}$ is a differential operator of order two. Using Proposition 11.24 and proceeding as in (11.75), (11.76), we deduce from (12.110) that

$$\begin{aligned} &|L''_{0,2}(\lambda - L_{u,4})^{-1} L'_{u,4}(\lambda u^2 - L'''_{u,4})^{-1} L''_{0,3}\,\sigma|_{u,T/u,0,-1} \\ &\leqslant C u\,|(\lambda u^2 - L'''_{u,4})^{-1} L''_{0,3}\,\sigma|^{\sim}_{u,T/u,0,1}. \end{aligned} \tag{12.111}$$

It is now elementary to verify that as $u \to 0$

$$|(\lambda u^2 - L'''_{u,4})^{-1} L''_{0,3}\,\sigma|^{\sim}_{u,T/u,0,1} \tag{12.112}$$

remains uniformly bounded. Therefore the right-hand side of (12.111) tends to zero as $u \to 0$.

From (12.105)-(12.112), we deduce that as $u \to 0$

$$\left| \left(\frac{L''_{0,2}}{u} (\lambda - L_{u,4})^{-1} \frac{L''_{0,3}}{u} - L''_{0,2} (\lambda u^2 - L'''_{u,4})^{-1} L''_{0,3} \right) \sigma \right|_{u, T/u, 0, -1} \to 0. \tag{12.113}$$

Finally it is elementary to verify that, as $u \to 0$

$$| L''_{0,2} (\lambda u^2 - L'''_{u,4})^{-1} L''_{0,3} + L''_{0,2} (L'''_{0,4})^{-1} L''_{0,3}) \sigma |_{u, T/u, 0, -1} \to 0. \tag{12.114}$$

Equivalently, as $u \to 0$

$$\begin{aligned} &| (L''_{0,2} (u^2 \lambda - L'''_{u,4})^{-1} L''_{0,3} + P^{\xi^-_{y_0}} (i^* \nabla^\xi V)_{y_0} \\ &\qquad P^{\xi^+_{y_0}} [(V^+)^2]^{-1} (y_0) P^{\xi^+_{y_0}} (i^* \nabla^\xi V)_{y_0} P^{\xi^-_{y_0}}) \sigma |_{u, T/u, 0, -1} \to 0. \end{aligned} \tag{12.115}$$

From (12.36), (12.95), (12.99), (12.100), (12.103), (12.113), (12.115), we conclude that

$$\begin{aligned} &\Bigg| \Bigg(L_{u,1} + L_{u,2} (\lambda - L_{u,4})^{-1} L_{u,3} - \Bigg\{ -\frac{1}{2} \sum_1^{2l} \left(\nabla_{e_i} + \frac{1}{2} \langle i^* (\nabla^{TX})^2_{y_0} Z, e_i \rangle \right)^2 \\ &\quad + \frac{T^2}{2} | P^N Z |^2 + T S_{y_0} + \frac{T\sqrt{-1}}{\sqrt{2}} \hat{c} (J A P^{TY} Z)_{y_0} \\ &\quad + i^* \left(\frac{1}{2} \operatorname{Tr} [(\nabla^{TX})^2] \right)_{y_0} + i^* (P^{\xi^-} (\nabla^\xi)^2 P^{\xi^-} \\ &\quad - P^{\xi^-} \nabla^\xi V P^{\xi^+} [(V^+)^2]^{-1} P^{\xi^+} \nabla^\xi V P^{\xi^-})_{y_0} \Bigg\} \Bigg) \sigma \Bigg|_{u, T/u, 0, -1} \to 0. \end{aligned} \tag{12.116}$$

Now by Theorem 12.12, we know that

$$i^* (\tilde{\nabla}^{\xi^-})^2 = i^* (P^{\xi^-} (\nabla^\xi)^2 P^{\xi^-} - P^{\xi^-} \nabla^\xi V P^{\xi^+} [(V^+)^2]^{-1} P^{\xi^+} \nabla^\xi V P^{\xi^-}). \tag{12.117}$$

Also by Proposition 8.11, the connection $i^* \tilde{\nabla}^{\xi^-}$ on $\xi^- |_Y$ is exactly the holomorphic Hermitian connection on $\xi^- |_Y$. Finally we have the identifications of holomorphic Hermitian vector bundles on Y, $\xi^- |_Y = \Lambda N^* \otimes \eta$. Therefore if $\widehat{(\nabla^N)^2}$ is the natural action of the curvature $(\nabla^N)^2$ on ΛN^*, we find that

$$i^* (\tilde{\nabla}^{\xi^-})^2 = \widehat{(\nabla^N)^2} + (\nabla^\eta)^2. \tag{12.118}$$

Using (5.10), (12.116)-(12.118), it is now clear that (12.93) holds. Our Theorem is proved. □

i) Proof of Theorems 12.4 and 6.7.

We now prove Theorem 12.4, or equivalently (12.26) and (12.27). By Proposition 12.9 and by Theorem 12.14, it is clear that (12.27) holds.

Also by Theorem 11.27 and by its analogue for $\exp(-(\mathscr{B}^2_{T^2, y_0} + (\nabla^\eta)^2_{y_0}))$, we know that if Γ is the contour (11.115), then

$$\exp(-L^{3, y_0}_{u, T/u}) = \frac{1}{2\pi i}\int_\Gamma \exp(-\lambda)(\lambda - L^{3, y_0}_{u, T})^{-1}\, d\lambda, \tag{12.119}$$

$$\exp(-(\mathscr{B}^{2, y_0}_{T^2} + (\nabla^\eta)^2_{y_0})) = \frac{1}{2\pi i}\int_\Gamma \exp(-\lambda)(\lambda - \mathscr{B}^{2, y_0}_{T^2} - (\nabla^\eta)^2_{y_0})^{-1}\, d\lambda.$$

Also

$$\exp(-(\mathscr{B}^{2, y_0}_{T^2} + (\nabla^\eta)^2_{y_0})) = \exp(-\mathscr{B}^{2, y_0}_{T^2})\exp(-(\nabla^\eta)^2_{y_0}).$$

From the uniform inequality (12.71), from Theorem 12.16 and from (12.119), we find that as $u \to 0$

$$P^{3, y_0}_{u, T/u} \to Q^{y_0}_{T^2}\exp(-\nabla^\eta)^2_{y_0}) \quad \text{in the sense of distributions on } (T_{\mathbf{R}}X)_{y_0} \times (T_{\mathbf{R}}X)_{y_0}. \tag{12.120}$$

Using the uniform bounds of Theorem 12.14, we deduce from (12.120) that as $u \to 0$

$$P^{3, y_0}_{u, T/u}(Z, Z') \to Q^{y_0}_{T^2}(Z, Z')\exp(-(\nabla^\eta)^2_{y_0}) \quad \text{uniformly over compact sets of } (T_{\mathbf{R}}X)_{y_0} \times (T_{\mathbf{R}}X)_{y_0}. \tag{12.121}$$

From (12.121), we find that as $u \to 0$, for any $Z_0 \in N_{\mathbf{R}, y_0}$

$$\operatorname{Tr}_s[N_H[P^{3, y_0}_{u, T/u}(Z_0, Z_0)]^{\max}] \to \{\operatorname{Tr}_s[N_H Q^{y_0}_{T^2}(Z_0, Z_0)]\operatorname{Tr}[\exp(-(\nabla^\eta)^2_{y_0})]\}^{\max}. \tag{12.122}$$

Using Proposition 12.9 and (12.122), we see that as $u \to 0$, for any $Z_0 \in N_{\mathbf{R}, y_0}$

$$u^{2\dim N}\operatorname{Tr}_s[N_H P^{1, y_0}_{u, T/u}(uZ_0, uZ_0)] \to \{(-i)^{\dim Y}\operatorname{Tr}_s[N_H Q^{y_0}_{T^2}(Z_0, Z_0)]\operatorname{Tr}[\exp(-(\nabla^\eta)^2_{y_0})]\}^{\max}, \tag{12.123}$$

from which (12.26) follows. Theorem 12.4 is proved.

The proof of Theorem 6.7 is thus completed. □

j) A remark on Sobolev spaces with weights.

As pointed out in the introduction to this Section, the scaling on the Clifford variables $c(e_i)(2l'+1 \leqslant i \leqslant 2l)$ does not play any role in the proof of Theorem 6.7. We used the norm $| \ |_{u, T/u, 0, 0}$ essentially for convenience.

A more adapted choice of norms would have been as follows. Let $\mathbf{K}^p$ be the set of smooth sections of $(\Lambda^p(T^*_{\mathbf{R}} Y) \hat{\otimes} \Lambda(\bar{N}^*) \hat{\otimes} \xi)_{y_0}$ over $(T_{\mathbf{R}} X)_{y_0}$. For $u > 0$, if $s \in \mathbf{K}^p$ has compact support, set

$$|s|^2_{u,0} = \int_{(T_{\mathbf{R}} X)_{y_0}} |s|^2 \left(1 + |Z| \rho\left(\frac{uZ}{2}\right)\right)^{2(2l'-p)} dv_{TX}(Z), \tag{12.124}$$

$$|s|^2_{u,1} = |s|^2_{u,0} + \frac{1}{u^2} |s^+|^2_{u,0} + \frac{1}{u^2} |\rho(uZ) V^-(uZ) s^-|^2_{u,0} + \sum_{i=1}^{2l} |\nabla_{e_i} s|^2_{u,0}.$$

Note that these new norms do not depend on T. We could have used as well the associated function spaces to prove Theorem 6.7 for any $T > 0$.

XIII - THE ANALYSIS OF THE TWO PARAMETERS SEMI-GROUP $\exp(-(u\mathrm{D}^{\mathrm{X}}+\mathrm{TV})^2)$ IN THE RANGE $u\in]0, 1]$, $\mathrm{T}\geqslant 1/u$

The purpose of this Section is to prove Theorem 6.8, *i.e.* to show the existence of $\mathrm{C}>0$, $\delta\in]0,1]$ such that if $u\in]0,1]$, $\mathrm{T}\geqslant 1$

$$\left|\mathrm{Tr}_s\left[\mathrm{N_H}\exp\left(-\left(u\mathrm{D}^{\mathrm{X}}+\frac{\mathrm{T}}{u}\mathrm{V}\right)^2\right)\right]-\frac{1}{2}\dim\mathrm{N}\,\chi(\eta)\right|\leqslant\frac{\mathrm{C}}{\mathrm{T}^{\delta}}. \tag{13.1}$$

Let us point out that for a fixed $u\in]0,1]$, inequality (13.1) follows from Theorem 6.4. The whole point of (13.1) is to obtain uniformity in $u\in]0,1]$. On the other hand, the fact that for fixed $\mathrm{T}\in]0,+\infty[$, the left-hand side of (13.1) remains bounded as $u\to 0$ is non trivial, and of course follows from Theorem 6.7. This exactly means that to prove Theorem 6.8, we have to face the difficulties in the proofs of Theorems 6.4 and 6.7 simultaneously. We must in fact control the concentration of the considered supertraces as $\mathrm{T}\to+\infty$ near Y and also the local cancellations mechanism in these supertraces as $u\to 0$.

The first key idea of this Section is that the proof of (13.1) can be localized near any $y_0\in\mathrm{Y}$. To prove this, we use the finite propagation speed of solutions of hyperbolic equations in an essential way. This permits us to split

$$\mathrm{Tr}_s\left[\mathrm{N_H}\exp\left(-\left(u\,\mathrm{D^X}+\frac{\mathrm{T}}{u}\mathrm{V}\right)^2\right)\right]$$

into two pieces, the first piece being dealt with by the techniques of Sections 8 and 9, the second piece, being local near $y_0 \in \mathrm{Y}$, is accessible in principle to the techniques of Sections 11 and 12. That this is not directly the case comes as a bad but understandable surprise.

In fact, recall that in Sections 8 and 9, we trivialized the considered vector bundles along geodesics in X normal to the submanifold Y. In Section 12, these vector bundles were trivialized along geodesics in X starting from $y_0 \in \mathrm{Y}$. These two trivializations are not compatible. In order to reduce the proof of (13.1) to an infinite dimensional version of the simple problem on matrices considered in the introduction of Section 12, we must construct here a new trivialization of these vector bundles, along geodesics in X normal to Y, and along geodesics in Y starting at $y_0 \in \mathrm{Y}$.

This Section is very much organized as Section 11, to which the reader is referred when necessary. In a), we show that the proof of (13.1) can be localized globally near Y. In b), using finite propagation speed, we reduce the proof of (13.1) to a local problem near an arbitrary $y_0 \in \mathrm{Y}$. We thus construct a function $\lambda \in \mathbf{C} \to \tilde{\mathrm{F}}_u(\lambda) \in \mathbf{C}$, and we replace $\exp(-(u\mathrm{D^X}+(\mathrm{T}/u)\mathrm{V})^2)$ by $\tilde{\mathrm{F}}_u((u\mathrm{D^X}+(\mathrm{T}/u)\mathrm{V})^2)$ which is an operator which can be studied locally. In c), we describe the properties of the function $\tilde{\mathrm{F}}_u(\lambda)$ as $|\lambda| \to +\infty$.

In d) and e), we construct a coordinate system and a trivialization of $\Lambda(\mathrm{T}^{*(0,1)}\mathrm{X}) \hat{\otimes} \xi$ near a given $y_0 \in \mathrm{Y}$. In f), we reduce the proof of the inequality (13.1) to a uniform estimate on certain smooth kernels over $(\mathrm{T}_{\mathbf{R}}\mathrm{X})_{y_0}$. In g), we rescale the coordinate $\mathrm{Z} \in (\mathrm{T}_{\mathbf{R}}\mathrm{X})_{y_0}$ and also we use Getzler's rescaling [Ge] on certain Clifford variables. The operator $(u\mathrm{D^X}+(\mathrm{T}/u)\mathrm{V})^2$ is now replaced by an operator $\mathscr{L}^{3,y_0}_{u,\mathrm{T}}$. We will prove (13.1) by establishing uniform estimates on the smooth kernel of the operator $\tilde{\mathrm{F}}_u(\mathscr{L}^{3,y_0}_{u,\mathrm{T}})$.

In h), we write an explicit formula for $\mathscr{L}^{3,y_0}_{u,\mathrm{T}}$. In i), we briefly study the asymptotics as $u \to 0$ of $\mathscr{L}^{3,y_0}_{u,\mathrm{T}}$. This permits us to recover the results of Section 12 in a different trivialization.

In j), we calculate the asymptotics as $\mathrm{T} \to +\infty$ of the (3, 3) matrix of the operator $\mathscr{L}^{3,y_0}_{u,\mathrm{T}}$ with respect to a natural orthogonal splitting of the Hilbert space $\mathbf{K}^0_{y_0}$, on which $\mathscr{L}^{3,y_0}_{u,\mathrm{T}}$ acts as an unbounded operator. Theorem 13.22, in which the asymptotics of the operator $\mathscr{L}^{3,y_0}_{u,\mathrm{T}}$ is described, plays a crucial role in the whole Section.

In k), we introduce a new family of Sobolev norms which depend on u, T and we prove certain uniform estimates on the operator $\mathscr{L}^{3,y_0}_{u,\mathrm{T}}$. These estimates, which are established in Theorem 13.27, depend in a crucial way on the results of j). Their proof is long, and sometimes painful. In l), we derive from k) natural uniform estimates on the resolvent of $\mathscr{L}^{3, y_0}_{u, \mathrm{T}}$. In m), we prove uniform regularizing properties of this resolvent. The proof involves estimates on commutators of $\mathscr{L}^{3, y_0}_{u, \mathrm{T}}$ with a natural class of operators. These estimates are not trivial.

In n), we finally obtain our first uniform estimates on the smooth kernel of $\tilde{\mathrm{F}}_u(\mathscr{L}^{3, y_0}_{u, \mathrm{T}})$. In o), we calculate the asymptotics as $\mathrm{T} \to +\infty$ of the operator $\tilde{\mathrm{F}}_u(\mathscr{L}^{3, y_0}_{u, \mathrm{T}})$. This is done by studying the (3, 3) matrix of the resolvent of $\mathscr{L}^{3, y_0}_{u, \mathrm{T}}$. This study is slightly complicated by the need to split certain Sobolev spaces of negative index. We also have to control a very large number of terms in $\mathscr{L}^{3, y_0}_{u, \mathrm{T}}$ and this makes the proofs relatively technical. The main outcome of this subsection is the production of a mysterious second order elliptic operator $\Xi^{y_0}_u$, such that as $\mathrm{T} \to +\infty$, $\tilde{\mathrm{F}}_u(\mathscr{L}^{3, y_0}_{u, \mathrm{T}})$ converges uniformly to $\tilde{\mathrm{F}}_u(\Xi^{y_0}_u)$ in norm theoretic sense. In p), by using the results of Section 8, we identify $\Xi^{y_0}_u$ as being essentially the operator $(u\mathrm{D}^{\mathrm{Y}})^2$ written in a natural trivialization of the vector bundles under consideration near $y_0 \in \mathrm{Y}$.

In q), comes the long awaited and yet happy end, *i.e.* the proof of (13.1).

a) The problem is localizable globally near Y.

Proposition **13.1.** — *Take* $\alpha > 0$. *There exist* $c > 0$, $\mathrm{C} > 0$ *such that for any* $x \in \mathrm{X}$, *with* $d(x, \mathrm{Y}) \geqslant \alpha$, *and any* $u \in]0, 1]$, $\mathrm{T} \geqslant 1$, *then*

$$|\mathrm{P}_{u, \mathrm{T}/u}(x, x)| \leqslant c \exp(-\mathrm{CT}). \tag{13.2}$$

Proof. — The operator $(u\mathrm{D}^{\mathrm{X}} + (\mathrm{T}/u)\mathrm{V})^2$ is self-adjoint and nonnegative. Using spectral theory, we find that for any $x \in \mathrm{X}$, $\beta \in \mathbf{R}^*_+ \to \mathrm{Tr}[\mathrm{P}_{u\beta, \mathrm{T}\beta/u}(x, x)]$ is a decreasing function. Since $\mathrm{P}_{u, \mathrm{T}/u}(x, x)$ is self-adjoint and positive, we find that if $|\ \ |$ denotes the norm associated with the trace on elements of $\mathrm{End}(\Lambda(\mathrm{T}^{*(0,1)}\mathrm{X}) \hat{\otimes} \xi)$, for any $x \in \mathrm{X}$, $\beta \in]0, 1]$

$$|\mathrm{P}_{u, \mathrm{T}/u}(x, x)| \leqslant |\mathrm{P}_{u\beta, \mathrm{T}\beta/u}(x, x)|.$$

Assume that $u \in]0, 1]$, $\mathrm{T} \geqslant 1$. By taking $\beta = 1/\sqrt{\mathrm{T}}$, we obtain

$$|\mathrm{P}_{u, \mathrm{T}/u}(x, x)| \leqslant |\mathrm{P}_{u/\sqrt{\mathrm{T}}, \sqrt{\mathrm{T}}/u}(x, x)|. \tag{13.3}$$

Now observe that $u/\sqrt{\mathrm{T}} \in]0, 1]$. Using Proposition 12.1, we get

$$|\mathrm{P}_{u/\sqrt{\mathrm{T}}, \sqrt{\mathrm{T}}/u}(x, x)| \leqslant c \exp\left(-\frac{\mathrm{CT}}{u^2}\right). \tag{13.4}$$

Then (13.2) follows from (13.3), (13.4). □

Remark **13.2.** – One can also give a proof of Proposition 13.1 which does not use the positivity of $(u D^X + (T/u) V)^2$, by directly imitating the proof of Proposition 12.1.

We here use the notation of Section 8 e). Take $\varepsilon \in]0, (\varepsilon_0/2)]$. From Proposition 13.1, one finds that there exist $c > 0$, $C > 0$ such that for $u \in]0, 1]$, $T \geqslant 1$

$$\left| \int_{X \setminus \mathcal{U}_{\varepsilon/8}} \operatorname{Tr}_s [N_H P_{u, T/u}(x, x)] \frac{dv_X(x)}{(2\pi)^{\dim X}} \right| \leqslant c \exp(-CT). \tag{13.5}$$

It is now clear that to prove Theorem 6.8, we only need to show that there exist $C > 0$, $\delta \in]0, 1]$ such that for $u \in]0, 1]$, $T \geqslant 1$

$$\left| \int_{\mathcal{U}_{\varepsilon/8}} \operatorname{Tr}_s [N_H P_{u, T/u}(x, x)] \frac{dv_X(x)}{(2\pi)^{\dim X}} - \frac{1}{2} \dim N \chi(\eta) \right| \leqslant \frac{C}{T^\delta}. \tag{13.6}$$

We have thus shown that the problem is **globally** local near Y, *i.e.* depends only on the kernel $P_{u, T/u}(x, x)$ near Y.

Showing that the problem can be localized near any arbitrary $y_0 \in Y$ is much subtler. This question will be dealt with in the next subsection.

b) Finite propagation speed and localization.

Recall that $\varepsilon_0 > 0$ was determined in Section 8 e) and that a is the injectivity radius of X. Let b be the injectivity radius of Y.

We now fix $\varepsilon \in]0, \inf(\varepsilon_0/2, a/2, b/2)]$. Let α be a positive constant. The precise value of α will be determined in Section 13 e).

Let f be a smooth even function defined on **R** with values in [0, 1] such that

$$\begin{aligned} f(t) &= 1 \quad \text{if} \quad |t| \leqslant \frac{\alpha}{2}, \\ &= 0 \quad \text{if} \quad |t| > \alpha. \end{aligned} \tag{13.7}$$

Set

$$g(t) = 1 - f(t). \tag{13.8}$$

Definition **13.3.** — If $u \in]0, 1]$, $a \in \mathbf{C}$, set

$$(13.9)\qquad \begin{aligned} F_u(a) &= \int_{-\infty}^{+\infty} \exp(it\sqrt{2}a)\exp\left(\frac{-t^2}{2}\right) f(ut)\frac{dt}{\sqrt{2\pi}}, \\ G_u(a) &= \int_{-\infty}^{+\infty} \exp(it\sqrt{2}a)\exp\left(\frac{-t^2}{2}\right) g(ut)\frac{dt}{\sqrt{2\pi}}. \end{aligned}$$

Then

$$(13.10)\qquad \exp(-a^2) = F_u(a) + G_u(a).$$

Since f is even, F_u and G_u are even functions, which take real values on $\mathbf{R}$. Moreover F_u and G_u lie in the Schwartz space $S(\mathbf{R})$.

From (13.10), we deduce that

$$(13.11)\qquad \exp\left(-\left(uD^X + \frac{T}{u}V\right)^2\right) = F_u\left(uD^X + \frac{T}{u}V\right) + G_u\left(uD^X + \frac{T}{u}V\right).$$

Since F_u, $G_u \in S(\mathbf{R})$ and since $uD^X + (T/u)V$ is an elliptic operator, using integration by parts, it is clear that the operators $F_u(uD^X + (T/u)V)$, $G_u(uD^X + (T/u)V)$ are given by smooth kernels, and so are trace class.

Our first fundamental result is as follows.

Theorem **13.4.** — *There exist* $c > 0$, $C > 0$ *such that for any* $u \in]0, 1]$, $T \geq 1$

$$(13.12)\qquad \left|\mathrm{Tr}_s\left[N_H G_u\left(uD^X + \frac{T}{u}V\right)\right] - \frac{\dim N}{2}\chi(\eta) G_u(0)\right| \leq \frac{c}{\sqrt{T}}\exp\left(\frac{-C}{u^2}\right).$$

Proof. — Set

$$(13.13)\qquad H_u(a) = \int_{-\infty}^{+\infty} \exp(it\sqrt{2}a)\exp\left(\frac{-t^2}{2u^2}\right) g(t)\frac{dt}{u\sqrt{2\pi}}.$$

Then

$$(13.14)\qquad G_u(a) = H_u\left(\frac{a}{u}\right).$$

Recall that $g(t)$ vanishes near $t=0$. For $p \in \mathbf{N}$, set

$$(13.15)\qquad H_{u,p}(a) = (p-1)! \int_{-\infty}^{+\infty} \exp(it\sqrt{2}a)\exp\left(\frac{-t^2}{2u^2}\right)\frac{g(t)}{(it\sqrt{2})^{p-1}}\frac{dt}{u\sqrt{2\pi}}.$$

Clearly

$$\frac{H_{u,p}^{(p-1)}(a)}{(p-1)!} = H_u(a). \tag{13.16}$$

Then $a \in \mathbf{C} \to H_{u,p}(a)$ is holomorphic. Moreover for any $c > 0$, if $|\operatorname{Im} a| \leqslant c$, as $|a| \to +\infty$, $H_{u,p}(a)$ decays faster than any $|a|^{-m}$ $(m \in \mathbf{N})$.

Let Δ, δ be the contours in $\mathbf{C}$ considered in Section 9 g). From the previous considerations, we deduce that for any $a \in \mathbf{C}$ lying inside the domain bounded by $\Delta \cup \delta$, then

$$H_u(a) = \frac{1}{2\pi i} \int_{\Delta \cup \delta} H_u(\lambda)(\lambda - a)^{-1} d\lambda. \tag{13.17}$$

Equivalently, for any $p \in \mathbf{N}$

$$H_u(a) = \frac{1}{2\pi i} \int_{\Delta \cup \delta} H_{u,p}(\lambda)(\lambda - a)^{-p} d\lambda. \tag{13.18}$$

We now use the notation of Section 9. Observe that

$$\frac{1}{u}\left(u D^X + \frac{T}{u} V\right) = A_{T/u^2}. \tag{13.19}$$

From (13.14), (13.19), we get

$$G_u\left(u D^X + \frac{T}{u} V\right) = H_u(A_{T/u^2}). \tag{13.20}$$

Take now $p \in \mathbf{N}$, $p \geqslant 2 \dim X + 2$. From (13.18), (13.20), we find that

$$G_u\left(u D^X + \frac{TV}{u}\right) = \frac{1}{2\pi i} \int_{\Delta \cup \delta} H_{u,p}(\lambda)(\lambda - A_{T/u^2})^{-p} d\lambda. \tag{13.21}$$

For $T > 0$, let U_T be the subset of $\mathbf{C}$ defined in (9.113) (the value of $c_1 > 0$ is determined in Theorem 9.21). Using Theorem 9.24, we find that if T/u^2 is large enough, if $\lambda \in U_{T/u^2}$,

$$|\operatorname{Tr}_s[N_H(\lambda - A_{T/u^2})^{-p}] - \operatorname{Tr}_s[N_H^\theta(\lambda - D^Y)^{-p}]| \leqslant \frac{Cu}{\sqrt{T}}(1 + |\lambda|)^{p+1}. \tag{13.22}$$

On the other hand, since $g(t)$ vanishes near 0, we deduce from (13.15) that for any $m \in \mathbf{N}$, there exists $c_m > 0$, $C_m > 0$ such that if $\lambda \in \Delta \cup \delta$,

$$|\lambda^m H_{u,p}(\lambda)| \leqslant c_m \exp\left(\frac{-C_m}{u^2}\right). \tag{13.23}$$

From (13.22), (13.23), it is clear that

$$\begin{aligned} \Bigg| \mathrm{Tr}_s\left[N_H \frac{1}{2\pi i} \int_{(\Delta \cup \delta) \cap U_{T/u^2}} H_{u,p}(\lambda)(\lambda - A_{T/u^2})^{-p} d\lambda\right] \\ - \mathrm{Tr}_s\left[N_H^\theta \frac{1}{2\pi i} \int_{(\Delta \cup \delta) \cap U_{T/u^2}} H_{u,p}(\lambda)(\lambda - D^Y)^{-p} d\lambda\right]\Bigg| \\ \leqslant c \frac{u}{\sqrt{T}} \exp\left(\frac{-C}{u^2}\right). \end{aligned} \tag{13.24}$$

Also, for T/u^2 large enough, if $\lambda \in (\Delta \cup \delta) \cap {}^c U_{T/u^2}$, then $|\lambda| \geqslant c_1 \sqrt{T}/u$. Using (9.170) and (13.23), we find that for any $m \in \mathbf{N}$, if T/u^2 is large enough, then

$$\begin{aligned} \left|\mathrm{Tr}_s\left[N_H \frac{1}{2\pi i} \int_{(\Delta \cup \delta) \cap {}^c U_{T/u^2}} H_{u,p}(\lambda)(\lambda - A_{T/u^2})^{-p} d\lambda\right]\right| \\ \leqslant c\left(\frac{u^2}{T}\right)^m \exp\left(\frac{-C}{u^2}\right), \\ \left|\mathrm{Tr}_s\left[N_H \frac{1}{2\pi i} \int_{(\Delta \cup \delta) \cap {}^c U_{T/u^2}} H_{u,p}(\lambda)(\lambda - D^Y)^{-p} d\lambda\right]\right| \\ \leqslant c\left(\frac{u^2}{T}\right)^m \exp\left(\frac{-C}{u^2}\right). \end{aligned} \tag{13.25}$$

From (13.24), (13.25), we deduce

$$\left|\mathrm{Tr}_s\left[N_H G_u\left(u D^X + \frac{T}{u} V\right)\right] - \mathrm{Tr}_s[N_H^\theta G_u(u D^Y)]\right| \leqslant \frac{cu}{\sqrt{T}} \exp\left(\frac{-C}{u^2}\right). \tag{13.26}$$

Now by Proposition 8.4, $N_H^\theta = 1/2 \dim N$. On the other hand, since f is an even function, $G_u(a)$ is a smooth function of a^2. By the analogue of the McKean-Singer formula [MKS], we find that for $1 \leqslant j \leqslant d$

$$\mathrm{Tr}_s[G_u(u D^{Y_j})] = \chi(\eta_j) G_u(0). \tag{13.27}$$

Using (13.26), (13.27), we obtain (13.12). □

Remark **13.5.** — By (13.10), we know that

$$F_u(0)+G_u(0)=1. \tag{13.28}$$

In view of Theorem 13.4 and of (13.28), we see that to prove Theorem 6.8, we only need to show there exist $C>0$, $\delta\in]0,1]$ such that for $u\in]0,1]$, $T\geqslant 1$

$$\left|\operatorname{Tr}_s\left[N_H F_u\left(u D^X+\frac{T}{u}V\right)\right]-\frac{1}{2}\dim N\,\chi(\eta)\,F_u(0)\right|\leqslant\frac{C}{T^\delta}. \tag{13.29}$$

Since $f(t)$ vanishes for $|t|\geqslant\alpha$, we find that

$$F_u(a)=\int_{-\alpha/u}^{\alpha/u}\exp(it\sqrt{2}\,a)\exp\left(\frac{-t^2}{2}\right)f(ut)\frac{dt}{\sqrt{2\pi}}. \tag{13.30}$$

In particular

$$F_u\left(u D^X+\frac{T}{u}V\right)=\int_{-\alpha/u}^{\alpha/u}\exp\left(it\sqrt{2}\left(u D^X+\frac{T}{u}V\right)\right)\exp\left(\frac{-t^2}{2}\right)f(ut)\frac{dt}{\sqrt{2\pi}}. \tag{13.31}$$

Note that since f is even, (13.31) can also be written in the form

$$F_u\left(u D^X+\frac{T}{u}V\right)=\int_{-\alpha/u}^{\alpha/u}\cos\left(t\sqrt{2}\left|u D^X+\frac{T}{u}V\right|\right)\exp\left(\frac{-t^2}{2}\right)f(ut)\frac{dt}{\sqrt{2\pi}}. \tag{13.32}$$

By general results on hyperbolic equations [CP, Section 7.8], [T, Section 4.4], for any $t\in\mathbf{R}$, $x\in X$, $h\in(\Lambda(T^{*(0,1)}X)\hat{\otimes}\xi)_x$,

$$\operatorname{supp}\exp\left(it\sqrt{2}\left(u D^X+\frac{T}{u}V\right)\right)h\,\delta_{\{x\}}\subset B^X(x,ut). \tag{13.33}$$

In particular if $|ut|\leqslant\alpha$, then

$$\operatorname{supp}\exp\left(it\sqrt{2}\left(u D^X+\frac{T}{u}V\right)\right)h\,\delta_{\{x\}}\subset B^X(x,\alpha). \tag{13.34}$$

Let $F_u(u D^X+(T/u)V)(x,x')$ $(x,x'\in X)$ be the smooth kernel of the operator $F_u(u D^X+(T/u)V)$ with respect to the volume form $dv_X/(2\pi)^{\dim X}$. From (13.31), (13.34), we conclude that for any $x\in X$, the map $x'\to F_u(u D^X+(T/u)V)(x,x')$ only depends on the restrictions of the operators D^X, V to the ball $B^X(x, \alpha)$, and that moreover if $x' \notin B^X(x, \alpha)$, $F_u(uD^X + (T/u)V)(x, x')$ vanishes. In particular $F_u(x, x)$ only depends on the restriction of the operators D^X, V to the ball $B^X(x, \alpha)$.

Now

$$\text{(13.35)} \qquad \operatorname{Tr}_s\left[N_H F_u\left(uD^X + \frac{T}{u}V\right)\right] = \int_X \operatorname{Tr}_s\left[N_H F_u\left(uD^X + \frac{T}{u}V\right)(x, x)\right] \frac{dv_X(x)}{(2\pi)^{\dim X}}.$$

Using (13.35) and the previous considerations, it is clear that the proof of (13.29) is local on X, *i. e.* it can be obtained by studying the restrictions of the operators D^X, V to an arbitrary open covering of the manifold X.

Observe that, by using (8.21), we get

$$\text{(13.36)} \qquad \int_{\mathscr{U}_{\varepsilon/8}} \operatorname{Tr}_s\left[N_H F_u\left(uD^X + \frac{T}{u}V\right)(x, x)\right] \frac{dv_X(x)}{(2\pi)^{\dim X}}$$
$$= \left(\frac{1}{2\pi}\right)^{\dim X} \int_Y \left\{\int_{|Z_0| \leqslant \varepsilon\sqrt{T}/8u} \left(\frac{u}{\sqrt{T}}\right)^{2\dim N} \operatorname{Tr}_s\left[N_H F_u\left(uD^X + \frac{T}{u}V\right)\left(\left(y_0, \frac{uZ_0}{\sqrt{T}}\right), \left(y_0, \frac{uZ_0}{\sqrt{T}}\right)\right)\right] k\left(y_0, \frac{uZ_0}{\sqrt{T}}\right) dv_N(Z_0)\right\} dv_Y(y_0).$$

Let $F_u(uD^Y)(y, y')$ $(y, y' \in Y)$ be the smooth kernel of the operator $F_u(uD^Y)$ with respect to the volume element $dv_Y(y')/(2\pi)^{\dim Y}$.

Recall that $\alpha > 0$ is a parameter which appears in the definition of f.

Theorem **13.6.** – *If* $\varepsilon \in]0, \inf(\varepsilon_0/2, a/2, b/2)]$, $\alpha > 0$ *are small enough, for any* $p \in \mathbf{N}$, *there exists* $C > 0$ *such that for any* $u \in]0, 1]$, $T \geqslant 1$, $y_0 \in Y$, $Z_0 \in N_{\mathbf{R}, y_0}$, $|Z_0| \leqslant \varepsilon\sqrt{T}/8u$, *then*

$$\text{(13.37)} \qquad (1 + |Z_0|)^p \left(\frac{u}{\sqrt{T}}\right)^{2\dim N} \left|\operatorname{Tr}_s\left[N_H F_u\left(uD^X + \frac{T}{u}V\right)\left(\left(y_0, \frac{uZ_0}{\sqrt{T}}\right), \left(y_0, \frac{uZ_0}{\sqrt{T}}\right)\right)\right]\right| \leqslant C.$$

There exist $C'>0$, $\delta'\in]0, 1/2]$ *such that for any* $u\in]0, 1]$, $T\geqslant 1$, $y_0\in Y$, $Z_0\in N_{\mathbf{R}, y_0}$, $|Z_0|\leqslant \varepsilon\sqrt{T}/8u$, *then*

$$(13.38)\qquad \left|\left(\frac{1}{2\pi}\right)^{\dim X}\left(\frac{u}{\sqrt{T}}\right)^{2\dim N} \operatorname{Tr}_s\left[N_H F_u\left(uD^X+\frac{T}{u}V\right)\left(\left(y_0, \frac{uZ_0}{\sqrt{T}}\right), \left(y_0, \frac{uZ_0}{\sqrt{T}}\right)\right)\right] - \frac{\exp(-|Z_0|^2)}{\pi^{\dim N}}\frac{\dim N}{2}\left(\frac{1}{2\pi}\right)^{\dim Y}\operatorname{Tr}_s[F_u(uD^Y)(y_0, y_0)]\right| \leqslant \frac{C'}{T^{\delta'}}.$$

Proof. — The proof of Theorem 13.6 is delayed to the next subsections. □

Remark **13.7.** — We now briefly show how to deduce (13.29) from Theorem 13.6, from which Theorem 6.8 follows.

In fact from (13.37), (13.38), we see that for any $p\in\mathbf{N}$

$$(13.39)\qquad (1+|Z_0|)^p\left|\left(\frac{1}{2\pi}\right)^{\dim X}\left(\frac{u}{\sqrt{T}}\right)^{2\dim N} \operatorname{Tr}_s\left[N_H F_u\left(uD^X+\frac{T}{u}V\right)\left(\left(y_0, \frac{uZ_0}{\sqrt{T}}\right), \left(y_0, \frac{uZ_0}{\sqrt{T}}\right)\right)\right] - \frac{\exp(-|Z_0|^2)}{\pi^{\dim N}}\frac{\dim N}{2}\left(\frac{1}{2\pi}\right)^{\dim Y}\operatorname{Tr}_s[F_u(uD^Y)(y_0, y_0)]\right| \leqslant \frac{C''}{T^{\delta'/2}}.$$

From (13.39), it is clear that for $u\in]0, 1]$, $T\geqslant 1$

$$(13.40)\qquad \left|\int_{\mathscr{U}_{\varepsilon/8}} \operatorname{Tr}_s\left[N_H F_u\left(uD^X+\frac{T}{u}V\right)(x, x)\right]\frac{dv_X(x)}{(2\pi)^{\dim X}} - \int_Y \frac{\dim N}{2}\operatorname{Tr}_s[F_u(uD^Y)(y_0, y_0)]\frac{dv_Y(y_0)}{(2\pi)^{\dim Y}}\right| \leqslant \frac{C'''}{T^{\delta'/2}}.$$

On the other hand, Theorem 13.6 also holds in the case where $Y=\varnothing$. More precisely, as the reader will easily check, the proof of Theorem 13.6 can be modified on $X\setminus\mathscr{U}_{\varepsilon/8}$ so that

$$(13.41)\qquad \left|\int_{X\setminus\mathscr{U}_{\varepsilon/8}} \operatorname{Tr}_s\left[N_H F_u\left(uD^X+\frac{T}{u}V\right)(x, x)\right]\frac{dv_X(x)}{(2\pi)^{\dim X}}\right| \leqslant \frac{C''''}{T^{\delta'/2}}.$$

Finally by using the McKean-Singer formula [MKS] as in (13.27), for $1 \leqslant j \leqslant d$, we get

$$\int_{Y_j} \operatorname{Tr}_s [F_u(u D^{Y_j})(y, y)] \frac{dv_{Y_j}(y)}{(2\pi)^{\dim Y_j}} = \chi(\eta_j) F_u(0). \tag{13.42}$$

From (13.40)-(13.42), we obtain (13.29).

c) The function $F_u(a)$ as a function of a^2.

Proposition **13.8.** — *For any* $c > 0$, $m \in \mathbf{N}$, $m' \in \mathbf{N}$, *there exists* $C > 0$ *such that for any* $u \in]0, 1]$

$$\sup_{\substack{a \in \mathbf{C} \\ |\operatorname{Im} a| \leqslant c}} |a|^m |F_u^{(m')}(a)| \leqslant C. \tag{13.43}$$

Proof. — Observe that if $a \in \mathbf{C}$, $|\operatorname{Im} a| \leqslant c$, then

$$\left| \exp(it\sqrt{2}\,a) \exp\left(\frac{-t^2}{2}\right) \right| \leqslant \exp\left(c\sqrt{2}|t| - \frac{t^2}{2}\right) \leqslant C \exp\left(\frac{-t^2}{4}\right). \tag{13.44}$$

Then (13.43) immediately follows from (13.9) and (13.44). □

Observe that since $F_u(a)$ is an even function of a, there exists a unique holomorphic function $\tilde{F}_u(a)$ on $\mathbf{C}$ such that

$$F_u(a) = \tilde{F}_u(a^2). \tag{13.45}$$

Definition **13.9.** — For $c > 0$, set

$$V_c = \left\{ \lambda \in \mathbf{C};\ \operatorname{Re}(\lambda) \geqslant \frac{(\operatorname{Im} \lambda)^2}{4c^2} - c^2 \right\}. \tag{13.46}$$

Proposition **13.10.** — *For any* $c > 0$, $m \in \mathbf{N}$, $m' \in \mathbf{N}$, *there exists* $C > 0$ *such that for any* $u \in]0, 1]$,

$$\sup_{a \in V_c} |a|^m |\tilde{F}_u^{(m')}(a)| \leqslant C. \tag{13.47}$$

Proof. — Observe that V_c is exactly the image of $U_c = \{\lambda \in \mathbf{C};\ |\operatorname{Im} \lambda| \leqslant c\}$ by the map $\lambda \to \lambda^2$. Our result now follows from Proposition 13.8. □

Using (13.45), we find that

$$\text{(13.48)}\qquad \begin{aligned} &F_u\left(u D^X + \frac{T}{u} V\right) = \tilde{F}_u\left(\left(u D^X + \frac{T}{u} V\right)^2\right),\\ &F_u(u D^Y) = \tilde{F}_u((u D^Y)^2). \end{aligned}$$

d) An orthogonal splitting of TX and a connection on TX.

We here use the notation of Section 8 e).

On Y, we have the splitting of C^∞ vector bundles

$$\text{(13.49)}\qquad TX|_Y = TY \oplus N.$$

We will extend the splitting (13.49) to $\mathcal{U}_\varepsilon$.

Definition **13.11.** – If $y_0 \in Y$, $Z_0 \in N_{\mathbf{R}, y_0}$, $|Z_0| \leq \varepsilon$, let $TX^1_{\exp^X_{y_0}(Z_0)}$, $TX^2_{\exp^X_{y_0}(Z_0)}$ be the subspaces of $TX_{\exp^X_{y_0}(Z_0)}$ which are obtained by parallel transport of TY_{y_0}, N_{y_0} with respect to the connection ∇^{TX} along the geodesic $t \in [0, 1] \to \exp^X_{y_0}(t Z_0)$.

Then TX^1, TX^2 are smooth vector subbundles of $TX|_{\mathcal{U}_\varepsilon}$ such that

$$\text{(13.50)}\qquad \begin{aligned} &TX^1|_Y = TY,\\ &TX^2|_Y = N. \end{aligned}$$

Moreover over $\mathcal{U}_\varepsilon$, TX splits orthogonally into

$$\text{(13.51)}\qquad TX = TX^1 \oplus TX^2.$$

Let P^{TX^1}, P^{TX^2} be the orthogonal projection operators from $TX|_{\mathcal{U}_\varepsilon}$ on TX^1, TX^2 respectively. Let ∇^{TX^1}, ∇^{TX^2} be the connections on TX^1, TX^2

$$\text{(13.52)}\qquad \begin{aligned} &\nabla^{TX^1} = P^{TX^1} \nabla^{TX},\\ &\nabla^{TX^2} = P^{TX^2} \nabla^{TX}. \end{aligned}$$

Then the restriction of the connection ∇^{TX^1} to Y coincides with ∇^{TY}.

