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M. J. DIAS CARNEIRO
JACOB PALIS

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BIFURCATIONS AND GLOBAL STABILITY OF FAMILIES OF GRADIENTS

by M. J. DIAS CARNEIRO *and* J. PALIS

Dedicated to René Thom on his sixty fifth anniversary.

In 1967 it was proved that among gradient vector fields on compact boundaryless manifolds, the elements of an open and dense subset are structurally stable: under small perturbations they have their orbit structure unchanged up to orbit preserving homeomorphisms [14], [16]. From this result it follows that the stability of a gradient flow is equivalent to the hyperbolicity of the singularities and transversality of their stable and unstable manifolds.

At the end of that decade, Thom was asking about the bifurcations and stability of families of gradients, specially about k-parameter families with $k \leqslant 4$. The question is very challenging and indeed it might ammount to a rather formidable program, since not even just locally near a singularity the question for $k = 4$ is solved (and this problem by itself is very interesting). A point to stress here is that the dynamic bifurcations of a gradient family are in general considerably richer than those of the corresponding family of potentials; see [3], [4], [21], [22] for comments. Also from the global point of view this comparison is relevant to understand Thom's question: often near a bifurcating singularity there appear secondary bifurcations due to tangencies between invariant (stable and unstable) manifolds from far away singularities. Finer dynamic analysis is then needed to describe the bifurcation diagram and to prove stability of a generic family. In this line, in 1983, in a paper dedicated to Thom on the occasion of his sixtieth birthday, the question for $k = 1$ was settled [15]: among one-parameter families of gradients, the stable ones are dense. These stable families can be characterized up to high codimension degeneracies; see Section I.

The purpose of the present paper is to provide a proof of a similar result for two-parameter families of gradients. New techniques, specially concerning singular invariant foliations, are introduced to study the bifurcation diagrams and to prove stability.

Let us state our result in a precise way. Let M be a compact boundaryless C^∞

manifold. Gradients of real functions can be considered either with respect to a fixed Riemannian metric or to all possible ones. Although our result is true in both cases, we will restrict ourselves to the last one. Let $\chi^2_g(M)$ denote the set of C^∞ two-parameter families of gradients endowed with the C^∞ Whitney topology, the parameter being taken in the unit disc D in $\mathbf{R}^2$, and denote by $\pi_2 : M \times D \to D$ the natural projection. We say that X_μ, $\hat{X}_\mu \in \chi^g_2(M)$ are equivalent if there are homeomorphisms $H : M \times D \to M \times D$ and $\varphi : D \to D$ such that $\pi_2 H = \varphi \pi_2$ and, for each $\mu \in D$, h_μ is an equivalence between X_μ and $\hat{X}_{\varphi(\mu)}$, where h_μ is defined by

$$H(x, \mu) = (h_\mu(x), \varphi(\mu)).$$

That is, h_μ sends orbits of X_μ onto orbits of $\hat{X}_{\varphi(\mu)}$ for each $\mu \in D$. The family X_μ is called (structurally) stable if it is equivalent to all nearby elements in $\chi^g_2(M)$. Our main result can now be presented as follows.

Theorem. — There is an open and dense subset $\mathscr{Y}$ in $\chi^g_2(M)$ whose elements are stable.

Several comments are in order. First of all, as we observe in Section IV, the parameter space in our theorem can be taken to be any compact surface. Second, while the result makes one hopeful of giving a similar positive answer about stability of k-parameter families for $k = 3$ and $k = 4$, it is known that this is not true for $k \geqslant 8$ [18]; actually, it is not true even locally near a singularity [19]. On the other hand, positive local results near a singularity were obtained for $k = 3$ and to some extent $k = 4$ in [22], [4]; however, the question for $k = 4$ is still essentially open and very interesting. We also point out that our result was obtained by Vegter [22], [23] for manifolds of dimension less than or equal to three. These papers and [15] were the starting point of our work. However, the analysis of codimension-two bifurcations in higher dimensions is considerably more elaborated and led us to introduce new kinds of singular invariant foliations (that might even be useful in other contexts); see, for instance, § 1 of Section III below.

The paper is organized in the following way. The first two sections are preparatory ones, so the reader gets acquainted with some basic concepts and tools and the previous result for one-parameter families. To serve as references for other cases, already in Section II we use these tools to exhibit the bifurcation diagrams and to prove local (in the parameter space) stability of families with quadratic and higher order contact between invariant manifolds. In Section III we complete the definition of the subset of families $\mathscr{Y}$ (up to a slight modification still to be performed in the last section) and prove the local stability of its elements, except for the ones already considered in the preceeding section. In Section IV we globalize the result to the whole parameter space.

To be more specific, in Section I we recall the characterization of the stable one-parameter families and from it infer what shall be a corresponding characterization for two parameters. This leads to a list of cases that begins with codimension-one bifurcations, namely a saddle-node and a quasi-transversal tangency. We then have combinations of these two cases, like the simultaneous occurence of two saddle-nodes. There are also the purely codimension-two cases: a codimension-two tangency (cubic contact or lack of dimensions) and a codimension-two singularity. Also, three cases arise from the degeneracy of one of the transversality conditions concerning center-stable and strong-stable, rep. unstable, manifolds that are required for one-parameter families. In Section II we present the basic concept of compatible systems of invariant foliations and provide a brief description of how they are constructed. This kind of foliations have been used in previous work like [6], [14], [15], [16] and [22]. Here, the concept has to be considerably extended to include several new types of singular foliations. Using these foliations, we treat in this section the initial cases of quadratic and cubic (or higher order) contact between invariant manifolds. Finally, in Section III we obtain $\mathcal{Y}$ as the intersection of several open and dense subsets of $\chi_2^g(M)$, each one corresponding to families that present one of the bifurcations listed in Section I. We prove that every family in $\mathcal{Y}$ is stable at every value of the parameter $\mu \in D$; i.e., the family restricted to a small neighbourhood of μ in D is globally stable on M. We then show in Section IV that we can piece together, in terms of the parameter space, our construction of the equivalence between two nearby families in $\mathcal{Y}$, thus proving the result.

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Section I. — Bifurcations of codimension two

We first recall the bifurcations of codimension one and some generic conditions that are imposed in order to obtain stable one-parameter families as in [15]. Let X_μ, $\mu \in \mathbf{R}$, be a family of gradients on M and dim $M = n$.

a) Saddle-nodes. — We say that $X_{\bar\mu}$ has a saddle-node singularity p if one of the eigenvalues of $dX_{\bar\mu}(p)$ is zero and all the others nonzero. Moreover, restricted to a center manifold through p, $X_{\bar\mu}$ has the form $Z_{\bar\mu}(x) = ax^2 \dfrac{\partial}{\partial x} + O(|x|^3)$ with $a \neq 0$ (about center manifolds see [10]). For each μ near $\bar\mu$, there is a μ-dependent center manifold restricted to which X_μ has the form

$$X_\mu(x) = (ax^2 + b(\mu - \bar\mu)) \frac{\partial}{\partial x} + O(|x|^3 + |x(\mu - \bar\mu)| + |(\mu - \bar\mu)|^2)$$

with $a \neq 0$. The saddle-node unfolds generically if also $b \neq 0$. This condition is satisfied by the elements of an open and dense subset of families.

b) Quasi-transversal orbits. — Let p, q be hyperbolic singularities of $X_{\bar\mu}$; that is, all eigenvalues of $dX_{\bar\mu}$ at p and q are nonzero. Let $W^u(p)$ and $W^s(q)$ be their unstable and stable manifolds. Suppose γ is an orbit of tangency between them and assume that $\dim T_r W^u(p) + \dim T_r W^s(q) = \dim M - 1$ for $r \in \gamma$. In local coordinates near r, we have

$$X_{\bar\mu} = \frac{\partial}{\partial x_1}.$$

$$W^u(p) = (x_1, \ldots, x_u, 0, \ldots, 0), \qquad u = \dim W^u(p),$$

$$W^s(q) = (x_1, \ldots, x_k, 0, \ldots, 0, x_{u+1}, \ldots, x_{n-1}, g(x_2, \ldots, x_k)),$$

where $n = \dim M$ and $k = \dim(T_r W^u(p) \cap T_r W^s(q))$. We say that γ is quasi-transversal if g is a Morse function. For each μ near $\bar\mu$, we can write similar expressions for X_μ, $W^u(p_\mu)$ and $W^s(q_\mu)$, g being replaced by a μ-dependent function g_μ. We then say that γ unfolds generically if $\frac{\partial g_\mu}{\partial \mu}(r)\big|_{\mu=\bar\mu} \neq 0$.

c) Generic conditions. — We now list a number of generic conditions concerning the stability of families of gradients.

