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CLOSED ORBITS IN HOMOLOGY CLASSES

by ATSUSHI KATSUDA *and* TOSHIKAZU SUNADA

Dedicated to Pr. Akihiko Morimoto for his 60th birthday

0. Introduction

Let $\{\varphi_t\}$ be a smooth, transitive and weakly mixing Anosov flow on a compact manifold X. In this paper, employing an idea in analytic number theory, we count the number of closed orbits in a homology class. An analogue of Dirichlet L-functions plays a crucial role in our argument.

Given a surjective homomorphism ψ of $H_1(X, \mathbf{Z})$ onto an abelian group H, we set, for each $\alpha \in H$ and positive number x,

$$\Pi(x, \alpha) = \{ \mathfrak{p};\ \text{closed orbits with } \psi[\mathfrak{p}] = \alpha \text{ and } \ell(\mathfrak{p}) < x \},$$
$$\pi(x, \alpha) = \text{the cardinality of } \Pi(x, \alpha),$$

where $[\mathfrak{p}]$ denotes the homology class and $\ell(\mathfrak{p})$ the least period of $\mathfrak{p}$. One of the results in the present paper is concerned with an asymptotic estimate of $\pi(x, \alpha)$ as x goes to infinity. The resemblance of our problem to a number theoretic problem suggests that an analogue of the density theorem for prime numbers holds. The " Galois group " H, however, is possibly of infinite order, so that some extra phenomenon will appear.

Before stating our results, we must introduce several dynamical quantities. We denote by h the *topological entropy* of the flow and by m a (unique) invariant probability measure on X of maximal entropy. Let Z be the vector field generating the flow. We define the *winding cycle* Φ, which is a linear functional on the space of closed one-forms on X, by

$$\Phi(\omega) = \int_X \langle \omega, Z \rangle \, dm.$$

Since $\Phi(\text{exact forms}) = 0$, the linear functional Φ yields a homology class in $H_1(X, \mathbf{R}) = \mathrm{Hom}(H^1(X, \mathbf{R}), \mathbf{R})$. The ergodicity of the flow leads to the equality

$$\Phi(\omega) = \lim_{t \to \infty} \frac{1}{t} \int_0^t \langle \omega, Z \rangle (\varphi_\tau x) \, d\tau \qquad \text{a.e. } x,$$

hence the winding cycle is regarded as the average of the " homological " direction in which the orbits are traveling. The central limit theorem (cf. Denker and Philipp [5]) guarantees the existence of the limit

$$\delta(\omega, \omega) = \lim_{t \to \infty} \frac{1}{t} \int_X dm(x) \left(\int_0^t \langle \omega, Z \rangle (\varphi_\tau x) \, d\tau - t\Phi(\omega) \right)^2,$$

which yields a positive semi-definite quadratic form on $H^1(X, \mathbf{R})$. We call δ the *covariance form.* As we will see later, δ is positive definite on Ker Φ, and hence gives rise to a Euclidean metric on Ker Φ. Consider the character group $\hat{H}$ of H. The tangent space $T_1 \hat{H}$ at the trivial character **1** is identified with the dual $H^\dagger = \mathrm{Hom}(H, \mathbf{R})$, which is also identified, in a natural manner, with a subspace in $H^1(X, \mathbf{R})$. Therefore if Φ vanishes on $H^\dagger$, the covariance form induces a flat metric on the group $\hat{H}$. We denote by vol$(\hat{H})$ the volume with respect to the metric.

Theorem **1** *(Density theorem). — If Φ vanishes on the dual* $H^\dagger$, *then*

$$\pi(x, \alpha) \sim C \frac{e^{hx}}{x^{(b/2)+1}} \quad as\ x \uparrow \infty, \tag{0.1}$$

where $b = \mathrm{rank}\, H$ *and* $C = (2\pi)^{-b/2} \mathrm{vol}(\hat{H})^{-1} h^{-1}$.

The above condition on the winding cycle is necessary for the asymptotic like (0.1). In fact we have the following

Theorem **2.** — *If* $\Phi(H^\dagger) \neq (0)$, *then for every positive integer* N

$$\frac{\pi(x, \alpha)}{e^{hx}} = o(x^{-N}) \quad as\ x \uparrow \infty.$$

An extreme case is given by a perturbation of the suspension flow of an Anosov diffeomorphism. In fact, we may construct, in this way, a weakly mixing flow (X, φ_t) such that each homology class $\alpha \in H_1(X, \mathbf{Z})$ contains only finitely many closed orbits.

The condition in Theorem 1 is valid for a finite group H since $H^\dagger = (0)$. In this case, the density theorem was established by W. Parry and M. Pollicott [17], and T. Adachi and T. Sunada [3].

A typical example of Anosov flows with vanishing winding cycle on the full cohomology group is the geodesic flow on the unit tangent sphere bundle over a negatively curved manifold. Thus Theorem 1 is a refinement of a result in T. Adachi and T. Sunada [2] on the existence of infinitely many closed geodesics in a fixed homology class and also a generalization of the density theorem for hyperbolic spaces established by R. Phillips and P. Sarnak [18], and ourselves [11]. It should be noticed that the method in [18] and [11] is applied only to symmetric spaces, because, in the general case, one can not exploit the Selberg zeta function (or trace formula) which enables one to relate the poles of dynamical L-functions and the spectra of twisted Laplacians.

One may ask how are closed orbits in a homology class distributed spacially in the manifold. Let us suppose that they are *equidistributed* in the sense that, for all $f \in C^{\infty}(X)$, the following equality holds:

$$\textbf{(0.2)} \qquad \lim_{x \uparrow \infty} \pi(x, \alpha)^{-1} \sum_{\mathfrak{p} \in \Pi(x, \alpha)} \ell(\mathfrak{p})^{-1} \int_{\mathfrak{p}} f = \int_X f \, dm.$$

Then $\pi(x, \alpha) \uparrow \infty$ as $x \uparrow \infty$, and the left hand side of (0.2) equals $\Phi(\omega)$ when $f = \langle \omega, Z \rangle$. If $\omega \in H^{\dagger}$, then $\int_{\mathfrak{p}} \langle \omega, Z \rangle = \int_{\mathfrak{p}} \omega$ does not depend on $\mathfrak{p} \in \Pi(x, \alpha)$, so that the right hand side of (0.2) equals

$$\left(\int_{\mathfrak{p}} \omega\right) \pi(x, \alpha)^{-1} \sum_{\mathfrak{p} \in \Pi(x, \alpha)} \ell(\mathfrak{p})^{-1},$$

which tends to zero as $x \uparrow \infty$. Therefore $\Phi \equiv 0$ on $H^{\dagger}$. We shall prove the converse, which generalizes a result of W. Parry [15].

Theorem **3** *(Equidistribution theorem).* — *If* $\Phi \equiv 0$ *on* $H^{\dagger}$, *then* (0.2) *holds for all* α.

The organization of this paper is as follows. On the whole, we take up a method parallel to the classical proof of the density theorem for primes in arithmetic progression. In Section 1, we set down the basic facts about dynamical L-functions. Employing the perturbation theory of Ruelle operators, we pay close attention to the poles of L-functions located in a neighborhood of the real axis. In Section 2, we give a criterion on the existence of poles on the critical line in terms of the covariance form. In Section 3, we examine the singularities of the integral of higher logarithmic derivative of L-functions over the character group $\hat{H}$. Though a basic idea of computations is already seen in [11], we are forced, in the general case, to make a careful analysis because of the poles possibly located off the real axis. If b is even, the singularity has the form to which we can apply the ordinary Wiener-Ikehara Tauberian theorem. For an odd b, we need to establish a modified Tauberian theorem. This and the proof of Theorem 1 are done in Section 4. Section 5 is devoted to the proof of Theorem 2. The proof of Theorem 3 is outlined in Section 6 since it is almost the same as the proof of Theorem 1.

It should be pointed out that C. Epstein [6] obtained an asymptotic formula for the case of non-compact hyperbolic spaces with finite volume, a case not covered by our results in the present form.

Acknowledgements: This work originated in 1986 during a visit of one of the authors (T. S.) to the Institut Henri Poincaré. The density theorem for geodesic flows was announced in [10]. The work for the general cases was accomplished during a visit of T. S. to the Institut des Hautes Études Scientifiques in 1987/1988. We would like to thank the IHP and the IHES for their hospitality. We also wish to thank T. Adachi and K. Tanaka for various enlightening remarks during the development of this work.

Recently the authors learned that M. Pollicott (for geodesic flows over negatively curved surfaces or manifolds with the even first betti number) and S. P. Lalley (for geodesic flows on negatively curved manifolds) obtained independently the density theorem.

1. Dynamical L-functions

The (dynamical) L-function associated with a unitary character $\chi : H_1(X, \mathbf{Z}) \to U(1)$ is defined by

$$L(s, \chi) = \prod_{p} (1 - \chi([p])\, e^{-s\ell(p)})^{-1}.$$

It is known (W. Parry and M. Pollicott [17], T. Adachi and T. Sunada [2]) that $L(s, \chi)$ converges absolutely and is holomorphic in $\operatorname{Re} s > h$, and has a nowhere vanishing meromorphic extension to an open neighborhood $\mathcal{D}$ (not depending on χ) of the closed region $\operatorname{Re} s \geqslant h$. Our primary concern in this section is in the location of poles of $L(s, \chi)$ near $s = h$, which turns out to be closely related to distribution of closed orbits in a fixed homology class.

Proposition **1.1.** — *There exists a smooth function $s = s(\chi)$ defined on an open neighborhood of the trivial character* **1** *in the character group of $H_1(X, \mathbf{Z})$ such that $s(\mathbf{1}) = h$ and $s(\chi)$ is a unique (simple) pole of $L(s, \chi)$ around $s = h$. Furthermore*

$$\nabla_{\chi=1} \operatorname{Re} s(\chi) = 0, \tag{1.1}$$

$$\nabla_{\chi=1} \operatorname{Im} s(\chi) = \Phi, \tag{1.2}$$

$$(\operatorname{Hess}_{\chi=1} \operatorname{Re} s(\chi)) = -4\pi^2\, \delta, \tag{1.3}$$

$$(\operatorname{Hess}_{\chi=1} \operatorname{Im} s(\chi)) = 0. \tag{1.4}$$

Proof. — For a real valued smooth function F on X, define

$$L(s, F) = \prod_{p} \left(1 - \exp\left(- s\ell(p) + \sqrt{-1}\int_p F\right)\right)^{-1},$$

where $$\int_p F = \int_0^{\ell(p)} F(\varphi_\tau x)\, d\tau, \quad x \in p.$$

If we put $F = 2\pi \langle \omega, Z \rangle$, $d\omega = 0$, we find that $L(s, F) = L(s, \chi_\omega)$, where χ_ω is the character defined by

$$\chi_\omega(\alpha) = \exp 2\pi\sqrt{-1}\int_{C_\alpha} \omega, \quad \alpha \in H_1(X, \mathbf{Z}),$$

C_α being a closed curve representing the homology class α. Note that every character in the identity component of the character group has the form χ_ω for some $\omega \in H^1(X, \mathbf{R})$.