Let ${}^0\nabla^{TX} = \nabla^{TX^1} \oplus \nabla^{TX^2}$ be the direct sum of the connections ∇^{TX^1}, ∇^{TX^2} on $TX = TX^1 \oplus TX^2$. Then the connection ${}^0\nabla^{TX}$ restricts on Y to the connection ${}^0\nabla^{TX|_Y}$ defined in Definition 8.7. Set

$$\text{(13.53)}\qquad A' = \nabla^{TX} - {}^0\nabla^{TX}.$$

Then A′ is a 1-form on $\mathcal{U}_\varepsilon$ taking values in skew-adjoint endomorphisms of TX which exchange TX^1 and TX^2.

Recall that A was defined in Definition 8.7 as a 1-form on Y taking values in skew-adjoint elements of $\mathrm{End}(\mathrm{TX}|_{\mathrm{Y}})$ exchanging TY and N. By construction

$$i^* \mathrm{A}' = \mathrm{A}. \tag{13.54}$$

Also if $y_0 \in \mathrm{Y}$, $\mathrm{Z}_0 \in \mathrm{N}_{\mathbf{R}, y_0}$, then

$$\mathrm{A}'_{y_0}(\mathrm{Z}_0) = 0. \tag{13.55}$$

We now prove a fundamental identity, which extends the well-known symmetry identity of the curvature of the Levi-Civita connection.

Proposition **13.12.** – *If* $y_0 \in \mathrm{Y}$, $\mathrm{U}, \mathrm{U}' \in (\mathrm{T}_{\mathbf{R}} \mathrm{Y})_{y_0}$, $\mathrm{Z}, \mathrm{Z}' \in (\mathrm{T}_{\mathbf{R}} \mathrm{X})_{y_0}$, *then*

$$\langle ({}^0\nabla^{\mathrm{TX}})^2_{y_0}(\mathrm{Z}, \mathrm{Z}') \mathrm{U}, \mathrm{U}' \rangle = \langle ((\nabla^{\mathrm{TX}})^2_{y_0} - \mathrm{P}^{\mathrm{TY}} \mathrm{A}^2_{y_0} \mathrm{P}^{\mathrm{TY}})(\mathrm{U}, \mathrm{U}') \mathrm{Z}, \mathrm{Z}' \rangle. \tag{13.56}$$

Proof. – From (13.53), we get

$$\begin{aligned} \langle ({}^0\nabla^{\mathrm{TX}})^2_{y_0}(\mathrm{Z}, \mathrm{Z}') \mathrm{U}, \mathrm{U}' \rangle \\ = \langle (\nabla^{\mathrm{TX}})^2_{y_0}(\mathrm{Z}, \mathrm{Z}') \mathrm{U}, \mathrm{U}' \rangle - \langle \mathrm{A}'^2_{y_0}(\mathrm{Z}, \mathrm{Z}') \mathrm{U}, \mathrm{U}' \rangle. \end{aligned} \tag{13.57}$$

Since the metric g^{TX} is Kähler, ∇^{TX} induces the Levi-Civita connection on $\mathrm{T}_{\mathbf{R}}\mathrm{X}$, and so

$$\langle (\nabla^{\mathrm{TX}})^2_{y_0}(\mathrm{Z}, \mathrm{Z}') \mathrm{U}, \mathrm{U}' \rangle = \langle (\nabla^{\mathrm{TX}})^2_{y_0}(\mathrm{U}, \mathrm{U}') \mathrm{Z}, \mathrm{Z}' \rangle. \tag{13.58}$$

Using (13.54), (13.55), we find that

$$\begin{aligned} \langle \mathrm{A}'^2_{y_0}(\mathrm{Z}, \mathrm{Z}') \mathrm{U}, \mathrm{U}' \rangle = & -\langle \mathrm{A}_{y_0}(\mathrm{P}^{\mathrm{TY}} \mathrm{Z}') \mathrm{U}, \mathrm{A}_{y_0}(\mathrm{P}^{\mathrm{TY}} \mathrm{Z}) \mathrm{U}' \rangle \\ & + \langle \mathrm{A}_{y_0}(\mathrm{P}^{\mathrm{TY}} \mathrm{Z}) \mathrm{U}, \mathrm{A}_{y_0}(\mathrm{P}^{\mathrm{TY}} \mathrm{Z}') \mathrm{U}' \rangle. \end{aligned} \tag{13.59}$$

Since the connection ∇^{TX} is torsion free, if $\mathrm{H}, \mathrm{H}' \in (\mathrm{T}_{\mathbf{R}} \mathrm{Y})_{y_0}$

$$\mathrm{A}_{y_0}(\mathrm{H}) \mathrm{H}' = \mathrm{A}_{y_0}(\mathrm{H}') \mathrm{H}. \tag{13.60}$$

Using (13.59), (13.60), we get

$$\begin{aligned} \langle \mathrm{A}'^2_{y_0}(\mathrm{Z}, \mathrm{Z}') \mathrm{U}, \mathrm{U}' \rangle = & -\langle \mathrm{A}_{y_0}(\mathrm{U}) \mathrm{P}^{\mathrm{TY}} \mathrm{Z}', \mathrm{A}_{y_0}(\mathrm{U}') \mathrm{P}^{\mathrm{TY}} \mathrm{Z} \rangle \\ & + \langle \mathrm{A}_{y_0}(\mathrm{U}) \mathrm{P}^{\mathrm{TY}} \mathrm{Z}, \mathrm{A}_{y_0}(\mathrm{U}') \mathrm{P}^{\mathrm{TY}} \mathrm{Z}' \rangle \\ = & \langle \mathrm{A}^2_{y_0}(\mathrm{U}, \mathrm{U}') \mathrm{P}^{\mathrm{TY}} \mathrm{Z}, \mathrm{P}^{\mathrm{TY}} \mathrm{Z}' \rangle. \end{aligned} \tag{13.61}$$

From (13.57), (13.58), (13.61), we obtain (13.56). □

e) A local coordinate system near $y_0 \in Y$ and a trivialization of $\Lambda(T^{*(0,1)}X) \hat{\otimes} \xi$.

For $\alpha > 0$, $y_0 \in Y$, let $B^Y(y_0, \alpha)$ (resp. $B^{TY}_{y_0}(0, \alpha)$) be the open ball in Y (resp. $(T_{\mathbf{R}} Y)_{y_0}$) of center y_0 (resp. 0) and radius α.

Take $y_0 \in Y$. If $U \in (T_{\mathbf{R}} Y)_{y_0}$, let $t \in \mathbf{R} \to y_t = \exp^Y_{y_0}(tU) \in Y$ be the geodesic in Y such that $y|_{t=0} = y_0$, $dy/dt|_{t=0} = U$. Since $\varepsilon \leqslant b/2$, the map $U \in B^{TY}_{y_0}(0, \varepsilon) \to \exp^Y_{y_0}(U) \in B^Y(y_0, \varepsilon)$ is a diffeomorphism.

If $U \in (T_{\mathbf{R}} Y)_{y_0}$, $|U| < \varepsilon$, $V \in N_{\mathbf{R}, y_0}$, let $\tau_U V \in N_{\mathbf{R}, \exp^Y_{y_0}(U)}$ be the parallel transport of V with respect to the connection ∇^N along the curve $t \in [0, 1] \to \exp^Y_{y_0}(tU)$.

Recall that $\tilde{\pi}$ is the projection $N \to Y$. Then the map

$$(U, V) \in B^{TY}_{y_0}(0, \varepsilon) \times N_{\mathbf{R}, y_0} \to (\exp^Y_{y_0}(U), \tau_U V) \in \tilde{\pi}^{-1}(B(y_0, \varepsilon))$$

is a trivialization of $N_{\mathbf{R}, y_0}$ over $B^Y(y_0, \varepsilon)$.

If $x \in X$, $Z \in (T_{\mathbf{R}} X)_x$, let $t \in \mathbf{R} \to x_t = \exp^X_x(tZ)$ be the geodesic in X such that $x_0 = x$, $dx/dt|_{t=0} = Z$.

Recall that N is identified with the orthogonal bundle to TY in $TX|_Y$. Let $B^N_{y_0}(0, \varepsilon)$ be the open ball in $N_{\mathbf{R}, y_0}$ of center 0 and radius ε.

If $Z \in (T_{\mathbf{R}} X)_{y_0}$, $Z = U + U'$, $U \in (T_{\mathbf{R}} Y)_{y_0}$, $U' \in N_{\mathbf{R}, y_0}$, $|U| < \varepsilon$, $|U'| < \varepsilon$, we identify Z with $\exp^X_{\exp^Y_{y_0} U}(\tau_U U') \in \mathscr{U}_\varepsilon$. This identification is in fact a diffeomorphism from $B^{TY}_{y_0}(0, \varepsilon) \times B^N_{y_0}(0, \varepsilon)$ into an open neighborhood $\mathscr{W}_\varepsilon(y_0)$ of y_0 in X contained in $\mathscr{U}_\varepsilon$. In Section 8 e), $\mathscr{U}_\varepsilon$ is itself identified with an open set B_ε in $N_{\mathbf{R}}$. The image of $B^{TY}_{y_0}(0, \varepsilon) \times B^N_{y_0}(0, \varepsilon)$ under the previous diffeomorphism is exactly $\tilde{\pi}^{-1}(B^Y(y_0, \varepsilon)) \cap B_\varepsilon \subset N_{\mathbf{R}}$. In particular

$$\mathscr{W}_\varepsilon(y_0) \cap Y = B^{TY}_{y_0}(0, \varepsilon) \times \{0\}.$$

From now on, we use indifferently the notation $B^{TY}_{y_0}(0, \varepsilon) \times B^N_{y_0}(0, \varepsilon)$ or $\mathscr{W}_\varepsilon(y_0)$, y_0 or $0 \ldots$

Clearly there exists $\alpha_0(\varepsilon) > 0$ such that for any $y_0 \in Y$, $Z_0 \in N_{\mathbf{R}, y_0}$, $|Z_0| \leqslant \varepsilon/8$, the open Riemannian ball in X, $B^X(Z_0, \alpha_0(\varepsilon))$, is contained in $\mathscr{W}_{\varepsilon/2}(y_0)$. In particular $0 < \alpha_0(\varepsilon) \leqslant \varepsilon/2 \leqslant b/4$.

We now fix $\alpha \in]0, \alpha_0(\varepsilon)]$. Recall that the precise value of ε will be determined in Theorem 13.27.

Let $k''(Z)$ be the function defined on $\mathscr{W}_\varepsilon(y_0)$ by

(13.62) $$dv_X(Z) = k''(Z)\, dv_{TX}(Z).$$

Then $k''(0) = 1$.

If $Z \in (T_{\mathbf{R}} X)_{y_0}$, $Z = U + U'$, $U \in (T_{\mathbf{R}} Y)_{y_0}$, $U' \in N_{\mathbf{R}, y_0}$, $|U| < \varepsilon$, $|U'| < \varepsilon$, we identify TX_Z, $\Lambda(T^{*(0,1)}X)_Z$ (resp. ξ_Z) with TX_{y_0}, $\Lambda(T^{*(0,1)}X)_{y_0}$ (resp. ξ_{y_0}) by parallel transport with respect to the connection ${}^0\nabla^{TX}$ (resp. $\tilde{\nabla}^{\xi}$) along the curve $t \in [0, 1] \to 2tU$ $(0 \leqslant t \leqslant 1/2)$, $U + (2t-1)U'$ $(1/2 \leqslant t \leqslant 1)$. It is very important to observe that for $1/2 \leqslant t \leqslant 1$, parallel transport with respect to the connection ${}^0\nabla^{TX}$ coincides with parallel transport with respect to the connection ∇^{TX}. Also note that under the identification of TX_Z with TX_{y_0}, TX_Z^1, TX_Z^2 are respectively identified with TY_{y_0}, N_{y_0}.

Let Γ_Z^{TX}, ${}^0\Gamma_Z^{TX}$, Γ_Z^{ξ}, $\tilde{\Gamma}_Z^{\xi}$ be the connection forms of the connections ∇^{TX}, ${}^0\nabla^{TX}$, ∇^{ξ}, $\tilde{\nabla}^{\xi}$ in the considered trivializations. By (8.33), (13.53), we know that

$$\begin{aligned} \Gamma_Z^{TX} &= {}^0\Gamma_Z^{TX} + A'_Z, \\ \Gamma_Z^{\xi} &= \tilde{\Gamma}_Z^{\xi} + B_Z. \end{aligned} \tag{13.63}$$

By [ABoP, Proposition 3.7], we know that

$$\begin{aligned} {}^0\Gamma_Z^{TX}(U) = \frac{1}{2}({}^0\nabla^{TX})^2_{y_0}(Z, U) \\ + O(|Z|^2)\,U \quad \text{if } Z, U \in (T_{\mathbf{R}} Y)_{y_0} \text{ or if } Z, U \in N_{\mathbf{R}, y_0}. \end{aligned} \tag{13.64}$$

By construction

$$\begin{aligned} {}^0\Gamma_Z^{TX}(U) &= 0 \quad \text{if } Z \in (T_{\mathbf{R}} Y)_{y_0}, \quad U \in N_{\mathbf{R}, y_0}; \\ \tilde{\Gamma}_Z^{\xi}(U) &= 0 \quad \text{if } Z \in (T_{\mathbf{R}} Y)_{y_0}, \quad U \in N_{\mathbf{R}, y_0}. \end{aligned} \tag{13.65}$$

Also by using the definition of curvature, we find easily that

$$ {}^0\Gamma_Z^{TX}(U) = ({}^0\nabla^{TX})^2_{y_0}(Z, U) + O(|Z|^2)\,U \quad \text{if } Z \in N_{\mathbf{R}, y_0}, \quad U \in (T_{\mathbf{R}} Y)_{y_0}. \tag{13.66}$$

Remark **13.13.** – The trivialization of ξ is imposed by the structure of the problem. However instead of the considered trivialization of TX, $\Lambda(T^{*(0,1)}X)$, we might as well trivialize TX, $\Lambda(T^{*(0,1)}X)$ by identifying $(TX)_Z$, $\Lambda(T^{*(0,1)}X)_Z$ with $(TX)_{y_0}$, $\Lambda(T^{*(0,1)}X)_{y_0}$ by parallel transport along the curve $t \in [0, 1] \to tZ$ with respect to the connection ${}^0\nabla^{TX}$. If ${}^0\Gamma_Z'^{TX}$ denotes the connection form of ${}^0\nabla^{TX}$ in this new trivialization, by [ABoP, Proposition 3.7]

$$ {}^0\Gamma_Z'^{TX} = \frac{1}{2}({}^0\nabla^{TX})^2_{y_0}(Z, \,.\,) + O(|Z|^2). \tag{13.67}$$

The main justification for the choice of the trivializations of $\Lambda(T^{*(0,1)}X)$ and ξ which were described before is that they are compatible with the trivializations of $\Lambda(T^{*(0,1)}X)$ and ξ near Y considered in Section 8 g). In fact the only new ingredient with respect to Section 8 g), is that, for $y \in Y$ near y_0, the fibres $\Lambda(T^{*(0,1)}X)_y$, ξ_y have been identified with the fibres $\Lambda(T^{*(0,1)}X)_{y_0}$, ξ_{y_0}.

f) Replacing the manifold X by $(T_{\mathbf{R}}X)_{y_0}$.

We use the trivialization of $\Lambda(T^{*(0,1)}X)\hat{\otimes}\xi$ described in Section 13 e). In particular the operators D^X, V now act on the set of smooth sections of $(\Lambda(T^{*(0,1)}X)\hat{\otimes}\xi)_{y_0}$ over $\mathscr{W}_\varepsilon(y_0)$.

If $U \in (T_{\mathbf{R}}X)_{y_0}$, $Z \in \mathscr{W}_\varepsilon(y_0)$, let ${}^0\tau U(Z)$ be the parallel transport of U with respect to the connection ${}^0\nabla^{TX}$ along the curve $t \in [0,1] \to 2t P^{TY} Z$, $0 \leqslant t \leqslant 1/2$, $P^{TY} Z + (2t-1) P^N Z$, $1/2 \leqslant t \leqslant 1$.

Let $e_1, \ldots, e_{2l}$ be an orthonormal base of $(T_{\mathbf{R}}X)_{y_0}$. Then by Proposition 8.5, we find that

$$D^X = \sum_1^{2l} \frac{c(e_i)}{\sqrt{2}} \nabla^X_{{}^0\tau e_i(Z)}. \tag{13.68}$$

Recall that b is the injectivity radius of Y. Let $a \in \mathbf{R} \to \gamma(a) \in [0,1]$ be the function considered in (9.1). If $y_0 \in Y$, $U \in (T_{\mathbf{R}}Y)_{y_0}$, set

$$\mu(U) = \gamma\left(\frac{4|U|}{3b}\right). \tag{13.69}$$

Then

$$\begin{aligned} \mu(U) &= 1 \quad \text{if} \quad |U| \leqslant \frac{3b}{8}, \\ &= 0 \quad \text{if} \quad |U| \geqslant \frac{3b}{4}. \end{aligned} \tag{13.70}$$

Let Δ^{TY} be the Euclidean Laplacian on $(T_{\mathbf{R}}Y)_{y_0}$.

Definition **13.14.** – Let L be the differential operator on $(T_{\mathbf{R}}X)_{y_0}$

$$L = (1 - \mu^2(P^{TY}Z))\Delta^{TY} + \mu^2(P^{TY}Z)\sum_1^{2l'} \nabla^2_{{}^0\tau e_i(P^{TY}Z)}. \tag{13.71}$$

Recall that $\xi^-_{y_0} = (\Lambda N^* \otimes \eta)_{y_0}$. If $e_{2l'+1}, \ldots, e_{2l}$ is an orthonormal base of $N_{\mathbf{R}, y_0}$, let $S \in \operatorname{End}(\Lambda(T^{*(0,1)}X)\hat{\otimes}\xi^-)_{y_0}$ be given by

$$S = \frac{\sqrt{-1}}{2} \sum_{2l'+1}^{2l} c(e_i)\hat{c}(J e_i). \tag{13.72}$$

Let $(a, b) \in \mathbf{R}^2 \to \kappa(a, b) \in [0, 1]$ be a smooth function such that

$$\begin{aligned} \kappa(a, b) &= 1 \quad \text{if} \quad |a| \leqslant \frac{1}{2}, \quad |b| \leqslant \frac{1}{2} \\ &= 0 \quad \text{if} \quad |a| \geqslant \frac{3}{4}, \quad \text{or } |b| \geqslant \frac{3}{4}. \end{aligned} \tag{13.73}$$

If $Z \in (T_{\mathbf{R}} X)_{y_0}$, set

$$\varphi(Z) = \kappa\left(\frac{|P^{TY} Z|}{\varepsilon}, \frac{|P^N Z|}{\varepsilon}\right). \tag{13.74}$$

Then

$$\begin{aligned} \varphi(Z) &= 1 \quad \text{if} \quad |P^{TY} Z| \leqslant \frac{\varepsilon}{2}, \quad |P^N Z| \leqslant \frac{\varepsilon}{2}, \\ &= 0 \quad \text{if} \quad |P^{TY} Z| \geqslant \frac{3\varepsilon}{4}, \quad \text{or } |P^N Z| \geqslant \frac{3\varepsilon}{4}. \end{aligned} \tag{13.75}$$

We still define the vector space $\mathbf{H}_{y_0}$ as in Definition 11.17. Let Δ^N be the Laplacian on $N_{\mathbf{R}, y_0}$.

Definition **13.15.** – For $u > 0$, $T > 0$, $y_0 \in Y$, let $\mathscr{L}_{u,T}^{1, y_0}$, $\mathscr{M}_{u,T}^{1, y_0}$ be the operators acting on $\mathbf{H}_{y_0}$

$$\begin{aligned} \mathscr{L}_{u,T}^{1, y_0} &= (1 - \varphi^2(Z)) \left(\frac{-u^2}{2} (L + \Delta^N) + TP^{\xi^-} SP^{\xi^-} \right. \\ &\quad \left. + \frac{T^2}{u^2} \left(P^{\xi^+} + \frac{|P^N Z|^2}{2} P^{\xi^-} \right) \right) + \varphi^2(Z) \left(u D^X + \frac{T}{u} V(Z) \right)^2, \\ \mathscr{M}_{u,T}^{1, y_0} &= -(1 - \varphi^2(Z)) \frac{u^2}{2} (L + \Delta^N) + \varphi^2(Z) (u D^X)^2. \end{aligned} \tag{13.76}$$

For $\delta > 0$, $A > 0$, let U, Γ be the subsets of $\mathbf{C}$

$$\begin{aligned} U &= \{\lambda \in \mathbf{C};\ \operatorname{Re}(\lambda) \leqslant \delta \operatorname{Im}^2(\lambda) - A\}, \\ \Gamma &= \{\lambda \in \mathbf{C};\ \operatorname{Re}(\lambda) = \delta \operatorname{Im}^2(\lambda) - A\}. \end{aligned} \tag{13.77}$$

By proceeding as in the proof of Theorem 11.27, we find that for given $u \in]0, 1]$, $T > 0$, if δ is small enough, and if A is large enough, if $\lambda \in U$, the resolvent $(\lambda - \mathscr{L}_{u,T}^{1, y_0})^{-1}$ exists.

Moreover by Proposition 13.10, the function $\tilde{F}_u(a)$ and its derivatives decay as $a \in \Gamma \to +\infty$ faster than any $|a|^{-m}$. Therefore if $\delta > 0$, $A > 0$ are chosen as before, we may define the operator $\tilde{F}_u(\mathscr{L}^{1,y_0}_{u,T})$ by the formula

$$\tilde{F}_u(\mathscr{L}^{1,y_0}_{u,T}) = \frac{1}{2\pi i}\int_\Gamma \tilde{F}_u(\lambda)(\lambda - \mathscr{L}^{1,y_0}_{u,T})^{-1}\, d\lambda. \tag{13.78}$$

Let $\tilde{F}_u(\mathscr{L}^{1,y_0}_{u,T})(Z, Z')$, $(Z, Z' \in (T_{\mathbf{R}}X)_{y_0})$ be the smooth kernel associated with the operator $\tilde{F}_u(\mathscr{L}^{1,y_0}_{u,T})$ with respect to the measure $dv_{TX}(Z')/(2\pi)^{\dim X}$.

Recall that the function k'' was defined in (13.62). Also if $Z_0 \in N_{\mathbf{R}, y_0}$, $|Z_0| \leqslant \varepsilon/8$, by construction $B^X(Z_0, \alpha) \subset \mathscr{W}_{\varepsilon/2}(y_0)$, so that φ^2 is equal to 1 on $B^X(Z_0, \alpha)$. By using finite propagation speed as in Section 13 b), we find that if $Z_0 \in N_{\mathbf{R}, y_0}$, $|Z_0| \leqslant \varepsilon/8$, then

$$\tilde{F}_u\left(\left(u D^X + \frac{T}{u} V\right)^2\right)((y_0, Z_0), (y_0, Z_0))\, k''(Z_0) = \tilde{F}_u(\mathscr{L}^{1,y_0}_{u,T})(Z_0, Z_0). \tag{13.79}$$

g) Rescaling of the variable Z and of the horizontal Clifford variables.

For $u > 0$, $T > 0$, let $G_{u,T}$ be the linear map $h \in \mathbf{H}_{y_0} \to G_{u,T} h \in \mathbf{H}_{y_0}$ with

$$G_{u,T}\, h(Z) = h\left(\frac{P^{TY} Z}{u} + \frac{\sqrt{T}\, P^N Z}{u}\right). \tag{13.80}$$

Set

$$\begin{aligned} \mathscr{L}^{2,y_0}_{u,T} &= G^{-1}_{u,T}\, \mathscr{L}^{1,y_0}_{u,T}\, G_{u,T}, \\ \mathscr{M}^{2,y_0}_{u,T} &= G^{-1}_{u,T}\, \mathscr{M}^{1,y_0}_{u,T}\, G_{u,T}. \end{aligned} \tag{13.81}$$

By proceeding as in Section 11 g), we know that $\mathscr{L}^{2,y_0}_{u,T}$, $\mathscr{M}^{2,y_0}_{u,T}$ lie in $(c(T_{\mathbf{R}}X) \hat{\otimes} \mathrm{End}(\xi))_{y_0} \otimes \mathrm{Op}$.

Let $e_1, \ldots, e_{2l'}$ be an orthonormal oriented base of $(T_{\mathbf{R}}Y)_{y_0}$, let $e_{2l'+1}, \ldots, e_{2l}$ be an orthonormal oriented base of $N_{\mathbf{R}, y_0}$. Let $e^1, \ldots, e^{2l'}$ and $e^{2l'+1}, \ldots, e^{2l}$ be the corresponding dual bases of $(T^*_{\mathbf{R}}Y)_{y_0}$ and $(N^*_{\mathbf{R}})_{y_0}$.

Recall that the vector spaces $\mathbf{K}_{y_0}$, $\mathbf{K}^{\pm}_{y_0}$ were defined in Definition 12.7. Also for $1 \leqslant i \leqslant 2l'$, the operators $c_u(e_i)$ were defined in Definition 12.8.

Definition **13.16.** – Let $\mathscr{L}^{3,y_0}_{u,T}$, $\mathscr{M}^{3,y_0}_{u,T} \in \mathrm{End}(\mathbf{K}_{y_0})$ be the operators obtained from $\mathscr{L}^{2,y_0}_{u,T}$, $\mathscr{M}^{2,y_0}_{u,T}$ be replacing the Clifford variables $c(e_i)$ by $c_u(e_i)$ for $1 \leqslant i \leqslant 2l'$, while leaving unchanged the operators $c(e_i)$ for $2l'+1 \leqslant i \leqslant 2l$.

Let $\tilde{F}_u(\mathscr{L}^{3,y_0}_{u,T})(Z, Z')$, $(Z, Z' \in (T_{\mathbf{R}}X)_{y_0})$ be the smooth kernel associated with the operator $\tilde{F}_u(\mathscr{L}^{3,y_0}_{u,T})$ with respect to the volume $dv_{TX}(Z')/(2\pi)^{\dim X}$.

We still define $[\tilde{F}_u(\mathscr{L}^{3,y_0}_{u,T})(Z, Z')]^{\max}$ as in (12.31).

Proposition **13.17.** — *For any* $u>0$, $T>0$, $y_0 \in Y$, $Z_0 \in N_{\mathbf{R}, y_0}$, $|Z_0| \leqslant \varepsilon \sqrt{T}/8u$, *the following identity holds*

$$\left(\frac{u}{\sqrt{T}}\right)^{2\dim N} \operatorname{Tr}_s\left[N_H F_u\left(u D^X + \frac{T}{u} V\right)\left(\left(y_0, \frac{u Z_0}{\sqrt{T}}\right), \left(y_0, \frac{u Z_0}{\sqrt{T}}\right)\right)\right] k''\left(\frac{u Z_0}{\sqrt{T}}\right) = (-i)^{\dim Y} \operatorname{Tr}_s[N_H [\tilde{F}_u(\mathscr{L}^{3, y_0}_{u, T})(Z_0, Z_0)]^{\max}]. \tag{13.82}$$

Proof. — We start from the identity (13.79) and we proceed as in the proof of Proposition 12.9. □

h) A formula for the operator $\mathscr{L}^{3, y_0}_{u, T}$.

Observe that by (9.50), (13.76), we have the identity

$$\begin{aligned}\mathscr{L}^{3, y_0}_{u, T} = {} & \mathscr{M}^{3, y_0}_{u, T} + \varphi^2\left(u P^{TY} Z + \frac{u}{\sqrt{T}} P^N Z\right) \\ & \left\{\frac{T}{u} \sum_1^{2l'}\left(e^i \wedge - \frac{u^2}{2} i_{e_i}\right)(\nabla^{\xi}_{{}^0\tau e_i} V)\left(u P^{TY} Z + \frac{u}{\sqrt{T}} P^N Z\right)\right. \\ & + T \sum_{2l'+1}^{2l} \frac{c(e_i)}{\sqrt{2}} (\nabla^{\xi}_{{}^0\tau e_i} V)\left(u P^{TY} Z + \frac{u}{\sqrt{T}} P^N Z\right) \\ & \left. + \frac{T^2}{u^2} V^2\left(u P^{TY} Z + \frac{u}{\sqrt{T}} P^N Z\right)\right\} \\ & + \left(1 - \varphi^2\left(u P^{TY} Z + \frac{u}{\sqrt{T}} P^N Z\right)\right) \\ & \left(T P^{\xi^-} S P^{\xi^-} + \frac{T^2}{u^2} P^{\xi^+} + T \frac{|P^N Z|^2}{2} P^{\xi^-}\right).\end{aligned} \tag{13.83}$$

Let us recall that if g is a smooth function from $\mathbf{R}^k$ into $\mathbf{R}$, then

$$g(x) - g(0) = \sum_1^k x^i \int_0^1 \frac{\partial g}{\partial x^i}(tx)\, dt. \tag{13.84}$$

We will write (13.84) in the form

$$g(x)-g(0)=\langle x, [g](x)\rangle. \tag{13.85}$$

In particular, $[g]$ (0) is the derivative of g at 0.

Then if h is a smooth function defined on $(\mathrm{T}_{\mathbf{R}}\mathrm{X})_{y_0}$ with values in $\mathbf{R}$, we use the notation

$$h\left(u\mathrm{P}^{\mathrm{TY}}\mathrm{Z}+\frac{u}{\sqrt{\mathrm{T}}}\mathrm{P}^{\mathrm{N}}\mathrm{Z}\right)-h(u\mathrm{P}^{\mathrm{TY}}\mathrm{Z}) = \frac{u}{\sqrt{\mathrm{T}}}\left\langle \mathrm{P}^{\mathrm{N}}\mathrm{Z}, [h]\left(u\mathrm{P}^{\mathrm{TY}}\mathrm{Z}+\frac{u}{\sqrt{\mathrm{T}}}\mathrm{P}^{\mathrm{N}}\mathrm{Z}\right)\right\rangle. \tag{13.86}$$

If $\mathrm{Z}\in(\mathrm{T}_{\mathbf{R}}\mathrm{X})_{y_0}$, let ∇_{Z} be the ordinary differentiation operator in the direction Z acting on $\mathbf{K}_{y_0}$.

Theorem **13.18.** — *The following identity holds*

$$\begin{aligned}\mathscr{M}^{3,y_0}_{u,\mathrm{T}} = & -\frac{1}{2}\left(1-\varphi^2\left(u\mathrm{P}^{\mathrm{TY}}\mathrm{Z}+\frac{u}{\sqrt{\mathrm{T}}}\mathrm{P}^{\mathrm{N}}\mathrm{Z}\right)\right)(\mathrm{L}_{u\mathrm{P}^{\mathrm{TY}}\mathrm{Z}}+\mathrm{T}\Delta^{\mathrm{N}})\\ & +\varphi^2\left(u\mathrm{P}^{\mathrm{TY}}\mathrm{Z}+\frac{u}{\sqrt{\mathrm{T}}}\mathrm{P}^{\mathrm{N}}\mathrm{Z}\right)\\ & \left\{-\frac{1}{2}\sum_{i=1}^{2l}\left(\sqrt{\mathrm{T}}\,\nabla_{\mathrm{P}^{\mathrm{N}}\,{}^0\tau e_i(u\mathrm{P}^{\mathrm{TY}}\mathrm{Z}+(u/\sqrt{\mathrm{T}})\mathrm{P}^{\mathrm{N}}\mathrm{Z})}+\nabla_{\mathrm{P}^{\mathrm{TY}}\,{}^0\tau e_i(u\mathrm{P}^{\mathrm{TY}}\mathrm{Z}+(u/\sqrt{\mathrm{T}})\mathrm{P}^{\mathrm{N}}\mathrm{Z})}\right.\right.\\ & +\frac{1}{2}\sum_{1\leqslant j,k\leqslant 2l'}\left\langle\left\langle \mathrm{P}^{\mathrm{TY}}\mathrm{Z}+\frac{\mathrm{P}^{\mathrm{N}}\mathrm{Z}}{\sqrt{\mathrm{T}}}, [{}^0\Gamma^{\mathrm{TX}}({}^0\tau e_i)]\left(u\mathrm{P}^{\mathrm{TY}}\mathrm{Z}\right.\right.\right.\\ & \left.\left.\left.+\frac{u}{\sqrt{\mathrm{T}}}\mathrm{P}^{\mathrm{N}}\mathrm{Z}\right)\right\rangle e_j, e_k\right\rangle\left(e^j\wedge-\frac{u^2}{2}i_{e_j}\right)\left(e^k\wedge-\frac{u^2}{2}i_{e_k}\right)\\ & +\frac{u}{4}\sum_{2l'+1\leqslant j,k\leqslant 2l}\left\langle {}^0\Gamma^{\mathrm{TX}}({}^0\tau e_i)\left(u\mathrm{P}^{\mathrm{TY}}\mathrm{Z}\right.\right.\\ & \left.\left.+\frac{u}{\sqrt{\mathrm{T}}}\mathrm{P}^{\mathrm{N}}\mathrm{Z}\right)e_j, e_k\right\rangle c(e_j)c(e_k)\\ & +\frac{1}{\sqrt{2}}\sum_{\substack{1\leqslant j\leqslant 2l'\\ 2l'+1\leqslant k\leqslant 2l}}\left\langle \mathrm{A}'({}^0\tau e_i)\left(u\mathrm{P}^{\mathrm{TY}}\mathrm{Z}+\frac{u}{\sqrt{\mathrm{T}}}\mathrm{P}^{\mathrm{N}}\mathrm{Z}\right)e_j, e_k\right\rangle\left(e^j\wedge-\frac{u^2}{2}i_{e_j}\right)c(e_k)\end{aligned} \tag{13.87}$$

$$+u\left(\Gamma^{\xi}+\frac{1}{2}\operatorname{Tr}[{}^{0}\Gamma^{\mathrm{TX}}]\right)({}^{0}\tau e_i)\left(u\mathrm{P}^{\mathrm{TY}}\mathrm{Z}+\frac{u}{\sqrt{\mathrm{T}}}\mathrm{P}^{\mathrm{N}}\mathrm{Z}\right)\Bigg)^2$$

$$+\Bigg[\frac{1}{2}u\Big(\sqrt{\mathrm{T}}\,\nabla_{\mathrm{P}^{\mathrm{N}}\sum_{i=1}^{2l}\nabla^{\mathrm{TX}}_{{}^{0}\tau e_i}{}^{0}\tau e_i}+\nabla_{\mathrm{P}^{\mathrm{TY}}\sum_{i=1}^{2l}\nabla^{\mathrm{TX}}_{{}^{0}\tau e_i}{}^{0}\tau e_i}\Big)$$

$$+\frac{1}{4}\sum_{1\leqslant j,k\leqslant 2l'}\left\langle \Gamma^{\mathrm{TX}}\left(\sum_{i=1}^{2l}\nabla^{\mathrm{TX}}_{{}^{0}\tau e_i}{}^{0}\tau e_i\right)e_j, e_k\right\rangle$$

$$\left(e^j\wedge-\frac{u^2}{2}i_{e_j}\right)\left(e^k\wedge-\frac{u^2}{2}i_{e_k}\right)$$

$$+\frac{u^2}{8}\sum_{2l'+1\leqslant j,k\leqslant 2l}\left\langle \Gamma^{\mathrm{TX}}\left(\sum_{i=1}^{2l}\nabla^{\mathrm{TX}}_{{}^{0}\tau e_i}{}^{0}\tau e_i\right)e_j, e_k\right\rangle c(e_j)c(e_k)$$

$$+\frac{u}{2\sqrt{2}}\sum_{\substack{1\leqslant j\leqslant 2l'\\ 2l'+1\leqslant k\leqslant 2l}}\left\langle \Gamma^{\mathrm{TX}}\left(\sum_{i=1}^{2l}\nabla^{\mathrm{TX}}_{{}^{0}\tau e_i}{}^{0}\tau e_i\right)e_j, e_k\right\rangle$$

$$\left(e^j\wedge-\frac{u^2}{2}i_{e_j}\right)c(e_k)+\frac{u^2}{8}\mathrm{K}$$

$$+\frac{1}{2}\sum_{1\leqslant j,k\leqslant 2l'}\left(e^j\wedge-\frac{u^2}{2}i_{e_j}\right)\left(e^k\wedge-\frac{u^2}{2}i_{e_k}\right)$$

$$\left((\nabla^{\xi})^2+\frac{1}{2}\operatorname{Tr}[(\nabla^{\mathrm{TX}})^2]\right)({}^{0}\tau e_j, {}^{0}\tau e_k)$$

$$+\frac{u^2}{4}\sum_{2l'+1\leqslant j,k\leqslant 2l}c(e_j)c(e_k)\left((\nabla^{\xi})^2+\frac{1}{2}\operatorname{Tr}[(\nabla^{\mathrm{TX}})^2]\right)({}^{0}\tau e_j, {}^{0}\tau e_k)$$

$$+\frac{u}{\sqrt{2}}\sum_{\substack{1\leqslant j\leqslant 2l'\\ 2l'+1\leqslant k\leqslant 2l}}\left(e^j\wedge-\frac{u^2}{2}i_{e_j}\right)c(e_k)$$

$$\left((\nabla^{\xi})^2+\frac{1}{2}\operatorname{Tr}[(\nabla^{\mathrm{TX}})^2]\right)({}^{0}\tau e_j, {}^{0}\tau e_k)\Bigg]\left(u\mathrm{P}^{\mathrm{TY}}\mathrm{Z}+\frac{u}{\sqrt{\mathrm{T}}}\mathrm{P}^{\mathrm{N}}\mathrm{Z}\right)\Bigg\}.$$

Proof. – Recall that on $\mathscr{W}_{\varepsilon}(y_0)$, TX has been identified with $(\mathrm{TX})_{y_0}$. Under this identification the connection ${}^{0}\nabla^{\mathrm{TX}}$ preserves the splitting $(\mathrm{TX})_{y_0}=(\mathrm{TY})_{y_0}\oplus \mathrm{N}_{y_0}$. Equivalently the connection form ${}^{0}\Gamma^{\mathrm{TX}}$ preserve this splitting. On the other hand, the 1-form A′ exchanges $(\mathrm{TY})_{y_0}$ and N_{y_0}.

If $C \in \mathrm{End}(TX)_{y_0}$ is skew-adjoint, the action $C^\wedge$ of C on $(\Lambda(T^{*(0,1)}X) \hat{\otimes} \xi)_{y_0}$ is given by

$$C^\wedge = \frac{1}{4} \sum_{1 \leqslant i, j \leqslant 2l} \langle C e_i, e_j \rangle c(e_i) c(e_j) + \frac{1}{2} \mathrm{Tr}[C]. \tag{13.88}$$

From (13.88), we deduce that the action of Γ^{TX} on $(\Lambda(T^{*(0,1)}X) \hat{\otimes} \xi)_{y_0}$ is given by

$$\begin{aligned} &\frac{1}{4} \sum_{1 \leqslant j, k \leqslant 2l'} \langle {}^0\Gamma^{TX} e_j, e_k \rangle c(e_j) c(e_k) \\ &+ \frac{1}{4} \sum_{2l'+1 \leqslant j, k \leqslant 2l} \langle {}^0\Gamma^{TX} e_j, e_k \rangle c(e_j) c(e_k) \\ &+ \frac{1}{2} \sum_{\substack{1 \leqslant j \leqslant 2l' \\ 2l'+1 \leqslant k \leqslant 2l}} \langle A' e_j, e_k \rangle c(e_j) c(e_k) + \frac{1}{2} \mathrm{Tr}[{}^0\Gamma^{TX}]. \end{aligned} \tag{13.89}$$

Also since ${}^0\Gamma_0^{TX} = 0$, by (13.86), we have the identity

$$\begin{aligned} &{}^0\Gamma^{TX}_{uP^{TY}Z + u/\sqrt{T} P^N Z} \\ &= \left\langle uP^{TY}Z + \frac{u}{\sqrt{T}} P^N Z, [{}^0\Gamma^{TX}]\left(uP^{TY}Z + \frac{u}{\sqrt{T}} P^N Z\right)\right\rangle. \end{aligned} \tag{13.90}$$

Then (13.87) follows from Proposition 11.4 and from (13.76), (13.88)-(13.90). □

Theorem **13.19.** — *For any* $y_0 \in Y$, $u \in]0, 1]$, $Z \in (T_{\mathbf{R}}X)_{y_0}$ *such that* $|P^{TY}Z| \leqslant 3\varepsilon/4u$, *as* $T \to +\infty$

$$\begin{aligned} &\frac{T}{u} \sum_1^{2l'} \left(e^i \wedge - \frac{u^2}{2} i_{e_i}\right) (\nabla^\xi_{{}^0\tau e_i} V)\left(uP^{TY}Z + \frac{u}{\sqrt{T}} P^N Z\right) \\ &= \frac{T}{u} \sum_1^{2l'} \left(e^i \wedge - \frac{u^2}{2} i_{e_i}\right) (\nabla^\xi_{{}^0\tau e_i} V)(uP^{TY}Z) \\ &+ \sqrt{T} \sum_1^{2l'} \left(e^i \wedge - \frac{u^2}{2} i_{e_i}\right) (\tilde{\nabla}^\xi_{P^N Z} \nabla^\xi_{\tau e_i} V)(uP^{TY}Z) + O(u|P^N Z|^2), \\ &T \sum_{2l'+1}^{2l} \frac{c(e_i)}{\sqrt{2}} (\nabla^\xi_{{}^0\tau e_i} V)\left(uP^{TY}Z + \frac{u}{\sqrt{T}} P^N Z\right) \\ &= T \sum_{2l'+1}^{2l} \frac{c(e_i)}{\sqrt{2}} (\nabla^\xi_{{}^0\tau e_i} V)(uP^{TY}Z) + u\sqrt{T} \sum_{2l'+1}^{2l} \frac{c(e_i)}{\sqrt{2}} \end{aligned} \tag{13.91}$$

$$\tilde{\nabla}^{\xi}_{\mathrm{P^N Z}} \nabla^{\xi}_{{}^0\tau e_i} \mathrm{V}\,(u\,\mathrm{P^{TY} Z}) + O(|u\,\mathrm{P^N Z}|^2),$$

$$\frac{\mathrm{T}^2}{u^2}(\mathrm{V}^-)^2\left(u\,\mathrm{P^{TY}Z} + \frac{u}{\sqrt{\mathrm{T}}}\mathrm{P^N Z}\right) = \mathrm{T}\frac{|\mathrm{P^N Z}|^2}{2}\mathrm{P}^{\xi^-} + \frac{u\sqrt{\mathrm{T}}}{2}$$

$$\left[\frac{\sqrt{-1}}{\sqrt{2}}\hat{c}(\mathrm{J\,P^N Z}),\ \tilde{\nabla}^{\xi}_{\mathrm{P^N Z}}\tilde{\nabla}^{\xi}_{\mathrm{P^N Z}}\mathrm{V}^-(u\,\mathrm{P^{TY}Z})\right] + O(u^2\,|\mathrm{P^N Z}|^4).$$

Moreover for any $y_0 \in \mathrm{Y}$, $u \in]0, 1]$, $\mathrm{Z} \in (\mathrm{T_{\mathbf{R}} X})_{y_0}$ such that $|\mathrm{P^{TY} Z}| \leqslant 3\,\varepsilon/4\,u$, the following identities hold

$$\text{(13.92)}\qquad \begin{aligned} &\mathrm{P}^{\xi^-}\sum_{1}^{2l'}\left(e^i \wedge -\frac{u^2}{2}i_{e_i}\right)(\tilde{\nabla}^{\xi}_{\mathrm{P^N Z}}\nabla^{\xi}_{{}^0\tau e_i}\mathrm{V})\,(u\,\mathrm{P^{TY}Z})\,\mathrm{P}^{\xi^-} = 0\\ &\mathrm{P}^{\xi^-}\sum_{2l'+1}^{2l}\frac{c(e_i)}{\sqrt{2}}(\nabla^{\xi}_{{}^0\tau e_i}\mathrm{V})\,(u\,\mathrm{P^{TY}Z})\,\mathrm{P}^{\xi^-} = \mathrm{P}^{\xi^-}\mathrm{S}_{y_0}\mathrm{P}^{\xi^-}.\end{aligned}$$

Proof. — Since the vector bundle ξ is trivialized along the curve $t \to u\,\mathrm{P^{TY}Z} + t\,\mathrm{P^N Z}$ by parallel transport with respect to the connection $\tilde{\nabla}^{\xi}$, the identities (13.91) follow by Taylor expansion.