*c.*1. *Distinct eigenvalues.* — We assume that the eigenvalues of $X_{\bar\mu}$ at the singularities associated with an orbit of tangency have multiplicity one. Hence, there exists a smallest expansion (respectively contraction) and we can consider the strong unstable manifold W^{uu} corresponding to all but the smallest positive eigenvalue (see [15]); similarly for the strong-stable manifold W^{ss}. And, corresponding to the smallest positive eigenvalue and all negative ones, we have a C^1 center-stable manifold W^{cs}, which is transversal to W^{uu}. Similarly for a center-unstable manifold W^{cu}. We observe that, in the presence of an orbit of tangency the assumption on the *multiplicity one of the eigenvalues* at singularities are generic (open and dense) for two-parameter families of gradients. In fact, a failure of these conditions gives rise to a subset of codimension at least 3: an orbit of tangency corresponds to subsets of codimension at least one and a multiple eigenvalue of the linear part to subsets of codimension at least two (since it is a symmetric operator).

*c.*2. *Noncriticality.* — We assume that the strong stable and strong unstable manifolds of a saddle-node are transversal to the unstable, resp. stable, manifolds of all other singularities. Similarly for the singularities associated to a tangency.

*c.*3. *Transversality of center-unstable and stable manifolds at a tangency.* — If γ is an orbit of tangency between $W^u(p)$ and $W^s(q)$, we require $W^{cu}(p)$ to be transversal to $W^s(q)$; similarly for $W^u(p)$ and $W^{cs}(q)$.

*c.*4. *Linearizability.* — For a family X_μ with one of the bifurcations of type I through IX below, we assume that $X_{\bar\mu}$ is C^m linearizable transversally to a center manifold of a saddle-node or near the singularities associated with an orbit of tangency. Actually, this linearization is also required for each μ near the bifurcation value $\bar\mu$. The integer m is taken to be bigger than $\rho + 2$, where ρ is the maximum ratio of positive, resp. negative, eigenvalues of $dX_{\bar\mu}$ at the singularity (cf. *c.*1). By [17], [20], the linearization may be taken to depend differentiably on the parameter and these conditions are generic for two-parameter families. We do not assume this hypothesis when we deal with strictly codimension-one cases: a quasi-transversal orbit of tangency, treated in Theorem A, Section II, or a saddle-node, treated in § 7.B, Section III.

Given a family of gradients, a ***parameter value*** is called ***regular*** if it corresponds to a stable field, in this case a Morse-Smale gradient field (***hyperbolic singularities*** and ***transversality*** between stable and unstable manifolds); otherwise, it is a ***bifurcation value.*** Now, for an open and dense subset of arcs each bifurcation value is isolated and it corresponds to a unique tangency or to a nonhyperbolic singularity, for which conditions *a), b)* and *c)* are satisfied. These arcs are stable [15].

What we prove in Sections II and III is the analogue of these results for two-parameter families of gradients. The subset of families we consider must now include codimension-two bifurcations; they are listed here and studied in detail in Section III except for case VIII (cubic contact) which is considered in Section II. We keep denoting a family of gradients by X_μ, but now μ varies in the unit disc D in $\mathbf{R}^2$.

I. *A quasi-transversal orbit of tangency with criticality.* — For some $\mu = \bar{\mu}$, there are singularities p, q such that $W^u(p)$ and $W^s(p)$ have a quasi-transversal tangency. However, unlike in *c*.2 above, there is another singularity s such that $W^u(s)$ and $W^{ss}(p)$ are nontransverse along a unique orbit, which is quasi-transversal, or similarly there is such an orbit in $W^{uu}(q)$. Except for that, all conditions in *a), b)* and *c)* above are satisfied.

II. *Two quasi-transversal orbits of tangency.* — Two orbits of tangency may occur simultaneously, but they must satisfy the generic conditions *b)* and *c)* above.

III. *A saddle-node with criticality.* — The unstable manifold of some singularity is nontransverse to the strong stable manifold of a saddle-node along a unique orbit which is quasi-transversal, or similarly with respect to the strong unstable manifold of a saddle-node. All other generic conditions in *a), b)* and *c)* are satisfied.

IV. *Two saddle-nodes.* — Two saddle-nodes occur simultaneously and both satisfy the generic conditions *a)* and *c)* above.

V. *A saddle-node and a quasi-transversal tangency.* — These two bifurcations may occur simultaneously; again we assume all generic conditions we have mentioned concerning hyperbolicity of the other singularities, linearizability and transversality in *a), b)* and *c).*

VI. *A quasi-transversal orbit of tangency along which the corresponding stable and a center-unstable manifolds are tangent.* — Since all center-unstable manifolds are tangent on each orbit of the unstable manifold, the condition does not depend on which center-unstable manifold we consider. We also require all singularities to be hyperbolic and the generic conditions in *b), c*.1, *c*.2 and *c*.4 to be satisfied. We will show in § 5 of Section III that the orbit of tangency may be taken to have quadratic contact.

VII. *Codimension-two tangency originating from lack of dimensions.* — A tangency occurs between $W^u(p)$ and $W^s(q)$, for some singularities p and q, so that the sum of their dimensions is equal to $(\dim M) - 2$. Several generic conditions are imposed including the ones already mentioned.

VIII. *Tangency corresponding to cubic contact.* — Similar to the previous case, but now $W^u(p)$ and $W^s(q)$ have cubic contact along a unique nontransversal orbit of intersection.

IX. *Codimension-two singularity.* — $X_{\bar{\mu}}$ has a unique nonhyperbolic singularity which has a single eigenvalue zero; restricted to the corresponding center manifold, $X_{\bar{\mu}}$ has the form $X_{\bar{\mu}}(x) = (x^3 + O(|x|^4)\dfrac{\partial}{\partial x}$. Actually, we will treat here the case of a codimension c singularity for all $c \geqslant 1$ under the hypothesis of a single eigenvalue zero.

Section II. — Invariant foliations and invariant manifolds

A basic tool in the proof of stability of a family of gradients is to construct invariant foliations which should be globally compatible: *they ought to be preserved so that we can fit together localized constructions of flow equivalences or conjugacies.* It is also helpful to restrict the family to invariant submanifolds in order to "reduce dimensions", for instance, to obtain the bifurcation diagram. The strategy has been successfully adopted in several previous papers [6], [11], [15] and we refer to them for more details. In this section we recall the notions of compatible system of (invariant) foliations and of center-unstable and center-stable foliations, applying them to prove local stability of families presenting either quadratic or cubic (or more generally *simple*) contact between stable and unstable manifolds of hyperbolic singularities as in *b)* and VIII of Section I. The first case has been proved in [15] since it corresponds to a typical codimension-one bifurcation. However, our treatment is different from the one in [15] and, in fact, it contains some of the main new and old arguments involved in the proof of several other bifurcations. For this reason it will be repeatedly quoted in subsequent cases.

Definition II.1. — Let σ be a hyperbolic singularity for $X_{\bar{\mu}}$, $\bar{\mu} \in \mathbf{R}^2$, and $U \subset \mathbf{R}^2$ a neighbourhood of $\bar{\mu}$ and V a neighbourhood of σ in M such that for each $\mu \in U$ there is a unique singularity $\sigma(\mu)$ of X_μ in V, with $\sigma(\bar{\mu}) = \sigma$. A (local) *unstable foliation* $F^u(\sigma)$ for X_μ is a continuous foliation such that

a) The leaves are C^m discs, $m \geqslant 2$, varying continuously in the C^m topology, with distinguished leaf $F^u(\sigma(\mu), \mu) = W^u(\sigma(\mu)) \cap V \times \{\mu\}$,

b) Each leaf $F^u(x, \mu)$ is contained in $V \times \{\mu\}$,

c) $F^u(\sigma)$ is invariant: $X_{\mu, t}(F^u(x, \mu)) \supset F^u(X_{\mu, t}(x), \mu)$, $t \geqslant 0$, *d)* For each $\mu \in U$, the intersection of a leaf of $F^u(\sigma(\mu))$ with $W^s(\sigma(\mu))$ is a point. A (global) unstable foliation $F^u(\sigma)$ is just the positive saturation by the flow of X_μ, $\mu \in U$, of the local unstable foliation. Similarly we define a *stable foliation* $F^s(\sigma)$.