Using a perturbation technique for Ruelle operators (see the discussion below), we observe that $L(s, F)$ has a unique pole $s = s(F)$ in a neighborhood of $s = s(0) = h$ if F is sufficiently closed to 0, and that $\operatorname{Re} s(F) \leqslant h$. Since $\overline{L(s, F)} = L(\bar{s}, -F)$, we obtain

(1.5) $$\overline{s(F)} = s(-F).$$

The equalities (1.1) — (1.4) in Proposition 1.1 are special cases of the following.

Proposition **1.2.** — (1) $\left.\dfrac{d}{dt}\right|_{t=0} \operatorname{Re} s(tF) = 0.$

(2) $\left.\dfrac{d}{dt}\right|_{t=0} \operatorname{Im} s(tF) = \int_X F\, dm.$

(3) $\left.\dfrac{d^2}{dt^2}\right|_{t=0} \operatorname{Re} s(tF) = -\lim_{t\to\infty} \dfrac{1}{t}\int_X dm(x) \left(\int_0^t F(\varphi_\tau x)\, d\tau - t\int_X F\, dm\right)^2.$

(4) $\left.\dfrac{d^2}{dt^2}\right|_{t=0} \operatorname{Im} s(tF) = 0.$

Proof. — (1) and (4) are clear from (1.5). From now on, we take up the notation used in M. Pollicott [21]. Due to R. Bowen [4], there exist a suspended flow $\sigma_t^r : \Sigma^r \to \Sigma^r$ and a Lipschitz surjective bounded-one map $p : \Sigma^r \to X$ such that $p \circ \sigma_t^r = \varphi_t \circ p$. Furthermore, if m is the measure of maximal entropy on Σ^r, then $p^* m$ is the measure of maximal entropy on X and p is an isomorphism with respect to these two measures.

Define $L(s, \cdot)$ for (Σ^r, σ_t^r) in the same way as for (X, φ_t). Using a routine technique in symbolic dynamics, we observe that $L(s, F \circ p)/L(s, F)$ is a non-vanishing holomorphic function in $\operatorname{Re} s > h - \varepsilon$, $\varepsilon > 0$, so that the problem reduces to the case of suspended flows. We set

$$f(x) = \int_0^{r(x)} F(x, \tau)\, d\tau, \quad x \in \Sigma, \quad \tau \in [0, r(x)].$$

We denote by μ the equilibrium state for the function $-hr$. Then

$$m = \mu \times \tau \Big/ \int_\Sigma r\, d\mu.$$

Let r^+ and f^+ be functions in F_θ, $0 < \theta < 1$, which are cohomologous to r, f respectively. Consider the (complex) Ruelle operator $\mathscr{L}_{s,t}$ ($s, t \in \mathbf{C}$) defined by

$$\mathscr{L}_{s,t}\, \varphi(x) = \sum_{\substack{y \\ \sigma^+ y = x}} e^{-sr^+(y) + \sqrt{-1}\, t f^+(y)}\, \varphi(y).$$

The Ruelle-Perron-Frobenius theorem says that for $(s, t) = (h, 0)$, $\mathscr{L}_{s,t}$ has a simple eigenvalue 1 with positive eigenfunction. Applying the perturbation theory, we can find a function $\lambda(s, t)$ and $\varphi_{s,t} \in F_\theta$ such that $\mathscr{L}_{s,t}\, \varphi_{s,t} = \lambda(s, t)\, \varphi_{s,t}$, $\lambda(h, 0) = 1$, $\varphi_{h,0} > 0$, and $\lambda(s, t)$ is holomorphic in s and t. Furthermore

(1.6) $$\left.\frac{\partial\lambda}{\partial s}\right|_{s=h,\, t=0} < 0.$$

It should be noted that if $P(u)$ denotes the pressure of u, then

$$e^{P(-hr^{+}+tf^{+})} = \lambda(h, -\sqrt{-1}t). \tag{1.7}$$

From (1.6), the equation $\lambda(s, t) = 1$ can be solved by the variable s around $(h, 0)$, that is, there exists a function $s = s(t)$ such that

$$\lambda(s(t), t) = 1, \qquad s(0) = h.$$

It is obvious that $s(t\mathrm{F}) = s(t)$.

We now prove (2). Taking the first derivative of $\lambda(s(t), t)$, we have

$$\frac{\partial\lambda}{\partial s}\frac{ds}{dt} + \frac{\partial\lambda}{\partial t} = 0.$$

Using the analyticity of λ and the fact (D. Ruelle [22]) that

$$\frac{\partial}{\partial t}\bigg|_{t=0} P(-hr^{+} + tf^{+}) = \int_{\Sigma^{+}} f^{+}\, d\mu_{+} = \int r\, d\mu \int_{X} F\, dm,$$

we have

$$\frac{\partial\lambda}{\partial t}\bigg|_{s=h,\, t=0} = \sqrt{-1}\int r\, d\mu \int_{X} F\, dm.$$

On the other hand, we have

$$\frac{\partial\lambda}{\partial s}\bigg|_{s=h,\, t=0} = -\int_{\Sigma^{+}} r^{+}\, d\mu^{+} = -\int_{\Sigma} r\, d\mu,$$

hence we get (2).

We now proceed to (3). For simplicity, we put $\tilde{F} = \int_{X} F\, dm$. Then

$$L(s - \sqrt{-1}\tilde{F}, F - \tilde{F}) = L(s, F),$$

so that $s(F) = s(F - \tilde{F}) + \sqrt{-1}\tilde{F}$. Thus we may assume $\tilde{F} = 0$. An easy calculation shows

$$\frac{d^2}{dt^2}\bigg|_{t=0} s(t) = \frac{1}{\int_{\Sigma} r\, d\mu}\frac{\partial^2}{\partial t^2}\lambda(h, 0).$$

Using [22] again, we find

$$\begin{aligned}\frac{\partial^2}{\partial t^2}\lambda\bigg|_{s=h,\, t=0} &= -\frac{d^2}{dt^2}\bigg|_{t=0} P(-hr^{+} + tf^{+}) \\ &= -\sum_{n\in\mathbf{Z}}\int f(x) f(\sigma^n x)\, d\mu(x).\end{aligned}$$

Thus it suffices to show that the last term is equal to

$$-\int r\, d\mu \times \lim_{t\to\infty}\frac{1}{t}\int_{X} dm(x)\left(\int_0^t F(\varphi_\tau x)\, d\tau\right)^2.$$

If we put

$$\rho_F(t) = \int_X F(x)\, F(\varphi_t\, x)\, dm(x),$$

then we have

$$\int_{-\infty}^{\infty} \rho_F(t)\, dt = \lim_{t\to\infty} \frac{1}{t} \int_X dm(x) \left(\int_0^t F(\varphi_\tau\, x)\, d\tau\right)^2.$$

Note the following identity

$$F\sigma_t^r(x, u) = \sum_{n\in \mathbf{Z}} \int_0^{r(\sigma^n x)} F(\sigma^n\, x, v)\, \delta(u + t - v - r^n(x))\, dv$$

(cf. [21]), where

$$\begin{aligned} r^n(x) &= r(x) + r(\sigma x) + \dots + r(\sigma^{n-1}\, x) \quad n \geqslant 0 \\ r^{-n}(x) &= -\, (r(\sigma^{-1}\, x) + \dots + r(\sigma^{-n}\, x)) \quad n \geqslant 1. \end{aligned}$$

Therefore

$$\begin{aligned} \int_{-\infty}^{\infty} \rho_F(t)\, dt &= \int_{-\infty}^{\infty} dt \int_{\Sigma^r} F(y)\, F(\sigma_t^r y)\, dm(y) \quad (y = (x, u)) \\ &= \left(\int r\, d\mu\right)^{-1} \sum_{n\in\mathbf{Z}} \int d\mu(x) \left(\int_0^{r(x)} F(x, u)\, du\right)\left(\int_0^{r(\sigma^n x)} F(\sigma^n x, v)\, dv\right) \\ &= \left(\int r\, d\mu\right)^{-1} \sum_{n\in\mathbf{Z}} \int f(x) f(\sigma^n\, x)\, dm(x). \end{aligned}$$

This completes the proof.

Let $\pi : X \to M$ be the unit tangent sphere bundle over a compact Riemannian manifold M, and (X, φ_t) be the geodesic flow. Assume that $\{\varphi_t\}$ is of Anosov type. It should be noted that closed orbits correspond to prime closed geodesics in M, and the least period is just the length of closed geodesics. We shall show that $\Phi \equiv 0$ on $H^1(X, \mathbf{R})$. In fact, a much stronger assertion can be verified. That is, $s(\chi)$ is real for all characters χ near the trivial one. From (1.5), we have

(1.8) $$\overline{s(\chi)} = s(\overline{\chi}).$$

On the other hand, by reversing the orientation of closed geodesics, we obtain an involution $\mathfrak{p} \to \mathfrak{p}'$ acting in the set of closed orbits. Since the character χ near the trivial one comes from a character of $H_1(M, \mathbf{Z})$ via the induced homomorphism $\pi_* : H_1(X, \mathbf{Z}) \to H_1(M, \mathbf{Z})$, one has

$$\overline{\chi([\mathfrak{p}])} = \chi(-\,[\mathfrak{p}]) = \chi([\mathfrak{p}'])$$

(note that, if $\dim M > 2$, then π_* is an isomorphism, and if $\dim M = 2$, then the kernel of π_* is isomorphic to $\mathbf{Z}/\chi(M)\,\mathbf{Z}$, where $\chi(M)$ is the Euler number of M). Therefore

$$L(s, \overline{\chi}) = L(s, \chi),$$

and $\overline{s(\chi)} = s(\overline{\chi}) = s(\chi)$.