By Proposition 8.13, we find that

$$\text{(13.93)}\qquad \mathrm{P}^{\xi^-}\sum_{2l'+1}^{2l}\frac{c(e_i)}{\sqrt{2}}(\nabla^{\xi}_{{}^0\tau e_i}\mathrm{V})\,(u\,\mathrm{P^{TY}Z})\,\mathrm{P}^{\xi^-} = \mathrm{P}^{\xi^-}\frac{\sqrt{-1}}{2}\sum_{2l'+1}^{2l}c(e_i)\,\hat{c}(\mathrm{J}\,{}^0\tau e_i)\,(u\,\mathrm{P^{TY}Z})\,\mathrm{P}^{\xi^-}.$$

For $2\,l'+1 \leqslant i \leqslant 2\,l$, ${}^0\tau e_i(u\,\mathrm{P^{TY}Z}) \in \mathrm{N}_{\mathbf{R},\,u\mathrm{P^{TY}Z}}$ is the parallel transport of e_i with respect to the connection ∇^{N} along the curve $t \in [0, 1] \to tu\,\mathrm{P^{TY}Z}$. On the other hand, recall that $\xi^-_{u\mathrm{P^{TY}Z}}$ is identified with $\xi^-_{y_0}$ by parallel transport with respect to the connection $\tilde{\nabla}^{\xi^-}$ along the curve $t \in [0, 1] \to tu\,\mathrm{P^{TY}Z}$. Finally by (8.31), we have the identification of holomorphic Hermitian vector bundles $\xi^-|_{\mathrm{Y}} \simeq \Lambda\mathrm{N}^* \otimes \eta$. It then follows that, under the identification $\xi^-_{u\mathrm{P^{TY}Z}} \simeq \xi^-_{y_0}$, for $2\,l'+1 \leqslant i \leqslant 2\,l$, $\hat{c}(\mathrm{J}\,{}^0\tau e_i)$ is identified with $\hat{c}(\mathrm{J}\,e_i)$. Using (13.72), (13.93), we get the second identity in (13.92).

Take now $i \in \{1, \ldots, 2\,l'\}$. If $|\mathrm{P^{TY}Z}| < 3\,\varepsilon/4$, then by construction, ${}^0\tau e_i(\mathrm{P^{TY}Z}) \in (\mathrm{T_{\mathbf{R}}Y})_{\mathrm{P^{TY}Z}}$. We can then use formula (12.54) with Z replaced by $\mathrm{P^N Z}$, and we find that

$$\text{(13.94)}\qquad \mathrm{P}^{\xi^-}\tilde{\nabla}^{\xi}_{\mathrm{P^N Z}}\nabla^{\xi}_{{}^0\tau e_i}\mathrm{V}\,(u\,\mathrm{P^{TY}Z})\,\mathrm{P}^{\xi^-} = \mathrm{P}^{\xi^-}\frac{\sqrt{-1}}{\sqrt{2}}\hat{c}(\mathrm{J\,P^N}\nabla^{\mathrm{TX}}_{\mathrm{P^N Z}}\,{}^0\tau e_i)\,\mathrm{P}^{\xi^-}.$$

Now by construction

$$\nabla^{TX}_{P^N Z}\, {}^0\tau e_i = 0. \tag{13.95}$$

The first identity in (13.92) follows from (13.94), (13.95). □

i) The algebraic structure of the operator $\mathscr{L}^{3,y_0}_{u,T}$ as $u \to 0$.

We first describe the behaviour of the operator $\mathscr{L}^{3,y_0}_{u,T}$ as $u \to 0$. This is not directly related to the main purpose of this Section, which is to obtain some sort of uniformity in $u \in]0,1]$ as $T \to +\infty$. Still the obtained formulas will be illuminating, and will also fit nicely with the conjugation formulas of Theorem 5.6.

Recall that by (13.54), (13.55), A'_{y_0} vanishes on vectors of $N_{\mathbf{R},y_0}$ and coincides with A_{y_0} on vectors of $(T_{\mathbf{R}}Y)_{y_0}$. Also by (13.60), if $U, U' \in (T_{\mathbf{R}}Y)_{y_0}$

$$A_{y_0}(U)\,U' = A_{y_0}(U')\,U.$$

Using these two facts together with Proposition 13.12 and with the identities (13.64)-(13.66), (13.87), we find that as $u \to 0$, the operator $\mathscr{M}^{3,y_0}_{u,T}$ converges to an operator $\mathscr{M}^{3,y_0}_{0,T}$ given by the formula

$$\begin{aligned}\mathscr{M}^{3,y_0}_{0,T} = -\frac{1}{2}\sum_1^{2l}\Bigg(&\sqrt{T}\,\nabla_{P^N e_i} + \nabla_{P^{TY} e_i} \\ &+\frac{1}{2}\left\langle ((\nabla^{TX})^2_{y_0} - P^{TY} A^2_{y_0} P^{TY})\left(P^{TY} Z + \frac{P^N Z}{\sqrt{T}}\right), e_i\right\rangle \\ &+\frac{1}{2}\left\langle (\nabla^{TX})^2_{y_0}\frac{P^N Z}{\sqrt{T}}, P^{TY} e_i\right\rangle \\ &-\frac{1}{2}\langle (\nabla^{TX})^2_{y_0} P^{TY} Z, P^N e_i\rangle - \frac{c(AP^{TY} e_i)}{\sqrt{2}}\Bigg)^2 \\ &+ i^*\left((\nabla^{\xi})^2 + \frac{1}{2}\operatorname{Tr}[(\nabla^{TX})^2]\right)_{y_0}.\end{aligned} \tag{13.96}$$

Also as $u \to 0$

$$\frac{1}{u}\sum_1^{2l'}\left(e^i \wedge -\frac{u^2}{2} i_{e_i}\right)(\nabla^{\xi}_{{}^0\tau e_i} V)\left(u P^{TY} Z + \frac{u}{\sqrt{T}} P^N Z\right) \tag{13.97}$$

$$= \frac{1}{u} i^* (\nabla^\xi V)(y_0) + \sum_1^{2l'} e^i \wedge (\tilde{\nabla}^\xi_{P^{TY} Z + P^N Z/\sqrt{T}} \nabla^\xi_{{}^0\tau e_i} V)(y_0)$$
$$+ \frac{1}{u} O\left(u^2 \left(1 + |P^{TY} Z|^2 + \frac{|P^N Z|^2}{T}\right)\right).$$

In (13.94), (13.95), we saw that for $1 \leqslant i \leqslant 2l'$

$$\text{(13.98)} \qquad P^{\xi^-} (\tilde{\nabla}^\xi_{P^N Z} \nabla^\xi_{{}^0\tau e_i} V)(y_0) P^{\xi^-} = 0.$$

Also by formula (12.54), we find that

$$\text{(13.99)} \qquad P^{\xi^-} (\tilde{\nabla}^\xi_{P^{TY} Z} \nabla^\xi_{{}^0\tau e_i} V)(y_0) P^{\xi^-}$$
$$= P^{\xi^-} \frac{\sqrt{-1}}{\sqrt{2}} \hat{c} (-J A(e_i) P^{TY} Z + J P^N \nabla^{TX}_{P^{TY} Z} {}^0\tau e_i) P^{\xi^-}.$$

Now for $1 \leqslant i \leqslant 2l'$, if $Z \in (T_{\mathbf{R}} Y)_{y_0}$, $|Z| \leqslant \varepsilon$, $({}^0\tau e_i)(Z) \in (T_{\mathbf{R}} Y)_Z$. Therefore

$$\text{(13.100)} \qquad P^N \nabla^{TX}_{P^{TY} Z} {}^0\tau e_i (y_0) = A_{y_0} (P^{TY} Z) e_i = A(e_i) P^{TY} Z.$$

From (13.99), (13.100), we deduce that

$$\text{(13.101)} \qquad P^{\xi^-} (\tilde{\nabla}^\xi_{P^{TY} Z} \nabla^\xi_{{}^0\tau e_i} V)(y_0) P^{\xi^-} = 0.$$

Using (13.98), (13.101), we get

$$\text{(13.102)} \qquad P^{\xi^-} \sum_1^{2l'} e^i \wedge (\tilde{\nabla}^\xi_{P^{TY} Z + P^N Z/\sqrt{T}} \nabla^\xi_{{}^0\tau e_i} V)(y_0) P^{\xi^-} = 0.$$

Identity (13.96) now plays the role of identity (12.36). Identity (13.97) replaces the first identity in (12.42). The second identity in (12.43) is replaced by (13.102), the third identity in (12.43) by the last identity in (13.92).

For $u > 0$, let $\mathscr{D}^2_u$ be the operator defined in Theorem 5.6 associated to the exact sequence of holomorphic Hermitian vector bundles on Y $0 \to TY \to TX|_Y \to N \to 0$. Using the previous formulas, the second identity in (5.12), and proceeding as in Section 12, we find that for any $T > 0$

$$\text{(13.103)} \qquad \lim_{u \to 0} \operatorname{Tr}_s \left[N_H \exp\left(- \left(u D^X + \frac{T}{u} V \right)^2 \right) \right]$$
$$= \int_Y \Phi \operatorname{Tr}_s [N_H \exp(-G^{-1}_{1,T} \mathscr{D}^2_{T^2} G_{1,T})] \operatorname{ch}(\eta, g^\eta).$$

Clearly

$$\mathrm{Tr}_s[\mathrm{N_H}\exp(-\mathrm{G}_{1,\mathrm{T}}^{-1}\mathscr{D}_{\mathrm{T}^2}^2\mathrm{G}_{1,\mathrm{T}})]=\mathrm{Tr}_s[\mathrm{N_H}\exp(-\mathscr{D}_{\mathrm{T}^2}^2)].$$

Then (13.103) is equivalent to

$$\lim_{u\to 0}\mathrm{Tr}_s\left[\mathrm{N_H}\exp\left(-\left(u\mathrm{D}^{\mathrm{X}}+\frac{\mathrm{T}}{u}\mathrm{V}\right)^2\right)\right] = \int_{\mathrm{Y}}\Phi\,\mathrm{Tr}_s[\mathrm{N_H}\exp(-\mathscr{D}_{\mathrm{T}^2}^2)]\,\mathrm{ch}(\eta, g^{\eta}). \tag{13.104}$$

Of course (13.104) is compatible with Theorem 6.7, since as we saw in (5.15)

$$\mathrm{Tr}_s[\mathrm{N_H}\exp(-\mathscr{B}_{\mathrm{T}^2}^2)]=\mathrm{Tr}_s[\mathrm{N_H}\exp(-\mathscr{D}_{\mathrm{T}^2}^2)]. \tag{13.105}$$

The second identity in (5.11), which defines $\mathscr{D}_{\mathrm{T}^2}^2$ in terms of the operator $\mathscr{B}_{\mathrm{T}^2}^2$ should also have a clear interpretation. In fact the operators $\mathscr{B}_{\mathrm{T}^2}^2$ and $\mathscr{D}_{\mathrm{T}^2}^2$ are the "limits" as $u\to 0$ of the same family of operators calculated in two different trivializations. Although the gauge transformation which passes from one trivialization to the other tends to the identity as $u\to 0$, the effect of the Clifford rescaling inflates the gauge transformation to such a point it survives as a non trivial gauge transformation in the limit. This is the geometric interpretation of identity (5.11) in this very special situation.

j) The matrix structure of the operator $\mathscr{L}_{u,\mathrm{T}}^{3,y_0}$ as $\mathrm{T}\to+\infty$.

Definition **13.20.** – Let $\mathbf{F}_{y_0}$ (resp. $\mathbf{F}_{y_0}^0$) be the vector space of smooth (resp. square integrable) sections of $(\Lambda(\mathrm{T}_{\mathbf{R}}^*\mathrm{Y})\otimes\eta)_{y_0}$ over $(\mathrm{T}_{\mathbf{R}}\mathrm{Y})_{y_0}$. Let $\mathbf{K}_{y_0}^0$, $\mathbf{K}_{y_0}^{\pm,0}$ be the vector spaces of square integrable sections of $(\Lambda(\mathrm{T}_{\mathbf{R}}^*\mathrm{Y})\hat{\otimes}\Lambda(\bar{\mathrm{N}}^*)\otimes\xi)_{y_0}$, $(\Lambda(\mathrm{T}_{\mathbf{R}}^*\mathrm{Y})\hat{\otimes}\Lambda(\bar{\mathrm{N}}^*)\hat{\otimes}\xi^{\pm})_{y_0}$ over $(\mathrm{T}_{\mathbf{R}}\mathrm{X})_{y_0}$.

We equip $\mathbf{F}_{y_0}^0$ with the Hermitian product

$$\sigma,\sigma'\in\mathbf{F}^0\to\langle\sigma,\sigma'\rangle=\int_{(\mathrm{T}_{\mathbf{R}}\mathrm{Y})_{y_0}}\langle\sigma,\sigma'\rangle(\mathrm{Z})\,\frac{dv_{\mathrm{TY}}(\mathrm{Z})}{(2\pi)^{\dim\mathrm{Y}}}. \tag{13.106}$$

We equip $\mathbf{K}_{y_0}^0$ with the Hermitian product

$$s,s'\in\mathbf{K}_{y_0}^0\to\langle s,s'\rangle=\int_{(\mathrm{T}_{\mathbf{R}}\mathrm{X})_{y_0}}\langle s,s'\rangle(\mathrm{Z})\,\frac{dv_{\mathrm{TX}}(\mathrm{Z})}{(2\pi)^{\dim\mathrm{X}}}. \tag{13.107}$$

We now use the notation of Sections 7, 8a) and 8i). In particular θ_{y_0} denotes the Kähler form of the fiber $N_{\mathbf{R}, y_0}$. Set for $Z \in (T_{\mathbf{R}} X)_{y_0}$

$$\beta_{y_0}(Z) = \exp\left(\theta_{y_0} - \frac{|P^N Z|^2}{2}\right). \tag{13.108}$$

Here $\beta_{y_0}(Z)$ is considered as a section of $(\Lambda(\bar{N}^*) \hat{\otimes} \Lambda(N^*))_{y_0}$. Recall that $\xi^-_{y_0} = (\Lambda N^* \otimes \eta)_{y_0}$.

Definition **13.21.** – Let ψ be the linear map : $\sigma \in \mathbf{F}^0 \to \sigma \beta_{y_0} \in \mathbf{K}^0_{y_0}$.

Let $\mathbf{K}'^{,0}_{y_0}$ be the image of $\mathbf{F}^0_{y_0}$ in $\mathbf{K}^{-,0}_{y_0}$. By Theorem 7.4, ψ is an isometry from $\mathbf{F}^0_{y_0}$ onto $\mathbf{K}'^{,0}_{y_0}$.

Let $\mathbf{K}'^{,0,\perp}_{y_0}$, $\mathbf{K}'^{,0,\perp,-}_{y_0}$ be the orthogonal vector spaces to $\mathbf{K}'^{,0}_{y_0}$ in $\mathbf{K}^0_{y_0}$, $\mathbf{K}^{-,0}_{y_0}$ respectively. We then have the orthogonal splittings

$$\begin{aligned} \mathbf{K}^0_{y_0} &= \mathbf{K}'^{,0}_{y_0} \oplus \mathbf{K}'^{,0,\perp}_{y_0}, \\ \mathbf{K}^{-,0}_{y_0} &= \mathbf{K}'^{,0}_{y_0} \oplus \mathbf{K}'^{,0,\perp,-}_{y_0}. \end{aligned} \tag{13.109}$$

Let p, $p^\perp$ denote the orthogonal projection operators from $\mathbf{K}^0_{y_0}$ on $\mathbf{K}'^{,0}_{y_0}$, $\mathbf{K}'^{,0,\perp}_{y_0}$ with respect to the Hermitian product (13.107).

Set

$$\begin{aligned} &A_{u,T} = p\,\mathscr{L}^{3,y_0}_{u,T}\,p; && B_{u,T} = p\,\mathscr{L}^{3,y_0}_{u,T}\,p^\perp P^{\xi^-}; && C_{u,T} = p\,\mathscr{L}^{3,y_0}_{u,T}\,P^{\xi^+}; \\ &D_{u,T} = P^{\xi^-} p^\perp \mathscr{L}^{3,y_0}_{u,T}\,p; && E_{u,T} = P^{\xi^-} p^\perp \mathscr{L}^{3,y_0}_{u,T}\,p^\perp P^{\xi^-}; && F_{u,T} = P^{\xi^-} p^\perp \mathscr{L}^{3,y_0}_{u,T}\,P^{\xi^+}; \\ &G_{u,T} = P^{\xi^+} \mathscr{L}^{3,y_0}_{u,T}\,p; && H_{u,T} = P^{\xi^+} \mathscr{L}^{3,y_0}_{u,T}\,p^\perp P^{\xi^-}; && I_{u,T} = P^{\xi^+} \mathscr{L}^{3,y_0}_{u,T}\,P^{\xi^+}. \end{aligned} \tag{13.110}$$

Then we write the operator $\mathscr{L}^{3,y_0}_{u,T}$ as a (3, 3) matrix with respect to the splitting $\mathbf{K}^0_{y_0} = \mathbf{K}'^{,0}_{y_0} \oplus \mathbf{K}'^{,0,\perp,-}_{y_0} \oplus \mathbf{K}^{+,0}_{y_0}$

$$\mathscr{L}^{3,y_0}_{u,T} = \begin{bmatrix} A_{u,T} & B_{u,T} & C_{u,T} \\ D_{u,T} & E_{u,T} & F_{u,T} \\ G_{u,T} & H_{u,T} & I_{u,T} \end{bmatrix}. \tag{13.111}$$

By proceeding as in Section 8h), we know that for $u \in]0, 1]$, as $T \to +\infty$, the differential operator $\mathscr{L}^{3,y_0}_{u,T}$ has an asymptotic expansion of the form

$$\mathscr{L}^{3,y_0}_{u,T} = \sum_{k \leqslant 4} \mathcal{O}_{u,k} T^{k/2}. \tag{13.112}$$

Therefore as $T \to +\infty$, the operators $A_{u,T}$, $B_{u,T}$, ... have asymptotic expansions similar to (13.112).

Recall that ν was defined in Definition 8.8. We now prove one of the central results of this Section.

Theorem **13.22.** — *For* $u \in]0, 1]$, *there exist operators* A_u, B_u, C_u, D_u, E, F_u, G_u, H_u, I_u *such that as* $T \to +\infty$,

$$(13.113)\qquad \begin{aligned} &A_{u,T} = A_u + O\left(\frac{1}{\sqrt{T}}\right); \quad B_{u,T} = \sqrt{T}\, B_u + O(1); \quad C_{u,T} = TC_u + O(\sqrt{T}); \\ &D_{u,T} = \sqrt{T}\, D_u + O(1); \quad E_{u,T} = TE + O(\sqrt{T}); \quad F_{u,T} = TF_u + O(\sqrt{T}); \\ &G_{u,T} = TG_u + O(\sqrt{T}); \quad H_{u,T} = TH_u + O(\sqrt{T}); \quad I_{u,T} = T^2 I_u + O(T^{3/2}). \end{aligned}$$

Let $\mathscr{P}_u$ *be the operator acting on* $\mathbf{K}_{y_0}$

$$(13.114)\qquad \begin{aligned} \mathscr{P}_u = u P^{\xi^-} \Big\{ &\varphi^2 (u P^{TY} Z) \nabla_{\dim Y v(u P^{TY} Z)} \\ &+ \sum_{2l'+1}^{2l} \frac{c(e_i)}{\sqrt{2}} \tilde{\nabla}^{\xi}_{P^N Z} (\varphi^2 \nabla^{\xi}_{{}^0\tau e_i} V)(u P^{TY} Z) \\ &+ \frac{1}{2} [\tilde{\nabla}^{\xi}_{P^N Z} (\varphi V^-)(u P^{TY} Z), \tilde{\nabla}^{\xi}_{P^N Z} \tilde{\nabla}^{\xi}_{P^N Z} (\varphi V^-)(u P^{TY} Z)] \\ &- (\nabla_{P^N Z} \varphi^2)(u P^{TY} Z) \left(S + \frac{|P^N Z|^2}{2} \right) \Big\} P^{\xi^-}. \end{aligned}$$

Then the following identities hold

$$(13.115)\qquad \begin{aligned} B_u &= p \mathscr{P}_u p^{\perp} P^{\xi^-}; \\ C_u &= p P^{\xi^-} \varphi^2 (u P^{TY} Z) \left(\frac{1}{u} \sum_{1}^{2l'} \left(e^i \wedge - \frac{u^2}{2} i_{e_i} \right) (\nabla^{\xi}_{{}^0\tau e_i} V)(u P^{TY} Z) \right. \\ &\quad \left. + \sum_{2l'+1}^{2l} \frac{c(e_i)}{\sqrt{2}} (\nabla^{\xi}_{{}^0\tau e_i} V)(u P^{TY} Z) \right) P^{\xi^+}, \\ D_u &= P^{\xi^-} p^{\perp} \mathscr{P}_u p, \\ E &= p^{\perp} P^{\xi^-} \left(-\frac{1}{2} \Delta^N + \frac{1}{2} |P^N Z|^2 + S \right) P^{\xi^-} p^{\perp}, \\ G_u &= P^{\xi^+} \varphi^2 (u P^{TY} Z) \left(\frac{1}{u} \sum_{1}^{2l'} \left(e^i \wedge - \frac{u^2}{2} i_{e_i} \right) (\nabla^{\xi}_{{}^0\tau e_i} V)(u P^{TY} Z) \right. \\ &\quad \left. + \sum_{2l'+1}^{2l} \frac{c(e_i)}{\sqrt{2}} (\nabla^{\xi}_{{}^0\tau e_i} V)(u P^{TY} Z) \right) p, \\ I_u &= \frac{1}{u^2} P^{\xi^+} ((\varphi V^+)^2 + (1 - \varphi^2) P^{\xi^+})(u P^{TY} Z) P^{\xi^+}. \end{aligned}$$

Proof. – Using formulas (13.83), (13.87), (13.91), (13.92), we will calculate the Taylor expansion of the operator $\mathscr{L}^{3,y_0}_{u,T}$ as $T \to +\infty$, and we will obtain (13.113), (13.115).

Inspection of (13.83), (13.87) shows that the coefficient of T^2 in the Taylor expansion of $\mathscr{L}^{3,y_0}_{u,T}$ is the operator

$$(13.116)\qquad \frac{1}{u^2} P^{\xi^+} (\varphi^2 (V^+)^2 + (1-\varphi^2) P^{\xi^+})(u P^{TY} Z) P^{\xi^+}.$$

We thus obtain the formula in (13.115) for I_u. The coefficient of $T^{3/2}$ maps $\mathbf{K}^+_{y_0}$ into itself. Therefore it can be disregarded.

Observe that if $|P^{TY} Z| \leqslant 3\varepsilon/4u$, ${}^0\tau e_{2l'+1}(u P^{TY} Z), \ldots, {}^0\tau e_{2l}(u P^{TY} Z)$ is an orthonormal base of $N_{\mathbf{R}, y_0}$ and so

$$(13.117)\qquad \sum_{2l'+1}^{2l} \nabla^2_{{}^0\tau e_i (u P^{TY} Z)} = \Delta^N.$$

If, in the coefficient of T, we eliminate the piece which obviously maps $\mathbf{K}^+_{y_0}$ into itself, we then obtain

$$(13.118)\qquad P^{\xi^-} (1 - \varphi^2 (u P^{TY} Z)) \left(-\frac{\Delta^N}{2} + \frac{|P^N Z|^2}{2} + S \right) P^{\xi^-}$$
$$+ \varphi^2 (u P^{TY} Z) \left(-\frac{\Delta^N}{2} + \frac{1}{u} \sum_1^{2l'} \left(e^i \wedge -\frac{u^2}{2} i_{e_i} \right) (\nabla^{\xi}_{{}^0\tau e_i} V)(u P^{TY} Z) \right.$$
$$\left. + \sum_{2l'+1}^{2l} \frac{c(e_i)}{\sqrt{2}} (\nabla^{\xi}_{{}^0\tau e_i} V)(u P^{TY} Z) + \frac{|P^N Z|^2}{2} P^{\xi^-} \right).$$

By Proposition 7.2 and Theorem 7.4, $\mathbf{K}'^{,0}_{y_0}$ is exactly the kernel of the operator

$$(13.119)\qquad P^{\xi^-} \left(\frac{-\Delta^N}{2} + \frac{|P^N Z|^2}{2} + S \right) P^{\xi^-},$$

which acts as an unbounded operator on $\mathbf{K}'^{,0,\perp,-}_{y_0}$. Using the first identity in (12.43), (13.92) and (13.118), we get all the coefficients of T in (13.113), together with the formulas for C_u, E, G_u in (13.115).

We now study the coefficient of $\sqrt{T}$. This is the most difficult term. In view of the previous results, we apply on both sides of this operator the operator P^{ξ^-}. In the sequel, $[\ ,\ ]_+$ denotes an anticommutator. We then obtain as the relevant coefficient of $\sqrt{T}$ the operator $\mathscr{P}_u$ given by

(13.120)
$$\begin{aligned}\mathscr{P}_u = \mathrm{P}^{\xi-} \Big\{ & \frac{u}{2} \langle d\varphi^2(u\mathrm{P}^{\mathrm{TY}}\mathrm{Z}), \mathrm{P}^{\mathrm{N}}\mathrm{Z}\rangle \Big(\Delta^{\mathrm{N}} - \sum_{2l'+1}^{2l} (\nabla_{{}^0\tau e_i(u\mathrm{P}^{\mathrm{TY}}\mathrm{Z})})^2\Big) \\
& - \frac{1}{2}\varphi^2(u\mathrm{P}^{\mathrm{TY}}\mathrm{Z})\Big\{\sum_{2l'+1}^{2l}\Big[\nabla_{{}^0\tau e_i(u\mathrm{P}^{\mathrm{TY}}\mathrm{Z})}, u\nabla_{\langle \mathrm{P}^{\mathrm{N}}\mathrm{Z}, [\mathrm{P}^{\mathrm{N}}{}^0\tau e_i](u\mathrm{P}^{\mathrm{TY}}\mathrm{Z})\rangle} \\
& + \frac{1}{2}\sum_{1\leqslant j,k\leqslant 2l'} \langle\langle \mathrm{P}^{\mathrm{TY}}\mathrm{Z}, [{}^0\Gamma^{\mathrm{TX}}({}^0\tau e_i)](u\mathrm{P}^{\mathrm{TY}}\mathrm{Z})\rangle e_j, e_k\rangle \Big(e^j\wedge - \frac{u^2}{2} i_{e_j}\Big) \\
& \Big(e^k\wedge - \frac{u^2}{2} i_{e_k}\Big) + \frac{u}{4}\sum_{2l'+1\leqslant j,k\leqslant 2l} \langle {}^0\Gamma^{\mathrm{TX}}({}^0\tau e_i)(u\mathrm{P}^{\mathrm{TY}}\mathrm{Z})e_j, e_k\rangle \\
& c(e_j)c(e_k) + \frac{1}{\sqrt{2}} \sum_{\substack{1\leqslant j\leqslant 2l \\ 2l'+1\leqslant k\leqslant 2l}} \langle \mathrm{A}'({}^0\tau e_i)(u\mathrm{P}^{\mathrm{TY}}\mathrm{Z})e_j, e_k\rangle \\
& \Big(e^j\wedge - \frac{u^2}{2} i_{e_j}\Big) c(e_k) + u\Big(\Gamma^{\xi} + \frac{1}{2}\mathrm{Tr}[{}^0\Gamma^{\mathrm{TX}}]\Big)({}^0\tau e_i)(u\mathrm{P}^{\mathrm{TY}}\mathrm{Z})\Big]_+ \\
& - u\nabla_{\mathrm{P}^{\mathrm{N}}\sum_1^{2l'}(\nabla^{\mathrm{TX}}_{{}^0\tau e_i}{}^0\tau e_i)(u\mathrm{P}^{\mathrm{TY}}\mathrm{Z})}\Big\} \\
& + \sum_1^{2l'}\Big(e^i\wedge - \frac{u^2}{2} i_{e_i}\Big)\tilde{\nabla}^{\xi}_{\mathrm{P}^{\mathrm{N}}\mathrm{Z}}(\varphi^2\nabla^{\xi}_{{}^0\tau e_i}\mathrm{V})(u\mathrm{P}^{\mathrm{TY}}\mathrm{Z}) \\
& + u\sum_{2l'+1}^{2l}\frac{c(e_i)}{\sqrt{2}}\tilde{\nabla}^{\xi}_{\mathrm{P}^{\mathrm{N}}\mathrm{Z}}(\varphi^2\nabla^{\xi}_{{}^0\tau e_i}\mathrm{V})(u\mathrm{P}^{\mathrm{TY}}\mathrm{Z}) \\
& + \frac{u}{2}[\tilde{\nabla}^{\xi}_{\mathrm{P}^{\mathrm{N}}\mathrm{Z}}(\varphi \mathrm{V}^-)(u\mathrm{P}^{\mathrm{TY}}\mathrm{Z}), (\tilde{\nabla}^{\xi}_{\mathrm{P}^{\mathrm{N}}\mathrm{Z}}\tilde{\nabla}^{\xi}_{\mathrm{P}^{\mathrm{N}}\mathrm{Z}}\varphi\mathrm{V}^-)(u\mathrm{P}^{\mathrm{TY}}\mathrm{Z})] \\
& - u(\nabla_{\mathrm{P}^{\mathrm{N}}\mathrm{Z}}\varphi^2)(u\mathrm{P}^{\mathrm{TY}}\mathrm{Z})\Big(\mathrm{S} + \frac{|\mathrm{P}^{\mathrm{N}}\mathrm{Z}|^2}{2}\Big)\Big\}\mathrm{P}^{\xi-}.\end{aligned}$$

Formula (13.120) for $\mathscr{P}_u$ can be simplified. In fact:

- By (8.80), we know that for $1 \leqslant i \leqslant 2l$

(13.121)
$$\begin{aligned}\frac{\partial}{\partial t}\mathrm{P}^{\mathrm{N}}({}^0\tau e_i)(u\mathrm{P}^{\mathrm{TY}}\mathrm{Z} + t\mathrm{P}^{\mathrm{N}}\mathrm{Z})|_{t=0} \\ = -\mathrm{P}^{\mathrm{N}}\mathrm{C}_{u\mathrm{P}^{\mathrm{TY}}\mathrm{Z}}(\mathrm{P}^{\mathrm{TY}}\,{}^0\tau e_i(u\mathrm{P}^{\mathrm{TY}}\mathrm{Z}))\,\mathrm{P}^{\mathrm{N}}\mathrm{Z},\end{aligned}$$

and so if $2l'+1 \leqslant i \leqslant 2l$

(13.122)
$$\frac{\partial}{\partial t}\mathrm{P}^{\mathrm{N}}({}^0\tau e_i)(u\mathrm{P}^{\mathrm{TY}} + t\mathrm{P}^{\mathrm{N}}\mathrm{Z})|_{t=0} = 0.$$

From (13.122), we deduce that for $2l'+1 \leqslant i \leqslant 2l$

$$\langle P^N Z, [P^{N\,0}\tau e_i](u P^{TY} Z)\rangle = 0. \tag{13.123}$$

• Using (13.65), it is clear that for $2l'+1 \leqslant i \leqslant 2l$

$$\begin{gathered} {}^0\Gamma^{TX}({}^0\tau e_i)(u P^{TY} Z) = 0, \\ \langle P^{TY} Z, [{}^0\Gamma^{TX}({}^0\tau e_i)](u P^{TY} Z)\rangle = 0. \end{gathered} \tag{13.124}$$

• From (13.55), we find that if $2l'+1 \leqslant i \leqslant 2l$

$$A'_{uP^{TY}Z}({}^0\tau e_i(u P^{TY} Z)) = 0. \tag{13.125}$$

• Using (13.63), (13.65) and the fact that $B_{uP^{YT}Z}$ exchanges $\xi^+_{y_0}$ and $\xi^-_{y_0}$, we get for $2l'+1 \leqslant i \leqslant 2l$

$$P^{\xi^-}\Gamma^{\xi}({}^0\tau e_i)(u P^{TY} Z) P^{\xi^-} = 0. \tag{13.126}$$

• Using the first identity in (12.43) and the first identity in (13.92), we also see that

$$P^{\xi^-}\sum_1^{2l'}\left(e^i\wedge - \frac{u^2}{2} i_{e_i}\right)\tilde{\nabla}^{\xi}_{P^N Z}(\varphi^2\nabla^{\xi}_{{}^0\tau e_i} V)(u P^{TY} Z) P^{\xi^-} = 0. \tag{13.127}$$

From (13.117), (13.120)-(13.127), we find that $\mathscr{P}_u$ is in fact given by formula (13.114). Also observe that if $2l'+1 \leqslant i, j, k \leqslant 2l$, then

$$\begin{gathered} \int_{N_{\mathbf{R},y_0}} \exp\left(-\frac{|Z_0|^2}{2}\right)\nabla_{e_i}\exp\left(-\frac{|Z_0|^2}{2}\right)\frac{dv_N(Z_0)}{(2\pi)^{\dim N}} = 0, \\ \int_{N_{\mathbf{R},y_0}} \exp\left(-\frac{|Z_0|^2}{2}\right) Z_0^i \exp\left(-\frac{|Z_0|^2}{2}\right)\frac{dv_N(Z_0)}{(2\pi)^{\dim N}} = 0, \\ \int_{N_{\mathbf{R},y_0}} \exp\left(-\frac{|Z_0|^2}{2}\right) Z_0^i Z_0^j Z_0^k \exp\left(-\frac{|Z_0|^2}{2}\right)\frac{dv_N(Z_0)}{(2\pi)^{\dim N}} = 0, \end{gathered} \tag{13.128}$$

From (13.120)-(13.128), we deduce that

$$p\mathscr{P}_u p = 0. \tag{13.129}$$

Using (13.114), (13.127), we also get the formulas in (13.115) for B_u and D_u.
Theorem 13.22 is proved. □

Remark **13.23.** — It is here time to relate the previous calculations to the asymptotic formulas (8.57), (8.58) of Theorem 8.18.

In fact recall that as pointed out in Remark 13.13, our trivialization of $\Lambda(T^{*(0,1)}X) \hat{\otimes} \xi$ is compatible with the trivialization of $\Lambda(T^{*(0,1)}X) \hat{\otimes} \xi$ considered in Section 8 g). The only minor difference is that on a neighborhood $\mathscr{V}$ of y_0 in Y, the fibres of the vector bundle $\Lambda(T^{*(0,1)}X) \hat{\otimes} \xi|_{\mathscr{V}}$ have been identified to $(\Lambda(T^{*(0,1)}X) \hat{\otimes} \xi)_{y_0}$ by parallel transport with respect to the connection ${}^0\tilde{\nabla}^Y$ defined in (8.50) along radial geodesics in Y. Also note that $G_{1,T} = F_T^{-1}$. Using Proposition 8.9 and Theorem 8.18, we find that as $T \to +\infty$

$$G_{1,T}^{-1} D^X G_{1,T} = \sqrt{T}\, D^N + D^H + M - \frac{\dim Y}{\sqrt{2}} c(v) + O\left(\frac{1}{\sqrt{T}}\right). \tag{13.130}$$

From (13.130), we deduce that as $T \to +\infty$

$$G_{1,T}^{-1} (D^X)^2 G_{1,T} = T(D^N)^2 + \sqrt{T}\left[D^N, D^H + M - \frac{\dim Y}{\sqrt{2}} c(v)\right] + O(1). \tag{13.131}$$

By proceeding as in (9.69), we find that

$$[D^N, D^H] = 0. \tag{13.132}$$

Also using (8.51), we get

$$\begin{aligned} &[D^N, M] = -\sum_{2l'+1}^{2l} B(e_i)\nabla_{e_i}, \\ &\left[D^N, \frac{-\dim Y}{\sqrt{2}} c(v)\right] = \nabla_{\dim Y v}. \end{aligned} \tag{13.133}$$

Using (13.131)-(13.133), we see that as $T \to +\infty$

$$G_{1,T}^{-1} (D^X)^2 G_{1,T} = T(D^N)^2 + \sqrt{T}\left(-\sum_{2l'+1}^{2l} B(e_i)\nabla_{e_i} + \nabla_{\dim Y v}\right) + O(1), \tag{13.134}$$

and so

$$\begin{aligned} P^{\xi^-} G_{1,T}^{-1} (D^X)^2 G_{1,T} P^{\xi^-} = {} & T P^{\xi^-} (D^{N,-})^2 P^{\xi^-} \\ & + \sqrt{T}\, P^{\xi^-} \nabla_{\dim Y v} P^{\xi^-} + O(1). \end{aligned} \tag{13.135}$$

The simplicity of formula (13.114) for $\mathscr{P}_1$ is now entirely explained by (13.135). By a simple scaling argument, we obtain the corresponding result for $\mathscr{P}_u$, $u \in]0, 1]$.

Of course at least when $\varphi(u P^{TY} Z) = 1$, the whole asymptotic expansion of the operator $\mathscr{L}_{u,T}^{3,y_0}$ as $T \to +\infty$ is entirely explained by formula (8.58) in Theorem 8.18.

We are anyway forced to forget Theorem 8.18 for the moment, because we also need to understand the cancellations which occur as $u \to 0$.

k) A family of Sobolev spaces with weights.

Let q be the orthogonal projection operator from $(\Lambda(T_{\mathbf{R}}^* Y) \otimes \Lambda(\bar{N}^*) \otimes \xi)_{y_0}$ on $(\Lambda(T_{\mathbf{R}}^* Y) \otimes \{\exp \theta\})_{y_0}$.

Recall that p is the orthogonal projection operator from $\mathbf{K}_{y_0}^0$ on $\mathbf{K}_{y_0}'^{,0}$ and that $p^{\perp} = 1 - p$. By an obvious analogue of (8.91), we know that if $s \in \mathbf{K}_{y_0}^0$

$$ps(Z) = \frac{1}{\pi^{\dim N}} \exp\left(\frac{-|P^N Z|^2}{2}\right) q \int_{N_{\mathbf{R}}, y_0} \exp\left(\frac{-|Z'|^2}{2}\right) s(P^{TY} Z + Z')\, dv_N(Z'). \tag{13.136}$$

Let ψ^* be the adjoint of the map $\psi : \mathbf{F}_{y_0}^0 \to \mathbf{K}_{y_0}^0$ defined in Definition 13.21 with respect to the Hermitian products (13.106), (13.107). Then

$$\psi^* = \psi^{-1} p. \tag{13.137}$$

Definition **13.24.** — If $Z \in (T_{\mathbf{R}} X)_{y_0}$, $U \in (T_{\mathbf{R}} Y)_{y_0}$, set

$$g_{u,T}(Z) = 1 + (1 + |P^{TY} Z|^2)^{1/2} \varphi\left(\frac{1}{2} u P^{TY} Z\right) + \left(1 + \frac{|P^N Z|^2}{T}\right)^{1/2} \varphi\left(\frac{u}{2\sqrt{T}} P^N Z\right), \tag{13.138}$$

$$\tilde{g}_u(U) = 1 + (1 + |U|^2)^{1/2} \varphi\left(\frac{uU}{2}\right).$$

The algebra $\Lambda(T_{\mathbf{R}}^* Y)_{y_0}$ splits into

$$\Lambda(T_{\mathbf{R}}^* Y)_{y_0} = \bigoplus_0^{2l'} \Lambda^r(T_{\mathbf{R}}^* Y)_{y_0}. \tag{13.139}$$

This splitting induces corresponding splittings

$$\mathbf{K}_{y_0}^0 = \bigoplus_0^{2l'} \mathbf{K}_{r, y_0}^0, \qquad \mathbf{F}_{y_0}^0 = \bigoplus_0^{2l'} \mathbf{F}_{r, y_0}^0. \tag{13.140}$$

Definition **13.25.** — If $s \in \mathbf{K}^0_{r, y_0}$, set

$$|s|^2_{u, \mathrm{T}, y_0, 0} = \int_{(\mathrm{T}_{\mathbf{R}} \mathrm{X})_{y_0}} |s|^2 \, [g_{u, \mathrm{T}}]^{2(2l'-r)}(\mathrm{Z}) \, dv_{\mathrm{TX}}(\mathrm{Z}). \tag{13.141}$$

Let $\langle \, , \, \rangle_{u, \mathrm{T}, y_0, 0}$ be the Hermitian product on $\mathbf{K}^0_{y_0}$ which is the direct sum of the Hermitian products on the $\mathbf{K}^0_{r, y_0}$'s associated with formula (13.141).

Observe that by (13.136)

$$\begin{aligned} &|s|_{u, \mathrm{T}, y_0, 0} \leqslant |ps|_{u, \mathrm{T}, y_0, 0} + |p^{\perp} s|_{u, \mathrm{T}, y_0, 0}, \\ &|ps|_{u, \mathrm{T}, y_0, 0} \leqslant \mathrm{C} \, |s|_{u, \mathrm{T}, y_0, 0}, \\ &|p^{\perp} s|_{u, \mathrm{T}, y_0, 0} \leqslant \mathrm{C} \, |s|_{u, \mathrm{T}, y_0, 0}. \end{aligned} \tag{13.142}$$

If $\mu \in \mathbf{R}$, let $\mathbf{K}^{\mu}_{y_0}$, $\mathbf{K}^{\pm, \mu}_{y_0}$ be the Sobolev spaces of order μ of sections of $(\Lambda(\mathrm{T}^*_{\mathbf{R}} \mathrm{Y}) \hat{\otimes} \Lambda(\bar{\mathrm{N}}^*) \hat{\otimes} \xi)_{y_0}$, $(\Lambda(\mathrm{T}^*_{\mathbf{R}} \mathrm{Y}) \hat{\otimes} \Lambda(\bar{\mathrm{N}}^*) \hat{\otimes} \xi^{\pm})_{y_0}$ over $(\mathrm{T}_{\mathbf{R}} \mathrm{X})_{y_0}$. If $s \in \mathbf{K}^{\mu}_{y_0}$, we write $s = s^+ + s^-$, $s^{\pm} \in \mathbf{K}^{\pm, \mu}_{y_0}$.