Let us suppose that the vector field $X_{\bar\mu}$ has only (finitely many) hyperbolic singularities with their unstable and stable manifolds intersecting transversally. We may order the singularities $\sigma_1(\mu) \leqslant \ldots \leqslant \sigma_\ell(\mu)$ for $\mu \in U$, U a small neighbourhood of $\bar\mu$ in $\mathbf{R}^2$, in such way that if $W^u(\sigma_i(\mu)) \cap W^s(\sigma_j(\mu)) \neq \emptyset$, then $\sigma_i(\mu) \leqslant \sigma_j(\mu)$, and if $i \neq j$, then $i < j$ whenever $\sigma_i(\mu) \leqslant \sigma_j(\mu)$. For each singularity $\sigma_i(\mu)$ which is not a sink, we consider an unstable foliation $F^u(\sigma_i(\mu))$.

Definition II.2. — The foliations $F^u(\sigma_1(\mu)), \ldots, F^u(\sigma_{k-1}(\mu))$ form a *compatible unstable system* if whenever a leaf F of $F^u(\sigma_i(\mu))$ intersects a leaf $\widetilde{F}$ of $F^u(\sigma_j(\mu))$, $i < j$, then $F \supset \widetilde{F}$.

A similar definition holds for a *compatible stable system*, $F^s(\sigma_{k+1}(\mu)), \ldots, F^s(\sigma_\ell(\mu))$. The construction of such systems is detailed in [6], [14], [15]; in [6] a *center-unstable foliation* $F^{cu}(\sigma_k(\mu))$ is also obtained which is compatible with the system $F^u(\sigma_1(\mu)), \ldots, F^u(\sigma_{k-1}(\mu))$ in the above sense: a leaf of $F^u(\sigma_i(\mu))$ that intersects a leaf of $F^{cu}(\sigma_k(\mu))$ actually contains this leaf. Each leaf of the center-unstable foliation is a C^1 disc and is the union of leaves of an unstable foliation F^u_k. *For fixed* μ *the foliation* $F^{cu}(\sigma_k(\mu))$ *is tangent to the vector field* X_μ. In order to construct $F^{cu}(\sigma_k(\mu))$ it is assumed that the linear part $DX_\mu(\sigma_k(\mu))$ has a smallest contraction, that is a negative eigenvalue of smallest absolute value, and hence we may take a center-unstable manifold $W^{cu}(\sigma_k(\mu))$ as the distinguished leaf of $F^{cu}(\sigma_k(\mu))$. Actually, since in the bifurcations of type I to VII we assume the linearizability condition *c*.4 for X_μ near the singularity $\sigma_k(\mu)$, there is a natural choice for $W^{cu}(\sigma_k(\mu))$ in this special coordinates, namely, $W^{cu}(\sigma_k(\mu))$ is linear.

Another important tool that we have often been using in bifurcation theory, as in the present work, is the following parametrized version of the well-known Isotopy Extension Theorem (see [12]).

Let N be a C^r compact manifold, $r \geqslant 1$, and A an open subset of $\mathbf{R}^s$. Let M be a C^∞ manifold with $\dim M > \dim N$. We indicate by $C^k_A(N \times A, M \times A)$ the set of C^k mappings $f: N \times A \to M \times A$ such that $\pi = \pi' f$, endowed with the C^k topology, $1 \leqslant k \leqslant r$. Here, π and π' denote the natural projections $\pi: N \times A \to A$, $\pi': M \times A \to A$. Let $\mathrm{Diff}^k_A(M \times A)$ be the set of C^k diffeomorphisms φ of $M \times A$ such that $\pi' = \pi' \varphi$, again with the C^k topology.

Isotopy Extension Theorem. — *Let* $i \in C^k_A(N \times A, M \times A)$ *be an embedding and* A' *a compact subset of* A. *Given neighbourhoods* U *of* $i(N \times A)$ *in* $M \times A$ *and* V *of the identity in* $\mathrm{Diff}^k_A(M \times A)$, *there exists a neighbourhood* W *of* i *in* $C^k_A(N \times A, M \times A)$ *such that for each* $j \in W$ *there exists* $\varphi \in V$ *satisfying* $\varphi i = j$ *restricted to* $N \times A'$ *and* $\varphi(x) = x$ *for all* $x \notin U$.

This theorem is used to extend homeomorphisms h which are defined on top dimension submanifolds with boundary $N \subset M$ whose restrictions to the boundary are C^1 diffeomorphisms, C^1 close to the identity. Hence, by applying the above theorem to $h \mid \partial N$ we obtain an extension $\overline{H}$ to all of M and defining $H : M \to M$ such that $H \mid N = h$ and $H \mid M \backslash N = \overline{H}$ we get the desired extension. One needs this parametrized version in order to obtain such *extensions on each leaf of an invariant foliation.* We refer to [15] and [11] for some applications of these ideas in very similar situations.

The use of the above invariant foliations is illustrated in Theorem A below. Before that, we recall the definition of local stability.

Definition II.3 (Local Stability). — A family $X_\mu \in \chi_2^g(M)$ is stable at $\bar{\mu} \in \mathbf{R}^2$ if there is a neighbourhood U of $\bar{\mu}$ in $\mathbf{R}^2$ and a neighbourhood $\mathcal{U}$ of X_μ in $\chi_2^g(M)$ such that for each family $X_\mu \in \mathcal{U}$ there is a value $\hat{\mu} \in U$ and a homeomorphism

$$H : U \times M \to \mathbf{R}^2 \times M$$

of the form $H(\mu, x) = (\varphi(\mu), h(\mu, x))$, with $\varphi : (U, \bar{\mu}) \to (\mathbf{R}^2, \hat{\mu})$ also a homeomorphism onto its image, such that $h_\mu : M \to M$ is a topological equivalence between X_μ and $\hat{X}_{\varphi(\mu)}$, where $h_\mu(x) = h(\mu, x)$ for $x \in M$.

Theorem A. — *Let* X_μ, $\mu \in \mathbf{R}^2$, *be a family of gradients and* $\bar{\mu}$ *a bifurcation value such that* $X_{\bar{\mu}}$ *presents exactly one orbit of quasi-transversal intersection between stable and unstable manifolds, which unfolds generically as described in b) of Section I. Suppose that all singularities of* $X_{\bar{\mu}}$ *are hyperbolic and the conditions described in c.1, c.2 and c.3 of Section I are satisfied. Then,* X_μ *is stable at* $\bar{\mu}$.

Proof. — First we describe the bifurcation set for X_μ, μ close to $\bar{\mu}$. For μ in a neighbourhood U of $\bar{\mu}$ in $\mathbf{R}^2$, we order the singularities of X_μ, $\sigma_1(\mu) \leqslant \ldots \leqslant \sigma_\ell(\mu)$, as above, and assume that the orbit of tangency γ belongs to the intersection of $W^u(\sigma_k(\bar{\mu}))$ and $W^s(\sigma_{k+1}(\bar{\mu}))$. Let us assume that $\dim W^u(\sigma_k(\bar{\mu})) + \dim W^s(\sigma_{k+1}(\bar{\mu})) \geqslant n+1$ for, otherwise, similar arguments apply. Let $\Sigma(\mu)$ be a small cross-section intersecting γ and $\Sigma = \bigcup_{\mu \in U} \Sigma(\mu)$. From the assumptions of quasi-transversality and generic unfolding, there are C^∞ coordinates $(\mu, x_1, \ldots, x_r, y_1, \ldots, y_{u-r}, z_1, \ldots, z_{s-r}, w_1)$ in Σ centered at $p = \gamma \cap \Sigma(\bar{\mu})$ such that

$$W^u(\sigma_k(\mu)) \cap \Sigma = (\mu, x, y, 0, 0),$$

$$W^s(\sigma_{k+1}(\mu)) \cap \Sigma = (\mu, x, 0, z, F(\mu, x))$$

with $r = \dim[T_p W^u(\sigma_k(\bar{\mu})) \cap T_p W^s(\sigma_{k+1}(\bar{\mu}))] - 1$ and $x \mapsto F(\bar{\mu}, x)$

is a C^∞ Morse function such that $\operatorname{rank} \begin{pmatrix} dF(\bar{\mu}, 0) \\ d\left[\frac{\partial F}{\partial x}(\bar{\mu}, 0)\right] \end{pmatrix} = r + 1$. Therefore, the tangency between $W^u(\sigma_k(\mu))$ and $W^s(\sigma_{k+1}(\mu))$, for μ near $\bar{\mu}$, is characterized by the equations $F(\mu, x) = 0$, $\frac{\partial F}{\partial x}(\mu, x) = 0$. By the implicit function theorem, there is a C^∞ curve Γ in $\mathbf{R}^2$, containing $\bar{\mu}$, such that $\mu \in \Gamma$ if and only if $W^u(\sigma_k(\mu))$ is quasi-transversal to $W^s(\sigma_{k+1}(\mu))$. Moreover, it is easy to see that the noncriticality and transversality conditions ($c.2$, $c.3$ of Section I) imply that these are the only bifurcations near $\bar{\mu}$.