Suppose now that M is a locally symmetric space of negative curvature. Let $\lambda_0(\chi)$ denote the first eigenvalue of the twisted Laplacian Δ_χ acting on sections of the flat line bundle associated with the character χ of $H_1(M, \mathbf{Z})$. In view of the relationships between the L-function $L(s, \chi)$ and the Selberg zeta function, we get

$$s(\chi) = \frac{h}{2} + \left(\frac{h^2}{4} - \lambda_0(\chi)\right)^{1/2}.$$

Thus, employing the result in [11], we obtain

Proposition **1.3.** — *Let* η *be a harmonic* 1*-form on* M. *Then*

$$\delta(\pi^* \eta, \pi^* \eta) = \frac{1}{(h/2)\,\mathrm{vol}(M)} \int_M |\eta|^2.$$

It should be noted, in this special case, that the measure m coincides with the Liouville measure on the unit tangent sphere bundle.

2. Singularities of L-functions on the critical line

In this section, we are concerned with poles of $L(s, \chi)$ on the line $\mathrm{Re}\, s = h$.

We denote by E^s (resp. E^u) the contracting subbundle (resp. the expanding subbundle) of TX.

Proposition **2.1.** — *The following three conditions are equivalent.*

(1) *There exists a non-trivial character* χ *such that* $L(s, \chi)$ *has a pole on the line* $\mathrm{Re}\, s = h$.

(2) *The covariance form* δ *is degenerate.*

(3) $E^s \oplus E^u$ *is integrable in the sense that it is the tangent bundle of a* C^1 *foliation.*

The following lemma is frequently used in the proof of Proposition 2.1.

Lemma **2.2.** — (1) *(V. Guillemin and D. Kazhdan* [8]*).* — *Given a smooth function* f *on* X, *satisfying*

$$\int_p f = 0$$

for all closed orbits p, *there exists a function* $u \in C^1(X)$ *such that, for* $t > 0$

$$u(\varphi_t\, x) - u(x) = \int_0^t f(\varphi_\tau\, x)\, d\tau.$$

(2) *Given a smooth function* f *on* X, *satisfying*

$$\exp 2\pi\sqrt{-1} \int_p f = 1$$

for all closed orbits p, *there exists a function* $u \in C^1(X)$ *such that* $|u(x)| = 1$ *and, for* $t > 0$,

$$u(\varphi_t\, x) = u(x) \exp\left(2\pi\sqrt{-1} \int_0^t f(\varphi_\tau\, x)\, d\tau\right).$$

See [8] for the proof of (1). As for (2), it is a multiplicative version of (1) and can be proven in much the same way. We omit the proof.

Proof of Proposition **2.1.** — We prove (1) $\to$ (2). Suppose that $L(s, \chi)$ has a pole on the line $\operatorname{Re} s = h$ for some non-trivial character χ. It is known ([3]) that $L(s, \chi)$ has a pole at $s = h + \sqrt{-1}\,a$ if and only if $\chi([\mathfrak{p}]) = \exp \sqrt{-1}\,a\ell(\mathfrak{p})$ for all closed orbits $\mathfrak{p}$. Note that $a \neq 0$ and $\chi^k \neq \mathbf{1}$ for every nonzero integer k (if $\chi^k = \mathbf{1}$, then $ka\ell(\mathfrak{p}) \in 2\pi\mathbf{Z}$ for all $\mathfrak{p}$, and hence the flow is isomorphic to the suspension of an Anosov diffeomorphism; this contradicts the assumption that the flow is weakly mixing). Hence we may assume, without loss of generality, that the character χ is in the identity component of the character group of $H_1(X, \mathbf{Z})$ and is non-trivial, so that

$$\chi([\mathfrak{p}]) = \exp 2\pi \sqrt{-1} \int_{\mathfrak{p}} \omega$$

for some closed (non-exact) 1-form ω. The function g defined by

$$g(x) = \frac{a}{2\pi} - \langle \omega, Z \rangle (x)$$

satisfies the condition in Lemma 2.2 (2). Hence there exists a C^1 function u such that

$$Zu = 2\pi \sqrt{-1}\, u \left(\frac{a}{2\pi} - \langle \omega, Z \rangle\right).$$

We now put

$$\rho = \omega + \frac{1}{2\pi\sqrt{-1}} u^{-1}\, du.$$

Then the continuous real 1-form ρ satisfies

$$\langle \rho, Z \rangle = \frac{a}{2\pi} = \text{constant},$$

and is closed in the sense that for every C^1 immersed two-disc σ with piecewise C^1 boundary $\partial\sigma$,

$$\int_{\partial\sigma} \rho = 0.$$

Take a smooth closed 1-form ω' which is cohomologous to ρ, so that

$$\int_{\mathfrak{p}} \omega' = \int_{\mathfrak{p}} \rho = \frac{a}{2\pi} \ell(\mathfrak{p})$$

for all closed orbits $\mathfrak{p}$. On the other hand, we observe that $\frac{a}{2\pi} = \int_X \langle \omega', Z \rangle\, dm$. In fact, applying Lemma 2.2 (1), we may find a C^1 function u such that

$$\langle \omega', Z \rangle - \frac{a}{2\pi} = Zu.$$

Integrating both sides, we obtain the desired identity. To show that ω' lies in the null space of δ, we shall prove that $\delta(\eta, \eta) = 0$ if and only if

$$\int_{\mathfrak{p}} \eta = \ell(\mathfrak{p})\, \Phi(\eta)$$

for all closed orbits $\mathfrak{p}$. Let $F = \langle \eta, Z \rangle$. By [22],

$$\lim_{t \to \infty} \frac{1}{t} \int_X dm(x) \left(\int_0^t F(\varphi_\tau x)\, d\tau - t \int F\, dm \right)^2 = 0 \tag{2.1}$$

if and only if there exists $u \in C^0(X)$ such that

$$\int_0^t F(\varphi_\tau x)\, d\tau - t \int F\, dm = u(\varphi_t x) - u(x). \tag{2.2}$$

Hence, for all closed orbits $\mathfrak{p}$,

$$\int_{\mathfrak{p}} F - \ell(\mathfrak{p}) \int F\, dm = 0. \tag{2.3}$$

Conversely, if $\int_{\mathfrak{p}} F = \ell(\mathfrak{p}) \int F\, dm$ for every closed orbit $\mathfrak{p}$, then again by Lemma 2.2 (2), the equality (2.2) holds for some $u \in C^1(X)$, from which (2.1) follows. This completes the proof of (1) $\to$ (2).

We now proceed to the proof of (2) $\to$ (3). Let ω be an element in the null space of the quadratic form δ. Then there exists a continuous closed 1-form ρ cohomologous to ω such that

$$\langle \rho, Z \rangle = \text{constant } (= \Phi(\omega)).$$

Moreover the one-form ρ is uniquely determined. To see this, take a function $u \in C^1(X)$ such that

$$\langle \omega, Z \rangle - \Phi(\omega) = Zu.$$

Putting $\rho = \omega - du$, we find $\langle \rho, Z \rangle = \Phi(\omega)$. To show the uniqueness of ρ, it suffices to prove that if $\langle du, Z \rangle = c$ ($=$ const), then $du = 0$. But this is clear from the fact that $c = \int \langle du, Z \rangle\, dm = 0$, and $Zu = \langle du, Z \rangle = 0$. Transitivity of the flow implies that $u =$ constant.

The 1-form ρ constructed above is closed. Therefore the subbundle $\operatorname{Ker} \rho \subset TX$ is integrable (see J. Plante [19]). To complete the proof of (2) $\to$ (3), we shall show that $\operatorname{Ker} \rho = E^s \oplus E^u$. Since $\varphi_t^* \omega = \varphi_t^* \rho + d(\varphi_t^* u)$ and $\varphi_t^* \omega = \omega + df$ for some smooth function f, the form $\varphi_t^* \rho$ is cohomologous to ω. We find that

$$\begin{aligned} \langle \varphi_t^* \rho, Z \rangle (x) &= \langle (\varphi_t^* \rho)(x), Z(x) \rangle \\ &= \langle \rho(\varphi_t x), (\varphi_{t*} Z)(x) \rangle \\ &= \langle \rho(\varphi_t x), Z(\varphi_t x) \rangle, \end{aligned}$$

from which it follows that $\langle \varphi_t^* \rho, Z \rangle =$ constant, and hence $\varphi_t^* \rho = \rho$. Now let v be an element in E^s. Then

$$\begin{aligned} |\langle \rho, v \rangle| &= |\langle \varphi_t^* \rho, v \rangle| \\ &= |\langle \rho(\varphi_t x), \varphi_{t*} v \rangle| \\ &\leqslant \text{const} \, \| \varphi_{t*} v \| \to 0, \end{aligned}$$

as t goes to $+\infty$. Thus we get $\langle \rho, v \rangle = 0$. In a similar way, we get $\langle \rho, v \rangle = 0$ for v in E^u. Therefore $\operatorname{Ker} \rho = E^s \oplus E^u$.

Finally we prove (3) $\to$ (1). By [19], Proposition 2.3, one can find a closed 1-form ρ such that $\operatorname{Ker} \rho = E^s \oplus E^u$ and

$$\int_s^t \langle \rho, Z \rangle (\varphi_\tau x) \, d\tau = t - s, \quad s, t \in \mathbf{R}; \ x \in X.$$

In particular,

$$\int_{\mathfrak{p}} \rho = \ell(\mathfrak{p})$$

for all closed orbits $\mathfrak{p}$. Since there exists a character χ satisfying

$$\chi([\mathfrak{p}]) = \exp 2\pi \sqrt{-1} \int_{\mathfrak{p}} \rho \equiv \exp 2\pi \sqrt{-1} \, \ell(\mathfrak{p})$$

for all closed orbits $\mathfrak{p}$, $L(s, \chi)$ has a pole $s = h + 2\pi\sqrt{-1}$. This completes the proof.

We do not know whether there exists a weakly mixing Anosov flow satisfying the conditions in Theorem 2.1. We propose

Conjecture. — If an Anosov flow is weakly mixing, then the quadratic form δ is non-degenerate.