Definition **13.26.** — If $s \in \mathbf{K}^1_{y_0}$, set

$$\begin{aligned} |s|^2_{u, \mathrm{T}, y_0, 1} = \frac{\mathrm{T}^2}{u^2} |s^+|^2_{u, \mathrm{T}, y_0, 0} + \mathrm{T} \, |p^{\perp} s^-|^2_{u, \mathrm{T}, y_0, 0} \\ + \mathrm{T} \, \| \mathrm{P}^{\mathrm{N}} \mathrm{Z} \, |p^{\perp} s^-|^2_{u, \mathrm{T}, y_0, 0} + |ps|^2_{u, \mathrm{T}, y_0, 0} + \sum_{1}^{2l'} |\nabla_{e_i} s|^2_{u, \mathrm{T}, y_0, 0} \\ + \mathrm{T} \sum_{2l'+1}^{2l} |\nabla_{e_i} p^{\perp} s|^2_{u, \mathrm{T}, y_0, 0}. \end{aligned} \tag{13.143}$$

Then (13.143) defines a Hilbert norm on $\mathbf{K}^1_{y_0}$. Let $\mathbf{K}^{-1}_{y_0}$ be the antidual of $\mathbf{K}^1_{y_0}$ and let $| \; |_{u, \mathrm{T}, y_0, -1}$ be the norm on $\mathbf{K}^{-1}_{y_0}$ associated with the norm $| \; |_{u, \mathrm{T}, y_0, 1}$ on $\mathbf{K}^1_{y_0}$. We identify $\mathbf{K}^0_{y_0}$ with its antidual by the Hermitian product $\langle \, , \, \rangle_{u, \mathrm{T}, y_0, 0}$.

We then have the family of continuous dense embeddings with uniformly bounded norms

$$\mathbf{K}^1_{y_0} \to \mathbf{K}^0_{y_0} \to \mathbf{K}^{-1}_{y_0}.$$

Theorem **13.27.** — *If* $\varepsilon \in]0, \inf(\varepsilon_0/2, a/2, b/2)]$ *is small enough, there exists constants* $\mathrm{C}_1 > 0$, $\mathrm{C}_2 > 0$, $\mathrm{C}_3 > 0$, $\mathrm{C}_4 > 0$, $\mathrm{T}_0 \geqslant 1$ *such that if* $y_0 \in \mathrm{Y}$, $u \in]0, 1]$, $\mathrm{T} \geqslant \mathrm{T}_0$, *for any* $s, s' \in \mathbf{K}_{y_0}$ *with compact support, then*

$$\begin{aligned} &\mathrm{Re} \langle \mathscr{L}^{3, y_0}_{u, \mathrm{T}} s, s \rangle_{u, \mathrm{T}, y_0, 0} \geqslant \mathrm{C}_1 |s|^2_{u, \mathrm{T}, y_0, 1} - \mathrm{C}_2 |s|^2_{u, \mathrm{T}, y_0, 0}, \\ &|\mathrm{Im} \langle \mathscr{L}^{3, y_0}_{u, \mathrm{T}} s, s \rangle_{u, \mathrm{T}, y_0, 0}| \leqslant \mathrm{C}_3 |s|_{u, \mathrm{T}, y_0, 1} \, |s|_{u, \mathrm{T}, y_0, 0}, \\ &|\langle \mathscr{L}^{3, y_0}_{u, \mathrm{T}} s, s' \rangle_{u, \mathrm{T}, y_0, 0}| \leqslant \mathrm{C}_4 |s|_{u, \mathrm{T}, y_0, 1} \, |s'|_{u, \mathrm{T}, y_0, 1}. \end{aligned} \tag{13.144}$$

Proof. – The proof of Theorem 13.27 is divided into two parts. In the first part, we construct an operator $\underline{\mathscr{L}}^{3, y_0}_{u, T}$ which is a "principal part" of $\mathscr{L}^{3, y_0}_{u, T}$ and we prove that it verifies estimates similar to (13.144). In a second part (which is the most difficult), we show that for ε small enough, $\mathscr{R}^{y_0}_{u, T} = \mathscr{L}^{3, y_0}_{u, T} - \underline{\mathscr{L}}^{3, y_0}_{u, T}$ is a "small" perturbation of $\underline{\mathscr{L}}^{3, y_0}_{u, T}$ with respect to these estimates.

In the whole proof the constants C, C'... are assumed to be positive, and independent of y_0, u, T. They may vary from line to line. Constants which may be chosen to be also independent of ε will be underlined like $\underline{C}$, $\underline{C}'$, ...

Theorem 13.22 and its proof will play an **essential role** in the proof of Theorem 13.27.

a) Estimates on the operator $\underline{\mathscr{L}}^{3, y_0}_{u, T}$.

We now fix ε for the moment. Set

$$\begin{aligned}
\underline{\mathscr{L}}^{3, y_0}_{u, T} = & -\frac{1}{2}(1-\varphi^2(u P^{TY} Z))\, L_{u P^{TY} Z} - \frac{1}{2}\varphi^2(u P^{TY} Z) \sum_1^{2l'} \nabla^2_{{}^0\tau e_i (u P^{TY} Z)} \\
& - \frac{1}{2} T \Delta^N + \frac{T^2}{u^2}(\varphi^2 (V^+)^2 + (1-\varphi^2) P^{\xi^+})\left(u P^{TY} + \frac{u}{\sqrt{T}} P^N Z\right) \\
& + T \frac{|P^N Z|^2}{2} P^{\xi^-} + T P^{\xi^-} S P^{\xi^-}, \\
\mathscr{R}^{y_0}_{u, T} = & \mathscr{L}^{3, y_0}_{u, T} - \underline{\mathscr{L}}^{3, y_0}_{u, T}.
\end{aligned} \tag{13.145}$$

Note the trivial inequalities

$$\begin{aligned}
& |\nabla_{e_i} g_{u, T}| \leqslant C; \quad |\nabla_{e_i} \tilde{g}_u| \leqslant C; \quad 1 \leqslant i \leqslant 2l' \\
& |\nabla_{e_i} g_{u, T}| \leqslant \frac{C}{\sqrt{T}}; \quad 2l'+1 \leqslant i \leqslant 2l.
\end{aligned} \tag{13.146}$$

Observe that $\mathbf{K}^{+, 0}_{y_0}$, $\mathbf{K}^{-, 0}_{y_0}$, and $\mathbf{K}^{+, 1}_{y_0}$, $\mathbf{K}^{-, 1}_{y_0}$ are mutually orthogonal with respect to the Hermitian products $\langle\, , \rangle_{u, T, y_0, 0}$ and $\langle\, , \rangle_{u, T, y_0, 1}$. Also the operator $\underline{\mathscr{L}}^{3, y_0}_{u, T}$ preserves $\mathbf{K}^+_{y_0}$ and $\mathbf{K}^-_{y_0}$. To verify the analogue of (13.144) for $\underline{\mathscr{L}}^{3, y_0}_{u, T}$, we can then assume that s, s' both lie in $\mathbf{K}^+_{y_0}$ or $\mathbf{K}^-_{y_0}$ and have compact support.

If $s, s' \in \mathbf{K}^+_{y_0}$, using (13.146), the inequalities (13.144) for the operator $\underline{\mathscr{L}}^{3, y_0}_{u, T}$ are trivial. Also the constant corresponding to C_1 can be fixed independently of ε. So we now assume that $s, s' \in \mathbf{K}^-_{y_0}$.

Let $\mathscr{S}_u$ be the operator

$$\mathscr{S}_u = -\frac{1}{2}(1-\varphi^2(u P^{TY} Z))\, L_{u P^{TY} Z} - \frac{1}{2}\varphi^2(u P^{TY} Z) \sum_1^{2l'} \nabla^2_{{}^0\tau e_i (u P^{TY} Z)}. \tag{13.147}$$

If $U \in T_{\mathbf{R}}Y$, $|U| \leqslant 3\varepsilon/4$, ${}^0\tau e_1(U), \ldots, {}^0\tau e_{2l'}(U)$ span $(T_{\mathbf{R}}Y)_{y_0}$. Using (13.71), (13.146), we find that

$$\text{(13.148)}\qquad \begin{aligned} &\operatorname{Re}\langle \mathscr{S}_u s, s\rangle_{u,T,y_0,0} \geqslant \underline{C} \sum_1^{2l'} |\nabla_{e_i} s|^2_{u,T,y_0,0} - C' |s|^2_{u,T,y_0,0}, \\ &|\operatorname{Im}\langle \mathscr{S}_u s, s\rangle_{u,T,y_0,0}| \leqslant C |s|_{u,T,y_0,1} |s|_{u,T,y_0,0}, \\ &|\langle \mathscr{S}_u s, s'\rangle_{u,T,y_0,0}| \leqslant C |s|_{u,T,y_0,1} |s'|_{u,T,y_0,1}. \end{aligned}$$

Let E be the operator

$$\text{(13.149)}\qquad E = -\frac{1}{2}\Delta^N + \frac{|P^N Z|^2}{2} P^{\xi^-} + P^{\xi^-} S P^{\xi^-}.$$

By Proposition 7.2 and Theorem 7.4, we get

$$\text{(13.150)}\qquad \langle E s, s'\rangle_{u,T,y_0,0} = \langle E p^\perp s, p^\perp s'\rangle_{u,T,y_0,0} + \langle E p^\perp s, p s'\rangle_{u,T,y_0,0}.$$

Using (13.146) and Cauchy-Schwarz's inequality, we find that, if T is large enough

$$\text{(13.151)}\qquad \operatorname{Re}\langle E p^\perp s, p^\perp s\rangle_{u,T,y_0,0} \geqslant \underline{C}\left(\sum_{2l'+1}^{2l} |\nabla_{e_i} p^\perp s|^2_{u,T,y_0,0} + \| P^N Z | p^\perp s|^2_{u,T,y_0,0}\right) - \underline{C}' |p^\perp s|^2_{u,T,y_0,0}.$$

We will show there exists $\underline{C}'' > 0$ such that if $T \geqslant 1$ is large enough, if $s \in \mathbf{K}^-_{y_0}$ has compact support, then

$$\text{(13.152)}\qquad \operatorname{Re}\langle E p^\perp s, p^\perp s\rangle_{u,T,y_0,0} \geqslant \underline{C}'' |p^\perp s|^2_{u,T,y_0,0}.$$

To prove (13.152), we may and we will assume that $s \in \mathbf{K}^-_{r,y_0}$ $(0 \leqslant r \leqslant 2l')$. If $r = 2l'$, then (13.152) follows from Theorem 7.4. So we suppose that $0 \leqslant r \leqslant 2l'-1$. We only need to show that there exist $c_0 > 0$, $\underline{C}'' > 0$, such that for T large enough

$$\text{(13.153)}\qquad \operatorname{Re}\langle (E + c_0 p) s, s\rangle_{u,T,y_0,0} \geqslant \underline{C}'' |s|^2_{u,T,y_0,0}.$$

Let E'_r be the operator

$$\text{(13.154)}\qquad E'_r = g^{2l'-r}_{u,T}(E + c_0 p)\, g^{r-2l'}_{u,T}.$$

Recall that $\langle\ ,\ \rangle$ is the ordinary unweighted L_2 Hermitian product (13.107) on $\mathbf{K}^0_{y_0}$. Let $|\ \ |$ denote the corresponding norm. Let E'^*_r be the adjoint of E'_r with respect to $\langle\ ,\ \rangle$. In the sequel, $E'_r + E'^*_r$ is considered as an unbounded operator acting on

$(\mathbf{K}_{y_0}^{-,0}, |\ |)$. Then (13.153) is equivalent to

$$\frac{1}{2}(\mathrm{E}'_r + \mathrm{E}'^*_r) \geqslant \underline{\mathrm{C}}''. \tag{13.155}$$

Now

$$\frac{1}{2}(\mathrm{E}'_r + \mathrm{E}'^*_r) = \mathrm{E} - \frac{1}{2}\sum_{2l'+1}^{2l}\left|\frac{\nabla_{e_i} g_{u,\mathrm{T}}^{2l'-r}}{g_{u,\mathrm{T}}^{2l'-r}}\right|^2 + \frac{1}{2}c_0\left(g_{u,\mathrm{T}}^{2l'-r}\, p g_{u,\mathrm{T}}^{r-2l'} + g_{u,\mathrm{T}}^{r-2l'}\, p g_{u,\mathrm{T}}^{2l'-r}\right). \tag{13.156}$$

By (13.146), we find that

$$\sum_{2l'+1}^{2l}\left|\frac{\nabla_{e_i} g_{u,\mathrm{T}}^{2l'-r}}{g_{u,\mathrm{T}}^{2l'-r}}\right|^2 \leqslant \frac{\mathrm{C}}{\mathrm{T}}. \tag{13.157}$$

If $s \in \mathbf{K}_{r,y_0}^0$, then by (13.136)

$$g_{u,\mathrm{T}}^{2l'-r}\, p g_{u,\mathrm{T}}^{r-2l'}\, s(\mathrm{Z}) = \frac{g_{u,\mathrm{T}}^{2l'-r}(\mathrm{Z})}{\pi^{\dim \mathrm{N}}}\exp\left(-\frac{|\mathrm{P}^{\mathrm{N}}\mathrm{Z}|^2}{2}\right) q\int_{\mathrm{N}_{\mathbf{R},y_0}}\exp\left(-\frac{|\mathrm{Z}'|^2}{2}\right)(g_{u,\mathrm{T}}^{r-2l'}\, s)(\mathrm{P}^{\mathrm{TY}}\mathrm{Z}+\mathrm{Z}')\, dv_{\mathrm{N}}(\mathrm{Z}'). \tag{13.158}$$

Also if $\mathrm{U} \in \mathrm{T}_{\mathbf{R}}\mathrm{Y}$, Z'_0, $\mathrm{Z}''_0 \in \mathrm{N}_{\mathbf{R},y_0}$, we have

$$\frac{g_{u,\mathrm{T}}(\mathrm{U}+\mathrm{Z}''_0)}{g_{u,\mathrm{T}}(\mathrm{U}+\mathrm{Z}'_0)} = \frac{\tilde{g}_u(\mathrm{U}) + \left(1+\frac{|\mathrm{Z}''_0|^2}{\mathrm{T}}\right)^{1/2}\varphi\left(\frac{1}{2}\frac{u\mathrm{Z}''_0}{\sqrt{\mathrm{T}}}\right)}{\tilde{g}_u(\mathrm{U}) + \left(1+\frac{|\mathrm{Z}'_0|^2}{\mathrm{T}}\right)^{1/2}\varphi\left(\frac{1}{2}\frac{u\mathrm{Z}'_0}{\sqrt{\mathrm{T}}}\right)}. \tag{13.159}$$

From (13.159), we deduce that

$$\left|\frac{g_{u,\mathrm{T}}(\mathrm{U}+\mathrm{Z}''_0)}{g_{u,\mathrm{T}}(\mathrm{U}+\mathrm{Z}'_0)} - 1\right| \leqslant \frac{\mathrm{C}}{\sqrt{\mathrm{T}}}|\mathrm{Z}''_0 - \mathrm{Z}'_0|, \tag{13.160}$$

and so if $r < 2l'$

$$\left|\frac{g_{u,\mathrm{T}}^{2l'-r}(\mathrm{U}+\mathrm{Z}''_0)}{g_{u,\mathrm{T}}^{2l'-r}(\mathrm{U}+\mathrm{Z}'_0)} - 1\right| \leqslant \frac{\mathrm{C}'}{\sqrt{\mathrm{T}}}(1+|\mathrm{Z}''_0 - \mathrm{Z}'_0|)^{2l'-r}. \tag{13.161}$$

From (13.158), (13.161), we see that if $0 \leqslant r \leqslant 2l'$, if $s \in \mathbf{K}^0_{r,y_0}$

$$\int_{(\mathrm{T}_{\mathbf{R}}\mathrm{X})_{y_0}} |(g^{2l'-r}_{u,\mathrm{T}}\, p g^{r-2l'}_{u,\mathrm{T}} - p)\, s|^2\, dv_{\mathrm{TX}} \leqslant \frac{\mathrm{C}}{\mathrm{T}} \int_{(\mathrm{T}_{\mathbf{R}}\mathrm{X})_{y_0}} |s|^2\, dv_{\mathrm{TX}}. \tag{13.162}$$

$$\int_{(\mathrm{T}_{\mathbf{R}}\mathrm{X})_{y_0}} |(g^{r-2l'}_{u,\mathrm{T}}\, p g^{2l'-r}_{u,\mathrm{T}} - p)\, s|^2\, dv_{\mathrm{TX}} \leqslant \frac{\mathrm{C}}{\mathrm{T}} \int_{(\mathrm{T}_{\mathbf{R}}\mathrm{X})_{y_0}} |s|^2\, dv_{\mathrm{TX}}.$$

Using Theorem 7.4, (13.156), (13.157), (13.162), we get (13.155).

We have thus proved (13.152). Combining (13.151) and (13.152), we find that for T large enough, for $0 < \eta < 1$

$$\operatorname{Re}\langle \mathrm{E}p^{\perp}s, p^{\perp}s\rangle_{u,\mathrm{T},y_0,0} \geqslant \underline{\mathrm{C}}\eta \Big(\sum_{2l'+1}^{2l} |\nabla_{e_i} p^{\perp}s|^2_{u,\mathrm{T},y_0,0} + \|\mathrm{P^N Z}|p^{\perp}s|^2_{u,\mathrm{T},y_0,0}\Big) + (\underline{\mathrm{C}}''(1-\eta) - \underline{\mathrm{C}}'\eta)\, |p^{\perp}s|^2_{u,\mathrm{T},y_0,0}. \tag{13.163}$$

By taking η small enough, we see that for T large enough

$$\operatorname{Re}\mathrm{T}\langle \mathrm{E}p^{\perp}s, p^{\perp}s\rangle_{u,\mathrm{T},y_0,0} \geqslant \underline{\mathrm{C}}''' \Big(\mathrm{T} \sum_{2l'+1}^{2l} |\nabla_{e_i} p^{\perp}s|^2_{u,\mathrm{T},y_0,0} + \mathrm{T}\,\|\mathrm{P^N Z}|p^{\perp}s|^2_{u,\mathrm{T},y_0,0} + \mathrm{T}\,|p^{\perp}s|^2_{u,\mathrm{T},y_0,0}\Big). \tag{13.164}$$

Let $\tilde{p}$, $\tilde{p}^{\perp}$ be the adjoints of $p, p^{\perp}$ with respect to the Hermitian product $\langle\ ,\ \rangle_{u,\mathrm{T},y_0,0}$. Then $\tilde{p}, \tilde{p}^{\perp}$ act on $\mathbf{K}^0_{r,y_0}$ as $g^{-2(2l'-r)} p g^{2(2l'-r)}$, $g^{-2(2l'-r)} p^{\perp} g^{2(2l'-r)}$ respectively.

Using Theorem 7.4, we find that if $s, s' \in \mathbf{K}^-_{y_0}$ have compact support,

$$\langle \mathrm{E}p^{\perp}s, ps'\rangle_{u,\mathrm{T},y_0,0} = \langle \mathrm{E}p^{\perp}s, \tilde{p}^{\perp}ps'\rangle_{u,\mathrm{T},y_0,0} \tag{13.165}$$

or equivalently

$$\langle \mathrm{E}p^{\perp}s, ps'\rangle_{u,\mathrm{T},y_0,0} = \langle \mathrm{E}p^{\perp}s, (p-\tilde{p})ps'\rangle_{u,\mathrm{T},y_0,0}. \tag{13.166}$$

Using (13.146), (13.161) and integration by parts, we deduce from (13.166) that

$$|\mathrm{T}\langle \mathrm{E}p^{\perp}s, ps'\rangle_{u,\mathrm{T},y_0,0}| \leqslant C\sqrt{\mathrm{T}}\, |p^{\perp}s|_{u,\mathrm{T},y_0,0}\, |ps'|_{u,\mathrm{T},y_0,0}. \tag{13.167}$$

In particular, we find from (13.167) that for any $\alpha > 0$

$$|\operatorname{Re}\mathrm{T}\langle \mathrm{E}p^{\perp}s, ps\rangle_{u,\mathrm{T},y_0,0}| \leqslant \frac{\mathrm{C}}{2}\Big(\alpha\mathrm{T}\,|p^{\perp}s|^2_{u,\mathrm{T},y_0,0} + \frac{1}{\alpha}|ps|^2_{u,\mathrm{T},y_0,0}\Big). \tag{13.168}$$

From (13.150), (13.164), (13.168), by taking $\alpha > 0$ small enough, we get for T large

$$(13.169)\quad \left|\operatorname{Re} T\langle Es, s\rangle_{u,T,y_0,0}\right| \geqslant \underline{C}\left(T\sum_{2l'+1}^{2l} |\nabla_{e_i} p^{\perp} s|^2_{u,T,y_0,0} + T\,\| P^N Z |\, p^{\perp} s|^2_{u,T,y_0,0} + T\,|p^{\perp} s|^2_{u,T,y_0,0}\right) - C'\,|ps|^2_{u,T,y_0,0}.$$

Using (13.146), (13.150), (13.167) and the fact that S and $|P^N Z|^2\, P^{\xi^-}$ are self-adjoint operators, we also find that if $s, s' \in \mathbf{K}^-_{y_0}$ have compact support, then

$$(13.170)\quad \begin{aligned} &\left|\operatorname{Im} T\langle Es, s\rangle_{u,T,y_0,0}\right| \leqslant C\,|s|_{u,T,y_0,1}\,|s|_{u,T,y_0,0},\\ &\left|T\langle Es, s'\rangle_{u,T,y_0,0}\right| \leqslant C\,|s|_{u,T,y_0,1}\,|s'|_{u,T,y_0,1}.\end{aligned}$$

From (13.145), (13.148), (13.169), (13.170), we finally find that there exists $\underline{C}'_1 > 0$, $C'_2 > 0$, $C'_3 > 0$, $C'_4 > 0$ such that if $s, s' \in \mathbf{K}_{y_0}$ have compact support, for T large enough

$$(13.171)\quad \begin{aligned} &\operatorname{Re}\langle \underline{\mathscr{L}}^{3,y_0}_{u,T} s, s\rangle_{u,T,y_0,0} \geqslant \underline{C}'_1\,|s|^2_{u,T,y_0,1} - C'_2\,|s|^2_{u,T,y_0,0},\\ &\left|\operatorname{Im}\langle \underline{\mathscr{L}}^{3,y_0}_{u,T} s, s\rangle_{u,T,y_0,0}\right| \leqslant C'_3\,|s|_{u,T,y_0,1}\,|s|_{u,T,y_0,0},\\ &\left|\langle \underline{\mathscr{L}}^{3,y_0}_{u,T} s, s'\rangle_{u,T,y_0,0}\right| \leqslant C'_4\,|s|_{u,T,y_0,1}\,|s'|_{u,T,y_0,1}.\end{aligned}$$

b) The operator $\mathbf{R}^{y_0}_{u,T}$ *is a small perturbation of the operator* $\underline{\mathscr{L}}^{3,y_0}_{u,T}$.

We will show that given $c > 0$, if $\varepsilon \in]0, \inf(\varepsilon_0/2, a/2, b/2)]$ is small enough, there exist $T_0 \geqslant 1$, $C > 0$ such that if $y_0 \in Y$, $u \in]0, 1]$, $T \geqslant T_0$, if $s, s' \in \mathbf{K}_{y_0}$ have compact support

$$(13.172)\quad \begin{aligned} &\left|\langle \mathscr{R}^{y_0}_{u,T} s, s'\rangle_{u,T,y_0,0}\right| \leqslant c\,|s|_{u,T,y_0,1}\,|s'|_{u,T,y_0,1}\\ &\qquad + C\left(|s|_{u,T,y_0,1}\,|s'|_{u,T,y_0,0} + |s|_{u,T,y_0,0}\,|s'|_{u,T,y_0,1}\right),\\ &\left|\operatorname{Im}\langle \mathscr{R}^{y_0}_{u,T} s, s\rangle_{u,T,y_0,0}\right| \leqslant C\,|s|_{u,T,y_0,1}\,|s|_{u,T,y_0,0}.\end{aligned}$$

Using (13.171), it is clear that the proof of Theorem 13.27 will then be completed.

In the proof of (13.172), the fact that the function $Z_0 \in N_{\mathbf{R},y_0} \to \exp(-|Z_0|^2/2)$ lies in the Schwartz space $S(N_{\mathbf{R},y_0})$ of the fibre $N_{\mathbf{R},y_0}$ will play a key role. Also observe that for $1 \leqslant i \leqslant 2l'$

$$(13.173)\quad [\nabla_{e_i}, p] = 0.$$

Finally, Theorem 13.22 will play a crucial role in establishing (13.172).

Note that if $|u\mathrm{P}^{\mathrm{TY}}\mathrm{Z}| \leqslant 3\varepsilon/4$, $|(u/\sqrt{\mathrm{T}})\mathrm{P}^{\mathrm{N}}\mathrm{Z}| \leqslant 3\varepsilon/4$, then $\varphi((u/2)\mathrm{P}^{\mathrm{TY}}\mathrm{Z})=1$, $\varphi((u/2\sqrt{\mathrm{T}})\mathrm{P}^{\mathrm{N}}\mathrm{Z})=1$. By proceeding as in the proof of Proposition 11.24, we find that the operators indexed by i, $1 \leqslant i \leqslant 2l'$

$$1_{|u\mathrm{P}^{\mathrm{TY}}\mathrm{Z}| \leqslant 3\varepsilon/4,\, |(u/\sqrt{\mathrm{T}})\mathrm{P}^{\mathrm{N}}\mathrm{Z}| \leqslant 3\varepsilon/4}\left(e^i \wedge -\frac{u^2}{2} i_{e_i}\right), \tag{13.174}$$

$$1_{|u\mathrm{P}^{\mathrm{TY}}\mathrm{Z}| \leqslant 3\varepsilon/4,\, |(u/\sqrt{\mathrm{T}})\mathrm{P}^{\mathrm{N}}\mathrm{Z}| \leqslant 3\varepsilon/4}\left(|\mathrm{P}^{\mathrm{TY}}\mathrm{Z}| + \frac{|\mathrm{P}^{\mathrm{N}}\mathrm{Z}|}{\sqrt{\mathrm{T}}}\right)\left(e^i \wedge -\frac{u^2}{2} i_{e_i}\right)$$

act as a uniformly bounded family of operators on the Hilbert space $(\mathbf{K}^0_{y_0}, |\ \ |_{u,\mathrm{T},y_0,0})$.

To prove (13.172), we will consider first the part of $\mathscr{R}^{y_0}_{u,\mathrm{T}}$ which belongs to $\mathscr{M}^{3,y_0}_{u,\mathrm{T}}$ and later the part which belongs to $\mathscr{L}^{3,y_0}_{u,\mathrm{T}} - \mathscr{M}^{3,y_0}_{u,\mathrm{T}}$.

1. *The contribution of* $\mathscr{M}^{3,y_0}_{u,\mathrm{T}}$ *to* $\mathscr{R}^{y_0}_{u,\mathrm{T}}$.

In the contribution of $\mathscr{M}^{3,y_0}_{u,\mathrm{T}}$ to $\mathscr{R}^{y_0}_{u,\mathrm{T}}$, we will consider in succession second derivatives in the direction of $(\mathrm{T}_{\mathbf{R}}\mathrm{Y})_{y_0}$, in the directions of $\mathrm{N}_{\mathbf{R},y_0}$, mixed derivatives, and remaining terms.

α) *Second derivatives in the directions of* $(\mathrm{T}_{\mathbf{R}}\mathrm{Y})_{y_0}$.

Let $\mathscr{A}_{u,\mathrm{T}}$ be the operator

$$\begin{aligned} \mathscr{A}_{u,\mathrm{T}} = {} & \frac{1}{2}\left(\varphi^2\left(u\mathrm{P}^{\mathrm{TY}}\mathrm{Z} + \frac{u}{\sqrt{\mathrm{T}}}\mathrm{P}^{\mathrm{N}}\mathrm{Z}\right) - \varphi^2(u\mathrm{P}^{\mathrm{TY}}\mathrm{Z})\right)\mathrm{L}_{u\mathrm{P}^{\mathrm{TY}}\mathrm{Z}} \\ & - \frac{1}{2}\varphi^2\left(u\mathrm{P}^{\mathrm{TY}}\mathrm{Z} + \frac{u}{\sqrt{\mathrm{T}}}\mathrm{P}^{\mathrm{N}}\mathrm{Z}\right)\sum_1^{2l}\nabla^2_{\mathrm{P}^{\mathrm{TY}}\,{}^0\tau e_i(u\mathrm{P}^{\mathrm{TY}}\mathrm{Z} + (u/\sqrt{\mathrm{T}})\mathrm{P}^{\mathrm{N}}\mathrm{Z})} \\ & + \frac{1}{2}\varphi^2(u\mathrm{P}^{\mathrm{TY}}\mathrm{Z})\sum_1^{2l'}\nabla^2_{{}^0\tau e_i(u\mathrm{P}^{\mathrm{TY}}\mathrm{Z})}. \end{aligned} \tag{13.175}$$

Clearly

$$\begin{aligned} \mathscr{A}_{u,\mathrm{T}} = {} & \frac{1}{2}\left(\varphi^2\left(u\mathrm{P}^{\mathrm{TY}}\mathrm{Z} + \frac{u}{\sqrt{\mathrm{T}}}\mathrm{P}^{\mathrm{N}}\mathrm{Z}\right) - \varphi^2(u\mathrm{P}^{\mathrm{TY}}\mathrm{Z})\right) \\ & \left(\mathrm{L}_{u\mathrm{P}^{\mathrm{TY}}\mathrm{Z}} - \sum_1^{2l'}\nabla^2_{{}^0\tau e_i(u\mathrm{P}^{\mathrm{TY}}\mathrm{Z})}\right) - \frac{1}{2}\varphi^2\left(u\mathrm{P}^{\mathrm{TY}}\mathrm{Z} + \frac{u}{\sqrt{\mathrm{T}}}\mathrm{P}^{\mathrm{N}}\mathrm{Z}\right) \\ & \sum_1^{2l}\left(\nabla^2_{\mathrm{P}^{\mathrm{TY}}\,{}^0\tau e_i(u\mathrm{P}^{\mathrm{TY}}\mathrm{Z} + (u/\sqrt{\mathrm{T}})\mathrm{P}^{\mathrm{N}}\mathrm{Z})} - \nabla^2_{\mathrm{P}^{\mathrm{TY}}\,{}^0\tau e_i(u\mathrm{P}^{\mathrm{TY}}\mathrm{Z})}\right). \end{aligned} \tag{13.176}$$

By (13.75), $\varphi^2(u\mathrm{P}^{\mathrm{TY}}\mathrm{Z}+(u/\sqrt{\mathrm{T}})\mathrm{P}^{\mathrm{N}}\mathrm{Z})-\varphi^2(u\mathrm{P}^{\mathrm{TY}}\mathrm{Z})$ is nonzero only if $|u\mathrm{P}^{\mathrm{TY}}\mathrm{Z}|\leqslant 3\varepsilon/4$. Recall that $\varepsilon\leqslant b/2$. By (13.70), we find that if $|u\mathrm{P}^{\mathrm{TY}}\mathrm{Z}|\leqslant 3\varepsilon/4$, then $\mu(u\mathrm{P}^{\mathrm{TY}}\mathrm{Z})=1$. From (13.71), (13.176), we deduce that

$$(13.177)\qquad \mathscr{A}_{u,\mathrm{T}}=-\frac{1}{2}\varphi^2\left(u\mathrm{P}^{\mathrm{TY}}+\frac{u}{\sqrt{\mathrm{T}}}\mathrm{P}^{\mathrm{N}}\mathrm{Z}\right)$$
$$\sum_1^{2l}(\nabla^2_{\mathrm{P}^{\mathrm{TY}}\,{}^0\tau e_i(u\mathrm{P}^{\mathrm{TY}}\mathrm{Z}+(u/\sqrt{\mathrm{T}})\mathrm{P}^{\mathrm{N}}\mathrm{Z})}-\nabla^2_{\mathrm{P}^{\mathrm{TY}}\,{}^0\tau e_i(u\mathrm{P}^{\mathrm{TY}}\mathrm{Z})}).$$

Also, $\varphi^2(u\mathrm{P}^{\mathrm{TY}}\mathrm{Z}+(u/\sqrt{\mathrm{T}})\mathrm{P}^{\mathrm{N}}\mathrm{Z})$ is nonzero only if $|u\mathrm{P}^{\mathrm{TY}}\mathrm{Z}|\leqslant 3\varepsilon/4$, $|(u/\sqrt{\mathrm{T}})\,\mathrm{P}^{\mathrm{N}}\mathrm{Z}|\leqslant 3\varepsilon/4$. Using (13.146), (13.177) and the previous considerations, we find that

$$(13.178)\qquad |\langle\mathscr{A}_{u,\mathrm{T}}\,s,\,s'\rangle_{u,\mathrm{T},y_0,0}|\leqslant \underline{\mathrm{C}}\,\varepsilon\,|s'|_{u,\mathrm{T},y_0,1}\,|s'|_{u,\mathrm{T},y_0,1}$$
$$+\mathrm{C}'(|s|_{u,\mathrm{T},y_0,1}\,|s|_{u,\mathrm{T},y_0,0}+|s'|_{u,\mathrm{T},y_0,0}\,|s'|_{u,\mathrm{T},y_0,1}),$$
$$|\mathrm{Im}\langle\mathscr{A}_{u,\mathrm{T}}\,s,\,s\rangle_{u,\mathrm{T},y_0,0}|\leqslant \mathrm{C}\,|s|_{u,\mathrm{T},y_0,1}\,|s|_{u,\mathrm{T},y_0,0}.$$

From (13.178), we deduce that $\mathscr{A}_{u,\mathrm{T}}$ is harmless for the estimates (13.172).

β) *Second derivatives in the directions of* $\mathrm{N}_{\mathbf{R},y_0}$.

Let $\mathscr{A}'_{u,\mathrm{T}}$ be the operator

$$(13.179)\qquad \mathscr{A}'_{u,\mathrm{T}}=\frac{\mathrm{T}}{2}\varphi^2\left(u\mathrm{P}^{\mathrm{TY}}\mathrm{Z}+\frac{u}{\sqrt{\mathrm{T}}}\mathrm{P}^{\mathrm{N}}\mathrm{Z}\right)$$
$$\left(-\sum_1^{2l}\nabla^2_{\mathrm{P}^{\mathrm{N}}\,{}^0\tau e_i(u\mathrm{P}^{\mathrm{TY}}\mathrm{Z}+(u/\sqrt{\mathrm{T}})\mathrm{P}^{\mathrm{N}}\mathrm{Z})}+\Delta^{\mathrm{N}}\right).$$

Using the identity (13.117), we find that

$$(13.180)\qquad \mathscr{A}'_{u,\mathrm{T}}=-\frac{\mathrm{T}}{2}\varphi^2\left(u\mathrm{P}^{\mathrm{TY}}\mathrm{Z}+\frac{u}{\sqrt{\mathrm{T}}}\mathrm{P}^{\mathrm{N}}\mathrm{Z}\right)$$
$$\sum_1^{2l}(\nabla^2_{\mathrm{P}^{\mathrm{N}}\,{}^0\tau e_i(u\mathrm{P}^{\mathrm{TY}}\mathrm{Z}+(u/\sqrt{\mathrm{T}})\mathrm{P}^{\mathrm{N}}\mathrm{Z})}-\nabla^2_{\mathrm{P}^{\mathrm{N}}\,{}^0\tau e_i(u\mathrm{P}^{\mathrm{TY}}\mathrm{Z})}).$$

Clearly if $s,s'\in\mathbf{K}_{y_0}$ have compact support, then

$$(13.181)\qquad \langle\mathscr{A}'_{u,\mathrm{T}}\,s,\,s'\rangle_{u,\mathrm{T},y_0,0}=\langle\mathscr{A}'_{u,\mathrm{T}}\,ps,\,ps'\rangle_{u,\mathrm{T},y_0,0}$$
$$+\langle\mathscr{A}'_{u,\mathrm{T}}\,p^{\perp}s,\,p^{\perp}s'\rangle_{u,\mathrm{T},y_0,0}+\langle\mathscr{A}'_{u,\mathrm{T}}\,p^{\perp}s,\,ps'\rangle_{u,\mathrm{T},y_0,0}$$
$$+\langle\mathscr{A}'_{u,\mathrm{T}}\,ps,\,p^{\perp}s'\rangle_{u,\mathrm{T},y_0,0}.$$

Recall that for $1 \leqslant i \leqslant 2l'$, $P^{N\,0}\tau e_i(uP^{TY}Z)=0$. From (13.146), from Theorem 13.22 and its proof (especially equations (13.122) and (13.123)) and from obvious properties of the function $Z_0 \in N_{\mathbf{R}, y_0} \to \exp(-|Z_0|^2/2)$, we find that

$$(13.182) \qquad |\langle \mathscr{A}'_{u,T}\, ps, ps' \rangle_{u,T,y_0,0}| \leqslant C\, |s|_{u,T,y_0,1}\, |s'|_{u,T,y_0,0}.$$

Using (13.146), (13.180) and proceeding as in (13.178), we get

$$(13.183) \qquad \begin{aligned} |\langle \mathscr{A}'_{u,T}\, p^{\perp} s, p^{\perp} s' \rangle_{u,T,y_0,0}| &\leqslant \underline{C}\varepsilon\, |s|_{u,T,y_0,1}\, |s'|_{u,T,y_0,1} \\ &+ C'(|s|_{u,T,y_0,0}\, |s'|_{u,T,y_0,1} + |s|_{u,T,y_0,1}\, |s'|_{u,T,y_0,0}). \\ |\operatorname{Im}\langle \mathscr{A}'_{u,T}\, p^{\perp} s, p^{\perp} s' \rangle_{u,T,y_0,0}| &\leqslant C\, |s|_{u,T,y_0,1}\, |s'|_{u,T,y_0,0}. \end{aligned}$$

Using again (13.146), Theorem 13.22 (in particular equations (13.122)-(13.123)) and elementary properties of the function $Z_0 \in N_{\mathbf{R}, y_0} \to \exp(-|Z_0{}^2/2)$, we find

$$(13.184) \qquad \begin{aligned} &|\langle \mathscr{A}'_{u,T}\, p^{\perp} s, ps' \rangle_{u,T,y_0,0}| + |\langle \mathscr{A}'_{u,T}\, ps, p^{\perp} s' \rangle_{u,T,y_0,0}| \\ &\qquad \leqslant C(|s|_{u,T,y_0,1}\, |s'|_{u,T,y_0,0} + |s|_{u,T,y_0,0}\, |s'|_{u,T,y_0,1}). \end{aligned}$$

From (13.181)-(13.184), we deduce that $\mathscr{A}'_{u,T}$ is harmless for the estimate (13.172).

γ) *Mixed derivatives.*

Let $\mathscr{A}''_{u,T}$ be the operator

$$(13.185) \qquad \begin{aligned} \mathscr{A}''_{u,T} = -\frac{\sqrt{T}}{2}\, \varphi^2 \left(uP^{TY}Z + \frac{u}{\sqrt{T}} P^N Z\right) \\ \sum_1^{2l} [\nabla_{P^{N\,0}\tau e_i(uP^{TY}Z + (u/\sqrt{T})P^N Z)}, \nabla_{P^{TY\,0}\tau e_i(uP^{TY}Z + (u/\sqrt{T})P^N Z)}]_+. \end{aligned}$$

Recall that for $1 \leqslant i \leqslant 2l$, ${}^0\tau e_i(uP^{TY}Z)$ lies in $(T_{\mathbf{R}}Y)_{y_0}$ or in $N_{\mathbf{R}, y_0}$. By proceeding in the same way as for the operator $\mathscr{A}'_{u,T}$, one finds easily that $\mathscr{A}''_{u,T}$ is also inoffensive for the estimate (13.172).

δ) *The remaining terms in* $\mathscr{M}^{3, y_0}_{u,T}$.

Using (13.128) and proceeding as in (13.166), we find that if $s, s' \in \mathbf{K}_{y_0}$ have compact support, then

$$(13.186) \qquad \begin{aligned} \langle \sqrt{T}\, \nabla_{v(uP^{TY}Z)}\, ps, ps' \rangle_{u,T,y_0,0} &= \sqrt{T}\, \langle p^{\perp}\, \nabla_{v(uP^{TY}Z)}\, ps, ps' \rangle_{u,T,y_0,0} \\ &= \sqrt{T}\, \langle \nabla_{v(uP^{TY}Z)}\, ps, (p-\tilde{p})\, ps \rangle_{u,T,y_0,0}. \end{aligned}$$

By (13.161), (13.186), we get

$$(13.187) \qquad |\langle \sqrt{T}\, \nabla_{v(uP^{TY}Z)}\, ps, ps' \rangle_{u,T,y_0,0}| \leqslant C\, |s|_{u,T,y_0,1}\, |s'|_{u,T,y_0,0}.$$

By using (13.128) again, we find easily that if $\mathscr{A}'''_{u,\mathrm{T}}$ is the remaining contribution of $\mathscr{M}^{3,y_0}_{u,\mathrm{T}}$ to $\mathscr{R}^{y_0}_{u,\mathrm{T}}$, $\mathscr{A}'''_{u,\mathrm{T}}$ is harmless for the estimate (13.172).

2. *The contribution of* $\mathscr{L}^{3,y_0}_{u,\mathrm{T}}-\mathscr{M}^{3,y_0}_{u,\mathrm{T}}$ *to* $\mathscr{R}^{y_0}_{u,\mathrm{T}}$.

