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ARITHMETIC INTERSECTION THEORY

by Henri GILLET* *and* Christophe SOULÉ

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Introduction

This paper describes an intersection theory for arithmetic varieties which generalizes the work of Arakelov and others on arithmetic surfaces. We develop a theory both of *arithmetic Chow groups*, and intersection products between them, for arithmetic varieties. By an arithmetic variety we mean a quasi-projective variety over an arithmetic ring, i.e. a regular noetherian ring equipped with a set of complex embeddings, see 3.1.3 below. Note that via the complex embeddings any arithmetic variety X determines an analytic space X_∞. In a second paper [G-S 4], using the results of the present paper, we develop a theory of characteristic classes for Hermitian vector bundles over arithmetic varieties. The main results were announced in [G-S 2].

The idea that one should compactify the prime spectrum of the ring of integers in a number field by adding the archimedean completions of the number field as primes at infinity has been known for a long time; for instance A. Weil, in his paper [We 1], discussed in some detail the analogy between function fields of curves and number fields, in which the role of the point(s) at infinity of an algebraic curve is taken by the archimedean completion(s) of a number field (in [We 2], p. 252, he attributes this idea to Hasse or Artin). Given the successes of intersection theory for varieties over fields, such as Weil's proof of the Riemann hypothesis for curves over finite fields, it is natural to look for an analogous theory for varieties over rings of algebraic integers. However, unless one has a theory which includes the prime at infinity the analogy will be incomplete and one will not have a good theory of intersection numbers. For example on Spec($\mathbf{Z}$) the degree of a zero cycle is not invariant under rational equivalence; indeed all such cycles are rationally equivalent to zero, while the natural definition of the degree of the divisor of a rational number q is $\log | q |$. However this defect is remedied if one adjoins a point v at infinity to Spec($\mathbf{Z}$) corresponding to the real completion of $\mathbf{Q}$, and defines the v-adic valuation of a rational number q to be $- \log | q |$. The assertion that a principal divisor has degree zero is then just the product formula.

Given an arithmetic surface X over the ring of integers in a number field F, S. J. Arakelov [Ar] " compactified " X by choosing a Hermitian metric d_∞ on each Riemann surface associated to X by the choice of an Archimedian place v of F. To the data $\overline{X} = (X, d_\infty)$ he associated a divisor class group $Cl(\overline{X})$ which is an extension of the usual class group Cl(X) by a real vector space. He then showed that a real valued pairing could be defined on $Cl(\overline{X})$; it was later shown by Hriljac [Hr] and Faltings [F] that from this pairing one could recover the Néron-Tate height pairing for divisors of degree zero. Arakelov also showed that the group $Cl(\overline{X})$ is isomorphic to the group of isomorphism classes of line bundles on X equipped with " admissible " Hermitian metrics at each infinite place (where " admissible " means that the curvature is a constant multiple of the volume form of the Riemann surface). In [De 2] P. Deligne showed that the intersection pairing of Arakelov could be extended to the full group of Hermitian line bundles on X. In the note [G-S 1], we announced an extension of Arakelov's theory to higher dimensional varieties X which have projective nonsingular generic fibers: once a Kähler metric is chosen on X_∞, one defines a codimension p cycle to be a pair (Z, h) consisting of a codimension p cycle Z on X and a harmonic $(p-1, p-1)$ real form on X_∞. This construction was also made by A. A. Beilinson in [Be 2] and was inspired in part by his extension to higher dimensions of the Neron height pairing in [Be 1].

The theory described in this paper builds on the results of Deligne as well as those of Arakelov. We develop an intersection theory on any arithmetic variety X which does not depend on the choice of a Kähler metric on X_∞. Define an *arithmetic cycle* on X to be a pair (Z, g) consisting of an algebraic cycle Z and a *Green current* for Z, i.e. a current on the complex manifold X_∞ satisfying the equation

$$dd^c g + \delta_Z = \omega,$$

with ω a smooth form (the cohomology class of which is then the Poincaré dual of the cycle Z). The classes of arithmetic cycles for an appropriate notion of linear equivalence form the *arithmetic Chow groups* $\widehat{CH}^\cdot(X)$. We prove that these groups have a product structure and functoriality properties which are analogous to those of the classical Chow groups of varieties over fields. The proofs use both complex geometry (one has to construct Green currents with reasonable growth) and K-theory of schemes (in order to get algebraic intersections to exist over Spec $\mathbf{Z}$, using the methods of [G-S 3]).

Let us give an outline of the paper. In section 1, we develop the basic existence theorem for Green currents on a complex manifold. In particular we show (Theorem 1.3.5) that a Green current for a cycle Z may always be represented (possibly after adding to it currents of the form $\partial u + \bar{\partial} v$) by a form which is C^∞ away from Z and which is "of logarithmic type" along Z (see 1.3 for the precise definition). In a previous version of this paper we used instead a notion of "logarithmic growth" along Z, but forms of logarithmic type are easier to work with. Our methods in this section are similar to those used by J. King in [K 2], except that he considers ∂ instead of dd^c.

In section 2, we examine the relationship between Green currents and intersecting cycles. In particular we define a product on Green currents, the $*$-product, which is compatible with the intersection product of cycles which meet properly. This $*$-product is analogous to the $*$-product of differential characters defined by J. Cheeger in [C]. We prove that the $*$-product is both associative and commutative. We also show that even when cycles Y and Z do not meet properly, the $*$-product of Green currents can be used to define a product current $\delta_Y . \delta_Z$ supported on $Y \cap Z$. A similar result had already been obtained by King in [K 3].

In section 3 we introduce the arithmetic Chow groups and describe their basic properties. We show that these groupes fit into a collection of short exact sequences which involve the Beilinson regulator maps for K_1 of a complex variety [Be 1], and we compute these groups in some simple cases. In section 4.3 we show that the arithmetic

Chow groups of a nonsingular arithmetic variety have, at least when tensored with the rational numbers, a commutative ring structure. If (Y, g) and (Z, h) are arithmetic cycles such that Y and Z intersect properly on X, then their product is defined as $(Y.Z, g * h)$, with Y.Z defined as in Serre [Se]. However, since there is not as yet an intersection theory with integer coefficients for arbitrary regular schemes, if Y and Z do not intersect properly we must appeal to the results of [G-S 3] on intersection theory with rational coefficients to define the product Y.Z. But in section 4.3 we show that, given any map $f: X \to Y$ of relative dimension d between nonsingular arithmetic varieties which is proper and induces a smooth map of complex manifolds $X_\infty \to Y_\infty$, one can define an intersection pairing from $\widehat{CH}^p(X) \otimes \widehat{CH}^{d-p+1}(X)$ to $\widehat{CH}^1(Y)$, without having to tensor with $\mathbf{Q}$. This pairing is a direct generalization of the pairing defined in [De 2]. In 4.4 we show that our theory is functorial, without having to tensor with $\mathbf{Q}$, while in 4.5 we show that if one restricts attention to varieties which are smooth over a Dedekind domain, then using work of Fulton in [Fu], one can define the product on $\widehat{CH}^*(X)$ without having to tensor with $\mathbf{Q}$.

Finally in section 5 we discuss two complements. First we show how the intersection pairing on the Arakelov Chow groups $CH^*(\overline{X})$ of a compactification $\overline{X} = (X, \omega)$ of an arithmetic variety X is induced by the pairing of section 4.3. We also show that given two different choices of Kähler metric on X, giving rise to compactifications $\overline{X}$ and $\overline{X}'$, there is an isomorphism $\theta : CH^p(\overline{X}) \to CH^p(\overline{X}')$ which identifies the intersection products. Second we describe the basic ingredients in a theory of correspondences between arithmetic varieties, and we observe that the change of metrics isomorphism θ mentioned above is induced by a correspondence.

Finally, let us mention a few topics which may be worth exploring. First, Green currents play a crucial role in Nevanlinna's value distribution theory (as in [Sha], [Co-G] for instance); our formalism might be of use there, for example see the discussion of Levine forms in [G-S 4]. Second, the analogy between arithmetic Chow groups and differential characters deserves further examination. Third, for noncomplete arithmetic varieties, our theory may well not be optimal, since it ignores the Hodge theory of such varieties; it may in fact be useful to impose on differential forms growth conditions such as those considered by M. Harris and D. H. Phong in [H-P]. Last of all, one could imagine an adelic intersection theory, in which cycles would be pairs $(Z, (g_v))$, with Z an algebraic cycle on a variety over a number field F, and g_v a choice of a Green current at each place v of F. However we do not know what the p-adic analogs of Green currents should be.

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1. Green currents

1.1. Currents on complex manifolds

1.1.1. We shall start with a brief review of currents on complex manifolds, following the original book [deRh] of deRham and the article [K 1].

Let X be a complex manifold of (complex) dimension d; for simplicity we suppose that all the components of X have the same dimension. The space $A^n(X)$ of C^∞ complex valued n-forms on X has a topology defined using the sup norms, on compact subsets of coordinate charts of X, of the k-fold partial derivatives of the coefficients of a form, for all $k \geqslant 0$, see [deRh] § 9 for details. Since the topology is defined by a family of semi-norms, $A^n(X)$ is a locally convex topological space. Let $A^n_c(X)$ be the subspace of compactly supported forms. We write $\mathscr{D}_n(X)$ for the bornological dual of $A^n_c(X)$ and $\mathscr{D}^c_n(X)$ for the dual of $A^n(X)$; these are the spaces of currents of dimension n on X, and of currents with compact support, respectively; note that $\mathscr{D}^c_n(X) \subset \mathscr{D}_n(X)$. Since X is a complex manifold, we have the decomposition

$$A^n_c(X) = \bigoplus_{p+q=n} A^{p,q}_c(X)$$

and the corresponding decomposition

$$\mathscr{D}_n(X) \simeq \bigoplus_{p+q=n} \mathscr{D}_{p,q}(X);$$

there is a similar decomposition of $\mathscr{D}^c_n(X)$. The exterior derivative

$$d = d' + d'' : A^n_c(X) \to A^{n+1}_c(X)$$

induces a dual homomorphism

$$b = b' + b'' : \mathscr{D}_{n+1}(X) \to \mathscr{D}_n(X),$$

which restricts to $b : \mathscr{D}^c_{n+1}(X) \to \mathscr{D}^c_n(X)$, i.e. if $T \in \mathscr{D}_{n+1}(X)$, $\alpha \in A^n_c(X) : bT(\alpha) = T(d\alpha)$. Note that b decomposes as the sum of the maps:

$$b' : \mathscr{D}_{p,q}(X) \to \mathscr{D}_{p-1,q}(X)$$
$$b'' : \mathscr{D}_{p,q}(X) \to \mathscr{D}_{p,q-1}(X).$$

1.1.2. *Examples.*

(i) *Chains.* — Any smooth oriented singular n-simplex

$$\sigma : \Delta^n \to X$$

or more generally any smooth n-chain $c = \Sigma a_i \sigma_i$ defines a current $\delta_c \in \mathscr{D}^c_n(X)$:

$$\delta_c(\varphi) = \int_c \varphi \overset{\text{def}}{=} \Sigma a_i \int_{\Delta^n} \sigma_i^*(\varphi).$$

By Stokes theorem

$$b\,\delta_c = \delta_{\partial c}.$$

(ii) *Analytic subspaces.* — We give $\mathbf{C}^k$, with coordinates $z_1, \ldots, z_k$, the orientation determined by the volume form

$$\left(\frac{i}{2}\right)^k dz_1 \wedge d\bar{z}_1 \wedge \ldots \wedge dz_k \wedge d\bar{z}_k = dx_1 \wedge dy_1 \wedge \ldots \wedge dx_k \wedge dy_k;$$

here $z_r = x_r + iy_r$ for $1 \leq r \leq k$. Note that any $\mathbf{C}$-linear automorphism $\mathbf{C}^k \to \mathbf{C}^k$ is orientation preserving, as are the natural isomorphisms $\mathbf{C}^k \times \mathbf{C}^\ell \to \mathbf{C}^{k+\ell}$, where $\mathbf{C}^k \times \mathbf{C}^\ell$ is given the exterior product of the orientations of $\mathbf{C}^k$ and $\mathbf{C}^\ell$.

Our choice of orientation on $\mathbf{C}^k$, for all $k \geq 0$, determines an orientation on any (finite-dimensional) complex manifold. Therefore, for any closed k-dimensional submanifold $M \subset X$, there is a current $\delta_M \in \mathcal{D}_{2k}(X)$, defined by

$$\delta_M(\alpha) = \int_M i^* \alpha$$

for $\alpha \in A_c^{2k}(X)$ and $i : M \to X$ the inclusion map. More generally, if $i : Y \to X$ is a k-dimensional analytic subspace of X, we can define a $2k$-dimensional current δ_Y (first introduced by Lelong [Le]) by

$$\delta_Y(\alpha) = \int_{Y^{ns}} i^* \alpha = \int_{\tilde{Y}} \pi^* i^* \alpha;$$

here Y^{ns} is the (dense, open) subset of smooth points in Y, while $\pi : \tilde{Y} \to Y$ is a resolution of singularities of Y. Note that $\delta_Y \in \mathcal{D}_{k,k}(X)$ since, if $\alpha^{p,q} \in A^{p,q}(X)$ for $p + q = 2k$, $i^* \alpha = 0$ unless $p = q = k$.

(iii) *Analytic cycles.* — If $Y = \sum_{i=1}^m n_i [Y_i]$ is an *analytic cycle* of dimension k, i.e. a finite formal sum of k-dimensional closed analytic subspaces $Y_i \subset X$, we define $\delta_Y = \sum_{i=1}^k n_i \, \delta_{Y_i}$.

(iv) L^1 *forms.* — There are products

$$\mathcal{D}_n(X) \otimes A^m(X) \to \mathcal{D}_{n-m}(X)$$

which decompose into

$$\mathcal{D}_{p,q}(X) \otimes A^{r,s}(X) \to \mathcal{D}_{p-r,q-s}(X).$$

If $T \in \mathcal{D}_n(X)$, $\alpha \in A^m(X)$, we denote their product by $T \wedge \alpha$, and if $\beta \in A^{n-m}(X)$, the product is defined by

$$(T \wedge \alpha)(\beta) = T(\alpha \wedge \beta).$$

In particular, since X is d dimensional, there is a map

$$A^{p,q}(X) \to \mathcal{D}_{d-p,d-q}(X)$$
$$\alpha \mapsto \delta_X \wedge \alpha.$$

Often we will write $[\alpha]$ for $\delta_X \wedge \alpha$, or, when the meaning is clear, simply α. More generally, if α is an L^1-form on X, i.e. in any coordinate patch α has coefficients which are locally L^1, the integral $\int_X \alpha \wedge \beta$ is well defined, for $\beta \in A_c^{d-p,d-q}(X)$, so that we have a map

$$L^1_{loc}(X, \Omega_X^{p,q}) \to \mathscr{D}_{d-p,d-q}(X).$$

N.B. — If α is not C^∞ on the whole of X, do not confuse α and $[\alpha]$, since in general $[d\alpha]$ and $d[\alpha]$ will not be equal. Their difference is called a residue.

1.1.3. The spaces $\mathscr{D}_{p,q}$ have a natural topology ([deRh] § 10), for which the maps $A^{p,q}(X) \to \mathscr{D}_{d-p,d-q}(X)$ are continuous, with dense image. This leads us to write

$$\mathscr{D}_{d-p,d-q}(X) = \mathscr{D}^{p,q}(X);$$

we may view $\mathscr{D}^{p,q}(X)$ as the space of forms of type (p, q) with distribution coefficients ([Sch], [deRh]). It is important to note, however, that this identification depends on the choice of orientation of X.

Furthermore, the map $\alpha \mapsto [\alpha]$ does not send d to b, but rather, if $\alpha \in A^n(X)$ and $\beta \in A_c^{d-n-1}(X)$,

$$[d\alpha]\,(\beta) = \int_X d\alpha \wedge \beta = \int_X d(\alpha \wedge \beta) - \int_X (-1)^n \alpha \wedge d\beta$$

which, by Stokes theorem, since $\alpha \wedge \beta \in A_c^*(X)$, is equal to

$$(-1)^{n+1} \int_X \alpha \wedge d\beta = (-1)^{n+1} (b[\alpha])\,(\beta).$$

So $[d\alpha] = (-1)^{n+1}\, b[\alpha]$. Therefore, if we define

$$d = (-1)^{n+1}\, b : \mathscr{D}^n(X) \to \mathscr{D}^{n+1}(X),$$

the inclusion $A^n(X) \to \mathscr{D}^n(X)$ commutes with d.

Similarly, we can embed

$$A_c^n(X) \subset \mathscr{D}_c^n(X) = \mathscr{D}_{d-n}^c(X).$$

1.1.4. If $f: X^d \to Y^{d-r}$ is a holomorphic map of compact complex manifolds we have maps

$$f^* : A^{p,q}(Y) \to A^{p,q}(X)$$

and dual maps

$$f_* : \mathscr{D}_{d-p,d-q}^c(X) \to \mathscr{D}_{d-p,d-q}^c(Y)$$

which may be viewed as maps

$$f_* : \mathscr{D}_c^{p,q}(X) \to \mathscr{D}_c^{p-r,q-r}(Y).$$

If f is smooth, then f_* extends the integration over the fibre homomorphism (see [G-H-V] Ch. IV for example)

$$\int_f : A_c^{p,q}(X) \to A_c^{p-r,q-r}(Y),$$

which itself induces a dual homomorphism

$$f^* : \mathscr{D}^{p,q}(Y) \to \mathscr{D}^{p,q}(X)$$

extending f^* on forms.

If f is proper, then we have maps

$$f^* : A_c^{p,q}(Y) \to A_c^{p,q}(X)$$

$$f_* : \mathscr{D}_{d-p,d-q}(X) \to \mathscr{D}_{d-p,d-q}(Y)$$

and

$$f_* : \mathscr{D}^{p,q}(X) \to \mathscr{D}^{p-r,q-r}(Y),$$

which, when f is smooth, extends to

$$\int_f : A^{p,q}(X) \to A^{p-r,q-r}(Y).$$

If f is birational, for any (locally) L^1-form α on Y, we have an equality of currents

$$[\alpha] = f_*([f^*(\alpha)].$$

1.1.5. The following remark will be useful (see also [K 2]). Let $f: X \to Y$ be a projective morphism between smooth quasi-projective complex varieties and α a smooth form on X which is locally L^1. Then the map $f: X \to f(X)$ is generically smooth ([Ha], III, Corollary 10.7), therefore, by Fubini theorem, there is a dense Zariski open set $U \subset f(X)$ and an L^1 form β on U such that the current $f_*[\alpha]$ is given by the following convergent indefinite integral

$$f_*[\alpha]\,(\eta) = \int_U \beta \wedge \eta$$

for every smooth form η with compact support on Y of the appropriate degree. In particular, when $f(X) = Y$, we get $f_*[\alpha] = [\beta]$. When the degree of α is less than $2(\dim(X) - \dim f(X))$, the current $f_*[\alpha]$ vanishes (since the degree of $f^*(\eta)$ is too small).

1.1.6. *Remarks.* — (i) If X is compact then $A^*(X) = A_c^*(X)$ and we can omit the distinction between $\mathscr{D}^*(X)$ and $\mathscr{D}_c^*(X)$.

(ii) From now on we will ignore $b = b' + b''$, and work solely with

$$d = d' + d'' = \partial + \bar\partial.$$

Note also that $d^c = (i/4\pi)\,(\bar\partial - \partial)$, so

$$dd^c = -\,d^c\,d = \frac{i}{2\pi}\,\partial\bar\partial.$$

1.2. Green currents

1.2.1. *Theorem.* — *Let* X *be a compact Kähler manifold. Suppose* $\eta \in \mathscr{D}^{p,q}(X)$, $p, q \geqslant 1$, *is d-closed and is either d,* ∂, *or* $\bar\partial$ *exact. Then there exists* $\gamma \in \mathscr{D}^{p-1,q-1}(X)$ *such that*

$$dd^c\,\gamma = \frac{i}{2\pi}\,\partial\bar\partial\gamma = \eta.$$

In addition:

(i) *If* $p = q$ *and* η *is real, then* γ *can be taken to be real.*

(ii) *If* $p = q$, $F : X \to X$ *is an antiholomorphic involution, and* $F^* \eta = (-1)^p \eta$, *then* γ *can be chosen so that* $F^* \gamma = (-1)^{p-1} \gamma$. *Similarly if* F *is holomorphic and* $F^* \eta = \eta$, *then* γ *can be chosen so that* $F^* \gamma = \gamma$.

(iii) *If* $U \subset X$ *is an open set, and* $\eta|_U$ *is* C^∞, *then* γ *may be chosen so that* $\gamma|_U$ *is* C^∞.

Proof. — This is well known. In [G-H], p. 149, it is shown that if η is C^∞, an explicit solution of $\partial\bar\partial u = \eta$ is given by $\pm\, \partial^* \bar\partial G_{\bar\partial}^2\, \eta$, $G_{\bar\partial}$ being the Green operator associated to the $\bar\partial$-Laplacian, which is a C^∞ form (see also [Wel], Ch. V, Prop. 2.2, for a simpler formula). Since the operators ∂^*, $\bar\partial^*$ and $G_{\bar\partial}$ extend to currents, the same formula gives a solution of the equation when η is a current. Since dd^c is real, if η is real, then $dd^c \gamma = \eta$ implies that $dd^c(\operatorname{Im} \gamma) = 0$ and we can replace γ by its real part, proving (i). For (ii), observe that if F is holomorphic, it commutes with dd^c and so if $F^* \eta = \eta$, $(1/2)\,(F^* \gamma + \gamma)$ will be an F-invariant solution, while if F is antiholomorphic, it anticommutes with dd^c and so $(1/2)\,(\gamma - (-1)^p F^* \gamma)$ is the desired solution. Finally, for (iii), we use the fact that $G_{\bar\partial}$ is represented by a kernel on $X \times X$ which is C^∞ away from the diagonal ([deRh]) and hence the singular set of $G_{\bar\partial}(T)$ is contained in the singular set of T.

1.2.2. *Theorem. — Let* X *be a complex manifold. Then:*

(i) *If* γ *is a current on* X *such that* $\partial\bar\partial\gamma$ *is smooth (i.e. equals* $[\varphi]$ *for some* C^∞ *form* φ*), then there exist currents* α *and* β *such that* $\gamma = \omega + \partial\alpha + \bar\partial\beta$ *with* ω *smooth.*

(ii) *If* ω *is a* C^∞ *form on* X *such that* $\omega = \partial u + \bar\partial v$ *for currents* u *and* v, *then there exist smooth currents* α *and* β *such that* $\omega = \partial\alpha + \bar\partial\beta$.

(iii) *If* X *is compact and Kähler, and* γ *is a current on* X *satisfying* $\partial\bar\partial\gamma = 0$, *then* $\gamma = \omega + \partial\alpha + \bar\partial\beta$ *with* ω *harmonic. Furthermore, if* γ *is smooth, both* α *and* β *can be chosen to be smooth.*

Proof. — First recall that by [Do] Th. 1.4 or p. 385 of [G-H], the d, ∂ and $\bar\partial$ cohomology of currents on X is the same as the d, ∂, and $\bar\partial$ cohomology of C^∞ forms on X. We now prove each part of the theorem in turn:

(i) If $\partial\bar\partial g = \eta$ with η smooth, then $\eta = \partial(\bar\partial g)$, hence $\eta = \partial\alpha$ for some C^∞ form α. So $\partial(\bar\partial g - \alpha) = 0$ and $\bar\partial g - \alpha = \beta + \partial g_1$, where β is smooth. This implies that $\partial\bar\partial g_1 = \eta_1 = \bar\partial(\alpha + \beta)$ is smooth. By repeating this argument we get a sequence of currents g_n such that $\bar\partial g_n = u_n + \partial g_{n+1}$ with u_n smooth.

But, since $\bar\partial$ (resp. ∂) has bidegree (0, 1), resp. (1, 0), one can choose g_n with no component of type (p, q), $p \leqslant n$. When n is big enough this implies $g_{n+1} = 0$. Therefore $\bar\partial g_n = u_n$ is smooth, hence $g_n = \omega_n + \bar\partial\beta_n$, with ω_n smooth. So

$$\bar\partial(g_{n-1} + \partial\beta_n) = \bar\partial u_{n-1} + \partial\omega_n$$

is smooth, and therefore

$$g_{n-1} = \omega_{n-1} + \partial\alpha_{n-1} + \bar\partial\beta_{n-1}$$

with ω_{n-1} smooth. By repeating this argument one concludes that $g = \omega + \partial\alpha + \bar\partial\beta$, with ω smooth.

(ii) If $\omega = \partial u + \bar\partial v$, then $\partial\omega = \partial\bar\partial v$ and hence, by part (i),

$$v = \alpha + \partial x + \bar\partial y$$

with α smooth. Therefore

$$\bar\partial v = \bar\partial\alpha + \bar\partial\partial x.$$

Similarly $\partial u = \partial\beta + \partial\bar\partial z$ with β smooth. Therefore $\omega = \partial\alpha + \bar\partial\beta + \partial\bar\partial(z - x)$. By part (i), $z - x = \gamma + \partial s + \bar\partial t$ with γ smooth. Therefore $\partial\bar\partial(z - x) = \partial\bar\partial(\gamma)$, so $\omega = \partial(\alpha + \bar\partial\gamma) + \bar\partial\beta$.

(iii) This is a corollary to Theorem 1.2.1. Since $\partial\bar\partial\gamma = 0$, $\bar\partial\gamma$ is both ∂ exact and d closed. Hence by the theorem, $\bar\partial\gamma = \partial\bar\partial\alpha$ for some α, which may be taken to be smooth if γ is. Therefore $\bar\partial(\gamma - \partial\alpha) = 0$. Similarly, there exists β (C^∞ if γ is) such that $\partial(\gamma - \bar\partial\beta) = 0$. Hence $d(\gamma - \partial\alpha - \bar\partial\beta) = 0$. Since the d-cohomology of currents and forms is the same, this implies that there exists φ, C^∞ if γ is, such that

$$\gamma - \partial\alpha - \bar\partial\beta = \omega + d\varphi$$

for ω harmonic, i.e.

$$\gamma = \omega + \partial(\alpha + \varphi) + \bar\partial(\beta + \varphi).$$

1.2.3. For X a complex manifold, let us write

$$\widetilde{A}^{p,q} = A^{p,q}(X)/(\partial A^{p-1,q} + \bar\partial A^{p,q-1})$$
$$\widetilde{\mathscr{D}}^{p,q}(X) = \mathscr{D}^{p,q}(X)/(\partial\mathscr{D}^{p-1,q} + \bar\partial\mathscr{D}^{p,q-1}).$$

Observe that by (ii) of Theorem 1.2.2, the natural map $\widetilde{A}^{p,q}(X) \to \widetilde{\mathscr{D}}^{p,q}(X)$ is an injection. The homomorphisms $\partial\bar\partial$ on both forms and currents factor through $\widetilde{A}^{p,q}(X)$ and $\widetilde{\mathscr{D}}^{p,q}(X)$. By (i) of Theorem 1.2.2, the kernel of $\partial\bar\partial : \widetilde{\mathscr{D}}^{p,q}(X) \to \mathscr{D}^{p+1,q+1}(X)$ is contained in $\widetilde{A}^{p,q}(X)$. If X is compact and Kähler, the space

$$H^{p,q}(X) = (\ker\bar\partial : A^{p,q}(X) \to A^{p,q+1}(X))/\bar\partial(A^{p,q-1}(X))$$

can be identified with the space of harmonic forms of type (p, q) on X, and by (iii) of Theorem 1.2.2, we have exact sequences, and a map between them:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^{p,q}(X) & \longrightarrow & \widetilde{A}^{p,q}(X) & \longrightarrow & B^{p+1,q+1}(X) & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & H^{p,q}(X) & \longrightarrow & \widetilde{\mathscr{D}}^{p,q}(X) & \longrightarrow & \mathscr{B}^{p+1,q+1}(X) & \longrightarrow & 0 \end{array}$$

Here $B^{*,*}(X)$ and $\mathscr{B}^{*,*}(X)$ are the spaces of exact forms and currents, respectively.

Definition. — If X is a complex manifold, and $Y = \Sigma n_i[Y_i]$ is a codimension p analytic cycle on X, a *Green current* for Y is an element $g \in \widetilde{\mathscr{D}}^{p-1,p-1}(X)$, which is the class of a real current, such that $dd^c g + \delta_Y = \omega$, with ω a C^∞ form (necessarily of type (p, p)). Such a Green current always exists if X is compact and Kähler, since if ω is any real (p, p) form representing the cohomology class of Y, $\delta_Y - \omega$ will be exact and we can apply Theorem 1.2.1. If $|Y|$ is the support of Y in X, we define a *Green form* for Y to be a C^∞ form g on $X - |Y|$, locally L^1 on X, for which $[g]$ is a Green current for Y.

If $F : X \to X$ is an antiholomorphic involution, such that $F(Y) = Y$, then $F^* \delta_Y = (-1)^p \delta_Y$, and g may be chosen so that $F^* g = (-1)^{p-1} g$.

1.2.4. *Lemma.* — *If* X *is a complex manifold, and* Y *a codimension* p *analytic cycle on* X, *any two Green currents for* Y *differ by an element of* $\widetilde{A}^{p-1,p-1}(X) \subset \widetilde{\mathscr{D}}^{p-1,p-1}(X)$.

Proof. — If $dd^c g_i + \delta_Y = \omega_i$ for $i = 1, 2$ with ω_i C^∞, then

$$\partial\bar\partial(g_1 - g_2) = 2\pi\sqrt{-1}(\omega_1 - \omega_2)$$

is a smooth form. Hence by Theorem 1.2.2 (i), $g_1 - g_2 = \gamma + \partial u + \bar\partial v$ with γ smooth, i.e. $g_1 - g_2 \in \widetilde{A}^{p-1,p-1}(X) \subset \widetilde{\mathscr{D}}^{p-1,p-1}(X)$.

1.3. Green forms of logarithmic type

1.3.1. We shall now give a geometric construction of Green currents. This construction will be used in 2.1 to define pull-backs and $*$-products of Green currents. The basic example is the case of a divisor Y in a complex manifold X. Then there exists a holomorphic line bundle $\mathscr{L}$ on X and a meromorphic section s of $\mathscr{L}$ such that Y is the divisor of s (see [G-H], Ch. 1.1 for a construction of $\mathscr{L}$ and s). Choose a smooth Hermitian metric on $\mathscr{L}$, i.e. a norm $\|\ \|$. Then $\log \|s\|^2$ is an L^1 function on X, and by the Poincaré-Lelong formula ([Le] and [G-H], Ch. 3),

(1.3.1.1) $$dd^c([\log \|s\|^2]) = \delta_Y - \beta$$

for β a closed smooth (1, 1)-form on X, namely the first Chern form of $(\mathscr{L}, \|\ \|)$. In other words $[-\log \|s\|^2]$ is a Green current for the divisor Y. It can be shown that all Green currents for divisors are obtained in this way ([G-S 4], 2.5).

We shall use Hironaka's resolution of singularities to construct Green currents in arbitrary codimension (when X is algebraic) from the case of divisors. But first we introduce the notion of forms of logarithmic type.

1.3.2. Let X be a quasi-projective complex manifold, and $Y \subset X$ a closed analytic subspace in X which does not contain any irreducible component of X.

Definition. — A smooth form η on $X - Y$ is said to be a *form of logarithmic type* (or log type) along Y if there exists a projective morphism

$$\pi : Z \to X$$

and a smooth form φ on $Z - \pi^{-1}(Y)$ such that

(i) Z is smooth, $\pi^{-1}(Y)$ is a divisor with normal crossings (d.n.c.), and π is smooth over $X - Y$;

(ii) η is the direct image by π of the restriction of φ to $Z - \pi^{-1}(Y)$;

(iii) for any point $x \in Z$, there is an open neighbourhood U of x, and a system of holomorphic coordinates $(z_1, \ldots, z_n)$ of U centered at x such that $\pi^{-1}(Y) \cap U$ has equation $z_1 \ldots z_k = 0$, for some $k \leqslant n$, and there exist smooth ∂ and $\bar{\partial}$-closed forms α_i on U, $i = 1, \ldots, k$ and a smooth form β on U with

$$\varphi_{|U} = \sum_{i=1}^{k} \alpha_i \log |z_i|^2 + \beta. \qquad (1.3.2.1)$$

1.3.3. When η is a form of log type along Y, as in Definition 1.3.2, it follows from the Fubini theorem (as in 1.1.4) that η is locally L^1 on X and that $[\eta] = \pi_*[\varphi]$. Furthermore:

Lemma. — (i) *Let $f: X' \to X$ be a morphism of smooth quasi-projective varieties and η a form of* log *type along $Y \subset X$. If $f^{-1}(Y)$ does not contain any component of X', the form $f^*(\eta)$ is of* log *type along $f^{-1}(Y)$.*

(ii) *Let $f: X \to X'$ be a projective morphism of smooth quasi-projective varieties and η a form of* log *type along $Y \subset X$. Assume that f is smooth outside Y, and that $f(Y)$ does not contain any component of X'. Then $f_*(\eta)$ has* log *type along $f(Y)$ and the equality of currents $[f_*(\eta)] = f_*[\eta]$ holds.*

(iii) *If η_1 is of* log *type along $Y_1 \subset X$ and η_2 is of* log *type along $Y_2 \subset X$, their sum $\eta = \eta_1 + \eta_2$ is of* log *type along $Y_1 \cup Y_2$.*

Proof. — To prove (i) consider a diagram

$$\begin{array}{ccc} Z' & \xrightarrow{f'} & Z \\ {\scriptstyle \pi'}\downarrow & & \downarrow{\scriptstyle \pi} \\ X' & \xrightarrow{f} & X \end{array}$$

where Z and π are given by Definition 1.3.2, π' is projective, Z' is smooth, $(f\pi')^{-1}(Y)$ is a d.n.c. in Z', and the induced diagram

$$\begin{array}{ccc} Z' - (f\pi')^{-1}(Y) & \xrightarrow{f'} & Z - \pi^{-1}(Y) \\ {\scriptstyle \pi'}\downarrow & & \downarrow{\scriptstyle \pi} \\ X' - f^{-1}(Y) & \xrightarrow{f} & X - Y \end{array}$$

is cartesian (in particular π' is smooth over $X' - f^{-1}(Y)$). Such a diagram can be obtained by resolving the singularities of the closure of $(X' - f^{-1}(Y) \times_X (Z - \pi^{-1}(Y))$ in the fiber product $X' \times_X Z$ (using [Hi] Theorem II; see also [De 1] (3.2.11) *c*). Since $\eta = \pi_*(\varphi)$ we deduce that

$$f^*(\eta) = f^* \pi_*(\varphi) = \pi'_* f'^*(\varphi)$$

on $X' - f^{-1}(Y)$. Let E_i be a component of $\pi^{-1}(Y)$ with local equation $z_i = 0$. Its pull-back by f' as a Cartier divisor is $\sum_j n_{ij} E'_j$, where E'_j is a component of $(f\pi')^{-1}(Y)$ with local equation $z'_j = 0$. Therefore, locally on Z', we may write

$$f'^*(\varphi) = \sum_i f'^*(\alpha_i) \sum_j n_{ij} \log |z'_j|^2 + f'^*(\beta).$$

We conclude that $f^*(\eta)$ is of log type along $f^{-1}(Y)$.

To prove (ii) let $\pi : Z \to X$ and φ be as in Definition 1.3.2. The d.n.c. $\pi^{-1}(Y)$ is contained in $(f\pi)^{-1}(f(Y))$. Using [Hi], Theorem II, we may find a projective map $p : Z' \to Z$ which is an isomorphism outside $(f\pi p)^{-1}(f(Y))$, and such that both $(\pi p)^{-1}(Y)$ and $(f\pi p)^{-1}(f(Y))$ are d.n.c. The form $\varphi' = p^*(\varphi)$ then satisfies the conditions of Definition 1.3.2, showing that $f_*(\eta)$ has log type along $f(Y)$.

To prove (iii) we consider $\pi_i : Z_i \to X$ and φ_i on $Z_i - \pi_i^{-1}(Y_i)$ satisfying the conditions of 1.3.2 with $\pi_*(\varphi_i) = \eta_i$, $i = 1, 2$. Then the form $\varphi = \varphi_1 + \varphi_2$ on the disjoint union $Z_1 \amalg Z_2$ is such that $\pi_*(\varphi) = \eta$, where π is equal to π_i on Z_i, $i = 1, 2$. Therefore η is log type along $Y_1 \cup Y_2$.

1.3.4. Let η be a form of log type along Y as in 1.3.2. We need to compute $dd^c[\eta]$. Let E_i, $i \in I$, be the (smooth) irreducible components of $\pi^{-1}(Y)$ and $\varepsilon_i : E_i \to Z$ the inclusion.

Lemma. — There exists a smooth form b on Z, and, for every $i \in I$, a ∂- and $\bar\partial$-closed smooth form a_i on E_i such that

$$dd^c[\eta] = \pi_*(\sum_{i \in I} \varepsilon_{i*}[a_i] + b).$$

If E_i has local equation $z_i = 0$, the form a_i is locally equal to the restriction of α_i to $E_i \cap U$ (see 1.3.2. (iii)).

Proof. — Of course $dd^c[\eta] = \pi_* \, dd^c[\varphi]$. Choose an open subset $U \subset Z$ as in 1.3.2 (iii) and write $\varphi_{|U} = \sum_{i=1}^{k} \alpha_i \log |z_i|^2 + \beta$ as in loc. cit. By the Poincaré-Lelong equation (1.3.1.1) we know that, on U,

$$dd^c \log |z_i|^2 = \delta_{E_i \cap U},$$

where E_i has local equation $z_i = 0$. Therefore, since α_i is ∂- and $\bar\partial$-closed,

$$dd^c[\varphi]_{|U} = \sum_{i \in I}^{k} \varepsilon_{i*}([\varepsilon_i^*(\alpha_i)]) + [dd^c \beta].$$

The forms $dd^c \beta$ and $\varepsilon_i^*(\alpha_i)$ are uniquely determined by this equation. Therefore these are restrictions to U of forms a_i (resp. b) defined on the whole of E_i (resp. Z).