In view Proposition 2.1, this conjecture is equivalent to the following one proposed by J. Plante in [19].

Conjecture. — If (X, φ_t) is an Anosov flow such that $E^s \oplus E^u$ is integrable, then it is (modulo change of time scale by a constant factor) the suspension of an Anosov diffeomorphism of a C^1 compact submanifold of codimension one in X.

In his paper [19], Plante showed that a flow satisfying the above condition is topologically conjugate to the suspension of an Anosov diffeomorphism. Especially, each $\alpha \in H_1(X, \mathbf{Z})$ contains only finitely many closed orbits.

We now relate the non-eixstence of poles on the critical line and the vanishing of the winding cycle.

Lemma **2.3.** — *If* $\Phi(H^\dagger) = (0)$ *and* $\chi(\neq \mathbf{1}) \in \hat{H}$, *then* $L(s, \chi)$ *has no poles on* $\operatorname{Re} s = h$.

Proof. — The arguments in the proof of Proposition 2.1 says that if $L(s, \chi)$ has a pole on $\operatorname{Re} s = h$ for a non-trivial character χ in $\hat{H}$, then there exists a closed, non-exact 1-form ω in $H^\dagger$ such that

$$\int_{\mathfrak{p}} \omega = \Phi(\omega)\, \ell(\mathfrak{p}) = 0$$

for all closed orbits $\mathfrak{p}$. Since closed orbits span the homology group $H_1(X, \mathbf{R})$ ([20]), the form ω must be exact. This is a contradiction.

Remark. — The above argument says that the quadratic form δ is positive definite on $\operatorname{Ker} \Phi$.

Proposition **2.4.** — *Suppose* $\Phi(H^\dagger) = (0)$. *Then there exists an open domain* $\mathscr{D}$ *containing* $\{\operatorname{Re} s \geqslant h\}$ *such that, for all* $\chi \in \hat{H}$, $L(s, \chi)$ *has no pole in* $\mathscr{D}$ *except for* $s(\chi)$.

Proof. — Suppose the contrary. Then there exist convergent sequences $\chi_k \in \hat{H}$ and $s_k \in \mathbf{C}$ such that $\operatorname{Re} \lim s_k = h$, s_k is a pole of $L(s, \chi_k)$, and $s_k \neq s(\chi_k)$. Put

$$s_0 = \lim s_k, \qquad \chi_0 = \lim \chi_k.$$

By T. Adachi [1], 1 is an eigenvalue of the twisted Ruelle operator $\mathscr{L}_{s_k, \chi_k}$. The perturbation theory leads to the conclusion that 1 is also an eigenvalue of $\mathscr{L}_{s_0, \chi_0}$, so that s_0 is a pole of $L(s, \chi_0)$. From Lemma 2.3, it follows that $\chi_0 = \mathbf{1}$ and $s_0 = h$. Since $s(\chi)$ is a unique pole of $L(s, \chi)$ around $s = h$ for χ near $\mathbf{1}$, one has $s_k = s(\chi_k)$, thus yielding a contradiction.

3. Integrals of higher logarithmic derivatives of L-functions over the character group

Throughout this section, we assume that $\Phi(H^\dagger) = (0)$, so that the hessian of $\operatorname{Re} s(\chi)$ at $\chi = \mathbf{1}$ is negative definite.

Our approach to Theorem 1 is to analyze the following function.

$$F_\alpha(s) = \int_{\hat{H}} \chi(-\alpha) \left(-\frac{d}{ds}\right)^g \frac{L'(s, \chi)}{L(s, \chi)}\, d\chi, \quad \alpha \in H,$$

where $d\chi$ denotes the normalized Haar measure on the character group $\hat{H}$ and $g = [b/2]$. We shall take up the method developed in [11], where we treated a special case corresponding to the situation that $\operatorname{Im} s(\chi) \equiv 0$ and b is even.

To avoid the repeated use of a lengthy statement, we make

Definition **3.1.** — A holomorphic function $F(s)$ defined in $\operatorname{Re} s > h$ is said to satisfy the property $*_1$ (resp. $*_2$) if the limit $\lim_{\varepsilon \downarrow 0} F(h + \varepsilon + \sqrt{-1}\, t)$ exists almost everywhere on $\mathbf{R} = \{ t \}$, and is in $L^1_{\mathrm{loc}}(\mathbf{R})$ (resp. $W^{1,1}_{\mathrm{loc}}(\mathbf{R})$, the Sobolev space consisting of locally integrable functions with locally integrable derivatives), and there exists a locally integrable function $h(t)$ such that

$$| F(s) | \leqslant h(t), \qquad s = h + \varepsilon + \sqrt{-1}\, t.$$

In what follows, we put, for brevity,

$$c = (2\pi)^{-b/2} \operatorname{vol}(\hat{H})^{-1} = Ch,$$

where C is the constant given in the statement of Theorem 1.

Proposition **3.2.** — *Assume that* Φ *vanishes on* $H^\dagger$.

(1) *If* b *is even, then* $F_\alpha(s) - \dfrac{c}{s-h}$ *satisfies the property* $*_1$.

(2) *If* b *is odd, then* $F_\alpha(s) - \dfrac{\sqrt{\pi}\, c}{\sqrt{s-h}}$ *satisfies the property* $*_2$.

An easy calculation yields the following technical lemma, which we shall use repeatedly in the proof of Proposition 3.2.

Lemma **3.3.** — *Let* μ, $a > 0$, *and let* b *be a real number. If* $m, n \geqslant 0$ *and* $\min\{ m, n\} \geqslant 1$, *then*

$$\int_{-\mu}^{\mu} \frac{1}{(t^2 + a^2)^m \, ((t-b)^2 + a^2)^n} \, dt = O(a^{-2(m+n)+1}) \quad \text{as } a \downarrow 0.$$

Proof of Proposition **3.2.** — By Proposition 2.4, we find that $F_\alpha(s)$ is holomorphic in $\mathscr{D} \backslash \{ s(\chi)\,;\, \chi \in U \}$, and hence the limit $\lim_{\varepsilon \downarrow 0} F_\alpha(h + \varepsilon + \sqrt{-1}\, t)$ exists and is smooth except for $t = 0$. We may write

$$F_\alpha(s) = \int_U \chi(-\alpha) \left(- \frac{d}{ds} \right)^g \frac{1}{s - s(\chi)} \, d\chi + h_1(s),$$

where U is a (small) open neighborhood of $\chi = \mathbf{1}$ in $\hat{H}$, and $h_1(s)$ is a holomorphic function in $\mathscr{D}$. Identifying U with a neighborhood of 0 in $H^\dagger$, we transform the integral into

$$\frac{1}{\operatorname{vol}(\hat{H})} \int_U \chi_\omega(-\alpha) \left(- \frac{d}{ds} \right)^g \frac{1}{s - s(\chi_\omega)} \, d\omega,$$

where $d\omega$ denotes the Lebesgue measure on $H^\dagger$ induced from the metric δ. We apply the following refinement of the Morse lemma to the even function $\operatorname{Re} s(\chi_\omega)$.

Lemma **3.4.** — *Let $f(x)$ be an even smooth function in a neighborhood of* 0 *in* $\mathbf{R}^n$, *with* $f(0) = 0$. *If* 0 *is a non-degenerate critical point for* f, *then there exists a local coordinate system* $y = (y_1, \ldots, y_n)$ *in a neighborhood of* 0 *with* $y(0) = 0$ *and such that* $y(-x) = -y(x)$ *and*

$$f(y) = y_1^2 + \ldots + y_k^2 - y_{k+1}^2 - \ldots - y_n^2.$$

The proof is carried out by a careful choice of the coordinate in the usual proof of the Morse lemma.

Thus one can find a local coordinate $x(\omega) = \{x_i(\omega)\}_{i=1}^{b}$ such that $x(-\omega) = -x(\omega)$ and

$$\mathrm{Re}\, s(\chi_\omega) = h - \| x \|^2, \quad (\| x \|^2 = \sum_{i=1}^{b} x_i^2).$$

By Proposition 1.1, the integral is transformed into

$$\textbf{(3.1)} \qquad \frac{1}{\mathrm{vol}(\hat{\mathrm{H}})} (2\pi^2)^{-b/2} \int_{\mathrm{U}} \left(-\frac{d}{ds}\right)^{g} \frac{1 + \mathrm{P}(x)}{s - h + \| x \|^2 + \sqrt{-1}\,\mathrm{Q}(x)}\, dx,$$

where $\mathrm{P}(x)$ is a complex valued function with $\mathrm{P}(0) = 0$ and $\mathrm{Q}(x)$ is a real valued odd function (remember that $\mathrm{Im}\, s(\chi)$ is an odd function). Hence, $\mathrm{Q}(0) = \mathrm{Hess}\, \mathrm{Q}(0) = 0$. From the assumption that $\Phi = 0$ on $\mathrm{H}^\dagger$, we have $\nabla \mathrm{Q}(0) = 0$. We may assume that $\mathrm{U} = \{x; \| x \| \leqslant a\}$. Changing to the polar coordinate, we find that (3.1) is equal to

$$\frac{1}{\mathrm{vol}(\hat{\mathrm{H}})} (2\pi^2)^{-b/2} \int_{\mathrm{S}^{b-1}} d\Omega \int_0^a \left(-\frac{d}{ds}\right)^{g} \frac{1 + \mathrm{P}(r\Omega)}{s - h + r^2 + \sqrt{-1}\,\mathrm{Q}(r\Omega)}\, r^{b-1}\, dr,$$

which we write $\mathrm{Y}(s)$.