By (13.83), (13.145), the contribution of $\mathscr{L}^{3,y_0}_{u,\mathrm{T}}-\mathscr{M}^{3,y_0}_{u,\mathrm{T}}$ to $\mathscr{R}^{y_0}_{u,\mathrm{T}}$ is the operator $\mathscr{C}_{u,\mathrm{T}}$ given by

$$\begin{aligned}\mathscr{C}_{u,\mathrm{T}}=\varphi^2\Big(\frac{\mathrm{T}}{u}\sum_1^{2l'}\Big(e^i\wedge-\frac{u^2}{2}i_{e_i}\Big)\nabla^{\xi}_{o_{\tau e_i}}\mathrm{V}\\ +\mathrm{T}\sum_{2l'+1}^{2l}\frac{c(e_i)}{\sqrt{2}}\nabla^{\xi}_{o_{\tau e_i}}\mathrm{V}+\frac{\mathrm{T}^2}{u^2}(\mathrm{V}^-)^2\Big)\Big(u\mathrm{P}^{\mathrm{TY}}\mathrm{Z}+\frac{u}{\sqrt{\mathrm{T}}}\mathrm{P}^{\mathrm{N}}\mathrm{Z}\Big)\\ +\Big(1-\varphi^2\Big(u\mathrm{P}^{\mathrm{TY}}\mathrm{Z}+\frac{u}{\sqrt{\mathrm{T}}}\mathrm{P}^{\mathrm{N}}\mathrm{Z}\Big)\Big(\mathrm{TP}^{\xi^-}\mathrm{SP}^{\xi^-}+\frac{\mathrm{T}|\mathrm{P}^{\mathrm{N}}\mathrm{Z}|^2}{2}\mathrm{P}^{\xi^-}\Big)\\ -\mathrm{T}\frac{|\mathrm{P}^{\mathrm{N}}\mathrm{Z}|^2}{2}-\mathrm{TP}^{\xi^-}\mathrm{SP}^{\xi^-}.\end{aligned}\tag{13.188}$$

Clearly

$$\begin{aligned}\mathscr{C}_{u,\mathrm{T}}=\varphi^2\Big(\frac{\mathrm{T}}{u}\sum_1^{2l'}\Big(e^i\wedge-\frac{u^2}{2}i_{e_i}\Big)\nabla^{\xi}_{o_{\tau e_i}}\mathrm{V}+\mathrm{T}\sum_{2l'+1}^{2l}\frac{c(e_i)}{\sqrt{2}}\nabla^{\xi}_{o_{\tau e_i}}\mathrm{V}\\ -\mathrm{TP}^{\xi^-}\mathrm{SP}^{\xi^-}\Big)\Big(u\mathrm{P}^{\mathrm{TY}}\mathrm{Z}+\frac{u}{\sqrt{\mathrm{T}}}\mathrm{P}^{\mathrm{N}}\mathrm{Z}\Big)+\varphi^2\Big(u\mathrm{P}^{\mathrm{TY}}\mathrm{Z}+\frac{u}{\sqrt{\mathrm{T}}}\mathrm{P}^{\mathrm{N}}\mathrm{Z}\Big)\\ \Big(\frac{\mathrm{T}^2}{u^2}(\mathrm{V}^-)^2\Big(u\mathrm{P}^{\mathrm{TY}}\mathrm{Z}+\frac{u}{\sqrt{\mathrm{T}}}\mathrm{P}^{\mathrm{N}}\mathrm{Z}\Big)-\mathrm{T}\frac{|\mathrm{P}^{\mathrm{N}}\mathrm{Z}|^2}{2}\mathrm{P}^{\xi^-}\Big).\end{aligned}\tag{13.189}$$

Take $s, s'\in\mathbf{K}_{y_0}$ with compact support. Then

$$\begin{aligned}\langle\mathscr{C}_{u,\mathrm{T}}s,s'\rangle_{u,\mathrm{T},y_0,0}=\langle\mathscr{C}_{u,\mathrm{T}}ps,ps'\rangle_{u,\mathrm{T},y_0,0}\\ +\langle\mathscr{C}_{u,\mathrm{T}}p^\perp s,p^\perp s'\rangle_{u,\mathrm{T},y_0,0}+\langle\mathscr{C}_{u,\mathrm{T}}p^\perp s,ps'\rangle_{u,\mathrm{T},y_0,0}\\ +\langle\mathscr{C}_{u,\mathrm{T}}ps,p^\perp s'\rangle_{u,\mathrm{T},y_0,0}.\end{aligned}\tag{13.190}$$

By using the first identity in (12.43), Theorem 13.19, Theorem 13.22 and its proof and more precisely equation (13.128), and also (13.160), we find that

$$|\langle\mathscr{C}_{u,\mathrm{T}}ps,ps'\rangle_{u,\mathrm{T},y_0,0}|\leqslant\mathrm{C}\,|s|_{u,\mathrm{T},y_0,1}\,|s'|_{u,\mathrm{T},y_0,0}.\tag{13.191}$$

Using the first identity in (12.43) and Theorem 13.19, we also find that

$$(13.192)\qquad \left|\left\langle\left(\varphi^2 \frac{\mathrm{T}}{u}\sum_1^{2l'}\left(e^i\wedge-\frac{u^2}{2}i_{e_i}\right)\nabla^{\xi}_{{}^0\tau e_i}\mathrm{V}\right)\left(u\mathrm{P}^{\mathrm{TY}}\mathrm{Z}+\frac{u}{\sqrt{\mathrm{T}}}\mathrm{P}^{\mathrm{N}}\mathrm{Z}\right)p^{\perp}s,\, p^{\perp}s'\right\rangle_{u,\mathrm{T},y_0,0}\right| \leqslant \mathrm{C}\,|s|_{u,\mathrm{T},y_0,1}\,|s'|_{u,\mathrm{T},y_0,0},$$

$$\left|\left\langle\varphi^2\left(\mathrm{T}\sum_{2l'+1}^{2l}\frac{c(e_i)}{\sqrt{2}}\nabla^{\xi}_{{}^0\tau e_i}\mathrm{V}-\mathrm{T}\mathrm{P}^{\xi^-}\mathrm{S}\mathrm{P}^{\xi^-}\right)\left(u\mathrm{P}^{\mathrm{TY}}\mathrm{Z}+\frac{u}{\sqrt{\mathrm{T}}}\mathrm{P}^{\mathrm{N}}\mathrm{Z}\right)p^{\perp}s,\, p^{\perp}s'\right\rangle_{u,\mathrm{T},y_0,0}\right| \leqslant \mathrm{C}\,|s|_{u,\mathrm{T},y_0,1}\,|s'|_{u,\mathrm{T},y_0,0}.$$

Recall that $\varphi^2(u\mathrm{P}^{\mathrm{TY}}\mathrm{Z}+(u/\sqrt{\mathrm{T}})\mathrm{P}^{\mathrm{N}}\mathrm{Z})$ vanishes for $|(u/\sqrt{\mathrm{T}})\mathrm{P}^{\mathrm{N}}\mathrm{Z}|\geqslant 3\varepsilon/4$. Using Theorem 13.19, we find that

$$(13.193)\qquad \left|\left\langle\varphi^2\left(u\mathrm{P}^{\mathrm{TY}}\mathrm{Z}+\frac{u}{\sqrt{\mathrm{T}}}\mathrm{P}^{\mathrm{N}}\mathrm{Z}\right)\left(\frac{\mathrm{T}^2}{u^2}(\mathrm{V}^-)^2\left(u\mathrm{P}^{\mathrm{TY}}\mathrm{Z}+\frac{u}{\sqrt{\mathrm{T}}}\mathrm{P}^{\mathrm{N}}\mathrm{Z}\right)-\mathrm{T}\frac{|\mathrm{P}^{\mathrm{N}}\mathrm{Z}|^2}{2}\mathrm{P}^{\xi^-}\right)p^{\perp}s,\, p^{\perp}s'\right\rangle_{u,\mathrm{T},y_0,0}\right| \leqslant \underline{\mathrm{C}}\varepsilon\,|s|_{u,\mathrm{T},y_0,1}\,|s'|_{u,\mathrm{T},y_0,1}.$$

Using again the first identity in (12.43), Theorem 13.19 and elementary properties of the function $\exp(-|\mathrm{Z}_0|^2/2)$, we also find that

$$(13.194)\qquad \begin{aligned}&|\langle\mathscr{C}_{u,\mathrm{T}}\,ps,\, p^{\perp}s'\rangle_{u,\mathrm{T},y_0,0}|\leqslant \mathrm{C}\,|s|_{u,\mathrm{T},y_0,0}\,|s'|_{u,\mathrm{T},y_0,1}\\ &|\langle\mathscr{C}_{u,\mathrm{T}}\,p^{\perp}s,\, ps'\rangle_{u,\mathrm{T},y_0,0}|\leqslant \mathrm{C}\,|s|_{u,\mathrm{T},y_0,0}\,|s'|_{u,\mathrm{T},y_0,1}.\end{aligned}$$

For any $\mathrm{Z}\in(\mathrm{T}_{\mathbf{R}}\mathrm{X})_{y_0}$, the operator

$$(13.195)\qquad \varphi^2\left(u\mathrm{P}^{\mathrm{TY}}\mathrm{Z}+\frac{u}{\sqrt{\mathrm{T}}}\mathrm{P}^{\mathrm{N}}\mathrm{Z}\right)\left(\frac{\mathrm{T}^2}{u^2}(\mathrm{V}^-)^2\left(u\mathrm{P}^{\mathrm{TY}}\mathrm{Z}+\frac{u}{\sqrt{\mathrm{T}}}\mathrm{P}^{\mathrm{N}}\mathrm{Z}\right)-\mathrm{T}\frac{|\mathrm{P}^{\mathrm{N}}\mathrm{Z}|^2}{2}\mathrm{P}^{\xi^-}\right)$$

acting on $(\Lambda(\mathrm{T}^*_{\mathbf{R}}\mathrm{Y})\,\widehat{\otimes}\,\Lambda(\overline{\mathrm{N}}^*)\,\widehat{\otimes}\,\xi)_{y_0}$ is self-adjoint. Therefore it does not contribute to $\mathrm{Im}\,\langle\mathscr{C}_{u,\mathrm{T}}\,p^{\perp}s,\, p^{\perp}s'\rangle_{u,\mathrm{T},y_0,0}$. In view of (13.190)-(13.195), we find that $\mathscr{C}_{u,\mathrm{T}}$ is also compatible with (13.172).

The proof of Theorem 13.27 is completed. □

l) Estimates on the resolvent of $\mathscr{L}^{3, y_0}_{u, T}$.

From now on, ε and $T_0 \geqslant 1$ are fixed as in Theorem 13.27.

In the sequel, we consider the operator $\mathscr{L}^{3, y_0}_{u, T}$ as an unbounded operator, with domain

$$D = \{s \in \mathbf{K}^2_{y_0}, \ |P^N Z|^2 s \in \mathbf{K}^0_{y_0}\}.$$

We use notations which are formally similar to the notation in Section 11 l). In particular if $A \in \mathscr{L}(\mathbf{K}^0_{y_0})$ (resp. $\mathscr{L}(\mathbf{K}^{-1}_{y_0}, \mathbf{K}^1_{y_0})$), $\| A \|^{0, 0}_{u, T, y_0}$ (resp. $\| A \|^{-1, 1}_{u, T, y_0}$) denotes the norm of A with respect to the norm $| \ |_{u, T, y_0, 0}$ on $\mathbf{K}^0_{y_0}$ (resp. to the norms $| \ |_{u, T, y_0, -1}$ and $| \ |_{u, T, y_0, 1}$).

Theorem **13.28.** – *There exist* $C > 0$, $A > 0$, $\delta > 0$ *such that if*

$$U = \{\lambda \in \mathbf{C}, \ \operatorname{Re}(\lambda) \leqslant \delta \operatorname{Im}^2(\lambda) - A\}, \tag{13.196}$$

if $u \in]0, 1]$, $T \geqslant T_0$, $y_0 \in Y$, $\lambda \in U$, *the resolvent* $(\lambda - \mathscr{L}^{3, y_0}_{u, T})^{-1}$ *exists, extends to a continuous linear map from* $\mathbf{K}^{-1}_{y_0}$ *into* $\mathbf{K}^1_{y_0}$ *and moreover*

$$\begin{aligned} &\| (\lambda - \mathscr{L}^{3, y_0}_{u, T})^{-1} \|^{0, 0}_{u, T, y_0} \leqslant C, \\ &\| (\lambda - \mathscr{L}^{3, y_0}_{u, T})^{-1} \|^{-1, 1}_{u, T, y_0} \leqslant C (1 + |\lambda|)^2. \end{aligned} \tag{13.197}$$

Proof. – Using Theorem 13.27 instead of Theorem 11.26, the proof of Theorem 13.28 is the same as the proof of Theorem 11.27. □

m) Regularizing properties of the resolvent of $\mathscr{L}^{3, y_0}_{u, T}$.

We consider the family of functions $f_1, \ldots, f_r$ defined on X with values in [0, 1] which has been constructed in Section 11 m).

Definition **13.29.** – Let $\mathscr{Q}_{u, T, y_0}$ be the family of operators acting on $\mathbf{K}_{y_0}$

$$\mathscr{Q}_{u, T, y_0} = \Big\{ \nabla_{e_i}, \ 1 \leqslant i \leqslant 2l'; \ p^\perp \nabla_{e_i} p^\perp, \ 2l' + 1 \leqslant i \leqslant 2l; \\ \frac{\sqrt{T}}{u} p^\perp (\varphi f_j) \Big(u P^{TY} Z + \frac{u}{\sqrt{T}} P^N Z \Big) p^\perp, \ 1 \leqslant j \leqslant r \Big\}. \tag{13.198}$$

For $m \in \mathbf{N}$, let $\mathcal{Q}^m_{u,\mathrm{T},y_0}$ be the family of operators Q which can be written in the form

$$\mathrm{Q}=\mathrm{Q}_1 \ldots \mathrm{Q}_m;\quad \mathrm{Q}_i \in \mathcal{Q}_{u,\mathrm{T},y_0}.$$

For $m \in \mathbf{N}$, we equip the Sobolev space $\mathbf{K}^m_{y_0}$ with the norm $\| \ \|_{u,\mathrm{T},y_0,m}$ such that if $s \in \mathbf{K}^m_{y_0}$, then

$$\|s\|^2_{u,\mathrm{T},y_0,m}=\sum_{p=0}^{m} \sum_{\mathrm{Q} \in \mathcal{Q}^p_{u,\mathrm{T},y_0}} |\mathrm{Q}s|^2_{u,\mathrm{T},y_0,0}. \tag{13.199}$$

The analogue of Proposition 11.29 is the following result.

Theorem **13.30.** — *Take* $m \in \mathbf{N}$. *There exists* $\mathrm{C}_m > 0$ *such that for any* $u \in]0, 1]$, $\mathrm{T} \geqslant \mathrm{T}_0$, $y_0 \in \mathrm{Y}$, $\mathrm{Q}_1, \ldots, \mathrm{Q}_m \in \mathcal{Q}_{u,\mathrm{T},y_0}$, *if* $s, s' \in \mathbf{K}_{y_0}$ *have compact support, then*

$$|\langle [\mathrm{Q}_1, [\mathrm{Q}_2 \ldots [\mathrm{Q}_m, \mathscr{L}^{3,y_0}_{u,\mathrm{T}}]] \ldots] s, s' \rangle_{u,\mathrm{T},y_0,0}| \leqslant \mathrm{C}_m |s|_{u,\mathrm{T},y_0,1} |s'|_{u,\mathrm{T},y_0,1}. \tag{13.200}$$

Proof. — We will use the notation and also the organization of the proof of Theorem 13.27.

a) The case where $\mathrm{Q}=\nabla_{e_i}$ $(1 \leqslant i \leqslant 2l')$.

We make the key observations that $[\mathrm{Q}, \mathrm{E}]=0$, and also that the properties of the various operators considered in the proof of Theorem 13.27, which lead in particular to the third inequality in (13.144), are invariant by translations by elements of $(\mathrm{T}_{\mathbf{R}}\mathrm{Y})_{y_0}$.

b) The case where $\mathrm{Q}=p^{\perp} \nabla_{e_i} p^{\perp}$ $(2l'+1 \leqslant i \leqslant 2l)$.

Clearly

$$\left[\mathrm{Q}, \frac{\mathrm{T}^2}{u^2}(\varphi^2(\mathrm{V}^+)^2+(1-\varphi^2)\mathrm{P}^{\xi^+})\left(u\mathrm{P}^{\mathrm{TY}}\mathrm{Z}+\frac{u}{\sqrt{\mathrm{T}}}\mathrm{P}^{\mathrm{N}}\mathrm{Z}\right)\right] = \frac{\mathrm{T}^{3/2}}{u} \nabla_{e_i}(\varphi^2(\mathrm{V}^+)^2+(1-\varphi^2)\mathrm{P}^{\xi^+})\left(u\mathrm{P}^{\mathrm{TY}}\mathrm{Z}+\frac{u}{\sqrt{\mathrm{T}}}\mathrm{P}^{\mathrm{N}}\mathrm{Z}\right). \tag{13.201}$$

We then easily find that the operator (13.201) is harmless in the proof of (13.200). Also

$$[p^{\perp} \nabla_{e_i} p^{\perp}, \mathscr{S}_u]=0. \tag{13.202}$$

By Theorem 7.4, we find that

$$(13.203)\qquad [\mathrm{Q}, \mathrm{E}]=p^{\perp}[\nabla_{e_i}, \mathrm{E}]\,p^{\perp}=\mathrm{P}^{\xi^-}p^{\perp}\langle \mathrm{P}^{\mathrm{N}}\mathrm{Z}, e_i\rangle p^{\perp}\mathrm{P}^{\xi^-}.$$

In the estimates which follow, we assume that s, $s'\in \mathbf{K}^-_{y_0}$ have compact support. Clearly

$$(13.204)\qquad \langle [\mathrm{Q}, \mathrm{TE}]\,s, s'\rangle_{u,\mathrm{T},y_0,0}=\mathrm{T}\langle p^{\perp}\langle \mathrm{P}^{\mathrm{N}}\mathrm{Z}, e_i\rangle p^{\perp}s, ps'\rangle_{u,\mathrm{T},y_0,0} +\mathrm{T}\langle p^{\perp}\langle \mathrm{P}^{\mathrm{N}}\mathrm{Z}, e_i\rangle p^{\perp}s, p^{\perp}s'\rangle_{u,\mathrm{T},y_0,0}.$$

By proceeding as in (13.165), (13.166), we find that

$$(13.205)\qquad \mathrm{T}\langle p^{\perp}\langle \mathrm{P}^{\mathrm{N}}\mathrm{Z}, e_i\rangle p^{\perp}s, ps'\rangle_{u,\mathrm{T},y_0,0} =\mathrm{T}\langle\langle \mathrm{P}^{\mathrm{N}}\mathrm{Z}, e_i\rangle p^{\perp}s, (p-\widetilde{p})\,ps'\rangle_{u,\mathrm{T},y_0,0}.$$

Using (13.161), (13.205), we get

$$(13.206)\qquad |\mathrm{T}\langle p^{\perp}\langle \mathrm{P}^{\mathrm{N}}\mathrm{Z}, e_i\rangle p^{\perp}s, ps'\rangle_{u,\mathrm{T},y_0,0}|\leqslant \mathrm{C}\,|s|_{u,\mathrm{T},y_0,1}\,|s'|_{u,\mathrm{T},y_0,1}.$$

From (13.142), we see that

$$(13.207)\qquad \sqrt{\mathrm{T}}\,|p^{\perp}\langle \mathrm{P}^{\mathrm{N}}\mathrm{Z}, e_i\rangle p^{\perp}s|_{u,\mathrm{T},y_0,0}\leqslant \mathrm{C}\,|s|_{u,\mathrm{T},y_0,1}.$$

By (13.204)-(13.207) we finally obtain

$$(13.208)\qquad |\langle [\mathrm{Q}, \mathrm{TE}]\,s, s'\rangle_{u,\mathrm{T},y_0,0}|\leqslant \mathrm{C}\,|s|_{u,\mathrm{T},y_0,1}\,|s'|_{u,\mathrm{T},y_0,1}.$$

i.e. [Q, TE] is harmless in our proof of (13.200).

Also by (13.128), $p\nabla_{e_i}p=0$, and so

$$(13.209)\qquad [\mathrm{Q}, \mathscr{A}_{u,\mathrm{T}}]=[\nabla_{e_i}-\nabla_{e_i}p-p\nabla_{e_i}, \mathscr{A}_{u,\mathrm{T}}].$$

Now $[\nabla_{e_i}, \mathscr{A}_{u,\mathrm{T}}]$ is a scalar second order differential operator which only involves differentials in the directions of $(\mathrm{T}_{\mathbf{R}}\mathrm{Y})_{y_0}$. Using (13.146) and integration by parts, we find that $[\nabla_{e_i}, \mathscr{A}_{u,\mathrm{T}}]$ is harmless for our estimates.

The remaining terms in the right-hand side of (13.209) are also inoffensive. In fact because of trivial properties of the function $\mathrm{Z}_0\in \mathrm{N}_{\mathbf{R},y_0}\to \exp(-|\mathrm{Z}_0|^2/2)$, the operators $\nabla_{e_i}p$, $p\nabla_{e_i}$ act as uniformly bounded operators on $(\mathbf{K}^0_{y_0}, |\ |_{u,\mathrm{T},y_0,0})$, and they commute with the ∇_{e_j}'s $(1\leqslant j\leqslant 2l')$. We can then use (13.146) and integration by parts to take care of the remaining terms in (13.209).

Similarly

$$(13.210)\qquad [\mathrm{Q}, \mathscr{A}'_{u,\mathrm{T}}]=[\nabla_{e_i}-\nabla_{e_i}p-p\nabla_{e_i}, \mathscr{A}'_{u,\mathrm{T}}].$$

Using (13.146), Theorem 13.22 and its proof (especially equations (13.122)-(13.123)), we find that $[\nabla_{e_i}, \mathscr{A}'_{u,\mathrm{T}}]$ is a second order operator only involving differentiation in the directions of $\mathrm{N}_{\mathbf{R},y_0}$, which is essentially of the same type as $\mathscr{A}'_{u,\mathrm{T}}$, with obvious modifications of the powers of T which appear as factors. By proceeding as in

(13.180)-(13.184), we find that $[\nabla_{e_i}, \mathscr{A}'_{u,\mathrm{T}}]$ is harmless. Using the same properties of $\mathscr{A}'_{u,\mathrm{T}}$ as before and also the fact that $\exp(-|Z_0|^2/2) \in S(N_{\mathbf{R},y_0})$, we find that the remaining terms in the right-hand side of (13.210) are also harmless.

Handling the operator $[Q, \mathscr{A}''_{u,\mathrm{T}}]$ is also easy and left to the reader. The commutators of Q with the remaining terms in $\mathscr{M}^{3,y_0}_{u,\mathrm{T}}$ can be easily dealt with as before.

Set

$$(13.211)\qquad \mathscr{C}'_{u,\mathrm{T}} = \left(\varphi^2 \frac{\mathrm{T}}{u} \sum_{1}^{2l'} \left(e^j \wedge -\frac{u^2}{2} i_{e_j}\right) \nabla^{\xi}_{{}^0\tau e_j} \mathrm{V}\right)\left(u \mathrm{P}^{\mathrm{TY}} \mathrm{Z} + \frac{u}{\sqrt{\mathrm{T}}} \mathrm{P}^{\mathrm{N}} \mathrm{Z}\right),$$

$$\mathscr{C}''_{u,\mathrm{T}} = \varphi^2 \left(\mathrm{T} \sum_{2l'+1}^{2l} \frac{c(e_j)}{\sqrt{2}} \nabla^{\xi}_{{}^0\tau e_j} \mathrm{V} - \mathrm{T}\mathrm{P}^{\xi^-} \mathrm{S} \mathrm{P}^{\xi^-}\right)$$
$$\left(u \mathrm{P}^{\mathrm{TY}} \mathrm{Z} + \frac{u}{\sqrt{\mathrm{T}}} \mathrm{P}^{\mathrm{N}} \mathrm{Z}\right) + \varphi^2 \left(u \mathrm{P}^{\mathrm{TY}} \mathrm{Z} + \frac{u}{\sqrt{\mathrm{T}}} \mathrm{P}^{\mathrm{N}} \mathrm{Z}\right)$$
$$\left(\frac{\mathrm{T}^2}{u^2} (\mathrm{V}^-)^2 \left(u \mathrm{P}^{\mathrm{TY}} \mathrm{Z} + \frac{u}{\sqrt{\mathrm{T}}} \mathrm{P}^{\mathrm{N}} \mathrm{Z}\right) - \mathrm{T} \frac{|\mathrm{P}^{\mathrm{N}} \mathrm{Z}|^2}{2} \mathrm{P}^{\xi^-}\right).$$

Then

$$(13.212)\qquad \mathscr{C}_{u,\mathrm{T}} = \mathscr{C}'_{u,\mathrm{T}} + \mathscr{C}''_{u,\mathrm{T}}.$$

Clearly

$$(13.213)\qquad [Q, \mathscr{C}'_{u,\mathrm{T}}] = [\nabla_{e_i} - \nabla_{e_i} p - p \nabla_{e_i}, \mathscr{C}'_{u,\mathrm{T}}].$$

The operator $[\nabla_{e_i}, \mathscr{C}'_{u,\mathrm{T}}]$ is a matrix valued operator with a coefficient $\sqrt{\mathrm{T}}$. By the first identity in (12.43) and by Theorem 13.19, the operator $\mathrm{P}^{\xi^-} [\nabla_{e_i}, \mathscr{C}'_{u,\mathrm{T}}] \mathrm{P}^{\xi^-}$ vanishes for $\mathrm{P}^{\mathrm{N}} \mathrm{Z} = 0$. By proceeding as in (13.180)-(13.184), we find that if $s, s' \in \mathbf{K}_{y_0}$ have compact support

$$(13.214)\qquad |\langle [\nabla_{e_i}, \mathscr{C}'_{u,\mathrm{T}}] s, s' \rangle_{u,\mathrm{T},y_0,0}| \leqslant \mathrm{C} |s|_{u,\mathrm{T},y_0,1} |s'|_{u,\mathrm{T},y_0,1}.$$

Using the same arguments as before and obvious properties of the function $\exp(-|Z_0|^2/2)$, we also see that the remaining two terms in (13.213) can be handled in the same way. Therefore the commutator $[Q, \mathscr{C}'_{u,\mathrm{T}}]$ is harmless for our estimates.

Take $s, s' \in \mathbf{K}_{y_0}$ with compact support. Then by proceeding as in (13.165)-(13.166), we find that

$$(13.215)\qquad \langle [Q, \mathscr{C}''_{u,\mathrm{T}}] ps, ps' \rangle_{u,\mathrm{T},y_0,0} = \langle \nabla_{e_i} p^{\perp} \mathrm{P}^{\xi^-} \mathscr{C}''_{u,\mathrm{T}} \mathrm{P}^{\xi^-} ps, (p - \tilde{p}) ps' \rangle_{u,\mathrm{T},y_0,0}.$$

Using Theorem 13.19 and (13.161), we get

(13.216) $$|\langle [Q, \mathscr{C}''_{u,T}]\, ps, ps' \rangle_{u,T,y_0,0}| \leqslant C\, |s|_{u,T,y_0,1}\, |s'|_{u,T,y_0,1}.$$

On the other hand, we have

(13.217) $$\langle [Q, \mathscr{C}''_{u,T}]\, ps, p^{\perp} s' \rangle_{u,T,y_0,0} = \langle p^{\perp} \nabla_{e_i} p^{\perp} \mathscr{C}''_{u,T}\, ps, p^{\perp} s' \rangle_{u,T,y_0,0}.$$

By proceeding as before, we find that

(13.218) $$|\langle [Q, \mathscr{C}''_{u,T}]\, ps, p^{\perp} s' \rangle_{u,T,y_0,0}| \leqslant C\, |s|_{u,T,y_0,1}\, |s'|_{u,T,y_0,1}.$$

Also

(13.219) $$\langle [Q, \mathscr{C}''_{u,T}]\, p^{\perp} s, ps' \rangle_{u,T,y_0,0} = \langle p^{\perp} \nabla_{e_i} p^{\perp} \mathscr{C}''_{u,T}\, p^{\perp} s, ps' \rangle_{u,T,y_0,0} - \langle p^{\perp} \nabla_{e_i} p^{\perp} s, \mathscr{C}''_{u,T}\, ps' \rangle_{u,T,y_0,0}.$$

Using (13.146) and integrating by parts in (13.219), we find that

(13.220) $$|\langle [Q, \mathscr{C}''_{u,T}]\, p^{\perp} s, ps' \rangle_{u,T,y_0,0}| \leqslant C\, |s|_{u,T,y_0,1}\, |s'|_{u,T,y_0,1}.$$

Finally

(13.221) $$[Q, \mathscr{C}''_{u,T}] = [\nabla_{e_i} - \nabla_{e_i} p - p \nabla_{e_i}, \mathscr{C}''_{u,T}].$$

Using obvious properties of the function $\exp(-|Z_0|^2/2)$, we easily deduce from (13.221) that

(13.222) $$|\langle [Q, \mathscr{C}''_{u,T}]\, p^{\perp} s, p^{\perp} s' \rangle_{u,T,y_0,0}| \leqslant C\, |s|_{u,T,y_0,1}\, |s'|_{u,T,y_0,1}.$$

From (13.215)-(13.222), we deduce that the commutator $[Q, \mathscr{C}''_{u,T}]$ is harmless.

c) The case where $Q = \frac{\sqrt{T}}{u} p^{\perp} (\varphi f_j) \left(u P^{TY} Z + \frac{u}{\sqrt{T}} P^N Z \right) p^{\perp}$ $(1 \leqslant j \leqslant r)$.

In this case

(13.223) $$\left[Q, \frac{T^2}{u^2} (\varphi^2 (V^+)^2 + (1-\varphi^2) P^{\xi^+}) \left(u P^{TY} Z + \frac{u}{\sqrt{T}} P^N Z \right) \right] = 0.$$

Also

(13.224) $$[Q, \mathscr{S}_u] = p^{\perp} \left[\frac{\sqrt{T}}{u} \varphi f_j \left(u P^{TY} Z + \frac{u}{\sqrt{T}} P^N Z \right), \mathscr{S}_u \right] p^{\perp}.$$

The operator $[\sqrt{T}/u\, \varphi f_j (u P^{TY} Z + (u/\sqrt{T}) P^N Z), \mathscr{S}_u]$ is a first order differential operator which only involves differentiation in the directions of $(T_{\mathbf{R}} Y)_{y_0}$, which comes with a coefficient $\sqrt{T}$. By using in particular (13.161) as in (13.167), we find that $[Q, \mathscr{S}_u]$ is harmless.

Also

$$[Q, TE] = p^{\perp}\left[\frac{u\sqrt{T}}{2}\Delta^{N}(\varphi f_j)\left(uP^{TY}Z+\frac{u}{\sqrt{T}}P^{N}Z\right) + T\sum_{i=2l'+1}^{2l}(\nabla_{e_i}\varphi f_j)\left(uP^{TY}Z+\frac{u}{\sqrt{T}}P^{N}Z\right)\nabla_{e_i}\right]p^{\perp}. \tag{13.225}$$

From (13.161), (13.225), we deduce as before that

$$|\langle [Q, TE]\, s, s'\rangle_{u,T,y_0,0}| \leqslant C\,|s|_{u,T,y_0,1}\,|s'|_{u,T,y_0,1}.$$

Also if $s, s' \in \mathbf{K}_{y_0}$ have compact support, then

$$\langle [Q, \mathscr{A}_{u,T}]\, ps, s'\rangle_{u,T,y_0,0} = \left\langle \frac{\sqrt{T}}{u}p^{\perp}\varphi f_j\left(uP^{TY}Z+\frac{u}{\sqrt{T}}P^{N}Z\right)p^{\perp}\mathscr{A}_{u,T}\,ps, s'\right\rangle_{u,T,y_0,0}. \tag{13.226}$$

Using the fact that φf_j vanishes for $P^{N}Z=0$, (13.146) and integration by parts, we find that

$$|\langle [Q, \mathscr{A}_{u,T}]\, ps, s'\rangle_{u,T,y_0,0}| \leqslant C\,|s|_{u,T,y_0,1}\,|s'|_{u,T,y_0,1}. \tag{13.227}$$

Also

$$[Q, \mathscr{A}_{u,T}] = \frac{\sqrt{T}}{u}\left(\left[\varphi f_j\left(uP^{TY}Z+\frac{u}{\sqrt{T}}P^{N}Z\right), \mathscr{A}_{u,T}\right] + \left[p\,\varphi f_j\left(uP^{TY}Z+\frac{u}{\sqrt{T}}P^{N}Z\right)p - p\,\varphi f_j\left(uP^{TY}Z+\frac{u}{\sqrt{T}}P^{N}Z\right) - \varphi f_j\left(uP^{TY}Z+\frac{u}{\sqrt{T}}P^{N}Z\right)p, \mathscr{A}_{u,T}\right]\right). \tag{13.228}$$

The operator

$$\frac{\sqrt{T}}{u}\left[\varphi f_j\left(uP^{TY}Z+\frac{u}{\sqrt{T}}P^{N}Z\right), \mathscr{A}_{u,T}\right]$$

is a first order differential operator only involving differentiations in the directions of $(T_{\mathbf{R}} Y)_{y_0}$ and a coefficient $\sqrt{T}$. Also since f_j vanishes for $P^N Z=0$, the operators

$$\frac{\sqrt{T}}{u} p \varphi f_j \left(u P^{TY} Z + \frac{u}{\sqrt{T}} P^N Z\right) p,$$

$$\frac{\sqrt{T}}{u} p \varphi f_j \left(u P^{TY} Z + \frac{u}{\sqrt{T}} P^N Z\right),$$

$$\frac{\sqrt{T}}{u} \varphi f_j \left(u P^{TY} Z + \frac{u}{\sqrt{T}} P^N Z\right) p$$

remain uniformly bounded on $(\mathbf{K}^0_{y_0}, |\ |_{u, T, y_0, 0})$ together with their derivatives in the variable $P^{TY} Z$. We thus deduce from (13.146), (13.228) that

$$|\langle [Q, \mathscr{A}_{u,T}] p^{\perp} s, s' \rangle_{u, T, y_0, 0}| \leqslant C |s|_{u, T, y_0, 1} |s'|_{u, T, y_0, 1}. \tag{13.229}$$

Also

$$\langle [Q, \mathscr{A}'_{u,T}] ps, s' \rangle = \left\langle p^{\perp} \frac{\sqrt{T}}{u} \varphi f_j \left(u P^{TY} Z + \frac{u}{\sqrt{T}} P^N Z\right) p^{\perp} \mathscr{A}'_{u,T}\, ps, s' \right\rangle_{u, T, y_0, 0}. \tag{13.230}$$

Since $\varphi f_j (u P^{TY} Z + (u/\sqrt{T}) P^N Z)$ vanishes for $P^N Z=0$, using the properties of $\mathscr{A}'_{u,T}$ which follow from (13.121)-(13.122) and from (13.180), we deduce from (13.230) that

$$|\langle [Q, \mathscr{A}'_{u,T}] ps, s' \rangle_{u, T, y_0, 0}| \leqslant C |s|_{u, T, y_0, 1} |s'|_{u, T, y_0, 1}. \tag{13.231}$$

We now use the analogue of (13.228) with $\mathscr{A}_{u,T}$ replaced by $\mathscr{A}'_{u,T}$. The operator

$$\frac{\sqrt{T}}{u} \left[\varphi f_j \left(u P^{TY} Z + \frac{u}{\sqrt{T}} P^N Z\right), \mathscr{A}'_{u,T}\right]$$

is a first order differential operator only involving differentiation in the directions of $N_{\mathbf{R}, y_0}$ which comes with a factor T, whose coefficients vanish for $P^N Z=0$, *i.e.* its coefficients grow at most like $u\sqrt{T}\, |P^N Z|$. One then easily finds that

$$\left|\left\langle \frac{\sqrt{T}}{u} \left[\varphi f_j \left(u P^{TY} Z + \frac{u}{\sqrt{T}} P^N Z\right), \mathscr{A}'_{u,T}\right] p^{\perp} s, s' \right\rangle_{u, T, y_0, 0}\right| \leqslant C |s|_{u, T, y_0, 1} |s'|_{u, T, y_0, 1}. \tag{13.232}$$

Because of the properties of f_j and of $\mathscr{A}'_{u,\mathrm{T}}$ stated before, the other commutators in the analogue of (13.228) do not raise any difficulty in establishing the estimate

$$|\langle [\mathrm{Q}, \mathscr{A}'_{u,\mathrm{T}}]\, p^{\perp} s, s' \rangle_{u,\mathrm{T}, y_0, 0}| \leqslant \mathrm{C}\, |s|_{u,\mathrm{T}, y_0, 1}\, |s'|_{u,\mathrm{T}, y_0, 1}. \tag{13.233}$$

From (13.230)-(13.233), we find that $[\mathrm{Q}, \mathscr{A}'_{u,\mathrm{T}}]$ is harmless.

The commutator $[\mathrm{Q}, \mathscr{A}''_{u,\mathrm{T}}]$ can be handled by the same techniques. In fact the operator $(\sqrt{\mathrm{T}}/u)\,[\varphi f_j, \mathscr{A}''_{u,\mathrm{T}}]$ is a first order differential operator, and each coefficient has to be analyzed. Details are left to the reader.

By using in particular (13.161) and by proceeding as in (13.215)-(13.222), the commutators of Q with the remaining terms in $\mathscr{L}^{3, y_0}_{u,\mathrm{T}}$ are easily dealt with by the same techniques as before.

We still define $\mathscr{C}'_{u,\mathrm{T}}$, $\mathscr{C}''_{u,\mathrm{T}}$ as in (13.211). Clearly

$$\begin{gathered}\left[\frac{\sqrt{\mathrm{T}}}{u}\, \varphi f_j \left(u \mathrm{P}^{\mathrm{TY}} \mathrm{Z} + \frac{u}{\sqrt{\mathrm{T}}} \mathrm{P}^{\mathrm{N}} \mathrm{Z}\right), \mathscr{C}'_{u,\mathrm{T}}\right] = 0,\\ \left[\frac{\sqrt{\mathrm{T}}}{u}\, \varphi f_j \left(u \mathrm{P}^{\mathrm{TY}} \mathrm{Z} + \frac{u}{\sqrt{\mathrm{T}}} \mathrm{P}^{\mathrm{N}} \mathrm{Z}\right), \mathscr{C}''_{u,\mathrm{T}}\right] = 0.\end{gathered} \tag{13.234}$$

By proceeding as in (13.213), (13.214) and using the properties of $\mathscr{C}'_{u,\mathrm{T}}$ and f_j which were listed before, we find that the commutator $[\mathrm{Q}, \mathscr{C}'_{u,\mathrm{T}}]$ is harmless. Also the commutator $[\mathrm{Q}, \mathscr{C}''_{u,\mathrm{T}}]$ can be dealt with by the same arguments as in (13.215)-(13.222).

We have thus shown that the commutator $[\mathrm{Q}, \mathscr{L}^{3, y_0}_{u,\mathrm{T}}]$ verifies the estimate (13.200).

d) Higher order commutators.

It appears from the analysis which has been done before that the commutators of length one are operators whose matrix and differential structures are roughly similar to the corresponding structure of $\mathscr{L}^{3, y_0}_{u,\mathrm{T}}$. One can then iterate the process which was described before and obtain Theorem 13.30 in full generality. □

If $\mathrm{A} \in \mathscr{L}(\mathbf{K}^m_{y_0}, \mathbf{K}^{m'}_{y_0})$, we denote by $|||\mathrm{A}|||^{m, m'}_{u,\mathrm{T}, y_0}$ the norm of A with respect to the norms $\|\ \|_{u,\mathrm{T}, y_0, m}$, $\|\ \|_{u,\mathrm{T}, y_0, m'}$ on $\mathbf{K}^m_{y_0}$, $\mathbf{K}^{m'}_{y_0}$.

Theorem **13.31.** — *For any* $m \in \mathbf{N}$, *there exists* $p_m \in \mathbf{N}$, $\mathrm{C}_m > 0$ *such that if* $u \in]0, 1]$, $\mathrm{T} \geqslant \mathrm{T}_0$, $y_0 \in \mathrm{Y}$, $\lambda \in \mathrm{U}$, *the resolvent* $(\lambda - \mathscr{L}^{3, y_0}_{u,\mathrm{T}})^{-1}$ *extends to a continuous linear map from* $\mathbf{K}^m_{y_0}$ *into* $\mathbf{K}^{m+1}_{y_0}$ *and moreover*

$$|||(\lambda - \mathscr{L}^{3, y_0}_{u,\mathrm{T}})^{-1}|||^{m, m+1}_{u,\mathrm{T}, y_0} \leqslant \mathrm{C}_m (1 + |\lambda|)^{p_m}. \tag{13.235}$$

Proof. — We first prove (13.235) when $m = 0$. By (13.142), if $s \in \mathbf{K}_{y_0}$ has compact support, then for $2l' + 1 \leqslant i \leqslant 2l$, $1 \leqslant j \leqslant r$

$$|p^{\perp} \nabla_{e_i} p^{\perp} s|_{u,\mathrm{T}, y_0, 0} \leqslant \mathrm{C}\, |\nabla_{e_i} p^{\perp} s|_{u,\mathrm{T}, y_0, 0} \leqslant \mathrm{C}'\, |s|_{u,\mathrm{T}, y_0, 1}, \tag{13.236}$$

$$\left|\frac{\sqrt{T}}{u} p^{\perp} \varphi f_j \left(u \mathrm{P}^{\mathrm{TY}} Z + \frac{u}{\sqrt{T}} \mathrm{P}^{\mathrm{N}} Z\right) p^{\perp} s\right|_{u, T, y_0, 0}$$
$$\leqslant C \frac{\sqrt{T}}{u} \left|\varphi f_j \left(u \mathrm{P}^{\mathrm{TY}} Z + \frac{u}{\sqrt{T}} \mathrm{P}^{\mathrm{N}} Z\right) p^{\perp} s\right|_{u, T, y_0, 0}$$
$$\leqslant C' \left(\frac{\sqrt{T}}{u} |s^+|_{u, T, y_0, 0} + \| \mathrm{P}^{\mathrm{N}} Z | p^{\perp} s^- |_{u, T, y_0, 0}\right) \leqslant C'' |s|_{u, T, y_0, 1}.$$

From (13.236), we deduce

$$\|s\|_{u, T, y_0, 1} \leqslant C''' |s|_{u, T, y_0, 1}. \tag{13.237}$$

When $m=0$, (13.235) follows from Theorem 13.28 and from (13.237).

Using Theorem 13.28 and Theorem 13.30 instead of Theorem 11.27 and Proposition 11.29, the proof then proceeds as the proof of Theorem 11.30. □

n) Uniform estimates on the kernel $\widetilde{\mathrm{F}}_u(\mathscr{L}^{3, y_0}_{u, T})$.

Theorem **13.32.** – *For any* $m \in \mathbf{N}$, *there exists* $C > 0$ *such that if* $u \in]0, 1]$, $T \geqslant T_0$, $y_0 \in Y$, *then*

$$\sup_{\substack{Z_0 \in N_{\mathbf{R}, y_0} \\ |Z_0| \leqslant \varepsilon \sqrt{T}/4u}} (1 + |Z_0|)^m |\widetilde{\mathrm{F}}_u(\mathscr{L}^{3, y_0}_{u, T})(Z_0, Z_0)| \leqslant C. \tag{13.238}$$

For any $M > 0$, $m' \in \mathbf{N}$, *there exists* $C' > 0$ *such that if* $u \in]0, 1]$, $T \geqslant T_0$, $y_0 \in Y$

$$\sup_{\substack{Z, Z' \in (T_{\mathbf{R}} X)_{y_0} \\ |\mathrm{P}^{\mathrm{TY}} Z|, |\mathrm{P}^{\mathrm{TY}} Z'| \leqslant \varepsilon/4 \\ |\mathrm{P}^{\mathrm{N}} Z|, |\mathrm{P}^{\mathrm{N}} Z'| \leqslant \varepsilon \sqrt{T}/4u \\ |\alpha|, |\alpha'| \leqslant m'}} \left|\frac{\partial^{|\alpha| + |\alpha'|}}{\partial Z^{\alpha} \partial Z'^{\alpha'}} \widetilde{\mathrm{F}}_u (\mathscr{L}^{3, y_0}_{u, T})(Z, Z')\right| \leqslant C'. \tag{13.239}$$

Proof. – We proceed very much as in the proofs of Theorems 11.31 and 12.14. We take $\delta > 0$, $A > 0$ as in Theorem 13.28. Let Γ be the contour in $\mathbf{C}$

$$\Gamma = \{\lambda \in \mathbf{C};\ \mathrm{Re}(\lambda) = \delta \, \mathrm{Im}^2(\lambda) - A\}. \tag{13.240}$$

By using Proposition 13.10 with $c^2 > \sup(A, 1/4\delta)$, we find that in the domain $\widetilde{\Gamma}$ of $\mathbf{C}$ which is limited by Γ, if $u \in]0, 1]$, the function $\widetilde{\mathrm{F}}_u(a)$ and its derivatives exhibit polynomial decay at infinity and this uniformly in $u \in]0, 1]$.