1.3.5. Let X be a smooth quasi-projective complex variety and $Y \subset X$ a closed irreducible variety of codimension $p > 0$. A *Green form of* log *type* for Y is a smooth real form $g \in A^{p-1,p-1}(X - Y)$ which is of log type along Y and whose associated current $[g]$ on X is a Green current for Y, i.e. such that $dd^c[g] + \delta_Y$ is smooth. For example, when $p = 1$, $-\log \| s \|^2$ is a Green form of log type for $Y = \operatorname{div}(s)$ (see 1.3.1).

Theorem. — *For any* $Y \subset X$ *as above, there exists a Green form of* log *type for* Y.

1.3.6. To prove Theorem 1.3.5 we first consider the case where X is projective, Y is smooth and the restriction map $H^{q,q}(X) \to H^{q,q}(Y)$ is surjective for all $q \geq 0$. Let Z be the blow up of X along Y and $E \subset Z$ the exceptional divisor, so that we have the diagram

$$\begin{array}{ccc} E & \xrightarrow{j} & Z \\ {\scriptstyle \pi_Y}\downarrow & & \downarrow{\scriptstyle \pi} \\ Y & \xrightarrow{i} & X \end{array}$$

If N is the normal bundle of Y in X, $E = \mathbf{P}(N)$ is a smooth divisor in Z.

Lemma. — *If* $n = \operatorname{codim}_X(Y)$, *there is a real, closed smooth form* α *of type* $(n-1, n-1)$ *on* Z *such that*

$$\pi_*(\delta_E \wedge \alpha) = \delta_Y.$$

Proof. — Let $[Y] \in H^{n,n}_Y(X, \mathbf{R})$ be the cohomology class with supports of Y. Since $E = \pi^{-1}(Y)$, $\pi^*[Y]$ lies in $H^{n,n}_E(X, \mathbf{R})$, a group isomorphic to $H^{n-1,n-1}(E, \mathbf{R})$ by the cap-product $x \mapsto x \cap [Z]$. But we claim that the restriction map

$$j^* : H^{q,q}(Z, \mathbf{R}) \to H^{q,q}(E, \mathbf{R})$$

is surjective for all $q \geq 0$. Indeed $\bigoplus_{q \geq 0} H^{q,q}(E, \mathbf{R})$ is a free module on $\bigoplus_{q \geq 0} H^{q,q}(Y, \mathbf{R})$ with basis ξ^i, $i = 0, \ldots, n-1$, where $\xi = j^*[E]$ is the pull-back of the cycle class of E in $H^{1,1}(Z, \mathbf{R})$. Since i^* is surjective and $\pi_Y^* i^* = j^* \pi^*$, we conclude that j^* is surjective.

Therefore we can choose a closed real form α of type $(n-1, n-1)$ on Z such that $j^*(\alpha)$ is a representative of $\pi^*[Y] \cap [Z]$ in $H^{n-1,n-1}(E, \mathbf{R})$. Then $\pi_{Y*}(j^* \alpha)$ represents $\pi_{Y*}(\pi^*[Y] \cap [Z])$ in $H^{0,0}(Y, \mathbf{R})$. By the projection formula

$$\pi_{Y*}(\pi^*[Y] \cap [Z]) = [Y] \cap \pi_*[Z] = [Y] \cap [X] = [Y],$$

in $H^{0,0}(Y, \mathbf{R})$. Since any closed current of type $(0, 0)$ on the compact manifold Y is determined by its cohomology class, we conclude that

$$\pi_*(\delta_E \cup \alpha) = i_*[\pi_{Y*}(j^* \alpha)] = i_*(1) = \delta_Y.$$

This proves the Lemma.

1.3.7. We keep the notations of 1.3.6 and construct a Green form of log type for Y as follows. Choose a line bundle $\mathscr{L}$ on Z, a holomorphic section s in $\mathscr{L}$ with divisor E, and a smooth Hermitian metric on $\mathscr{L}$. Let

$$\beta = \delta_E - dd^c[\log \| s \|^2]$$

be the first Chern form of $\mathscr{L}$, and α be as in Lemma 1.3.6. Then

$$dd^c([-\log \| s \|^2 \wedge \alpha]) + \delta_E \wedge \alpha = \beta \wedge \alpha.$$

Since $\delta_E \wedge \alpha$ represents $\pi^*[Y]$ in $H^{n,n}(Z, \mathbf{R})$ (see 1.3.6), so does $\beta \wedge \alpha$. Let ω be a closed form of type (n, n) on X representing [Y]. The closed forms $\pi^*(\omega)$ and $\beta \wedge \alpha$ are cohomologous on Z, hence, by 1.2.1, there exists a smooth $(n-1, n-1)$ form γ on Z such that

$$dd^c \gamma = \pi^*(\omega) - \beta \wedge \alpha.$$

Therefore

$$dd^c([-\log \| s \|^2 \wedge \alpha + \gamma]) + \delta_E \wedge \alpha = \pi^*(\omega),$$

and

$$dd^c([g]) + \delta_Y = \omega,$$

where $g = \pi_*(-\log \| s \|^2 \wedge \alpha + \gamma)$ is a Green form of log type for Y.

1.3.8. *Proposition.* — *Let* X *be a smooth projective complex manifold and* $i : Y \to X$ *a closed irreducible submanifold of codimension* n. *Let* α *be a closed* (p, p) *form on* Y. *Then there exists an* $(n+p-1, n+p-1)$ *form* g *of* log *type along* Y *such that* $dd^c([g]) + i_*[\alpha]$ *is smooth on* X.

Proof. — Let $\Gamma \subset Y \times X$ be the graph of the immersion $i : Y \to X$, and $p_1 : Y \times X \to Y$ and $p_2 : Y \times X \to X$ the two projections. For any $q \geq 0$, the composite map

$$H^{q,q}(Y) \xrightarrow{p_1^*} H^{q,q}(Y \times X) \longrightarrow H^{q,q}(\Gamma)$$

is an isomorphism, therefore, by 1.3.7, we know that Γ has a Green form of log type g_Γ on $(Y \times X) - \Gamma$. In particular $\beta' = dd^c[g_\Gamma] + \delta_\Gamma$ is smooth on $Y \times X$. By Lemma 1.3.3 (ii), the form $g = p_{2*}(g_\Gamma \wedge p_1^* \alpha)$ has log type along $p_2(\Gamma) = Y$. This form has the property that

$$dd^c[g] + i_*[\alpha] = p_{2*}(dd^c[g_\Gamma] \wedge p_1^* \alpha) + p_{2*}(\delta_\Gamma \wedge p_1^* \alpha) = p_{2*}(\beta' \wedge p_1^* \alpha)$$

is smooth over X.

1.3.9. Let now X be a smooth projective complex manifold and $Y \subset X$ a closed irreducible subvariety of codimension $p > 0$. By [Hi] we know that there is a proper map $\pi : \tilde{X} \to X$ with $\tilde{X}$ smooth and $E = \pi^{-1}(Y)$ a d.n.c. Using either deformation to the normal cone [Fu] 6.6 or algebraic K-theory [Gi 4], we know that $\pi^*[Y] \in CH^p(\tilde{X})$ is represented by an algebraic cycle class $\eta \in CH^{p-1}(E)$. The cycle class η is necessarily a sum of cycles $\eta = \sum_{i=1}^{k} \eta_i$ with $\eta_i \in \mathrm{CH}^p(\mathrm{E}_i)$, where $\mathrm{E}_1, \ldots, \mathrm{E}_k$ are the irreducible components of E. Hence, if $[\mathrm{Y}] \in \mathrm{H}_{\mathrm{Y}}^{p,p}(\mathrm{X}, \mathbf{R})$ is the fundamental class of Y in cohomology, then $\pi^*[\mathrm{Y}] \in \mathrm{H}_{\mathrm{E}}^{p,p}(\widetilde{\mathrm{X}}, \mathbf{R})$ is a sum of classes $[\eta_i] \in \mathrm{H}_{\mathrm{E}_i}^{p,p}(\widetilde{\mathrm{X}}, \mathbf{R})$ under the obvious map $\bigoplus_{i=1}^{k} \mathrm{H}_{\mathrm{E}_i}^{p,p}(\widetilde{\mathrm{X}}, \mathbf{R}) \to \mathrm{H}_{\mathrm{E}}^{p,p}(\widetilde{\mathrm{X}}, \mathbf{R})$. It follows that for $i = 1, \ldots, k$, there is a real, closed $(p-1, p-1)$ form α_i on E_i such that, if $\varepsilon_i : \mathrm{E}_i \to \widetilde{\mathrm{X}}$ is the inclusion, $\pi^*[\mathrm{Y}]$ is represented by the current $\sum_{i=1}^{k} \varepsilon_{i_*}[\alpha_i]$. Now, by Proposition 1.3.8, there exists a form g_i of log type along E_i such that

$$dd^c[g_i] + \varepsilon_{i_*}[\alpha_i] = \beta_i$$

is smooth on $\widetilde{\mathrm{X}}$. Since $\sum_{i=1}^{k} \beta_i$ is cohomologous to $\pi^*[\mathrm{Y}]$, if ω is a real closed (p, p) form on X cohomologous to [Y], there exists (see 1.2.1) a smooth real form γ of type $(p-1, p-1)$ on $\widetilde{\mathrm{X}}$ such that

$$dd^c \gamma = \pi^*(\omega) - \sum_{i=1}^{k} \beta_i.$$

Hence, if $g = \sum_{i=1}^{k} g_i + \gamma$, the form g is of log type along E (by Lemma 1.3.3 (iii)) and such that, on $\widetilde{\mathrm{X}}$,

$$dd^c[g] + \sum_{i=1}^{k} \varepsilon_{i_*}[\alpha_i] = \pi^*(\omega).$$

Since $\widetilde{\mathrm{X}} - \mathrm{E} = \mathrm{X} - \mathrm{Y}$ we may view $g = \pi_*(g)$ as a form of log type on $\mathrm{X} - \mathrm{Y}$ (Lemma 1.3.3 (ii)), whose associated current on X satisfies

$$dd^c[g] + \pi_*\Big(\sum_{i=1}^{k} \varepsilon_{i_*}[\alpha_i]\Big) = \omega.$$

It remains to show that

$$\pi_*\Big(\sum_{i=1}^{k} \varepsilon_{i_*}[\alpha_i]\Big) = \delta_{\mathrm{Y}}.$$

Let $p_i : \widetilde{\mathrm{Z}}_i \to \mathrm{Z}_i = \pi(\mathrm{E}_i)$ be a resolution of singularities of Z_i. Then we may construct a diagram

$$\begin{array}{ccccc} \widetilde{\mathrm{E}}_i & \xrightarrow{\tilde{\pi}} & \widetilde{\mathrm{Z}}_i & & \\ {\scriptstyle q_i}\downarrow & & \downarrow{\scriptstyle p_i} & & \\ \mathrm{E}_i & \xrightarrow[\pi]{} & \mathrm{Z}_i & \xrightarrow[\varepsilon]{} & \mathrm{X} \end{array}$$

with q_i birational and $\widetilde{\mathrm{E}}_i$ smooth. We have an equation of currents:

$$\pi_* \, \varepsilon_{i_*}[\alpha_i] = \varepsilon_* \, p_{i_*} \, \tilde{\pi}_*[q_i^* \, \alpha_i]$$

which may be verified using test forms on X. Now $[q_i^* \alpha_i]$ is a current of dimension $2(\dim(X) - \operatorname{codim}(Y))$; hence, since Y is irreducible, $\tilde{\pi}_*[q_i^* \alpha_i] = 0$ unless $Z_i = Y$. Therefore $\pi_*(\sum_{i=1}^{k} \varepsilon_{i_*}[\alpha_i]) = p_*(S)$, where $p : \tilde{Y} \to Y \to X$ is obtained by resolving the singularities of Y, and S is a closed current in $\mathscr{D}^{0,0}(\tilde{Y})$. Such an S is a constant, hence $\pi_*(\sum_{i=1}^{k} \varepsilon_{i_*}[\alpha_i]) = \mu\, \delta_Y$ for $\mu \in \mathbf{R}$. But $\mu\, \delta_Y$ represents $\pi_*\, \pi^*[Y]$ in $H_Y^{2p}(X, \mathbf{R})$ and therefore, since π is birational and so $\pi_*\, \pi^*[Y] = [Y]$, we have $\mu = 1$. Thus we have the equation of currents on X

$$dd^c[g] + \delta_Y = \omega$$

as desired.

Now suppose that X is quasi-projective. Then, by resolution of singularities [Hi], X has a smooth projective compactification $\overline{X}$. Let $\overline{Y} \subset \overline{X}$ be the Zariski closure of Y in $\overline{X}$. By the preceeding discussion $\overline{Y}$ has a Green form of log type $g_{\overline{Y}}$. The restriction g_Y of $g_{\overline{Y}}$ to X is then a Green form of log type for Y.

1.4. Examples

— For any point $z = (z_1, \ldots, z_n)$ in $\mathbf{C}^n$, let $\| z \| = | z_1 |^2 + \ldots + | z_n |^2$. The form

$$g = (-1)^n (\log \| z \|)\, (dd^c \log \| z \|)^{n-1}$$

on $\mathbf{C}^n - \{ 0 \}$ is a Green form of logarithmic type for the origin. This follows from the Bochner-Martinelli formula ([G-H], p. 372).

— Let $Y \subset \mathbf{P}^n(\mathbf{C})$ be the linear subspace with equation $x_0 = \ldots = x_{p-1} = 0$, where $(x_0, \ldots, x_n)$ are homogeneous coordinates. On $\mathbf{P}^n(\mathbf{C}) - Y$ define

$$\tau = \log(| x_0 |^2 + \ldots + | x_n |^2), \qquad \sigma = \log(| x_0 |^2 + \ldots + | x_{p-1} |^2),$$

$$\alpha = dd^c \tau, \qquad \beta = dd^c \sigma \qquad \text{and} \qquad \Lambda = (\tau - \sigma)\, (\sum_{\nu=0}^{p-1} \alpha^\nu \beta^{p-1-\nu}).$$

The form Λ, first introduced by H. Levine, is a Green form of logarithmic type for Y (see [G-S 4], Proposition 5.1).

— In [B-G-S], Theorems 3.14 and 3.15, one defines an explicit Green form for the zero section Y in the total space X of an arbitrary holomorphic vector bundle on a complex manifold. By blowing up Y, one checks that this form is of logarithmic type along Y.

2. Green currents and intersecting cycles

2.1. Pull-backs and the $*$-product

2.1.1. Before discussing the relationship between Green currents and intersecting algebraic cycles, it will be useful to understand the relationship between currents and cohomology with supports. Recall that if X is a complex (or more generally a C^∞ real and orientable) manifold, then the cohomology of X with complex coefficients, $H^*(X)$, may be computed as the cohomology of either the complex of C^∞ (complex valued) differential forms on X, $A^*(X)$, or the complex of currents on X, $\mathcal{D}^*(X)$. The inclusion $A^*(X) \subset \mathcal{D}^*(X)$ induces the canonical quasi-isomorphism between these complexes. If Y is a closed subset of X, then the relative cohomology groups $H^*(X, X - Y)$ (which are also written $H^*_Y(X)$ and called cohomology with supports in Y) may be computed as the cohomology of either the mapping cone $C(i^*)$ of the restriction map on forms, $i^* : A^*(X) \to A^*(X - Y)$, or the mapping cone $C_{\mathcal{D}}(i^*)$ of the restriction map $i^* : \mathcal{D}^*(X) \to \mathcal{D}^*(X - Y)$ on currents; here $i : (X - Y) \to X$ is the inclusion. Note that, by Alexander duality, $H^n_Y(X) \simeq H^c_{2d-n}(Y)$ for $d = \dim_{\mathbf{C}} X$, where $H^c_*(Y)$ denotes homology with locally compact supports ([E-S]) or Borel-Moore homology ([B-M]). A class in $H^n(X, X - Y)$ can be represented, therefore, by a pair $(\alpha, \beta) \in A^n(X) \oplus A^{n-1}(X - Y)$ (or $\mathcal{D}^n(X) \oplus \mathcal{D}^{n-1}(X - Y)$) such that $d_X \alpha = 0$ and $d_{X-Y} \beta = (-1)^{n-1} i^* \alpha$. Two such pairs, (α, β) and (α', β'), represent the same cohomology class if there is a pair $(\eta, \zeta) \in A^{n-1}(X) \oplus A^{n-2}(X - Y)$ (or $\mathcal{D}^{n-1}(X) \oplus \mathcal{D}^{n-2}(X - Y)$) such that $d(\eta, \zeta) = (\alpha - \alpha', \beta - \beta')$ with, by definition, $d(\eta, \zeta) = (d\eta, i^* \eta + (-1)^{n-2} d\zeta)$. A particularly simple set of representatives of classes in $H^*(X, X - Y)$ consists of the pairs $(T, 0)$ with T a closed current supported on Y; more precisely, if $\mathcal{D}^*_Y(X) = \ker(i^* : \mathcal{D}^*(X) \to \mathcal{D}^*(X - Y))$ is the complex of currents supported on Y, the map $T \to (T, 0)$ is the canonical factorization of the map $\mathcal{D}^*_Y(X) \to \mathcal{D}^*(X)$ through the mapping cone of i^*. When Y is a submanifold of codimension p with $j : Y \to X$ the inclusion, then we have the natural map $j_* : \mathcal{D}^*(Y) \to \mathcal{D}^{*+2p}_Y(X)$ and hence an induced map $j_* : H^*(Y) \to H^{*+2p}_Y(X)$. This map is the Alexander duality map composed with Poincaré duality, [B-M] or [Sp], and is an isomorphism. We leave to the reader the proof of the following lemma.

Lemma. — (i) *Let* $(\alpha, \beta) \in \mathcal{D}^n(X) \oplus \mathcal{D}^{n-1}(X)$ *be a pair such that* $d(\alpha, \beta) = 0$. *If* β *extends to a current* $\tilde{\beta}$ *on* X, *and we write* $d\tilde{\beta} = \alpha + R$, R *is a current supported on* Y *and* $(-R, 0)$ *represents the same class in* $H^n_Y(X)$ *as* (α, β). *In particular, if*

$$(\alpha, \beta) \in A^n(X) \oplus A^{n-1}(X - Y)$$

and β *is locally* L^1 *on* X, $(-\mathrm{Res}_Y(\beta), 0)$ *and* (α, β) *represent the same class in* $H^n_Y(X)$.

(ii) *Suppose that* Y *and* Z *are closed subsets of* X, $y \in H^m_Y(X)$ *is represented by*

$$(\alpha, \beta) \in A^m(X) \oplus A^{m-1}(X - Y)$$

and $z \in H^n_Z(X)$ *is represented by* $(T, 0)$ *with* $T \in \mathcal{D}^n_Z(X)$. *Then* $\beta \wedge T \in \mathcal{D}^{n+m-1}_{Z-(Y\cap Z)}(X - Y)$ *extends naturally to a current in* $\mathcal{D}^{n+m-1}_{Z-(Y\cap Z)}(X - (Y \cap Z))$ *and* $x \cup y \in H^{m+n}_{Y\cap Z}(X)$ *is represented by* $(\alpha \wedge T, (-1)^n \beta \wedge T)$.

(iii) *If* $f : Z \to X$ *is a map of complex manifolds,* $Y \subset X$ *is a closed subset, and* $y \in H^n_Y(X)$ *is represented by* $(\alpha, \beta) \in A^n(X) \oplus A^{n-1}(X - Y)$, *then* $(f^* \alpha, f^* \beta)$ *represents* $f^*(y) \in H^n_{f^{-1}(Y)}(Z)$.

2.1.2. For general background to the discussion of algebraic cycles in this section see [Fu] and § 3 below. Let $f: Z \to X$ be a map between quasi-projective varieties over C; we suppose that both X and Z are irreducible and that X is smooth. If $Y = \sum_{i=1}^{l} m_i[Y_i]$ is an algebraic cycle of codimension p on X, and we write $|Y|$ for the support of Y, one can define a rational equivalence class $f^*(Y)$ on $f^{-1}(|Y|)$ (the "pull-back" of Y) using either the construction of chapter 8 of [Fu] or higher algebraic K-theory [Gi 4]. Let us write $f^{-1}(|Y|) = S \cup T$, $S = \bigcup_{i=1}^{k} S_i$ being the union of the irreducible components $S_1, \dots, S_k$ of $f^{-1}(|Y|)$ which have codimension p in Z, and T being the union of the components of codimension strictly less than p. Then we have a unique decomposition

$$f^*(Y) = \sum_{i=1}^{k} n_i[S_i] + t$$

in which t is a rational equivalence class on T, i.e. an element of the Chow group with supports $\mathrm{CH}^p_T(Z)$. The intersection multiplicities $n_1, \dots, n_k$ can be computed using either the purely algebraic theory of multiplicities, as in [Se], Chapter V, or the following cohomological technique. The cycle Y has a cycle class $cl(Y) \in H^{2p}_{|Y|}(X)$, constructed by the methods of [B-M], [B-H]. We denote by $\gamma(Y) \in H_{2d-2p}(|Y|)$ the corresponding homology class. Using the description of $H^{2p}_{|Y|}(X)$ given in 2.1.1, $cl(Y)$ can be represented by $(\delta_Y, 0) \in \mathscr{D}^{2p}(X) \oplus \mathscr{D}^{2p-1}(X - |Y|)$. If $\dim_{\mathbf{C}} X = d$, then $f^*(Y)$ has a homology cycle class $\gamma(f^*(Y)) = \Sigma n_i \gamma(S_i) + \gamma(t) \in H_{2d-2p}(f^{-1}(|Y|))$. Using the Mayer-Vietoris sequence, one sees that

$$\begin{aligned} H_{2d-2p}(S \cup T) &\simeq H_{2d-2p}(S) \oplus H_{2d-2p}(T) \\ &\simeq \bigoplus_{i=1}^{k} \mathbf{R}cl(S_i) \oplus H_{2d-2p}(T). \end{aligned}$$

Hence the integers n_i are determined by $\gamma(f^*(Y))$. But by 19.2 of [Fu],

$$\gamma(f^*(Y)) = f^* cl(Y) \cap [Z].$$

Hence the n_i, and the homology class of t, are determined by $f^* cl(Y) \in H^{2p}_{f^{-1}(|Y|)}(Z)$.

We are interested in two special cases of this construction: either when Z is smooth or when $f: Z \to X$ is a closed immersion (i.e. Z is a closed subvariety of X). In the second case, note that

$$f_!(f^* cl(Y) \cap [Z]) = \gamma(Y.[Z]) = (cl(Y) \cup cl(Z)) \cap [Z]$$

in $H^{2p+2q}_{|Y| \cap Z}(X)$; here $q = \mathrm{codim}_X(Z)$.

2.1.3. We want to define a pull-back operation on Green currents which is compatible with the pull-back operation on cycles discussed previously. Let $f: Z \to X$ be a map of quasi-projective varieties over $\mathbf{C}$, with X smooth and Z irreducible. Let Y be an irreducible closed subvariety of X of codimension p such that $f^{-1}(Y) \neq Z$. Suppose that

g_Y is a Green form of log type for Y, and let $[g_Y]$ be the associated current. When Z is smooth, we know from Lemma 1.3.3 (i) that $f^*(g_Y)$ is a form of log type on $Z - f^{-1}(Y)$. We define

(2.1.3.1) $$f^*[g_Y] = [f^*(g_Y)] \in \mathscr{D}^{p-1,p-1}(Z).$$

The pull-back map extends to Green currents of cycles. If Z is smooth, $Y = \Sigma a_i[Y_i]$ and g_{Y_i} is a Green form of logarithmic type for Y_i, and if $f^{-1}(|Y|) \neq Z$, then we define $f^*[g_Y] = \Sigma a_i f^*[g_{Y_i}]$.

In general, there is a resolution of singularities $\pi : \widetilde{Z} \to Z$, where $\widetilde{Z}$ is nonsingular and π is projective and birational. Let $\psi = f \circ \pi$ be the composite map. Then $\psi^*(g_Y)$ is of log type along $\psi^{-1}(Y)$. If $f: Z \to X$ is a closed immersion of codimension q, we define a current $[g_Y] \wedge \delta_Z = \delta_Z \wedge [g_Y]$ in $\mathscr{D}^{p+q-1,p+q-1}(X)$ by

(2.1.3.2) $$[g_Y] \wedge \delta_Z = \psi_*[\psi^* g_Y].$$

This current does not depend on the choice of the resolution π (see 1.1.4).

More generally, if $Z = \Sigma b_i[Z_i]$ and $Y = \Sigma a_j[Y_j]$ are such that $Z_i \nsubseteq |Y|$ for all i, we define

$$[g_Y] \wedge \delta_Z = \sum_{i,j} a_j[g_{Y_j}] \wedge \delta_{Z_i}.$$

Now, if g_Z is an arbitrary Green current for Z, we define the $*$-product of $[g_Y]$ and g_Z by

$$[g_Y] * g_Z = [g_Y] \wedge \delta_Z + \omega_Y \wedge g_Z \in \widetilde{\mathscr{D}}^{m+n-1,m+n-1}(X).$$

(Recall that we are, temporarily, writing g_Y for the Green form and $[g_Y]$ for the Green current of Y.)

Remarks. — (i) Though the current $[g_Y] * g_Z$ depends, a priori, on the choice of form g_Y, we shall prove in § 2.2.9 that, in fact, it only depends on the class of $[g_Y]$ in $\widetilde{\mathscr{D}}(X)$.

(ii) If g'_Z is another choice of a Green current for Z,

$$[g_Y] * g_Z - [g_Y] * g'_Z = \omega_Y \wedge (g_Z - g'_Z) \in \widetilde{A}(X).$$

(iii) This $*$-product is analogous to the product on differential characters, which has the same name, defined by Cheeger in [C].

2.1.4. *Theorem.* — *Let* X *be a nonsingular quasi-projective variety over* **C**, *and* $Y = \sum_{i=1}^{\ell} a_i[Y_i]$ *a codimension n cycle on* X *and, for* $i = 1, \ldots, \ell$, *let* g_{Y_i} *be a Green form of logarithmic type for* Y_i; *we write* $g_Y = \sum_{i=1}^{\ell} g_{Y_i}$. *Then:*

(i) *If* $Z = \Sigma b_j[Z_j]$ *is a codimension m cycle on* X *such that* $Z_j \nsubseteq |Y|$ *for all j, and* g_Z *is a Green current for* Z, *we have*

$$dd^c([g_Y] * g_Z) + \sum_{i=1}^{k} \mu_i \delta_{S_i} + \tau = \omega_Y \omega_Z.$$

Here $|Y| \cap |Z| = S \cup T$ *with* S *the union of the components* $S_1, \ldots, S_k$ *of* $|U| \cap |Z|$ *of codimension* $m+n$, T *is the union of the components of codimension* $< m+n$,

$$[Y].[Z] = \sum_{i=1}^{k} \mu_i [S_i] + t,$$

and τ *is a closed current associated to an* L^1 *form on a closed analytic subset of* T, *representing the homology class of* t.

(ii) *Let* $f: Z \to X$ *be a map between quasi-projective smooth varieties, and let* $Y = \sum_{i=1}^{\ell} a_i [Y_i]$ *be a codimension* p *cycle on* X *for which* $f^{-1}(|Y|) \neq Z$. *If we write*

$$f^{-1}(|Y|) = S \cup T$$

as above, and if g_Y *is a Green form of logarithmic type for* Y (i.e. *a sum of such forms for each* i), *then*

$$dd^c f^*[g_Y] + \sum_{i=1}^{k} \mu_i \, \delta_{S_i} + \tau = f^* \omega_Y.$$

Here $f^*(Y) = \Sigma \mu_i [S_i] + t$ *is the pull-back cycle class as in* 2.1.2, *and* τ *is a current supported on* T *representing the homology class of* t. (*Note that* $\omega_Y = \delta_Y + dd^c g_Y$ *and* $\omega_Z = \delta_Z + dd^c g_Z$).

Proof. — We start with (i). It is enough to show that

$$dd^c([g_Y] \wedge \delta_Z) + \Sigma_{i=1}^{k} \mu_i \, \delta_{S_i} + \tau = \omega_Y \, \delta_Z.$$

We shall consider the case in which Y and Z are prime cycles, i.e. irreducible subvarieties of X; the general case follows from this one by additivity.

Let $f: Z \to X$ be the inclusion, $\pi: \tilde{Z} \to Z$ a resolution of singularities of Z, and $\psi = f \circ \pi$. Since $\psi^*(g_Y)$ is of log type along $\psi^{-1}(Y)$, there is a projective morphism

$$\pi': Z' \to \tilde{Z},$$

smooth outside $\psi^{-1}(Y)$, such that $E = r^{-1}(Y)$ is a d.n.c., where $r = \psi \circ \pi'$, and a form φ on $Z' - E$ such that $\psi^*(g_Y) = \pi'_*(\varphi)$ and φ can be written locally as in Definition 1.3.2 Following Lemma 1.3.4 we write

$$dd^c[\varphi] = \sum_{i \in I} \varepsilon_{i_*}[a_i] + [b],$$

where a_i is smooth and closed on the component E_i of E, and b is smooth on Z'. Since

$$[g_Y] \wedge \delta_Z = \psi_*[\psi^* g_Y] = r_*[\varphi],$$

we get

$$dd^c([g_Y] \wedge \delta_Z) = \sum_{i \in I} r_* \, \varepsilon_{i_*}[a_i] + r_*[b].$$

Let ω_Y be the smooth form $dd^c[g_Y] + \delta_Y$. Outside $\psi^{-1}(Y)$ we have $dd^c \psi^*(g_Y) = \psi^*(\omega_Y)$. Therefore

$$r_*[b] = \psi_* \, \psi^*(\omega_Y) = \omega_Y \wedge \delta_Z.$$

On the other hand, we know, from 1.1.5, that for every i in I, the current $r_* \, \varepsilon_{i_*}[a_i]$ is equal to $[\beta_i]$, where β_i is a closed L^1 form on $r(E_i) \subset X$. We may now distinguish between several cases, according to the codimension of $r(E_i)$ in X. If this codimension is less than $p+q$, $r(E_i)$ is contained in T, and we denote by τ the sum of the currents $r_* \varepsilon_{i_*}[a_i]$ with $r(E_i)$ contained in T. If the codimension of E_i is bigger than $p+q$, the degree of a_i is less that $2(\dim(E_i) - \dim(r(E_i)))$, therefore $r_* \varepsilon_{i_*}[a_i] = 0$ (see 1.1.5). Finally when $r(E_i)$ is equal to one of the component S_i, the L^1 form β_i has degree zero. Since a_i is closed, the distribution $[\beta_i] = r_* \varepsilon_{i_*}[a_i]$ is closed, so it is equal to a constant μ_i (apply 1.2.2 (i) to a resolution of the singularities of $r(E_i)$).

Now we claim that the current $R = \sum_{i \in I} r_* \varepsilon_{i_*}[a_i]$ represents $cl([Y].[Z])$ in $H^{p+q,\, p+q}_{Y \cap Z}(X)$. Recall from 2.1.1 that $cl(Y) \in H^{2p}_Y(X)$ can be represented by

$$(\delta_Y, 0) \in \mathscr{D}^{2p}(X) \oplus \mathscr{D}^{2p-1}(X - Y).$$

Therefore, by Lemma 2.1.1 (i), it is also represented by $(\omega_Y, d^c g_Y)$. Note also that $cl(Z) \in H^{2q}_Z(X)$ is represented by $(\delta_Z, 0)$. Since $\delta_Z \wedge d^c g_Y$ extends to a current $d^c(\delta_Z \wedge [g_Y])$ on X, we conclude, by Lemma 2.1.1 (ii), that

$$cl([Y], [Z]) = cl(Y) \cup cl(Z) \in H^{p+q,\, p+q}_{Y \cap Z}(X)$$

is represented by $(\omega_Y \delta_Z, d^c(\delta_Z \wedge [g_Y]))$. Applying part (i) of Lemma 2.1.1 again, we find that this cohomology class is also represented by $(R, 0)$. Therefore R is a current supported on $Y \cap Z$ which represents $cl([Y].[Z])$ in $H^{p+q,\, p+q}_{Y \cap Z}(X)$. It follows that μ_i is equal to Serre's intersection multiplicity of Y and Z on S_i, and the closed current $\tau = R - \sum_{i=1}^{k} \mu_i \delta_{S_i}$ represents the cohomology class of t in $H^{2(p+q)}_T(X)$.

The proof of (ii) follows essentially the same pattern, with part (iii) of Lemma 2.1.1 replacing part (ii), and $R = dd^c(f^*[g_Y]) - f^*(\omega_Y)$ being computed by the same arguments as above.

2.1.5. If X is a nonsingular quasi-projective variety over **C** and $Y = \sum_{i=1}^{k} a_i[Y_i]$ is a codimension n cycle on X, we can approximate an L^1 Green form g_Y (C^∞ on $X - |Y|$) for Y by C^∞ forms on X as follows. Choose a locally finite open covering of X by coordinate charts and, for each $\varepsilon > 0$, let ρ_ε be a C^∞ real valued function on X which is:

(i) $\geqslant 0$ on all of X,
(ii) $\leqslant 1$ on all of X,
(iii) $= 1$ outside the neighbourhood $N_\varepsilon(Y)$ of radius ε of $|Y|$ in each coordinate chart,
(iv) $= 0$ in some open neighbourhood of $|Y|$.

Lemma. — For each $\varepsilon > 0$, $g_Y^\varepsilon = \rho_\varepsilon g_Y \in \mathscr{D}^{n-1,\, n-1}(X)$ *satisfies:*

(i) g_Y^ε *is a* C^∞ *form on* X.
(ii) $dd^c g_Y^\varepsilon + \omega_Y^\varepsilon = \omega_Y$, ω_Y^ε *a* C^∞ *form supported in the union of the closures of the* $N_\varepsilon(Y)$.
(iii) $\lim_{\varepsilon \to 0} [g_Y^\varepsilon] = [g_Y]$.
(iv) $\lim_{\varepsilon \to 0} [\omega_Y^\varepsilon] = \delta_Y$.

Proof. — (i) and (ii) are obvious. (iii) is a basic property of L^1 forms and (iv) follows from (iii).

Keeping the above notation, we have the following corollary to theorem 2.14.

Corollary. — *Let* X, Y, Z, g_Y *be as in Theorem* 2.1.4 (i). *Then we have an equality of currents, with the limit taken in the space of currents of order* $\geqslant 2$,

$$\lim_{\varepsilon \to 0} \omega_Y^\varepsilon \wedge \delta_Z = \sum_{i=1}^{k} \mu_i \, \delta_{S_i} + \tau.$$

In particular, if Y *and* Z *meet properly,* $\lim_{\varepsilon \to 0} \omega_Y^\varepsilon \wedge \delta_Z = \delta_{Y.Z}$. *Similarly if* $f : Z \to X$ *is as in part* (ii) *of the theorem,*

$$\lim_{\varepsilon \to 0} f^*(\omega_Y^\varepsilon) = \sum_{i=1}^{k} \mu_i \, \delta_{S_i} + \tau.$$

Proof. — Clearly $[g_Y] \wedge \delta_Z = \lim_{\varepsilon \to 0} g_Y^\varepsilon \wedge \delta_Z$, hence

$$\begin{aligned} \lim_{0 \to 0} \omega_Y^\varepsilon \wedge \delta_Z &= \omega_Y \wedge \delta_Z - \lim_{\varepsilon \to 0} (dd^c g_Y^\varepsilon) \wedge \delta_Z \\ &= \omega_Y \wedge \delta_Z - \lim_{\varepsilon \to 0} dd^c(g_Y^\varepsilon \wedge \delta_Z) \\ &= \omega_Y \wedge \delta_Z - dd^c([g_Y] \wedge \delta_Z) \\ &= \sum_{i=1}^{k} \mu_i \, \delta_{S_i} + \tau. \end{aligned}$$

The proof of the second part of the corollary is similar.

2.2. Associativity and commutativity of the $*$-product

2.2.1. We want to prove that the $*$-product of Green currents, defined in the previous paragraph, is both commutative and associative in $\widetilde{\mathscr{D}}$, i.e. modulo the sum of the images of ∂ and $\bar{\partial}$. These properties can easily be checked formally. Indeed, let g_Y, g_Z, g_W be Green currents for irreducible closed subvarieties Y, Z and W in a complex manifold X. We get

$$\begin{aligned} g_Y * g_Z &= g_Y \, \delta_Z + \omega_Y \, g_Z = g_Y \, \delta_Z + (dd^c g_Y + \delta_Y) \, g_Z \\ &= g_Y \, \delta_Z + \delta_Y \, g_Z + dd^c(g_Y) \, g_Z, \end{aligned}$$

and Stokes formula implies that $g_Y * g_Z = g_Z * g_Y$ in $\widetilde{\mathscr{D}}(X)$.

Similarly

$$(g_Y * g_Z) * g_W = (g_Y \, \delta_Z + \omega_Y \, g_Z) * g_W = g_Y \, \delta_Z \, \delta_W + \omega_Y g_Z \, \delta_W + \omega_Y \, \omega_Z \, g_W$$

while

$$g_Y * (g_Z * g_W) = g_Y * (g_Z \, \delta_W + \omega_Z g_W) = g_Y \, \delta_Z \, \delta_W + \omega_Y g_Z \, \delta_W + \omega_Y \, \omega_Z g_W.$$

But these formal computations need to be justified since the $*$-product $[g_Y] * g_Z$ has been defined only when Y and Z meet properly and $[g_Y]$ is the current associated to a

Green form g_Y of log type for Y. Furthermore, the use of Stokes' formula for currents and forms requires some work to take care of residues (see 1.1.2). For instance the commutativity of the $*$-product will follow from the formulae

$$\partial[g_Y \bar{\partial}(g_Z)] = [\partial g_Y \bar{\partial} g_Z] + [g_Y \bar{\partial}\partial(g_Z)] + 2\pi i g_Y \delta_Z$$

and

$$\bar{\partial}[g_Z \partial(g_Y)] = [\bar{\partial} g_Z \partial g_Y] + [g_Z \bar{\partial}\partial(g_Y)] - 2\pi i g_Z \delta_Y,$$

when Y and Z meet properly and g_Y (resp. g_Z) is a Green form of log type for Y (resp. Z). For technical reasons, associativity will be shown only when the ambiant variety X is projective.