We first show (1). In this case, $g = b/2$, and $\mathrm{Y}(s)$ equals

$$\textbf{(3.2)} \qquad \frac{1}{\mathrm{vol}(\hat{\mathrm{H}})} (2\pi^2)^{-b/2} \Bigg(\Omega_{b-1} \int_0^a \left(-\frac{d}{ds}\right)^{g} \frac{r^{b-1}}{s - h + r^2}\, dr$$
$$+ \int_{\mathrm{S}^{b-1}} d\Omega \int_0^a \frac{\mathrm{R}_0(r\Omega)}{(s - h + r^2)^{g+1}}\, dr$$
$$+ \sum_{i=1}^{g+1} \int_{\mathrm{S}^{b-1}} d\Omega \int_0^a \frac{\mathrm{R}_i(r\Omega)}{(s - h + r^2)^i \,(s - h + r^2 + \sqrt{-1}\,\mathrm{Q}(r\Omega))^{g+1}}\, dr \Bigg),$$

where $\Omega_{b-1} = \int_{\mathrm{S}^{b-1}} d\Omega$ denotes the volume of the unit $(b-1)$-sphere, and

$$\mathrm{R}_0(r\Omega) = g!\, r^{b-1}\, \mathrm{P}(r\Omega),$$
$$\mathrm{R}_i(r\Omega) = g!\, r^{b-1} (1 + \mathrm{P}(r\Omega)) \binom{g+1}{i} \left(\sqrt{-1}\,\mathrm{Q}(r\Omega)\right)^i \quad \text{for } i \geqslant 1.$$

Note that $| \mathrm{R}_0(r\Omega) | \leqslant \mathrm{C} r^b$, $| \mathrm{R}_i(r\Omega) | \leqslant \mathrm{C} r^{b-1+3i}$ $(i \geqslant 1)$. By a straightforward computation, we find that the first term of (3.2) equals $c/(s - h)$ + (a holomorphic function in $\operatorname{Re} s \geqslant h$). To show that the other terms of (3.2) converge to locally integrable functions as ε tends to zero, we set

$$h_2(t) = \int_0^a \frac{r^b}{(t^2 + r^4)^{(g+1)/2}}\, dr,$$

$$h_3^i(t) = \int_0^a \frac{r^{b-1+3i}}{(t^2 + r^4)^{i/2}\,((t + Q(r\Omega))^2 + r^4)^{(g+1)/2}}\, dr, \quad \text{for } i \geqslant 1,$$

which, up to multiplication by a constant, dominate the second and third terms respectively. By Lemma 3.3, we have

$$\int_{-\mu}^{\mu} \frac{r^b}{(t^2 + r^4)^{(g+1)/2}}\, dt = O(1)$$

$$\int_{-\mu}^{\mu} \frac{r^{b-1+3i}}{(t^2 + r^4)^{i/2}\,((t + Q(r\Omega))^2 + r^4)^{(g+1)/2}}\, dt = O(r^{i-1}),$$

hence

$$\int_0^a \int_{-\mu}^{\mu} \frac{r^b}{(t^2 + r^4)^{(g+1)/2}}\, dt\, dr < \infty$$

$$\int_0^a \int_{-\mu}^{\mu} \frac{r^{b-1+3i}}{(t^2 + r^4)^{i/2}\,((t + Q(r\Omega))^2 + r^4)^{(g+1)/2}}\, dt\, dr < \infty.$$

By Fubini's theorem (P. R. Halmos [9], p. 147, Theorem B), the integrals $\int_{-\mu}^{\mu} h_2(t)\, dt$ and $\int_{-\mu}^{\mu} h_3^i(t)\, dt$ are finite. Thus $Y(s) - c/(s-h)$ satisfies the property $*_1$. This proves (1).

We now proceed to the proof of (2). In this case, $g = (b-1)/2$. We write

$$Y(s) = \frac{1}{\operatorname{vol}(\hat{H})} (2\pi^2)^{-b/2} (Y^1(s) + Y^2(s) + Y^3(s) + Y^4(s)),$$

where

$$Y^1(s) = \int_U \left(-\frac{d}{ds}\right)^g \frac{1}{s - h + \|x\|^2}\, dx,$$

$$Y^2(s) = \int_U \left(-\frac{d}{ds}\right)^g \frac{\nabla P(0)\, x}{s - h + \|x\|^2}\, dx,$$

$$Y^3(s) = \int_U \left(-\frac{d}{ds}\right)^g \frac{P(x) - \nabla P(0)\, x}{s - h + \|x\|^2}\, dx,$$

$$Y^4(s) = \int_U \left(-\frac{d}{ds}\right)^g \left(\frac{1 + P(x)}{s - h + \|x\|^2 + \sqrt{-1}\,Q(x)} - \frac{1 + P(x)}{s - h + \|x\|^2}\right) dx.$$

The first term $Y^1(s)$ is equal to

$$\begin{aligned}
&g!\,\Omega_{b-1}\int_0^a \frac{r^{b-1}}{(s-h+r^2)^{g+1}}\,dr\\
&= g!\,\Omega_{b-1}\left(\frac{b-2}{2g}\int_0^a \frac{r^{b-3}}{(s-h+r^2)^g}\,dr - \left[\frac{r^{b-2}}{2g(s-h+r^2)^g}\right]_0^a\right)\\
&= g!\,\Omega_{b-1}\frac{(b-2)!!}{2^g g!}\int_0^a \frac{1}{s-h+r^2}\,dr + (\text{a holomorphic function in } \operatorname{Re} s \geqslant h)\\
&= 2(2\pi)^{(b-1)/2}\,2^{-g}\left[\frac{1}{\sqrt{s-h}}\tan^{-1}\frac{r}{\sqrt{s-h}}\right]_0^a + (\text{the same one as above})\\
&= 2(2\pi)^{(b-1)/2}\,2^{-g}\frac{1}{\sqrt{s-h}}\left(\frac{\pi}{2} - \tan^{-1}\frac{\sqrt{s-h}}{a}\right) + (\text{the same one as above})\\
&= (2\pi)^{(b-1)/2}\,2^{-g}\,\pi\frac{1}{\sqrt{s-h}} + h_4(s)
\end{aligned}$$

where $h_4(s)$ satisfies the property $*_2$. Therefore $\dfrac{1}{\mathrm{vol}(\hat{\mathrm{H}})}(2\pi^2)^{-b/2}\,Y^1(s)$ is equal to

$$\frac{\sqrt{\pi}\,c}{\sqrt{s-h}} + h_5(s),$$

where $h_5(s)$ satisfies the property $*_2$. The second term $Y^2(s)$ is zero. Therefore it suffices to show that $Y^3(s)$ and $Y^4(s)$ satisfy the property $*_2$.

Since $P(x) - \nabla P(0)\,x = O(r^2)$, we obtain

$$\begin{aligned}
|\,Y^3(s)\,| &\leqslant \text{const}\int_U \frac{r^{b+1}}{((\varepsilon+r^2)^2+t^2)^{(g+1)/2}}\,dr\\
&\leqslant \text{const}\int_0^a r^{b+1-2(g+1)}\,dr < \infty,
\end{aligned}$$

from which it follows that $\lim_{\varepsilon\downarrow 0} Y^3(s)$ exists and is locally integrable. Note that the limit can be viewed as one in the distribution sense. We also have

$$\begin{aligned}
\left|\frac{d}{dt}Y^3(s)\right| &\leqslant \text{const}\int_0^a \frac{r^{b+1}}{((\varepsilon+r^2)^2+t^2)^{(g+2)/2}}\,dr\\
&\leqslant \text{const}\int_0^a \frac{r^{b+1}}{(r^4+t^2)^{(g+2)/2}}\,dr.
\end{aligned}$$

Using again Lemma 3.3, we find

$$\int_0^a \int_{-\mu}^{\mu} \frac{r^{b+1}}{(r^4 + t^2)^{(g+2)/2}} \, dt \, dr < \infty,$$

so that $\lim_{\varepsilon \downarrow 0} \frac{d}{dt} Y^3(s)$ exists and is locally integrable. Since, in the distribution sense, we get

$$\frac{d}{dt} \lim Y^3(s) = \lim \frac{d}{dt} Y^3(s),$$

we conclude that $\lim Y^3(s)$ is in $W^{1,1}_{\mathrm{loc}}(\mathbf{R})$. Therefore, $Y^3(s)$ satisfies the property $*_2$.

As for the fourth term $Y^4(s)$, we write

$$Y^4(s) = \sum_{i=0}^{g+1} I_i(s),$$

where

$$I_0(s) = \int_U \frac{T_0(x)}{(s - h + \| x \|^2)\,(s - h + \| x \|^2 + \sqrt{-1}\,Q(x))^{g+1}} \, dx,$$

$$I_i(s) = \int_U \frac{T_i(x)}{(s - h + \| x \|^2)^i\,(s - h + \| x \|^2 + \sqrt{-1}\,Q(x))^{g+1}} \, dx, \quad \text{for } i \geqslant 1.$$

and

$$T_0(x) = (g+1)!\,\sqrt{-1}\,Q_3(x),$$

$$T_1(x) = (g+1)!\,\sqrt{-1}\,(P(x)\,Q(x) + (Q - Q_3)\,(x)),$$

$$T_i(x) = g!(1 + P(x))\,\binom{g+1}{i}\,(\sqrt{-1}\,Q(x))^i \quad (i \geqslant 2).$$

Here, $Q_3(x)$ denotes the term of degree three in the Taylor expansion of $Q(x)$ at $x = 0$. Note that $T_1(x) = O(\| x \|)^4)$ and $T_i(x) = O(\| x \|^{3i})$ for $i \geqslant 2$. We also write

$$I_0(s) = \sum_{i=0}^{g+1} I_0^i(s),$$

where

$$I_0^0(s) = \int_U \frac{T_0(x)}{(s - h + \| x \|^2)^{(g+2)}} \, dx$$

$$I_0^1(s) = \int_U \frac{T_0(x)\,(T_0(x) + T_1(x))}{(s - h + \| x \|^2)^2\,(s - h + \| x \|^2 + \sqrt{-1}\,Q(x))^{g+1}} \, dx,$$

$$I_0^i(s) = \int_U \frac{T_0(x)\,T_i(x)}{(s - h + \| x \|^2)^{(i+1)}\,(s - h + \| x \|^2 + \sqrt{-1}\,Q(s))^{g+1}} \, dx, \quad i \geqslant 2.$$

Clearly, $I_0^0(s) = 0$. We shall show that $I_i(s)$ and $I_0^i(s)$, $i \geqslant 1$, satisfy the property $*_2$. Since the proof is almost the same, we treat only $I_0^1(s)$. It is immediate that

$$|\,I_0^1(s)\,| \leqslant \text{const} \int_0^a \frac{r^{b+5}}{(t^2 + r^4)\,((t + Q(r\Omega))^2 + r^4)^{(\sigma+1)/2}}\,dr$$

and

$$\left|\frac{d}{dt}\,I_0^1(s)\right| \leqslant \text{const} \int_0^a \frac{r^{b+5}}{(t^2 + r^4)\,((t + Q(r\Omega))^2 + r^4)^{(\sigma+1)/2}} \times \left(\frac{1}{(t^2 + r^4)^{1/2}} + \frac{1}{((t + Q)^2 + r^4)^{1/2}}\right) dr,$$

from which, in the same way as above, we conclude that $I_0^1(s)$ satisfies $*_2$.