By Theorem 13.28, we see that if $u \in]0, 1]$, $T \geqslant T_0$ then

$$\tilde{F}_u(\mathscr{L}^{3, y_0}_{u, T}) = \frac{1}{2\pi i} \int_\Gamma \tilde{F}_u(\lambda)(\lambda - \mathscr{L}^{3, y_0}_{u, T})^{-1} d\lambda. \tag{13.241}$$

For $p \in \mathbf{N}$, let $\tilde{F}_{u, p}(\lambda)$ be the unique holomorphic function defined on a neighborhood of $\tilde{\Gamma}$, which is such that

- $\tilde{F}_{u, p}(\lambda) \to 0$ as $\lambda \in \tilde{\Gamma} \to +\infty$.
- The following equation holds

$$\frac{\tilde{F}^{(p-1)}_{u, p}(\lambda)}{(p-1)!} = F_u(\lambda). \tag{13.242}$$

Of course, for $u \in]0, 1]$, $\tilde{F}_{u, p}(\lambda)$ and its derivatives decay polynomially at infinity in $\tilde{\Gamma}$, and this uniformly in $u \in]0, 1]$. From (13.241), (13.242), we deduce that

$$\tilde{F}_u(\mathscr{L}^{3, y_0}_{u, T}) = \frac{1}{2\pi i} \int_\Gamma \tilde{F}_{u, 2p}(\lambda)(\lambda - \mathscr{L}^{3, y_0}_{u, T})^{-2p} d\lambda. \tag{13.243}$$

By Theorem 13.31, we know there exist $C > 0$, $q \in \mathbf{N}$ such that if $Q \in \mathscr{Q}^l_{u, T, y_0, 0}$, $l \leqslant p$, then

$$\| Q(\lambda - \mathscr{L}^{3, y_0}_{u, T})^{-p} \|^{0, 0}_{u, T, y_0} \leqslant C(1 + |\lambda|)^q. \tag{13.244}$$

By introducing the adjoint operator $\mathscr{L}^{3, y_0 *}_{u, T}$ of $\mathscr{L}^{3, y_0}_{u, T}$ with respect to the ordinary Hermitian product (13.107) on $\mathbf{K}^0_{y_0}$ and the corresponding adjoint Sobolev weights as in the proof of Theorem 11.31, we also find that if $Q \in \mathscr{Q}^l_{u, T, y_0}$, $l \leqslant p$, then

$$\| (\lambda - \mathscr{L}^{3, y_0}_{u, T})^{-p} Q \|^{0, 0}_{u, T, y_0} \leqslant C(1 + |\lambda|)^q. \tag{13.245}$$

From (13.244)-(13.245), we find that $Q \in \mathscr{Q}^l_{u, T, y_0}$, $Q' \in \mathscr{Q}^{l'}_{u, T, y_0}$, $l, l' \leqslant p$, then

$$\| Q(\lambda - \mathscr{L}^{3, y_0}_{u, T})^{-2p} Q' \|^{0, 0}_{u, T, y_0} \leqslant C(1 + |\lambda|)^{2q}. \tag{13.246}$$

Using (13.243), (13.246), we deduce that if $l, l' \in \mathbf{N}$, $Q \in \mathscr{Q}^l_{u, T, y_0, 0}$, $Q' \in \mathscr{Q}^{l'}_{u, T, y_0, 0}$ then

$$\| Q \tilde{F}_{u, T}(\mathscr{L}^{3, y_0}_{u, T}) Q' \|^{0, 0}_{u, T, y_0} \leqslant C. \tag{13.247}$$

Let R be one of the operators $\nabla_{e_i} (2l' + 1 \leqslant i \leqslant 2l)$, $(\sqrt{T}/u)\, \varphi f_j (u P^{TY} Z + (u/\sqrt{T}) P^N Z)$. If $Q_1, \ldots, Q_l \in \mathscr{Q}_{u, T, y_0}$, set

$$H = Q_1 \ldots Q_l\, p^\perp R p. \tag{13.248}$$

If all the Q_i's are of the form $p^\perp \nabla_{e_i} p^\perp$ $(2l' + 1 \leqslant i \leqslant 2l)$, $(\sqrt{T}/u)\, p^\perp (\varphi f_j)\, (u P^{TY} Z$

$+(u/\sqrt{T})\, P^N Z)\, p^{\perp} (1 \leqslant j \leqslant r)$, it is clear that

$$\| H \|^{0,0}_{u, T, y_0} \leqslant C. \tag{13.249}$$

Equation (13.249) still holds if $p^{\perp}$ and p are interchanged in (13.248).

If some Q_i's are equal to $\nabla_{e_i} (1 \leqslant i \leqslant 2l')$, we can commute such operators in order to put them at the very right. We thus express H in the form

$$H = \sum U_i V_i, \tag{13.250}$$

where the U_i's are such that $\| U_i \|^{0,0}_{u, T, y_0} \leqslant C$, and the V_i's are products of operators $\nabla_{e_j} (1 \leqslant j \leqslant 2l')$. The same result holds if in (13.248), p and $p^{\perp}$ are interchanged.

Let $\mathscr{Q}'_{u, T, y_0}$ be the family of operators

$$\mathscr{Q}'_{u, T, y_0} = \{ \nabla_{e_i},\ 1 \leqslant i \leqslant 2l;\ (\sqrt{T}/u)\, (\varphi f_j)\, (u P^{TY} Z + (u/\sqrt{T})\, P^N Z),\ 1 \leqslant j \leqslant r \}. \tag{13.251}$$

For $l \in \mathbf{N}$, let $\mathscr{Q}'^{l}_{u, T, y_0}$ be the set of operators which are products of l operators in $\mathscr{Q}'_{u, T, y_0}$. From (13.247) and from the previous considerations, we deduce that if $l, l' \in \mathbf{N}$, if $Q \in \mathscr{Q}'^{l}_{u, T, y_0}$, $Q' \in \mathscr{Q}'^{l'}_{u, T, y_0}$, then

$$\| Q \tilde{F}_{u, T} (\mathscr{L}^{3, y_0}_{u, T})\, Q' \|^{0,0}_{u, T, y_0} \leqslant C. \tag{13.252}$$

Using (13.252), the proof of Theorem 13.32 continues very much as the proof of Theorem 11.31. Details are easy to fill and are left to the reader. □

o) The asymptotics of the operator $\tilde{F}_u (\mathscr{L}^{3, y_0}_{u, T})$ as $T \to +\infty$.

We now will calculate the asymptotics of the operator $\tilde{F}_u (\mathscr{L}^{3, y_0}_{u, T})$, and this uniformly in $u \in]0, 1]$. The general organization of this Section is closely related to the proof of Theorem 12.16.

Proposition **13.33.** — *There exists* $C > 0$ *such that if* $u \in]0, 1]$, $T \geqslant T_0$, $y_0 \in Y$, $\lambda \in U$, *if* $s' \in \mathbf{K}_{y_0}$ *has compact support, then*

$$| p^{\perp} (\lambda - \mathscr{L}^{3, y_0}_{u, T})^{-1} s' |_{u, T, y_0, 0} \leqslant \frac{C}{\sqrt{T}} (1 + |\lambda|)^2\, | s' |_{u, T, y_0, 0}. \tag{13.253}$$

Proof. — By Theorem 13.28, we know that

$$\begin{aligned} | (\lambda - \mathscr{L}^{3, y_0}_{u, T})^{-1} s' |_{u, T, y_0, 1} &\leqslant C (1 + |\lambda|)^2\, | s |_{u, T, y_0, -1} \\ &\leqslant C (1 + |\lambda|)^2\, | s |_{u, T, y_0, 0}. \end{aligned} \tag{13.254}$$

Now by (13.143)

$$|p^{\perp}(\lambda-\mathscr{L}^{3,y_0}_{u,T})^{-1}s'|_{u,T,y_0,0}\leqslant\frac{C'}{\sqrt{T}}|(\lambda-\mathscr{L}^{3,y_0}_{u,T})^{-1}s'|_{u,T,y_0,1}. \tag{13.255}$$

From (13.254), (13.255), we get (13.253). □

Set

$$\begin{aligned}\mathbf{K}'^{,1}_{y_0}&=\mathbf{K}^1_{y_0}\cap\mathbf{K}'^{,0}_{y_0},\\ \mathbf{K}'^{,1,\perp}_{y_0}&=\mathbf{K}^1_{y_0}\cap\mathbf{K}'^{,0,\perp}_{y_0}.\end{aligned} \tag{13.256}$$

Then $\mathbf{K}'^{,1}_{y_0}$, $\mathbf{K}'^{,1,\perp}_{y_0}$ are closed subspaces of $\mathbf{K}^1_{y_0}$, which inherit the norm $|\ |_{u,T,y_0,1}$.

By (13.142), (13.143), the linear map $\rho: s\in\mathbf{K}^0_{y_0}\to(ps,p^{\perp}s)\in\mathbf{K}'^{,0}_{y_0}\oplus\mathbf{K}'^{,0,\perp}_{y_0}$ induces a linear map from $\mathbf{K}^1_{y_0}$ into $\mathbf{K}'^{,1}_{y_0}\oplus\mathbf{K}'^{,1,\perp}_{y_0}$. Moreover ρ acts in both cases as an invertible operator, and the norms of ρ and ρ^{-1} with respect to the norms $|\ |_{u,T,y_0,0}$ and $|\ |_{u,T,y_0,1}$ are uniformly bounded.

Let $\mathbf{K}'^{,-1}_{y_0}$, $\mathbf{K}'^{,-1,\perp}_{y_0}$ be the antiduals of $\mathbf{K}'^{,1}_{y_0}$, $\mathbf{K}'^{,1,\perp}_{y_0}$. We identify $\mathbf{K}'^{,0}_{y_0}$, $\mathbf{K}'^{,0,\perp}_{y_0}$ with their antiduals by the Hermitian product $\langle\ ,\ \rangle_{u,T,y_0,0}$. Let $|\ |'_{u,T,y_0,-1}$, $|\ |'^{,\perp}_{u,T,y_0,-1}$ be the norms on $\mathbf{K}'^{,-1}_{y_0}$, $\mathbf{K}'^{,-1,\perp}_{y_0}$ associated with the restriction of $|\ |_{u,T,y_0,1}$ to $\mathbf{K}'^{,1}_{y_0}$, $\mathbf{K}'^{,1,\perp}_{y_0}$. We then have the continuous dense embeddings with bounded norm

$$\begin{aligned}&\mathbf{K}'^{,1}_{y_0}\to\mathbf{K}'^{,0}_{y_0}\to\mathbf{K}'^{,-1}_{y_0},\\ &\mathbf{K}'^{,1,\perp}_{y_0}\to\mathbf{K}'^{,0,\perp}_{y_0}\to\mathbf{K}'^{,-1,\perp}_{y_0}.\end{aligned}$$

The main purpose of the next Proposition is to show that for T large enough, $\mathbf{K}'^{,-1}_{y_0}$, $\mathbf{K}'^{,-1,\perp}_{y_0}$ can be considered as closed vector subspaces of $\mathbf{K}^{-1}_{y_0}$.

Proposition **13.34.** — *The linear map* $\rho:\mathbf{K}^0_{y_0}\to\mathbf{K}'^{,0}_{y_0}\oplus\mathbf{K}'^{,0,\perp}_{y_0}$ *extends to a continuous linear map from* $\mathbf{K}^{-1}_{y_0}$ *into* $\mathbf{K}'^{,-1}_{y_0}\oplus\mathbf{K}'^{,-1,\perp}_{y_0}$. *There exists* $T'_0\geqslant 1$ *such that for* $u\in]0,1]$, $T\geqslant T'_0$, $y_0\in Y$, *the linear map* $\rho:\mathbf{K}^{-1}_{y_0}\to\mathbf{K}'^{,-1}_{y_0}\oplus\mathbf{K}'^{,-1,\perp}_{y_0}$ *is invertible, and the norms of* ρ *and* ρ^{-1} *with respect to the norms* $|\ |_{u,T,y_0,-1}$, $|\ |'_{u,T,y_0,-1}$, $|\ |'^{,\perp}_{u,T,y_0,-1}$ *are uniformly bounded.*

Proof. — As in the proof of Theorem 13.27, we denote by $\tilde{p}$, $\tilde{p}^{\perp}$ the adjoints of p, $p^{\perp}$ with respect to the Hermitian product $\langle\ ,\ \rangle_{u,T,y_0,0}$. By the explicit formula for $\tilde{p}$, $\tilde{p}^{\perp}$ given in the proof of Theorem 13.27 after (13.164), we find that $\tilde{p}$, $\tilde{p}^{\perp}$ map $\mathbf{K}^1_{y_0}$ into itself with a uniformly bounded norm with respect to $|\ |_{u,T,y_0,1}$. It is now clear that ρ extends by continuity to a linear map from $\mathbf{K}^{-1}_{y_0}$ into $\mathbf{K}'^{,-1}_{y_0}\oplus\mathbf{K}'^{,-1,\perp}_{y_0}$.

We claim that for T large enough, the operators $\tilde{p}+p^{\perp}$ and $\tilde{p}^{\perp}+p$ are continuous invertible operators from $\mathbf{K}^1_{y_0}$ into itself, and that the norms of the inverses are uniformly bounded with respect to the norm $|\ |_{u,\mathrm{T},y_0,1}$. In fact

$$\begin{aligned}\tilde{p}^{\perp}+p&=1+p-\tilde{p}\\ \tilde{p}+p^{\perp}&=1+\tilde{p}-p.\end{aligned}\tag{13.257}$$

Using (13.159), (13.160) and corresponding identities for derivatives, we find easily that if s, $s'\in\mathbf{K}_{y_0}$ have compact support, then

$$\begin{aligned}|p(\tilde{p}-p)ps|_{u,\mathrm{T},y_0,1}&\leqslant\frac{\mathrm{C}}{\sqrt{\mathrm{T}}}|ps|_{u,\mathrm{T},y_0,1},\\ |p(\tilde{p}-p)p^{\perp}s|_{u,\mathrm{T},y_0,1}&\leqslant\frac{\mathrm{C}}{\mathrm{T}}|p^{\perp}s|_{u,\mathrm{T},y_0,1},\\ |p^{\perp}(\tilde{p}-p)ps|_{u,\mathrm{T},y_0,1}&\leqslant\mathrm{C}\,|ps|_{u,\mathrm{T},y_0,1},\\ |p^{\perp}(\tilde{p}-p)p^{\perp}s|_{u,\mathrm{T},y_0,1}&\leqslant\frac{\mathrm{C}}{\sqrt{\mathrm{T}}}|p^{\perp}s|_{u,\mathrm{T},y_0,1}.\end{aligned}\tag{13.258}$$

Recall that ρ is an isomorphism from $\mathbf{K}^1_{y_0}$ into $\mathbf{K}'^{,1}_{y_0}\oplus\mathbf{K}'^{,1,\perp}_{y_0}$ and that the corresponding norms $|\ |_{u,\mathrm{T},y_0,1}$ on these two vector spaces are uniformly equivalent. The matrices of the operators $\tilde{p}^{\perp}+p$ or $\tilde{p}+p^{\perp}$ acting on $\mathbf{K}'^{,1}_{y_0}\oplus\mathbf{K}'^{,1,\perp}_{y_0}$ are then of the form

$$\begin{bmatrix}1+O\left(\frac{1}{\sqrt{\mathrm{T}}}\right) & O\left(\frac{1}{\mathrm{T}}\right)\\ O(1) & 1+O\left(\frac{1}{\sqrt{\mathrm{T}}}\right)\end{bmatrix}.\tag{13.259}$$

It is now clear that for T large enough, $\tilde{p}+p^{\perp}$ and $\tilde{p}+p^{\perp}$ act as invertible operators on $\mathbf{K}^1_{y_0}$, and that their norms and the norms of their inverses are uniformly bounded.

The Hermitian product $\langle\ ,\ \rangle_{u,\mathrm{T},y_0,0}$ extends to a map from $\mathbf{K}^1_{y_0}\times\mathbf{K}^{-1}_{y_0}$ into $\mathbf{C}$, which is linear in the first variable, and antilinear in the second variable.

Let σ be the linear map from $\mathbf{K}'^{,-1}_{y_0}\oplus\mathbf{K}'^{,-1,\perp}_{y_0}$ into $\mathbf{K}^{-1}_{y_0}$, which is such that if $(\beta,\gamma)\in\mathbf{K}'^{,-1}_{y_0}\oplus\mathbf{K}'^{,-1,\perp}_{y_0}$, if $s\in\mathbf{K}^1_{y_0}$, then

$$\begin{aligned}\langle s,\sigma(\beta,\gamma)\rangle_{u,\mathrm{T},y_0,0}&=\langle p(\tilde{p}+p^{\perp})^{-1}\tilde{p}s,\beta\rangle_{u,\mathrm{T},y_0,0}\\ &\quad+\langle p^{\perp}(\tilde{p}^{\perp}+p)^{-1}\tilde{p}^{\perp}s,\gamma\rangle_{u,\mathrm{T},y_0,0}.\end{aligned}\tag{13.260}$$

For T large enough, σ is a continuous linear map whose norm is uniformly bounded. We claim that σ is an inverse of ρ. In fact $\tilde{p}^2=\tilde{p}$, $(\tilde{p}^{\perp})^2=\tilde{p}^{\perp}$, $\tilde{p}\tilde{p}^{\perp}=0$, $\tilde{p}^{\perp}\tilde{p}=0$. Also if $s\in\mathbf{K}'^{,1}_{y_0}$, $p^{\perp}s=0$, and so

$$p(\tilde{p}+p^{\perp})^{-1}\tilde{p}s=p(\tilde{p}+p^{\perp})^{-1}(\tilde{p}+p^{\perp})s=ps=s.\tag{13.261}$$

Similarly, if $s' \in \mathbf{K}'^{,1,\perp}_{y_0}$,

$$p^\perp(\tilde{p}^\perp + p)^{-1}\tilde{p}^\perp s' = s'. \tag{13.262}$$

From the previous considerations, we deduce easily that $\rho \circ \sigma$ is the identity.

Also if $s \in \mathbf{K}^1_{y_0}$, $\alpha \in \mathbf{K}^{-1}_{y_0}$, then

$$\begin{aligned}&\langle s, (\sigma \circ \rho)(\alpha) \rangle_{u,\mathrm{T},y_0,0} \\ &\qquad = \langle (\tilde{p}p(\tilde{p}+p^\perp)^{-1}\tilde{p} + \tilde{p}^\perp p^\perp (\tilde{p}^\perp + p)^{-1}\tilde{p}^\perp)s, \alpha \rangle_{u,\mathrm{T},y_0,0}.\end{aligned} \tag{13.263}$$

We claim that

$$\tilde{p}p(\tilde{p}+p^\perp)^{-1}\tilde{p} = \tilde{p}. \tag{13.264}$$

To verify (13.264), since $\tilde{p}^\perp + p$ is invertible, we only need to show that

$$\tilde{p}p(\tilde{p}+p^\perp)^{-1}\tilde{p}(\tilde{p}^\perp + p) = \tilde{p}(\tilde{p}^\perp + p), \tag{13.265}$$

or equivalently that

$$\tilde{p}p(\tilde{p}+p^\perp)^{-1}\tilde{p}p = \tilde{p}p. \tag{13.266}$$

Now (13.266) is an obvious consequence of the fact that $\tilde{p}p = (\tilde{p}+p^\perp)p$. Therefore (13.264) holds. Similarly

$$\tilde{p}^\perp p^\perp (\tilde{p}^\perp + p)^{-1}\tilde{p}^\perp = \tilde{p}^\perp. \tag{13.267}$$

From (13.263)-(13.267), we find that $\sigma \circ \rho$ is the identity.

The proof of Proposition 13.34 is thus completed. □

Of course we can always assume that in Theorem 13.27, T_0 is larger than T'_0, so that Proposition 13.34 in fact holds with $\mathrm{T}'_0 = \mathrm{T}_0$.

Since $\tilde{p}$, $\tilde{p}^\perp$ map $\mathbf{K}^1_{y_0}$ into itself with uniformly bounded norms with respect to $|\ |_{u,\mathrm{T},y_0,1}$, p, $p^\perp$ map $\mathbf{K}^{-1}_{y_0}$ into itself with uniformly bounded norms with respect to $|\ |_{u,\mathrm{T},y_0,-1}$. Set

$$\begin{aligned}{}^0\mathbf{K}'^{,-1}_{y_0} &= p\,\mathbf{K}^{-1}_{y_0} \\ {}^0\mathbf{K}'^{,-1,\perp}_{y_0} &= p^\perp\,\mathbf{K}^{-1}_{y_0}.\end{aligned} \tag{13.268}$$

Then ${}^0\mathbf{K}'^{,-1}_{y_0}$, ${}^0\mathbf{K}'^{,-1,\perp}_{y_0}$ inherit the norm $|\ |_{u,\mathrm{T},y_0,-1}$. Let ${}^0\rho$ be the linear map $s \in \mathbf{K}^{-1}_{y_0} \to {}^0\rho s = (ps, p^\perp s) \in {}^0\mathbf{K}'^{,-1}_{y_0} \oplus {}^0\mathbf{K}'^{,-1,\perp}_{y_0}$. Then ${}^0\rho$ is a continuous linear isomorphism and the norms of ${}^0\rho$ and $({}^0\rho)^{-1}$ are uniformly bounded with respect to the norm $|\ |_{u,\mathrm{T},y_0,-1}$.

We now identify $\mathbf{K}^{-1}_{y_0}$ with ${}^0\mathbf{K}'^{,-1}_{y_0} \oplus {}^0\mathbf{K}'^{,-1,\perp}_{y_0}$ by ${}^0\rho$. Then Proposition 13.34 exactly says that for $\mathrm{T} \geqslant \mathrm{T}_0$, ρ is a linear isomorphism from $\mathbf{K}^{-1}_{y_0} = {}^0\mathbf{K}'^{,-1}_{y_0} \oplus {}^0\mathbf{K}'^{,-1,\perp}_{y_0}$ into $\mathbf{K}'^{,-1}_{y_0} \oplus \mathbf{K}'^{,-1,\perp}_{y_0}$, and that the norms of ρ and ρ^{-1} are uniformly bounded.

Therefore, for $T \geqslant T_0$, we see that $\mathbf{K}'^{,-1}_{y_0}$, $\mathbf{K}'^{,-1,\perp}_{y_0}$ can be safely considered as closed subspaces of $\mathbf{K}^{-1}_{y_0}$, that $\mathbf{K}^{-1}_{y_0} = \mathbf{K}'^{,-1}_{y_0} \oplus \mathbf{K}'^{,-1,\perp}_{y_0}$ and that $\rho = (p, p^\perp)$ provides the canonical projection from $\mathbf{K}^{-1}_{y_0}$ on $\mathbf{K}'^{,-1}_{y_0}$ and $\mathbf{K}'^{,-1,\perp}_{y_0}$. This fact plays a crucial role in the sequel.

Set

$$\begin{aligned} \mathscr{L}_{u,T,1} &= p\,\mathscr{L}^{3,y_0}_{u,T}\,p, & \mathscr{L}_{u,T,2} &= p\,\mathscr{L}^{3,y_0}_{u,T}\,p^\perp, \\ \mathscr{L}_{u,T,3} &= p^\perp\,\mathscr{L}^{3,y_0}_{u,T}\,p, & \mathscr{L}_{u,T,4} &= p^\perp\,\mathscr{L}^{3,y_0}_{u,T}\,p^\perp. \end{aligned} \tag{13.269}$$

The $\mathscr{L}_{u,T,j}$'s $(1 \leqslant j \leqslant 4)$ will now be considered as the matrix components of $\mathscr{L}^{3,y_0}_{u,T}$. Also for $T \geqslant T_0$, these operators map $\mathbf{K}^1_{y_0}$ into $\mathbf{K}^{-1}_{y_0}$.

Proposition **13.35.** – *There exist* $C > 0$, $T_0 \geqslant 1$ *such that if* $u \in]0, 1]$, $T \geqslant T_0$, $y_0 \in Y$, $\lambda \in U$, $|\lambda| \leqslant T^{1/8}$, *the resolvent* $(\lambda - \mathscr{L}_{u,T,4})^{-1}$ *exists, extends to a linear continuous map from* $\mathbf{K}'^{,-1,\perp}_{y_0}$ *into* $\mathbf{K}'^{,1,\perp}_{y_0}$, *and is such that*

$$\| (\lambda - \mathscr{L}_{u,T,4})^{-1} \|^{-1,1}_{u,T,y_0} \leqslant C(1 + |\lambda|)^2. \tag{13.270}$$

Proof. – We use the notation of the proof of Theorem 13.27. Let $p^\perp_{u,T}$ be the orthogonal projection operator from $\mathbf{K}^0_{y_0}$ on $\mathbf{K}'^{,0,\perp}_{y_0}$ with respect to the Hermitian product $\langle\ ,\ \rangle_{u,T,y_0,0}$. Set

$$\mathscr{L}'_{u,T,4} = p^\perp_{u,T}\,\mathscr{L}^{3,y_0}_{u,T}\,p^\perp. \tag{13.271}$$

Then if $s, s' \in \mathbf{K}_{y_0}$ have compact support

$$\langle \mathscr{L}^{3,y_0}_{u,T}\,p^\perp s, p^\perp s' \rangle_{u,T,y_0,0} = \langle \mathscr{L}'_{u,T,4}\,p^\perp s, p^\perp s' \rangle_{u,T,y_0,0}. \tag{13.272}$$

From (13.272) and Theorem 13.27, it is clear that the operator $\mathscr{L}'_{u,T,4}$ verifies estimates similar to (13.144). By proceeding as in (12.76) and in Theorem 13.28, we find that if $\lambda \in U$, $(\lambda - \mathscr{L}'_{u,T,4})^{-1}$ exists and also that

$$\| (\lambda - \mathscr{L}'_{u,T,4})^{-1} \|^{-1,1}_{u,T,y_0} \leqslant C(1 + |\lambda|^2). \tag{13.273}$$

Clearly

$$\mathscr{L}'_{u,T,4} - \mathscr{L}_{u,T,4} = (p^\perp_{u,T} - p^\perp)\,\mathscr{L}^{3,y_0}_{u,T}\,p^\perp. \tag{13.274}$$

Therefore if $s, s' \in \mathbf{K}_{y_0}$ have compact support, then

$$\begin{aligned} &\langle (\mathscr{L}'_{u,T,4} - \mathscr{L}_{u,T,4})\,p^\perp s, p^\perp s' \rangle_{u,T,y_0,0} \\ &\quad = \langle (1 - p^\perp)\,\mathscr{L}^{3,y_0}_{u,T}\,p^\perp s, p^\perp s' \rangle_{u,T,y_0,0} = \langle p\,\mathscr{L}^{3,y_0}_{u,T}\,p^\perp s, p^\perp s' \rangle_{u,T,y_0,0}. \end{aligned} \tag{13.275}$$

By proceeding as in (13.165)-(13.166), we deduce from (13.275) that

$$\langle (\mathscr{L}'_{u,T,4} - \mathscr{L}_{u,T,4})\,p^\perp s, p^\perp s' \rangle_{u,T,y_0,0} \tag{13.276}$$

$$=\langle \mathscr{L}^{3,y_0}_{u,\mathrm{T}}\, p^{\perp} s,\ (\tilde{p}-p)\, p^{\perp} s' \rangle_{u,\mathrm{T},y_0,0}.$$

From (13.144) and (13.276), we find that

$$(13.277)\qquad \left|\langle (\mathscr{L}'_{u,\mathrm{T},4}-\mathscr{L}_{u,\mathrm{T},4})\, p^{\perp} s,\ p^{\perp} s' \rangle_{u,\mathrm{T},y_0,0}\right| \leqslant \mathrm{C}\, |p^{\perp} s|_{u,\mathrm{T},y_0,1}\, |(\tilde{p}-p)\, p^{\perp} s'|_{u,\mathrm{T},y_0,1}.$$

Now using (13.159), (13.160) and analogue inequalities for derivatives with respect to U and Z''_0, we find that

$$(13.278)\qquad |(\tilde{p}-p)\, p^{\perp} s'|_{u,\mathrm{T},y_0,1} \leqslant \frac{\mathrm{C}'}{\sqrt{\mathrm{T}}}\, |p^{\perp} s'|_{u,\mathrm{T},y_0,1}.$$

By (13.275)-(13.278), we deduce that

$$(13.279)\qquad \|(\mathscr{L}'_{u,\mathrm{T},4}-\mathscr{L}_{u,\mathrm{T},4})\|^{1,-1}_{u,\mathrm{T},y_0} \leqslant \frac{\mathrm{C}}{\sqrt{\mathrm{T}}}.$$

From (13.273), (13.279), we find that for T large enough, if $\lambda\in \mathrm{U}$, $|\lambda|\leqslant \mathrm{T}^{1/8}$, $(\lambda-\mathscr{L}_{u,\mathrm{T},4})^{-1}$ exists and is such that (13.270) holds.

The proof of Proposition 13.35 is completed. □

In the sequel, we take $\mathrm{T}_0\geqslant 1$ large enough so that Theorems 13.27, 13.28, 13.30, 13.31, 13.32, Propositions 13.33, 13.34 and 13.35 are simultaneously verified for $\mathrm{T}\geqslant \mathrm{T}_0$.

If $s\in \mathbf{K}^0_{y_0}$, set

$$(13.280)\qquad s^{\|}=ps,\quad s^{\perp}=p^{\perp} s.$$

If $\lambda\in \mathrm{U}$, the equation

$$(13.281)\qquad s=(\lambda-\mathscr{L}^{3,y_0}_{u,\mathrm{T}})^{-1} s'$$

is equivalent to

$$(13.282)\qquad \begin{aligned} &(\lambda-\mathscr{L}_{u,\mathrm{T},1})\, s^{\|}-\mathscr{L}_{u,\mathrm{T},2}\, s^{\perp}=s'^{\|}\\ &-\mathscr{L}_{u,\mathrm{T},3}\, s^{\|}+(\lambda-\mathscr{L}_{u,\mathrm{T},4})\, s^{\perp}=s'^{\perp}.\end{aligned}$$

By Proposition 13.33, we already know that for $\mathrm{T}\geqslant \mathrm{T}_0$, $\lambda\in \mathrm{U}$

$$(13.283)\qquad |s^{\perp}|_{u,\mathrm{T},y_0,0} \leqslant \frac{\mathrm{C}}{\sqrt{\mathrm{T}}}\,(1+|\lambda|)^2\, |s'|_{u,\mathrm{T},y_0,0}.$$

Definition **13.36.** — If $\mathrm{T}\geqslant \mathrm{T}_0$, $\lambda\in \mathrm{U}$, let $\mathscr{E}_{u,\mathrm{T}}(\lambda)$ be the operator

$$(13.284)\qquad \mathscr{E}_{u,\mathrm{T}}(\lambda)=\lambda-\mathscr{L}_{u,\mathrm{T},1}-\mathscr{L}_{u,\mathrm{T},2}\,(\lambda-\mathscr{L}_{u,\mathrm{T},4})^{-1}\,\mathscr{L}_{u,\mathrm{T},3}.$$

Proposition **13.37.** – *There exists* $C > 0$ *such that if* $u \in]0, 1]$, $T \geqslant T_0$, $y_0 \in Y$, $\lambda \in U$, *then*

$$\begin{aligned} &\|\mathscr{E}_{u,T}^{-1}(\lambda)\|_{u,T,y_0}^{0,0} \leqslant C \\ &\|\mathscr{E}_{u,T}^{-1}(\lambda)\|_{u,T,y_0}^{-1,1} \leqslant C(1+|\lambda|)^2. \end{aligned} \tag{13.285}$$

Proof. – Equation (13.285) follows from Theorem 13.28 and from (13.282). □

Proposition **13.38.** – *There exists* $C > 0$ *such that for any* $u \in]0, 1]$, $T \geqslant T_0$, $y_0 \in Y$, $\lambda \in U$, $|\lambda| \leqslant T^{1/8}$, *then*

$$\|\mathscr{E}_{u,T}^{-1}(\lambda)\,\mathscr{L}_{u,T,2}(\lambda-\mathscr{L}_{u,T,4})^{-1}\|_{u,T,y_0}^{0,0} \leqslant \frac{C}{\sqrt{T}}(1+|\lambda|)^4. \tag{13.286}$$

Proof. – By Theorem 13.27, $\mathscr{L}_{u,T,2}$ is a uniformly bounded operator from $\mathbf{K}_{y_0}^1$ into $\mathbf{K}_{y_0}^{-1}$. From Propositions 13.35 and 13.37, we find that

$$\|\mathscr{E}_{u,T}^{-1}(\lambda)\,\mathscr{L}_{u,T,2}(\lambda-\mathscr{L}_{u,T,4})^{-1}\|_{u,T,y_0}^{-1,1} \leqslant C(1+|\lambda|)^4. \tag{13.287}$$

Then (13.286) follows from (13.143), (13.287). □

We now obtain an essential result on three matrix components of the operator $\tilde{F}_u(\mathscr{L}_{u,T}^{3,y_0})$.

Theorem **13.39.** – *There exists* $C > 0$ *such that for any* $u \in]0, 1]$, $T \geqslant T_0$, $y_0 \in Y$, *then*

$$\begin{aligned} &\|p^\perp \tilde{F}_u(\mathscr{L}_{u,T}^{3,y_0})p^\perp\|_{u,T,y_0}^{0,0} \leqslant \frac{C}{\sqrt{T}}, \\ &\|p^\perp \tilde{F}_u(\mathscr{L}_{u,T}^{3,y_0})\,p\|_{u,T,y_0}^{0,0} \leqslant \frac{C}{\sqrt{T}}, \\ &\|p\,\tilde{F}_u(\mathscr{L}_{u,T}^{3,y_0})p^\perp\|_{u,T,y_0}^{0,0} \leqslant \frac{C}{\sqrt{T}}. \end{aligned} \tag{13.288}$$

Proof. – The first two lines in (13.288) follow from (13.47), (13.241) and (13.283). We rewrite (13.241) in the form

$$\begin{aligned} \tilde{F}_u(\mathscr{L}_{u,T}^{3,y_0}) = &\frac{1}{2\pi i}\int_{\Gamma\cap\{\lambda;\,|\lambda|\leqslant T^{1/8}\}} \tilde{F}_u(\lambda)(\lambda-\mathscr{L}_{u,T}^{3,y_0})^{-1}\,d\lambda \\ &+\frac{1}{2\pi i}\int_{\Gamma\cap\{\lambda;\,|\lambda|\geqslant T^{1/8}\}} \tilde{F}_u(\lambda)(\lambda-\mathscr{L}_{u,T}^{3,y_0})^{-1}\,d\lambda. \end{aligned} \tag{13.289}$$

Using (13.47), (13.282), (13.284), (13.286), we find that

$$\left\| p \frac{1}{2\pi i} \int_{\Gamma \cap \{\lambda;\, |\lambda| \leqslant T^{1/8}\}} \tilde{F}_u(\lambda)(\lambda - \mathscr{L}^{3, y_0}_{u, T})^{-1} \, d\lambda \, p^{\perp} \right\|^{0,0}_{u, T, y_0} \leqslant \frac{C}{\sqrt{T}}. \tag{13.290}$$

Also by (13.47) and (13.197), we see that

$$\left\| \frac{1}{2\pi i} \int_{\Gamma \cap \{\lambda;\, |\lambda| \geqslant T^{1/8}\}} \tilde{F}_u(\lambda)(\lambda - \mathscr{L}^{3, y_0}_{u, T})^{-1} \, d\lambda \right\|^{0,0}_{u, T, y_0} \leqslant \frac{C}{\sqrt{T}}. \tag{13.291}$$

Using (13.142), (13.289)-(13.291), we get the last inequality in (13.288). □

We now use the notation of Theorem 13.22. Observe that E is an unbounded invertible operator from $\mathbf{K}'^{,0,\perp,-}_{y_0}$ into itself.

Definition **13.40.** – For $u > 0$, $y_0 \in Y$, let $\Xi^{y_0}_u$ be the operator from $\mathbf{F}_{y_0}$ into itself

$$\Xi^{y_0}_u = \psi^{-1} (A_u - B_u E^{-1} D_u - C_u I_u^{-1} G_u) \psi. \tag{13.292}$$

In view of Theorem 13.22, one easily verifies that $\Xi^{y_0}_u$ is a second order elliptic differential operator.

To study the last matrix component of the operator $\tilde{F}_u(\mathscr{L}^{3, y_0}_{u, T})$, we now establish an important result.

Theorem **13.41.** – *There exists* $\alpha \in]0, 1/8]$, $C > 0$ *such that if* $u \in]0, 1]$, $T \geqslant T_0$, $\lambda \in U$, $|\lambda| \leqslant T^{\alpha}$, *then*

$$\| \mathscr{L}_{u, T, 1} + \mathscr{L}_{u, T, 2} (\lambda - \mathscr{L}_{u, T, 4})^{-1} \mathscr{L}_{u, T, 3} - p \psi \Xi^{y_0}_u \psi^{-1} p \|^{1, -1}_{u, T, y_0} \leqslant \frac{C}{T^{1/4}}. \tag{13.293}$$

Proof. – In view of the properties of the function $Z_0 \in N_{\mathbf{R}, y_0} \to \exp(-|Z_0|^2/2)$, using Theorem 13.22 and by proceeding as in Theorem 13.27, we find that

$$\| \mathscr{L}_{u, T, 1} - A_u \|^{1, -1}_{u, T, y_0} \leqslant \frac{C}{\sqrt{T}}. \tag{13.294}$$

By Theorem 13.27 and by Proposition 13.35, we know that for $T \geqslant T_0$, $\lambda \in U$, $|\lambda| \leqslant T^{1/8}$, then

$$\begin{gathered} \| (\lambda - \mathscr{L}_{u, T, 4})^{-1} \mathscr{L}_{u, T, 3} \|^{1, 1}_{u, T, y_0, 0} \leqslant C(1 + |\lambda|)^2, \\ \| \mathscr{L}_{u, T, 2} (\lambda - \mathscr{L}_{u, T, 4})^{-1} \|^{-1, -1} \leqslant C(1 + |\lambda|)^2. \end{gathered} \tag{13.295}$$

Let $\underline{\mathscr{L}}^{3, y_0}_{u, T}$ be the operator considered in equation (13.145). As we saw in the proof of Theorem 13.27, the (2, 2) matrix of the operator $\underline{\mathscr{L}}^{3, y_0}_{u, T}$ with respect to the splitting $\mathbf{K}^0_{y_0} = \mathbf{K}'^{,0}_{y_0} \oplus \mathbf{K}'^{,0,\perp}_{y_0}$ is diagonal. If $\mathscr{R}^{y_0}_{u, T} = \mathscr{L}^{3, y_0}_{u, T} - \underline{\mathscr{L}}^{3, y_0}_{u, T}$, only $\mathscr{R}^{y_0}_{u, T}$ contributes to $\mathscr{L}_{u, T, 2}$ and $\mathscr{L}_{u, T, 3}$

We can expand $\mathscr{L}_{u,T,j}$ according to its partial degree in the differentiation operators ∇_{e_i} $(1 \leqslant i \leqslant 2l')$, *i.e.*

$$\mathscr{L}_{u,T,j} = \sum_{p=0}^{2} \mathscr{L}^p_{u,T,j}. \tag{13.296}$$

Also by (13.112), each $\mathscr{L}^p_{u,T,j}$ has an asymptotic expansion as $T \to +\infty$ of the type

$$\mathscr{L}^p_{u,T,j} = \sum_{k \leqslant 4} \mathscr{L}^{p,k}_{u,j} T^{k/2}. \tag{13.297}$$

By (13.87), (13.177), (13.185) if $j=2, 3$

$$\begin{aligned} \mathscr{L}^{p,k}_{u,T,j} &= 0 \quad \text{if } p=2, \ k \geqslant 0; \\ &= 0 \quad \text{if } p=1, \ k > 0. \end{aligned} \tag{13.298}$$

From (13.295), (13.298), we deduce that if $\lambda \in U$, $|\lambda| \leqslant T^{1/8}$

$$\begin{aligned} &\| \mathscr{L}^2_{u,T,2} (\lambda - \mathscr{L}_{u,T,4})^{-1} \mathscr{L}_{u,T,3} \|^{1,-1}_{u,T,y_0} \\ &\quad \leqslant \frac{C}{\sqrt{T}} \| (\lambda - \mathscr{L}_{u,T,4})^{-1} \mathscr{L}_{u,T,3} \|^{1,1}_{u,T,y_0} \leqslant \frac{C'}{\sqrt{T}} (1+|\lambda|)^2, \\ &\| \mathscr{L}^0_{u,T,2} (\lambda - \mathscr{L}_{u,T,4})^{-1} \mathscr{L}^2_{u,T,3} \|^{1,-1}_{u,T,y_0} \\ &\quad \leqslant \frac{C}{\sqrt{T}} \| \mathscr{L}^0_{u,T,2} (\lambda - \mathscr{L}_{u,T,4})^{-1} \|^{-1,-1}_{u,T,y_0} \leqslant \frac{C'}{\sqrt{T}} (1+|\lambda|)^2, \\ &\| \mathscr{L}^1_{u,T,2} (\lambda - \mathscr{L}_{u,T,4})^{-1} \mathscr{L}_{u,T,3} \|^{1,-1}_{u,T,y_0} \leqslant C \| (\lambda - \mathscr{L}_{u,T,4})^{-1} \mathscr{L}_{u,T,3} \|^{1,0} \\ &\quad \leqslant \frac{C}{\sqrt{T}} \| (\lambda - \mathscr{L}_{u,T,4})^{-1} \mathscr{L}_{u,T,3} \|^{1,1} \leqslant \frac{C'}{\sqrt{T}} (1+|\lambda|)^2, \\ &\| \mathscr{L}^0_{u,T,2} (\lambda - \mathscr{L}_{u,T,4})^{-1} \mathscr{L}^1_{u,T,3} \|^{1,-1}_{u,T,y_0} \leqslant C \| \mathscr{L}^0_{u,T,2} (\lambda - \mathscr{L}_{u,T,4})^{-1} \|^{0,-1}_{u,T,y_0} \\ &\quad \leqslant \frac{C}{\sqrt{T}} \| \mathscr{L}^0_{u,T,2} (\lambda - \mathscr{L}_{u,T,4})^{-1} \|^{1,-1}_{u,T,y_0} \leqslant \frac{C}{\sqrt{T}} (1+|\lambda|)^2. \end{aligned} \tag{13.299}$$

By (13.270), (13.294), (13.299) and the properties of the function $Z_0 \in N_{\mathbf{R},y_0} \to \exp(-|Z_0|^2/2)$, we see that to prove (13.293), we only need to show that if $\alpha \in]0, 1/8]$ is small enough, if $\lambda \in U$, $|\lambda| \leqslant T^{\alpha}$, then

$$\left\| \sum_{k=0}^{2} \mathscr{L}^{0,k}_{u,2} T^{k/2} (\lambda - \mathscr{L}_{u,T,4})^{-1} \sum_{k'=0}^{2} \mathscr{L}^{0,k'}_{u,3} T^{k'/2} + B_u E^{-1} D_u + C_u I_u^{-1} G_u \right\|^{1,-1}_{u,T,y_0} \leqslant \frac{C}{T^{1/4}}. \tag{13.300}$$

a) The terms with $k=2$, $k'=2$.