2.2.2. It turns out that the proofs of commutativity and associativity will both make use of the following statement.

Let X be a smooth quasi-projective variety over **C**. Suppose that Y, Z and W are irreducible closed subvarieties of X which have codimensions p, q and r respectively, with $p > 0$ and $q > 0$. We assume that $Y \cap Z$, $Y \cap W$ and $Y \cap Z \cap W$ have codimensions $p+q$, $p+r$ and $p+q+r$ respectively; this implies that if we write $Z \cap W = S \cup T$ with $S = S_1 \cup \ldots \cup S_k$ the union of the components of $Z \cap W$ of codimension $q+r$, and with T the union of the components of codimension $< q+r$, then $Y \cap T = \emptyset$ and Y meets the S_i's properly. As discussed in 2.1, $[Z].[W] = \sum_{i=1}^{k} \mu_k [S_k] + \tau$ with τ a rational equivalence class on T. Let g_Y and g_Z be Green forms of logarithmic type for Y and Z respectively. Recall from 2.1.3 and Theorem 2.1.4 that $[g_Z] \wedge \delta_W$ is defined, and that

$$dd^c([g_Z] \wedge \delta_W) + \sigma + \tau = \omega_Z \wedge \delta_W,$$

with $\sigma = \sum \mu_i \delta_{S_i}$ and with τ a closed current of order 0 supported on T representing the homology class of t on T. Note also that, since $Y \cap T = \emptyset$, $[g_Y] \wedge \tau$ is well defined, and that, since Y intersects S_i properly, $[g_Y] \wedge \delta_{S_i}$ is defined, for all i, by 2.1.3.

Theorem. — With the notation above, we have an equality of currents in $\widetilde{\mathscr{D}}^{p+q+r-1,\,p+q+r-1}(X)$:

$$[g_Y] \wedge (\sigma + \tau) + \omega_Y \wedge [g_Z] \wedge \delta_W = \delta_{[Y].[W]} \wedge g_Z + [g_Y] \wedge \omega_Z \wedge \delta_W.$$

2.2.3. To prove Theorem 2.2.2 we need a Lemma about blow ups. Let X be a smooth quasi-projective variety over **C**, $Y \subset X$ a closed irreducible smooth subvariety of codimension $p > 0$, $\pi : \widetilde{X} \to X$ the blow up of X along Y, $D = \pi^{-1}(Y)$ the exceptional divisor, and $f : Z \to \widetilde{X}$ a projective morphism such that Z is smooth and $f^{-1}(D) = E$ is a d.n.c. Let $z = 0$ be a local equation of one component of E in some open set $U \subset Z$. Let $\psi = \pi \circ f$ be the projection from Z to X and $r = |z|$ the modulus of z.

Lemma. — *Assume* η *is a smooth form with compact support on* X *of degree greater than* $2\operatorname{codim}_X(Y)+1$. *Then, locally on* Z, *the form* $\psi^*(\eta)$ *is the sum of smooth multiples of* r^2 *and* rdr.

Proof. — This is a local question on Z so we may assume that $X = \mathbf{C}^d$ with coordinates $z_1, \ldots, z_d$ and that $Y \subset \mathbf{C}^d$ is given by the equations $z_1 = \ldots = z_p = 0$. The blow up $\widetilde{X}$ can be covered by affine subsets $U_i \cong \mathbf{C}^d$, with coordinates

$$(u_1, \ldots, u_{i-1}, z_i, u_{i+1}, \ldots, u_p, z_{p+1}, \ldots, z_d), \quad 1 \leqslant i \leqslant p,$$

where the map π is given by

(2.2.3.1)
$$\pi(u_1, \ldots, u_{i-1}, z_i, u_{i+1}, \ldots, u_p, z_{p+1}, \ldots, z_d) = (u_1 z_i, \ldots, u_{i-1} z_i, z_i, u_{i+1} z_i, \ldots, u_p z_i, z_{p+1}, \ldots, z_d).$$

An equation of $D \cap U_i$ is $z_i = 0$. Since η has degree at least $2(d-p)+2$, when written in local coordinates in X, each of its components will involve at least two of the differentials dz_j and $d\bar{z}_j$, $j \leqslant p$. Locally in $f^{-1}(U_i)$, the function $f^*(z_i)$ is divisible by z. Since $\pi^*(z_j) = u_j z_i$ when $j \leqslant p$ (by (2.2.3.1)), we see that $\psi^*(\eta)$ is a sum of forms divisible by the product of two terms among z, $\bar{z}$, dz, and $d\bar{z}$, hence by r^2 or rdr.

2.2.4. We now return to the notation in 2.2.2 and make geometric constructions based upon Hironaka's resolution of singularities in its precise form ([Hi], Theorem II). First we can resolve $Y \cap Z \cap W$ in X by a succession of blow ups with smooth centers in the proper transform of $Y \cap Z \cap W$ (hence the codimension of these centers is at least $p+q+r$), till $Y \cap Z \cap W$ becomes a d.n.c. Let W′ be the inverse image of W in this resolution and $W'' \to W'$ be a smooth projective resolution of W′ where the inverse image of $Y \cap Z \cap W$ is still a d.n.c. (loc. cit). Let $\psi : W'' \to X$ be the obvious map. The forms $\psi^*(g_Y)$ and $\psi^*(g_Z)$ have log type along $\psi^{-1}(Y)$ and $\psi^{-1}(Z)$ respectively. So let $\pi_1 : W_1 \to W''$ be a projective morphism with W_1 smooth such that $D_1 = (\psi\pi_1)^{-1}(Y)$ is a d.n.c., and φ_1 be a smooth form on $W_1 - D_1$ such that $\pi_{1*}(\varphi_1) = \psi^*(g_Y)$ outside $\psi^{-1}(Y)$ and φ_1 can be written locally as in Definition 1.3.2. Similarly we define $\pi_2 : W_2 \to W''$, $D_2 = (\psi\pi_2)^{-1}(Z)$ and φ_2 with $\pi_{2*}(\varphi_2) = \psi^*(g_Z)$.

Now we consider the closure $\overline{W}$ of $(W_1 - D_1) \times_{W''} (W_2 - D_2)$ in the fiber product $W_1 \times_{W''} W_2$ and resolve its singularities (these are over $Y \cap Z$) to get a smooth variety $\widetilde{W}$ and a commutative diagram

$$\begin{array}{ccc} \widetilde{W} & \xrightarrow{p_1} & W_1 \\ {\scriptstyle p_2}\downarrow & & \downarrow{\scriptstyle \pi_1} \\ W_2 & \xrightarrow[\pi_2]{} & W'' \end{array}$$

such that, if $h = \psi\pi_2 p_2 = \psi\pi_1 p_1 : \widetilde{W} \to X$, then $E_Y = p_1^{-1}(D_1) = h^{-1}(Y)$, $E_Z = p_2^{-1}(D_2) = h^{-1}(Z)$, $E_Y \cap E_Z$ and $E_Y \cup E_Z$ are d.n.c. (see loc. cit. and [De 1]

(3.2.11) *c)*). The map $\widetilde{W} \to W_1 \times_{W''} W_2$ is an isomorphism outside the inverse image of $Y \cap Z$ and there is a cartesian square of smooth maps

$$\begin{array}{ccc} \widetilde{W} - (E_Y \cup E_Z) & \xrightarrow{p_1} & W_1 - D_1 \\ {\scriptstyle p_2}\downarrow & & \downarrow{\scriptstyle \pi_1} \\ W_2 - D_2 & \xrightarrow[\pi_2]{} & W'' - \psi^{-1}(Y \cap Z) \end{array}$$

We define $g_1 = p_1^*(\varphi_1)$ (resp. $g_2 = p_2^*(\varphi_2)$) on $\widetilde{W} - E_Y$ (resp. $\widetilde{W} - E_Z$).

2.2.5. We want to show that $g_1 \wedge \bar{\partial} g_2$ is L^1 on $\widetilde{W}$ and compute the current $\partial h_*[g_1 \wedge \bar{\partial} g_2]$ on X.

If we write $\widetilde{W}$ as a union $\bigcup_\alpha \Omega_\alpha$ of open sets Ω_α isomorphic to the polydisc

$$\Delta^n = \{ \underline{z} = (z_1, \ldots, z_n) \in \mathbf{C}^n \mid \sum_{i=1}^n \mid z_i \mid^2 < 1 \},$$

we can choose a partition of unity $\sum_\alpha \lambda_\alpha = 1$ subordinate to $\{ \Omega_\alpha \}$. Given any smooth form η with compact support on X, we must have

$$\partial h_*[g_1 \wedge \bar{\partial} g_1] (\eta) = \sum_\alpha \partial[g_1 \wedge \bar{\partial} g_1] (\lambda_\alpha h^*(\eta)).$$

So, for our computations, we may replace $\widetilde{W}$ by Δ^n, denote by E_i the divisor of equation $z_i = 0$ in Δ^n, and assume that

$$E_Y = \bigcup_{1 \leqslant i \leqslant k_2} E_i \quad \text{and} \quad E_Z = \bigcup_{k_1 \leqslant i \leqslant k} E_i.$$

Finally we may assume that

$$g_1 = \sum_{i=1}^{k_2} \alpha_i \log \mid z_i \mid^2 + \beta \tag{2.2.5.1}$$

and
$$g_2 = \sum_{i=k_1}^{k} \alpha_i' \log \mid z_i \mid^2 + \beta',$$

where α_i, β, α_i', β' are smooth forms on Δ^n, α_i and α_i' being ∂ and $\bar{\partial}$-closed. Now the most divergent terms in $g_1 \wedge \bar{\partial} g_2$ when we apply (2.2.5.1) are smooth multiples of

$$\log \mid z_i \mid d\bar{z}_i, \, k_1 \leqslant i \leqslant k_2.$$

These are L^1 on Δ^n, therefore $g_1 \wedge \bar{\partial} g_2$ is L^1 on $\widetilde{W}$.

To compute the derivative of $[g_1 \wedge \bar{\partial} g_2]$, we let U_ε^i, for any small $\varepsilon > 0$, be the set of $\underline{z} \in \Delta^n$, such that $\mid z_i \mid \leqslant \varepsilon$. Let $U_\varepsilon = \bigcup_{i=1}^k U_\varepsilon^i$, $W_\varepsilon = \partial U_\varepsilon$ be its boundary, and W_ε^i be the set of $\underline{z} \in W_\varepsilon$ such that $\mid z_i \mid = \varepsilon$. Let λ be a compactly supported function on Δ^n and η a smooth form with compact support on X. From Stokes' formula for forms we get

(2.2.5.2) $$\partial[g_1 \wedge \bar{\partial} g_2]\,(\lambda h^*(\eta)) = \lim_{\varepsilon \to 0} - \int_{\Delta^n - U_\varepsilon} (g_1 \wedge \bar{\partial} g_2) \wedge \partial(\lambda h^*(\eta))$$
$$= \lim_{\varepsilon \to 0} \left[\int_{\Delta^n - U_\varepsilon} \partial g_1 \wedge \bar{\partial} g_2 \wedge \lambda h^*(\eta) + \int_{\Delta^n - U_\varepsilon} g_1 \wedge \partial\bar{\partial}(g_2) \wedge \lambda h^*(\eta) + \sum_{i=1}^{k} \int_{W_\varepsilon^i} g_1 \wedge \bar{\partial} g_2 \wedge \lambda h^*(\eta) \right].$$

We shall see that each term in this sum has a limit when ε goes to zero, and we shall compute it.

2.2.6. First let $k_1 \leqslant i \leqslant k_2$. We claim that

(2.2.6.1) $$\lim_{\varepsilon \to 0} \int_{W_\varepsilon^i} g_1 \wedge \bar{\partial} g_2 \wedge \lambda h^*(\eta) = 0.$$

By hypothesis $h(E_i)$ is contained in $Y \cap Z \cap W$. Therefore, by the construction in 2.2.4, the map h admits a factorization

$$\widetilde{W} \to X'' \to X' \to X$$

where $X'' \to X'$ is a blow up with smooth center of codimension at least $p + q + r$ containing the image of E_i, and $X' \to X$ is birational. The total degree of the direct image of $g_1 \wedge \bar{\partial} g_2$ on X'' is $2(p + q) - 3$, therefore each integral in (2.2.6.1) vanishes unless η has total degree $2\dim(X) - 2(p + q) + 2$. Consequently, by Lemma 2.2.3, $h^*(\eta)$ restricted to W_ε^i is $O(\varepsilon^2)$. On the other hand, the most divergent term in $g_1 \wedge \bar{\partial} g_2$ near W_ε^i is a smooth multiple of $\log |z_i| = d\bar{z}_i/\bar{z}_i$, i.e. $O(\log \varepsilon)$. This proves (2.2.6.1).

Furthermore, when $k_1 \leqslant i \leqslant k_2$, the form $\partial g_1 \wedge \bar{\partial} g_2 \wedge \lambda h^*(\eta)$ is integrable on U_ε^i. Indeed, when applying (2.2.5.1), the only summand in $\partial g_1 \wedge \bar{\partial} g_2$ which is not L^1 is a smooth multiple of $dz_i\, d\bar{z}_i/|z_i|^2$. Because of Lemma 2.2.3, its product with $\lambda h^*(\eta)$ is also integrable. It follows that the limit

$$\lim_{\varepsilon \to 0} \int_{\Delta^n - U_\varepsilon} \partial g_1 \wedge \bar{\partial} g_2 \wedge \lambda h^*(\eta)$$

exists, and hence so does

$$\int_{\widetilde{W}} \partial g_1 \wedge \bar{\partial} g_2 \wedge \lambda h^*(\eta).$$

With some abuse of notation, we may denote it

$$h_*[\partial g_1 \wedge \bar{\partial} g_2]\,(\eta).$$

2.2.7. Now assume that $1 \leqslant i < k_1$. We claim again that

(2.2.7.1) $$\lim_{\varepsilon \to 0} \int_{W_\varepsilon^i} g_1 \wedge \bar{\partial} g_2 \wedge \lambda h^*(\eta) = 0.$$

This is clear, because the most divergent term of $g_1 \wedge \bar{\partial} g_2$ on W^i_ε is a smooth multiple of $\log |z_i|$.

Finally, when $k_2 < i \leqslant k$, there is only one term contributing to

$$\lim_{\varepsilon \to 0} \int_{W^i_\varepsilon} g_1 \wedge \bar{\partial} g_2 \wedge \lambda h^*(\eta),$$

namely, using (2.2.5.1),

$$\lim_{\varepsilon \to 0} \int_{W^i_\varepsilon} g_1 \wedge \alpha'_i \wedge (d\bar{z}_i/\bar{z}_i) \wedge \lambda h^*(\eta).$$

Notice that g_1 is L^1 on W^i_ε when ε is small enough, and on E_i. The limit above is equal to

$$-\, 2\pi i \int_{E_i} g_1 \wedge \alpha'_i \wedge \lambda h^*(\eta).$$

Let a' be the closed form on the smooth part of $h^{-1}(Z) = E_Z$ whose restriction to E_i, for $k_2 < i \leqslant k$, is equal to the restriction of α'_i (see Lemma 1.3.4). We conclude from (2.2.5.2), (2.2.6.1) and (2.2.7.1) that

(2.2.7.2)
$$\partial[g_1 \wedge \bar{\partial} g_2]\,(h^*(\eta)) = \int_{\widetilde{W}} \partial g_1 \wedge \bar{\partial} g_2 \wedge h^*(\eta) + \int_{\widetilde{W}} g_1 \wedge \partial\bar{\partial}(g_2) \wedge h^*(\eta) - 2\pi i \int_{h^{-1}(Z)} g_1 \wedge a' \wedge h^*(\eta).$$

To compute the third integral in (2.2.7.2) we may restrict to $h^{-1}(Z) - h^{-1}(Y)$. This subvariety of $\widetilde{W}$ is isomorphic to its image in the fiber product $(W_1 - D_1) \times_{W''} W_2$. By Lemma 1.3.4,

$$dd^c([\varphi_2]) = \sum_j \varepsilon_{j*}[a_j] + [b],$$

where the sum runs over components of D_2, a_j is a ∂ and $\bar{\partial}$-closed form on the j-th component, and b is smooth. Denote by a the smooth form on the smooth locus of D_2 equal to a_j on the j-th component. Then

(2.2.7.3)
$$\begin{aligned}\int_{h^{-1}(Z)} g_1 \wedge a' \wedge h^*(\eta) &= \int_{D_2 - (\psi\pi_2)^{-1}(Y)} p_2^*(g_1) \wedge a \wedge (\psi\pi_2)^*(\eta)\\ &= \int_{D_2 - (\psi\pi_2)^{-1}(Y)} \pi_2^*(\pi_{1*}(\varphi_1)) \wedge a \wedge (\psi\pi_2)^*(\eta)\\ &= \int_{D_2 - (\psi\pi_2)^{-1}(Y)} a \wedge (\psi\pi_2)^*(g_Y \wedge \eta).\end{aligned}$$

As in 2.1.5, let $\{ g^\delta_Y \}$, $\delta > 0$, be a sequence of smooth forms on X such that $\lim_{\delta \to 0} [g^\delta_Y] = [g_Y]$.

For any δ the integral

$$\int_{D_2} a \wedge (\psi\pi_2)^*(g^\delta_Y \wedge \eta) = (\psi\pi_2)_*(a \wedge \delta_{D_2})\,(g^\delta_Y \wedge \eta)$$

was computed in the proof of Theorem 2.1.3 (when we studied $\sum_i r_*\, \varepsilon_{i*}[a_i]$). Since (by (2.1.4)) $\psi^*(g_Z) = \pi_{2*}(\varphi_2)$ is a Green form of log type for the cycle $\psi^*(Z)$, we find

$$(\psi\pi_2)_*(a \wedge \delta_{D_2}) = -\, \psi_*[\delta_{\psi^*[Z]}] = -\,\sigma - \tau.$$

Therefore

$$\text{(2.2.7.4)} \qquad \int_{D_2} a \wedge (\psi\pi_2)^* (g_Y \wedge \eta) = \lim_{\delta \to 0} \int_{D_2} a \wedge (\psi\pi_2)^* (g_Y^\delta \wedge \eta)$$
$$= - \lim_{\delta \to 0} [(\sigma + \tau) \wedge g_Y^\delta] (\eta) = - ((\sigma + \tau) \wedge [g_Y]) (\eta).$$

Here we used definition (2.1.3.2) for $\sigma \wedge [g_Y]$ and the fact that g_Y is smooth on the support of τ, since $Y \cap Z \cap W$ has codimension $p + q + r$.

2.2.8. The integral of forms $\int_{\widetilde{W}} g_1 \wedge \partial\bar{\partial}(g_Z) \wedge h^*(\eta)$ may be computed on $(W_1 - D_1) \times_{W''} (W_2 - D_2)$. It is equal to

$$\int_{W'' - \psi^{-1}(Y \cup Z)} \psi^*(g_Y) \wedge \partial\bar{\partial}(\psi^*(g_Z)) \wedge \psi^*(\eta) = - 2\pi i([g_Y] \wedge \delta_W \wedge \omega_Z) (\eta)$$

by the definition of $[g_Y] \wedge \delta_W$. Therefore, using (2.2.7.3) and (2.2.7.4), we may rewrite (2.2.7.2) as follows:

$$\partial[g_1 \wedge \bar{\partial}(g_2)] \, (h^*(\eta))$$
$$= \int_{\widetilde{W}} \partial g_1 \wedge \bar{\partial} g_2 \wedge h^*(\eta) + 2\pi i(- [g_Y] \wedge \delta_W \wedge \omega_Z + [g_Y] \wedge (\sigma + \tau)) (\eta).$$

Assume now that we interchange Y and Z, and ∂ and $\bar{\partial}$ in this formula. Since Y meets W properly, we get

$$\bar{\partial}[g_2 \wedge \partial(g_1)] \, (h^*(\eta))$$
$$= \int_{\widetilde{W}} \bar{\partial} g_2 \wedge \partial g_1 \wedge h^*(\eta) + 2\pi i(- [g_Z] \wedge \delta_W \wedge \omega_Y + g_Z \wedge \delta_{[Y].[W]}) (\eta).$$

Adding up these two equations (and recalling that $dd^c = i\partial\bar{\partial}/2\pi$) we obtain Theorem 2.2.2.

2.2.9. *Corollary. — Let* X *be a smooth quasi-projective variety over* **C**, Y *and* Z *irreducible subvarieties of* X *which intersect properly (i.e. such that the codimension of their intersection is the sum of their codimensions). Then if* g_Y *and* g_Z *are Green forms with logarithmic growth for* Y *and* Z,

$$g_Y * g_Z = g_Z * g_Y.$$

Proof. — Take W = X in the thoerem.

2.2.10. *Corollary. — If* X, Y, Z *are as above, and* g_Y, g'_Y *are two Green forms with logarithmic growth for* Y, *then*

$$g_Y * g_Z - g'_Y * g_Z = (g_Y - g'_Y) \wedge \omega_Z.$$

Proof. — We know that, as in 2.1.3 (ii),

$$g_Z * g_Y - g_Z * g'_Y = g_Z \wedge [Y] + \omega_Z \wedge g_Y - g_Z \wedge [Y] - \omega_Z \wedge g'_Y$$
$$= \omega_Z \wedge (g_Y - g'_Y).$$

2.2.11. *Corollary. — Suppose that* X *is a smooth quasi-projective variety, that* Y *and* Z *are cycles on* X *which intersect properly and that* g'_Y *(respectively* g'_Z*) and* g''_Y *(respectively* g''_Z*)*

are Green forms with logarithmic growth for Y *(resp.* Z*). If* $[g'_Y] = [g''_Y]$ *and* $[g'_Z] = [g''_Z]$ *in* $\widetilde{\mathscr{D}}(X)$*, then* $g'_Y * g'_Z = g''_Y * g''_Z$ *in* $\widetilde{\mathscr{D}}(X)$.

Proof. — We need only show that $g'_Y * g'_Z = g'_Y * g''_Z$ in $\widetilde{\mathscr{D}}(X)$, since $g'_Y * g''_Z = g''_Y * g''_Z$ will then follow by Corollary 2.2.9. By Remark 2.1.3 (ii)

$$g'_Y * g'_Z - g'_Y * g''_Z = \omega_Y \wedge (g'_Z - g''_Z).$$

By assumption, $g'_Z - g''_Z = \partial u + \bar\partial v$; since ω_Y is closed,

$$\omega_Y \wedge (g'_Z - g''_Z) = \partial(\omega_Y u) + \bar\partial(\omega_Y v),$$

which represents 0 in $\widetilde{\mathscr{D}}(X)$, q.e.d.

Definition. — Suppose that X is a quasi-projective variety, smooth over **C**, and that Y and Z are cycles on X which intersect properly. If g_Y and g_Z are Green currents for Y and Z, then by theorem 1.3.4 we can represent g_Y and g_Z by Green forms with logarithmic growth, $\tilde g_Y$ and $\tilde g_Z$. We define the " star product ", $g_Y * g_Z$ as the class, in $\widetilde{\mathscr{D}}(X)$, of $\tilde g_Y * \tilde g_Z$; by the corollary, this class is independent of the representatives of g_Y and g_Z chosen. One can define in the same way $g_Y \wedge \delta_Z$.

2.2.12. Let X be a smooth projective variety over the complex numbers, and Y, Z, W three irreducible closed subvarieties intersecting properly. By this we mean that, if p, q, r are the codimensions of Y, Z, W respectively, $\mathrm{codim}_X(Y \cap Z) = p + q$, $\mathrm{codim}_X(Y \cap W) = p + r$, $\mathrm{codim}_X(Z \cap W) = q + r$ and $\mathrm{codim}_X(Y \cap Z \cap W) = p + q + r$. We assume that $p > 0$ and $r > 0$. Let g_Y be an arbitrary Green current for Y and g_W a Green form of log type for W. Then, as in 2.1.5, we may write $[g_W] = \lim_{\varepsilon \to 0} [g^\varepsilon_W]$, where g^ε_W is a smooth form on X, equal to g_W outside an ε-neighborhood of W; then $\omega^\varepsilon_W = \omega_W - dd^c g^\varepsilon_W$ is such that $\lim_{\varepsilon \to 0} [\omega^\varepsilon_W] = \delta_W$. Let $[Z].[W]$ be the intersection cycle of Z and W.

Corollary. — *The following equality holds in* $\widetilde{\mathscr{D}}(X)$:

$$\lim_{\varepsilon \to 0} g_Y \wedge \delta_Z \wedge \omega^\varepsilon_W = g_Y \wedge \delta_{[Z].[W]}.$$

In particular, when $Z = X$,

$$\lim_{\varepsilon \to 0} g_Y \wedge \omega^\varepsilon_W = g_Y \wedge \delta_W.$$

Proof. — Since X is projective the images of ∂ and $\bar\partial$ are closed (since ∂ and $\bar\partial$ are continuous and their cohomology groups are finite dimensional). Therefore $\widetilde{\mathscr{D}}(X)$ is separated and, in this group, we can compute

$$\begin{aligned} \lim_{\varepsilon \to 0} g_Y \wedge \delta_Z \wedge \omega^\varepsilon_W &= \lim_{\varepsilon \to 0} (g_Y \wedge \delta_Z \wedge \omega_W - g_Y \wedge \delta_Z \wedge dd^c(g^\varepsilon_W)) \\ &= \lim_{\varepsilon \to 0} (g_Y \wedge \delta_Z \wedge \omega_W - dd^c(g_Y \wedge \delta_Z) \wedge g^\varepsilon_W) \\ &= g_Y \wedge \delta_Z \wedge \omega_W + \delta_{[Y].[Z]} \wedge g_W - \omega_Y \wedge \delta_Z \wedge g_W. \end{aligned}$$

This is equal to $g_Y \wedge \delta_{[Y].[W]}$ by Theorem 2.2.2 (with the roles of Z and W switched).

2.2.13. *Corollary.* — *Let* X *be a smooth projective irreducible complex variety,* Y *and* W *two closed irreducible subvarieties intersecting properly. Let* $\widetilde{g}_Y \in \mathscr{D}(X)$ *be any representative of the Green current* $g_Y \in \widetilde{\mathscr{D}}(X)$ *such that* $\lim_{\varepsilon \to 0} \widetilde{g}_Y \wedge \omega_W^\varepsilon$ *exists, where* ω_W^ε *is defined as in* 2.1.5 *and* 2.2.12. *Then the class of* $\lim_{\varepsilon \to 0} \widetilde{g}_Y \wedge \omega_W^\varepsilon$ *in* $\widetilde{\mathscr{D}}(X)$ *is equal to* $g_Y \wedge \delta_Z$.

2.2.14. *Theorem.* — *Let* X *be a smooth projective variety of* **C**, *and let* Y, Z, W *be three cycles intersecting properly on* X. *Then we have an equality in* $\widetilde{\mathscr{D}}(X)$, *for any choice of Green currents for* Y, Z *and* W:

$$(g_Y * g_Z) * g_W = g_Y * (g_Z * g_W).$$

Proof. — Without loss of generality we may assume that Y, Z, W, X are irreducible of positive codimension. The right hand side of the equation is equal to

$$g_Y \wedge \delta_{[Z].[W]} + \omega_Y \wedge g_Z \wedge \delta_W + \omega_Y \wedge \omega_Z \wedge g_W.$$

Choose a representative of g_W by a Green form of log type along W, and a sequence g_W^ε of smooth forms converging to g_W as in 2.1.5, with $dd^c g_W^\varepsilon = \omega_W - \omega_W^\varepsilon$. Then, according to Corollary 2.2.13, the left-hand side of the equation we look for is equal to

$$\lim_{\varepsilon \to 0} (g_Y * g_Z) \wedge \omega_W^\varepsilon + \omega_Y \wedge \omega_Z \wedge g_W,$$

if the limit exists. But $g_Y * g_Z = \omega_Y \wedge g_Z + g_Y \wedge \delta_Z$ and, from Corollary 2.2.12, we know that

$$\lim_{\varepsilon \to 0} \omega_Y \wedge g_Z \wedge \omega_W^\varepsilon = \omega_Y \wedge g_Z \wedge \delta_W$$

and

$$\lim_{\varepsilon \to 0} g_Y \wedge \delta_Z \wedge \omega_W^\varepsilon = g_Y \wedge \delta_{[Z].[W]}.$$

This proves the Theorem.

2.2.15. *Lemma.* — *Let* X *be a smooth quasi-projective variety over* **C**, *let* $W \subset X \times \mathbf{P}^1$ *be a closed subvariety of codimension* p *which is flat over* $\mathbf{P}^1$, *and let* Z *be a codimension* q *closed subvariety of* X. *Suppose that*

$$(Z \times \mathbf{P}^1) \cap W = S \cup T,$$

with S *of codimension* $p + q$, T *of codimension* $p + q - e$ $(e > 0)$ *and*

$$T \subset X \times \{a_1, \ldots, a_k\} \subset X \times (\mathbf{P}^1 - \{0, \infty\}) \simeq X \times \mathbf{C}^*.$$

As in (4.1.2), $[Z \times \mathbf{P}^1].[W] = \sigma + \tau$ *with* σ *a uniquely determined codimension* $p + q$ *cycle on* S, *and* τ *a rational equivalence class on* T; *note that we can write* $\tau = \sum_{i=1}^k \tau_i$ *with* τ_i *supported in* $T \cap (X \times \{a_i\})$. *Then if* t *is the rational function on* $\mathbf{P}^1$ *identifying* $\mathbf{P}^1 - \{\infty\}$ *with* **C**, *so* $\operatorname{div}(t) = [0] - [\infty]$, *we have an equation of currents*

$$[\log |t|^2|_W] \wedge \delta_{Z \times \mathbf{P}^1} = \log |t|^2 \delta_\sigma + \sum_{i=1}^k \log |a_i|^2 \theta_i,$$

with θ_i *a current on* $X \times \{a_i\}$ *representing* τ_i *in cohomology. (Note that* $\log |t|^2|_W$ *is a Green current for* $[W \times \{\infty\}] - [W \times \{0\}]$, *so the left-hand side of the equation is defined by* 2.2.11.*)*

Proof. — First observe that the cycles $\operatorname{div}(t).[W]$ and $Z \times \mathbf{P}^1$ meet properly, hence if $g_{Z \times \mathbf{P}^1}$ is any Green form for $Z \times \mathbf{P}^1$, by 2.2.9

$$[\log | t |^2 |_W] \wedge \delta_{Z \times \mathbf{P}^1} = \delta_{\operatorname{div}(t).[W]} \wedge g_{Z \times \mathbf{P}^1} + \log | t |^2 \wedge \delta_W \wedge \omega_{Z \times \mathbf{P}^1}.$$

If we apply Theorem 2.2.2, with $Y = \operatorname{div}(t)$, $g_Y = \log | t |^2$ and Z replaced by $Z \times \mathbf{P}^1$, we find that

$$[\log | t |^2 |_W] \wedge \delta_{Z \times \mathbf{P}^1} = \log | t |^2 \delta_\sigma + \log | t |^2 \theta$$

with θ a closed current represented by an L^1 form on a subset of $\bigcup_{i=1}^{k} X \times \{ a_1, \ldots, a_k \}$. We can write $\theta = \sum_{i=1}^{k} \theta_i$ with each θ_i a current supported on $X \times \{ a_i \}$. Because $\log | t |^2$ is a continuous function in an open neighbourhood of $X \times \{ a_i \}$ for each $i = 1, \ldots, k$, $\log | t |^2 \theta_i = \log | a_i |^2 \theta_i$. Therefore

$$[\log | t |^2 |_W] \wedge \delta_{Z \times \mathbf{P}^1} = \log | t |^2 \delta_\sigma + \sum_{i=1}^{k} \log | a_i |^2 \theta_i$$

as desired.

3. Arithmetic Chow groups

3.1. Arithmetic varieties

3.1.1. *Definition.* — An *arithmetic ring* is a triple (A, Σ, F_∞) consisting of an excellent regular Noetherian integral domain A, a finite nonempty set Σ of monomorphisms $\sigma : A \to \mathbf{C}$, and a conjugate-linear involution of $\mathbf{C}$-algebras, $F_\infty : \mathbf{C}^\Sigma \to \mathbf{C}^\Sigma$, such that the diagram

$$\begin{array}{ccc} A & \xrightarrow{\delta} & \mathbf{C}^\Sigma \\ \| & & \downarrow F_\infty \\ A & \xrightarrow{\delta} & \mathbf{C}^\Sigma \end{array}$$

commutes. Here δ is the natural map to the product induced by the family of maps $\{ \sigma : A \to \mathbf{C} \}_{\sigma \in \Sigma}$. Note that we have an induced commutative diagram:

$$\begin{array}{ccc} \mathbf{C} \otimes_{\mathbf{Z}} A & \xrightarrow{\delta'} & \mathbf{C}^\Sigma \\ {\scriptstyle c \otimes \mathrm{Id}} \downarrow & & \downarrow F_\infty \\ \mathbf{C} \otimes_{\mathbf{Z}} A & \xrightarrow{\delta'} & \mathbf{C}^\Sigma \end{array}$$

where $c(z) = \bar{z}$ and $\delta' = \{ \mathrm{Id} \otimes \sigma \}_{\sigma \in \Sigma}$. We shall write $\mathbf{C}^\Sigma = \prod_{\sigma \in \Sigma} \mathbf{C}_\sigma$, so that $\sigma : A \to \mathbf{C}_\sigma$.

3.1.2. *Examples.*

1) If the field of fractions of A is a number field F, let $\Sigma = \operatorname{Hom}(A, \mathbf{C})$ be the set of all embeddings of A into $\mathbf{C}$ and let F_∞ be the usual Frobenius on $\mathbf{C}^\Sigma \simeq \mathbf{C} \otimes_{\mathbf{Q}} F$ induced

by the complex conjugation. In particular A could be F itself or any localization of the ring of integers $\mathcal{O}_F$ in F. Unless we state otherwise, we shall always use this arithmetic ring structure for subrings of number fields.

2) $A = \mathbf{R}$ or any subring of $\mathbf{R}$, Σ consists of the obvious embedding $A \to \mathbf{C}$ and F_∞ is the complex conjugation.

3) Let $A = \mathbf{C}$ itself. Then there is an isomorphism $\mathbf{C} \otimes_{\mathbf{R}} A \simeq \mathbf{C} \times \mathbf{C}$ sending $z \otimes w$ to $(zw, z\bar{w})$. The composition of this map with the natural map $A \to \mathbf{C} \otimes_{\mathbf{R}} A$ sending a to $1 \otimes a$ is the sesquidiagonal map $\delta : a \mapsto (a, \bar{a})$. It will be convenient to view $\mathbf{C}$ as an arithmetic ring via this map, i.e. we shall write $\mathbf{C}$ for the triple $\{ \mathbf{C}, \{ \mathrm{Id}, c \}, F_\infty \}$ where $c : \mathbf{C} \to \mathbf{C}$ is the complex conjugation and $F_\infty(a, b) = (\bar{b}, \bar{a})$.

3.1.3. A *homomorphism* of arithmetic rings $f: (A, \Sigma, f_\infty) \to (A', \Sigma', F'_\infty)$ is a pair $f_1 : A \to A'$ and $f_2 : \mathbf{C}^\Sigma \to \mathbf{C}^{\Sigma'}$ with f_2 a homomorphism of $\mathbf{C}$ algebras, such that $f_2.\delta = \delta'.f_1$ and $F'_\infty.f_2 = f_1.F_\infty$. Since A and A′ are subrings of $\mathbf{C}$, f is necessarily injective.

Note that there is a natural homomorphism of arithmetic rings $\mathcal{O}_F \to \mathcal{O}_E$ corresponding to each extension E/F of number fields.

Also observe that $\mathbf{Z}$, viewed as an arithmetic ring via either of the two equivalent structures of examples 1) and 2) above, is an initial object in the category of arithmetic rings.

3.2. Arithmetic varieties

3.2.1. *Definition.* — If (A, Σ, F_∞) is an arithmetic ring, an *arithmetic variety over* A is a scheme which is flat and of finite type over $S = \mathrm{Spec}(A)$, $\pi : X \to S$. If F is the fraction field of A, let us write X_F for the generic fibre of X; we shall, in addition, always suppose that X_F is smooth. If $s \in S$, then we write $X(s) = \pi^{-1}(s)$ for the fibre over s while, if $\sigma \in \Sigma$, we write $X_\sigma = X \otimes_\sigma \mathbf{C}$ and $X_\Sigma = \coprod_{\sigma \in \Sigma} X_\sigma = X \otimes_A \mathbf{C}^\Sigma$. Finally, we write $X_\infty = X_\Sigma(\mathbf{C})$ for the analytic space associated with the scheme X_Σ.

The conjugate-linear automorphism F_∞ of $\mathbf{C}^\Sigma$ induces a continuous involution of X_∞. Since X_F is a smooth variety over F, X_∞ is a complex manifold. We shall write $A^{p,q}(X)$ for the space of (p, q) forms on X_∞, and similarly we shall write $\mathcal{D}^{p,q}(X)$ in place of $\mathcal{D}^{p,q}(X_\infty)$. Observe that F_∞ acts on both $A^{*,*}(X)$ and $\mathcal{D}^{*,*}(X)$. We define $A^{p,p}(X_{\mathbf{R}})$ (respectively $\mathcal{D}^{p,p}(X_{\mathbf{R}})$) to be the subspace of $A^{p,p}(X)$ (resp. $\mathcal{D}^{p,p}(X)$) consisting of real forms (resp. currents) satisfying $F_\infty^* \alpha = (-1)^p \alpha$. Note that dd^c maps $A^{p,p}(X_{\mathbf{R}})$ to $A^{p+1,p+1}(X_{\mathbf{R}})$ and $\mathcal{D}^{p,p}(X_{\mathbf{R}})$ to $\mathcal{D}^{p+1,p+1}(X_{\mathbf{R}})$. Similarly we define:

$$\widetilde{A}^{p,p}(X_{\mathbf{R}}) = A^{p,p}(X_{\mathbf{R}})/(\mathrm{Im}\,\partial + \mathrm{Im}\,\bar{\partial}),$$

$$\widetilde{A}(X_{\mathbf{R}}) = \bigoplus_{p \geqslant 0} \widetilde{A}^{p,p}(X_{\mathbf{R}}),$$

and, if X_F is projective,

$$H^{p,p}(X_{\mathbf{R}}) = \{ \alpha \in H^{p,p}(X, \mathbf{R}) \mid F_\infty^* \alpha = (-1)^p \alpha \}.$$

Note that when $(A, \Sigma, F_\infty) = \mathbf{C}$, as in example 3.1.2.3) above, X is a complex variety and $A^{p,p}(X_{\mathbf{R}})$ is just the space of real (p, p)-forms on the complex manifold $X(\mathbf{C})$ of complex points of X.