4. Tauberian theorems and the proof of Theorem 1

We first recall the classical Tauberian theorem due to Wiener and Ikehara (cf. S. Lang [12]).

Proposition **4.1.** — *Let* $\varphi(x)$ *be a monotone nondecreasing function with* $\varphi(x) = 0$ *for* $x \leqslant 0$. *Define* $f(s)$ *via*

$$f(s) = \int_0^\infty e^{-sx}\,d\varphi(x).$$

Assume that

(1) $f(s)$ *is holomorphic in* $\operatorname{Re} s > 1$,

(2) $f(s) - \dfrac{1}{s-1}$ *satisfies the property* $*_1$ *with* $h = 1$.

Then $\lim_{x \to \infty} \varphi(x)/e^x = 1$.

We shall prove the following Tauberian theorem, which is required in the proof of Theorem 1 for an odd b (cf. [26]).

Proposition **4.2.** — *Let* $\varphi(x)$ *be a monotone nondecreasing function with* $\varphi(x) = 0$ *for* $x \leqslant \sigma$, $\sigma > 0$. *Put*

$$f(s) = \int_0^\infty \frac{1}{\sqrt{x}}\,e^{-sx}\,d\varphi(x).$$

Assume that

(1) $f(s)$ *is holomorphic in* $\operatorname{Re} s > 1$,

(2) $f(s) - \dfrac{1}{\sqrt{s-1}}$ *satisfies the property* $*_2$ *with* $h = 1$.

Then $\lim_{x \to \infty} \varphi(x)/e^x = 1/\sqrt{\pi}$.

To prove this proposition, we need several lemmas.

Lemma **4.3.** — *If we put*

$$Z(s) = s\int_0^\infty \frac{1}{\sqrt{x}} e^{-sx}\,\varphi(x)\,dx,$$

then $f(s) - Z(s)$ *satisfies the property* $*_2$, *and hence so does* $Z(s) - \dfrac{1}{\sqrt{s-1}}$.

Proof. — Integration by parts leads to

$$\int_0^\infty \frac{1}{\sqrt{x}} e^{-sx}\,d\varphi(x) = \left[\frac{1}{\sqrt{x}} e^{-sx}\,\varphi(x)\right]_0^\infty + s\int_0^\infty \frac{1}{\sqrt{x}} e^{-sx}\,\varphi(x)\,dx$$
$$+ \frac{1}{2}\int_0^\infty x^{-(3/2)}\,e^{-sx}\,\varphi(x)\,dx.$$

From the fact that $\varphi(x)/\sqrt{x}e^{(1+\varepsilon)x} \leqslant f(1+\varepsilon) < \infty$, we see that

$$\lim_{x\to\infty} \frac{1}{\sqrt{x}} e^{-sx}\,\varphi(x) = 0 \quad \text{for } \operatorname{Re} s > 1.$$

Thus if write $F(s) = \int_0^\infty x^{-(3/2)}\,e^{-sx}\,\varphi(x)\,dx$, then we find

$$\textbf{(4.1)}\qquad -sF'(s) + \frac{1}{2}F(s) = f(s),$$
$$Z(s) = -sF'(s).$$

Since $\lim_{\varepsilon\downarrow 0} f(s)$ is locally integrable (note that $(s-1)^{-1/2}$ is locally integrable), by solving the differential equation (4.1), we find that $\lim_{\varepsilon\downarrow 0} F(s)$ is in $W^{1,1}_{\mathrm{loc}}(\mathbf{R})$. It is also easy to see that $F(s)$ is dominated by a locally integrable function.

The rest of our argument will be a modification of the argument used in the proof of the ordinaty Tauberian theorem. Put $H(x) = \varphi(x)\,e^{-x}$.

Lemma **4.4.** — $\displaystyle\lim_{v\to\infty} \int_{-\infty}^{\lambda v} H\left(y - \frac{w}{\lambda}\right) \frac{\sqrt{y}}{\sqrt{y - w/\lambda}}\,\frac{\sin^2 w}{w^2}\,dw = \sqrt{\pi}.$

Proof. — Put $s = 1 + \varepsilon + \sqrt{-1}\,t$ and

$$K_\varepsilon(t) = \frac{1}{s}\left(Z(s) - \left(\frac{1}{\sqrt{s-1}} + \sqrt{s-1}\right)\right).$$

We then have

$$\begin{aligned} K_\varepsilon(t) &= \int_0^\infty \frac{1}{\sqrt{x}} e^{-sx} \varphi(x)\, dx - \frac{1}{\sqrt{s-1}} \\ &= 2\int_0^\infty e^{-sv^2} \varphi(v^2)\, dv - \frac{1}{\sqrt{s-1}} \qquad (x = v^2) \\ &= 2\int_0^\infty e^{-(s-1)v^2} H(v^2)\, dv - \frac{2}{\sqrt{\pi}}\int_0^\infty e^{-(s-1)v^2}\, dv \\ &= 2\int_0^\infty e^{-(s-1)v^2}\left(H(v^2) - \frac{1}{\sqrt{\pi}}\right) dv. \end{aligned}$$

Thus, we get

$$K_\varepsilon(t) = \lim_{\xi\to\infty} 2\int_0^\xi \left(H(v^2) - \frac{1}{\sqrt{\pi}}\right) e^{-(s-1)v^2}\, dv$$

uniformly in $|t| \leqslant 2\lambda$ when ε is fixed. We multiply the last expression by the function $\sqrt{y}\, e^{\sqrt{-1}tv}\left(1 - \frac{|t|}{2\lambda}\right)$ and integrate over t from -2λ to 2λ. Then, we have

$$\begin{aligned} &\int_{-2\lambda}^{2\lambda} \sqrt{y}\, e^{\sqrt{-1}tv}\left(1 - \frac{|t|}{2\lambda}\right) K_\varepsilon(t)\, dt \\ &\quad = \lim_{\xi\to\infty} 2\int_{-2\lambda}^{2\lambda} \sqrt{y}\, e^{\sqrt{-1}ty}\left(1 - \frac{|t|}{2\lambda}\right)\left(\int_0^\xi \left(H(v^2) - \frac{1}{\sqrt{\pi}}\right) e^{-\varepsilon v^2 - \sqrt{-1}tv^2}\, dv\right) dt. \end{aligned}$$

Since the convergence is uniform, we can interchange the integral and the limit. Thus the last expression equals

$$\begin{aligned} &2\int_0^\infty \left(H(v^2) - \frac{1}{\sqrt{\pi}}\right) e^{-\varepsilon v^2}\left(\int_{-2\lambda}^{2\lambda} \sqrt{y}\left(1 - \frac{|t|}{2\lambda}\right) e^{\sqrt{-1}(y-v^2)t}\, dt\right) dv \\ &= 2\int_0^\infty \left(H(v^2) - \frac{1}{\sqrt{\pi}}\right) e^{-\varepsilon v^2}\left(\int_{-2}^{2} \lambda\sqrt{y}\left(1 - \frac{|u|}{2}\right) e^{\sqrt{-1}\lambda(y-v^2)u}\, du\right) dv \\ &\qquad\qquad (u = t/\lambda). \end{aligned}$$

Replacing the variable v by $w = \lambda(y - v^2)$ (i.e. $v = \sqrt{y - w/\lambda}$), we find that the last term equals

$$\begin{aligned} &\int_{-\infty}^{\lambda y} \left(H\left(y - \frac{w}{\lambda}\right) - \frac{1}{\sqrt{\pi}}\right) e^{-\varepsilon(y-(w/\lambda))}\sqrt{y}\left(\int_{-2}^{2}\left(1 - \frac{|u|}{2}\right) e^{\sqrt{-1}wu}\, du\right)\frac{dw}{\sqrt{y-(w/\lambda)}} \\ &= 2\int_{-\infty}^{\lambda y} H\left(y - \frac{w}{\lambda}\right) e^{-\varepsilon(y-(w/\lambda))}\frac{\sin^2 w}{w^2}\,\frac{\sqrt{y}}{\sqrt{y-(w/\lambda)}}\, dw \\ &\quad - \frac{2}{\sqrt{\pi}}\int_{-\infty}^{\lambda y} \frac{\sqrt{y}}{\sqrt{y-(w/\lambda)}}\,\frac{\sin^2 w}{w^2}\, e^{-\varepsilon(y-(w/\lambda))}\, dw. \end{aligned}$$

We now take the limit as $\varepsilon \to 0$. Since the function

$$\mathrm{H}\left(y - \frac{w}{\lambda}\right) e^{-\varepsilon(y-(w/\lambda))} \frac{\sin^2 w}{w^2} \frac{\sqrt{y}}{\sqrt{y-(w/\lambda)}}$$

is positive, increases as ε tends to zero, and $\mathrm{Z}(s) - \dfrac{1}{\sqrt{s-1}}$ satisfies the property $*_2$, we have

$$\int_{-2\lambda}^{2\lambda} \mathrm{K}_0(t)\left(1 - \frac{|t|}{2\lambda}\right)\sqrt{y}\, e^{\sqrt{-1}ty}\, dy + \frac{2}{\sqrt{\pi}}\int_{-\infty}^{\lambda y} \frac{\sqrt{y}}{\sqrt{y-(w/\lambda)}} \frac{\sin^2 w}{w^2}\, dw = 2\int_{-\infty}^{\lambda y} \mathrm{H}\left(y - \frac{w}{\lambda}\right)\frac{\sin^2 w}{w^2}\frac{\sqrt{y}}{\sqrt{y-(w/\lambda)}}\, dw, \tag{4.2}$$

where $\mathrm{K}_0(t) = \lim_{\varepsilon \downarrow 0} \mathrm{K}_\varepsilon(t)$ and we have applied the Lebesgue convergence theorem to the integral containing K_0. The second term in (4.2) is evaluated in