Let X_μ be a nearby family so that all conditions described above are also satisfied. We get a curve $\hat{\Gamma}$ near Γ which represents the bifurcations of the family $\hat{X}_\mu$. Let $\Sigma^c = (\mu, x, 0, 0, w_1)$ be a normal section in Σ. From Morse's Lemma with parameters (see [13] or [5]), there is a diffeomorphism $h^c : \Sigma^c \to \Sigma^c$ of the form $h^c(\mu, x, w_1) = (\varphi(\mu), h_1^c(\mu, x, w_1))$ which sends $W^u(\sigma_k(\mu)) \cap \Sigma^c$ to $W^u(\hat{\sigma}_k(\varphi(\mu))) \cap \Sigma^c$, $W^s(\sigma_{k+1}(\mu)) \cap \Sigma^c$ to $W^s(\hat{\sigma}_{k+1}(\varphi(\mu))) \cap \Sigma^c$. Here φ is a diffeomorphism defined in a neighbourhood of $\bar{\mu}$ such that $\varphi(\Gamma) = \hat{\Gamma}$ and which is close to the identity.

Let us assume that f_μ, the potential function of X_μ, has distinct critical values for μ near $\bar{\mu}$. If $\hat{f}_\mu$ is the potential function of $\hat{X}_\mu$, then, since f_μ and $\hat{f}_\mu$ are C^∞ close Morse functions, there are C^∞ families of diffeomorphisms $H_\mu : M \to M$, $\lambda_\mu : I \to I$ such that $f_\mu \circ H_\mu^{-1} = \lambda_\mu \circ \hat{f}_\mu$, and so $\operatorname{grad}_{H_\mu^*(g_\mu)} f_\mu$ is equivalent to $\operatorname{grad}_{\hat{g}_\mu} \hat{f}_\mu = \hat{X}_\mu$. (Here, g_μ and $\hat{g}_\mu$ are the respective metrics and I an interval.) Hence, there is no loss of generality if we assume that X_μ and $\hat{X}_{\varphi(\mu)}$ have the same potential.

The equivalence between X_μ and $\hat{X}_{\varphi(\mu)}$ will be a conjugacy outside a neighbourhood V of the closure of the orbit of tangency γ in $M \times \mathbf{R}^2$. Inside V it will preserve the level sets of f_μ. Let us now describe this *distinguished neighbourhood* V. Let $c_k(\mu) = f_\mu(\sigma_k(\mu))$ and $f_\mu(\sigma_{k+1}(\mu)) = c_{k+1}(\mu)$. If $\varepsilon > 0$ is small and $\tilde{V}_i(\mu)$ is an open neighbourhood of $\sigma_i(\mu)$ in M, we consider $A_i(\mu) = f_\mu^{-1}(c_i(\mu) - \varepsilon) \cap \tilde{V}_i(\mu)$ and $B_i(\mu) = f_\mu^{-1}(c_i(\mu) + \varepsilon) \cap \tilde{V}_i(\mu)$ for $i = k, k+1$. Let

$$V_i(\mu) = \{ x \in \tilde{V}_i(\mu) ; X_{\mu,t}(x) \cap B_i(\mu) \neq \emptyset \text{ for } t > 0 \text{ or } X_{\mu,t}(x) \cap A_i(\mu) \neq \emptyset \text{ for } t < 0 \} \cup \{ \sigma_i(\mu) \}, \quad i = k, k+1,$$

be neighbourhoods of $\sigma_k(\mu)$ and $\sigma_{k+1}(\mu)$, respectively. We connect $V_k(\mu)$ to $V_{k+1}(\mu)$ along γ in the following way. We consider $D \subset B_k(\bar{\mu})$, a small closed disc centered at $\gamma \cap B_k(\bar{\mu})$ such that $\bigcup_{t \geq 0} X_{\bar{\mu},t}(D)$ does not intersect the boundary of the closure of $A_{k+1}(\mu)$ in M, and define $D(\mu) = \{ x \in M ; X_{\mu,t}(x) \in D \text{ for some } t \leq 0 \text{ and } X_{\mu,t}(x) \in A_{k+1}(\mu) \text{ for some } t > 0 \}$. Let $V(\mu) = V_k(\mu) \cup D(\mu) \cup V_{k+1}(\mu)$ and $V = \bigcup_{\mu \in U} V(\mu)$. Observe that, in order to glue continuously a conjugacy in the complement of $V(\mu)$ with a level preserving equivalence, we adjust the metric in a neighbourhood of the part of $\partial V(\mu)$ which is a union of trajectories, in such way that $\| X_\mu \|_{g_\mu} = 1$. Moreover, since the critical levels in V will be preserved, this ajustment is such that the time it takes to go from $f_\mu^{-1}(c_i(\mu))$ to $A_i(\mu)$ and from $f_\mu^{-1}(c_i(\mu))$ to $B_i(\mu)$ is constant.

We now briefly describe the construction of a center-unstable foliation $F^{cu}(\sigma_k(\mu))$; we refer to [15] for more details. Let $A_k^s(\mu)$ denote the sphere $A_k(\mu) \cap W^s(\sigma_k(\mu))$ which is transversal to X_μ, and intersects every nonsingular orbit in $W^s(\sigma_k(\mu))$, i.e. it is a *fundamental domain* for $W^s(\sigma_k(\mu))$. It contains $A_k^{ss}(\mu) = A_k(\mu) \cap W^{ss}(\sigma_k(\mu))$ as a codimension-one (equatorial) sphere. Recall that $W^s(\sigma_k(\mu))$ is foliated by a unique codimension-one C^∞ foliation $F^{ss}(\sigma_k(\mu))$ — the strong stable foliation. We can write

$A^s_k(\mu) = D^+_s(\mu) \cup C_s(\mu) \cup D^-_\mu(\mu)$. Here $C_s(\mu)$ is a small tubular neighbourhood of $A^{ss}_k(\mu)$ in $A^s_k(\mu)$; $D^+_s(\mu)$ and $D^-_s(\mu)$ are closed discs whose respective boundaries $\partial D^+_s(\mu)$ and $\partial D^-_s(\mu)$ are the intersection of leaves F^+ and F^- of $F^{ss}(\sigma_k(\mu))$ with $A^s_k(\mu)$. The subset $C_s(\mu)$ is taken in such way that if $W^u(\sigma_i(\mu)) \cap C_s(\mu) \neq \emptyset$, then $W^u(\sigma_i(\mu)) \cap A^{ss}_k(\mu) \neq \emptyset$ and $W^u(\sigma_i(\mu)) \cap C_s(\mu)$ is transversal to the induced foliation $F^{ss}(\sigma_k(\mu)) \cap C_s(\mu)$. This is possible because of the noncriticality assumption $(c.2)$ of Section I. The condition also allows one to construct a one-dimensional C^1 foliation $F^c(\sigma_k(\mu))$ on $C_s(\mu)$ which is compatible with the system $F^u(\sigma_1(\mu)), \ldots, F^u(\sigma_{k-1}(\mu))$ and is transversal to $F^{ss}(\sigma_k(\mu)) \cap C_s(\mu)$ and to $\partial D^-_s(\mu) \cup \partial D^+_s(\mu)$.