3.3. Chow groups of arithmetic varieties

3.3.1. Let X be a Noetherian scheme; following [EGA IV] $Z^p(X)$, the group of cycles of codimension p on X, is the free abelian group on the set of codimension p integral subschemes of X. An equivalent definition is:

$$Z^p(X) = \bigoplus_{x \in X^{(p)}} \mathbf{Z},$$

where $X^{(p)} = \{ x \in X \mid \mathcal{O}_{X,x} \text{ has Krull dimension } p \}$. If T is a codimension p integral subscheme, we write [T] for the associated cycle. If $Y \subset X$ is an integral subscheme of codimension $(p - 1)$, with generic point $y \in X^{(p-1)}$, then for any $f \in k(y)^*$ (note that $k(y)$ is the function field of y) we define a codimension p cycle

$$\operatorname{div}(f) = \sum_V \operatorname{ord}_V(f)\, [V].$$

Here the sum is over all integral subschemes V of Y of codimension p in X, and the definition of the order function $\operatorname{ord}_V(\ )$ may be found in [Fu] A.3. Note that in contrast with [Fu] Chapter 1, we have graded our cycles by codimension rather than dimension. Finally we set

$$CH^p(X) = Z^p(X)/\mathrm{Rat}^p(X),$$

where $\mathrm{Rat}^p(X) \subset Z^p(X)$ is the subgroup generated by all cycles of the form $\operatorname{div}(f)$.

3.3.2. If X is an arithmetic variety over $A = (A, \Sigma, F_\infty)$ with smooth, quasi-projective generic fibre and Y is a codimension p integral subscheme of X, Y_∞ is an analytic subspace of X_∞ (which will be empty if $Y \cap X_F = \emptyset$, where F is the fraction field of A). Since Y is a subscheme of X, Y_∞ is invariant under F_∞, hence integration over Y_∞ defines a current in $\mathcal{D}^{p,p}(X_{\mathbf{R}})$, which we denote by δ_Y. Extending by linearity, we obtain a map

$$Z^p(X) \to \mathcal{D}^{p,p}(X_{\mathbf{R}}).$$

3.3.3. With X and (A, Σ, F_∞) as above, let $\hat{Z}^p(X)$ be the subgroup of

$$Z^p(X) \oplus \tilde{\mathcal{D}}^{p-1,p-1}(X_{\mathbf{R}})$$

consisting of pairs $(Z = \Sigma n_i [Z_i], g)$ such that g is a Green current for Z, i.e.

$$dd^c g + \delta_Z = \omega(Z, g)$$

where $\omega_Z = \omega(Z, g) \in A^{p,p}(X_{\mathbf{R}})$.

If $Y \subset X$ is a reduced irreducible subscheme of codimension $p-1$, let $\pi : \widetilde{Y}_\infty \to Y_\infty$ be a resolution of singularities of Y_∞ with π proper. For $f \in k(Y)^*$, f restricts to a rational function $\widetilde{f}$ on $\widetilde{Y}_\infty$. The function $\log|\widetilde{f}|^2$ is real valued and L^1 on $\widetilde{Y}_\infty$; it therefore defines a current in $\mathscr{D}^{0,0}(\widetilde{Y})$. If $\widetilde{i} : \widetilde{Y}_\infty \to X_\infty$ is the natural map, then $\widetilde{i}_*[\log|\widetilde{f}|^2] \in \mathscr{D}^{p-1,p-1}(X)$. This current is independent of the choice of $\widetilde{Y}$ (see 1.4.4); we will therefore write it as $i_*[\log|f|^2]$, for $i : Y \to X$ the inclusion. Since both Y and f are invariant under F_∞, $i_*[\log|f|^2] \in \mathscr{D}^{p-1,p-1}(X_{\mathbf{R}})$. Its class in $\widetilde{\mathscr{D}}^{p-1,p-1}(X_{\mathbf{R}})$ will also be denoted $i_* \log|f|^2$. By the Poincaré-Lelong lemma ([G-H], [Le]) applied to X_∞, $dd^c\, i_*[\log|f|^2] = \delta_{\operatorname{div}(f)}$, the current associated to the restriction to X_∞ of $\operatorname{div}(f)$, viewed as a codimension p cycle on X. In other words, $\widehat{\operatorname{div}}(f) = (\operatorname{div}(f), -i_*\log|f|^2)$ is an element of $\hat{Z}^p(X)$.

3.3.4. *Definition.* — Let X be as in 3.3.2, then we define $\widehat{CH}^p(X) = \hat{Z}^p(X)/\hat{R}^p(X)$ where $\hat{R}^p(X)$ is the subgroup generated by all pairs $\widehat{\operatorname{div}}(f) = (\operatorname{div}(f), -i_*\log|f|^2)$ for $f \in k(W)^*$, W a codimension $p-1$ integral subscheme, as above. We call the groups $\widehat{CH}^p(X)$ for $p \geqslant 0$, the *arithmetic Chow groups* of X.

We write

$$\widehat{CH}^*(X) = \bigoplus_{p\geqslant 0} \widehat{CH}^p(X)$$

and
$$\widetilde{A}(X_{\mathbf{R}}) = \bigoplus_{p\geqslant 0} \widetilde{A}^{p,p}(X_{\mathbf{R}}),$$

where
$$\widetilde{A}^{p,p}(X_{\mathbf{R}}) = A^{p,p}(X_{\mathbf{R}})/(\operatorname{Im}\partial + \operatorname{Im}\bar{\partial}).$$

If $\eta \in \widetilde{A}(X_{\mathbf{R}})$, we shall also write η in place of $(0, \eta) \in \widehat{CH}^*(X)$.

We can define several maps involving $\widehat{CH}^*(X)$.

(i) $\zeta : \widehat{CH}^p(X) \to CH^p(X)$, $(Z, g) \mapsto Z$.

Since $\zeta(\widehat{\operatorname{div}}(f)) = \operatorname{div}(f)$, this map preserves rational equivalence, so is well defined.

(ii) $a : \widetilde{A}^{p-1,p-1}(X_{\mathbf{R}}) \to \widehat{CH}^p(X)$, $\alpha \mapsto (0, \alpha)$.

(iii) $\omega : \widehat{CH}^p(X) \to A^{p,p}(X_{\mathbf{R}})$, $(Z, g) \mapsto \omega(Z, g) = \delta_Z + dd^c g$.

Note that ω is well defined since $dd^c\, i_*[\log|f|^2] = \delta_{\operatorname{div}(f)}$, so that $\omega(\widehat{\operatorname{div}}(f)) = 0$.

3.3.5. We need some notation in order to state the next theorem. We let

$\widehat{CH}^p(X)_0 = \operatorname{Ker}(\omega : \widehat{CH}^p(X) \to A^{p,p}(X_{\mathbf{R}}))$

$Z^{p,p}(X_{\mathbf{R}})$ = subspace of $A^{p,p}(X_{\mathbf{R}})$ consisting of closed forms

$CH^p(X)_0$ = subgroup of $CH^p(X)$ consisting of cycles homologically equivalent to zero in the generic fibre.

If X_F is projective,

$c : CH^p(X) \to H^{p,p}(X_{\mathbf{R}})$ is the cycle class map,

and $h : Z^{p,p}(X_{\mathbf{R}}) \to H^{p,p}(X_{\mathbf{R}})$ sends a closed form to its cohomology class.

Also, recall that we have groups, for any Noetherian scheme X,

$$\mathrm{CH}^{p,p-1}(\mathrm{X}) = \frac{\mathrm{Ker}\{ d^{p-1} : \bigoplus_{x \in \mathrm{X}^{(p-1)}} k(x)^* \to \bigoplus_{x \in \mathrm{X}^{(p)}} \mathbf{Z} \}}{\mathrm{Im}\{ d^{p-2} : \bigoplus_{x \in \mathrm{X}^{(p-2)}} \mathrm{K}_2\, k(x) \to \bigoplus_{x \in \mathrm{X}^{(p-1)}} k(x)^* \}}.$$

Here d^{p-1} and d^{p-2} are the differentials in the E_1 term of the spectral sequence of [Q] § 7, so that $\mathrm{CH}^{p,p-1}$ is the $\mathrm{E}_2^{p-1,-p}$ term of that spectral sequence (the notation $\mathrm{CH}^{p,p-1}$ is that of [Gi 1]). The differential d^{p-1} sends $f \in k(x)^*$ to $\mathrm{div}(f)$ and d^{p-2} is essentially the tame symbol: see [Q] § 7, [Gi 1], [Gr 1] and [Gr 2].

Theorem. — Let X *be an arithmetic variety over* $\mathrm{A} = (\mathrm{A}, \Sigma, \mathrm{F}_\infty)$. *Then there are exact sequences, where we assume that* X_F *is projective in* (ii) *and* (iii):

(i) $\mathrm{CH}^{p,p-1}(\mathrm{X}) \xrightarrow{\rho} \widetilde{\mathrm{A}}^{p-1,p-1}(\mathrm{X}_\mathbf{R}) \xrightarrow{a} \widehat{\mathrm{CH}}^p(\mathrm{X}) \xrightarrow{\zeta} \mathrm{CH}^p(\mathrm{X}) \longrightarrow 0;$

(ii) $\mathrm{CH}^{p,p-1}(\mathrm{X}) \xrightarrow{\rho} \mathrm{H}^{p-1,p-1}(\mathrm{X}_\mathbf{R}) \xrightarrow{a} \widehat{\mathrm{CH}}^p(\mathrm{X})$
$\xrightarrow{(\zeta,-\omega)} \mathrm{CH}^p(\mathrm{X}) \oplus \mathrm{Z}^{p,p}(\mathrm{X}_\mathbf{R}) \xrightarrow{c+h} \mathrm{H}^{p,p}(\mathrm{X}_\mathbf{R}) \longrightarrow 0;$

(iii) $\mathrm{CH}^{p,p-1}(\mathrm{X}) \xrightarrow{\rho} \mathrm{H}^{p-1,p-1}(\mathrm{X}_\mathbf{R}) \xrightarrow{a} \widehat{\mathrm{CH}}^p(\mathrm{X})_0 \longrightarrow \mathrm{CH}^p(\mathrm{X})_0 \longrightarrow 0.$

Proof. — (i) If Z is a codimension p algebraic cycle on X, Z_∞ admits a real valued Green current g by Theorem 1.3.5, and replacing g by $(g + (-1)^{p-1}\mathrm{F}_\infty^* g)/2$ if necessary, we may suppose that $g \in \widetilde{\mathscr{D}}^{p-1,p-1}(\mathrm{X}_\mathbf{R})$; hence ζ is surjective. Clearly $\mathrm{Im}(a) \subset \mathrm{Ker}\,\zeta$. If $[(\mathrm{Z}, g)] \in \mathrm{Ker}(\zeta)$, then $\mathrm{Z} = \Sigma_\alpha \mathrm{div}(f_\alpha)$ for $f_\alpha \in k(\mathrm{W}_\alpha)^*$ with $\{\mathrm{W}_\alpha\}$ a finite set of a codimension $p-1$ cycles. Hence

$$(\mathrm{Z}, g) = (0, g + \sum_\alpha i_{\alpha *}[\log |f_\alpha|^2]) + \sum_\alpha \widehat{\mathrm{div}}(f_\alpha).$$

Now $dd^c(g + \Sigma_\alpha i_{\alpha *} \log(|f_\alpha|^2)) = \omega(\mathrm{Z}, g) \in \mathrm{A}^{p,p}(\mathrm{X}_\mathbf{R})$, therefore, by theorem 1.2.2,

$$g + \Sigma_\alpha i_{\alpha *}[\log |f_\alpha|^2] \in \widetilde{\mathrm{A}}^{p-1,p-1}(\mathrm{X}_\mathbf{R}) \subset \widetilde{\mathscr{D}}^{p-1,p-1}(\mathrm{X}_\mathbf{R}),$$

and so $\mathrm{Ker}(\zeta) \subset \mathrm{Im}(a)$. A form $\beta \in \widetilde{\mathrm{A}}^{p-1,p-1}(\mathrm{X}_\mathbf{R})$ lies in $\mathrm{Ker}(a)$ if and only if

$$(0, \beta) = \sum_\alpha \widehat{\mathrm{div}}(f_\alpha),$$

i.e.
$$\sum_\alpha \mathrm{div}(f_\alpha) = 0,$$

and
$$\beta = -\sum_\alpha i_{\alpha *} \log |f_\alpha|^2 \quad \text{in } \widetilde{\mathscr{D}}^{p-1,p-1}(\mathrm{X}_\mathbf{R}).$$

The first of these equations says that

$$\bigoplus_\alpha \{f_\alpha\} \in \mathrm{Ker}\{ d^{p-1} : \bigoplus_{x \in \mathrm{X}^{(p-1)}} k(x)^* \to \bigoplus_{x \in \mathrm{X}^{(p)}} \mathbf{Z} \}.$$

It follows that $dd^c(\beta) = 0$, hence we may view β as an element of $\widetilde{\mathrm{A}}^{p-1,p-1}(\mathrm{X}_\mathbf{R})$. To finish the proof of (i) it suffices to prove that the map

$$\rho : \mathrm{Ker}\, d^{p-1} \to \widetilde{\mathrm{A}}^{p-1,p-1}(\mathrm{X}_\mathbf{R})$$
$$\bigoplus \{f_\alpha\} \mapsto -\sum_\alpha i_{\alpha *} \log |f_\alpha|^2$$

vanishes on $\operatorname{Im} d^{p-2}$. Since this is a question about currents on X_∞, we may suppose that $(A, \Sigma, F_\infty) = \mathbf{C}$. Let $Y \subset X$ be an integral closed subscheme of codimension $p-2$, and let $f, g \in k(Y)^*$; it suffices to show that $\rho(d^{p-2}(\{f, g\})) = 0$ for all such Y, f, g. Let $\tilde{\pi} : \tilde{Y} \to Y$ be a resolution of singularities of Y such that $D = \operatorname{div}(f) \cup \operatorname{div}(g)$ is a d.n.c. on $\tilde{Y}$. From the covariance of the Quillen spectral sequence [Gi 1] we have a commutative diagram, where $\pi : \tilde{Y} \to X$ is the composition of $\tilde{\pi}$ with the inclusion $Y \to X$:

$$\begin{array}{ccccc} K_2 k(\tilde{Y}) & \xrightarrow{d^0_{\tilde{Y}}} & \bigoplus_{y \in \tilde{Y}^{(1)}} k(y)^* & \xrightarrow{\rho_{\tilde{Y}}} & \tilde{\mathscr{D}}^{1,1}(\tilde{Y}) \\ \downarrow \pi_* & & \downarrow \pi_* & & \downarrow \pi_* \\ \bigoplus_{x \in X^{(p-2)}} K_2 k(x) & \xrightarrow{d^{p-2}_X} & \bigoplus_{x \in X^{(p-1)}} k(x)^* & \xrightarrow{\rho_X} & \tilde{\mathscr{D}}^{p-1,p-1}(X) \end{array}$$

It suffices therefore to show that the composition $\rho_{\tilde{Y}} . d^0_{\tilde{Y}} = 0$. By ([Gi 1], [Gr 2]) we know that the "y" component of $d^0_{\tilde{Y}}$ is the tame-symbol

$$t_v : K_2 k(\tilde{Y}) \to k(y)^*$$
$$t_v : \{f, g\} \quad \mapsto (-1)^{v(f)\, v(g)} \overline{\left\{ \frac{f^{v(g)}}{g^{v(f)}} \right\}}$$

associated to the valuation v of $k(\tilde{Y})$ corresponding to the prime divisor y.

Let $D = \operatorname{div}(f) \cup \operatorname{div}(g)$; by assumption $D = \bigcup_{i=1}^k D_i$ is a divisor with normal crossings on $\tilde{Y}$, with irreducible components $\{D_i\}$ for $i = 1, \ldots, k$. Consider the C^∞ forms

$$\alpha = \log |f|^2 \, \bar{\partial} \log |g|^2$$

and

$$\beta = \log |g|^2 \, \partial \log |f|^2$$

on $\tilde{Y} - D$. Since both α and β are $O(r^{-1} \log r)$ near D, they are L^1 on $\tilde{Y}$ and so define currents $[\alpha]$ and $[\beta]$. Since the form $\partial\alpha + \bar{\partial}\beta$ vanishes on $\tilde{Y} - D$, the current $\partial[\alpha] + \bar{\partial}[\beta]$ is supported on D. If we can show that

$$\frac{i}{2\pi} \{ \partial[\alpha] + \bar{\partial}[\beta] \} = -\tilde{\rho}(d^0_{\tilde{Y}}\{f, g\}),$$

$\tilde{\rho}$ being the map with values in $\mathscr{D}^{1,1}(\tilde{Y})$ defined by the same formula as the one defining ρ, we will be done, since $\partial[\alpha] + \bar{\partial}[\beta] \equiv 0$ in $\tilde{\mathscr{D}}^{1,1}(\tilde{Y})$. This equation may be checked locally on $\tilde{Y}$ (note that being $\equiv 0$ in $\tilde{\mathscr{D}}^{1,1}$ is not a local question), so we may suppose that $\tilde{Y} = \Delta^m = \{ \underline{z} \in \mathbf{C}^m \mid |\underline{z}| < 1 \}$. Furthermore, both sides of the equation are biadditive in (f, g), so it will be enough to consider two cases:

1) $f = z_1$, $g = z_1$, so $D = D_1 = \{ \underline{z} \in \Delta^m \mid z_1 = 0 \}$. If $\varphi \in A_c^{m-1, m-1}(\Delta^m)$, then (since $\partial\alpha + \bar{\partial}\beta = 0$)

$$(\partial[\alpha] + \bar{\partial}[\beta])(\varphi) = \lim_{\varepsilon \to 0} \int_{|z_1| = \varepsilon} (\alpha + \beta) \wedge \varphi.$$

But if we write $z_1 = re^{i\theta}$, $\alpha + \beta = (2/r) \log(r^2)\, dr$, and hence $(\alpha + \beta) \wedge \varphi \,|_{r=\varepsilon}$ vanishes, so $\partial[\alpha] + \bar{\partial}[\beta] = 0$. On the other hand, $d\{z_1, z_1\} = \{-1\}_{D_1}$, so $\rho(d\{z_1, z_1\}) = 0$.

2) $f = z_1$, $g = z_2$, so $D = D_1 \cup D_2$, or $f = z_1$ and g is a unit, or f is a unit and $g = z_2$, or f and g are units. Then $\operatorname{div}(f)$ and $\operatorname{div}(g)$ meet properly, so

$$\begin{aligned} -\rho(d\{f, g\}) &= \log |f|^2 \delta_{\operatorname{div}(g)} - \log |g|^2 \delta_{\operatorname{div}(f)} \\ &= \log |f|^2 * \log |g|^2 - \log |g|^2 * \log |f|^2. \end{aligned}$$

A direct computation shows that this is equal to $(i/2\pi)\,(\partial[\alpha] + \bar\partial[\beta])$ when $f = z_1$ and $g = z_2$; this is clearly true in the other cases.

This equality was also shown during the proof of Theorem 2.2.2. We could also deduce (i) from Theorem 3.5.4 below, in which for X_F projective we identify ρ with the Beilinson regulator map [Be 1].

Turning to part (ii) of the theorem, observe that, if X_F is projective, the image of ρ is contained in

$$\operatorname{Ker}(dd^c : \tilde{A}^{p-1,p-1}(X_{\mathbf{R}}) \to A^{p,p}(X_{\mathbf{R}})) \simeq H^{p-1,p-1}(X_{\mathbf{R}}),$$

and the image of ρ is equal to the kernel of a restricted to $H^{p-1,p-1}(X_{\mathbf{R}})$. Next observe that $(Z, g) \in \widehat{CH}^p(X)$ is in the kernel of $(\zeta, -\omega) : \widehat{CH}^p(X) \to CH^p(X) \oplus Z^{p,p}(X_{\mathbf{R}})$ if and only if $(Z, g) \in \operatorname{Ker}(\zeta) \cap \operatorname{Ker}(\omega)$ or, equivalently, $(Z, g) \equiv (0, \alpha)$ and $dd^c \alpha = 0$, i.e. $\alpha \in H^{p-1,p-1}(X_{\mathbf{R}})$. If $(Z, g) \in \widehat{CH}^p(X)$, then $dd^c g + \delta_Z = \omega(Z, g)$ hence δ_Z and $\omega(Z, g)$ are cohomologous as currents, therefore $c(Z) = h(\omega(Z, g))$. Conversely if $Z \in CH^p(X)$ and ω is a closed (p, p) form representing its cohomology class, then, by Theorem 1.3.5, there is a g such that $dd^c g + \delta_Z = \omega$. We have shown that $\operatorname{Ker}(c + h) = \operatorname{Im}(\zeta, -\omega)$; next observe that h, and therefore $c + h$, is trivially surjective. This finishes the proof of (ii), and (iii) is a subexact sequence of (ii).

Remark. — If in parts (ii) and (iii) of the theorem we do not assume that X_F is projective, the sequences remain exact if we replace $H^{p-1,p-1}(X_{\mathbf{R}})$ by the group

$$H_I^{p-1,p-1}(X_{\mathbf{R}}) = \frac{\operatorname{Ker}\{\partial\bar\partial : A^{p-1,p-1}(X_{\mathbf{R}}) \to A^{p,p}(X_{\mathbf{R}})\}}{\operatorname{Im}\partial + \operatorname{Im}\bar\partial}$$

and $H^{p,p}(X_{\mathbf{R}})$ by the quotient of $Z^{p,p}(X_{\mathbf{R}})$ by the image of dd^c.

However, such a group need not be finite-dimensional over $\mathbf{R}$. On the other hand, it can be shown (by extending the " Dolbeault lemma " of [H-P] to forms of arbitrary type) that if growth conditions at infinity are imposed to smooth forms on $X_{\mathbf{R}}$, this group becomes isomorphic to the Deligne cohomology group $H_{\mathscr{D}}^{2p-1}(X, \mathbf{R}(p))$ (see 3.5 below). It would be interesting to redefine the groups $\widehat{CH}^p(X)$ by imposing such growth conditions on forms.

3.4. Computations

In this section we shall compute $\widehat{CH}^*(X)$ in some simple cases.

3.4.1. Note that $\widehat{CH}^0(X) = CH^0(X)$ is the free Abelian group on the irreducible components of X.

3.4.2. If $p = 1$ and X is projective over A and irreducible, then $CH^{1,0}(X) = A^*$ and $A^{0,0}(X_{\mathbf{R}})$ is the space of F_∞-invariant continuous real-valued functions on X_∞; therefore by Theorem 3.3.5, exact sequence (i), we have

$$A^* \xrightarrow{\amalg_{\sigma\in\Sigma} \log|\ |^2_\sigma} C^\infty(X_\infty, \mathbf{R})^{F_\infty} \longrightarrow \widehat{CH}^1(X) \longrightarrow CH^1(X) \longrightarrow 0.$$

We shall see in [G-S 4] proposition 2.5 that there is a canonical isomorphism between $\widehat{CH}^1(X)$ and the group $\widehat{\mathrm{Pic}}(X)$ of isometric isomorphism classes of line bundles on X equipped with F_∞-invariant Hermitian metrics over X_∞.

3.4.3. In particular, if $X = A$, then we have

$$A^* \xrightarrow{\amalg_{\sigma\in\Sigma} \log|\ |^2_\sigma} (\bigoplus_{\sigma\in\Sigma} \mathbf{R})^{F_\infty} \longrightarrow \widehat{CH}^1(X) \longrightarrow CH^1(X) \longrightarrow 0$$

which, for a Dedekind domain, may be rewritten

$$A^* \to (\bigoplus_{\sigma\in\Sigma} \mathbf{R})^{F_\infty} \to \widehat{CH}^1(X) \to \mathrm{Cl}(A) \to 0.$$

Here Cl(A) is the ideal class group of A ([L 2]). Specializing even further, if $A = \mathcal{O}_F$ is the ring of integers in a number field, we get

$$\{1\} \to \mu(F) \to \mathcal{O}_F^* \xrightarrow{\rho} \mathbf{R}^{r_1+r_2} \to \widehat{CH}^1(X) \to \mathrm{Cl}(\mathcal{O}_F) \to 0$$

and we can identify

$$\widehat{CH}^1(X) \approx F^*\backslash J(F)/U_F,$$

where J(F) is the idele group of F, see [L 2], F^* is the multiplicative group $F - \{0\}$, and U_F is the maximal compact subgroup of J(F). This isomorphism is obtained by sending the idele (α_v) to the arithmetic cycle (Z, g), where $Z = \sum_v v(\alpha_v)[v]$ and $g = -\log|\alpha_v|^2$ when v is real and $g = -2\log|\alpha_v|^2$ when v is complex. Notice that ρ is, up to a factor -2, the classical Dirichlet regulator map ([L 2]).

In this case $A = \mathcal{O}_F$, there is a homomorphism

$$\deg : \widehat{CH}^1(\mathrm{Spec}(\mathcal{O}_F)) \to \mathbf{R}$$
$$(Z, g) \mapsto \log \#(Z) + \frac{1}{2}\int_X g$$

(here, if $Z = \bigoplus n_i[\wp_i]$ is a divisor, with $\wp_i$ prime ideals in $\mathcal{O}_F$, then

$$\#(Z) = \sum n_i \, \#(\mathcal{O}_F/\wp_i),$$

and if $g = \{g_\sigma\}_{\sigma\in\Sigma}$, $g_\sigma \in \mathbf{R}$, then $\int_X g = \sum_{\sigma\in\Sigma} g_\sigma$).

Finally, if $A = \mathbf{Z}$, then

$$\deg : \widehat{CH}^1(\mathrm{Spec}\,\mathbf{Z}) \to \mathbf{R}$$

is an isomorphism, since $\mathbf{Z}^* = \mu(\mathbf{Q})$, and $\mathrm{Cl}(\mathbf{Z}) = \{1\}$.

3.5. The map ρ and the Beilinson regulator

3.5.1. Let X be a compact complex manifold. If $A \subset \mathbf{C}$ is a subring and for $q \in \mathbf{Z}$, $\underline{A}(q) = (2\pi i)^q \underline{A} \subset \underline{\mathbf{C}}$ is the associated constant sheaf, define, for $p \geqslant 0$,

$$H^p_{\mathscr{D}}(X, A(q)) = \mathbf{H}^p(X, \underline{A}(q) \to \mathscr{O}_X \to \dots \to \Omega_X^{q-1}).$$

These are the Deligne cohomology groups of X with coefficients in A; see [Be 1] and [B-G]. In [Be 1] Beilinson extended the definition to all pairs (X, U) with $U \subset X$ an open subvariety, in such a way that there are exact sequences

$$(3.5.1.1) \qquad 0 \to \frac{H^{p-1}(X, U; \mathbf{C})}{F^q H^{p-1}(X, U; \mathbf{C}) + H^{p-1}(X, U; A(q))}$$
$$\to H^p_{\mathscr{D}}(X, U; A(q)) \to H^p(X, U; A(q)) \cap F^q H^p(X, U; \mathbf{C}) \to 0,$$

where $F^{\bullet} H(X, U; \mathbf{C})$ is the Hodge filtration constructed in [De 1]. As one would expect, there are also long exact sequences:

$$\dots \to H^p_{\mathscr{D}}(X, U; A(q)) \to H^p_{\mathscr{D}}(X; A(q)) \to H^p_{\mathscr{D}}(U; A(q))$$
$$\xrightarrow{\partial} H^{p+1}_{\mathscr{D}}(X, U; A(q)) \to \dots$$

and a purity theorem for $Y \subset X$ a closed subvariety of codimension d in a smooth variety:

$$H^p_{\mathscr{D}}(X, X - Y; A(q)) \simeq \begin{cases} 0 & \text{if } p < 2d \\ H^{p-2d}_{\mathscr{D}}(Y; A(q-d)) & \text{if Y is smooth.} \end{cases}$$

3.5.2. If X is a connected compact complex manifold, let $C_*(X, A(q))$ be the complex of differentiable singular chains with values in $A(q)$; there is a natural map of complexes $C_*(X, A(q)) \to \mathscr{D}^*(X)$ in which we assign degree $2d - i$ to $C_i(X, A(q))$ ($d = \dim X$). If $F^{\bullet} \mathscr{D}^*(X)$ is the Hodge filtration,

$$F^p \mathscr{D}^n(X) \simeq \bigoplus_{\substack{i \geqslant p \\ i+j=n}} \mathscr{D}^{i,j}(X),$$

let

$$u : C_*(X, A(q)) \oplus F^q \mathscr{D}^*(X) \to \mathscr{D}^*(X)$$

be the difference of the two natural maps. Then one sees easily that

$$H^p_{\mathscr{D}}(X, A(q)) \simeq H^p(X, C^*(u)\,[-1]),$$

in which $C^*(u)$ is the mapping cone of u. If $Y \subset X$ is a smooth subvariety of codimension n, the map $j_* : H^p_{\mathscr{D}}(Y, A(q)) \to H^{p+2n}_{\mathscr{D}}(X, A(q+n))$ is the map induced by the natural maps $j_* : C_*(Y, A(q)) \to C_*(X, A(q))$ and $j_* : \mathscr{D}^*(Y) \to \mathscr{D}^*(X)\,[2n]$.

More generally, let X be smooth and compact, let $Y \subset X$ be a closed analytic subspace of X, and let U be the smooth locus of Y. Let $\mathscr{D}^*_Y(X)$ be the subcomplex of $\mathscr{D}^*(X)$ consisting of currents supported in Y. Let $\varphi \in \mathscr{D}^{2p-2}_Y(X)$ be a current which is represented by a form ψ on U which is L^1 on Y in the sense that it is L^1 as a form on some resolution of singularities of Y. If φ satisfies the following conditions: (i) $\partial\varphi = [\alpha]$ for

$\partial\psi = \alpha$ a form lying in the $(p - n)$-th stage, for $n = \operatorname{codim}_X(Y)$, of the Hodge filtration on the complex of forms with logarithmic poles on a normal crossings compactification of the smooth locus $U \subset Y$; and (ii) $\bar{\partial}\partial\psi = 0$, then φ defines a class in the Deligne cohomology of U, hence in $H^{2p-2}_{\mathscr{D}}(X, X - Y; \mathbf{R}(p))$ by the isomorphisms at the end of 3.5.1. Its image in the Deligne cohomology of X can be represented by the current φ.

3.5.3. Two additional pieces of information that we shall need are:

1) If X is compact and smooth

$$H^{2p-1}_{\mathscr{D}}(X, \mathbf{R}(p)) \simeq H^{p-1,\,p-1}(X, \mathbf{R}(p-1)).$$

This follows from the exact sequence (3.5.1.1).

2) If X is smooth but noncompact, and $X \subset \overline{X}$ with $\overline{X}$ smooth and compact, and $\overline{X} - X = Y$ is a normal crossing divisor on $\overline{X}$, then ([Be 1], 1.5.2)

$$H^1_{\mathscr{D}}(X, \mathbf{R}(1)) \simeq \{\, \varphi \in \Gamma(X, \mathscr{O}_X/\underline{\mathbf{R}}(1)) \mid \partial\varphi \text{ has logarithmic poles on } Y \,\}.$$

3.5.4. In [Be 1], if $\mathbf{Q} \subset A$, Beilinson defines Chern characters, for $p, i \geqslant 0$ and $Y \subset X$:

$$\mathrm{ch}_i : K^Y_p(X) \to H^{2i-p}_{\mathscr{D}}(X, X - Y; A(i)).$$

In particular, for $p = 1$ there are maps

$$K_1(X) \to H^{2i-1}_{\mathscr{D}}(X; \mathbf{R}(q)) \simeq H^{i-1,\,i-1}(X, \mathbf{R}(i-1)).$$

Using the Brown-Gersten-Quillen spectral sequence and the Riemann-Roch theorem one can show ([She], [So]) that there is a canonical isomorphism

$$K_1(X)_{\mathbf{Q}} \simeq \bigoplus_{p\geqslant 0} \mathrm{CH}^{p,\,p-1}(X)_{\mathbf{Q}}.$$

Composing this isomorphism with the Chern character, we get, for each $p \geqslant 1$, a map induced by ch_p,

$$r_p : \mathrm{CH}^{p,\,p-1}(X) \to H^{p-1,\,p-1}(X, \mathbf{R}(p-1)).$$

Theorem. — Let X *be a smooth projective variety over* $\mathbf{C}$. *Then if*

$$\rho_p : \mathrm{CH}^{p,\,p-1}(X) \to H^{p-1,\,p-1}(X, \mathbf{R}(p-1))$$

is the map constructed in 3.3.5, $\rho_p = -2r_p$.

Proof. — Let W be a reduced closed subscheme of codimension $p - 1$ in X, and let $W_1, \dots, W_n$ be its irreducible components. Suppose $f_i \in k(W_i)^*$ for $i = 1, \dots, k$ are such that $\sum_i \operatorname{div}(f_i) = 0$ in $Z^p(X)$, so $\sum_i \{f_i\} = \varphi$ represents a class in $\mathrm{CH}^{p,\,p-1}(X)$. Since φ is supported on W, φ determines a class in $\mathrm{CH}^{1,0}(W)$. If W^0 is the non-singular locus of $W - \operatorname{div}(f)$, then, under the restriction map ($W^0_i = W^0 \cap W_i$)

$$\mathrm{CH}^{1,0}(W) \to \mathrm{CH}^{1,0}(W^0) \simeq \bigoplus_i H^0(W^0_i, \mathscr{O}^*_{W^0}),$$

φ maps to $\bigoplus \{f_i\}$. We have a commutative diagram

$$\begin{array}{ccccc}
CH^{p,p-1}(X)_{\mathbf{Q}} & \longrightarrow & K_1(X)_{\mathbf{Q}} & \longrightarrow & H^{2p-1}_{\mathcal{D}}(X, \mathbf{R}(p)) \\
\uparrow & & \uparrow & & \uparrow \\
CH^{1,0}(W)_{\mathbf{Q}} & \longrightarrow & K_1^W(X)_{\mathbf{Q}} & \longrightarrow & H^{2p-1}_{\mathcal{D}}(X, X - W; \mathbf{R}(p)) \\
\Big| & & \downarrow & & \downarrow \\
\Big| & & K_1^{W^0}(X)_{\mathbf{Q}} & \longrightarrow & H^{2p-1}_{\mathcal{D}}(X - (W - W^0), X - W^0; \mathbf{R}(p)) \\
\downarrow & & \wr\Vert & & \wr\Vert \\
H^0(W^0, \mathcal{O}^*_{W^0})_{\mathbf{Q}} & \longrightarrow & K_1(W_0)_{\mathbf{Q}} & \longrightarrow & H^1(W^0; \mathbf{R}(1)).
\end{array}$$

The isomorphisms between Deligne cohomology groups follow from purity for Deligne cohomology, while the isomorphism $K_1^{W^0}(X) \simeq K_1(W^0)$ follows from Quillen's localization and devissage theorems ([Q] §7). Now consider the current $[\log |\varphi|] = \sum_{\alpha=1}^{n} i_{\alpha *}[\log |f_\alpha|]$; by the remarks at the end of 3.5.2, $[\log |\varphi|]$ defines a class in $H^{2p-1}_{\mathcal{D}}(X, X - W; \mathbf{R}(p))$. Under the isomorphism

$$H^{2p-1}_{\mathcal{D}}(X, X - W; \mathbf{R}(p)) \simeq \bigoplus_i H^1(W_i^0, \mathbf{R}(1)),$$

$[\log |\varphi|]$ corresponds to $\bigoplus_i [\log |f_i|]$. Note that $[\log |f_i|] \in \Gamma(W_i^0, \mathcal{O}_{W_i^0}/\underline{\mathbf{R}}(1))$ satisfies $\partial[\log |f_i|] = \left[\frac{df_i}{2f_i}\right]$. By [Be 1] § 2.3, $ch_1 : K_1(W_i^0) \to H^1_{\mathcal{D}}(W_i^0, \mathbf{R}(1))$ sends $\{f_i\}$ to $[\log |f_i|]$, while by the Riemann Roch theorem (ibid.) we have a commutative diagram:

$$\begin{array}{ccc}
K_1(X) & \longrightarrow & H^{2p-1}_{\mathcal{D}}(X, \mathbf{R}(p)) \\
\downarrow & & \downarrow \\
K_1^W(X) & \xrightarrow{ch_p} & H^{2p-1}_{\mathcal{D}}(X, X - W; \mathbf{R}(p)) \\
\downarrow & & \downarrow \cong \\
K_1(W^0) & \xrightarrow{ch_1} & H^1_{\mathcal{D}}(W^0; \mathbf{R}(1))
\end{array}$$

Hence $ch_p(\varphi) = [\log |\varphi|]$. Finally $\rho(\varphi) = - \Sigma i_{\alpha *}[\log |f_\alpha|^2] = - 2r_p(\varphi)$.

3.6. Flat-pullback and pushforward

3.6.1. *Theorem.* — *Let $f : X \to Y$ be a morphism between arithmetic varieties over an arithmetic ring $A = (A, \Sigma, F_\infty)$. Writing F for the fraction field of A, suppose that f induces a smooth map $X_F \to Y_F$ between the generic fibres of X and Y. Then:*

(i) *If f is flat, for all $p \geqslant 0$, there is a natural homomorphism*

$$f^* : \widehat{CH}^p(Y) \to \widehat{CH}^p(X).$$

(ii) *If f is proper, and* X *and* Y *are equidimensional, there is a map*

$$f_* : \widehat{\mathrm{CH}}^p(\mathrm{X}) \to \widehat{\mathrm{CH}}^{p-d}(\mathrm{Y})$$

for $d = \dim \mathrm{X} - \dim \mathrm{Y}$.