Sub-Lemma **4.5.** — $\displaystyle\lim_{y\to\infty}\int_{-\infty}^{\lambda y} \frac{\sqrt{y}}{\sqrt{y-(w/\lambda)}}\frac{\sin^2 w}{w^2}\, dw = \int_{-\infty}^{\infty}\frac{\sin^2 w}{w^2}\, dw = \pi.$

Proof. — Put $w = \lambda y \sigma$. We have

$$\int_{-\infty}^{\lambda y} \frac{\sqrt{y}}{\sqrt{y-(w/\lambda)}}\frac{\sin^2 w}{w^2}\, dw = \int_{-\infty}^{1}\frac{1}{\sqrt{1-\sigma}}\frac{\sin^2(\lambda y\sigma)}{\sigma^2}\frac{d\sigma}{\lambda y}.$$

Fix $\varepsilon > 0$. Take a constant $\delta > 0$ so that if $|\sigma| < \delta$, then

$$\left|1 - 1/\sqrt{1-\sigma}\right| < \varepsilon.$$

Thus, we see that

$$\left|\int_{-\infty}^{1}\frac{1}{\sqrt{1-\sigma}}\frac{\sin^2(\lambda y\sigma)}{\sigma^2}\frac{d\sigma}{\lambda y} - \int_{-\delta}^{\delta}\frac{\sin^2(\lambda y\sigma)}{\sigma^2}\frac{d\sigma}{\lambda y}\right| \leqslant \varepsilon\int_{-\delta}^{\delta}\frac{\sin^2(\lambda y\sigma)}{\sigma^2}\frac{d\sigma}{\lambda y} + \int_{-\infty}^{-\delta}\frac{1}{\sqrt{1-\sigma}}\frac{\sin^2(\lambda y\sigma)}{\sigma^2}\frac{d\sigma}{\lambda y} + \int_{\delta}^{1}\frac{1}{\sqrt{1-\sigma}}\frac{\sin^2(\lambda y\sigma)}{\sigma^2}\frac{d\sigma}{\lambda y}.$$

The second term in the right hand side can be estimated as follows.

$$\left|\int_{-\infty}^{-\delta}\frac{1}{\sqrt{1-\sigma}}\frac{\sin^2(\lambda y\sigma)}{\sigma^2}\frac{d\sigma}{\lambda y}\right| \leqslant \int_{-\infty}^{-\delta}\frac{1}{\lambda y\sigma^2}\frac{1}{\sqrt{1-\sigma}}\, d\sigma \to 0 \quad \text{as } y \to \infty.$$

The third term is estimated in a similar way. Hence the conclusion follows from

$$\lim_{y\to\infty}\int_{-\delta}^{\delta}\frac{\sin^2(\lambda y\sigma)}{\sigma^2}\frac{d\sigma}{\lambda y} = \lim_{y\to\infty}\int_{-\lambda y\delta}^{\lambda y\delta}\frac{\sin^2 t}{t^2}\, dt = \pi.$$

Sub-Lemma **4.6.** — *If* $j(t) \in W^{1,1}_{loc}(\mathbf{R})$, *then*

$$\lim_{y\to\infty} \int_0^{2\lambda} \sqrt{y}\, e^{\sqrt{-1}ty} j(t)\, dt = 0.$$

Proof. — This is an easy conclusion from integration by parts.

Applying this lemma to the first term in the left hand side of (4.2) (note that $K_0(t)\left(1 - \dfrac{|t|}{2\lambda}\right)$ is in $W^{1,1}_{loc}(\mathbf{R})$), we have Lemma 4.3.

Lemma **4.7.** — $\overline{\lim_{y\to\infty}}\, H(y) \leqslant \dfrac{1}{\sqrt{\pi}}$.

Proof. — Fix $\lambda > 0$. Note that the integrand is positive. Cutting down the domain of integration, we get

$$\sqrt{\pi} \geqslant \overline{\lim_{y\to\infty}} \int_{-\sqrt{\lambda}}^{\sqrt{\lambda}} H\left(y - \frac{w}{\lambda}\right) \frac{\sqrt{y}}{\sqrt{y - (w/\lambda)}} \frac{\sin^2 w}{w^2}\, dw.$$

Using the monotony of φ and the corresponding property of H, we have in the interval $[-\lambda, \lambda]$,

$$H\left(y - \frac{w}{\lambda}\right) \geqslant H(y - 1/\sqrt{\lambda})\, e^{-2/\sqrt{\lambda}}.$$

Hence

$$\sqrt{\pi} \geqslant \overline{\lim_{y\to\infty}} \int_{-\sqrt{\lambda}}^{\sqrt{\lambda}} H\left(y - \frac{1}{\sqrt{\lambda}}\right) e^{-2/\sqrt{\lambda}} \frac{\sqrt{y}}{\sqrt{y - w/\lambda}} \frac{\sin^2 w}{w^2}\, dw.$$

Since λ is fixed, y can be replaced by $y + 1/\sqrt{\lambda}$. Hence

$$\overline{\lim_{y\to\infty}}\, H(y) \leqslant \frac{\sqrt{\pi}\, e^{-2/\lambda}}{\displaystyle\int_{-\sqrt{\lambda}}^{\sqrt{\lambda}} \frac{\sqrt{y}}{\sqrt{y - w/\lambda}} \frac{\sin^2 w}{w^2}\, dw}.$$

By an argument similar to that of Sub-Lemma 4.5, we get

$$\lim_{\lambda\to\infty} \int_{-\sqrt{\lambda}}^{\sqrt{\lambda}} \frac{\sqrt{y}}{\sqrt{y - w/\lambda}} \frac{\sin^2 w}{w^2}\, dw = \pi.$$

Hence, letting $\lambda \to \infty$, we have

$$\overline{\lim_{y\to\infty}}\, H(y) \leqslant \frac{1}{\sqrt{\pi}}.$$

Lemma **4.8.** — $\underline{\lim_{y\to\infty}}\, H(y) \geqslant \dfrac{1}{\sqrt{\pi}}$.

Proof. — By the monotony of φ, we see that, in the interval $[-\sqrt{\lambda}, \sqrt{\lambda}]$,

$$e^{-2/\sqrt{\lambda}} \mathrm{H}\left(y - \frac{w}{\lambda}\right) \leqslant \mathrm{H}(y + 1/\sqrt{\lambda}).$$

By Lemma 4.7, there exists a constant $\mathrm{Q} > 0$ such that $\mathrm{H}(y) \leqslant \mathrm{Q}$. If we fix $\lambda > 0$, we get

$$\sqrt{\pi} = \varliminf_{y\to\infty} \int_{-\infty}^{\lambda y} \mathrm{H}\left(y - \frac{w}{\lambda}\right) \frac{\sqrt{y}}{\sqrt{y - w/\lambda}} \frac{\sin^2 w}{w^2}\, dw$$

$$\leqslant \mathrm{Q}\int_{-\infty}^{-\sqrt{\lambda}} \frac{dw}{w^2} + \varliminf_{y\to\infty} \int_{-\sqrt{\lambda}}^{\sqrt{\lambda}} \mathrm{H}\left(y + \frac{1}{\sqrt{\lambda}}\right) e^{2/\sqrt{\lambda}} \frac{\sqrt{y}}{\sqrt{y - w/\lambda}} \frac{\sin^2 w}{w^2}\, dw$$

$$+ \mathrm{Q} \lim_{y\to\infty} \int_{\sqrt{\lambda}}^{\lambda y} \frac{\sqrt{y}}{\sqrt{y - w/\lambda}} \frac{\sin^2 w}{w^2}\, dw$$

$$\leqslant \frac{\mathrm{Q}}{\sqrt{\lambda}} + \varliminf_{y\to\infty} \mathrm{H}(y)\, e^{2/\sqrt{\lambda}} \pi + \mathrm{Q} \lim_{y\to\infty} \int_{\sqrt{\lambda}}^{\lambda y} \frac{\sqrt{y}}{\sqrt{y - w/\lambda}} \frac{\sin^2 w}{w^2}\, dw.$$

The third term can be estimated as follows.

$$\int_{\sqrt{\lambda}}^{\lambda y} \frac{\sqrt{y}}{\sqrt{y - w/\lambda}} \frac{\sin^2 w}{w^2}\, dw = \int_{\sqrt{\lambda}}^{\lambda y/2} + \int_{\lambda y/2}^{\lambda y}$$

$$\leqslant \int_{\sqrt{\lambda}}^{\lambda y/2} \frac{1}{\sqrt{2}} \frac{\sin^2 w}{w^2}\, dw + \frac{4\sqrt{y}}{(\lambda y)^2} \int_{\lambda y/2}^{\lambda y} \frac{1}{\sqrt{y - w/\lambda}}\, dw$$

$$\leqslant \frac{1}{\sqrt{2}} \int_{\sqrt{\lambda}}^{\infty} \frac{\sin^2 w}{w^2}\, dw + \frac{4\sqrt{2}}{\lambda y}.$$

Therefore, if we let λ tend to infinity, we get

$$\varliminf_{y\to\infty} \mathrm{H}(y) \geqslant \lim_{\lambda\to\infty} e^{-2/\sqrt{\lambda}} \left(\frac{1}{\sqrt{\pi}} - \frac{\mathrm{Q}}{\pi\sqrt{\lambda}} - \frac{4\sqrt{2}\mathrm{Q}}{\pi\lambda y} - \frac{\mathrm{Q}}{\sqrt{2}\pi} \int_{\sqrt{\lambda}}^{\infty} \frac{\sin^2 w}{w^2}\, dw\right)$$

$$= \frac{1}{\sqrt{\pi}}.$$

The proof of Proposition 4.2 is complete.

We are now in a position to prove Theorem 1. Put

$$\varphi_\alpha(x) = \sum_{k=1}^{\infty} \sum_{\substack{\mathfrak{p} \\ k\ell(\mathfrak{p}) < x \\ k[\mathfrak{p}] = \alpha}} k^{b/2} \ell(\mathfrak{p})^{1+b/2}.$$

By the orthogonal relation of the characters, we obtain

$$\begin{aligned}
F_\alpha(s) &= \sum_{k=1}^{\infty} \sum_{\mathfrak{p}} \int_{\widehat{H}} \chi(-\alpha)\, \chi(k[\mathfrak{p}])\, k^g\, \ell(\mathfrak{p})^{g+1}\, e^{-sk\ell(\mathfrak{p})}\, d\chi \\
&= \sum_{k=1}^{\infty} \sum_{\substack{\mathfrak{p} \\ k[\mathfrak{p}]=\alpha}} k^g\, \ell(\mathfrak{p})^{g+1}\, e^{-sk\ell(\mathfrak{p})} \\
&= \begin{cases} \displaystyle\int_0^\infty e^{-sx}\, d\varphi_\alpha(x) & \text{when } b \text{ is even} \\ \displaystyle\int_0^\infty \frac{1}{\sqrt{x}}\, e^{-sx}\, d\varphi_\alpha(x) & \text{when } b \text{ is odd.} \end{cases}
\end{aligned}$$

Applying the Tauberian theorem (Proposition 4.1 and 4.2), we get

$$\varphi_\alpha(x) \sim C e^{hx},$$

where the constant C is the same one as in the statement of Theorem 1. By the same argument as in [11], the proof of Theorem 1 is now complete.