Let $F^u_k(\mu)$ be a u-dimensional continuous foliation with C^1 leaves on $A_k(\mu)$ which is compatible with the system $F^u_1(\sigma_1(\mu)), \ldots, F^u(\sigma_{k-1}(\mu))$ and is transversal to $A^s_k(\mu)$. We now point out the following key fact: if $P^k_\mu : A_k(\mu) \backslash A^s_k(\mu) \to B_k(\mu)$ is the Poincaré map between the non-critical levels $A_k(\mu)$, $B_k(\mu)$ and $b_\mu : B_k(\mu) \hookleftarrow$ is a homeomorphism preserving leaves of $P^k_\mu(F^u(\sigma_k(\mu)))$, then the induced map $(P^k_\mu)^{-1} \circ b_\mu \circ P^k_\mu$ extends continuously to a full homeomorphism of $A_k(\mu)$. We observe that *preserving leaves* means that the map sends a leaf of the foliation into another one. This motivates the definition of a center-unstable foliation as $F^{cu}(\sigma_k(\mu)) = \bigcup_{t \geq 0} X_{\mu, t}(F^u_k(\mu))$, a distinguished leaf being a center-unstable manifold $W^{cu}(\sigma_k(\mu))$. We distinguish two parts in F^{cu} with different types of leaves. One, denoted by $F^{cu}_1(\sigma_k(\mu))$, has $(u+1)$-dimensional leaves, with $u = \dim(W^u(\sigma_k(\mu)))$, each leaf corresponding to a point of $D^+_s(\mu) \cup D^-_s(\mu)$. The other part of the foliation, denoted by $F^{cu}_2(\sigma_k(\mu))$, has a typical leaf of the form $\bigcup_{x \in F^c_\mu} F^{cu}(\sigma_k(\mu))_x$, where F^c_μ is a leaf of $F^c(\sigma_k(\mu))$ and $F^{cu}(\sigma_k(\mu))_x$ is the leaf of $F^{cu}(\sigma_k(\mu))$ through the point x. Notice that the leaves in $F^{cu}_2(\sigma_k(\mu))$ have dimension $u + 2$. The fact pointed out above concerning extensions of homeomorphisms together with the existence of a weakest contraction (see condition $(c.1)$) imply the following stronger property. A homeomorphism $b_\mu : B_k(\mu) \hookleftarrow$ that preserves $F^{cu}_2(\sigma_k(\mu))$ and the portion of $F^{cu}_1(\sigma_k(\mu))$ inside a conic region which contains the center-unstable manifold *induces*, as before, a homeomorphism on all of $A_k(\mu)$. Our conic region corresponds to a bundle of solid cones, with constant width over the sphere $B_k(\mu) \cap W^u(\sigma_k(\mu))$. The construction of a center-stable foliation $F^{cs}(\sigma_{k+1}(\mu))$ is dual to this one. We proceed in the same way to construct a center-unstable foliation $\hat{F}^{cu}(\hat{\sigma}_k(\varphi(\mu)))$ and a center-stable foliation $\hat{F}^{cs}(\hat{\sigma}_{k+1}(\varphi(\mu)))$ for the vector field $\hat{X}_\varphi(\mu)$.

Let us now construct an equivalence between X_μ and $\hat{X}_{\varphi(\mu)}$. We start by sending sources to sources and sinks to sinks. Then, proceeding by induction on j and using the Isotopy Extension Theorem as in [15], pages 413 and 414, we obtain a continuous family of homeomorphisms

$$h^s_\mu : \bigcup_{j \leq k-1} W^s(\sigma_j(\mu)) \to \bigcup_{j \leq k-1} W^s(\hat{\sigma}_j(\varphi(\mu)))$$

and

$$h^u_\mu : \bigcup_{j \geq k+2} W^u(\sigma_j(\mu)) \to \bigcup_{j \geq k+2} W^u(\hat{\sigma}_j(\varphi(\mu))).$$

On each step, say from $i-1$ to i, we use the unstable system in order to go from a fundamental domain of $W^s(\sigma_{i-1}(\mu))$ to a fundamental domain of $W^s(\sigma_i(\mu))$, and use also the fact that $W^s(\sigma_i(\mu))$ and $W^s(\hat{\sigma}_i(\varphi(\mu)))$ are C^1 close on compact parts. By imposing that the equivalence preserves the unstable system $F^u(\sigma_1(\mu)), \ldots, F^u(\sigma_{k-1}(\mu))$, we see that h^s_μ induces a homeomorphism in part of the sphere $A^{ss}_k(\mu)$. By using induction on the indices j, starting with $j = k - 1$, and using again the Isotopy Extension Theorem, we extend this homeomorphism to the whole of $A^{ss}_k(\mu)$ (space of leaves of $F^{cu}_2(\sigma_k(\mu))$. By preserving the central foliation $F^c(\sigma_k(\mu))$, we define a homeomorphism on $\partial D^+_s(\mu) \cup \partial D^-_s(\mu)$. Using once more the Isotopy Extension Theorem and induction we extend this homeomorphism to $D^+_s(\mu) \cup D^-_s(\mu)$. In this way, we obtain a homeomorphism on the space of leaves of the center-unstable foliation $F^{cu}(\sigma_k(\mu))$. We proceed dually to get a homeomorphism in the space of leaves of the center-stable foliation $F^{cs}(\sigma_{k+1}(\mu))$. Our next task is to construct a homeomorphism in the cross sections $\Sigma(\mu)$ (which we assume to be contained in a level set of f_μ) preserving the center-unstable and the center-stable foliations. We first construct this homeomorphism on the section $\Sigma^{cs}(\mu) = W^{cs}(\sigma_{k+1}(\mu)) \cap \Sigma(\mu)$. Let $F^{ss}_\mu \subset \Sigma^{cs}(\mu)$ be a C^1 foliation compatible with $W^s(\sigma_{k+1}(\mu)) \cap \Sigma(\mu)$ whose leaves have complementary dimension and are transversal to $\Sigma^c(\mu)$, where as above $\Sigma^c(\mu)$ is a smooth cross section which is tangent to

$$W^{cs}(\sigma_{k+1}(\mu)) \cap W^{cu}(\sigma_k(\mu)) \cap \Sigma(\mu).$$

Let (v_1, v_I, w_L) be coordinates for $\Sigma^{cs}(\mu)$ such that $W^u(\sigma_{k-1}(\mu)) \cap \Sigma^{cs}(\mu) = \{v_1 = v_I = 0\}$ and $W^{cu}(\sigma_{k-1}(\mu)) \cap \Sigma^{cs}(\mu) = \{v_I = 0\}$ where $v_I = (v_2, \ldots, v_{s-r})$; $w_L = (w_1, \ldots, w_r)$ and $s = \dim W^s(\sigma_{k+1}(\mu))$.

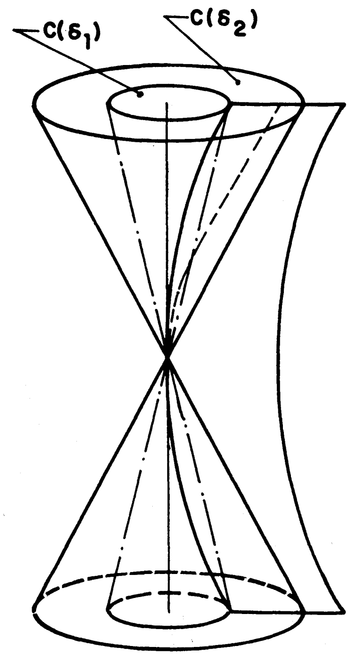

FIG. I

In $\Sigma^{cs}(\mu)$, we consider two conic regions $C(\delta_1) = \{ v_1^2 - \delta_1(\mu) \| v_I \|^2 \geq 0 \}$ and $C(\delta_2) = \{ v_1^2 - \delta_2(\mu) \| v_I \|^2 \leq 0 \}$, where $0 < \delta_1(\mu) < \delta_2(\mu)$ are chosen so that $\partial C(\delta_1)$ and $\partial C(\delta_2)$ are transversal to $F^{cu}(\sigma_k(\mu))$. The intersection of $F_1^{cu}(\sigma_k(\mu))$ with $\Sigma^{cs}(\mu)$ gives rise to a foliation in $C(\delta_1)$ which is singular along $W^u(\sigma_k(\mu)) \cap \Sigma^{cs}(\mu)$. This foliation is extended continuously on each leaf, say F_2, of $F_2^{cu}(\sigma_k(\mu)) \cap \Sigma^{cs}(\mu)$ in such way that $\partial C(\delta_2) \cap F_2$ is a leaf and it is non-singular in the interior of $C(\delta_2) \cap F_2$. We denote by $F_0^{cu}(\sigma_k(\mu))$ this new foliation. By construction $F_0^{cu}(\sigma_k(\mu))$ is topologically transversal to F_μ^{ss}, so the projection $\Sigma^{cs}(\mu) \to \Sigma^c(\mu)$ along the leaves of F_μ^{ss}, restricted to each leaf of $F_0^{cu}(\sigma_k(\mu))$, is a homeomorphism. Hence, by performing the same construction for $\hat{X}_{\varphi(u)}$, since we already have a homeomorphism $h_\mu^c : \Sigma^c(\mu) \to \Sigma^c(\mu)$, we can define h_μ^{cs} by sending F_μ^{ss} to $\hat{F}_{\varphi(\mu)}^{ss}$ and $F_0^{cu}(\mu)$ to $\hat{F}_0^{cu}(\varphi(\mu))$ preserving leaves of the center-unstable foliations. The main property of h_μ^{cs} is that it preserves the leaves of type $F_1^{cu}(\sigma_k(\mu))$ inside the conic region $C(\delta_1)$. Therefore, as we pointed out above, the induced homeomorphism $(\hat{P}_{\varphi(\mu)}^{cs})^{-1} \circ h_\mu^{cs} \circ P_\mu^{cs}$ automatically defines an extension to the fundamental domain $A_k^s(\mu)$. By proceeding analogously in the section