If $f : \mathrm{X} \to \mathrm{Y}$, $g : \mathrm{Y} \to \mathrm{Z}$ *are two maps inducing smooth maps between generic fibres, then* $f^* g^* = (gf)^*$ *and* $(gf)_* = g_* f_*$ *when either composition makes sense.*

Proof. — First suppose that f is flat. If $\mathrm{Z} = \Sigma n_i[\mathrm{Z}_i] \in \mathrm{Z}^p(\mathrm{Y})$, then $f^*\mathrm{Z} = \Sigma n_i[f^{-1}\mathrm{Z}_i]$ (see [Fu] 1.7) is a codimension p cycle on X. Since $f : \mathrm{X}_\mathrm{F} \to \mathrm{Y}_\mathrm{F}$ is smooth, so also is $f_\infty : \mathrm{X}_\infty \to \mathrm{Y}_\infty$; hence for any current $\mathrm{T} \in \mathscr{D}^{p,q}(\mathrm{Y})$, $f_\infty^* \mathrm{T}$ may be defined by $f_\infty^* \mathrm{T}(\varphi) = \mathrm{T}(f_{\infty *}\varphi)$ for φ any compactly supported form on X_∞, i.e. as the adjoint of the "integration over the fibre" map $f_{\infty *} : \mathrm{A}_c^{p,q}(\mathrm{X}_\infty) \to \mathrm{A}_c^{p-d,q-d}(\mathrm{Y}_\infty)$. If $\mathrm{Z} \subset \mathrm{Y}_\mathrm{F}$ is an integral subscheme, we have a Cartesian square

$$\begin{array}{ccc} f^{-1}(\mathrm{Z}) & \longrightarrow & \mathrm{X}_\mathrm{F} \\ {\scriptstyle f_\mathrm{Z}}\downarrow & & \downarrow{\scriptstyle f} \\ \mathrm{Z} & \longrightarrow & \mathrm{Y}_\mathrm{F} \end{array}$$

in which f_Z is a smooth map. If φ is a compactly supported C^∞ form on X_∞,

$$f_{\infty *}(\varphi)\,|_\mathrm{Z} = f_{\mathrm{Z}*}(\varphi\,|_{f^{-1}(\mathrm{Z})});$$

hence the current of integration over $f^{-1}(\mathrm{Z})$, $\delta_{f^{-1}(\mathrm{Z})}$, is $f^*\delta_\mathrm{Z}$, the pull-back of the current of integration over Z. Since integration over the fibre commutes with ∂ and $\bar\partial$, hence with dd^c, $f_\infty^* : \mathscr{D}^{p,q}(\mathrm{Y}_\infty) \to \mathscr{D}^{p,q}(\mathrm{X}_\infty)$ commutes with dd^c. Now suppose that $(\mathrm{Z}, g) \in \hat{\mathrm{Z}}^p(\mathrm{Y})$. Then

$$\begin{aligned} dd^c f_\infty^* g &= f_\infty^* dd^c g \\ &= f_\infty^*(\omega(\mathrm{Z}, g) - \delta_\mathrm{Z}) \\ &= f_\infty^* \omega(\mathrm{Z}, g) - f_\infty^* \delta_\mathrm{Z} \\ &= f_\infty^* \omega(\mathrm{Z}, g) - \delta_{f^*\mathrm{Z}}. \end{aligned}$$

Hence $f_\infty^* g$ is a Green current for $f^*\mathrm{Z}$, and we can define $f^*(\mathrm{Z}, g) = (f^*\mathrm{Z}, f_\infty^* g)$. If $i : \mathrm{W} \to \mathrm{Y}_\infty$ and φ is an L^1 form on W and $i_*\varphi$ the associated current on Y_∞, then writing $f_\infty^*(\varphi)$ for the pullback of φ to $f^{-1}(\mathrm{W}) \subset \mathrm{X}_\infty$, we have by the Fubini theorem for L^1 functions:

$$f_\infty^*(i_*\varphi) = i_{f^{-1}(\mathrm{W})*} f_\infty^*(\varphi).$$

Hence, if $g \in k(\mathrm{W})^*$ is a rational function on a codimension $p - 1$ subvariety of X,

$$\begin{aligned} f^*\widehat{\mathrm{div}}(g) &= (f^*\mathrm{div}(g), f^* i_{\mathrm{W}*}[-\log|g|^2]) \\ &= (\mathrm{div}(f^*(g)), i_{f^{-1}(\mathrm{W})*}[-\log|f_\infty^* g|^2]) \\ &= \widehat{\mathrm{div}}(f^*\{g\}). \end{aligned}$$

Therefore $f^* : \widehat{CH}^p(Y) \to \widehat{CH}^p(X)$ is well defined; compatibility with composition may be checked by the reader. Now suppose that X and Y are equidimensional and that f is proper of relative dimension d. Then

$$f_* : Z^p(X) \to Z^{p-d}(Y)$$

is well defined, see [Fu] 1.4. If Z is a codimension p integral subscheme of X,

$$f_*([Z]) = n_Z[f(Z)]$$
$$n_Z = \begin{cases} [k(Z) : k(f(Z))] & \text{if this is finite,} \\ 0 & \text{otherwise.} \end{cases}$$

Now observe that if $\varphi \in A_c^{n-p,n-p}(Y_\infty)$ (for $n = \dim X_\infty$),

$$f_* \delta_Z(\varphi) = \int_Z f^* \varphi = \int_Z f^*(\varphi \mid_{f(Z)})$$
$$= \begin{cases} \deg(Z/f(Z)) \int_{f(Z)} \varphi & \text{if } Z \to f(Z) \text{ finite} \\ 0 & \text{if } \dim f(Z) < n-p, \text{ since then } \varphi \mid_{f(Z)} \equiv 0. \end{cases}$$

Hence f_* on cycles is compatible with f_* on currents. Now set $f_*(Z, g) = (f_* Z, f_* g)$. Then

$$dd^c f_* g + f_* \delta_Z = f_* \omega(Z, g).$$

Since f_∞ is smooth, $f_* \omega(Z, g)$ is C^∞, so $f_*(Z, g) \in \hat{Z}^{p-d}(Y)$. Finally, if $h \in k(W)^*$ is a rational function on a codimension $(p-1)$ integral subscheme of X, we must show that $f_* \widehat{\operatorname{div}}(h)$ is itself the arithmetic divisor of a rational function, or is zero.

If $f_W = f|_W : W \to W' = f(W)$ is generically finite, then ([Fu] 1.4)

$$f_* \operatorname{div}(h) = \operatorname{div}(Nm_{k(W)|k(W')}(h)).$$

This proves our assertion when W, hence W′, does not meet the generic fiber. When char $k(W') = 0$, there is a dense open subset $U \subset W'$ such that $f_W : f_W^{-1}(U) \to U$ is finite and étale. If φ is an L^1 function on W, then for $y \in U$, $f_{W*}(\varphi)(y) = \sum_{x \in f^{-1}(y)} \varphi(x)$. Therefore $f_* \log |h| = \log |Nm_{k(W)/k(W')}(h)|$, and so $f_* \widehat{\operatorname{div}}(h) = \widehat{\operatorname{div}}(Nm_{k(W)/k(W')}(h))$.

If f_W is not finite, then $f_* \operatorname{div}(h) = 0$ by [Fu] Thm. 1.4, while $f_* \log |h| = 0$ by the argument involving dimensions used when defining f_* on $\hat{Z}^*(X)$.

4. Cup products and pull-backs

4.1. Chow groups with supports

4.1.1. If X is a Noetherian scheme and $Y \subset X$ is a closed subset, $Z_Y^p(X)$, the group of codimension p cycles on X with supports in Y is the free Abelian group on the set of codimension p integral subschemes of X which are contained in Y; i.e., $\zeta \in Z_Y^p(X)$ if and only if $\zeta = \sum n_i[Z_i] \in Z^p(X)$ and for all i, $Z_i \subset Y$. We define $CH_Y^p(X)$, the Chow group of codimension p cycles with supports in Y, to be the quotient of $Z_Y^p(X)$ by the subgroup generated by cycles of the form $\mathrm{div}(f)$, for $f \in k(\mathrm{W})^*$ with W a codimension $p-1$ integral subscheme of X contained in Y. Observe that there is an exact sequence

$$\mathrm{CH}^p_{\mathrm{Y}}(\mathrm{X}) \to \mathrm{CH}^p(\mathrm{X}) \to \mathrm{CH}^p(\mathrm{X}-\mathrm{Y}) \to 0.$$

More generally, let φ be a family of supports on X, i.e. a family of closed subsets of X such that the union of any two elements of φ is again an element of φ. Define

$$\mathrm{Z}^p_\varphi(\mathrm{X}) = \varinjlim_{\mathrm{Y}\in\varphi} \mathrm{Z}^p_{\mathrm{Y}}(\mathrm{X})$$

and

$$\mathrm{CH}^p_\varphi(\mathrm{X}) = \varinjlim_{\mathrm{Y}\in\varphi} \mathrm{CH}^p_{\mathrm{Y}}(\mathrm{X}).$$

Note that $\mathrm{CH}^p_{\mathrm{Y}}(\mathrm{X}) = \mathrm{CH}^p_{\{\mathrm{Y}\}}(\mathrm{X})$.

Finally, observe that if $\mathrm{Y} \subset \mathrm{X}$ is a closed subset of codimension p, then

$$\mathrm{CH}^p_{\mathrm{Y}}(\mathrm{X}) \simeq \mathrm{Z}^p_{\mathrm{Y}}(\mathrm{X})$$

while if $\mathrm{X}\{p\}$ is the family of all closed subsets of X of codimension at least p, then $\mathrm{CH}^p_{\mathrm{X}\{p\}}(\mathrm{X}) \simeq \mathrm{Z}^p(\mathrm{X})$.

By [So] Theorem 4 and [G-S 3] Theorems 8.2 and C, if X is a regular Noetherian scheme, there is an isomorphism

$$\mathrm{CH}^p_{\mathrm{Y}}(\mathrm{X})_{\mathbf{Q}} \simeq \mathrm{Gr}^p_\gamma\, \mathrm{K}^{\mathrm{Y}}_0(\mathrm{X})_{\mathbf{Q}}$$

($\mathrm{Gr}^\bullet_\gamma$ is the graded group associated to the γ-filtration on K-theory with supports, [So] and [G-S 3]). This isomorphism allows us to define products

$$\mathrm{CH}^p_{\mathrm{Y}}(\mathrm{X})_{\mathbf{Q}} \otimes \mathrm{CH}^q_{\mathrm{Z}}(\mathrm{X})_{\mathbf{Q}} \to \mathrm{CH}^{p+q}_{\mathrm{Y}\cap\mathrm{Z}}(\mathrm{X})_{\mathbf{Q}}$$

for all pairs Y, Z of closed subsets of X. If $\eta \in \mathrm{Z}^p_{\mathrm{Y}}(\mathrm{X})$ and $\zeta \in \mathrm{Z}^q_{\mathrm{Z}}(\mathrm{X})$ are cycles which intersect properly, then by [G-S 3] Theorem C, their product $\eta.\zeta$ under this pairing is given by Serre's tor-formula for intersection multiplicities ([Fu] 20.4, [Se]).

This product extends to Chow groups with supports in families of supports. If φ, ψ are two such families, define

$$\varphi \cap \psi = \{\bigcup_i (\mathrm{Y}_i \cap \mathrm{Z}_i) \mid \mathrm{Y}_i \in \varphi, \mathrm{Z}_i \in \psi\}.$$

Taking the direct limit over all $\mathrm{Y} \in \varphi$ and $\mathrm{Z} \in \psi$ of the product defined above, we obtain products

$$\mathrm{CH}^p_\varphi(\mathrm{X})_{\mathbf{Q}} \otimes \mathrm{CH}^q_\psi(\mathrm{X})_{\mathbf{Q}} \to \mathrm{CH}^{p+q}_{\varphi\cap\psi}(\mathrm{X})_{\mathbf{Q}}.$$

4.1.2. *Remark.* — If X is of finite type over a field, this product exists for Chow groups with integral coefficients; cf. [Fu], [Gi 4].

4.2. Cup products on arithmetic varieties

4.2.1. Let $\mathrm{A} = (\mathrm{A}, \Sigma, \mathrm{F}_\infty)$ be an arithmetic ring with fraction field F, and suppose that X is an arithmetic variety over A. If

$$\mathrm{fin} = \{\mathrm{Y} \subset \mathrm{X} \mid \mathrm{Y} \text{ closed}, \mathrm{Y} \cap \mathrm{X}_{\mathrm{F}} = \emptyset\},$$

there are exact sequences

$$\bigoplus_{\substack{x \in X^{(p-1)} \\ \overline{\{x\}} \cap X_F = \emptyset}} k(x)^* \xrightarrow{\text{div}} Z^p(X) \longrightarrow Z^p(X_F) \oplus CH^p_{\text{fin}}(X) \longrightarrow 0$$

and

$$\bigoplus_{x \in X_F^{(p-1)}} k(x)^* \xrightarrow{\text{div}} Z^p(X_F) \oplus CH^p_{\text{fin}}(X) \longrightarrow CH^p(X) \longrightarrow 0.$$

Also observe that if $\tau(p) = \{ Y \subset X \mid Y \text{ closed}, Y \cap X_F \text{ has codimension} \geqslant p \}$, then there is a canonical isomorphism

$$CH^p_{\tau(p)}(X) \simeq Z^p(X_F) \oplus CH^p_{\text{fin}}(X).$$

Suppose in addition that X is regular and that X_F is quasi-projective. If $x \in X^{(p-1)}$ with $\overline{\{x\}} \cap X_F = \emptyset$ and $f \in k(x)^*$, then $\text{div}(f) \cap X_F = \emptyset$. Therefore $[\log |f|^2] = 0$, and so $\widehat{\text{div}}(f) = (\text{div}(f), 0)$. Hence there is an exact sequence

$$\bigoplus_{x \in X_F^{(p-1)}} k(x)^* \xrightarrow{\widehat{\text{div}}} \hat{Z}^p(X_F) \oplus CH^p_{\text{fin}}(X) \longrightarrow \widehat{CH}^p(X) \longrightarrow 0,$$

i.e. in the definition of $\widehat{CH}^*(X)$ we can divide out by rational equivalence in the closed fibres before taking Green currents into account. (Notice that $\hat{Z}^p(X_F)$ makes sense because X_F is an arithmetic variety over the arithmetic ring F.)

4.2.2. Now suppose that Y and Z are integral subschemes of codimensions p and q respectively, which intersect properly on X_F (i.e. $\text{codim}(Y \cap Z) = \text{codim}(Y) + \text{codim}(Z)$). Then $[Y].[Z]$ is not necessarily well defined as a cycle on X since Y and Z may not intersect properly on the whole of X. However $[Y].[Z]$ is well defined as a class in $CH^{p+q}_{Y \cap Z}(X)_{\mathbf{Q}}$, and since $Y \cap Z \in \tau(p+q)$, $[Y].[Z]$ defines a class in

$$(Z^{p+q}(X_F) \oplus CH^{p+q}_{\text{fin}}(X))_{\mathbf{Q}}.$$

We can therefore define, if g_Y and g_Z are Green currents for Y and Z respectively,

(4.2.2.1) $$([Y], g_Y).([Z], g_Z) = ([Y].[Z], g_Y * g_Z) \in (Z^{p+q}(X_F) \oplus CH^{p+q}_{\text{fin}}(X))_{\mathbf{Q}} \oplus \tilde{\mathscr{D}}^{p+q-1,\, p+q-1}(X_{\mathbf{R}}).$$

Note that if Y and Z intersect properly on the whole of X, not just on X_F, then we can define

(4.2.2.2) $$([Y], g_Y).([Z], g_Z) = ([Y].[Z], g_Y * g_Z) \in \hat{Z}^{p+q}(X).$$

4.2.3. *Theorem.* — *Let* $A = (A, \Sigma, F_\infty)$ *be an arithmetic ring with fraction field* F. *Suppose that* X *is an arithmetic variety over* A *which is regular and has quasi-projective generic fibre* X_F. *Then:*

(i) *For each pair of non-negative integers* (p, q), *there is a pairing*

(4.2.3.1) $$\widehat{CH}^p(X) \otimes \widehat{CH}^q(X) \to \widehat{CH}^{p+q}(X)_{\mathbf{Q}}$$
$$\alpha \otimes \beta \mapsto \alpha\beta$$

which is uniquely determined by the following condition:

If Y *and* Z *are integral subschemes of* X *which intersect properly on* X_F, *if* g_Y *and* g_Z *are Green currents for* Y *and* Z, *and if* α *(resp.* β*) is the class of* $([Y], g_Y)$ *(resp.* $([Z], g_Z)$*), then* $\alpha\beta$ *is the class of the element* $([Y], g_Y) \cdot ([Z], g_Z)$ *defined by the formula* (4.2.2.1).

(ii) *The product* (4.2.3.1) *makes* $\widehat{CH}^*(X)_{\mathbf{Q}}$ *into a commutative, associative ring.*

(iii) *If* p *or* $q = 1$, *there is a unique pairing*

(4.2.3.2) $$\widehat{CH}^p(X) \otimes \widehat{CH}^q(X) \to \widehat{CH}^{p+q}(X)$$

which is given by formula (4.2.2.2) *for cycles meeting properly on* X. *This pairing induces the pairing* (4.2.3.1) *taking values in* $\widehat{CH}^{p+q}(X)_{\mathbf{Q}}$. *When* $\alpha, \beta \in \widehat{CH}^1(X)$ *and* $\gamma \in \widehat{CH}^q(X)$, $q \geqslant 0$, *we have*

(4.2.3.3) $$\alpha\gamma = \gamma\alpha \in \widehat{CH}^{q+1}(X)$$

and

(4.2.3.4) $$\alpha(\beta\gamma) = \beta(\alpha\gamma) \in \widehat{CH}^{q+2}(X).$$

Before proving the theorem, we shall prove three lemmas. We continue using the same notation.

4.2.4. *Lemma.* — *Let* Y, Z *and* W *be integral subschemes of* X *which intersect properly on* X_F *and which have codimensions* p, q *and* r *respectively. Then, considering* Y *and* Z *as a pair of integral subschemes which intersect properly on* X_F, *we have*

$$(Y, g_Y) \cdot (Z, g_Z) = (Z, g_Z) \cdot (Y, g_Y).$$

Furthermore, if X_F *is projective, and if* g_Y, g_Z *and* g_W *are Green currents for* Y, Z *and* W *respectively,*

(4.2.4.1) $$((Y, g_Y) \cdot (Z, g_Z)) \cdot (W, g_W) = (Y, g_Y) \cdot ((Z, g_Z) \cdot (W, g_W))$$

in $(Z^s(X_F) \oplus CH^s_{\mathrm{fin}}(X))_{\mathbf{Q}} \oplus \widetilde{\mathscr{D}}^{s-1, s-1}(X)$. *(Here* $s = p + q + r$.*)*

Proof. — By [G-S 3] 1.4 and 8.3, we have

$$([Y] \cdot [Z]) \cdot [W] = [Y] \cdot ([Z] \cdot [W])$$

and $$[Y] \cdot [Z] = [Z] \cdot [Y]$$

in $(Z^*(X_F) \oplus CH^*_{\mathrm{fin}}(X))_{\mathbf{Q}}$, while by 2.2.9 and 2.2.14,

$$g_Y * g_Z = g_Z * g_Y$$

in $\widetilde{\mathscr{D}}(X)$, while, if X_F is projective,

$$(g_Y * g_Z) * g_W = g_Y * (g_Z * g_W).$$

4.2.5. Before stating the next lemma, we must examine the properties of rational equivalence more closely. If X is a scheme, let us write $R^i_p(X)$ for the group $\bigoplus_{x \in X^{(i)}} K_{p-i}(k(x))$. For $i = p$, we get $R^p_p(X) = Z^p(X)$, while for $i = p - 1$, we get $R_p^{p-1}(X) = \bigoplus_{x \in X^{(p-1)}} k(x)^*$; let us call the elements of the latter group K_1-*chains*, and let us write a typical K_1-chain as $f = \Sigma[f_W]$ with $f_W \in k(W)^*$ as W runs through a finite set of integral codimension $(p-1)$ closed subschemes of X. For each $p \geqslant 1$, we define a homomorphism

$$\operatorname{div} : R_p^{p-1}(X) \to Z^p(X)$$
$$\Sigma[f_W] \mapsto \Sigma \operatorname{div}(f_W)$$

and, if X is arithmetic,

$$\widehat{\operatorname{div}} : R_p^{p-1}(X) \to \hat{Z}^p(X)$$
$$\Sigma[f_W] \mapsto (\Sigma \operatorname{div}(f_W), -\log|f|^2).$$

Note that, to simplify notation, we have written $\log|f|^2$ for the current $\Sigma i_{W*} \log|f_W|^2$. By the *support* of a K_1-chain $f = \Sigma[f_W]$ we mean the Zariski closed subset of X which is the union of all W for which $f_W \neq 1$. If $f \in R_p^{p-1}(X)$ is a K_1-chain and $\mathscr{Z} = \{Z_1 \dots Z_n\}$ is a collection of integral closed subschemes of X, we say that $f = \Sigma[f_W]$ meets $\mathscr{Z}$ properly if for all $Z \in \mathscr{Z}$: (i) each W for which $f_W \neq 1$ meets Z properly, and (ii) $\operatorname{div}(f_W)$ meets Z properly for all W. We say that f meets $\mathscr{Z}$ *almost properly* if condition (ii) alone is satisfied; note that this condition is strictly stronger than requiring that $\operatorname{div}(f)$ meet $\mathscr{Z}$ properly. Recall that the excess of a cycle $Y = \Sigma n_i[Y_i]$ with respect to $\mathscr{Z} = \{Z_1 \dots Z_n\}$ is the supremum, over all i and all j, of the excess of Y_i with respect to Z_j, which is itself the maximum of $\operatorname{codim}(Y_i) + \operatorname{codim}(Z_j) - \operatorname{codim}(T)$ as T runs through the irreducible components of $Y_i \cap Z_j$. We define the *excess* $e(f) \in \mathbf{N}$ of f, with respect to $\mathscr{Z}$, to be the supremum, over all W for which $f_W \neq 1$, of the excess of $\operatorname{div}(f_W)$ with respect to $\mathscr{Z}$; thus $e(f) = 0$ if and only if f meets $\mathscr{Z}$ almost properly. Suppose that W and Z are integral closed subschemes of X, of codimensions p and q respectively; let us write $W \cap Z = S \cup T$ with S and T closed, S of codimension $p + q$, and T of codimension $< p + q$. If an element $f \in k(W)^*$, viewed as a K_1-chain, meets Z almost properly, then $\operatorname{div}(f) \cap T = \emptyset$; otherwise $\operatorname{div}(f) \cap Z$ would have components with excess dimension. It follows that, since $\operatorname{div}(f) \cap T = \emptyset$, f restricted to T is a global unit (on T).

We now want to study products of cycles and of K_1-chains on a regular scheme X. If W is a closed integral subscheme of codimension $p - 1$ in X, $f \in k(W)^*$, and Z is a codimension q algebraic cycle meeting both W and $\operatorname{div}(f)$ properly then we can define a K_1-chain $[f].Z$ as follows. Since W and Z meet properly, we have a cycle $[W].Z = \sum_{i=1}^{k} n_i[S_i]$; since $\operatorname{div}(f)$ also meets Z properly, f is a unit at the generic point of each S_i, so we have rational functions $f|_{S_i} \in k(S_i)^*$, and then we define

$$[f].[Z] = \Sigma n_i[f|_{S_i}] = \Sigma[f^{n_i}|_{S_i}].$$

This definition extends by linearity to give a product $f.Z$ whenever f is a K_1-chain meeting an algebraic cycle Z properly.

Suppose now that $f \in k(\mathrm{W})^*$ with $\operatorname{div}(f)$ meeting Z properly. Then, as above, $\mathrm{W} \cap |\mathrm{Z}| = \mathrm{S} \cup \mathrm{T}$ with S of codimension $p+q-1$ and $\operatorname{div}(f) \cap \mathrm{T} = \emptyset$. As in 2.1.2 we can write $[\mathrm{W}].[\mathrm{Z}] = \Sigma n_i[\mathrm{S}_i] + t$, the S_i being the irreducible components of S, and t being a rational equivalence class supported on T. Unfortunately, unless T is a variety over a field, t is defined only as a class in $\mathrm{CH}_{\mathrm{T}}^{p+q-1}(\mathrm{X})_{\mathbf{Q}}$ rather than in $\mathrm{CH}_{\mathrm{T}}^{p+q-1}(\mathrm{X})$. Since $f|_{\mathrm{T}}$ is a unit, we have a class $[f].t \in \mathrm{CH}^{p+q,\, p+q-1}(\mathrm{X})_{\mathbf{Q}}$ (or in $\mathrm{CH}^{p+q,\, p+q-1}(\mathrm{X})$ if X is a variety over a field). Finally we define

$$[f].[\mathrm{Z}] = \Sigma n_i[f|_{\mathrm{S}_i}] + [f].t,$$

while this is not a well defined K_1-chain, it is well defined as an element of $\mathrm{R}_p^{p-1}/d(\mathrm{R}_p^{p-2})$ ($\otimes\, \mathbf{Q}$ if X is not a variety over a field), i.e. up to the image of $\bigoplus_{x \in \mathrm{X}^{(p-2)}} \mathrm{K}_2(k(x)) \to \bigoplus_{x \in \mathrm{X}^{(p-1)}} k(x)^*$. For our purposes, this is sufficient since, by 3.3.5, both div and $\widehat{\operatorname{div}}$ vanish on $d(\mathrm{R}_p^{p-2})$. Having defined $f.\mathrm{Z}$ for $f \in k(\mathrm{W})^*$, we extend to general f meeting Z almost properly by linearity.

More generally, if $g : \mathrm{Y} \to \mathrm{X}$ is a map of schemes then $g(\mathrm{Y})$ is a finite union of locally closed subsets ("strata") Z_i, $i = 1, \ldots, \mathrm{N}$, such that, for $i = 1, \ldots, \mathrm{N}$, the fibres of g have the same dimension at all points of Z_i. If $f \in \mathrm{R}_p^{p-1}(\mathrm{X})$ is a K_1-chain on X which meets the Zariski closure of each Z_i almost properly, then by a similar method to that used above, we can define a pull-back K_1-chain $g^*(f) \in \mathrm{R}_p^{p-1}(\mathrm{Y})/d(\mathrm{R}_p^{p-2}(\mathrm{Y}))$ ($\otimes\, \mathbf{Q}$ if X is not a variety over a field).

Lemma. — If f is a codimension $(p-1)$ K_1-chain on X, and Z is a codimension q cycle which meets f almost properly, then:

1) *We have an equality of cycles (in $\mathrm{Z}^{p+q}(\mathrm{X})_{\mathbf{Q}}$ if X is not a variety over a field):*

$$\operatorname{div}(f.\mathrm{Z}) = \operatorname{div}(f).\mathrm{Z}.$$

2) *If X is an arithmetic variety, we have an equality of currents in $\widetilde{\mathscr{D}}^{p+q-1,\, p+q-1}(\mathrm{X}_{\mathbf{R}})$:*

(4.2.5.1) $$\log|f|^2 \wedge \delta_{\mathrm{Z}} = \log|f.\mathrm{Z}|^2.$$

More generally, if $g : \mathrm{Y} \to \mathrm{X}$ is a morphism, and f meets each stratum of $g(\mathrm{Y})$ almost properly, we have

1′) $\operatorname{div}(g^*(f)) = g^*(\operatorname{div}(f))$,
2′) $\log|g^*(f)|^2 = g^* \log|f|^2$.

Proof. — Since the proofs of 1′) and 2′) are similar to those of 1) and 2), we only give the proof for 1) and 2). Without loss of generality, $f = [f]$ for $f \in k(\mathrm{W})^*$, and $\mathrm{W} \subset \mathrm{X}$ an integral subscheme. 1) Choose $\tilde{f} \in k(\mathrm{X})^*$ such that $\tilde{f}|_{\mathrm{W}} = f$ and $\operatorname{div}(\tilde{f})$ meets Z properly; then $\operatorname{div}(f.\mathrm{Z}) = \operatorname{div}(\tilde{f}).([\mathrm{W}].[\mathrm{Z}])$ by [Fu] Chapter 2, hence the equality follows from the associativity of intersection products. 2) Since $-\log|f|^2$ is a Green current for $\operatorname{div}(f)$, $\log|f|^2 \wedge \delta_{\mathrm{Z}} = \log|f|^2 * g_{\mathrm{Z}}$ for any choice of g_{Z}. By the commutativity of the $*$-product, the left hand side of equation 4.2.5.1 can be written $\log|f|^2 \wedge \omega_Z - \delta_{\mathrm{div}(f)} \wedge g_Z$. Choosing $\tilde{f}$ as in part 1) above we have, by definition, that

$$f.[Z] = \tilde{f}.[W]\,[Z] = \tilde{f}.(s+t)$$

for $[W].[Z] = \Sigma n_i[S_i] + t = s + t$ as above. Hence

$$\log|f.Z|^2 = \log|\tilde{f}|^2 \wedge (\delta_s + \delta_t),$$

which, by Theorem 2.2.1, is equal to

$$\log|\tilde{f}|^2 \wedge \omega_Z \wedge \delta_W - \delta_{\mathrm{div}(\tilde{f}).W} \wedge g_Z = \log|f|^2 \wedge \omega_Z - \delta_{\mathrm{div}(f)} \wedge g_Z$$
$$= \log|f|^2 \wedge \delta_Z.$$

4.2.6. Let us temporarily fix a field F of characteristic zero. If $L \subset \mathbf{P}^n = \mathbf{P}^n_F$ is a linear subspace, and $Z \subset \mathbf{P}^n$ is a subvariety for which $L \cap Z = \emptyset$, then there is a subvariety $C_L(Z) \subset \mathbf{P}^n$, called the *cone over* Z *with vertex* L, one definition of which may be found in [R]. A more geometric formulation of the definition is to say that $C_L(Z) = \overline{\pi_L^{-1}(\pi_L(Z))}$, $\pi_L : \mathbf{P}^n - L \to \mathbf{P}^r_F$ being the projection map, and $r+1$ being the codimension of L in $\mathbf{P}^n$. As is well known, if $\dim(Z) < r$, then for L belonging to a dense Zariski open set in the Grassmannian of all $(n-r-1)$-planes in $\mathbf{P}^n$, the map $Z \to \pi_L(Z)$ is birational, and hence there is a canonical inclusion $k(Z) \subset k(C_L(Z))$. Therefore, if $f \in R^p_{p+1}(\mathbf{P}^n)$ and $p \geqslant n - r + 1$, then for the generic L we have a well defined K_1-chain $C_L(f) \in R^{p+r-n}_{p+r-n+1}(\mathbf{P}^n)$. It follows directly from the definition of a cone that $\mathrm{div}(C_L(f)) = C_L(\mathrm{div}(f))$. Also, if f is supported on a subvariety $X \subset \mathbf{P}^n$ and $L \cap X = \emptyset$, then $C_L(f)$ meets X properly.

Lemma (Moving lemma for K_1*-chains). — Let* X *be a smooth quasi-projective variety over an infinite field* F. *Suppose that* $f \in R^{p-1}_p(X)$ *is a* K_1*-chain such that* $\mathrm{div}(f)$ *meets a finite collection* $\mathscr{Z} = \{Z_1, \ldots, Z_k\}$ *of subvarieties of* X *properly. Then there exists a* K_1*-chain* g, *such that*

(i) $\mathrm{div}(g) = \mathrm{div}(f)$,
(ii) $g - f$ *represents* 0 *in* $\mathrm{CH}^{p,p-1}(X)$,
(iii) g *meets* $\mathscr{Z}$ *almost properly.*

Proof. — Embed X in $\mathbf{P}^n$. If the dimension of X is r, applying the main lemma of [R], we can find a codimension $r+1$ linear space $L \subset \mathbf{P}^n$ such that $L \cap X = \emptyset$ and

(i) π_L is generically finite on the support of f so that $C_L(f)$ is defined;
(ii) if e = excess with respect to $\mathscr{Z}$, $e(C_L(f).X - f) \leqslant \mathrm{Max}(e(f) - 1, 0)$;
(iii) $C_L(\mathrm{div}(f)) = \mathrm{div}(C_L(f))$ meets $\mathscr{Z}$ properly.

Therefore, as in *op. cit.*, we can find linear subspaces $L_1, \ldots, L_e$, such that

$$f = \sum_{i=1}^{e} (-1)^{i-1} C_{L_i}(f_i).X + (-1)^e f_e,$$

with f_e meeting $\mathscr{Z}$ almost properly. We can find elements $g_i \in \mathrm{Aut}(\mathbf{P}^n)$ such that $g_i\, \mathrm{C}_{\mathrm{L}_i}(f_i)$ meets C and $\mathscr{Z}$ properly. Joining each g_i to the identity in $\mathrm{Aut}(\mathbf{P}^n)$ by a rational curve, we obtain, just as in the theorem of section 3 of *op. cit.*, a family f_t of K_1-chains on X, parameterized by $t \in \mathbf{P}^1$, such that

(i) $f_0 = f$;

(ii) f_∞ meets $\mathscr{Z}$ almost properly;

(iii) $\mathrm{div}(f_t)$, for all but a finite number of values of t, meets $\mathscr{Z}$ properly.

The family $\{f_t\}$ forms a K_1-chain $\tilde{f} = \sum_{\mathrm{V}} [\tilde{f}_{\mathrm{V}}]$ on $\mathrm{X} \times \mathbf{P}^1$, with each V flat over $\mathbf{P}^1$ and $\tilde{f}$ meeting $\mathrm{div}(t)$ properly. Since each V is flat over $\mathbf{P}^1$ and meets $\mathrm{div}(t)$ properly, we have an element

$$\sum_{\mathrm{W}} \{ t, f_{\mathrm{W}} \} \in \bigoplus_{x \in (\mathrm{X} \times \mathbf{P}^1)^{(p-1)}} \mathrm{K}_2\, k(x).$$

Under the differential

$$d: \bigoplus_{x \in (\mathrm{X} \times \mathbf{P}^1)^{(p-1)}} \mathrm{K}_2(k(x)) \to \bigoplus_{x \in (\mathrm{X} \times \mathbf{P}^1)^{(p)}} \mathrm{K}_1(k(x)),$$

this element maps to

$$\begin{aligned} d(\sum_{\mathrm{W}} \{ t, f_{\mathrm{W}} \}) &= \mathrm{div}(t) . \tilde{f} - \{ t \} \mathrm{div}(\tilde{f}) \\ &= f_0 \times \{ 0 \} - f_\infty \times \{ \infty \} - \{ t \} \mathrm{div}(\tilde{f}). \end{aligned}$$

By (iii) above, for each $\mathrm{Z} \in \mathscr{Z}$, the K_1-chain $\{ t \} \mathrm{div}(\tilde{f})$ meets $\mathrm{Z} \times \mathbf{P}^1$ almost properly. Hence, if $p : \mathrm{X} \times \mathbf{P}^1 \to \mathrm{X}$ is the projection, $p_*(\{ t \} \mathrm{div}(\tilde{f}))$ (p_* is defined as in [Gi 1] and commutes with d) meets Z almost properly and

$$d(p_*(\sum_{\mathrm{W}} \{ t, f_{\mathrm{W}} \})) = f - (f_\infty + p_*(\{ t \} \mathrm{div}(\tilde{f}))).$$

Therefore $g = f_\infty + p_*(\{ t \} \mathrm{div}(\tilde{f}))$ satisfies the conditions of the lemma.

Remarks. — 1) If $f: \mathrm{X} \to \mathrm{Y}$ is a morphism between quasi-projective nonsingular varieties over a field, let $\mathrm{R}_p^{p-1}(\mathrm{X})_f$ denote the group of K_1-chains $h = \sum_{\mathrm{W}} [h_{\mathrm{W}}]$ for which, for all W, $f^{-1}(\mathrm{div}(h_{\mathrm{W}}))$ has codimension at least $(p-1)$. Then taking 4.2.5 and lemma 4.2.6 together, we see that there is a well defined map

$$f^* : \mathrm{R}_p^{p-1}(\mathrm{Y})_f / d\mathrm{R}_p^{p-2}(\mathrm{Y}) \to \mathrm{R}_p^{p-1}(\mathrm{X}) / d\mathrm{R}_p^{p-2}(\mathrm{X}).$$

2) This lemma and its proof are closely related to Corollary 2.5 of [Bl 3].