By Theorem 13.22, we find that

$$\mathscr{L}^{0,2}_{u,2}=\mathrm{C}_u, \qquad \mathscr{L}^{0,2}_{u,3}=\mathrm{G}_u, \tag{13.301}$$

and so

$$\mathrm{T}^2\,\mathscr{L}^{0,2}_{u,2}(\lambda-\mathscr{L}_{u,\mathrm{T},4})^{-1}\,\mathscr{L}^{0,2}_{u,3}=\mathrm{T}^2\,\mathrm{C}_u\,\mathrm{P}^{\xi^+}(\lambda-\mathscr{L}_{u,\mathrm{T},4})^{-1}\,\mathrm{P}^{\xi^+}\,\mathrm{G}_u. \tag{13.302}$$

We now write $\mathscr{L}_{u,\mathrm{T},4}$ as a $(2,2)$ matrix acting as an unbounded operator on $\mathbf{K}'^{,0,\perp}_{y_0}=\mathbf{K}'^{,0,\perp,-}_{y_0}\oplus\mathbf{K}^{+,0}_{y_0}$. Using the notation of (13.111), we get

$$\mathscr{L}_{u,\mathrm{T},4}=\begin{bmatrix}\mathrm{E}_{u,\mathrm{T}} & \mathrm{F}_{u,\mathrm{T}}\\ \mathrm{H}_{u,\mathrm{T}} & \mathrm{I}_{u,\mathrm{T}}\end{bmatrix}. \tag{13.303}$$

Then if $s=s^++s^-$, $s'=s'^++s'^-$, the equation

$$(\lambda-\mathscr{L}_{u,\mathrm{T},4})\,s=s' \tag{13.304}$$

is equivalent to

$$\begin{aligned}(\lambda-\mathrm{E}_{u,\mathrm{T}})\,s^--\mathrm{F}_{u,\mathrm{T}}\,s^+&=s'^-,\\ -\mathrm{H}_{u,\mathrm{T}}\,s^-+(\lambda-\mathrm{I}_{u,\mathrm{T}})\,s^+&=s'^+.\end{aligned} \tag{13.305}$$

If $\lambda\in\mathrm{U}$, the operator $\lambda-\mathrm{I}_{u,\mathrm{T}}$ is invertible. More precisely using Theorem 13.27 and proceeding as in the proof of (12.76) or of Theorem 13.28, we get

$$\|(\lambda-\mathrm{I}_{u,\mathrm{T}})^{-1}\|^{-1,1}_{u,\mathrm{T},y_0}\leqslant \mathrm{C}(1+|\lambda|)^2. \tag{13.306}$$

Moreover the principal symbol of the operator $\mathscr{L}^{3,y_0}_{u,\mathrm{T}}$ is scalar, and the operator V^2 preserves the splitting $\xi_{y_0}=\xi^+_{y_0}\oplus\xi^-_{y_0}$. By proceeding as in the proof of Theorem 13.27, we deduce that

$$\|\mathrm{H}_{u,\mathrm{T}}\|^{0,-1}\leqslant \mathrm{C}. \tag{13.307}$$

Assume that $s'^-=0$. Then if $s'^+\in\mathbf{K}_{y_0}$ has compact support, we find from (13.305) that

$$s^+=(\lambda-\mathrm{I}_{u,\mathrm{T}})^{-1}\,(s'^++\mathrm{H}_{u,\mathrm{T}}\,s^-). \tag{13.308}$$

From (13.270), (13.306)-(13.308), we see that if $\lambda\in\mathrm{U}$, $|\lambda|\leqslant \mathrm{T}^{1/8}$, then

$$|(\lambda-\mathrm{I}_{u,\mathrm{T}})^{-1}\,\mathrm{H}_{u,\mathrm{T}}\,s^-|_{u,\mathrm{T},y_0,0}\leqslant\frac{\mathrm{C}u}{\mathrm{T}}\,|(\lambda-\mathrm{I}_{u,\mathrm{T}})^{-1}\,\mathrm{H}_{u,\mathrm{T}}\,s^-|_{u,\mathrm{T},y_0,1} \tag{13.309}$$

$$\leqslant \frac{Cu}{T}(1+|\lambda|)^2 |s^-|_{u,T,y_0,0}$$

$$\leqslant \frac{Cu}{T^{3/2}}(1+|\lambda|)^2 |(\lambda-\mathscr{L}_{u,T,4})^{-1} s'^+|_{u,T,y_0,1}$$

$$\leqslant \frac{Cu}{T^{3/2}}(1+|\lambda|)^4 |s'^+|_{u,T,y_0,-1} \leqslant \frac{Cu^2}{T^{5/2}}(1+|\lambda|)^4 |s'^+|_{u,T,y_0,0}.$$

From (13.308), (13.309), we deduce that if $\lambda\in U$, $|\lambda|\leqslant T^{1/8}$

$$\|P^{\xi^+}(\lambda-\mathscr{L}_{u,T,4})^{-1}P^{\xi^+}-(\lambda-I_{u,T})^{-1}\|^{0,0}_{u,T,y_0} \leqslant \frac{Cu^2}{T^{5/2}}(1+|\lambda|)^4. \tag{13.310}$$

Also using formula (13.115) for $\mathscr{L}^{0,2}_{u,3}=G_u$, by proceeding as in Proposition 11.24 and in (13.174) and also by using again the fact that $\exp(-|Z_0|^2/2)\in S(N_{\mathbf{R},y_0})$, we find that if $s\in \mathbf{K}_{y_0}$ has compact support, then

$$|\mathscr{L}^{0,2}_{u,3}\, s|_{u,T,y_0,0} \leqslant \frac{C}{u}|s|_{u,T,y_0,0}. \tag{13.311}$$

Using formula (13.115) for $\mathscr{L}^{0,2}_{u,2}=C_u$ and (13.310), (13.311), we find that if $\lambda\in U$, $|\lambda|\leqslant T^{1/8}$

$$\begin{aligned} T^2 \|\mathscr{L}^{0,2}_{u,2}((\lambda-\mathscr{L}_{u,T,4})^{-1}-(\lambda-I_{u,T})^{-1})\mathscr{L}^{0,2}_{u,3}\|^{1,-1}_{u,T,y_0,0} \\ \leqslant T^2\|\mathscr{L}^{0,2}_{u,2}((\lambda-\mathscr{L}_{u,T,4})^{-1}-(\lambda-I_{u,T})^{-1})\mathscr{L}^{0,2}_{u,3}\|^{0,0}_{u,T,y_0,0} \\ \leqslant \frac{C}{T^{1/2}}(1+|\lambda|)^4. \end{aligned} \tag{13.312}$$

Also

$$(\lambda-I_{u,T})^{-1}-(\lambda-T^2 I_u)^{-1}=(\lambda-I_{u,T})^{-1}(I_{u,T}-T^2 I_u)(\lambda-T^2 I_u)^{-1}. \tag{13.313}$$

Using formula (13.115) for $\mathscr{L}^{0,2}_{u,3}=G_u$ and the fact that for any $m\in\mathbf{N}$, $|Z_0|^m\exp(-|Z_0|^2/2)$ is square integrable in $N_{\mathbf{R},y_0}$, by formula (13.87) for $\mathscr{M}^{3,y_0}_{u,T}$ and proceeding as in the proof of Theorem 13.27, we find that if $\lambda\in U$, $|\lambda|\leqslant T^{1/8}$

$$\|(I_{u,T}-T^2 I_u)(\lambda-T^2 I_u)^{-1}\mathscr{L}^{0,2}_{u,3}\|^{1,-1}_{u,T,y_0} \leqslant C\frac{u}{T^{3/2}}. \tag{13.314}$$

From (13.306), (13.311), (13.313), (13.314), if $\lambda\in U$, $|\lambda|\leqslant T^{1/8}$, we get

$$T^2\|\mathscr{L}^{0,2}_{u,2}((\lambda-I_{u,T})^{-1}-(\lambda-T^2 I_u)^{-1})\mathscr{L}^{0,2}_{u,3}\|^{1,-1}_{u,T,y_0} \tag{13.315}$$

$$\leqslant \frac{\mathrm{CT}^2}{u} \| (\lambda - \mathrm{I}_{u,\mathrm{T}})^{-1} (\mathrm{I}_{u,\mathrm{T}} - \mathrm{T}^2 \mathrm{I}_u) (\lambda - \mathrm{T}^2 \mathrm{I}_u)^{-1} \mathscr{L}^{0,2}_{u,3} \|^{1,0}_{u,\mathrm{T},y_0}$$

$$\leqslant \mathrm{CT} \| (\lambda - \mathrm{I}_{u,\mathrm{T}})^{-1} (\mathrm{I}_{u,\mathrm{T}} - \mathrm{T}^2 \mathrm{I}_u) (\lambda - \mathrm{T}^2 \mathrm{I}_u)^{-1} \mathscr{L}^{0,2}_{u,3} \|^{1,1}_{u,\mathrm{T},y_0}$$

$$\leqslant \mathrm{C} (1 + |\lambda|)^2 \frac{u}{\mathrm{T}^{1/2}}.$$

Also

$$(13.316) \qquad (\lambda - \mathrm{T}^2 \mathrm{I}_u)^{-1} + \frac{\mathrm{I}_u^{-1}}{\mathrm{T}^2} = \lambda (\lambda - \mathrm{T}^2 \mathrm{I}_u)^{-1} \frac{\mathrm{I}_u^{-1}}{\mathrm{T}^2}.$$

From (13.306), (13.316), we find that if $\lambda \in \mathrm{U}$, $|\lambda| \leqslant \mathrm{T}^{1/8}$

$$(13.317) \qquad \mathrm{T}^2 \left\| \mathscr{L}^{0,2}_{u,2} \left((\lambda - \mathrm{T}^2 \mathrm{I}_u)^{-1} + \frac{\mathrm{I}_u^{-1}}{\mathrm{T}^2} \right) \mathscr{L}^{0,2}_{u,3} \right\|^{1,-1}_{u,\mathrm{T},y_0}$$

$$\leqslant \mathrm{T}^2 \left\| \mathscr{L}^{0,2}_{u,2} \left((\lambda - \mathrm{T}^2 \mathrm{I}_u)^{-1} + \frac{\mathrm{I}_u^{-1}}{\mathrm{T}^2} \right) \mathscr{L}^{0,2}_{u,3} \right\|^{0,0}_{u,\mathrm{T},y_0} \leqslant \frac{\mathrm{C} |\lambda| u^2}{\mathrm{T}^2} \leqslant \frac{\mathrm{C} u^2}{\mathrm{T}}.$$

Using (13.301), (13.312), (13.315), (13.317), we find that if $\lambda \in \mathrm{U}$, $|\lambda| \leqslant \mathrm{T}^{1/8}$

$$(13.318) \qquad \| \mathrm{T}^2 \mathscr{L}^{0,2}_{u,2} (\lambda - \mathscr{L}_{u,\mathrm{T},4})^{-1} \mathscr{L}^{0,2}_{u,3} + \mathrm{C}_u \mathrm{I}_u^{-1} \mathrm{G}_u \|^{1,-1}_{u,\mathrm{T},y_0} \leqslant \frac{\mathrm{C}}{\mathrm{T}^{1/2}} (1 + |\lambda|)^4.$$

b) The terms with $k=2$, $k' < 2$ or $k < 2$, $k' = 2$.

Consider again the system (13.304), (13.305). We then find

$$(13.319) \qquad \begin{aligned} & s^+ = (\lambda - \mathrm{I}_{u,\mathrm{T}})^{-1} (s'_+ + \mathrm{H}_{u,\mathrm{T}} s^-), \\ & (\lambda - \mathrm{E}_{u,\mathrm{T}} - \mathrm{F}_{u,\mathrm{T}} (\lambda - \mathrm{I}_{u,\mathrm{T}})^{-1} \mathrm{H}_{u,\mathrm{T}}) s^- = s'^- + \mathrm{F}_{u,\mathrm{T}} (\lambda - \mathrm{I}_{u,\mathrm{T}})^{-1} s'^+. \end{aligned}$$

By (13.307) and its analogue for $\mathrm{F}_{u,\mathrm{T}}$, we know that

$$(13.320) \qquad \| \mathrm{H}_{u,\mathrm{T}} \|^{0,-1} \leqslant \mathrm{C}; \ \| \mathrm{F}_{u,\mathrm{T}} \|^{0,-1} \leqslant \mathrm{C} \frac{\sqrt{\mathrm{T}}}{u}.$$

From (13.270), (13.306), (13.319), (13.320), we get for $\lambda \in \mathrm{U}$, $|\lambda| \leqslant \mathrm{T}^{1/8}$

$$(13.321) \qquad |s^+|_{u,\mathrm{T},y_0,0} \leqslant \frac{u}{\mathrm{T}} |s^+|_{u,\mathrm{T},y_0,1}$$

$$\leqslant \frac{\mathrm{C} u}{\mathrm{T}} (1 + |\lambda|)^2 (|s'_+|_{u,\mathrm{T},y_0,-1} + |\mathrm{H}_{u,\mathrm{T}} s^-|_{u,\mathrm{T},y_0,-1})$$

$$\leqslant C'(1+|\lambda|)^2\left(\frac{u^2}{T^2}|s'^+|_{u,T,y_0,0}+\frac{u}{T}|s^-|_{u,T,y_0,0}\right),$$

$$|s^-|_{u,T,y_0,0}\leqslant\frac{1}{\sqrt{T}}|s^-|_{u,T,y_0,1}$$

$$\leqslant\frac{C}{\sqrt{T}}(1+|\lambda|)^2(|s'^-|_{u,T,y_0,-1}+|F_{u,T}(\lambda-I_{u,T})^{-1}s'^+|_{u,T,y_0,-1})$$

$$\leqslant C(1+|\lambda|)^2\left(\frac{1}{T}|s'^-|_{u,T,y_0,0}+\frac{1}{u}|(\lambda-I_{u,T})^{-1}s'^+|_{u,T,y_0,0}\right)$$

$$\leqslant C(1+|\lambda|)^2\left(\frac{1}{T}|s'^-|_{u,T,y_0,0}+(1+|\lambda|)^2\frac{u}{T^2}|s'^+|_{u,T,y_0,0}\right).$$

By (13.321), we deduce that if $\lambda\in U$, $|\lambda|\leqslant T^{1/8}$

$$(13.322)\qquad \begin{aligned}|s^+|_{u,T,y_0,0}&\leqslant C(1+|\lambda|)^6\left(\frac{u^2}{T^2}|s'^+|_{u,T,y_0,0}+\frac{u}{T^2}|s'^-|_{u,T,y_0,0}\right)\\ |s^-|_{u,T,y_0,0}&\leqslant C(1+|\lambda|)^4\left(\frac{u}{T^2}|s'^+|_{u,T,y_0,0}+\frac{1}{T}|s'^-|_{u,T,y_0,0}\right).\end{aligned}$$

Using (13.87), (13.91), the fact that $\mathscr{L}^{0,2}_{u,2}=\mathscr{L}^{0,2}_{u,2}P^{\xi^+}$ and obvious properties of the function $\exp(-|Z_0|^2/2)$, we deduce from the first inequality in (13.322) that

$$(13.323)\qquad \left\|T\mathscr{L}^{0,2}_{u,2}(\lambda-\mathscr{L}_{u,T,4})^{-1}\sum_{k'=0}^{1}\mathscr{L}^{0,k'}_{u,3}T^{k'/2}\right\|^{0,0}_{u,T,y_0}\leqslant\frac{C}{\sqrt{T}}(1+|\lambda|)^6.$$

Similarly since $\mathscr{L}^{0,2}_{u,3}=P^{\xi^+}\mathscr{L}^{0,2}_{u,3}$, we deduce from the second inequality in (13.322) that

$$(13.324)\qquad \left\|\sum_{k=0}^{1}\mathscr{L}^{0,k}_{u,2}T^{k/2}(\lambda-\mathscr{L}_{u,T,4})^{-1}T\mathscr{L}^{0,2}_{u,3}\right\|^{0,0}\leqslant\frac{C}{\sqrt{T}}(1+|\lambda|)^6.$$

From (13.323), (13.324), we find that the terms with $k=2$, $k'<2$ or $k<2$, $k'=2$ are harmless for the estimate (13.300).

c) The terms with $k=1$, $k'=1$.

Using (13.115), (13.322), we find that in $T\mathscr{L}^{0,1}_{u,2}(\lambda-\mathscr{L}_{u,T,4})^{-1}\mathscr{L}^{0,1}_{u,3}$, the only relevant term to be considered is given by

$$(13.325)\qquad \begin{aligned}&T\mathscr{L}^{0,1}_{u,2}P^{\xi^-}p^{\perp}(\lambda-\mathscr{L}_{u,T,4})^{-1}p^{\perp}P^{\xi^-}\mathscr{L}^{0,1}_{u,3}\\ &\qquad=TB_uP^{\xi^-}p^{\perp}(\lambda-\mathscr{L}_{u,T,4})^{-1}p^{\perp}P^{\xi^-}D_u.\end{aligned}$$

Now by (13.319), we find that

$$P^{\xi^-} p^{\perp} (\lambda - \mathscr{L}_{u,T,4})^{-1} p^{\perp} P^{\xi^-} = (\lambda - E_{u,T} - F_{u,T} (\lambda - I_{u,T})^{-1} H_{u,T})^{-1}. \tag{13.326}$$

Set

$$J_{u,T}(\lambda) = (\lambda - E_{u,T})^{-1} F_{u,T} (\lambda - I_{u,T})^{-1} H_{u,T}. \tag{13.327}$$

From (13.326), we obtain

$$P^{\xi^-} p^{\perp} (\lambda - \mathscr{L}_{u,T,4})^{-1} p^{\perp} P^{\xi^-} = (1 - J_{u,T}(\lambda))^{-1} (\lambda - E_{u,T})^{-1}. \tag{13.328}$$

By the analogue of Proposition 13.35 for $E_{u,T}$ (with practically the same proof), we know that if $\lambda \in U$, $|\lambda| \leqslant T^{1/8}$, then

$$\|(\lambda - E_{u,T})^{-1}\|_{u,T,y_0}^{-1,1} \leqslant C(1 + |\lambda|)^2. \tag{13.329}$$

Using (13.306), (13.320), (13.329), we find that if $\lambda \in U$, $|\lambda| \leqslant T^{1/8}$, then

$$\begin{aligned} |J_{u,T}(\lambda) s^-|_{u,T,y_0,0} &\leqslant \frac{1}{\sqrt{T}} |J_{u,T}(\lambda) s^-|_{u,T,y_0,1} \\ &\leqslant \frac{C}{\sqrt{T}} (1 + |\lambda|)^2 |F_{u,T} (\lambda - I_{u,T})^{-1} H_{u,T} s^-|_{u,T,y_0,-1} \\ &\leqslant \frac{C}{u} (1 + |\lambda|)^2 |(\lambda - I_{u,T})^{-1} H_{u,T} s^-|_{u,T,y_0,0} \\ &\leqslant \frac{C}{T} (1 + |\lambda|)^4 |H_{u,T} s^-|_{u,T,y_0,-1} \leqslant \frac{C}{T} (1 + |\lambda|)^4 |s^-|_{u,T,y_0,0}. \end{aligned} \tag{13.330}$$

From (13.330), we deduce that for $T \geqslant 1$ large enough, if $\lambda \in U$, $|\lambda| \leqslant T^{1/8}$

$$\|(1 - J_{u,T}(\lambda))^{-1} - 1\|_{u,T,y_0}^{0,0} \leqslant \frac{C}{\sqrt{T}}. \tag{13.331}$$

In view of (13.328)-(13.331), we find that if $\lambda \in U$, $|\lambda| \leqslant T^{1/8}$

$$\begin{aligned} T\|B_u (P^{\xi^-} p^{\perp} (\lambda - \mathscr{L}_{u,T,4})^{-1} p^{\perp} P^{\xi^-} - (\lambda - E_{u,T})^{-1}) D_u\|_{u,T,y_0}^{1,-1} \\ = T\|B_u ((1 - J_{u,T}(\lambda))^{-1} - 1)(\lambda - E_{u,T})^{-1} D_u\|_{u,T,y_0}^{1,-1} \\ \leqslant CT\|B_u ((1 - J_{u,T}(\lambda))^{-1} - 1)(\lambda - E_{u,T})^{-1} D_u\|_{u,T,y_0}^{0,0} \\ \leqslant C\sqrt{T}\|(\lambda - E_{u,T})^{-1}\|_{u,T,y_0}^{0,0} \leqslant \frac{C}{\sqrt{T}} \|(\lambda - E_{u,T})^{-1}\|_{u,T,y_0}^{-1,1} \end{aligned} \tag{13.332}$$

$$\leqslant \frac{C}{\sqrt{T}}(1+|\lambda|)^2.$$

Also

$$(\lambda - E_{u,T})^{-1} - (\lambda - TE)^{-1} = (\lambda - E_{u,T})^{-1}(E_{u,T} - TE)(\lambda - TE)^{-1}. \tag{13.333}$$

Using (13.115), (13.329) we find that if $\lambda \in U$, $|\lambda| \leqslant T^{1/8}$

$$\begin{aligned} T \| B_u (\lambda - E_{u,T})^{-1} (E_{u,T} - TE)(\lambda - TE)^{-1} D_u \|^{1,-1} \\ \leqslant CT \| (\lambda - E_{u,T})^{-1} (E_{u,T} - TE)(\lambda - TE)^{-1} D_u \|^{1,0} \\ \leqslant C\sqrt{T}(1+|\lambda|)^2 \| (E_{u,T} - TE)(\lambda - TE)^{-1} D_u \|^{1,-1}. \end{aligned} \tag{13.334}$$

Let $\mathbf{L}_{y_0}$, $\mathbf{L}_{y_0}^{\pm}$ be the sets of smooth sections of $(\Lambda(T_{\mathbf{R}}^* Y) \hat{\otimes} \Lambda(\bar{N}^*) \hat{\otimes} \xi)_{y_0}$, $(\Lambda(T_{\mathbf{R}}^* Y) \hat{\otimes} \Lambda(\bar{N}^*) \otimes \xi^{\pm})_{y_0}$ over $N_{\mathbf{R}, y_0}$. For $\mu \in \mathbf{R}$, we also introduce the corresponding Sobolev spaces $\mathbf{L}_{y_0}^{\mu}$, $\mathbf{L}_{y_0}^{\pm,\mu}$. We equip $\mathbf{L}_{y_0}^0$ with its standard L_2 Hermitian product. Let $|\ |$ be the corresponding norm. Let $\mathbf{L}_{y_0}^{\prime,0}$ be the finite dimensional subvector space of $\mathbf{L}_{y_0}^{-,0}$, $\Lambda(T_{\mathbf{R}}^* Y) \hat{\otimes} \{\exp(\theta_{y_0} - |Z_0|^2/2)\} \otimes \eta$, and let $\mathbf{L}_{y_0}^{\prime,0,\perp,-}$ be its orthogonal in $\mathbf{L}_{y_0}^{-,0}$. Then if $T \geqslant 1$, $\lambda \in U$, $|\lambda| \leqslant T^{1/8}$, $(\lambda - TE)^{-1}$ acts on $\mathbf{L}_{y_0}^{\prime,0,\perp,-}$. If $\|(\lambda - TE)^{-1}\|$ denotes the norm of $(\lambda - TE)^{-1}$ with respect to the norm $|\ |$ on $\mathbf{L}_{y_0}^{\prime,0,\perp,-}$, then

$$\|(\lambda - TE)^{-1}\| \leqslant \frac{C}{T}. \tag{13.335}$$

By fixing for the moment $u P^{TY} Z$, using (13.114), (13.115), it is clear that the operator D_u maps $\mathbf{L}_{y_0}^{\prime,0}$ into functions in $\mathbf{L}_{y_0}$ which exhibit polynomial decay at infinity together with their derivatives. Also for any $k \in \mathbf{N}$, $E^k(\lambda - TE)^{-1} = (\lambda - TE)^{-1} E^k$. If $\sigma \in \mathbf{L}_{y_0}^{\prime,0}$, $E^k D_u \sigma \in \mathbf{L}_{y_0}^0$, and so $E^k(\lambda - TE)^{-1} D_u \sigma \in \mathbf{L}_{y_0}^0$. By a simple property of the harmonic oscillator, we thus deduce that if $\sigma \in \mathbf{L}_{y_0}^{\prime,0}$, $(\lambda - TE)^{-1} D_u \sigma$ is smooth and decays polynomially at infinity together with its derivatives. From (13.335), we find that if P is any differential operator with polynomial coefficients on $N_{\mathbf{R}, y_0}$, then if $\lambda \in U$, $|\lambda| \leqslant T^{1/8}$

$$|P(\lambda - TE)^{-1} D_u \sigma| \leqslant \frac{C}{T} |\sigma|. \tag{13.336}$$

Using (13.336), taking the obvious Taylor expansion of the operator $E_{u,T} - TE$ as $T \to +\infty$ and by proceeding as in Theorem 13.27, we find that if $\lambda \in U$, $|\lambda| \leqslant T^{1/8}$

$$\|(E_{u,T} - TE)(\lambda - TE)^{-1} D_u\|_{u,T,y_0}^{1,-1} \leqslant \frac{C}{T}. \tag{13.337}$$

Using (13.333)-(13.337), we find that if $\lambda \in U$, $|\lambda| \leqslant T^{1/8}$

$$T \| B_u ((\lambda - E_{u,T})^{-1} - (\lambda - TE)^{-1}) D_u \|_{u,T,y_0}^{1,-1} \leqslant \frac{C}{\sqrt{T}} (1 + |\lambda|)^2. \tag{13.338}$$

Finally

$$T(\lambda - TE)^{-1} + E^{-1} = \lambda E^{-1} (\lambda - TE)^{-1}. \tag{13.339}$$

Using (13.335), (13.336), (13.339), we find that if $\lambda \in U$, $|\lambda| \leqslant T^{1/8}$

$$\| B_u (T(\lambda - TE)^{-1} + E^{-1}) D_u \|_{u,T,y_0}^{1,-1} \leqslant \frac{C}{\sqrt{T}}. \tag{13.340}$$

By (13.332), (13.338), (13.340), we find that if $\alpha \in]0, 1/8]$ is small enough, $\lambda \in U$, $|\lambda| \leqslant T^{1/8}$, then

$$\| TB_u P^{\xi^-} p^{\perp} (\lambda - \mathscr{L}_{u,T,4})^{-1} p^{\perp} P^{\xi^-} D_u + B_u E^{-1} D_u \|_{u,T,y_0}^{1,-1} \leqslant \frac{C}{T^{1/4}}. \tag{13.341}$$

d) The terms with $k < 1$ or $k' < 1$.

In view of (13.322), we find these terms are harmless.

By (13.318), (13.323), (13.324), (13.341), we obtain (13.300). The proof of Theorem 13.41 is completed. □

Recall that Γ is the contour in $\mathbf{C}$ defined in (13.240). Before we proceed, let us just say one can easily show that the operator $\Xi_u^{y_0}$ verifies estimates which are very similar to the estimates (13.144). We may assume that, in Theorem 13.28, $\delta > 0$ is small enough, and $A > 0$ is large enough, so that if $\lambda \in U$, the resolvent $(\lambda - \Xi_u^{y_0})^{-1}$ is well defined, and also

$$\begin{aligned} &\| p \psi (\lambda - \Xi_u^{y_0})^{-1} \psi^{-1} p \|_{u,T,y_0}^{0,0} \leqslant C, \\ &\| p \psi (\lambda - \Xi_u^{y_0})^{-1} \psi^{-1} p \|_{u,T,y_0}^{-1,1} \leqslant C (1 + |\lambda|)^2, \\ &\tilde{F}_u (\Xi_u^{y_0}) = \frac{1}{2\pi i} \int_{\Gamma} \tilde{F}_u (\lambda) (\lambda - \Xi_u^{y_0})^{-1} d\lambda. \end{aligned} \tag{13.342}$$

Also $\tilde{F}_u(\Xi_u^{y_0})$ has a smooth kernel $\tilde{F}_u(\Xi_u^{y_0})(U, U')$ $(U, U' \in (T_{\mathbf{R}} Y)_{y_0})$ with respect to the volume $dv_{TY}(U')/(2\pi)^{\dim Y}$.

We now prove the following essential result.

Theorem **13.42.** – *There exists* $C > 0$ *such that if* $u \in]0, 1]$, $T \geqslant T_0$ *then*

$$\| \tilde{F}_u (\mathscr{L}_{u,T}^{3,y_0}) - p \psi \tilde{F}_u (\Xi_u^{y_0}) \psi^{-1} p \|_{u,T,y_0}^{0,0} \leqslant \frac{C}{T^{1/4}}. \tag{13.343}$$

Proof. — In view of Theorem 13.39, we only need to show that

$$\|p\,\tilde{\mathrm{F}}_u(\mathscr{L}^{3,y_0}_{u,\mathrm{T}})\,p-p\,\psi\,\tilde{\mathrm{F}}_u(\Xi^{y_0}_u)\,\psi^{-1}p\|^{0,0}_{u,\mathrm{T},y_0}\leqslant\frac{\mathrm{C}}{\mathrm{T}^{1/4}}.\tag{13.344}$$

By (13.241), (13.282), (13.284), (13.285), we know that

$$p\,\tilde{\mathrm{F}}_u(\mathscr{L}^{3,y_0}_{u,\mathrm{T}})\,p=\frac{1}{2\pi i}\int_\Gamma\tilde{\mathrm{F}}_u(\lambda)\,\mathscr{E}^{-1}_{u,\mathrm{T}}(\lambda)\,d\lambda.\tag{13.345}$$

Now if $\lambda\in\Gamma$

$$\begin{aligned}&\mathscr{E}^{-1}_{u,\mathrm{T}}(\lambda)-p\,\psi\,(\lambda-\Xi^{y_0}_u)^{-1}\,\psi^{-1}p\\&\quad=\mathscr{E}^{-1}_{u,\mathrm{T}}(\lambda)\,(\mathscr{L}_{u,\mathrm{T},1}+\mathscr{L}_{u,\mathrm{T},2}\,(\lambda-\mathscr{L}_{u,\mathrm{T},4})^{-1}\,\mathscr{L}_{u,\mathrm{T},3}\\&\qquad\qquad-p\,\psi\,\Xi^{y_0}_u\,\psi^{-1}p)\,p\,\psi\,(\lambda-\Xi^{y_0}_u)^{-1}\,\psi^{-1}p.\end{aligned}\tag{13.346}$$

We take $\alpha\in]0,1/8]$ as in Theorem 13.41. Using Proposition 13.10 and also (13.285), (13.293), (13.342), (13.346), we get

$$\left\|\frac{1}{2\pi i}\int_{\Gamma\cap\{\lambda;\,|\lambda|\leqslant\mathrm{T}^\alpha\}}\tilde{\mathrm{F}}_u(\lambda)\,(\mathscr{E}^{-1}_{u,\mathrm{T}}(\lambda)-p\,\psi\,(\lambda-\Xi^{y_0}_u)^{-1}\,\psi^{-1}p)\,d\lambda\right\|^{0,0}_{u,\mathrm{T},y_0}\leqslant\frac{\mathrm{C}}{\mathrm{T}^{1/4}}.\tag{13.347}$$

On the other hand, by Proposition 13.10, by (13.285), (13.342), we find that for any $q\in\mathbf{N}$

$$\begin{aligned}&\left\|\frac{1}{2\pi i}\int_{\Gamma\cap\{\lambda;\,|\lambda|\geqslant\mathrm{T}^\alpha\}}\tilde{\mathrm{F}}_u(\lambda)\,\mathscr{E}^{-1}_{u,\mathrm{T}}(\lambda)\,d\lambda\right\|^{0,0}_{u,\mathrm{T},y_0}\leqslant\frac{\mathrm{C}_q}{\mathrm{T}^q},\\&\left\|\frac{1}{2\pi i}\int_{\Gamma\cap\{\lambda;\,|\lambda|\geqslant\mathrm{T}^\alpha\}}\tilde{\mathrm{F}}_u(\lambda)\,p\,\psi\,(\lambda-\Xi^{y_0}_u)^{-1}\,\psi^{-1}p\,d\lambda\right\|^{0,0}_{u,\mathrm{T},y_0}\leqslant\frac{\mathrm{C}_q}{\mathrm{T}^q}.\end{aligned}\tag{13.348}$$

From (13.344)-(13.348), we get (13.343). □

p) Identification of the operator $\Xi^{y_0}_u$.

If $\mathrm{U}\in\mathrm{B}^{\mathrm{TY}}_{y_0}(0,\varepsilon)$, we identify $(\mathrm{TY})_\mathrm{U}$, η_U with $(\mathrm{TY})_{y_0}$, η_{y_0} by parallel transport along the geodesic in Y, $t\in[0,1]\to t\mathrm{U}$, with respect to the connections ∇^{TY}, ∇^η. Therefore if $\mathrm{U}\in\mathrm{B}^{\mathrm{TY}}_{y_0}(0,\varepsilon)$, $(\Lambda(\mathrm{T}^{*(0,1)}\mathrm{Y})\otimes\eta)_\mathrm{U}$ is identified with $(\Lambda(\mathrm{T}^{*(0,1)}\mathrm{Y})\otimes\eta)_{y_0}$.

Using the previous trivialization, we find that the operator $(u\,\mathrm{D}^{\mathrm{Y}})^2$ now acts on smooth sections of $(\Lambda(\mathrm{T}^{*(0,1)}\mathrm{Y})\otimes\eta)_{y_0}$ over $\mathrm{B}^{\mathrm{TY}}_{y_0}(0, \varepsilon)$.

Let G_u be the linear map

$$h\in\mathbf{F}_{y_0}\to\mathrm{G}_u h\in\mathbf{F}_{y_0};\qquad \mathrm{G}_u h(\mathrm{U})=h\left(\frac{\mathrm{U}}{u}\right),\quad \mathrm{U}\in(\mathrm{T}_{\mathbf{R}}\mathrm{Y})_{y_0}. \tag{13.349}$$

Let Σ^{2,y_0}_u be the operator acting on smooth sections of $\Lambda(\mathrm{T}^{*(0,1)}\mathrm{Y})\otimes\eta$ over $\mathrm{B}^{\mathrm{TY}}_{y_0}(0, \varepsilon/u)$

$$\Sigma^{2,y_0}_u=\mathrm{G}_u^{-1}(u\,\mathrm{D}^{\mathrm{Y}})^2\,\mathrm{G}_u. \tag{13.350}$$

As in (11.51), we find that

$$\Sigma^{2,y_0}_u\in(c(\mathrm{T}_{\mathbf{R}}\mathrm{Y})\otimes\mathrm{End}\,\eta)_{y_0}\otimes\mathrm{Op}. \tag{13.351}$$

Let Σ^{3,y_0}_u be the operator obtained from Σ^{2,y_0}_u by replacing the Clifford variables $c(e_i)\,(1\leqslant i\leqslant 2l')$ by $\sqrt{2}\,(e^i/u)\wedge-(u/\sqrt{2})\,i_{e_i}$. Then Σ^{3,y_0}_u is a differential operator acting on smooth sections of $(\Lambda(\mathrm{T}^*_{\mathbf{R}}\mathrm{Y})\otimes\eta)_{y_0}$ over $\mathrm{B}^{\mathrm{TY}}_{y_0}(0, \varepsilon/u)$. Of course $\Xi^{y_0}_u$ also acts on smooth sections of $(\Lambda(\mathrm{T}^*_{\mathbf{R}}\mathrm{Y})\otimes\eta)_{y_0}$ over $\mathrm{B}^{\mathrm{TY}}_{y_0}(0, \varepsilon/u)$.

Theorem **13.43.** — *Over* $\mathrm{B}^{\mathrm{TY}}_{y_0}(0, \varepsilon/2u)$ *we have the identity of differential operators*

$$\Sigma^{3,y_0}_u=\Xi^{y_0}_u. \tag{13.352}$$

Proof. — Clearly, we only need to prove (13.352) for $u=1$. In the identities which follow, the Clifford variables $c(e_i)$ $(1\leqslant i\leqslant 2l')$ are replaced by $\sqrt{2}\,e^i\wedge-i_{e_i}/\sqrt{2}$.

As pointed out in Remark 13.13, the trivialization of $\Lambda(\mathrm{T}^{*(0,1)}\mathrm{X})\,\hat{\otimes}\,\xi$ considered in Section 8 g) is compatible with the trivialization of $\Lambda(\mathrm{T}^{*(0,1)}\mathrm{X})\,\hat{\otimes}\,\xi$ of Section 13 e).

By the methods used in the proof of Theorem 8.18, we find that as $\mathrm{T}\to+\infty$, the operator $\mathrm{G}^{-1}_{1,\mathrm{T}}(\mathrm{D}^{\mathrm{X}}+\mathrm{TV})\,\mathrm{G}_{1,\mathrm{T}}$ has a Taylor expansion

$$\mathrm{G}^{-1}_{1,\mathrm{T}}(\mathrm{D}^{\mathrm{X}}+\mathrm{TV})\,\mathrm{G}_{1,\mathrm{T}}=\sum_{k\leqslant 2}\mathscr{P}_{1,k}\,\mathrm{T}^{k/2}. \tag{13.353}$$

Using Theorem 8.18, and also equations (8.65), (8.71), we obtain the first coefficients in (13.353)

$$\begin{aligned}\mathrm{G}^{-1}_{1,\mathrm{T}}(\mathrm{D}^{\mathrm{X}}+\mathrm{TV})\,\mathrm{G}_{1,\mathrm{T}}&=\mathrm{TV}^{+}(\mathrm{P}^{\mathrm{TY}}\mathrm{Z})+\sqrt{\mathrm{T}}\,(\mathrm{D}^{\mathrm{N}}+\tilde{\nabla}^{\xi}_{\mathrm{P}^{\mathrm{N}}\mathrm{Z}}\mathrm{V}(\mathrm{P}^{\mathrm{TY}}\mathrm{Z}))\\&\quad+\mathrm{D}^{\mathrm{H}}+\mathrm{M}+\frac{1}{2}\tilde{\nabla}^{\xi}_{\mathrm{P}^{\mathrm{N}}\mathrm{Z}}\tilde{\nabla}^{\xi}_{\mathrm{P}^{\mathrm{N}}\mathrm{Z}}\mathrm{V}(\mathrm{P}^{\mathrm{TY}}\mathrm{Z})-\frac{\dim\mathrm{Y}}{\sqrt{2}}\,c(\nu_{\mathrm{P}^{\mathrm{TY}}\mathrm{Z}})+O\left(\frac{1}{\sqrt{\mathrm{T}}}\right).\end{aligned} \tag{13.354}$$

By squaring (13.354), we thus obtain the expansion of the operator $\mathrm{G}^{-1}_{1,\mathrm{T}}(\mathrm{D}^{\mathrm{X}}+\mathrm{TV})^2\,\mathrm{G}_{1,\mathrm{T}}$ as $\mathrm{T}\to+\infty$.

By (13.75), (13.76), for $|P^{TY}Z| \leqslant \varepsilon/2$, $|P^N Z| \leqslant \varepsilon\sqrt{T}/2$, the operators $G_{1,T}^{-1}(D^X+TV)^2 G_{1,T}$ and $\mathscr{L}_{1,T}^{2,y_0}$ coincide. If $Z \in (T_{\mathbf{R}}X)_{y_0}$, $|P^{TY}Z| \leqslant \varepsilon/2$, we thus find that the asymptotic expansion (13.112) of $\mathscr{L}_{1,T}^{3,y_0}$ is simply obtained by squaring (13.354).

Let $\tilde{D}$ be the operator

$$\tilde{D} = D^H + M + \frac{1}{2}\tilde{\nabla}^{\xi}_{P^N Z}\tilde{\nabla}^{\xi}_{P^N Z} V(P^{TY}Z) - \frac{\dim Y}{\sqrt{2}} c(\nu_{P^{TY}Z}). \tag{13.355}$$

Using (13.354), (13.355), we find that in the asymptotic expansion as $T \to +\infty$ of the operator $G_{1,T}^{-1}(D^X+TV)^2 G_{1,T}$

- The coefficient of T^2 is $(V^+)^2(P^{TY}Z)$.
- The coefficient of $T^{3/2}$ is the supercommutator

$$[V^+(P^{TY}Z), D^N + \tilde{\nabla}^{\xi}_{P^N Z} V(P^{TY}Z)].$$

- The coefficient of T is given by

$$(D^N + \tilde{\nabla}^{\xi}_{P^N Z} V(P^{TY}Z))^2 + [V^+(P^{TY}Z), \tilde{D}].$$

- The coefficient of $\sqrt{T}$ is the sum of

$$[D^N + \tilde{\nabla}^{\xi}_{P^N Z} V(P^{TY}Z), \tilde{D}]$$

and of the supercommutator of $V^+(P^{TY}Z)$ with a differential operator of order one.

- The constant coefficient is the sum of $(\tilde{D})^2$, of a supercommutator with $V^+(P^{TY}Z)$ and of a supercommutator with $D^N + \tilde{\nabla}^{\xi}_{P^N Z} V(P^{TY}Z)$.