$$\Sigma^{cu}(\mu) = W^{cu}(\sigma_k(\mu)) \cap \Sigma(\mu),$$

we obtain a homeomorphism h_μ^{cu} which preserves a foliation F_μ^{uu} compatible with $W^{cs}(\sigma_k(\mu))$ and the center-stable foliation $F^{cs}(\sigma_{k+1}(\mu)) \cap \Sigma^{cu}(\mu)$. Finally, we match h_μ^{cs} and h_μ^{cu} to obtain a homeomorphism on the whole section $\Sigma(\mu)$. We do this by first considering a C^1 foliation $F^{cu}(\mu)$ transversal to $\Sigma^c(\mu)$ and of complementary dimension, such that $F^{su}(\mu) \cap \Sigma^{cs}(\mu) = F_\mu^{ss}$ and $F^{su}(\mu) \cap \Sigma^{cu}(\mu) = F_\mu^{uu}$. We then require the homeomorphism to preserve this foliation, as well as the center-unstable and center-stable foliations.

The homeomorphism extends to $V(\mu) \cap (\bigcup_{t \in \mathbf{R}} X_{t,\mu}(\Sigma(\mu)))$ just by preserving the level sets of f_μ and by sending orbits of X_μ to orbits of $\hat{X}_{\varphi(\mu)}$. In particular, it defines a homeomorphism on a closed disc $D_k(\mu)$ contained in the level set $B_k(\mu)$ and on a closed disc $D_{k+1}(\mu)$ contained in the level set $A_{k+1}(\mu)$. By construction, near the boundary of these discs the homeomorphisms are actually C^1 diffeomorphisms close to the identity. Hence, since all stable manifolds $W^s(\sigma_i(\mu))$, $k+1 \leq i \leq \ell$ are transversal to $W^u(\sigma_k(\mu))$ outside $D_k(\mu)$, we can proceed by induction on i and apply again the Isotopy Extension Theorem to get a homeomorphism on all of $B_k(\mu)$ which preserves the intersections of the stable system $F^{cs}(\sigma_{k+1}(\mu))$, $F^s(\sigma_{k+2}(\mu))$, ..., $F^s(\sigma_\ell(\mu))$ with $B_k(\mu)$. Similarly, we obtain a homeomorphism on the level $A_{k+1}(\mu)$ which preserves the unstable system. We complete the definition of the equivalence inside $V(\mu)$ by preserving the level sets of f_μ and of course sending orbits of X_μ to orbits of $\hat{X}_{\varphi(\mu)}$.

Thus, we have defined two families of homeomorphisms which depend continuously on μ,

$$h_\mu^s : \bigcup_{j \leq k+1} W^s(\sigma_j(\mu)) \to \bigcup_{j \leq k+1} W^s(\hat{\sigma}_j(\varphi(\mu)))$$

and

$$h_\mu^u : \bigcup_{i \geq k} W^u(\sigma_i(\mu)) \to \bigcup_{i \geq k} W^u(\hat{\sigma}_i(\varphi(\mu))).$$

Let F_c be the level set of f_μ that contains the cross-section $\Sigma(\mu)$ and suppose that f_μ is nondecreasing with respect to the ordering of the singularities. Observe that if $x \notin [\bigcup_{j \leqslant k+1} W^s(\sigma_j(\mu))] \cup [\bigcup_{i \geqslant k} W^u(\sigma_i(\mu))]$, then the orbit of x intersects F_c. Therefore, to extend the equivalence to all of M, we just construct a continuous family of homeomorphisms on F_c which are compatible with h^s_μ and h^u_μ and which preserves the stable and unstable systems. This is done by using once more the Isotopy Extension Theorem and induction on the dimension of the stable manifolds that intersect F_c, exactly as in [15]. ∎

Remark. — Let $\varphi : (U, \bar{\mu}) \to (\varphi(U), \varphi(\bar{\mu}))$ be the reparametrization obtained in the proof of Theorem A. If $D_1 \subset D_2 \subset U$ are closed discs centered at $\bar{\mu}$ and $\rho : \mathbf{R}^2 \to \mathbf{R}$ is a C^∞ function with supp $\rho \subset U$, $\rho \mid D_1$, $0 \leqslant \rho \leqslant 1$ and $\rho = 0$ outside U, then by defining $F^*_\mu = (1 - \rho(\mu)) f_\mu + \rho(\mu) \hat{f}_\mu$, $g^*_\mu = [1 - \rho(\mu)]\, g_\mu + \rho(\mu)\, \hat{g}_\mu$ and $X^*_\mu = \mathrm{grad}_{g^*_\mu} f^*_\mu$, we obtain a two-parameter family such that $X^*_\mu = X_\mu$ for $\mu \notin U$, $\hat{X}_\mu = X_\mu$ for $\mu \in D_1$. Since φ is C^1 close to the identity, there is another reparametrization $\psi : (U, \bar{\mu}) \to (\psi(U), \psi(\bar{\mu}))$ such that ψ restricted to D_1 is equal to φ and is equal to the identity outside a neighbourhood U_2 of the disc D_2. Observe that the system of foliations constructed in the proof of Theorem A can be taken to be the same for X_μ and X^*_μ when $\mu \notin U_2$. Hence, having the identity as the homeomorphism on the space of leaves of these foliations and repeating the proof of Theorem A, we obtain an equivalence h_μ between X_μ and $X^*_{\psi(\mu)}$ such that $h_\mu = \mathrm{id}_M$ for $\mu \notin U_2$. *This fact is very relevant in order to prove global stability, see Section IV, and it applies to all bifurcation cases treated in this and the next sections.*

We now prove local stability of bifurcations of type VIII, an orbit of tangency with cubic contact. Actually, as M. Khesin pointed out to us, using the theory of V-equivalence (or contact equivalence) as in [1], [8], [9] and the arguments in the proof of Theorem A, one can show the local stability of a much wider class of families of gradients. Recall that two germs f_1 and $f_2 : (\mathbf{R}^r, 0) \to (\mathbf{R}, 0)$ are V-equivalent if there is a germ of diffeomorphism $h : (\mathbf{R}^r, 0) \to (\mathbf{R}^r, 0)$ and a smooth germ $M : (\mathbf{R}^r, 0) \to \mathbf{R}$ such that $f_1(x) = M(x) . f_2(h(x))$ (so, h sends the " variety " $f_1^{-1}(0)$ to $f_2^{-1}(0)$). Let $\bar{\mu}$ be a bifurcation value for a family X_μ such that $\gamma \subset W^u(\sigma(\bar{\mu})) \cap W^s(\sigma'(\bar{\mu}))$ is an orbit of tangency along which the manifolds have simple contact of type A_k, D_k, E_6, E_7, E_8 as in Arnold's list [9]. By that we mean that for $p \in \gamma$,

$$\dim[T_p W^u(\sigma(\bar{\mu})) + T_p W^s(\sigma'(\bar{\mu}))] = \dim M - 1$$

and if Σ is a smooth cross section at p, there are μ-dependent coordinates (x, y, z, w_1) centered at p such that $W^u(\sigma(\mu)) \cap \Sigma = \{ z = 0, w_1 = 0 \}$, $W^s(\sigma'(\mu)) \cap \Sigma = \{ y = 0, w_1 = F(\mu, x) \}$ with $f(x) = F(0, x)$ being equivalent to one of the following normal forms: $A_k = x_1^{k+1} + Q$, $k \geqslant 1$, $Q(x_2, \ldots, x_r)$ a non-degenerate quadratic form; $D_k : x_1^2 x_2 \pm x_2^{k-1} + \bar{Q}$, $k \geqslant 4$; $E_6 : x_1^3 \pm x_2^4 + \bar{Q}$; $E_7 : x_1^3 + x_1 x_2^3 + \bar{Q}$; $E_8 : x_1^3 + x_2^5 + \bar{Q}$, $\bar{Q}(x_3, \ldots, x_r)$ a non-degenerate quadratic form. We require $F(\mu, x)$ to be a V-versal unfolding of $f(x)$ ($\bar{\mu} = 0$) [9]. So, for a nearby family $\hat{X}_\mu$ with corresponding unfolding $\hat{F}(\mu, x)$ we have