4.2.7. *Proof of Theorem* **4.2.3.** — Suppose that X is a nonsingular arithmetic variety over A, with quasi-projective generic fibre X_{F}. If $\alpha \in \widehat{\mathrm{CH}}^p(\mathrm{X})$, $\beta \in \widehat{\mathrm{CH}}^q(\mathrm{X})$, then by the moving lemma for cycles (Section 3, Theorem, [R]), applied to X_{F}, we can represent α and β by cycles $(\mathrm{Y}, g_{\mathrm{Y}})$ and $(\mathrm{Z}, g_{\mathrm{Z}})$ (respectively), which meet properly in the generic fibre X_{F}. We want to set $\alpha . \beta$ equal to the class represented by

$(Y, g_Y).(Z, g_Z)$; therefore we must show that this class is independent of the representatives of α and β chosen. Using the moving lemma again, together with the commutativity of the product (lemma 4.2.4), it suffices to show that if (Y', g'_Y) is another representative of α, such that Y' meets Z properly in X_F, then

$$(Y, g_Y).(Z, g_Y) - (Y', g'_Y)(Z, g_Y) \in \hat{R}^{p+q}(X)_Q.$$

By assumption, $(Y, g_Y) - (Y', g'_Y) = \widehat{\mathrm{div}}(f)$ for some K_1-chain $f = \Sigma[f_W] \in R_p^{p-1}(X)$. While f itself may not meet Z almost properly in X_F, we know by Lemma 4.2.6 that there exists an element $\gamma \in R_p^{p-2}(X_F) \subset R_p^{p-2}(X)$ such that, if $h = f + d\gamma$, h meets Z almost properly in X_F. Since $\mathrm{div} \circ d = 0$, we see that $\mathrm{div}(h) = \mathrm{div}(f)$, while Theorem 3.3.5 tells us that $\log |h|^2 = \log |f|^2 \in \tilde{\mathscr{D}}^{p-1,p-1}(X)$ and hence $\widehat{\mathrm{div}}(h) = \widehat{\mathrm{div}}(f)$. Now by Lemma 4.2.5

$$\begin{aligned}\widehat{\mathrm{div}}(h.Z) &= (\mathrm{div}(h).Z, -\log |h|^2 \wedge \delta_Z)\\ &= \widehat{\mathrm{div}}(h).(Z, g_Z)\\ &= \widehat{\mathrm{div}}(f).(Z, g_Z).\end{aligned}$$

This completes the proof of (i). To prove (ii), first observe that, by Lemma 4.2.4, the product is commutative in general and associative when X_F is projective, and has $(X, 0)$ as unit. To check associativity when X_F is quasi-projective, observe that by 2.1.3 we can explicitly compute the change in the cup product resulting from a change in Green currents, hence given three classes α, β, and γ in $\widehat{CH}(X)$, we may check the associativity of the product $\alpha\beta\gamma$ by replacing the three classes by three classes having the same image in the ordinary Chow groups of X; i.e. we may make arbitrary choices of Green currents and we can change algebraic cycles by rational equivalences. Using resolution of singularities on the generic fibre we can construct a regular scheme $\bar{X}$, containing X as an open subscheme, such that $\bar{X}_F$ is projective. Let Y, Z, W be three irreducible cycles on X meeting properly on X_F. Choose Green currents $g_{\bar{Y}}$, $g_{\bar{Z}}$, $g_{\bar{W}}$ for their closure in $\bar{X}_F$, with restriction g_Y, g_Z, g_W to X_F. The product of the three classes $(\bar{Y}, g_{\bar{Y}})$, $(\bar{Z}, g_{\bar{Z}})$, $(\bar{W}, g_{\bar{W}})$ in $\widehat{CH}(\bar{X})_Q$ is associative, hence the product of their restrictions to X is too.

Turning to (iii), we first remark that if D is a divisor on X, then given any finite set of points $\{y_1, \ldots, y_n\} \subset X$, D is rationally equivalent to a divisor D' such that for all i, $x_i \notin |D'|$ ($|D'|$ = the support of D'); this is because any regular semi-local ring is a unique factorization domain. Hence given any finite set $\{Y_1, \ldots, Y_n\}$ of subschemes of X, D is a rationally equivalent to a divisor D' which meets all the Y_i properly, i.e. so that $Y_i \not\subseteq D'$. Hence given $\alpha \in \widehat{CH}^1(X)$ and $\beta \in CH^q(X)$ we can choose representatives $\alpha = (D, g_D)$ and $\beta = \sum_i (Y_i, g_i)$ such that Y_i is integral for all i and D meets Y_i properly for all i, so we can define

$$\alpha.\beta = (\sum_i D.Y_i, \sum_i g_D * g_i).$$

If D_1 and D_2 are two rationally equivalent divisors meeting a codimension q subscheme $Y \subset X$ properly, then $D_1 - D_2 = \mathrm{div}(f)$ and $f|Y$ is well defined, hence by Lemma 4.2.5

$$\widehat{\mathrm{div}}(f).(Y, g_Y) = \widehat{\mathrm{div}}(f|_Y)$$

for any choice of g_Y, hence $\alpha.\beta$ is independent of the choice of representative of α. If W is an integral subscheme of codimension $q-1$ of X and $g \in k(W)^*$, then if $\beta = \widehat{\mathrm{div}}(g)$, we can choose a representative (D, g_D) for $\alpha \in \widehat{CH}^1(X)$ such that D meets W and $\mathrm{div}(g)$ properly. Then by Lemma 4.2.5 $(D, g_D).\widehat{\mathrm{div}}(g)$ represents zero in $\widehat{CH}^{q+1}(X)$. Therefore $\widehat{CH}^1(X) \otimes \widehat{CH}^q(X) \to \widehat{CH}^{q+1}(X)$ is well defined. Similarly there is a well defined product $\widehat{CH}^p(X) \otimes \widehat{CH}^1(X) \to \widehat{CH}^{p+1}(X)$.

To prove (4.2.3.4) suppose that $\alpha = (D, g_D)$, $\beta = (E, g_E)$ and $\gamma = (Z, g_Z)$. Since Cartier divisors can be moved in their rational equivalence class until they meet any cycle properly, we can assume that D, E and Z meet properly. Then, when X_F is projective,

$$\begin{aligned}\alpha(\beta\gamma) &= ([D].([E].[Z]), g_D * (g_E * g_Z)) \\ &= ([D].([E].[Z]), g_E * (g_D * g_Z))\end{aligned}$$

by the associativity and commutativity of the $*$-product. Since the intersection product for cycles which meet properly is also associative and commutative ([Se] V) we obtain

$$\alpha(\beta\gamma) = ([E].([D].[Z]), g_E * (g_D * g_Z)) = \beta(\alpha\gamma).$$

The general case of (4.2.3.4) follows as in the proof of (ii). The proof of (4.2.3.3), which is similar, is left to the reader.

Remark. — In [G-S 4], Theorem 7.3.4, the group $\widehat{CH}^p(X)_{\mathbf{Q}}$ is shown to be isomorphic to the weight p part $\hat{K}_0(X)^{(p)}$ of the Grothendieck group $\hat{K}_0(X)$ of Hermitian vector bundles on X. This identification uses 4.2.3 (iii) (to define characteristic classes) but not 4.2.3 (i). Since $\hat{K}_0(X)^{(p)}$ has a graded ring structure coming from the tensor product of Hermitian vector bundles, this gives another proof of 4.2.3 (i) and 4.2.3 (ii).

4.2.9. *Theorem. — Let* X *be as in Theorem* 4.2.3, *then with respect to the ring structure constructed above, the maps*

$$\zeta : \widehat{CH}^*(X)_{\mathbf{Q}} \to CH^*(X)_{\mathbf{Q}}$$

and

$$\omega : \widehat{CH}^*(X)_{\mathbf{Q}} \to A(X_{\mathbf{R}}) = \bigoplus_{p \geq 0} A^{p,p}(X_{\mathbf{R}})$$

defined in 3.3.4 *are both ring homomorphisms. In addition, the product* 4.2.3.2 *is compatible with* ζ *and* ω.

Proof. — That ζ is a ring homomorphism is essentially a tautology, for if $\alpha = (Y, g_Y)$ and $\beta = (Z, g_Z)$ then $\alpha.\beta = ([Y].[Z], g_Y * g_Z)$ with $[Y].[Z]$ defined using the intersection product $CH^p_Y(X) \otimes CH^q_Z(X) \to CH^{p+q}_{Y \cap Z}(X)_Q$. Turning to ω, observe that

$$\begin{aligned}\omega(\alpha.\beta) &= \omega(([Y].[Z], g_Y * g_Z)) \\ &= dd^c(g_Y * g_Z) - \delta_{[Y].[Z]} \\ &= \omega_Y \wedge \omega_Z \text{ by Theorem } 2.1.4 \\ &= \omega(\alpha) \wedge \omega(\beta).\end{aligned}$$

That the product $\widehat{CH}^1(X) \otimes \widehat{CH}^q(X) \to \widehat{CH}^{q+1}(X)$ of (4.2.3.2) is compatible with ω follows from the discussion above, since $A(X_R)$ is a $\mathbf{Q}$-vector space. Compatibility with ζ is again implicit in the construction of the product. An immediate consequence of the theorem is

Corollary. — (i) $\widehat{CH}^*(X)_{0,Q} = \mathrm{Ker}(\omega)$ *is an ideal in* $\widehat{CH}(X)_Q$.

(ii) $(\mathrm{Ker}\,\zeta)_Q = (\widetilde{A}(X_R)/\mathrm{Image}(\rho))_Q$ *is an ideal in* $\widehat{CH}(X)_Q$.

4.2.10. If X is projective we can in fact do better than Corollary 4.2.9; we consider $\mathrm{Ker}(\omega)$ first, and recall from 3.3.5 that $CH^*(X)_0$ is the subgroup of $CH^*(X)$ consisting of cycles homologically equivalent to zero in X_∞, so that ζ induces a surjective map from $\widehat{CH}^*(X)_0$ to $CH^*(X)_0$.

Theorem. — *Let* X *be as in Theorem* 4.2.3 *and suppose also that the generic fibre* X_F *is projective. Then the* $\widehat{CH}^*(X)_Q$ *module structure on the ideal* $\widehat{CH}^*(X)_{0,Q}$ *is induced by a* $CH^*(X)_Q$ *module structure*, i.e., *we have a factorization*

$$\begin{array}{ccc}\widehat{CH}^p(X)_0 \otimes \widehat{CH}^q(X) & \longrightarrow & \widehat{CH}^{p+q}(X)_{0,Q} \\ \downarrow{\scriptstyle \mathrm{Id}\otimes\zeta} & \nearrow & \\ \widehat{CH}^p(X)_0 \otimes CH^q(X) & & \end{array}$$

Proof. — It suffices to observe for $(Y, g_Y) \in \widehat{CH}^p(X)_0$ and $(Z, g_Z) \in \widehat{CH}^q(X)$, that $g_Y * g_Z = g_Y \wedge \delta_Z + \omega_Y \wedge g_Z = g_Y \wedge \delta_Z$ is independent of g_Z.

Corollary. — *If* X *has projective generic fibre, the product of Theorem* 4.2.3, *restricted to* $\widehat{CH}^*(X)_0$, *factors through* $CH^*(X)_0$, i.e. *we have a well defined product:*

$$CH^p(X)_0 \otimes CH^q(X)_0 \to \widehat{CH}^{p+q}(X)_0.$$

Proof. — This follows from the fact that the product of 4.2.3 is commutative.

4.2.11. Turning to $\mathrm{Ker}(\zeta)_Q$, observe that its $\widehat{CH}^*(X)_Q$ module structure is induced by a $\widehat{CH}^*(X)_Q$ module structure on $\widetilde{A}(X_R)$; if $\alpha \in \widehat{CH}^p(X)$ and $\theta \in \widetilde{A}^{q,q}(X_R)$, $\alpha.\theta = \omega(\alpha) \wedge \theta$. Note that $\mathrm{Ker}(\zeta)$ is not a square zero ideal, but rather its product is induced by the non-unitary associative ring structure on $\widetilde{A}(X_R)$ defined by the product $\alpha.\beta = (dd^c\,\alpha) \wedge \beta$. Note that this product is both commutative and well defined because

$$dd^c\,\alpha \wedge \beta - \alpha \wedge dd^c\,\beta = (i/2\pi)\,(\partial(\bar\partial\alpha \wedge \beta) + \bar\partial(\alpha \wedge \partial\beta)).$$

However, if X_F is projective and we look at the subgroups $(H^{p,p}(X_{\mathbf{R}})/\mathrm{Image}(\rho)) \subset \mathrm{Ker}(\zeta)$, we have:

Corollary. — The quotient $\bigoplus_{p \geqslant 0} (H^{p,p}(X_{\mathbf{R}})/\mathrm{Image}(\rho))_{\mathbf{Q}}$ *is a square zero ideal in* $\widehat{CH}^*(X)_{\mathbf{Q}}$.

Proof. — The group in question is the intersection of the ideals $\mathrm{Ker}(\zeta)$ and $\mathrm{Ker}(\omega)$, and hence is an ideal; the vanishing of the product follows immediately from Corollary 4.2.10.

Remark. — Let $H^*(X_{\mathbf{R}})_{\mathrm{alg}}$ be the subring of $\bigoplus_{p \geqslant 0} H^{p,p}(X_{\mathbf{R}})$ consisting of algebraic cohomology classes; i.e. it is the image of the cycle class map $CH^*(X) \to H^{*,*}(X_{\mathbf{R}})$. Then the $\widehat{CH}^*(X)_{\mathbf{Q}}$ module structure on $\bigoplus_{p \geqslant 0} (H^{p,p}(X_{\mathbf{R}})/\mathrm{Im}(\rho))$ is induced by an $H^*(X_{\mathbf{R}})_{\mathrm{alg}}$ module structure.

4.3. Intersection numbers

4.3.1. Let $\pi : X \to S$ be a proper map between nonsingular arithmetic varieties over an arithmetic ring A such that the map $\pi : X_F \to S_F$ (F is the fraction field of A) on generic fibres is smooth. If $d = \dim X - \dim S$ and $p + q = d + 1$, then we can construct a pairing

$$\textbf{(4.3.1)} \qquad \langle\ ,\ \rangle : \widehat{CH}^p(X) \otimes \widehat{CH}^q(X) \to \widehat{CH}^1(S)_{\mathbf{Q}}$$

by composing the product (4.2.3.1) with the direct image map

$$\pi_* : \widehat{CH}^{d+1}(X)_{\mathbf{Q}} \to \widehat{CH}^1(S)_{\mathbf{Q}}.$$

In particular, if X is a projective nonsingular arithmetic variety over $\mathbf{Z}$, of dimension $d + 1$ (so $\dim X_{\mathbf{Q}} = d$), there is a pairing

$$\widehat{CH}^p(X) \otimes \widehat{CH}^q(X) \to \mathbf{R}$$

since $\widehat{CH}^1(\mathrm{Spec}(\mathbf{Z})) \simeq \mathbf{R}$ by 3.4.3. However if we have an arithmetic variety over a more general base ring, for example the ring of integers in a number field, this construction does not provide the maximum amount of information possible, since it neglects torsion. Note however that the pairing above is enough to recover the intersection pairing of Arakelov ([Ar 1]) as well as its generalization to higher dimensions; see 5.1.4 below, [Be 1] and [G-S 1].

4.3.2. *Theorem. — Let* $\pi : X \to S$ *be a proper map between equidimensional nonsingular arithmetic varieties over an arithmetic ring* A *such that the map* $\pi : X_F \to S_F$ *on generic fibres is smooth. If* $d = \dim X - \dim S$ *and* $p + q = d + 1$, *there is a bi-additive pairing*

$$\langle\ ,\ \rangle : \widehat{CH}^p(X) \times \widehat{CH}^q(X) \to \widehat{CH}^1(S)$$

which induces the pairing (4.3.1).

Proof. — If $\alpha \in \widehat{CH}^p(X)$ and $\beta \in \widehat{CH}^q(X)$, we can use the moving lemma to choose representatives (Y, g_Y) and (Z, g_Z) for α and β, such that Y and Z meet properly in the generic fibre X_F. Let $|Y|$ and $|Z|$ be the supports in X of Y and Z respectively. Then $T = \pi(|Y| \cap |Z|) \subset S$ is a closed subset of codimension ≥ 1 in S. Since we want $\langle\ ,\ \rangle$ to be bi-additive, to define $\langle (Y, g_Y), (Z, g_Z)\rangle$ it is enough to consider the case in which Y and Z are "prime" cycles i.e. integral closed subschemes of X. The coherent sheaves $\mathcal{O}_Y$ and $\mathcal{O}_Z$ determine classes $[\mathcal{O}_Y] \in K_0^Y(X)$ and $[\mathcal{O}_Z] \in K_0^Z(X)$, and taking their cup product we obtain a class $[\mathcal{O}_Y] \cup [\mathcal{O}_Z] \in K_0^{Y \cap Z}(X) \simeq K'_0(Y \cap Z)$; see [G-S 3] Chapter 1 for the definition of K_0 with supports and the associated cup products. Since $\pi : X \to S$ is proper, it induces a proper map $Y \cap Z \to T$ and hence a direct image map $\pi_* : K_0^{Y\cap Z}(X) \simeq K'_0(Y \cap Z) \to K'_0(T) \simeq K_0^T(S)$. Therefore we have a class $\pi_*([\mathcal{O}_Y] \cup [\mathcal{O}_Z])$ in $K_0^T(S)$. To pass from this class to a cycle, we need:

4.3.3. *Lemma.* — *Let* T *be a closed subset, of codimension greater than or equal to n, of a Noetherian regular scheme* S. *Then there is a natural map* $\zeta_T^n : K_0^T(S) \to Z_T^n(S)$, *which, if* $W \subset T$ *is an integral subscheme of codimension n in* S, *sends* $[\mathcal{O}_W]$ *to* [W].

Proof. — Let $\mathbf{M}_T(S)$ be the category of coherent sheaves of $\mathcal{O}_S$ modules supported on T. For each point $t \in S^{(n)} \cap T$, the functor which sends $\mathcal{F}$ to its stalk $\mathcal{F}_t$ at t is an exact functor from $\mathbf{M}_T(S)$ to the category of $\mathcal{O}_{S,t}$ modules of finite length. But for any local ring R, K_0 of the category of R modules of finite length is isomorphic to $\mathbf{Z}$, the isomorphism being given by the map sending [M] to the length $\ell(M)$ of M. Hence we have a map

$$K_0^T(S) \to Z_T^n(S) \simeq \bigoplus_{t \in S^{(n)} \cap T} \mathbf{Z}$$
$$\mathcal{F} \mapsto \bigoplus_t \ell(\mathcal{F}_t).$$

Returning to the proof of the theorem, we can now define

$$\langle (Y, g_Y), (Z, g_Z)\rangle = (\zeta_T^1(\pi_*([\mathcal{O}_Y] \cup [\mathcal{O}_Z])), \pi_*(g_Y * g_Z))$$

and having defined the intersection pairing for prime cycles Y and Z, we can extend to arbitrary cycles by bi-additivity. We must now prove that $\langle \alpha, \beta \rangle$ is independent of the choices of representatives of α and β.

4.3.4. *Lemma.* — *Suppose that* $W \subset X$ *is an integral subscheme of codimension* $p - 1$, *that* $f \in k(W)^*$ *and that* $Z \subset X$ *is an integral subscheme of* X *meeting* div(f) *properly in the generic fibre* X_F. *Then* $\langle \widehat{\mathrm{div}}(f), (Z, g_Z)\rangle = 0$ *for any choice of Green current* g_Z *for* Z.

Proof. — First we must show that if $\mathrm{div}(f) = \sum_{i=1}^{k} n_i [Y_i]$, then $\sum n_i \zeta_T^1(\pi_*([\mathcal{O}_{Y_i}] \cup [\mathcal{O}_Z]))$ is a principal divisor. Now f induces a class $\{f\} \in K'_1(W - Y)$ $(Y = \mathrm{Support}(\mathrm{div}(f)))$ such that in the localization sequence

$$\ldots \to K'_1(W) \to K'_1(W - Y) \xrightarrow{\partial} K'_0(Y) \to K'_0(W) \to \ldots$$

we have $\partial\{f\} = \sum_{i=1}^{k} n_i[\mathcal{O}_{Y_i}] + \lambda \in K'_0(Y) \simeq K_0^Y(X)$, with $\lambda \in F^{p+1} K_0^Y(X)$; (recall that $F^{\bullet} K_0^Y(X)$ is the filtration by codimension of support, [G-S 3] Chapter 5). Then

$$\partial\{f\} \cup [\mathcal{O}_Z] = \sum_{i=1}^{k} n_i([\mathcal{O}_{Y_i}] \cup [\mathcal{O}_Z]) + (\lambda \cup [\mathcal{O}_Z])$$

so $$\pi_*(\Sigma n_i([\mathcal{O}_{Y_i}] \cup [\mathcal{O}_Z])) = \pi_*((\partial\{f\}) \cup [\mathcal{O}_Z]) - \pi_*(\lambda \cup [\mathcal{O}_Z]).$$

Next, we make two observations.

4.3.5. (I) $\zeta_T^1(\pi_*(\lambda \cup [\mathcal{O}_Z])) = 0$ in $Z^1(S)$.

Proof of I. Since $Z^1(S)$ is torsion free, it is enough to prove that this equation holds in $Z^1(S)_{\mathbf{Q}}$; from the exact sequence (Lemma 5.2 of [G-S 3])

$$F^2 K_0^T(S) \to K_0^T(S) \to \Big(\bigoplus_{t \in S^{(1)} \cap T} K_0^t(\mathrm{Spec}(\mathcal{O}_{S,t})) \simeq Z_T^1(S)\Big) \to 0$$

we see that it is enough to show that $\pi_*(\lambda \cup [\mathcal{O}_Z]) \in F^2 K_0^T(S)_{\mathbf{Q}}$. By Proposition 5.5, *ibidem*, $\lambda \cup [\mathcal{O}_Z] \in F^{p+q+1} K_0^{Y \cap Z}(X)_{\mathbf{Q}}$; but $\pi : X \to Y$ is of relative dimension $d = p + q - 1$, hence $\pi_* F^{p+q+1} K_0^{Y \cap Z}(X) \subset F^2 K_0^T(X)$ and we are done.

4.3.6. (II) The divisor $\zeta_T^1(\pi_*(\partial\{f\} \cup [\mathcal{O}_Z]))$ is principal.

Proof of II. There is a commutative diagram, induced by exact functors between categories of coherent and locally free sheaves

$$\begin{array}{ccccc}
 & & K'_1(W - Y) \otimes K_0^Z(W) & \xrightarrow{\partial \otimes \mathrm{Id}} & K'_0(Y) \otimes K_0^Z(X) \\
 & & \cap \downarrow & & \downarrow \cap \\
 & & K'_1(W - Y \cap Z) & \xrightarrow{\partial} & K'_0(Y \cap Z) \\
 & \overset{u^*}{\nearrow} & \downarrow & & \downarrow j_* \\
K'_1(W \cap Z - \pi^{-1}(T)) & & & & \\
 & \underset{i_*}{\searrow} & \downarrow & & \downarrow \\
 & & K'_1(X - \pi^{-1}(T)) & \xrightarrow{\partial} & K'_0(\pi^{-1}(T)) \\
 & & \pi_* \downarrow & & \downarrow \pi_* \\
 & & K'_1(S - T) & \xrightarrow{\partial} & K'_0(T) \simeq K_0^T(S) \\
 & & v^* \downarrow & & \downarrow \zeta_T^1 \\
 & & k(S)^* & \xrightarrow{\mathrm{div}} & Z^1(S)
\end{array}$$

in which

$$\begin{aligned} j &: Y \cap Z \to \pi^{-1}(T), \\ u &: (W \cap Z - \pi^{-1}(T)) \to ((W - Y \cap Z)), \\ i &: ((W \cap Z) - \pi^{-1}(T)) \to X - \pi^{-1}(T), \\ v &: \mathrm{Spec}(k(S)) \to S - T, \end{aligned}$$

are all the obvious inclusions. Hence

$$\zeta^1_T(\pi_*(\partial\{f\} \cap [\mathcal{O}_Z])) = \mathrm{div}(v^* \pi_*(\{f\} \cap [\mathcal{O}_Z]))$$

and in particular it is principal.

We want to compute the rational function $v^* \pi_*(\{f\} \cap [\mathcal{O}_Z])$ more explicitly. Since the generic point s of S is contained in the generic fibre S_F of S, we shall work entirely in the generic fibres X_F and S_F. $f \in k(W)^*$ defines a K_1-chain $[f]$ which meets Z almost properly in X_F. Following 4.2.5, write $W_F \cap Z_F = U \cup V$ with U of codimension $p + q - 1$ in X_F and $V \cap \mathrm{div}(f) = \emptyset$. If we write $[W].[Z] = \sum_{i=1}^{\ell} n_i [U_i] + b$, with $U_1, \ldots, U_\ell$ the irreducible components of U, and b a cycle class supported in V, then (as *ibidem*) we have an equation of K_1-chains on X_F:

$$[f].[Z] = \sum_{i=1}^{\ell} n_i [f|_{U_i}] + [f]\, b$$

with $[f]\, b \in CH_V^{p+q,\, p+q-1}(X_F)$. If $W \cap Z \cap \pi^{-1}(s) = \{u_1, \ldots, u_m\} \cup V_s$ (note that $m \leqslant \ell$ since $\pi(U_i)$ may be a proper subscheme of S, and also that $u_i \notin V_s$ for all i) then:

$$v^* \pi_*([f].[Z]) = \left(\prod_{i=1}^{m} Nm_{k(u_i)/k(s)}(f(u_i)^{n_i})\right) . v^* \pi_*([f]\, b).$$

Hence

$$\log |v^* \pi_*([f].[Z])|^2 = \sum_{i=1}^{m} n_i \log |Nm_{k(u_i)/k(s)}(f(u_i))|^2 - \rho(\pi_*([f]\, b)).$$

By Lemma 4.2.5, if g_Z is a Green current for Z, then

$$\begin{aligned} \log |f|^2 * g_Z &= \log |f|^2 \wedge \delta_Z \\ &= \sum_{i=1}^{\ell} n_i \log |f|_{U_i}|^2 - \rho([f].b). \end{aligned}$$

If $u_i \notin \pi^{-1}(s)$ then, by a dimension argument, we see that $\pi_* \log |f|_{U_i}| = 0$; hence

$$\pi_*(\log |f|^2 * g_Z) = \sum_{i=1}^{m} n_i\, \pi_* \log |f|_{U_i}|^2 - \pi_*\, \rho([f]\, b).$$

Since $\pi : U_i \to S$ is generically finite for $i = 1, \ldots, m$,

$$\pi_* \log |f|_{U_i}|^2 = \log |Nm_{k(u_i)/k(s)}(f|_{U_i})|^2$$

(see the proof of 3.6.1), while by the Riemann-Roch theorem for the Beilinson regulator [Be 1], [Gi 1])

$$\pi_*\, \rho([f].b) = \rho(\pi_*([f].b)).$$

Hence $\pi_*(\log |f|^2 * g_Z) = \log |\pi_*([f]\cdot[Z])|^2$
$= \log |v^* \pi_*(\{f\} \cap [\mathcal{O}_Z])|^2.$

and so $\pi_*(\widehat{\operatorname{div}}(f).(Z, g_Z)) = \widehat{\operatorname{div}}(v^* \pi_*(\{f\} \cap [\mathcal{O}_Z]))$. Finally, to show that $\langle\ ,\ \rangle$ preserves rational equivalence, we use the moving lemma for K_1-chains (4.2.6) following the same pattern as in 4.2.7.

4.3.8. *Remarks and examples.* — (i) The proof of this theorem is based on Beilinson's proof that his height pairing is compatible with rational equivalence ([Be 1]). The result itself generalizes both the height pairing of Beilinson (see (iii) below) and the Arakelov-Deligne intersection product, which is the case of $d = 1$. Note that Arakelov considered only admissible Green currents (5.1 below) and Y equal to the spectrum of a ring of integers; the extension to arbitrary Green currents for divisors and arbitrary Y is in Deligne's paper [De 2]. However Deligne's construction gives more than just the intersection pairing at the level of $\widehat{CH}^*$, for to every pair of metrized line bundles on X, he associates a metrized line bundle on Y, not just an isomorphism class of such line bundles. In the situation of Theorem (4.3.2) above, one can perform a related construction. To every $(Y, g_Y) \in \hat{Z}^p(X)$ and $(Z, g_Z) \in \hat{Z}^q(X)$ which intersect properly on X_F, with $p + q = d + 1$, consider the line bundle, unique up to canonical isomorphism, $\mathcal{L} = \det(R\pi_*(\mathcal{O}_Y \otimes^L_{\mathcal{O}_X} \mathcal{O}_Z))$; see [K-M], [De 2] for details on the functor det. Since $T = \pi(Y \cap Z) \neq S$, the line bundle $\mathcal{L}$ has a canonical nonvanishing section σ on $S - T$. Then it is not difficult to show that there is a unique C^∞ metric $\|\ \|$ on $\mathcal{L}$ such that $-\log \|\sigma\|^2 = \pi_*(g_Y * g_Z)$. Then the following equality holds in $\hat{Z}^1(X)$

$$(\operatorname{div}(\sigma), -\log \|\sigma\|^2) = (\zeta^1_T(\pi_*([\mathcal{O}_Y] \cup [\mathcal{O}_Z])), \pi_*(g_Y * g_Z)).$$

(ii) In the case when $Y = \operatorname{Spec}(\mathcal{O}_F)$ for $\mathcal{O}_F$ the ring of integers in a number field, so that X is a nonsingular projective arithmetic variety of dimension $d + 1$, one obtains the pairing, described in [Gi 3]

$$\widehat{CH}^p(X) \otimes \widehat{CH}^q(X) \to F^* \backslash J(F)/U_F$$

($J(F)$ is the idele group of F, $U_F \subset J(F)$ the maximal compact subgroup).

(iii) Following the method of 4.2.10 we obtain, in the situation of the theorem, a pairing, for $p + q = d + 1$,

$$\langle\ ,\ \rangle : CH^p(X)_0 \otimes CH^q(X)_0 \to \widehat{CH}^1(Y).$$

In particular, if X is a nonsingular projective arithmetic variety over **Z**, one obtains a pairing, for $p + q = \dim(X_{\mathbf{Q}}) + 1$,

$$\langle\ ,\ \rangle\ CH^p(X)_0 \otimes CH^q(X)_0 \to \mathbf{R}.$$

This is the Beilinson height pairing ([Be 1], [Be 2]), which is, presumably, the same as the height pairing defined by Bloch ([Bl 1]), and which leads to a generalization of the Neron-Tate height pairing.

(iv) One can give a more explicit formula for the cycle $\zeta^1(\pi_*([\mathcal{O}_Y] \cup [\mathcal{O}_Z]))$; it is equal to $\Sigma n_D[D]$, the sum being over all prime divisors on S, with

$$n_D = \sum_{i,j} (-1)^{i+j} \ell_{\mathcal{O}_{S,d}}((R^i \pi_* \mathcal{T}or_j^{\mathcal{O}_X}(\mathcal{O}_Y, \mathcal{O}_Z))_d),$$

d being the generic point of D. This is because the cup product $K_0^Y(X) \otimes K_0^Z(X)$, for X regular, sends $[\mathcal{O}_Y] \otimes [\mathcal{O}_Z]$ to $\sum_{j \geq 0} (-1)^j [\mathcal{T}or_j^{\mathcal{O}_X}(\mathcal{O}_Y, \mathcal{O}_Z)]$. (See [G-S 3]).

(v) In the case $Y = \mathrm{Spec}(\mathbf{Z})$, we can simplify the expression for the intersection pairing even further. Recall that $\widehat{CH}^1(\mathrm{Spec}(\mathbf{Z})) \simeq \mathbf{R}$, the isomorphism being given by

$$(\Sigma\, n_p[p], g) \mapsto \sum_p n_p \log p + \frac{1}{2} g.$$

Then if $\alpha = (Y, g_Y) \in \widehat{CH}^p(X)$, $\beta = (Z, g_Z) \in \widehat{CH}^q(X)$, with Y and Z prime cycles intersecting properly, i.e. not at all, on $X_{\mathbf{Q}}$, we have:

$$\langle \alpha, \beta \rangle = \langle Y, Z \rangle_f + \frac{1}{2} \int_{X(\mathbf{C})} g_Y * g_Z$$

with $$\langle Y, Z \rangle = \sum_{i,j \geq 0} (-1)^{i+j} \log \# H^i(X, \mathcal{T}or_j^{\mathcal{O}_X}(\mathcal{O}_Y, \mathcal{O}_Z)).$$

Here, if A is a finite set, $\# A$ is the cardinality of A.

(vi) With a little bit more work, one could show, if $p_1, \dots, p_k$ are non-negative integers satisfying $p_1 + \dots + p_k = d + 1$, that there is a k-fold product:

$$\langle, \dots, \rangle : \widehat{CH}^{p_1}(X) \otimes \dots \otimes \widehat{CH}^{p_k}(X) \to \widehat{CH}^1(S).$$

4.3.9. *Theorem. — Let* $\pi_1 : X \to S$, $\pi_2 : Y \to S$ *be a pair of maps satisfying the conditions of Theorem* 4.3.2, *and suppose that* $f : X \to Y$ *is a flat map, with* $\pi_1 = \pi_2 f$, *such that the induced map* $X_F \to Y_F$ *is smooth. If* $e = \dim X - \dim Y$, $d = \dim Y - \dim S$, $\alpha \in \widehat{CH}^p(Y)$, $\beta \in \widehat{CH}^q(X)$ *and* $p + q = d + e + 1$, *then*

$$\langle f^* \alpha, \beta \rangle = \langle \alpha, f_* \beta \rangle \in \widehat{CH}^1(S).$$

Proof. — Suppose that $\alpha = (Z, g_Z)$ and $\beta = (W, g_W)$ with Z and W prime cycles such that $f(W)$ and Z meet properly on Y_F. Then $[\mathcal{O}_{f^*Z}] = f^*[\mathcal{O}_Z]$ since f is flat, and

(4.3.9.1) $$\begin{aligned} \langle f^* \alpha, \beta \rangle &= (\zeta^1(\pi_{2*} f_*(f^*[\mathcal{O}_Z] \cup [\mathcal{O}_W])), \pi_{2*} f_*(f^* g_Z * g_W)) \\ &= (\zeta^1(\pi_{2*}([\mathcal{O}_Z] \cup f_*[\mathcal{O}_W])), \pi_{2*} f_*(f^* g_Z * g_W)) \end{aligned}$$

by the projection formula for K-theory. Since $f : X_\infty \to Y_\infty$ is smooth, it is straightforward to show, using the Fubini theorem, that $f_*(f^* \delta_Z \wedge g_W) = \delta_Z \wedge f_* g_W$ and that $f_*(f^* g_Z \wedge \omega_W) = g_Z \wedge f_* \omega_W$; hence $f_*(f^* g_Z * g_W) = g_Z * f^* g_W$. Combined with the equality above, 4.3.9.1, this gives

$$\langle f^* \alpha, \beta \rangle = \langle \alpha, f_* \beta \rangle$$

as desired.

4.4. Pull-backs

4.4.1. So far we have only discussed pull-backs for flat maps $f: X \to Y$ between arithmetic varieties. We turn now to arbitrary morphisms f between arithmetic varieties which are regular and quasi-projective over an arithmetic ring A (which we fix for the rest of the discussion). Any such morphism can be factored as the composition of a closed immersion $i: X \to \mathbf{P}^n_Y$, for some $n \geqslant 0$, and the projection map $\pi : \mathbf{P}^n_Y \to Y$. Since π is smooth, a pull-back map $\pi^* : \widehat{CH}^*(Y) \to \widehat{CH}^*(\mathbf{P}^n_Y)$ has already been defined. Since $\mathbf{P}^n_Y$ is regular, i is a regular immersion, so we shall now discuss pull-back maps for regular immersions.

Let $\mathbf{V}_A$ be the category of regular schemes which are flat and of finite type over a fixed excellent regular Noetherian domain A.

Theorem. — Let $f: X \to Y$ be a closed immersion between schemes in $\mathbf{V}_A$; note that f is necessarily regular. If $T \subset Y$ is a closed subset, there is a map

$$i^* : CH^n_T(Y) \to CH^n_{X \cap T}(X)$$

such that:

(i) *If $\alpha \in Z^n_T(Y)$ is an algebraic cycle supported on T which meets X properly, then $i^*(\alpha)$ is given by Serre's multiplicity formula* ([Fu] 20.4 *and* [Se] V).

(ii) *If $g : Y \to Z$ is another regular closed immersion with $Z \in \mathbf{V}_A$, and $S \subset Z$ is a closed subset, then*

$$f^* g^* = (gf)^* : CH^n_S(Z) \to CH^n_{S \cap X}(X).$$

(iii) *Suppose that $g : Y \to W$ is a flat map with $W \in \mathbf{V}_A$, so that if $S \subset W$ is a closed subset, $g^* : CH^n_S(W) \to CH^n_{g^{-1}(S)}(Y)$ is defined as in* 3.6.1. *Then if either $h = g.f$ is flat or g is smooth and h is a regular closed immersion,*

$$h^* = f^* g^* : CH^n_S(W) \to CH^n_{h^{-1}(S)}(X).$$

(iv) *Suppose that $g : W \to Y$ is flat with $W \in \mathbf{V}_A$ and form the Cartesian square:*

$$\begin{array}{ccc} W \times_Y X & \xrightarrow{\tilde{g}} & X \\ {\scriptstyle \tilde{f}}\downarrow & & \downarrow{\scriptstyle f} \\ W & \xrightarrow[g]{} & Y \end{array}$$

Observe that $\tilde{g}$ is flat and that $\tilde{f}$ is a regular immersion; suppose that the fibre product $W \times_Y X$ is in $\mathbf{V}_A$. Then if $S = \tilde{g}^{-1} f^{-1}(T)$, $\tilde{g}^* f^* = \tilde{f}^* g^* : CH^n_T(Y) \to CH^n_S(W \times_Y X)$.

(v) *Suppose that D is a Cartier divisor on Y, the support $|D|$ of which meets both T and X properly. Then if $\alpha \in CH^n_T(Y)$,*

$$f^*([D].\alpha) = [f^*(D)].f^* \alpha$$

in $CH^n_{|D| \cap X \cap T}(X)$. Here $f^*(D)$ is the pull-back divisor in the sense of [EGA] IV 21.4 *while the intersection product between cycles and divisors is that of* [Fu] 2.1 *and* 2.2.

(vi) *The map* $f^*: CH^n_T(Y) \to CH^n_{X\cap T}(X)$ *induces the same map* $CH^n_T(Y)_Q \to CH^n_T(X)_Q$ *as that induced by the isomorphism* $CH^*_T(X)_Q \simeq Gr^*_Y K^T_0(X)_Q$ *discussed in* [G-S 3] Ch. 5.