Remark **1.** — The appearance of Theorem 1 might remind the reader of some number-theoretic density theorems which could be proved by Landau's method (cf. J.-P. Serre [24]). But in order to apply the method, we must prove that $(s - h)\, F_\alpha(s)$ $\big(\text{or } \sqrt{s-h}\, F_\alpha(s)\big)$ has an analytic continuation to an open domain containing $\operatorname{Re} s \geqslant h$, which remains to be proved.

Remark **2.** — Proposition 4.2 is generalized in the following way: Given a monotone nondecreasing function $\varphi(x)$ with $\varphi(x) = 0$ for $x \leqslant \sigma$, $\sigma > 0$, we suppose that $\int_0^\infty x^{-\theta}\, e^{-sx}\, d\varphi(x)$ is holomorphic in $\operatorname{Re} s > 1$, and $\int_0^\infty x^{-\theta}\, e^{-sx}\, d\varphi(x) - (s-1)^{\theta-1}$ satisfies the property $*_2$ for some positive $\theta < 1$. Then $\varphi(x) \sim \Gamma(1-\theta)^{-1}\, e^x$ as $x \uparrow \infty$.

5. Proof of Theorem 2

We first show that the limit

$$\lim_{s \downarrow h} \sum_{\mathfrak{p} \in \Pi(x,\alpha)} \ell(\mathfrak{p})^n\, e^{-s\ell(\mathfrak{p})} \quad (s \in \mathbf{R})$$

exists for every nonnegative integer n. In view of the argument in Section 3, it suffices to show that the following integral

$$\int_U \frac{\chi(-\alpha)}{(\varepsilon + h - s(\chi))^n}\, d\chi$$

is bounded when ε tends to zero. Since the winding cycle Φ does not vanish, choosing a suitable coordinate $x = \{x_i\}_{i=1}^{b}$ in U, we may assume that

$$(5.1)\qquad \frac{\partial f}{\partial x_1} \neq 0 \quad \text{and} \quad \operatorname{Re} f \geqslant 0 \text{ on } U,$$

where $f(x) = h - s(\chi)$. Our assertion reduces to the following one:

Lemma **5.1.** — *Under the assumption* (5.1), *for all* $\varphi \in C_0^\infty(U)$ *and any* $n \geqslant 0$, *the limit*

$$\lim_{\varepsilon \to 0} \int_U \frac{\varphi(x)}{(\varepsilon + f(x))^n}\, dx$$

exists.

Proof. — Integration by parts yields the following equalities:

$$\int_U \frac{\varphi(x)}{(\varepsilon + f(x))^{n+1}}\, dx = \frac{1}{n}\int_U \frac{\partial}{\partial x_1}\left(\left(\frac{\partial f}{\partial x_1}\right)^{-1} \varphi(x)\right) \frac{1}{(\varepsilon + f(x))^n}\, dx \quad (n \geqslant 1),$$

$$\int_U \frac{\varphi(x)}{\varepsilon + f(x)}\, dx = -\int_U \frac{\partial}{\partial x_1}\left(\left(\frac{\partial f}{\partial x_1}\right)^{-1} \varphi(x)\right) \log(\varepsilon + f(x))\, dx,$$

$$\int_U \varphi(x) \log(\varepsilon + f(x))\, dx = -\int_U \frac{\partial}{\partial x_1}\left(\left(\frac{\partial f}{\partial x_1}\right)^{-1} \varphi(x)\right) \Phi(\varepsilon + f(x))\, dx,$$

where $\Phi(z) = \int_0^z \log\tau\, d\tau = \int_0^{|z|} \log\tau\, d\tau + \sqrt{-1}(\arg z)\,|z|$. Then our assertion is derived from the fact that the limit

$$\lim_{\varepsilon \to 0} \int_U \varphi(x)\, \Phi(\varepsilon + f(x))\, dx$$

exists. In fact, $\operatorname{Re}(\varepsilon + f(x)) \geqslant 0$, so that $|\arg(\varepsilon + f(x))| \leqslant \pi/2$. Since the function $\Phi(z)$ on the region $\{z; \operatorname{Re} z \geqslant 0\}$ is continuous, it follows that the above limit exists.

We now consider a Dirichlet series

$$\varphi(s) = \sum_{k=1}^{\infty} a_k\, e^{-\lambda_k s}$$

with $a_k > 0$ and $0 < \lambda_1 < \lambda_2 < \ldots \uparrow \infty$, and assume

$$\pi(x) := \sum_{\lambda_k < x} a_k \leqslant \text{const}\, e^{hx}.$$

Theorem 2 is a consequence of the following lemma.

Lemma **5.2.** — *Suppose that, for every* $n \geqslant 0$,

$$\lim_{s \downarrow h} \left(-\frac{d}{ds}\right)^n \varphi(s) = \lim_{s \downarrow h} \sum_{k=1}^{\infty} a_k\, \lambda_k^n\, e^{-\lambda_k s}$$

exists. Then $\pi(x)/e^{hx} = o(x^{-N})$ *as* $x \uparrow \infty$ *for every positive integer* N.

Proof. — Suppose that the conclusion is not true. Then there exist an integer $M \geqslant 0$ and a sequence $0 < x_1 < x_2 < \ldots \uparrow \infty$ such that

$$\pi(x_i) \geqslant e^{hx_i}/x_i^{M} \quad i = 1, 2, \ldots .$$

We put

$$K = \lim_{s \downarrow h} \sum_{k=1}^{\infty} a_k \lambda_k^{M+1} e^{-\lambda_k s}.$$

Then, for all $s > h$ and i, we have

$$\begin{aligned} K &\geqslant \Sigma_{x_i - m \log x_i \leqslant \lambda_k \leqslant x_i} a_k \lambda_k^{M+1} e^{-\lambda_k s} \\ &\geqslant (x_i - m \log x_i)^{M+1} e^{-x_i s} \Sigma_{x_i - m_i \log x_i \leqslant \lambda_k \leqslant x_i} a_k \\ &\geqslant (x_i - m \log x_i)^{M+1} e^{-x_i s}(e^{hx_i} x_i^{-M} - c e^{h(x_i - m \log x_i)}) \\ &= e^{(h-s)x_i} x_i \left(1 - \frac{m \log x_i}{x_i}\right)^{M+1} - \frac{c(x_i - m \log x_i)^{M+1}}{x_i^{mh}} e^{(h-s)x_i}. \end{aligned}$$

Letting $s \downarrow h$, we have

$$K \geqslant x_i \left(1 - \frac{m \log x_i}{x_i}\right)^{M+1} - \frac{c(x_i - m \log x_i)^{M+1}}{x_i^{mh}}.$$

If we take m with $mh > M + 1$, then the right hand side goes to ∞ when $x_i \uparrow \infty$, whence a contradiction.

6. Equidistribution theorem

Given $f \in C^{\infty}(X)$ and $\alpha \in H$, we put

$$\pi(x, \alpha : f) = \sum_{\substack{\ell(\mathfrak{p}) < x \\ \mathfrak{p} \in \Pi(x, \alpha)}} \ell_f(\mathfrak{p})/\ell(\mathfrak{p}), \quad \ell_f(\mathfrak{p}) = \int_{\mathfrak{p}} f.$$

Theorem 3 is a consequence of Theorem 1 and the following proposition.

Proposition **6.1.** — *Under the same assumption as in Theorem 3, we have*

$$\pi(x, \alpha : f) \sim C \int_X f\, dm \frac{e^{hx}}{x^{(b/2)+1}},$$

where C *is the same constant as in Theorem 1.*

Proof. — This comes from a combination of the argument in W. Parry [15] and Theorem 1. We define a modified L-function by

$$L(s, z, \chi) = \prod_{\mathfrak{p}} (1 - \chi([\mathfrak{p}]) e^{-s\ell(\mathfrak{p}) + z\ell_f(\mathfrak{p})})^{-1}.$$

By differentiating logarithmically with respect to the second variable at $z = 0$, we obtain

$$\text{(6.1)} \qquad \frac{L_z(s, 0, \chi)}{L(s, 0, \chi)} = \frac{s_z(0, \chi)}{s - s(\chi)} + h(s),$$

where $s(z, \chi)$ is the pole of $L(s, z, \chi)$ in a neighborhood of $s = h$, $s_z(0, \chi)$ denotes the derivative with respect to z at $z = 0$, and $h(s)$ is a holomorphic function in $\mathcal{D}$. By [15], $s_z(0, \chi)$ is written as

$$s_z(0, \chi) = \int_X f\, dm + k(\chi),$$

where $k(\chi)$ is a function with $k(\chi_\omega) = O(\|\omega\|)$. Thus if we replace $L'(s, \chi)/L(s, \chi)$ by (6.1) in the proof of Theorem 1, then we get

$$\sum_{\substack{k \\ k\ell(\mathfrak{p}) < x \\ k[\mathfrak{p}] = \alpha}} \sum_{\mathfrak{p}} k^{b/2} \ell_f(\mathfrak{p})\, \ell(\mathfrak{p})^{1+b/2} \sim C \left(\int_X f\, dm \right) e^{hx},$$

from which the assertion follows.

Remark. — It would be interesting to know whether $\pi(x, \alpha, f)$ has an asymptotic expansion. R. Phillips and P. Sarnak [18] (for $f \equiv 1$) and S. Zelditch [25] (for the general case) established an asymptotic formula for a compact hyperbolic space. If one could apply Landau's method (J.-P. Serre [24]) to $L(s, z, \chi)$ an asymptotic expansion could be obtained.

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