$$F(\mu, x) = M(\mu, x)\, \hat{F}(\varphi(\mu), h(\mu, x))$$

with $M(0, 0) \neq 0$, φ and h_μ being local diffeomorphisms. Therefore, there is a local diffeomorphism $\Sigma^c \to \hat{\Sigma}^c$ of the form $(\varphi(\mu), h(\mu, x), M(\mu, x)^{-1} w_1)$ which sends $W^u(\sigma(\mu)) \cap \Sigma^c$ to $W^u(\hat{\sigma}(\varphi(\mu))) \cap \hat{\Sigma}^c$ and $W^s(\sigma'(\mu)) \cap \Sigma^c$ to $W^s(\hat{\sigma}'(\varphi(\mu))) \cap \hat{\Sigma}^c$ (as in the proof of Theorem A, Σ^c denotes a smooth cross-section tangent to

$$W^{cu}(\sigma(\mu)) \cap W^{cs}(\sigma'(\mu)) \cap \Sigma).$$

Hence, from the non-criticality condition $c.2$ and the transversality between $W^{cu}(\sigma(\bar{\mu}))$ and $W^s(\sigma'(\bar{\mu}))$ and between $W^{cs}(\sigma'(\bar{\mu}))$ and $W^u(\sigma(\bar{\mu}))$ (condition $c.3$), we construct compatible unstable and stable systems and proceed exactly as in Theorem A to get an equivalence between X_μ and $\hat{X}_\mu$. ■

Thus, we have the following

Theorem B. — Let $\bar{\mu}$ be a bifurcation value for a family X_μ such that $X_{\bar{\mu}}$ presents exactly one orbit of tangency with cubic contact, or more generally simple contact, which unfolds generically. Suppose that all singularities are hyperbolic and conditions $c.1$, $c.2$, $c.3$ of Section I are satisfied. Then, X_μ is stable at $\bar{\mu}$.

We will see in § 6 of Section III that if $\dim[T_p W^u(\sigma) + T_p W^s(\sigma')] \leqslant \dim M - 2$, the family may present other tangencies *(secondary bifurcations)* besides the tangencies between $W^u(\sigma(\mu))$ and $W^s(\sigma'(\mu))$. This will impose several delicate adjustments in order to extend a local equivalence (in a neighbourhood of the orbit of tangency) to an equivalence on all of M.

Section III. — Local stability

In this section we continue to prove local stability of the bifurcations mentioned in Section I; in the previous one we have already studied the families that exhibit one orbit of *simple* contact between stable and unstable manifolds. As mentioned before, by local we mean that we only consider the parameter varying in some small neighbourhood in $\mathbf{R}^2$ of an initial bifurcation value $\bar{\mu}$. In each case, we start by requiring several additional generic assumptions for the family X_μ, then we obtain the bifurcation set near $\bar{\mu}$ and finally we prove stability.

§ 1. Bifurcations of type I: one orbit of tangency with criticality

(1.**A**) *Generic conditions describing the bifurcation.* — A family X_μ in $\chi^2_\sigma(M)$ has a bifurcation value $\bar{\mu}$ of type I if the following holds:

(1.1) The vector field $X_{\overline{\mu}}$ presents a unique orbit γ of quasi-transversal intersection between say, the unstable manifold $W^u(p(\overline{\mu}))$ and the unstable manifold $W^s(q(\overline{\mu}))$ of hyperbolic singularities $p(\overline{\mu})$, $q(\overline{\mu})$,

(1.2) Linearizability of the family near $p(\overline{\mu})$ and $q(\overline{\mu})$ with the respective linear part with distinct eigenvalues (conditions *c*.1 and *c*.4 of Section I),

(1.3) There is a unique orbit γ' of quasi-transversal intersection between the unstable manifold $W^u(\sigma(\overline{\mu}))$ of a hyperbolic singularity $\sigma(\overline{\mu})$ and the strong stable manifold of $p(\overline{\mu})$, $W^{ss}(p(\overline{\mu}))$,

(1.4) Let $W^{vs}(p(\overline{\mu})) \subset W^{ss}(p(\overline{\mu}))$ be the codimension two submanifold of $W^s(p(\overline{\mu}))$, invariant by X_μ, which corresponds (in linearized coordinates near $p(\overline{\mu})$) to the negative eigenvalues of $dX_{\overline{\mu}}(p(\overline{\mu}))$ except for the two ones of smallest norm $\alpha_1(\overline{\mu})$, $\alpha_2(\overline{\mu})$. Then $W^u(\sigma(\overline{\mu}))$ is transversal to $W^{vs}(p(\overline{\mu}))$; in particular, the orbit of tangency γ' does not belong to $W^{vs}(p(\overline{\mu}))$,

(1.5) The orbit of tangency γ unfolds generically as in *b)* of Section I, so there is a C^1 curve Γ in the parameter space such that $\mu \in \Gamma$ if and only if $W^u(p(\mu))$ is not transversal to $W^s(q(\mu))$,

(1.6) The orbit of tangency γ' also unfolds generically so that there is a C^1 curve Γ' in the parameter space such that $\mu \in \Gamma'$ if and only if $W^u(\sigma(\mu))$ is tangent to $W^{ss}(p(\mu))$. Moreover, the curves Γ and Γ' intersect transversally at $\overline{\mu}$,

(1.7) The vector field $X_{\overline{\mu}}$ satisfies the linearizability conditions near $\sigma(\overline{\mu})$ and the eigenvalues of $dX_{\overline{\mu}}(\sigma(\overline{\mu}))$ have multiplicity one so that if we take a C^1 center-unstable manifold $W^{cu}(\sigma(\overline{\mu}))$ for $\sigma(\overline{\mu})$, then it is transversal to $W^{ss}(p(\overline{\mu}))$. We also assume the non-criticality conditions *c*.2 of Section I, and that every unstable manifold is transversal to $W^{ss}(\sigma(\overline{\mu}))$ and every stable manifold is transversal to $W^{uu}(q(\overline{\mu}))$ and the hypothesis *c*.3 which says that $W^{cu}(p(\overline{\mu}))$ is transversal to $W^s(q(\overline{\mu}))$ and $W^{cs}(q(\overline{\mu}))$ is transversal to $W^u(p(\overline{\mu}))$,

(1.8) Let $n_{k-1} = \dim[T_r W^u(\sigma(\overline{\mu})) \cap T_r W^s(p(\overline{\mu}))]$, for $r \in \gamma'$, and consider the invariant manifold $V^s \subset W^s(p(\overline{\mu}))$ of dimension n_{k-1} corresponding to the n_{k-1} negative eigenvalues of $dX_{\overline{\mu}}(p(\overline{\mu}))$ of biggest norm. There is a subspace $E(r) \subset T_r M$ such that $\lim_{t \to -\infty} dX_{\overline{\mu}, t}(r).E(r) = T_{\sigma(\overline{\mu})} W^{ss}(\sigma(\overline{\mu}))$ and $\lim_{t \to +\infty} dX_{\overline{\mu}, t}(r).E(r) = T_{p(\overline{\mu})} V^s$,

(1.9) According to (1.2) there are C^m μ-dependent coordinates $(x_1, \ldots, x_s, y_1, \ldots, y_u)$ in a neighbourhood of $p(\overline{\mu})$ in M such that

$$X_\mu = - \sum_{i=1}^{s} \alpha_i(\mu)\, x_i \frac{\partial}{\partial x_i} + \sum_{j=1}^{u} \beta_j(\mu)\, y_j \frac{\partial}{\partial y_j},$$

where $0 < \alpha_1(\mu) < \ldots < \alpha_s(\mu)$ and $0 < \beta_1(\mu) < \ldots < \beta_u(\mu)$.