Proof. — Let us start by observing that any scheme in $\mathbf{V}_A$ is the disjoint union of its irreducible components. Therefore, since CH^* turns finite disjoint unions into direct sums, it is enough to prove the theorem for X and Y irreducible. Since a scheme in $\mathbf{V}_A$ is of finite type over a universally catenary ring, all schemes in V_A are themselves catenary. Let $a = \dim(X/S)$ and $b = \dim(Y/S)$ for $S = \operatorname{Spec} A$, be the relative dimensions of X and Y over S in the sense of [Fu] 20.1. Then since X and Y are flat over A, $CH^n_T(Y) \simeq CH_{b-n}(T/S)$ and $CH^n_{T\cap X}(X) \simeq CH_{a-n}((T\cap X)/S)$; here $CH_*(Z/S)$ denotes the Chow groups of Z graded by relative dimension over S rather than by codimension (in [Fu] 20.1, Fulton uses the notation $A_*(Z/S)$ for these groups). Therefore if $d = b - a$, it suffices to construct a map $f^*: CH_n(T/S) \to CH_{n-d}((T\cap X)/S)$ satisfying properties (i)-(vi). As discussed *ibidem*, the results of [Fu] § 2-§ 6 and § 7.1, which are stated for varieties over a field, carry through without change to varieties over A. In this form, the theorem is the conjunction of several results of *op. cit.* Specifically, (i) is example 7.1.2, (ii) is Theorem 6.5, (iii) is Proposition 6.5, (iv) is Theorem 6.2 *b*) and (v) is theorems 6-1 *c*) and 6.4. Finally, to prove (vi) we observe that the Gysin map f^* constructed in [Fu] is completely determined by two operations on Chow groups: pull-backs via flat maps and pull-backs via codimension one regular immersions. For both types of maps it is straightforward to check that the pull-backs defined in [Fu] agree with the pull-back on $CH^*(X)_Q$ defined in [G-S 3] using K-theory.

4.4.2. We must also understand how the map f^* defined in 4.4.1 behaves with respect to rational equivalence; to do this we will define a pull-back map on K_1-chains. If $\varphi \in R^{n-1}_n(Y)$ is a K_1-chain, and we write $Z = \operatorname{Support}(\varphi)$, $T = \operatorname{Support}(\operatorname{div}(\varphi))$ and $U = Z - T$, then φ determines, and is determined by, the class $\{\varphi\} \in CH^{n-1,n}_U(Y - T)$; furthermore, the image of $\{\varphi\}$ under the boundary map $\partial: CH^{n-1,n}_U(Y-T) \to CH^n_T(Y)$ is the class of $\operatorname{div}(\varphi)$. Here ∂ is the boundary map induced by the exact sequence of complexes:

$$0 \to R^*_n(Y)_T \to R^*_n(Y)_Z \to R^*_n(Y-T)_U \to 0,$$

in which we use the notation, for $V \subset W$,

$$R^*_n(W)_V = \operatorname{kernel}(R^*_n(W) \to R^*_n(W - V)),$$

and $CH^{i,j}_V(W) = H^i(R^*_j(W)_V)$. In [Gi 1] § 8 Gysin maps on the $CH^{i,j}$ were associated to regular immersions; we shall review the construction here for the case of a closed immersion $f: X \to Y$ in $\mathbf{V}_A$. We use, with some modifications, the deformation to the normal cone technique developed by Baum, Fulton and Macpherson [B-F-M] and Verdier [V]; see [Fu] Ch. 5-8. Let $\mathbf{A}^1_A = \operatorname{Spec} A[t]$, $\mathbf{A}^1_Y = T \times_A \mathbf{A}^1_A$; and let $\widetilde{M}$ be $\mathbf{A}^1_Y$ blown up along $X \times \{0\} \subset Y \times_A \mathbf{A}^1_A = \mathbf{A}^1_Y$. If $\widetilde{p}: \widetilde{M} \to \mathbf{A}^1_A$ is the projection map, $\widetilde{p}^{-1}(0)$ is the union of two divisors, $\mathbf{P}(N_X(Y) \oplus 1)$, which is the projective completion of the normal bundle to X in Y, and $\tilde{Y}$, which is Y blown up along X. We define $M = \tilde{M} - \tilde{Y}$; then as in [Fu] Ch. 5, one sees that:

(i) the projection map $p : M \to \mathbf{A}^1_A$ is flat,
(ii) $M_0 = p^{-1}(\{0\}) \simeq N_X(Y)$, the normal bundle of X in Y,
(iii) $M - M_0 \simeq Y \times \mathbf{G}_m = Y \times_A \operatorname{Spec}(A[t, t^{-1}])$,
(iv) $f: X \to Y$ induces a map $\tilde{f}: X \times \mathbf{A}^1 \to M$, such that for $t = 0$,

$$X \times \{0\} \to M_0 = N_X(Y)$$

is the zero section, while for $t \neq 0$, $X \times \mathbf{G}_m \to Y \times \mathbf{G}_m$ is the map induced by base change from $f: X \to Y$. Observe that t is a unit on $M - M_0$, so that it defines a class $\{t\}$ in $H^0(M - M_0, K_1(\mathcal{O}_M) = \mathcal{O}^*_M) = A^{0,1}(M - M_0)$ in the notation of [Gi 1] § 8. For the construction of $f^*\{\varphi\}$, it will initially be convenient to consider also the deformation to the normal cone construction for the inclusion of $X - (X \cap T)$ into $Y - T$. Let us write $f' : X' \to Y'$ for this, and M' for the corresponding scheme flat over $\mathbf{A}^1_A$; since $p' : Y' \times \mathbf{G}_m = M' - M'_0 \to Y'$ is flat, there is a pull-back map $p'^* : R^*_n(Y') \to R^*_n(Y' \times \mathbf{G}_m)$ and hence a map

$$p'^* : CH^{n-1,n}_U(Y') \to CH^{n-1,n}_{U \times \mathbf{G}_m}(M' - M'_0).$$

Associated to the short exact sequence

$$R^*_n(M'_0)\,[1] \to R^*_{n-1}(M') \to R^*_{n+1}(M' - M'_0)$$

we have a long exact sequence

$$CH^{i,j}_V(M'_0) \to CH^{i+1,j+1}_{\overline{U}}(M') \to CH^{i+1,j+1}_{U \times \mathbf{G}_m}(M' - M'_0) \xrightarrow{\partial} CH^{i+1,j}_V(M'_0) \to \dots$$

Here $\overline{U}$ is the Zariski closure of $U \times \mathbf{G}_m$ in M' and $V = \overline{U} \cap M'_0 = C_{U \cap X'}(U)$ is the normal cone of $U \cap X'$ in U (cf. [Fu] 4.2 and 5.2 and [Ha] Cor. 7.15). Next, recall from [Gi 1], pp. 276-277, that there is a product, for any Noetherian scheme S

$$\mathbf{R}^*_{m,S} \otimes K_n(\mathcal{O}_S) \to \mathbf{R}^*_{m+n,S}[-n]$$

where $\mathbf{R}^*_{m,S}$ is the complex of sheaves $U \mapsto R^*_m(U)$ on S. Hence there are products

$$H^0(M' - M'_0, K_1(\mathcal{O}_M) = \mathcal{O}^*_M) \otimes CH^{n-1,n}_{p^{-1}(U)} \to CH^{n-1,n+1}_{p^{-1}(U)}(M' - M'_0).$$

Putting the boundary map and the product together, we get a map (since $V \subset \pi^{-1}(X' \cap U)$, where $\pi : M'_0 \to X'$ is the projection)

$$\sigma_t : CH^{n-1,n}_U(Y') \to CH^{n-1,n}_V(M'_0) \to CH^{n-1,n}_{\pi^{-1}(X' \cap U)}(M_0)$$
$$\{\varphi\} \mapsto \partial(\{t\} * \{\varphi\}).$$

Finally, we observe that

$$\pi^* : CH^{n-1,n}_{X' \cap U}(X') \to CH^{n-1,n}_{\pi^{-1}(X' \cap U)}(M_0)$$

is an isomorphism ([Gi 1] Theorem 8.3). Composing this with the map σ_t above, we get a map

$$f^* = (\pi^*)^{-1}\,\sigma_t : \mathrm{CH}_{\mathrm{U}}^{n-1,n}(\mathrm{Y}') \to \mathrm{CH}_{\mathrm{U}\cap\mathrm{X}'}^{n-1,n}(\mathrm{X}').$$

Theorem. — Suppose that $f: \mathrm{X} \to \mathrm{Y}$ *is a closed immersion, which is necessarily regular, in* $\mathbf{V}_{\mathrm{A}}$. *If* $\varphi \in \mathrm{R}_n^{n-1}(\mathrm{Y})$ *is a* K_1*-chain with support* Z, *support* $(\mathrm{div}(\varphi)) = \mathrm{T}$ *and* $\mathrm{U} = \mathrm{Z} - \mathrm{T}$, *then:*

(i) $$\partial(f^*\{\varphi\}) = f^*(\partial\{\varphi\}) \in \mathrm{CH}_{\mathrm{X}\cap\mathrm{T}}^{n}(\mathrm{X})$$

where ∂ *is the boundary map* $\mathrm{CH}_{\mathrm{B}-\mathrm{C}}^{n,n-1}(\mathrm{A}-\mathrm{C}) \to \mathrm{CH}_{\mathrm{C}}^{n}(\mathrm{A})$ *for* $\mathrm{C} \subset \mathrm{B} \subset \mathrm{A}$.

(ii) *If* φ *meets* X *almost properly*, i.e. $\varphi = \Sigma\,[g_{\mathrm{W}}]$ *for* $g_{\mathrm{W}} \in k(\mathrm{W})^*$ *with* $\mathrm{W} \cap \mathrm{X} = \mathrm{S} \cup \mathrm{T}$, W *and* X *meeting properly at the generic points of* S, g_{W} *regular at the generic points of* S, *and* $\mathrm{div}(g_{\mathrm{W}}) \cap \mathrm{T}$ *empty, then*

$$f^*[g_{\mathrm{W}}] = \sum_i \mu_i [g_{\mathrm{W}}\,|\,_{\mathrm{S}_i}] + \sum_j g_{\mathrm{W}} \cdot \tau_j,$$

and $f^*(\varphi) = \Sigma_{\mathrm{W}} f^*[g_{\mathrm{W}}]$. *Here* μ_i *is the intersection multiplicity of* Z *and* W *at the generic point of the irreducible component* S_i *of* S, *and* τ_j *is the cycle class on the connected component* T_j *of* T *representing the component of* $f^*[\mathrm{W}]$ *in* $\mathrm{CH}_{\mathrm{T}_j}^{n-1}(\mathrm{X}) \subset \mathrm{CH}_{\mathrm{S}\cup\mathrm{T}}^{n-1}(\mathrm{X})$, *and the product* $g_{\mathrm{W}}\,\tau_j$ *is defined since* g_{W} *is a regular function on* T_j.

Before proving the theorem we need:

Lemma. — Let S *be a Noetherian scheme,* A *and* B *closed subsets of* S *and write* $\mathrm{C} = \mathrm{A} \cap \mathrm{B}$. *Let* $\mathrm{D} \subset \mathrm{S}$ *be any closed subset. Then the square*

$$\begin{array}{ccc}
\mathrm{CH}^{i,j}_{\mathrm{D}-\mathrm{D}\cap(\mathrm{A}\cup\mathrm{B})}(\mathrm{S}-(\mathrm{A}\cup\mathrm{B})) & \xrightarrow{\partial} & \mathrm{CH}^{i+1,j}_{(\mathrm{B}-\mathrm{C})\cap\mathrm{D}}(\mathrm{S}-\mathrm{A}) \\
\downarrow{\scriptstyle\partial} & & \downarrow{\scriptstyle\partial} \\
\mathrm{CH}^{i+1,j}_{\mathrm{D}\cap(\mathrm{A}-\mathrm{C})}(\mathrm{S}-\mathrm{B}) & \xrightarrow{\partial} & \mathrm{CH}^{i+2,j}_{\mathrm{C}\cap\mathrm{D}}(\mathrm{S})
\end{array}$$

(where the ∂*'s are the boundary maps in the appropriate long exact sequences) commutes up to a factor* -1.

Proof. — We have a diagram of complexes with exact rows and columns:

$$\begin{array}{ccccc}
\mathrm{R}_j^*(\mathrm{S})_{\mathrm{C}\cap\mathrm{D}} & \longrightarrow & \mathrm{R}_j^*(\mathrm{S})_{\mathrm{A}\cap\mathrm{D}} & \longrightarrow & \mathrm{R}_j^*(\mathrm{S}-\mathrm{B})_{\mathrm{A}-\mathrm{C}\cap\mathrm{D}} \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{R}_j^*(\mathrm{S})_{\mathrm{B}\cap\mathrm{D}} & \longrightarrow & \mathrm{R}_j^*(\mathrm{S})_{\mathrm{D}} & \longrightarrow & \mathrm{R}_j^*(\mathrm{S}-\mathrm{B})_{\mathrm{D}\cap(\mathrm{S}-\mathrm{B})} \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{R}_j^*(\mathrm{S})_{(\mathrm{B}-\mathrm{C})\cap\mathrm{D}} & \longrightarrow & \mathrm{R}_j^*(\mathrm{S}-\mathrm{A})_{\mathrm{D}\cap(\mathrm{S}-\mathrm{A})} & \longrightarrow & \mathrm{R}_j^*(\mathrm{S})_{\mathrm{D}-\mathrm{D}\cap(\mathrm{A}\cup\mathrm{B})}
\end{array}$$

Using this the lemma follows from a straightforward diagram chase.

Proof of theorem. — In the lemma, set $S = M$, $D =$ the Zariski closure of $Z \times \mathbf{G}_m$ in M, $A =$ the Zariski closure of $T \times \mathbf{G}_m$ in M, $B = M_0$. Hence $C = C_{T \cap X}(T)$ and is contained in $D \cap M_0 = C_{Z \cap X}(Z)$. We have a diagram:

$$\begin{array}{ccc}
CH_U^{n-1,n}(Y') & \xrightarrow{\ \partial\ } & CH_T^n(Y) \\
{\scriptstyle p^*}\downarrow & 1 & \downarrow{\scriptstyle p^*} \\
CH_{p^{-1}(U)}^{n-1,n}(Y' \times \mathbf{G}_m) & \xrightarrow{\ \partial\ } & CH_{p^{-1}(T)}^{n}(Y \times \mathbf{G}_m) \\
{\scriptstyle \{t\}*(\ )}\downarrow & 2 & \downarrow{\scriptstyle \{t\}*(\ )} \\
CH_{p^{-1}(U)}^{n-1,n+1}(Y' \times \mathbf{G}_m) & \xrightarrow{\ \partial\ } & CH_{p^{-1}(T)}^{n,n+1}(Y \times \mathbf{G}_m) \\
{\scriptstyle \partial}\downarrow & 3 & \downarrow{\scriptstyle \partial} \\
CH_{D \cap M_0'}^{n-1,n}(M_0') & \xrightarrow{\ \partial\ } & CH_{C_{T \cap X}(T)}^{n}(M_0)
\end{array}$$

In this diagram, square 1 commutes because p is flat ([Gi 1] proof of Theorem 8.3). It can be checked at the level of complexes that square 2 anti-commutes, while square 3 anti-commutes by the lemma. It follows that if $\varphi \in CH_U^{n-1,n}(Y')$, then

$$\partial\sigma_t(\varphi) = \sigma_t(\partial\varphi) \in CH^n_{\pi^{-1}(T \cap X)}(M_0),$$

proving (i). Turning to (ii), suppose $\varphi = [g_W]$ for $g_W \in k(W)^*$, and let $\tilde{g} \in k(Y)^*$ be a rational function which is regular at the generic points of W and X, and such that $\tilde{g}|_W = g_W$. Write $\tilde{D} = \operatorname{div}(\tilde{g})$, so $\tilde{D} \cap W = D$. Then $\varphi = \{\tilde{g}\}.[W]$ under the product

$$H^0(Y - \tilde{D}, K_1(\mathcal{O}_Y)) \otimes CH_W^{n-1}(Y) \to CH_{W-D}^{n-1,n}(Y - D).$$

One can see from the construction of $f^* : CH^{i,j}_{W-D}(Y - D) \to CH^{i,j}_{X \cap (W-D)}(X \cap (Y - D))$ that if $\alpha \in H^r(Y - \tilde{D}, K_s(\mathcal{O}_Y))$ and $\beta \in CH^{i,j}_{W-D}(Y - D)$, then $f^*(\alpha.\beta) = f^*(\alpha) f^*(\beta)$ where $f^*(\alpha)$ is the pull-back on sheaf cohomology induced by the pull-back on K-theory. Hence if $X.W = \sum_i \mu_i[S_i] + \Sigma_j \tau_j$,

$$\begin{aligned}
f^*[g_W] &= f^*\{\tilde{g}\}.f^*[W] \\
&= \{\tilde{g}|_X\}(\sum_i \mu_i[S_i] + \sum_j \tau_j) \\
&= \sum_i \mu_i[\tilde{g}|_{S_i}] + \sum_j \tilde{g}.\tau_j \\
&= \sum_i \mu_i[g_W|_{S_i}] + \sum_j g_W.\tau_j.
\end{aligned}$$

4.4.3. Let A be an arithmetic ring with fraction field F. Let $\mathbf{V}_A$ be the category of regular quasi-projective arithmetic varieties over A. Let $f: X \to Y$ be a map between varieties in $\mathbf{V}_A$. Then f may be factored $f = \pi.i$ with $i: X \to \mathbf{P}^n_Y$ an immersion and $\pi: \mathbf{P}^n_Y \to Y$ the natural projection, which is smooth and projective. By Theorem 3.6.1, there is a map $\pi^*: \widehat{CH}^*(Y) \to \widehat{CH}^*(P = \mathbf{P}^n_Y)$. If $i: X \to P$ is any immersion between regular quasi-projective varieties over A, it follows [Se] IV Proposition 22 that i is necessarily regular. Hence by Theorem 4.4.1 if Z is a codimension p cycle on P which meets X properly in the generic fibre there is a well defined cycle

$$i^*[Z] \in \mathrm{CH}^p_{\tau(p)}(X) \simeq Z^p(X_F) \oplus \mathrm{CH}^p_{\mathrm{fin}}(X),$$

in the notation of 4.2.1, since $\mathrm{Supp}(Z) \cap X \in \tau(p)$. Since Z_F meets X_F properly, if g_Z is a Green current for Z on Y, then $i^* g_Z$ is a Green current for $i^*[Z]$ by Theorem 2.1.4 and so we can define $i^*(Z, g_Z) = (i^*[Z], i^* g_Z) \in \hat{Z}^p(X_F) \oplus \mathrm{CH}^p_{\mathrm{fin}}(X)$. Suppose now that $\varphi \in R^{p-1}_p(P_F)$ is a K_1-chain such that $\mathrm{div}(\varphi)$ meets X_F properly. By the moving lemma for K_1-chains, Lemma 4.2.6, there is a K_1-chain ψ such that $\mathrm{div}(\psi) = \mathrm{div}(\varphi)$ and ψ meets X_F almost properly. Furthermore, $\psi - \varphi$ represents zero in $\mathrm{CH}^{p-1,p}(P_F)$, hence $\log|\psi|^2 = \log|\varphi|^2$, and therefore $\widehat{\mathrm{div}}(\psi) = \widehat{\mathrm{div}}(\varphi) \in \hat{Z}^p(P_F) \oplus \mathrm{CH}^p_{\mathrm{fin}}(P)$. By Theorem 4.4.2, if $Z = \mathrm{Supp}(\mathrm{div}(\psi))$, $i^*(\psi)$ is well defined in $\mathrm{CH}^{p-1,p}(X - (X \cap Z))$. By Theorem 4.4.2, $\mathrm{div}(i^*(\psi)) = i^* \mathrm{div}(\psi) \in \mathrm{CH}^p_{\tau(p)}(X)$, while by Lemma 4.2.5, since ψ meets X_F properly, $\log|i^*(\psi)|^2 = i^* \log|\psi|^2$, and hence $i^* \widehat{\mathrm{div}}(\psi) = \widehat{\mathrm{div}}(i^*(\psi))$. Therefore i^* induces a map $\widehat{\mathrm{CH}}^p(P) \to \widehat{\mathrm{CH}}^p(X)$. Composing with π^*, we obtain a map $i^* \pi^* : \widehat{\mathrm{CH}}^p(Y) \to \widehat{\mathrm{CH}}^p(X)$.

Theorem. — Let $f: X \to Y$ *be a map in* $\mathbf{V}_A$, *then:*

1) *If* f *is factored* $f = \pi . i$ *as above, the composition* $f^* = i^* \pi^* : \widehat{\mathrm{CH}}^p(Y) \to \widehat{\mathrm{CH}}^p(X)$ *does not depend on the particular factorization chosen.*

2) *If* $f: X \to Y$ *is flat and smooth on generic fibres over* A, *then* f *agrees with the map defined in* 3.6.1.

3) *If* $g: Y \to Z$ *is another map in* V_A, *then* $(gf)^* = f^* g^* : \widehat{\mathrm{CH}}^p(Z) \to \widehat{\mathrm{CH}}^p(Y)$.

4) *If* $(Z, g) \in \hat{Z}^p(Y)$ *and* $f^{-1}(Z)$ *has* codim p *in* X *then* $f^*(Z, g) = (f^*[Z], f^* g)$ *with* $f^*[Z]$ *defined as in* ([Se] V.C.).

5) f^* *induces a ring homomorphism* $\widehat{\mathrm{CH}}^*(Y)_{\mathbf{Q}} \to \widehat{\mathrm{CH}}^*(X)_{\mathbf{Q}}$ *with respect to the product defined in Theorem* 4.2.3.

6) *If* $\alpha \in \widehat{\mathrm{CH}}^1(Y)$ *and* $\beta \in \widehat{\mathrm{CH}}^p(Y)$ *then* $f^*(\alpha . \beta) = f^*(\alpha) . f^*(\beta) \in \widehat{\mathrm{CH}}^{p+1}(Y)$.

7) *If* f *is projective and smooth on generic fibres over* A, *of relative dimension* d, *so that a map* $f_* : \widehat{\mathrm{CH}}^p(X) \to \widehat{\mathrm{CH}}^{p-d}(Y)$ *is defined as in* 3.6.1, *and if* $\alpha \in \widehat{\mathrm{CH}}^p(X)$, $\beta \in \widehat{\mathrm{CH}}^q(Y)$ *we have:*

$$f_*(\alpha . f^* \beta) = f_*(\alpha) . \beta$$

with equality in $\widehat{\mathrm{CH}}^{p+q-d}(Y)$ *if* p *or* $q = 1$ *and* $\widehat{\mathrm{CH}}^{p+q-d}(Y)_{\mathbf{Q}}$ *in general.*

Proof.

Part (1). — Given $\alpha \in \widehat{\mathrm{CH}}^p(Y)$, since $f^* = i^* \pi^*$ preserves rational equivalence, in order to show that $f^*(\alpha)$ does not depend on the factorization $f = \pi . i$, we may suppose that $\alpha = (Z, g)$, with $f^{-1}(Z) \cap X_F$ of codimension p. Hence $i^* \pi^*(Z, g) = (i^* \pi^*[Z], \pi^* f^* g)$. First observe that g may be represented by an L^1 form $\tilde{g}$ which is C^∞ on $X_\infty - Z_\infty$, and that $i^* \pi^* g$ is represented by $f^* \tilde{g} = i^* \pi^* \tilde{g}$, hence $f^* g$ is independent of the factorization $f = \pi . i$. By [Fu] Proposition 6.6 *a*), cf. the discussion in 4.4.1, the cycle class $f^*[Z] \in \mathrm{CH}^p_{\tau(p)}(X)$ does not depend on the factorization either. Hence $f^*(\alpha) = (f^*[Z], f^* g)$ is well defined.

Parts (2) and (3). — By a similar argument, parts (2) and (3) follow from [Fu] Proposition 6.6 *b*) and *c*).

Part (4). — Follows from [Fu] 7.1.2.

Before proving parts (5) and (6), we need a lemma.

Lemma. — *Let $f: X \to Y$ be a map between smooth projective varieties over* **C**. *If* Z *and* W *are cycles on* Y *of codimensions m and n respectively which intersect properly on* Y *and such that $f^{-1}(Z)$, $f^{-1}(W)$ and $f^{-1}(Z \cap W)$ have codimensions m, n and $m + n$ respectively, then $f^*(g_Z * g_W) = f^*(g_Z) * f^*(g_W)$ for any choices of Green currents.*

Proof. — Let $\gamma : X \to X \times Y$ be the graph of f, with image $\Gamma \approx X$. If $p : X \times Y \to X$ and $q : X \times Y \to Y$ are the projections, T is a cycle on Y for which $\operatorname{codim} f^{-1}\, T = \operatorname{codim} T$ and g_T is a Green current for T, then $q^*\, T$ meets Γ properly and

$$f^*(g_T) = p_*(\delta_\Gamma \wedge q^*\, g_T).$$

Note that the lemma is easily checked for a smooth map, and hence we may suppose that it holds for q. Hence

$$\begin{aligned} f^*(g_Z * g_W) &= p_*(g_\Gamma * q^*(g_Z * g_W) - g_\Gamma \wedge q^*(\omega_Z \wedge \omega_W)) \\ &= p_*(g_\Gamma * (q^*(g_Z) * q^*(g_W)) - g_\Gamma \wedge q^*(\omega_Z) \wedge q^*\, \omega_W) \\ &= p_*((g_\Gamma * q^*\, g_Z) * q^*\, g_W - g_\Gamma \wedge q^*\, \omega_Z \wedge q^*\, \omega_W) \\ &= p_*(\delta_{\Gamma . q^*(Z)} \wedge q^*\, g_W + (g_\Gamma * q^*\, g_Z) \wedge q^*\, \omega_W - g_\Gamma \wedge q^*\, \omega_Z \wedge q^*\, \omega_W) \\ &= p_*(\delta_{\Gamma . q^*(Z)} \wedge q^*\, g_W + (\delta_\Gamma \wedge q^*\, g_Z) \wedge q^*\, \omega_W) \\ &= (\delta_{f^* Z} \wedge f^*\, g_W) + (f^*\, g_Z \wedge f^*\, \omega_W) \\ &= (f^*\, g_Z) * (f^*\, g_W), \end{aligned}$$

by the associativity of the $*$-product (Theorem 2.2.12), q.e.d.

Given the lemma, in order to prove (5) and (6), it suffices to prove the corresponding statements for cycles. These are straightforward if f is smooth, so we may suppose that f is a regular immersion. For (5) we must show that f^* defined by deformation to the normal cone is compatible with the intersection product defined via K-theory. Since the pull-back map on K-theory preserves products, it is enough to check that f^* defined via deformation to the normal cone agrees with f^* defined via K-theory. This may be checked directly using the deformation construction, or by means of the uniqueness of intersection products, as discussed in [Fu] Example 6.1.9. For (6), we observe that if $\alpha \in \widehat{\mathrm{CH}}^1(Y)$ and $\beta \in \widehat{\mathrm{CH}}^p(Y)$ with $\beta = (W, g_W)$, and W meeting X_F properly, then we can represent α by (D, g_D) with D meeting X, W and $X \cap W$ properly. By [Fu] 6.4 and 5.2.1 *d*), $i^*(D.W) = i^*(D).i^*(W)$ in $CH^{p+1}_{\tau(p+1)}(X)$.

Finally for (7), the projection formula, given $\alpha \in \widehat{CH}^p(Y)$, $\beta \in \widehat{CH}^q(X)$ we can suppose that $\alpha = (Z, g_Z) \in \hat{Z}^p(Y)$ and $\beta = (W, g_W)$ with Z and W meeting properly in the generic fibre X_F (i.e. $f^* Z$ and W meet properly); if $p = 1$ we can suppose that Z and W meet properly on X. We first check the formula on Green currents; by assumption $f: X_F \to Y_F$ is a proper smooth map. Now

$$f_*(f^*(g_Z) * g_W) = f_*(f^* g_Z \wedge \delta_W + f^* \omega_Z \wedge g_W)$$
$$= h_*[h^* \tilde{g}] + \omega_Z \wedge f_* g_W.$$

Here $h: W \to Y$ is the induced map and $\tilde{g}$ is a representative of g_Z of logarithmic type, while $f_*(f^* \omega_Z \wedge g_W) = \omega_Z \wedge f^* g_W$ by definition of the direct image of a current. By the argument of 3.6.1 one sees that $h_*[h^* g_Z] = g_Z \wedge \delta_{f_* W}$. Hence $f_*(f^* g_Z * g_W) = g_Z * f_* g_W$. Next we consider $f_*(f^* Z.W)$. If Z is a divisor, then working locally we can assume that $Z = \mathrm{div}(t)$ is principal and effective and that W is prime. Then at a generic point ξ of $Z \cap f(W)$, the multiplicity of ξ in $f_*(f^* Z.W)$ is

$$\ell(\mathcal{O}_{X,\xi}/t\mathcal{O}_{X,\xi} \otimes_{\mathcal{O}_{X,\xi}} \mathcal{O}_W) = \ell(\mathcal{O}_{Y,\xi}/t\mathcal{O}_{Y,\xi} \otimes_{\mathcal{O}_{Y,\xi}} f_* \mathcal{O}_W)$$

= the multiplicity of ξ in $[Z].f_*[W]$. In general, we can appeal to the projection formula for $Gr^*_\gamma K_0$ with supports. This follows from the projection formula for K_0 together with the Riemann-Roch theorem for the γ-filtration on K_0 with supports. This last is proved by combining the Riemann-Roch theorem for a regular immersion in [So], [G-S 3] with explicit calculations on $\mathbf{P}^n$.

4.4.4. *Remarks.* — One can also construct pull-back maps for arbitrary maps of regular arithmetic varieties $f: X \to Y$ without the quasi-projective assumption. The key point is to have a pull-back map for cycles for an arbitrary morphism of finite type between regular schemes. There are two methods for constructing such maps. The first is that of [Gi 2]. The second is to reduce to the affine case. Specifically, given $f: X \to Y$, there is a commutative diagram:

$$\begin{array}{ccc} U & \xrightarrow{\tilde{f}} & V \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ X & \xrightarrow[f]{} & Y \end{array} \qquad (4.4.4.1)$$

in which U and V are affine and regular, and p and q are torsors under vector bundles. Hence if $T \subset Y$ is a closed set, we have:

$$q^*: CH^*_T(Y) \simeq CH^*_{q^{-1}(T)}(V), \qquad p^*: CH^*_{f^{-1}(T)}(X) \simeq CH^*_{\tilde{f}^{-1}q^{-1}(T)}(U)$$

and since $\tilde{f}$ is quasi-projective, we have a map $\tilde{f}^*: CH^*_{q^{-1}(T)}(V) \to CH^*_{\tilde{f}^{-1}q^{-1}(T)}(U)$. Finally, set $f^* = p^{*-1} \tilde{f}^* q^*$. The existence of the commutative diagram (4.4.4.1) is a generalization, due to Thomason, to arbitrary divisorial schemes of Jouanalou's trick [J]. See Proposition 4.4 of [W].

4.4.5. If $f: (A, \Sigma, F_\infty) \to (A', \Sigma', F'_\infty)$ is a homomorphism between arithmetic rings, and $\pi : X \to \mathrm{Spec}(A)$ is an arithmetic variety over A, let $X' = X \times_{\mathrm{Spec}(A)} \mathrm{Spec}(A')$. If F and F' are the fraction fields of A and A' respectively note that $F \subset F'$, and hence that the map $X'_{F'} \to X_F$ is flat; also note that since $\mathbf{C}^\Sigma$ and $\mathbf{C}^{\Sigma'}$ are both isomorphic to products of finitely many copies of $\mathbf{C}$, the induced map $X'_\infty \to X_\infty$, when restricted to any component of X'_∞, is an isomorphism onto a component of X_∞. Hence we have pull-backs maps $f^* : Z^p(X_F) \to Z^p(X'_{F'})$ and, if X has a nonsingular, complete generic fibre, $f^* : \widetilde{\mathscr{D}}(X_{\mathbf{R}}) \to \widetilde{\mathscr{D}}(X'_{\mathbf{R}})$, which have the properly that if $Z \in Z^p(X_F)$ and g_Z is a Green current for Z, then $f^* g_Z$ is a Green current for $f^* Z$. Assuming that X has a smooth complete generic fibre, there will be pull-back homomorphisms $f^*_X : \widehat{CH}^*(X) \to \widehat{CH}^*(X')$, in either of the following two situations:

1) If $\pi : X \to \mathrm{Spec}(A)$ is smooth, so X and X' are both regular, and $f: A \to A'$ is of finite type, so that $f_X : X' \to X$ is quasi-projective. Then f^*_X is defined by the method of 4.4.2.

2) If $f: A \to A'$ is flat, for example if A is a Dedekind domain, then $f_X : X' \to X$ is also flat. Then a pull-back homomorphism $f^*_X : Z^p(X) \to Z^p(X')$ can be defined, for all $p \geqslant 0$, by the method used in [Fu] 1.7, which is compatible with the homomorphism $f^* : \widetilde{\mathscr{D}}(X_{\mathbf{R}}) \to \widetilde{\mathscr{D}}(X'_{\mathbf{R}})$, and which therefore induces a homomorphism $f^*_X : \widehat{CH}^p(X) \to \widehat{CH}^p(X')$. If X' is regular, for example if X is smooth over A, then one can show that f^*_X preserves both the product on the Chow groups with supports and the $*$-product of Green currents and hence that it preserves the ring structure on $\widehat{CH}^*(X)_{\mathbf{Q}}$.

4.4.6. We finish our discussion of pull-backs with a result which will be useful in [G-S 4].

Theorem. — Let X *be a nonsingular arithmetic variety with projective generic fibre. Let* $\alpha \in \widehat{CH}^p(X \times \mathbf{P}^1)$; *then if t is the parameter on* $\mathbf{P}^1$ (i.e. $t = T/S$, T *and* S *being homogeneous coordinates on* $\mathbf{P}^1$) *and* $i_t : X = X \times \{t\} \to X \times \mathbf{P}^1$, *for* $t = 0, \infty$, *are the two inclusions, then*

$$i_0^*(\alpha) - i_\infty^*(\alpha) = a\left(\int_{\mathbf{P}^1} \omega(\alpha) \log |t|^2\right).$$

($a : \widetilde{A}^{p-1,p-1} \to \widehat{CH}^p$ *and* $\omega : \widehat{CH}^p \to A^{p,p}$ *are the maps defined in* 3.3.4).

Proof. — Note first that by the homotopy property for CH^*, $\zeta(i_0^*(\alpha) - i_\infty^*(\alpha)) = 0$. By the moving lemma, we may choose a representative (Z, g) for α, for which Z meets the cycle $X \times \{0\} - X \times \{\infty\}$ properly in the generic fibre $X_F \times \mathbf{P}^1$. Then

$$\begin{aligned} i_t^*(\alpha) &= (i_t^*(Z), i_t^*(g)) \\ &= (\pi_*([Z].[X \times \{t\}]), \pi_*(g \wedge \delta_{[X \times \{t\}]})) \end{aligned}$$

(π is the projection $X \times \mathbf{P}^1 \to X$). Now in $\widehat{CH}^p(X)$ we have the equation

$$\begin{aligned} 0 &= \pi_*(\alpha.\widehat{\operatorname{div}}(t)) \\ &= \pi_*((Z, g).([X \times \{0\}] - [X \times \{\infty\}], -\log|t|^2)) \\ &= \pi_*(i_{0*}\, i_0^*(Z) - i_{\infty *}\, i_\infty^*(Z), g \wedge (\delta_{[X\times\{0\}]} - \delta_{[X\times\{\infty\}]}) - \omega(\alpha)\log|t|^2)) \\ &= i_0^*(\alpha) - i_\infty^*(\alpha) - a\int_{\mathbf{P}^1}\omega(\alpha)\log|t|^2. \end{aligned}$$

4.5. Arithmetic varieties smooth over a Dedekind domain

4.5.1. *Theorem. — Let* $A = (A, \Sigma, F_\infty)$ *be an arithmetic ring with* A *Dedekind. Suppose that* X *is an arithmetic variety, with projective generic fibre, which is smooth over* A. *Then the product structure on* $\widehat{CH}^*(X)_{\mathbf{Q}}$ *defined in* 4.2.3 *is induced by a product structure on* $\widehat{CH}^*(X)$ *with the property that if* (Y, g_Y) *and* (Z, g_Z) *are cycles on* X *which meet properly on* X_F *then* $(Y, g_Y) \cup (Z, g_Z) = ([Y].[Z], g_Y * g_Z)$ *where* $[Y].[Z]$ *is the intersection product of* [Fu] 20.2. *Furthermore, if* $f: Y \to X$ *is a map of nonsingular arithmetic varieties over* A *with* X *smooth over* A, *the map* $f^*: \widehat{CH}^*(X)_{\mathbf{Q}} \to \widehat{CH}^*(Y)_{\mathbf{Q}}$ *of* § 4.4 *is induced by a map* $f^*: \widehat{CH}^*(X) \to \widehat{CH}^*(Y)$. *If both* X *and* Y *are smooth over* A, f^* *is a ring homomorphism.*

Proof. — The only reason for introducing rational coefficients in the construction of the product on $\widehat{CH}^*$ is the lack of a product on the Chow groups with integral coefficients of a general regular scheme. So long as one has available, for a scheme X, products $\cup: CH^p_Y(X) \otimes CH^q_Z(X) \to CH^{p+q}_{Y\cap Z}(X)$, which satisfy the usual rules and which coincide with the usual product on cycles which meet properly, then the construction of 4.2.3 applies with integral rather than rational coefficients. Therefore it will suffice to show, following [Fu] 20.2 that if X is a scheme smooth over a Dedekind domain A and $Y \in Z^p(X)$, $Z \in Z^q(X)$, then there is a well defined intersection cycle

$$Y.Z \in CH^{p+q}_{Y\cap Z}(X) \simeq CH_{n-(p+q)}(Y \cap Z), \quad \text{for } n = \dim X.$$

Since X is smooth over $S = \operatorname{Spec}(A)$ so is $X \times_S X$, and the diagonal $\Delta: X \to X \times_S X$ is a regular codimension $n - 1$ immersion. First we construct an external product:

$$\times: Z^p(X) \otimes Z^q(X) \to Z^{p+q}(X \times_S X)$$

if Y and Z are prime cycles:

$$[Y] \otimes [Z] \mapsto \begin{cases} 0 & \text{if both Y and Z are contained} \\ & \text{in closed fibres} \\ [Y \times_S Z] & \text{otherwise.} \end{cases}$$

This external product is associative; if Y, Z, W are prime cycles on X, then we have an equality of cycles on $X \times_S X \times_S X$:

$$(Y \times Z) \times W = Y \times (Z \times W) = \begin{cases} 0 & \text{if any two of Y, Z, W contained} \\ & \text{in closed fibres} \\ [Y \times_S Z \times_S W] & \text{otherwise.} \end{cases}$$

It is also commutative in the sense that if $i : X \times_S X \to X \times_S X$ is the switch map,

$$[Y] \times [Z] = i^*([Z] \times [Y]).$$

Now we appeal to ([Fu] 6.2) and the discussion in 4.4.1 and 4.4.2.