Let $D^{N,-}$ be the restriction of D^N to smooth sections of $(\Lambda(T^{*(0,1)}X) \hat{\otimes} \xi^-)_{y_0}$. Using the previous considerations, Theorem 7.4, (7.23), and comparing with Theorem 13.22, we easily deduce that

$$\begin{aligned}
&A_1 = p(\tilde{D})^2 p,\\
&B_1 = p[D^{N,-} + \tilde{\nabla}^{\xi}_{P^N Z} V^-(P^{TY}Z), \tilde{D}] P^{\xi^-} p^{\perp},\\
&C_1 = p[V^+(P^{TY}Z), \tilde{D}] P^{\xi^+},\\
&D_1 = p^{\perp} P^{\xi^-}[D^{N,-} + \tilde{\nabla}^{\xi}_{P^N Z} V^-(P^{TY}Z), \tilde{D}] p,\\
&E = p^{\perp} P^{\xi^-}(D^{N,-} + \tilde{\nabla}^{\xi}_{P^N Z} V^-(P^{TY}Z))^2 P^{\xi^-} p^{\perp},\\
&G_1 = P^{\xi^+}[V^+(P^{TY}Z), \tilde{D}] p,\\
&I_1 = (V^+(P^{TY}Z))^2.
\end{aligned} \tag{13.356}$$

Using Theorem 7.4, (7.23), Proposition 8.13 and (13.356), we get

$$\begin{aligned}
&B_1 = p\tilde{D}\left(D^{N,-} + \frac{\sqrt{-1}}{\sqrt{2}}\hat{c}(J P^N Z)\right) P^{\xi^-} p^\perp,\\
&C_1 = p\tilde{D}V^+(P^{TY}Z)P^{\xi^+},\\
(13.357)\qquad &D_1 = p^\perp P^{\xi^-}\left(D^{N,-} + \frac{\sqrt{-1}}{\sqrt{2}}\hat{c}(J P^N Z)\right)\tilde{D}p,\\
&E = p^\perp P^{\xi^-}\left(D^{N,-} + \frac{\sqrt{-1}}{2}\hat{c}(J P^N Z)\right)^2 P^{\xi^-} p^\perp,\\
&G_1 = P^{\xi^+} V^+ (P^{TY}Z)\tilde{D}p.
\end{aligned}$$

From (13.356), (13.357), we find that

$$(13.358)\qquad A_1 - B_1 E^{-1} D_1 - C_1 I_1^{-1} G_1 = \tilde{p}(\tilde{D})^2 p - p\tilde{D}p^\perp P^{\xi^-}\tilde{D}p - p\tilde{D}P^{\xi^+}\tilde{D}p.$$

Now

$$(13.359)\qquad p^\perp = p^\perp P^{\xi^-} + P^{\xi^+}.$$

From (13.358), (13.359), we get

$$(13.360)\qquad A_1 - B_1 E^{-1} D_1 - C_1 I^{-1} G_1 = (p\tilde{D}p)^2.$$

Since $c(\nu_{P^{TY}Z})$ is the sum of two operators, one which increases the total degree in $\Lambda(\bar{N}^*)\hat{\otimes}\Lambda(N^*)$ by one, and the other which decreases the total degree by one, since $\exp(\theta_{y_0})$ is of total degree zero, then, as in (8.46), we get

$$(13.361)\qquad pc(\nu_{P^{TY}Z})p = 0.$$

Using Theorem 8.21, and (13.361) we find that

$$(13.362)\qquad \psi^{-1} p\tilde{D}p\psi = D^Y.$$

From (13.360), (13.362), we find that

$$(13.363)\qquad \Xi_1^{y_0} = (D^Y)^2,$$

which is exactly the identity (13.352) for $u=1$. Our Theorem is proved. □

Remark **13.44.** – Needless to say, Theorem 13.43 can also be obtained by completely calculating the operator $\Xi_u^{y_0}$ and by comparing with Lichnerowicz's formula for $(D^Y)^2$. In Section 14 d), we will make this calculation in the very degenerate case where u is equal to zero.

Let now $\tilde{F}_u(\Xi_u^{y_0})(U, U')$, $\tilde{F}_u(\Sigma_u^{3,y_0})(U, U')$ $(U, U' \in (T_{\mathbf{R}} Y)_{y_0})$ be the smooth kernels of the operators $\tilde{F}_u(\Xi_u^{y_0})$, $F_u(\Sigma_u^{3,y_0})$ calculated with respect to the volume element $dv_{TY}(U')/(2\pi)^{\dim Y}$.

Theorem **13.45.** – *For any* $u \in]0, 1]$, *the following identity holds*

$$\tilde{F}_u(\Xi_u^{y_0})(0, 0) = \tilde{F}_u(\Sigma_u^{3,y_0})(0, 0). \tag{13.364}$$

Proof. – By Theorem 13.43, the operators $\Xi_u^{y_0}$ and Σ_u^{3,y_0} coincide on $B_{y_0}^{TY}(0, \varepsilon/2u)$. As we saw in Section 13e), $\alpha \in]0, \varepsilon/2]$. Using the analogue of formula (13.32) and finite propagation speed, Theorem 13.45 follows. □

q) Proof of Theorem 13.6.

For $T \geqslant T_0$, (13.37) immediately follows from Proposition 13.17 and Theorem 13.32.

Using Theorem 13.32, Theorem 13.42, and by proceeding as in Section 11p), we find there exists $C > 0$, $\delta' \in]0, 1/2]$ such that if $u \in]0, 1]$, $T \geqslant T_0$, $Z_0 \in N_{\mathbf{R}, y_0}$, $|Z_0| \leqslant \varepsilon\sqrt{T}/8u$

$$\left| \left(\frac{1}{2\pi}\right)^{\dim X} \tilde{F}_u(\mathscr{L}_{u,T}^{3,y_0})(Z_0, Z_0) - \frac{\exp(-|Z_0|^2)}{\pi^{\dim N}} \left(\frac{1}{2\pi}\right)^{\dim Y} \tilde{F}_u(\Xi_u^{y_0})(0, 0)\, q \right| \leqslant \frac{C}{T^{\delta'}}. \tag{13.365}$$

Using (13.364), (13.365), we find that if $|Z_0| \leqslant \varepsilon\sqrt{T}/8u$

$$\left| \left(\frac{1}{2\pi}\right)^{\dim X} \tilde{F}_u(\mathscr{L}_{u,T}^{3,y_0})(Z_0, Z_0) - \frac{\exp(-|Z_0|^2)}{\pi^{\dim N}} \left(\frac{1}{2\pi}\right)^{\dim Y} \tilde{F}_u(\Sigma_u^{3,y_0})(0, 0)\, q \right| \leqslant \frac{C}{T^{\delta'}}. \tag{13.366}$$

Let $\tilde{F}_u((uD^Y)^2)(y, y')$ $(y, y' \in Y)$ be the smooth kernel of the operator $\tilde{F}_u((uD^Y)^2)$ with respect to the volume element $dv_{TY}(y')/(2\pi)^{\dim Y}$. By the analogue of Proposition 11.21, we know that

$$\mathrm{Tr}_s[\tilde{F}_u((uD^Y)^2)(y_0, y_0)] = (-i)^{\dim Y}\, \mathrm{Tr}\,[[\tilde{F}_u(\Sigma_u^{3,y_0})(0, 0)]^{\max}]. \tag{13.367}$$

Using (13.48), (13.367), we get

$$\mathrm{Tr}_s[F_u(uD^Y)(y_0, y_0)] = (-i)^{\dim Y}\, \mathrm{Tr}\,[[\tilde{F}_u(\Sigma_u^{3,y_0})(0, 0)]^{\max}]. \tag{13.368}$$

From Proposition 8.4 and from (13.368), we find that

$$\text{(13.369)} \qquad \mathrm{Tr}_s\left[\frac{\mathrm{N_H}\exp(-|\mathrm{Z}_0|^2)}{\pi^{\dim \mathrm{N}}}\left(\frac{1}{2\pi}\right)^{\dim \mathrm{Y}}[\tilde{\mathrm{F}}_u(\Sigma_u^{3,y_0})(0,0)]^{\max} q\right]$$
$$= (i)^{\dim \mathrm{Y}}\,\frac{\exp(-|\mathrm{Z}_0|^2)}{\pi^{\dim \mathrm{N}}}\,\frac{\dim \mathrm{N}}{2}\left(\frac{1}{2\pi}\right)^{\dim \mathrm{Y}}\mathrm{Tr}_s[\mathrm{F}_u(u\mathrm{D^Y})(y_0, y_0)].$$

We now use Proposition 13.17, (13.366)-(13.369), and we obtain for $\mathrm{T} \geqslant \mathrm{T}_0$, $|\mathrm{Z}_0| \leqslant \varepsilon\sqrt{\mathrm{T}}/8u$

$$\text{(13.370)} \qquad \left|\left(\frac{1}{2\pi}\right)^{\dim \mathrm{X}}\left(\frac{u}{\sqrt{\mathrm{T}}}\right)^{2\dim \mathrm{N}}\mathrm{Tr}_s\left[\mathrm{N_H}\,\mathrm{F}_u\left(u\mathrm{D^X}+\frac{\mathrm{T}}{u}\mathrm{V}\right)\right.\right.$$
$$\left.\left(\left(y_0, \frac{u\mathrm{Z}_0}{\sqrt{\mathrm{T}}}\right), \left(y_0, \frac{u\mathrm{Z}_0}{\sqrt{\mathrm{T}}}\right)\right)\right] k''\left(\frac{u\mathrm{Z}_0}{\sqrt{\mathrm{T}}}\right)$$
$$\left. - \frac{\exp(-|\mathrm{Z}_0|^2)}{\pi^{\dim \mathrm{N}}}\,\frac{\dim \mathrm{N}}{2}\left(\frac{1}{2\pi}\right)^{\dim \mathrm{Y}}\mathrm{Tr}_s[\mathrm{F}_u(u\mathrm{D^Y})(y_0, y_0)]\right| \leqslant \frac{\mathrm{C}}{\mathrm{T}^{\delta'}}.$$

Recall that $k''(0)=1$. Therefore if $|\mathrm{Z}_0| \leqslant \varepsilon\sqrt{\mathrm{T}}/8u$

$$\text{(13.371)} \qquad \left|k''\left(\frac{u\mathrm{Z}_0}{\sqrt{\mathrm{T}}}\right)-1\right| \leqslant \mathrm{C}'\,\frac{u|\mathrm{Z}_0|}{\sqrt{\mathrm{T}}}.$$

From (13.37), (13.370), (13.371), we obtain (13.38) for $\mathrm{T} \geqslant \mathrm{T}_0$.

We have thus established Theorem 13.6 when $\mathrm{T} \geqslant \mathrm{T}_0$.

On the other hand by local index theory, $\mathrm{Tr}_s[\mathrm{F}_u(u\mathrm{D^Y})] = \mathrm{Tr}_s[\tilde{\mathrm{F}}_u((u\mathrm{D^Y})^2)]$ remains uniformly bounded for $u \in]0, 1]$. So to prove Theorem 13.6, we only need to establish the estimate (13.37) in the range $1 \leqslant \mathrm{T} \leqslant \mathrm{T}_0$. However this estimate easily follows from the techniques used in Section 12g). We have thus established Theorem 13.6 in full generality.

The proof of Theorem 6.8 is finally completed.

XIV - A NEW DERIVATION OF THE ASYMPTOTICS OF THE ANALYTIC TORSION FORMS OF A SHORT EXACT SEQUENCE

To conclude this paper on a lighter note, we here give a new derivation of the second half of Theorem 5.9, which concerns the asymptotics of the forms $\Phi\,\mathrm{Tr}_s[\exp(-\mathscr{B}^2_u)]$ and $\Phi\,\mathrm{Tr}_s[\mathrm{N_H}\exp(-\mathscr{B}^2_u)]$ as $u \to +\infty$.

This result was first obtained in [B3, Theorem 7.7] by an explicit evaluation of the considered forms. We here obtain the asymptotics of these forms by using the techniques of Section 13. The identification of the limit of these forms as $u \to +\infty$ relies on a remarkable algebraic identity, which is proved in Theorem 14.12.

This Section is organized as follows. In a), we introduce our main assumptions and notation. We then construct from the operator $\mathscr{D}^{2,y_0}_{T^2}$ considered in Theorem 5.6 an operator $\mathscr{L}^{y_0}_T$, which we write in b) as a (2, 2) matrix. In c), if E is any matrix, we calculate the asymptotics of $\mathrm{Tr}_s[\mathrm{E}\exp(-\mathscr{D}^2_{T^2})]$ as $T \to +\infty$, by means of a limit operator Ξ, which we identify in d). Finally in e), we give a slightly weaker version of [B3, Theorem 7.7].

An initial version of this Section was written using the operator $\mathscr{C}^{2,y_0}_{T^2}$ instead of $\mathscr{D}^{2,y_0}_{T^2}$, which led to slightly more complicate calculations.

a) Assumptions and notation

We make the same assumptions and we use the same notation as in Section 5. In particular

$$0 \to \mathrm{L} \to \mathrm{M} \to \mathrm{N} \to 0 \tag{14.1}$$

denotes an exact sequence of holomorphic vector bundles on a complex manifold B, g^{M} is a Hermitian metric on M, and g^{L}, g^{N} are the induced metrics on L, N.

Definition **14.1.** — If $y_0 \in \mathrm{B}$, let $\mathbf{K}_{y_0}$ (resp. $\mathbf{K}^0_{y_0}$) be the set of smooth (resp. square integrable) sections of $\Lambda(\mathrm{T}^*_{\mathbf{R}}\mathrm{B})\,\hat{\otimes}\,\Lambda(\bar{\mathrm{N}}^*)\,\hat{\otimes}\,\Lambda(\mathrm{N}^*)$ over the fibre $\mathrm{M}_{\mathbf{R},y_0}$.

We equip $\mathbf{K}^0_{y_0}$ with the Hermitian product

(14.2) $$k, k' \in \mathbf{K}^0_{y_0} \to \langle k, k' \rangle = \int_{\mathrm{M}_{\mathbf{R}, y_0}} \langle k, k' \rangle (\mathrm{Z}) \frac{dv_{\mathrm{M}}(\mathrm{Z})}{(2\pi)^{\dim \mathrm{M}}}.$$

For any $u > 0$, the operator $\mathscr{D}^{2, y_0}_u$ defined in Theorem 5.6 acts on $\mathbf{K}_{y_0}$.

Definition **14.2.** — For $\mathrm{T} > 0$, let $\mathrm{F_T}$ be the linear map $h \in \mathbf{K}_{y_0} \to \mathrm{F_T} h \in \mathbf{K}_{y_0}$, with

(14.3) $$\mathrm{F_T} h(\mathrm{Z}) = h\left(\mathrm{P^L Z} + \frac{\mathrm{P^N Z}}{\sqrt{\mathrm{T}}}\right).$$

Let Δ^{L}, Δ^{M}, Δ^{N} be the Euclidean Laplacians on the fibres of L, M, N. Then

(14.4) $$\Delta^{\mathrm{M}} = \Delta^{\mathrm{L}} + \Delta^{\mathrm{N}}.$$

Recall that $\widehat{\mathrm{R^N}}$ is the natural action of $\mathrm{R^N}$ on $\Lambda(\mathrm{N}^*)$. Similarly, $\mathrm{P^N A^2 P^N}$ is a 2-form on B with values in skew-adjoint endomorphisms of N. Let $\widehat{\mathrm{P^N A^2 P^N}}$ be the natural action of $\mathrm{P^N A^2 P^N}$ on $\Lambda(\bar{\mathrm{N}}^*)$.

Then $\widehat{\mathrm{R^N}}$ and $\widehat{\mathrm{P^N A^2 P^N}}$ act naturally on $\Lambda(\mathrm{T^*_{\mathbf{R}} B}) \widehat{\otimes} \Lambda(\bar{\mathrm{N}}^*) \widehat{\otimes} \Lambda(\mathrm{N}^*)$.

Let $e_1, \ldots, e_{2l}$ be an orthonormal base of $\mathrm{L}_{\mathbf{R}, y_0}$ and let $e_{2l+1}, \ldots, e_{2m}$ be an orthonormal base of $\mathrm{N}_{\mathbf{R}, y_0}$.

Theorem **14.3.** — *For* $\mathrm{T} > 0$, *let* $\mathscr{L}_{\mathrm{T}}$ *be the operator*

(14.5) $$\mathscr{L}_{\mathrm{T}} = \mathrm{F_T} \mathscr{D}^2_{\mathrm{T}^2} \mathrm{F_T}^{-1}.$$

Then, the following identity holds

(14.6) $$\begin{aligned} \mathscr{L}_{\mathrm{T}} = {} & \mathrm{T}\left(-\frac{\Delta^{\mathrm{N}}}{2} + \frac{|\mathrm{P^N Z}|^2}{2} + \mathrm{S}\right) - \frac{1}{2}\Delta^{\mathrm{L}} \\ & - \frac{1}{2}\nabla_{(\mathrm{P^L R^M + P^L R^M P^N - P^L A^2 P^L})(\mathrm{P^L Z} + (\mathrm{P^N Z}/\sqrt{\mathrm{T}})) + \mathrm{P^N R^M P^N Z}} + \frac{1}{\sqrt{2}} \sum_1^{2l} c(\mathrm{A} e_i) \nabla_{e_i} \\ & - \frac{1}{8}\left|(\mathrm{R^M - P^L A^2 P^L + P^L R^M P^N - P^N R^M P^L})\left(\mathrm{P^L Z} + \frac{\mathrm{P^N Z}}{\sqrt{\mathrm{T}}}\right)\right|^2 \\ & + \frac{1}{2\sqrt{2}} c\left(\mathrm{A P^L (R^M + R^M P^N - P^L A^2 P^L)}\left(\mathrm{P^L Z} + \frac{\mathrm{P^N Z}}{\sqrt{\mathrm{T}}}\right)\right) \\ & + \widehat{\mathrm{R^N}} + \widehat{\mathrm{P^N A^2 P^N}} + \frac{1}{2}\mathrm{Tr}[\mathrm{R^M}] - \frac{1}{2}\mathrm{Tr}[\mathrm{P^N A^2 P^N}]. \end{aligned}$$

Proof. — From identity (5.12), we find that

$$\mathscr{D}_{T^2}^2 = -\frac{1}{2}\Big\{\Delta^M + \frac{1}{4}|(R^M - P^L A^2 P^L + P^L R^M P^N - P^N R^M P^L) Z|^2 + \frac{1}{2}\sum_1^{2l} (c(A e_i))^2 + \nabla_{(R^M + P^L R^M P^N - P^N R^M P^L - P^L A^2 P^L) Z} - \sqrt{2}\sum_1^{2l} c(A e_i)\nabla_{e_i} - \frac{1}{\sqrt{2}} c(A P^L (R^M + R^M P^N - P^L A^2 P^L) Z)\Big\} + T^2 \frac{|P^N Z|^2}{2} + TS + \widehat{R^N} + \frac{1}{2}\operatorname{Tr}[R^M]. \tag{14.7}$$

Then A is a 1-form on B taking values in skew-adjoint endomorphisms of M which exchange L and N. Therefore

$$\begin{aligned}\sum_1^{2l} (c(A e_i))^2 &= \sum_{\substack{1 \leqslant i \leqslant 2l \\ 2l+1 \leqslant j, k \leqslant 2m}} \langle A e_i, e_j\rangle c(e_j) \langle A e_i, e_k\rangle c(e_k) \\ &= -\sum_{2l+1 \leqslant j, k \leqslant 2m} \langle A e_j, A e_k\rangle c(e_j) c(e_k) \\ &= -\sum_{2l+1 \leqslant j, k \leqslant 2m} \langle P^N A^2 P^N e_j, e_k\rangle c(e_j) c(e_k).\end{aligned} \tag{14.8}$$

Now one easily verifies that

$$\widehat{P^N A^2 P^N} = \frac{1}{4}\sum_{2l+1 \leqslant j, k \leqslant 2m} \langle P^N A^2 P^N e_j, e_k\rangle c(e_j) c(e_k) + \frac{1}{2}\operatorname{Tr}[P^N A^2 P^N]. \tag{14.9}$$

Then (14.6) follows from (14.7)-(14.9). □

b) The operator $\mathscr{L}_T$ as a (2, 2) matrix

Definition **14.4.** — If $y_0 \in B$, let $\mathbf{F}_{y_0}$ (resp. $\mathbf{F}^0_{y_0}$) be the set of smooth (resp. square integrable) sections of $(\Lambda(T^*_{\mathbf{R}} B))_{y_0}$ over the fibre $\mathbf{L}_{\mathbf{R}, y_0}$.

We equip $\mathbf{F}^0_{y_0}$ with the Hermitian product

$$f, f' \in \mathbf{F}^0_{y_0} \to \langle f, f'\rangle = \int_{\mathbf{L}_{\mathbf{R}, y_0}} \langle f, f'\rangle (Z) \frac{dv_L(Z)}{(2\pi)^{\dim L}}. \tag{14.10}$$

Let θ_{y_0} be the Kähler form of the fibre $N_{\mathbf{R}, y_0}$.

Definition **14.5.** — Let φ be the linear map

$$a \in \mathbf{C} \to \frac{a \exp(\theta)}{2^{\dim N/2}} \in \Lambda(\bar{N}^*) \otimes \Lambda(N^*).$$

Let ψ be the linear map

$$f \in \mathbf{F}_{y_0}^0 \to f \exp\left(\theta_{y_0} - \frac{|P^N Z|^2}{2}\right) \in \mathbf{K}_{y_0}^0. \tag{14.11}$$

Let q be the orthogonal projection operator from $\Lambda(\bar{N}^*) \hat{\otimes} \Lambda(N^*)$ on the image of φ.

Let $\mathbf{K}_{y_0}^{\prime, 0}$ be the image of ψ in $\mathbf{K}_{y_0}^0$. By Theorem 7.4, one verifies that ψ is an isometry from $\mathbf{F}_{y_0}^0$ into $\mathbf{K}_{y_0}^{\prime, 0}$.

Let $\mathbf{K}_{y_0}^{\prime, 0, \perp}$ be the orthogonal space to $\mathbf{K}_{y_0}^{\prime, 0}$ in $\mathbf{K}_{y_0}^0$. Let p, $p^\perp$ be the orthogonal projection operators from $\mathbf{K}_{y_0}^0$ on $\mathbf{K}_{y_0}^{\prime, 0}$, $\mathbf{K}_{y_0}^{\prime, 0, \perp}$ respectively.

Set

$$\begin{aligned} \mathscr{L}_{T,1}^{y_0} &= p \mathscr{L}_T^{y_0} p, \quad \mathscr{L}_{T,2}^{y_0} = p \mathscr{L}_T^{y_0} p^\perp, \\ \mathscr{L}_{T,3}^{y_0} &= p^\perp \mathscr{L}_T^{y_0} p, \quad \mathscr{L}_{T,4}^{y_0} = p^\perp \mathscr{L}_T^{y_0} p^\perp. \end{aligned} \tag{14.12}$$

Then we can write $\mathscr{L}_T^{y_0}$ as a matrix with respect to the splitting $\mathbf{K}_{y_0}^0 = \mathbf{K}_{y_0}^{\prime, 0} \oplus \mathbf{K}_{y_0}^{\prime, 0, \perp}$

$$\mathscr{L}_T^{y_0} = \begin{bmatrix} \mathscr{L}_{T,1}^{y_0} & \mathscr{L}_{T,2}^{y_0} \\ \mathscr{L}_{T,3}^{y_0} & \mathscr{L}_{T,4}^{y_0} \end{bmatrix} \tag{14.13}$$

By (14.6), $\mathscr{L}_T^{y_0}$ can be written in the form

$$\mathscr{L}_T^{y_0} = \sum_{k=-2}^{2} \mathcal{O}_k T^{k/2}. \tag{14.14}$$

Therefore, for $j = 1, \ldots, 4$, the operators $\mathscr{L}_{T,j}^{y_0}$ have a similar decomposition.

Theorem **14.6.** — *There exist operators* $\mathscr{L}_1^{y_0}$, $\mathscr{L}_4^{y_0}$ *such that as* $T \to +\infty$

$$\begin{aligned} \mathscr{L}_{T,1}^{y_0} &= \mathscr{L}_1^{y_0} + O\left(\frac{1}{\sqrt{T}}\right), \\ \mathscr{L}_{T,2}^{y_0} &= O(1), \\ \mathscr{L}_{T,3}^{y_0} &= O(1), \\ \mathscr{L}_{T,4}^{y_0} &= T \mathscr{L}_4^{y_0} + O(\sqrt{T}). \end{aligned} \tag{14.15}$$

Moreover, the following identities hold

$$(14.16)\qquad \begin{aligned}\mathscr{L}_1 &= -\frac{\Delta^{L}}{2}-\frac{1}{2}\nabla_{R^L P^L Z}-\frac{1}{8}|R^L P^L Z|^2+\frac{1}{2}\operatorname{Tr}[R^L],\\ \mathscr{L}_4 &= p^{\perp}\left(-\frac{\Delta^{N}}{2}+\frac{|P^N Z|^2}{2}+S\right)p^{\perp}.\end{aligned}$$

Proof. – By Theorem 7.4, we know that

$$(14.17)\qquad K'^{,0}_{y_0}=\operatorname{Ker}(\mathscr{L}_4^{y_0}).$$

Moreover, using the fact that $P^N R^M P^N$ takes values in skew-adjoint endomorphisms of N, we find

$$(14.18)\qquad p\,\nabla_{P^N R^M P^N Z}\,p=0.$$

Also, we have the trivial identity

$$(14.19)\qquad P^L(R^M-A^2)P^L=P^L R^L P^L.$$

For $1 \leqslant i \leqslant 2l$, $c(A e_i)$ is a 1-form with values in operators which are sums of operators increasing or decreasing the total degree in $\Lambda(\bar{N}^*)\hat{\otimes}\Lambda(N^*)$ by one. Since $\exp(\theta)$ is of total degree zero, we find that, if $1 \leqslant i \leqslant 2l$, $pc(A e_i)p=0$. Therefore

$$(14.20)\qquad p\sum_{1}^{2l} c(A e_i)\nabla_{e_i}p=0.$$

The same argument shows that

$$(14.21)\qquad pc(AP^L(R^M+R^M P^N-P^L A^2 P^L)P^L Z)p=0.$$

Also, by proceeding as in the proof of Proposition 8.4, we find that

$$(14.22)\qquad \begin{aligned} p\widehat{R^N}p &= -\frac{1}{2}\operatorname{Tr}[R^N],\\ p\widehat{\widehat{P^N A^2 P^N}}p &= \frac{1}{2}\operatorname{Tr}[P^N A^2 P^N].\end{aligned}$$

Finally, we have the trivial

$$(14.23)\qquad \operatorname{Tr}[R^M]=\operatorname{Tr}[R^L]+\operatorname{Tr}[R^N].$$

Theorem 14.6 immediately follows from Theorem 14.3 and from (14.17)-(14.23). □

Remark **14.7.** – We can also derive Theorem 14.6 by making $u=0$ in Theorem 13.22. The fact that $\mathscr{P}_u$ vanishes at $u=0$ is directly related to the simple asymptotics of $\mathscr{L}^{y_0}_{T,2}$ and $\mathscr{L}^{y_0}_{T,3}$.

c) The asymptotics of $\mathrm{Tr}_s[\mathrm{E}\exp(-\mathscr{D}^2_{\mathrm{T}^2})]$

Let $\mathrm{P}^{y_0}_{\mathrm{T}}(\mathrm{Z}, \mathrm{Z}')$, $\mathrm{P}'^{y_0}_{\mathrm{T}}(\mathrm{Z}, \mathrm{Z}')$ $(\mathrm{Z}, \mathrm{Z}' \in \mathrm{M}_{\mathbf{R}, y_0})$ be the smooth kernels of the operators $\exp(-\mathscr{D}^{2, y_0}_{\mathrm{T}^2})$, $\exp(-\mathscr{L}^{y_0}_{\mathrm{T}})$ with respect to the volume element $dv_{\mathrm{M}}/(2\pi)^{\dim \mathrm{M}}$.

By (5.11), (5.13), we know that for any $\mathrm{T} > 0$, there exist c, $\mathrm{C} > 0$ such that if $\mathrm{Z}_0 \in \mathrm{N}_{\mathbf{R}, y_0}$

$$\begin{aligned} &|\mathrm{P}^{y_0}_{\mathrm{T}}(\mathrm{Z}_0, \mathrm{Z}_0)| \leqslant c\exp(-\mathrm{C}|\mathrm{Z}_0|^2), \\ &|\mathrm{P}'^{y_0}_{\mathrm{T}}(\mathrm{Z}_0, \mathrm{Z}_0)| \leqslant c\exp(-\mathrm{C}|\mathrm{Z}_0|^2). \end{aligned} \tag{14.24}$$

Let E be a smooth section of End $(\Lambda(\bar{\mathrm{N}}^*) \hat{\otimes} \Lambda(\mathrm{N}^*))$ on B.

Definition **14.8.** – The generalized supertraces of $\mathrm{E}\exp(-\mathscr{D}^2_{\mathrm{T}^2})$, $\mathrm{E}\exp(-\mathscr{L}_{\mathrm{T}})$ are defined by the formulas

$$\begin{aligned} &\mathrm{Tr}_s[\mathrm{E}\exp(-\mathscr{D}^2_{\mathrm{T}^2})] = \int_{\mathrm{N}_{\mathbf{R}}} \mathrm{Tr}_s[\mathrm{E}\mathrm{P}_{\mathrm{T}}(\mathrm{Z}_0, \mathrm{Z}_0)] \frac{dv_{\mathrm{N}}(\mathrm{Z}_0)}{(2\pi)^{\dim \mathrm{N}}}, \\ &\mathrm{Tr}_s[\mathrm{E}\exp(-\mathscr{L}_{\mathrm{T}})] = \int_{\mathrm{N}_{\mathbf{R}}} \mathrm{Tr}_s[\mathrm{E}\mathrm{P}'_{\mathrm{T}}(\mathrm{Z}_0, \mathrm{Z}_0)] \frac{dv_{\mathrm{N}}(\mathrm{Z}_0)}{(2\pi)^{\dim \mathrm{N}}}. \end{aligned} \tag{14.25}$$

Clearly

$$\mathrm{Tr}_s[\mathrm{E}\exp(-\mathscr{D}^{2; y_0}_{\mathrm{T}^2})] = \mathrm{Tr}_s[\mathrm{E}\exp(-\mathscr{L}^{y_0}_{\mathrm{T}})]. \tag{14.26}$$

Definition **14.9.** – Let Ξ^{y_0} be the second order differential operator acting on $\mathbf{F}_{y_0}$

$$\Xi^{y_0} = \psi^{-1} \mathscr{L}^{y_0}_1 \psi. \tag{14.27}$$

Let $\mathrm{Q}^{y_0}(\mathrm{U}, \mathrm{U}')(\mathrm{U}, \mathrm{U}' \in \mathrm{L}_{\mathbf{R}, y_0})$ be the smooth kernel associated to the operator $\exp(-\Xi^{y_0})$, calculated with respect to the volume $dv_{\mathrm{L}}(\mathrm{U}')/(2\pi)^{\dim \mathrm{L}}$.

Definition **14.10.** – Let E^{θ} be the smooth function on B

$$\mathrm{E}^{\theta} = \varphi^{-1} q \mathrm{E} q \varphi. \tag{14.28}$$

Theorem **14.11.** – *There exists* $\delta \in]0, 1]$ *such that, as* $\mathrm{T} \to +\infty$

$$\mathrm{Tr}_s[\mathrm{E}\exp(-\mathscr{D}^2_{\mathrm{T}^2})] = \mathrm{E}^{\theta} \mathrm{Q}^{y_0}(0, 0) + O\left(\frac{1}{\mathrm{T}^{\delta}}\right), \tag{14.29}$$

and $O(1/T^\delta)$ *is uniform over the compact sets in* B.

Proof. — By Theorem 14.6, we find that the algebraic structure of the (2, 2) matrix of $\mathscr{L}_T^{y_0}$ as $T \to +\infty$ is very similar to the corresponding structure of $\mathscr{L}_{u,T}^{3,y_0}$ in Theorem 13.22 when $\xi^+ = \{0\}$. The analogues of the norms $|\ |_{u,T,y_0,0}$, $|\ |_{u,T,y_0,1}$, defined in (13.141), (13.143), have to be used here, with $\xi^+ = \{0\}$, $\eta = \mathbf{C}$, $u=0$. The only minor difference is that the norms $|\ |_{0,T,y_0,0}$ are norms on a smaller space than $\mathbf{K}_{y_0}^0$. This is a situation we already met in Section 11 o). The proof then proceeds as the proof of Theorem 13.6, by following the strategy indicated in Sections 13 j)-13 o) and in Section 13 q). Details are easy to fill and are left to the reader. □

d) A formula for the operator Ξ

If $y_0 \in Y$, U denotes the generic element of $\mathbf{L}_{\mathbf{R}, y_0}$.

We now prove a remarkable identity. Let $e_1, \ldots, e_{2l}$ be an orthonormal base of $\mathbf{L}_{\mathbf{R}}$.

Theorem **14.12.** — *The following identity holds*

$$\Xi = -\frac{1}{2}\sum_1^{2l}\left(\nabla_{e_i} + \frac{1}{2}\langle R^L U, e_i\rangle\right)^2 + \frac{1}{2}\operatorname{Tr}[R^L]. \tag{14.30}$$

Proof. — Using (14.16), (14.30) follows. □

Remark **14.13.** — As explained in [B3, Remark 3.7], the operator in the right-hand side of (14.30) is a generalization of the Getzler operator [Ge] in local index theory.

e) A new proof of equation (5.18)

Theorem **14.14.** — *There exists* $\delta \in]0, 1]$ *such that as* $T \to +\infty$

$$\Phi \operatorname{Tr}_s[E\exp(-\mathscr{D}_{T^2}^2)] = E^\theta \operatorname{Td}(L, g^L) + O\left(\frac{1}{T^\delta}\right) \tag{14.31}$$

and $O(1/T^\delta)$ *is uniform on the compact subsets of* B.

Proof. — Using Theorem 14.12 and equation (5.17) (with $N = \{0\}$), we find that

$$Q(0, 0) = \operatorname{Td}(-R^L). \tag{14.32}$$

Then (14.31) follows from Theorem 14.11 and from (14.32). □

Remark **14.15.** – Clearly $1^\theta = 1$. Also, by Proposition 8.4, we know that $N_H^\theta = \dim N/2$. We thus deduce from Theorem 14.14 that as $T \to +\infty$

$$
\begin{aligned}
&\Phi \operatorname{Tr}_s[\exp(-\mathscr{D}_{T^2}^2)] = \operatorname{Td}(L, g^L) + O\left(\frac{1}{T^\delta}\right), \\
&\Phi \operatorname{Tr}_s[N_H \exp(-\mathscr{D}_{T^2}^2)] = \frac{\dim N}{2} \operatorname{Td}(L, g^L) + O\left(\frac{1}{T^\delta}\right).
\end{aligned}
\tag{14.33}
$$

Using (5.15), we obtain the same asymptotics for $\Phi \operatorname{Tr}_s[\exp(-\mathscr{B}_{T^2}^2)]$ and $\Phi \operatorname{Tr}_s[N_H \exp(-\mathscr{B}_{T^2}^2)]$.

REFERENCES

[AlGBMNV] ALVAREZ-GAUMÉ L., BOST J. B., MOORE G., NELSON P., VAFA C., Bosonization on higher genus Riemann surfaces, *Comm. Math. Phys.*, 112 (1987), 503-552.

[Ar] ARAKELOV S., Intersection theory of divisors on an arithmetic surface, *Izv. Akad. Nauk SSSR.*, Ser. Mat., 38 (1974), n° 6. *AMS Translation*, 8 (1974), n° 6, 1167-1180.

[ABo] ATIYAH M. F., BOTT R., A Lefschetz fixed point formula for elliptic complexes, I, *Ann. Math.*, 86 (1967), 374-407; II, *Ann. of Math.*, 88 (1968), 451-491.

[ABoP] ATIYAH M. F., BOTT R., PATODI V. K., On the heat equation and the Index Theorem, *Invent. Math.*, 19 (1973), 279-330.

[AS] ATIYAH M. F., SINGER I. M., The index of elliptic operators, III, *Ann. of Math.*, 87 (1968), 546-604.

[BaFM] BAUM P., FULTON W., MACPHERSON R., Riemann-Roch for singular varieties, *Publ. Math. IHES*, 45 (1975), 101-146.

[BeV] BERLINE N., VERGNE M., A proof of Bismut local index theorem for a family of Dirac operators, *Topology*, 26 (1987), 435-463.

[B1] BISMUT J. M., The index Theorem for families of Dirac operators: two heat equation proofs, *Invent. Math.*, 83 (1986), 91-151.

[B2] BISMUT J. M., Superconnection currents and complex immersions, *Invent. Math.*, 99 (1990), 59-113.

[B3] BISMUT J. M., Koszul complexes, harmonic oscillators and the Todd class, *J.A.M.S.*, 3 (1990), 159-256.

[B4] BISMUT J. M., *Large deviations and the Malliavin calculus*, Prog. Math. n° 45, Basel-Boston-Stuttgart, Birkhäuser, 1984.

[B5] BISMUT J. M., The Atiyah-Singer Index Theorems: a probabilistic approach, I, *J. Funct. Anal.*, 57 (1984), 56-99.

[B6] BISMUT J. M., Demailly's asymptotic Morse inequalities: a heat equation proof, *J. Funct. Anal.*, 72 (1987), 263-278.

[B7] BISMUT J. M., The Witten complex and the degenerate Morse inequalities, *J. of Diff. Geom.*, 23 (1986), 207-240.

[BGS1] BISMUT J. M., GILLET H., SOULÉ C., Analytic torsion and holomorphic determinant bundles, I, *Comm. Math. Phys.*, 115 (1988), 49-78.

[BGS2] BISMUT J. M., GILLET H., SOULÉ C., Analytic torsion and holomorphic determinant bundles, II, *Comm. Math. Phys.*, 115 (1988), 79-126.

[BGS3] BISMUT J. M., GILLET H., SOULÉ C., Analytic torsion and holomorphic determinant bundles, III, *Comm. Math. Phys.*, 115 (1988), 301-351.

[BGS4] BISMUT J. M., GILLET H., SOULÉ C., Bott-Chern currents and complex immersions. *Duke Math. Journal*, 60 (1990), 255-284.

[BGS5] BISMUT J. M., GILLET H., SOULÉ C., Complex immersions and Arakelov geometry, *The Grothendieck Festschrift*, P. Cartier and al. ed., pp. 249-331. Prog. Math. n° 86, Boston-Basel-Berlin, Birkhaüser, 1990.

[BL] BISMUT J. M., LEBEAU G., Immersions complexes et métriques de Quillen. *C.R. Acad. Sci. Paris*, 309, Série I (1989), 487-491.

[BoC] BOTT R., CHERN S. S., Hermitian vector bundles and the equidistribution of the zeros of their holomorphic sections, *Acta Math.*, 114 (1968), 71-112.

[CP] CHAZARAIN J., PIRIOU A., *Introduction à la théorie des équations aux dérivées partielles linéaires*, Paris, Gauthier-Villars, 1981.

[Ch] CHEEGER J., Analytic torsion and the heat equation, *Ann. of Math.*, 109 (1979), 259-322.

[De] DELIGNE P., Le déterminant de la cohomologie, *Contemporary mathematics*, 67 (1987), 93-177.

[Do] DONALDSON S., Anti-self dual Yang-Mills connections over complex algebraic surfaces and stable vector bundles, *Proc. London Math. Soc.*, 50 (1985), 1-26.

[Dy] DYNKIN E. B., Diffusion of tensors, *Soviet Math Doklady*, 9 (1968), 532-535.

[E] EILENBERG S., Homological dimension and local syzygies, *Ann. of Math.*, 64 (1956), 328-336.

[F] FALTINGS G., Calculus on arithmetic surfaces, *Ann. of Math.*, 119 (1984), 387-424.

[Ge] GETZLER E, A short proof of the Atiyah-Singer Index Theorem, *Topology*, 25 (1986), 111-117.

[GS1] GILLET H., SOULÉ C., Arithmetic Intersection Theory, *Publ. Math. IHES*, 72 (1990), 93-174.

[GS2] GILLET H., SOULÉ C., Characteristic classes for algebraic vector bundles with Hermitian metric, *Ann. of Math.*, I, 131 (1990), 163-203, II, 131 (1990), 205-238.

[GS3] GILLET H., SOULÉ C., Analytic torsion and the arithmetic Todd genus. *Topology*, 30 (1991), 21-54.

[GS4] GILLET H., SOULÉ C., Un théorème de Rieman-Roch-Grothendieck arithmétique. *C.R. Acad. Sci. Paris*, 309, Série I (1989), 929-932.

[GlJ] GLIMM J., JAFFE A., *Quantum physics*, Berlin, Heidelberg, NewYork, Springer, 1987.

[GrH] GRIFFITHS P., HARRIS J., *Principles of algebraic geometry*, New York, Wiley, 1978.

[HeSj1] HELFFER B., SJÖSTRAND J., Puits multiples en limite semi-classique. IV, Étude du complexe de Witten, *Comm. in P.D.E.*, 10 (1985), 245-340.

[HeSj2] HELFFER B., SJÖSTRAND J., Puits multiples en limite semi-classique. VI, Cas des puits sous-variétés, *Ann. Inst. H. Poincaré*, 46 (1987), 353-372.

[HeSj3] HELFFER B., SJÖSTRAND J., A proof of the Bott inequalities, in *Algebraic Analysis*, Vol. I, M. Kashiwara, T. Kawai ed., New York, Academic Press, 1988, 171-183.

[H] HIRZEBRUCH F., Arithmetic genera and the theorem of Riemann-Roch for algebraic varieties. *Proc. Natl. Acad. Sci.*, USA, 40 (1954), 110-114.

[Hi] HITCHIN N. J., Harmonic spinors. *Adv. in Math.*, 14 (1974), 1-55.

[I] ITÔ K., Stochastic parallel displacement, in *Probabilistic methods in differential equations*, Lecture Notes in Mathematics n° 45, Berlin, Heidelberg, New York, Springer, 1975, 1-7.

[KnM] KNUDSEN F. F., MUMFORD D., The projectivity of the moduli space of stable curves, I, Preliminaries on "det" and "div", *Math. Scand.*, 39 (1976), 19-55.

[K] KOBAYASHI S., *Differential geometry of complex vector bundles*, Iwanami Shoten and Princeton University Press, 1987.

[KN] KOBAYASHI S., NOMIZU K., *Foundations of differential geometry*, II, New York, Interscience, 1969.

[La] LANG S., *Introduction to Arakelov theory*, Berlin, Heidelberg, New York, Springer, 1988.

[Li] LICHNEROWICZ A., Spineurs harmoniques, *C.R. Acad. Sci. Paris*, Série A, 257 (1963), 7-9.

[MKS] MCKEAN H., SINGER I. M., Curvature and the eigenvalues of the Laplacian, *J. of Diff. Geom.*, 1 (1967), 43-69.

[Mü] MÜLLER W., Analytic torsion and R.torsion of Riemannian manifolds, *Adv. in Math.*, 28 (1978), 233-305.

[Q1] QUILLEN D., Superconnections and the Chern character, *Topology*, 24 (1985), 89-95.

[Q2] QUILLEN D., Determinants of Cauchy-Riemann operators over a Riemann surface, *Funct. Anal. Appl.*, 14 (1985), 31-34.

[RS1] RAY D. B., SINGER I. M., R.torsion and the Laplacian on Riemannian manifolds, *Adv. in Math.*, 7 (1971), 145-210.

[RS2] RAY D. B., SINGER I. M., Analytic torsion for complex manifolds, *Ann. of Math.*, 98 (1973), 154-177.

[ReSi] REED M., SIMON B., *Methods of modern mathematical physics*, Vol. I, Functional Analysis, New York, Academic Press, 1972. Vol. II, Fourier analysis, self-adjointness, New York, Academic Press, 1975.

[Se] SEELEY R. T., Complex powers of an elliptic operator. *Proc. Symp. Pure and Appl. Math. AMS*, 10 (1967), 288-307.

[Ser] SERRE J. P., *Algèbre locale. Multiplicités*. Lecture Notes in Math. n° 11, Berlin, Heidelberg, New York, Springer, 1965.

[Ta] TANGERMAN F. M., To appear.

[T] TAYLOR M., *Pseudodifferential operators*, Princeton, Princeton Univ. Press, 1981.

[W] WITTEN E., Supersymmetry and Morse theory, *J. Differential Geom.*, 17 (1982), 661-692.

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