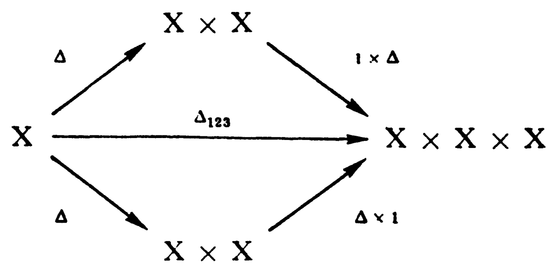
If $j : Y \to X$ is a regular immersion, and $Z \subset X$ there is a natural map

$$j^* : CH^p_Z(X) \to CH^p_{Z \cap Y}(Y)$$

with the property that if $T \in Z^p(X)$ is a cycle, contained in Z, meeting Y properly, then $j^*(T)$ is the usual inverse image. Furthermore, if $i : W \to Y$ is a second regular immersion such that $j.i : W \to X$ is a regular immersion, then $(j.i)^* = i^* j^* : CH^p_Z(X) \to CH^p_{Z \cap W}(X)$ ([Fu] 6.5). Given $Y \in Z^p(X)$, $Z \in Z^q(X)$, we define

$$[Y] \cup [Z] = \Delta^*(Y \times Z) \in CH^{p+q}_{Y \cap Z}(X).$$

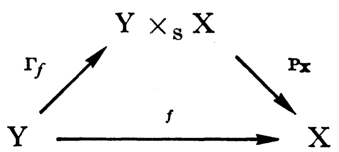
Since $i : X \times_S X \to X \times_S X$ is an isomorphism, and $\Delta i = \Delta$, $[Y] \cup [Z] = [Z] \cup [Y]$. Given $Y \in Z^p(X)$, $Z \in Z^q(X)$, $W \in Z^r(X)$, consider the diagram:



Then

$$\begin{aligned} ([Y] \cup [Z]) \cup [W] &= \Delta^*(\Delta^*(Y \times_S Z) \times_S W) \\ &= \Delta^*((\Delta \times 1)^* (Y \times_S Z \times_S W)) \\ &= \Delta^*_{123}(Y \times_S Z \times_S W) \\ &= [Y] \cup ([Z] \cup [W]) \text{ by symmetry.} \end{aligned}$$

If $f : Y \to X$ is an arbitrary map of varieties over A, with X smooth over A, then f factorizes:



where Γ_f is the graph of f. Since Y is flat over S and Y_F is smooth over Spec(F) by assumption, $p_X : Y \times_S X \to X$ is flat and smooth over X_F. Since X is smooth over S, the projection $p_Y : Y \times_S X \to Y$ is smooth and hence Γ_f is a regular immersion. We therefore define $f^* = \Gamma^*_f . p^*_X : CH^p_Z(X) \to CH^p_{f^{-1}(Z)}(Y)$. If Y is also smooth over S, then an argument similar to that for the associativity of the cup product above, shows that f^* is a ring homomorphism. To extend this construction to $\widehat{CH}^*$, i.e. including information about Green currents one uses the same methods as those of 4.4. It follows from the compatibility of Γ_f^* with Serre's product, [Fu] 20.4, that for cycles $Z \in Z^p(X)$ such that $f^{-1}(Z)$ has codimension p in Y, the two definitions of f^* agree.

4.5.2. Let us summarize. The arithmetic Chow groups $\widehat{CH}^p(X)$, $p \geqslant 0$ are contravariant in the quasi-projective regular arithmetic variety X (4.4.3 and 3.6.1 (i) for the flat case). They are covariant for generically smooth proper maps (3.6.1 (ii)). There is a commutative and associative graded product

$$\widehat{CH}^p(X) \otimes \widehat{CH}^q(X) \to \widehat{CH}^{p+q}(X)_{\mathbf{Q}}$$

(4.2.3 (i)). One may avoid tensoring with $\mathbf{Q}$ when $p \leqslant 1$ (4.2.3 (iii)), when X is smooth over a Dedekind ring (4.5.1), or when considering intersections numbers (4.3.2). These notions coincide when they are simultaneously defined. The formulas $(fg)^* = g^* f^*$, $(fg)_* = f_* g_*$, $f^*(\alpha\beta) = f^*(\alpha) f^*(\beta)$ and $f_*(\alpha f^*(\beta)) = f_*(\alpha)\, \beta$ are true whenever both sides are defined.

5. Complements

5.1. Chow groups of Arakelov varieties

5.1.1. Fix an arithmetic ring $A = (A, \Sigma, F_\infty)$. An *Arakelov variety* $\overline{X} = (X, \omega)$ is a pair consisting of an arithmetic variety X over A, regular with projective generic fibre, and a Kähler form ω on X_∞ satisfying $F_\infty^* \omega = - \omega$ (this is equivalent to requiring that the corresponding Kähler metric is invariant under F_∞). Let us write $\mathscr{H}^{p,p}(X_{\mathbf{R}})$ for the space of harmonic (with respect to ω) (p, p)-forms α on X_∞ satisfying $F_\infty^* \alpha = (-1)^p \alpha$. If $Y \in Z^p(X)$, we say that a Green current g_Y for Y is *admissible* if $\omega(Y, g_Y)$ is harmonic. Let $Z^p(\overline{X}) \subset \hat{Z}^p(X)$ be the subgroup consisting of all (Z, g) with g admissible. Note that if $W \subset X$ is an integral subscheme of X of codimension $p-1$ and $f \in k(W)^*$, then $\omega(\widehat{\mathrm{div}}(f)) = 0$ is trivially harmonic. Hence $\hat{R}^p(X) \subset Z^p(\overline{X})$ (see 3.3.4), and we can define $CH^p(\overline{X})$ as the quotient group. Equivalently,

$$CH^p(\overline{X}) = \omega^{-1}(\mathscr{H}^{p,p}(X_{\mathbf{R}})) \subset \widehat{CH}^p(X).$$

Let $H : A^{p,p}(X_{\mathbf{R}}) \to \mathscr{H}^{p,p}(X_{\mathbf{R}})$ be the orthogonal projection operator associated to the Kähler metric on X_∞. Then $H(\partial\alpha) = H(\bar{\partial}\beta) = 0$ for any forms α, β, hence

$$H : \widetilde{\mathscr{D}}^{p-1,p-1}(X_{\mathbf{R}}) \to \mathscr{H}^{p-1,p-1}(X_{\mathbf{R}})$$

is well defined, and may be viewed as a projection onto the subspace

$$H^{p-1,p-1}(X_{\mathbf{R}}) = \mathrm{Ker}(dd^c) \subset \widetilde{\mathscr{D}}^{p-1,p-1}(X_{\mathbf{R}}).$$

It follows that an admissible Green current g_Z for Z is determined by its harmonic projection $H(g_Z)$. This leads to:

Lemma. — *Let* $\overline{X} = (X, \omega)$ *be an Arakelov variety over the arithmetic ring* $A = (A, \Sigma, F_\infty)$. *Then*

$$Z^p(\overline{X}) \simeq Z^p(X) \oplus \mathscr{H}^{p-1,p-1}(X_{\mathbf{R}})$$

and

$$CH^p(\overline{X}) \simeq \frac{Z^p(X) \oplus \mathscr{H}^{p-1,p-1}(X_{\mathbf{R}})}{R^p(\overline{X})}$$

where $R^p(\overline{X}) \simeq \hat{R}^p(X)$ *is the subgroup generated by classes of the form:*

$$\overline{\mathrm{div}}(f) = \mathrm{div}(f) \oplus (-\,\mathrm{H}i_* \log |f|^2)$$

for $f \in k(W)^*$, $i : W \to X$ *an integral codimension* $p - 1$ *subscheme.*

Proof. — The map

$$Z^p(\overline{X}) \to Z^p(X) \oplus \mathscr{H}^{p-1,p-1}(X_{\mathbf{R}})$$
$$(Z, g) \mapsto Z \oplus (\mathrm{H}(g))$$

is an isomorphism by the remarks preceding the lemma. The identification of $\hat{R}_p(X)$ as the subgroup $R_p(\overline{X})$ of $Z^p(X) \oplus \mathscr{H}^{p-1,p-1}(X)_{\mathbf{R}}$ is obvious.

5.1.2. *Theorem.* — *Let* $\overline{X} = (X, \omega)$ *be as above. Then there is an exact sequence*

$$CH^{p,p-1}(X) \xrightarrow{\rho} H^{p-1,p-1}(X_{\mathbf{R}}) \xrightarrow{a} CH^p(\overline{X}) \xrightarrow{\zeta} CH^p(X) \longrightarrow 0.$$

Proof. — We have a commutative diagram, in which the top row is exact by 3.3.5:

$$\begin{array}{ccccccccc}
CH^{p,p-1}(X) & \xrightarrow{\rho} & \widetilde{A}^{p-1,p-1}(X_{\mathbf{R}}) & \xrightarrow{a} & \widehat{CH}^p(X) & \xrightarrow{\zeta} & CH^p(X) & \longrightarrow & 0 \\
\| & & \uparrow & & \uparrow & & \| & & \\
CH^{p,p-1}(X) & \longrightarrow & a^{-1}(CH^p(\overline{X})) & \longrightarrow & CH^p(\overline{X}) & \longrightarrow & CH^p(X) & \longrightarrow & 0.
\end{array}$$

The bottom row of the diagram is exact by construction; it suffices to show that

$$a^{-1}(CH^p(\overline{X})) \simeq H^{p-1,p-1}(X_{\mathbf{R}}).$$

As remarked above, $CH^p(\overline{X}) = \omega^{-1}(\mathscr{H}^{p,p}(X_{\mathbf{R}}))$; now

$$\omega . a = dd^c : \widetilde{A}^{p-1,p-1}(X_{\mathbf{R}}) \to A^{p,p}(X_{\mathbf{R}}),$$

therefore $$a^{-1}(CH^p(\overline{X})) = (dd^c)^{-1}(\mathscr{H}^{p,p}(X_{\mathbf{R}})).$$

But if $g \in \widetilde{A}^{p-1,p-1}(X_{\mathbf{R}})$, $dd^c g$ is harmonic if and only if it is zero, therefore

$$a^{-1}(CH^p(\overline{X})) = \mathrm{Ker}(dd^c) = H^{p-1,p-1}(X_{\mathbf{R}})$$

as desired.

Note that $CH^*(\overline{X})$ is a direct summand of $\widehat{CH}^*(X)$ with projection:

$$\widehat{CH}^*(X) \to CH^*(\overline{X})$$
$$(Z, g) \mapsto Z \oplus (\mathrm{H}(g)).$$

5.1.3. *Remarks.* — (i) Since we are using dd^c rather than $1/(\pi i)\, \partial\bar\partial$ our normalization in $\overline{\text{div}}$ is slightly different from that of [G-S 1].

(ii) If $\overline{X} = (X, \omega)$ is an arithmetic surface over the ring of integers in a number field F, then $CH^1(\overline{X})$ is the group introduced by Arakelov in [Ar 1]. In codimension two, the exact sequence 5.1.2 reads

$$\mathcal{O}_F^* \xrightarrow{\rho} \mathbf{R}^{r_1+r_2} \xrightarrow{a} CH^2(\overline{X}) \xrightarrow{\zeta} CH^2(X) \to 0$$

where ρ is the classical regulator multiplied by -2 (see 3.4.3), and $CH^2(X)$ is known to be finite [Bl 2].

(iii) If $X = \text{Spec}(\mathcal{O}_F)$, $\mathcal{O}_F$ the ring of integers in a number field F, then $X(\mathbf{C})$ is zero dimensional, hence all forms are harmonic and $\widehat{CH}^*(X) = CH^*(\overline{X})$.

(iv) One may view $\overline{X} = (X, \omega)$ as a relative compactification of X over $\overline{\text{Spec}(\mathcal{O}_F)}$ ($= \text{Spec}(\mathcal{O}_F) \cup \{$ the places at $\infty \}$), ω corresponding to the choice of an integral model at ∞, i.e. over the archimedean places of F; $\mathcal{H}^{p-1,p-1}(X_{\mathbf{R}})$ may then be interpreted as the Chow group of codimension p cycles on $\overline{X}$ supported in the closed fibre at ∞. See [De 2] for a detailed discussion of this analogy in the 2 dimensional case.

5.1.4. If $\overline{X} = (X, \omega)$ is an equidimensional projective nonsingular Arakelov variety over $\mathbf{Z}$, then the intersection pairing of 4.3.2, restricted to $CH^*(\overline{X})$, gives pairings, or all pairs (p, q) of nonnegative integers such that $p + q = \dim X$,

$$CH^p(\overline{X}) \otimes CH^q(\overline{X}) \to \mathbf{R}.$$

Using the description of $CH^*(\overline{X})$ given in Lemma 5.1.1, we can see, as follows, that this pairing coincides with the pairing defined in [G-S 1]. If $\alpha_1 \in CH^p(X)$, $\alpha_2 \in CH^q(X)$, then we can write $\alpha_i = Z_i \oplus h_i \in Z^p(\overline{X})$, Z_i being an algebraic cycle and h_i a harmonic form. Recall that if $\bar g_i$ is the antiharmonic (i.e. with harmonic projection zero) Green current for Z_i whose associated closed form is harmonic, then α_i corresponds to $(Z_i, \bar g_i + h_i) \in \hat Z^p(X)$. By the moving lemma, we can choose Z_1 and Z_2 so that they do not intersect in the generic fibre X_F, and hence so that $Z_1(\mathbf{C})$ and $Z_2(\mathbf{C})$ do not intersect in $X_\infty = X(\mathbf{C})$. By the isomorphism of Lemma 5.1.1, and 3.4.3, we see that

$$\begin{aligned}\langle \alpha_1, \alpha_2 \rangle &= \langle (Z_1, \bar g_1 + h_1), (Z_1, \bar g_2 + h_2) \rangle \\ &= \langle Z_1, Z_2 \rangle_f + \frac{1}{2}\int_{X(\mathbf{C})} (\bar g_1 + h_1) * (\bar g_2 + h_2) \\ &= \langle Z_1, Z_2 \rangle_f + \frac{1}{2}\int_{X(\mathbf{C})} (\bar g_1 + h_1) \wedge \delta_{Z_2} + \frac{1}{2}\int_{X(\mathbf{C})} H(\delta_{Z_1}) \wedge (\bar g_2 + h_2) \\ &= \langle Z_1, Z_2 \rangle_f + \frac{1}{2}\int_{X(\mathbf{C})} h_1 \wedge \delta_{Z_2} + \frac{1}{2}\int_{X(\mathbf{C})} H(\delta_{Z_1}) \wedge h_2 \\ &\qquad + \frac{1}{2}\int_{X(\mathbf{C})} \bar g_1 \wedge \delta_{Z_2}.\end{aligned}$$

Following the discussion in 4.3.8,

$$\langle Z_1, Z_2 \rangle_f = \sum_{i,j} (-1)^{i+j} \log \#(H^i(X, \mathcal{T}or_j^{\mathcal{O}_X}(\mathcal{O}_{Z_1}, \mathcal{O}_{Z_2})))$$

if Z_1, Z_2 are prime cycles, and for general pairs of cycles (Z_1, Z_2) meeting properly in X_F, we can extend this formula using the bi-additivity of $\langle\ ,\ \rangle$. Let us make the following definitions:

$$\langle Z_i, h_j \rangle = \langle h_j, Z_i \rangle = \frac{1}{2}\int_{X(C)} H(Z_i) \wedge h_j = \frac{1}{2}\int_{Z_i(C)} h_j = \frac{1}{2}\int_{X(C)} \delta_{Z_i} \wedge h_j,$$

and
$$\langle Z_1, Z_2 \rangle_\infty = \frac{1}{2}\int_{X(C)} \delta_{Z_1} \wedge \bar{g}_2 = \frac{1}{2}\int_{Z_1(C)} \bar{g}_2;$$

note that by Stokes' theorem $\langle Z_1, Z_2 \rangle = \langle Z_2, Z_1 \rangle$. Again, because of our use of dd^c rather than $\frac{1}{\pi i}\partial\bar{\partial}$, these formulae differ from those of [G-S 1] by a factor of 2; apart from this factor, we then obtain the same formula that appears in *op. cit.*,

$$\langle \alpha_1, \alpha_2 \rangle = \langle Z_1, Z_2 \rangle_f + \langle Z_1, Z_2 \rangle_\infty + \langle Z_1, h_2 \rangle + \langle h_1, Z_2 \rangle.$$

5.1.5. In general $CH^*(\overline{X})_Q$ cannot be a subring of $\widehat{CH}^*(X)_Q$, since that would require that $\omega(CH^*(\overline{X})) = \bigoplus_{p \geq 0} \mathscr{H}^{p,p}(X_R)$ be a subring of $A(X_R)$, which is not true in general. However if X_∞ is a complex symmetric space (for example if it is a Grassmannian, or product of Grassmannians) or an abelian variety, then the harmonic forms with respect to an invariant Kähler metric on X_∞ will be a subring of $A(X_R)$; for such an X, $CH^*(\overline{X})_Q$ will be ring, and if the base ring is a Dedekind domain, $CH^*(\overline{X})$ will be a ring. See [G-S 4] for further details on the structure of $CH^*(\overline{X})$ for X a product of Grassmannians.

Suppose now that (X, ω) and (X, ω') are two different Arakelov compactifications of the same nonsingular arithmetic variety X, which is projective over **Z**. We wish to compare the intersection pairing on $CH^*(X, \omega)$ to that on $CH^*(X, \omega')$, in a fashion similar that of section 5 of Arakelov's paper [Ar 1]. In order to do this we shall need the following

Lemma. — *Let* X *be an* $(n-1)$*-dimensional, compact Kähler manifold; for* $0 \leq p \leq n-1$, *let* $\alpha \in A^{p,p}(X)$ *be an exact form, and let* $\omega_1, \ldots, \omega_k$ *be closed* $(n-p, n-p)$ *forms representing a basis* $\bar{\omega}_1, \ldots, \bar{\omega}_k$ *of* $H^{n-p,n-p}(X; \mathbf{C})$. *Then there exists a unique form*

$$\varphi \in \widetilde{A}^{p-1,p-1}(X) = A^{p-1,p-1}(X)/(\operatorname{Im}\partial + \operatorname{Im}\bar{\partial}),$$

such that:

(i) $dd^c\varphi = \alpha$;

(ii) $\int_X \varphi \wedge \omega_i = 0$ *for all* $i = 1, \ldots, k$.

Proof. — From Theorem 1.2.1 we know that a solution of $dd^c u = \alpha$ exists and that any two solutions, in $A^{p-1,p-1}(X)$ differ by an element of

$H^{p-1,p-1}(X, \mathbf{C})$ $(\subset \widetilde{A}^{p-1,p-1}(X))$. Picking one solution, u_0 say, we obtain numbers $a_i = \int_X u_0 \wedge \omega_i$, for $i = 1, \ldots, k$. If $\overline{\omega}_i^*$, $i = 1, \ldots, k$, is the basis of $H^{p-1,p-1}(X, \mathbf{C})$ Poincaré dual to ω_i, $i = 1, \ldots, k$, then $\varphi = u_0 - \sum_{i=1}^{k} a_i \overline{\omega}_i^*$ is the (necessarily unique) solution to (i) and (ii).

If $x \in H^*(X(\mathbf{C}))$, let us write $\omega(x)$ and $\omega'(x)$ respectively, for the forms representing x which are harmonic with respect to ω and ω', respectively; while if Y is an algebraic cycle, $\omega(Y)$ and $\omega'(Y)$ are the corresponding forms representing the fundamental class of Y. Observing that if $\dim(X) = n$, and $x_1, \ldots, x_k$ is a basis of $H^{n-p,n-p}(X(\mathbf{C}), \mathbf{C})$, for $0 \leqslant p \leqslant n-1$, then, by the lemma, given an algebraic cycle $Y \in Z^p(X)$, there exists a unique $\varphi_Y \in \widetilde{A}^{p-1,p-1}(X_{\mathbf{R}})$ such that:

(i) $dd^c \varphi_Y = \omega(Y) - \omega'(Y)$;

(ii) $\int_{X(\mathbf{C})} \varphi_Y \wedge (\omega(x_i) + \omega'(x_i)) = 0$ for $i = 1, \ldots, k$.

5.1.6. *Lemma. — With the notation above, we have, if* $Y \in Z^p(X)$ *and* $Z \in Z^{n-p}(X)$:

(i) $\int_{X(\mathbf{C})} \varphi_Y \wedge (\omega(x) + \omega'(x)) = 0$ *for all* $x \in H^{n-p,n-p}(X(\mathbf{C}), \mathbf{C})$;

(ii) $\varphi_{Y_1} + \varphi_{Y_2} = \varphi_{Y_1+Y_2}$ *if* $Y_1, Y_2 \in Z^p(X)$;

(iii) $\varphi_Y = 0$, *if* Y *is rationally equivalent to zero*;

(iv) $\int_{X(\mathbf{C})} \varphi_Y \wedge \omega(Z) = \int_{X(\mathbf{C})} \omega(Y) \wedge \varphi_Z$;

(v) $g_Y * \varphi_Z = \omega(Y) \wedge \varphi_Z$ *if* g_Y *is a Green current for* Y, *which is admissible with respect to* ω;

(vi) $\varphi_Y * \varphi_Z = \varphi_Y \wedge (\omega_Z - \omega'_Z)$.

Proof. — (i) Let $(x_1, \ldots, x_k)$ be the basis of $H^{n-p,n-p}(X(\mathbf{C}), \mathbf{C})$ chosen in 5.1.5; then, if $x = \sum_{i=1}^{k} a_i x_i$,

$$\int_{X(\mathbf{C})} \varphi_Y \wedge (\omega(x) + \omega'(x)) = \sum_{i=1}^{k} a_i \int_{X(\mathbf{C})} \varphi_Y \wedge (\omega(x_i) + \omega'(x_i))$$
$$= 0.$$

(ii) This is obvious.

(iii) More generally, by the uniqueness of φ_Y, if Y is homologically equivalent to zero, so that $\omega(Y) = \omega'(Y) = 0$, then $\varphi_Y = 0$.

(iv) By (i) above,

$$\int_{X(\mathbf{C})} \varphi_Y \wedge (\omega(Z) + \omega'(Z)) = 0$$

therefore $$\int_{X(\mathbf{C})} \varphi_Y \wedge \omega(Z) = -\int_{X(\mathbf{C})} \varphi_Y \wedge \omega'(Z)$$

and so
$$2\int_{X(C)} \varphi_Y \wedge \omega(Z) = \int_{X(C)} \varphi_Y \wedge (\omega(Z) - \omega'(Z))$$
$$= \int_{X(C)} \varphi_Y \wedge dd^c \varphi_Z$$
which by Stokes' theorem
$$= \int_{X(C)} dd^c \varphi_Y \wedge \varphi_Z$$
$$= 2\int_{X(C)} \omega(Y) \wedge \varphi_Z, \text{ by symmetry.}$$

(v) and (vi) follow immediately from the definition of the $*$-product, together with its commutativity.

We can now compare the two intersection pairings.

Theorem. — *For all* p, $0 \leqslant p \leqslant n$, *consider the map*
$$\theta : \hat{Z}^p(X) \to \hat{Z}^p(X), \qquad \theta(Y, g_Y) = (Y, g_Y - \varphi_Y).$$
Then θ *induces an automorphism of* $\widehat{CH}^p(X)$ *for* $0 \leqslant p \leqslant n$ *such that*:

(i) θ *restricts to an isomorphism* $CH^p(X, \omega) \to CH^p(X, \omega')$;

(ii) *if* $\alpha \in CH^p(X, \omega)$ *and* $\beta \in CH^{n-p}(X, \omega)$ *then* $\langle \theta(\alpha), \theta(\beta) \rangle = \langle \alpha, \beta \rangle$.

Proof. — The fact that θ is a group homomorphism, and respects rational equivalence, follows from (ii) and (iii) of the lemma. It is an automorphism since it has an inverse, $\theta^{-1}(Y, g_Y) = (Y, g_Y + \varphi_Y)$. If $(Y, g_Y) \in CH^p(X, \omega)$, then $dd^c g_Y + \delta_Y = \omega(Y)$; therefore $dd^c(g_Y - \varphi_Y) + \delta_Y = \omega'(Y)$, i.e. $(Y, g_Y - \varphi_Y) \in CH^p(X, \omega')$, from which we deduce (i). Turning to (ii), it is sufficient to show that if $\alpha = (Y, g_Y)$ and $\beta = (Z, g_Z)$, then
$$\int_{X(C)} (g_Y - \varphi_Y) * (g_Z - \varphi_Z) = \int_{X(C)} g_Y * g_Z,$$
or equivalently,
$$\int_{X(C)} (g_Y * \varphi_Z + \varphi_Y * g_Z - \varphi_Y * \varphi_Z) = 0.$$
By (v) and (vi) of the lemma, we may rewrite this integral as:
$$\int_{X(C)} \omega(Y) \wedge \varphi_Z + \varphi_Y \wedge \omega(Z) + \varphi_Y \wedge (\omega'(Z) - \omega(Z))$$
$$= \int_{X(C)} \omega(Y) \wedge \varphi_Z + \varphi_Y \wedge \omega'(Z)$$
which, by part (iv) of the lemma, equals
$$\int_{X(C)} \varphi_Y \wedge (\omega(Z) + \omega'(Z))$$
which is zero by part (i) of the lemma.

5.1.7. We can also express the isomorphism θ by means of the description of $\mathrm{CH}^*(\overline{\mathrm{X}})$ given in Lemma 5.1.5, obtaining the version of Theorem 5.1.6 stated in [Gi 3].

Theorem. — *Let* X, ω *and* ω' *be as above, and let* H *and* H′ *be the harmonic projection operators corresponding to* ω *and* ω'. *Then, for* $0 \leqslant p \leqslant -1$, *the map* $\theta : \mathrm{CH}^p(\mathrm{X}, \omega) \to \mathrm{CH}^p(\mathrm{X}, \omega')$, *of Theorem* 5.1.6, *may be written*

$$\theta(\mathrm{Z} \oplus \alpha) = \mathrm{Z} \oplus \left(\mathrm{H}'(\alpha) + \frac{1}{2}\,\mathrm{H}'(\mathrm{H}(g'_{\mathrm{Z}}) - g_{\mathrm{Z}})\right);$$

here $\mathrm{Z} \in \mathrm{Z}^p(\mathrm{X})$, $\alpha \in \mathscr{H}^{p-1,\,p-1}(\mathrm{X}_{\mathbf{R}})$ *is harmonic with respect to* ω, *and* g_{Z} *(respectively* g'_{Z}*) is the Green current for* Z *which is anti-harmonic with respect to* ω *(respectively* ω'*).*

Proof. — Since

$$\mathrm{Z} \oplus \alpha = (\mathrm{Z}, g_{\mathrm{Z}} + \alpha) \in \hat{\mathrm{Z}}^p(\mathrm{X}),$$

$$\theta(\mathrm{Z} \oplus \alpha) = \mathrm{Z} \oplus (\mathrm{H}'(g_{\mathrm{Z}} + \alpha - \varphi_{\mathrm{Z}})) = \mathrm{Z} \oplus (\mathrm{H}'(\alpha) + \mathrm{H}'(g_{\mathrm{Z}}) - \mathrm{H}'(\varphi_{\mathrm{Z}})).$$

Therefore we must show that $\mathrm{H}'(\varphi_{\mathrm{Z}}) = \frac{1}{2}\,\mathrm{H}'(g_{\mathrm{Z}} + \mathrm{H}'(g_{\mathrm{Z}}))$, which is equivalent to showing that, if $x_1, \ldots, x_k$ is a basis of $\mathrm{H}^{n-p,\,n-p}(\mathrm{X}(\mathbf{C}), \mathbf{C})$,

$$\textbf{(5.1.7.1)} \qquad \int_{\mathrm{X}(\mathbf{C})} \varphi_{\mathrm{Z}} \wedge \omega'(x_i) = \frac{1}{2}\int_{\mathrm{X}(\mathbf{C})} (g_{\mathrm{W}} \wedge \omega'(x_i) + g'_{\mathrm{Z}} \wedge \omega(x_i)).$$

To do this, consider $\gamma = \varphi_{\mathrm{Z}} + g'_{\mathrm{Z}} - g_{\mathrm{Z}}$, which is an element of

$$\widetilde{\mathrm{A}}^{p-1,\,p-1}(\mathrm{X}_{\mathbf{R}}) \subset \widetilde{\mathscr{D}}^{p-1,\,p-1}(\mathrm{X}_{\mathbf{R}});$$

the dd^c operator maps γ to zero, hence it lies in the subspace $\mathrm{H}^{p-1,\,p-1}(\mathrm{X}_{\mathbf{R}})$ of $\widetilde{\mathrm{A}}^{p-1,\,p-1}(\mathrm{X}_{\mathbf{R}})$. Therefore, if β is a closed form in $\mathrm{A}^{n-p,\,n-p}(\mathrm{X}(\mathbf{C}))$, the integral $\int_{\mathrm{X}(\mathbf{C})} \gamma \wedge \beta$ depends only on the cohomology class of β. Hence, for $i = 1, \ldots, k$,

$$\int_{\mathrm{X}(\mathbf{C})} \gamma \wedge \omega'(x_i) = \frac{1}{2}\int_{\mathrm{X}(\mathbf{C})} \gamma \wedge (\omega'(x_i) + \omega(x_i)),$$

and using the conditions characterizing φ_{Z}, g_{Z} and g'_{Z}, we deduce that

$$\int_{\mathrm{X}(\mathbf{C})} \varphi_{\mathrm{Z}} \wedge \omega'(x_i) - \int_{\mathrm{X}(\mathbf{C})} g_{\mathrm{Z}} \wedge \omega'(x_i) = \frac{1}{2}\int_{\mathrm{X}(\mathbf{C})} (g'_{\mathrm{Z}} \wedge \omega(x_i) - g_{\mathrm{Z}} \wedge \omega'(x_i)).$$

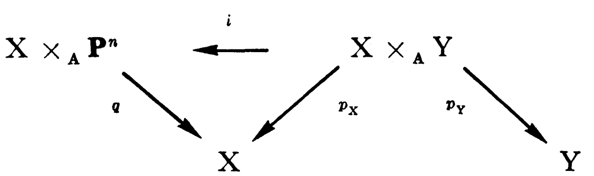
Upon simplifying this equation, we obtain equation 5.1.7.1 as desired.

5.2. Correspondences

5.2.1. Once one has constructed pull-back and push-forward maps for the arithmetic Chow groups, one can ask whether these maps form part of a theory of correspondences. (For a discussion of correspondences and Grothendieck's theory of motives, see [Kl].) However it seems too restrictive to consider only those correspondences between arithmetic varieties X and Y (over some base A) which arise from classes in $\widehat{CH}^*(X \times_A Y)$; for example, the identity on $\widehat{CH}^*(X)$ is not represented by such a class. Define a *correspondence from* X *to* Y, to be a pair (Γ, γ) consisting of an algebraic cycle Γ on $X \times_A Y$ and a current $\gamma \in \widetilde{\mathscr{D}}((X \times_A Y)_{\mathbf{R}})$ such that, if $\omega_\Gamma = \delta_\Gamma + dd^c \gamma$, the operator $\alpha \mapsto p_{Y*}(p_X^*(\alpha) \wedge \omega_\Gamma)$ maps C^∞ forms on X to C^∞ forms on Y. Here p_X and p_Y are the natural projections from $X \times Y$ to X and Y, respectively. Notice that this definition is not symmetric in X and Y.

Lemma. — *If* X *and* Y *are regular projective arithmetic varieties over an arithmetic ring* A, *a cycle* Γ *on* $X \times_A Y$ *defines a natural map* $x \mapsto \Gamma_*(x)$ *from* $CH_T^*(X)$ *to* $CH_S^*(Y)_{\mathbf{Q}}$ *for any pair of closed sets* $T \subset X$, $S \subset Y$ *such that* $|\Gamma| \cap p_X^{-1}(T) \subset p_Y^{-1}(S)$.

Proof. — Suppose $Y \subset \mathbf{P}_A^n$, and consider the diagram



Since $\mathbf{P}_A^n$ is smooth over A, $X \times_A \mathbf{P}^n$ is regular, hence if $x \in CH_T^*(X)$ and

$$\Gamma \in CH_{|\Gamma|}^*(X \times_A Y) \simeq CH_{|\Gamma|}^*(X \times_A \mathbf{P}^n)$$

there is a product class $p_X^*(x) \cup \Gamma$ in $CH^*_{|\Gamma| \cap p_X^{-1}(T)}(X \times_A \mathbf{P}^n)_{\mathbf{Q}} \simeq CH^*_{|\Gamma| \cap p_X^{-1}(T)}(X \times_A Y)_{\mathbf{Q}}$, which is mapped by p_{Y*} into $CH_S^*(Y)_{\mathbf{Q}}$.

Note that if Γ intersects $p_X^{-1}(Z)$ properly for every cycle Z on X, then Γ_* takes values in $CH_S^*(Y)$; i.e. we need not tensor with $\mathbf{Q}$.

Theorem. — *Let* X *and* Y *be regular projective arithmetic varieties over* A; *as usual we write* F *for the fraction field of* A. *If* $\Gamma = (\Gamma, \gamma)$ *is a correspondence from* X *to* Y, *there is a map*

$$\Gamma_* : \widehat{CH}^*(X) \to \widehat{CH}^*(Y)_{\mathbf{Q}}$$

such that, whenever $(Z, g) \in \hat{Z}^*(X)$ *is an arithmetic cycle for which* $p_X^*(Z)$ *meets* Γ *properly in* $(X \times_A Y)_F$,

$$\Gamma_*(Z, g) = (\Gamma_*(Z), p_{Y*}(\delta_\Gamma \wedge p_X^* g + \gamma \wedge p_X^* \omega_Z)).$$

Proof. — If $(Z, g) \in \hat{Z}^*(X)$ is a cycle such that $p_X^*(Z)$ meets Γ properly in $(X \times_A Y)_F$ then we define $\Gamma_*(Z, g)$ using the formula above, noting that $\Gamma_*(Z)$ is defined using the lemma. Given a general (Z, g), since X_F is a nonsingular projective variety, Z is rationally equivalent, in X, to a cycle Z′, such that $p_X^*(Z')$ meets Γ properly in X_F. Using the same argument as in Section 4.2 one shows that the rational equivalence class of $\Gamma_*(Z, g)$ is independent of the choice of Z′ and hence also of the rational equivalence class of (Z, g).

Note that if Γ meets $p_X^*(Z)$ properly in $X \times_X Y$, not just in the generic fibre, for all Z, then Γ_* takes values in $\widehat{CH}^*(X)$.

5.2.2. *Examples.* — (i) If $f: X \to Y$ is a morphism for which $f: X_F \to Y_F$ is smooth, then $f_*: \widehat{CH}^*(X) \to \widehat{CH}^*(Y)$ is induced by the correspondence (Γ, δ_Γ), where Γ is the graph of f.

(ii) For any $f, f^*: \widehat{CH}^*(Y) \to \widehat{CH}^*(X)_Q$ is induced by the transpose of the correspondence of (i).

(iii) If X is a compact complex manifold equipped with a Kähler form ω, then $p_1^* \omega + p_2^* \omega$ is a Kähler form on $X \times X$. If α and β are harmonic forms on X, then $p_1^*(\alpha) \wedge p_2^*(\beta)$ is harmonic on $X \times X$, and every harmonic form on $X \times X$ is a sum of such forms. Suppose that X is a nonsingular, projective arithmetic variety over A, and that ω is a Kähler form on X_∞ such that $F_\infty^* \omega = -\omega$. Then if $\Delta \subset X \times_A X$ is the diagonal, let $g(\omega)$ be the unique Green current for Δ which is antiharmonic with respect to $p_1^* \omega + p_2^* \omega$ and such that $\omega(\Delta, g(\omega))$ is harmonic. Then, writing $\Delta(\omega)$ for $(\Delta, g(\omega))$, one may easily check that $\Delta(\omega)_*$ is the projection operator from $\widehat{CH}^*(X)$ onto $CH^*(\overline{X})$ discussed in 5.1.2. One can also show that if Z is a cycle on X, then $p_{1*}(g(\omega) \wedge p_2^* \delta_Z)$ is the antiharmonic admissible Green current for Z.

(iv) It is also true that if ω and ω' are two Kähler forms on X_∞, then the automorphism θ of $\widehat{CH}^*(X)$ defined in Theorem 5.1.6 is induced by a correspondence. For every p, $0 \leq p \leq n$, let y_i and y_i^* be dual bases of $H^{p,p}(X(\mathbf{C}), \mathbf{C})$ and $H^{n-1-p,n-1-p}(X(\mathbf{C}), \mathbf{C})$, and x_i as in 5.1.5. Define φ_i in $\widetilde{A}^{p-1,p-1}(X_\mathbf{R})$ by the two conditions

$$dd^c \varphi_i = \omega(y_i) - \omega'(y_i)$$

and

$$\int_{X(\mathbf{C})} \varphi_i \wedge (\omega(x_i) + \omega'(x_i)) = 0$$

for $i = 1, \ldots, k$. Then one can check that θ is induced by the identity minus the sum of $p_1^*(\varphi_i) \wedge p_2^*(\omega(y_i^*))$, for all i and p.

(v) We shall not discuss composition of correspondences, though we believe that composition is well defined and associative.

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