The manifold $W^s(q(\bar{\mu}))$ is transversal to the plane $(0, x_2, 0, \ldots, 0, y_1, \ldots, y_u)$. This can be formulated intrinsically by saying that $W^s(q(\bar{\mu}))$ is transversal to any C^m invariant manifold that contains the orbit of tangency γ' and the unstable manifold $W^u(p(\bar{\mu}))$. Similarly, the unstable manifold $W^u(\sigma(\bar{\mu}))$ is transversal to the plane $(0, x_2, \ldots, x_s, y_1, 0, \ldots, 0)$.

(1.B) *The bifurcation diagram.* — Let $\sigma_1(\mu) \leqslant \ldots \leqslant \sigma_k(\mu) \leqslant \sigma_{k+1}(\mu) \leqslant \ldots \leqslant \sigma_\ell(\mu)$ be the ordering of the singularities of the vector field X_μ for μ near $\bar{\mu}$ as in the previous section. We assume $p(\mu) = \sigma_k(\mu)$, $q(\mu) = \sigma_{k+1}(\mu)$ and without loss of generality we may also assume that $\sigma(\mu) = \sigma_{k-1}(\mu)$. First observe that since $W^{cu}(\sigma_k(\bar{\mu}))$ is transversal to $W^s(\sigma_{k+1}(\bar{\mu}))$ and $W^{cs}(\sigma_{k+1}(\bar{\mu}))$ is transversal to $W^u(\sigma_k(\bar{\mu}))$ the only possibility for non-stability of the vector field X_μ comes from either the tangency between $W^u(\sigma_k(\mu))$ and $W^s(\sigma_{k+1}(\mu))$ or between $W^u(\sigma_{k-1}(\mu))$ and $W^s(\sigma_{k+1}(\mu))$. This follows from the fact that if $W^u(\sigma_j(\bar{\mu}))$ is transversal to $W^{ss}(\sigma_k(\bar{\mu}))$, then $W^u(\sigma_j(\mu))$ is transversal to $W^s(\sigma_{k+1}(\mu))$ for μ near $\bar{\mu}$.

Proposition. — *Let X_μ be a family presenting a bifurcation value $\bar{\mu}$ of type I as described above. Then, there is a neighbourhood U of $\bar{\mu}$ in $\mathbf{R}^2$ such that the bifurcation set for X_μ in U is the union of two C^1 curves $\Gamma \cup \Gamma_0$, such that $\mu \in \Gamma$ if and only if X_μ presents a unique orbit of quasi-transversality $\gamma_\mu \subset W^u(\sigma_k(\mu)) \cap W^s(\sigma_{k+1}(\mu))$ and $\mu \in \Gamma_0$ if and only if there is a unique orbit of quasi-transversality $\gamma'_u \subset W^u(\sigma_{k-1}(\mu)) \cap W^s(\sigma_{k+1}(\mu))$. The relative position of Γ and Γ_0 is illustrated in Fig. III.*

Proof. — Using a μ-dependent C^m ($m \geqslant 3$) linearization for X_μ near $\sigma_k(\mu)$ and the transversality between $W^u(\sigma_k(\bar{\mu}))$ and $W^{cs}(\sigma_{k+1}(\bar{\mu}))$, we may construct a C^1 submanifold $W^{cs}_{k+1} \subset M \times \mathbf{R}^2$, $W^{cs}_{k+1} = \bigcup_\mu W^{cs}(\sigma_{k+1}(\mu)) \times \{\mu\}$ such that $W^{cs}(\sigma_{k+1}(\bar{\mu}))$ contains the closure of γ. Moreover, W^{cs}_{k+1} and $W^{cs}(\sigma_{k+1}(\mu))$ admit smoothing C^r structures, $r \geqslant 3$ (see Chapter II.1 of [15]). In the sequel, we shall take $r = m \geqslant 3$. Analogously, from (1.7) the center-unstable manifold $W^{cu}(\sigma_{k-1}(\bar{\mu}))$ can be extended to a C^1 manifold W^{cu}_{k-1} that contains the closure of the orbit γ' which also admits a C^r smoothing structure. Let $W^c = W^{cu}_{k-1} \cap W^{cs}_{k+1}$ and consider the restriction of X_μ to $W^c(\mu) = W^{cu}(\sigma_{k-1}(\mu)) \cap W^{cs}(\sigma_{k+1}(\mu))$. In this setting both $W^u(\sigma_{k-1}(\mu))$ and $W^s(\sigma_{k+1}(\mu))$ have codimension one and we may also assume that $X_\mu \mid W^c(\mu)$ is of class C^{m-1} having a C^m μ-dependent linearization near $\sigma_{k+1}(\mu)$ for μ near $\bar{\mu}$:

$$X_\mu \mid W^c(\mu) = -\sum_{j=1}^{n_{k-1}} \alpha_i(\mu)\, x_i \frac{\partial}{\partial x_i} + \sum_{j=1}^{n_k} \beta_j(\mu)\, y_j \frac{\partial}{\partial y_j}$$

with $\quad 0 < \alpha_1(\mu) < \ldots < \alpha_{n_{k-1}}(\mu), \qquad 0 < \beta_1(\mu) < \ldots < \beta_{n_k}(\mu)$

and $\quad n_{k-1} = \dim[T_{z'} W^u(\sigma) \cap T_{z'} W^s(p(\bar{\mu}))]$ for $z' \in \gamma'$,

$\quad n_k = \dim[T_z W^u(p(\bar{\mu})) \cap T_z W^s(q(\bar{\mu}))]$ for $z \in \gamma$.

If $\Sigma^c \subset \{x_2 = 1\}$ and $S^c \subset \{y_1 = 1\}$ are two cross-sections with coordinates $(\mu, x_1, x_I, y_1, y_L)$ and $(\mu, v_1, v_2, v_I, w_L)$, respectively, such that $\gamma' \cap \Sigma^c(\bar{\mu}) = (\bar{\mu}, 0, a_I, 0, 0)$, $\gamma \cap S^c(\bar{\mu}) = (\bar{\mu}, 0, 0, 0)$ where $x_I = (x_3, \ldots, x_{n_{k-1}})$ and $y_L = (y_2, \ldots, y_{n_k})$ then

$$W^s(\sigma_{k+1}(\mu)) \cap S(\mu) = \{v_1 = F(\mu, v_2, v_I, w_L)\}$$

and
$$W^u(\sigma_{k-1}(\mu)) \cap \Sigma^c(\mu) = \{x_1 = G(\mu, x_I, y_1, y_L)\}$$

where F and G are C^m functions, $m \geqslant 3$. The quasi-transversality assumptions mean that the functions $w_L \mapsto F(\bar{\mu}, 0, 0, w_L)$ and $x_I \mapsto G(\bar{\mu}, x_I, 0, 0)$ have non-degenerate (Morse) critical points at $w_L = 0$ and $x_I = a_I$, respectively. Hence, $W^u(\sigma_k(\mu))$ is tangent to $W^s(\sigma_{k+1}(\mu))$ if and only if $F(\mu, 0, 0, w_L) = 0$ and $\frac{\partial F}{\partial w_j}(\mu, 0, 0, w_L) = 0$ for $j = 2, \ldots n_k$. By the generic unfolding of the orbit of tangency γ and the implicit function theorem, we get a C^{m-1} curve Γ in the parameter space such that the corresponding vector field X_μ, $\mu \in \Gamma$, presents an orbit of quasi-transversality between $W^u(\sigma_k(\mu))$ and $W^s(\sigma_{k+1}(\mu))$. Therefore, Γ belongs to the bifurcation diagram. Analogously, solving the equations $G(\mu, x_I, 0, 0) = 0$, $\frac{\partial G}{\partial x_i}(\mu, x_I, 0, 0) = 0$, we obtain a curve Γ' containing $\bar{\mu}$ such that, for $\mu \in \Gamma'$, the vector field X_μ has an orbit of quasi-transversal intersection between $W^u(\sigma_{k-1}(\mu))$ and $W^{ss}(\sigma_k(\mu))$. The condition (I.6) says that Γ and Γ' are transversal at $\bar{\mu}$ (the tangencies have independent unfolding). We may suppose that $\bar{\mu} = 0$, $\Gamma = \{\mu_2 = 0\}$ and $\Gamma' = \{\mu_1 = 0\}$.

To study the tangencies between $W^u(\sigma_{k-1}(\mu))$ and $W^s(\sigma_{k+1}(\mu))$, we write in the above coordinates

$$W^u(\sigma_{k-1}(\mu)) \cap S^c(\mu)$$
$$= \{(e^{-\alpha_1(\mu)t}\, G(\mu, x_I, e^{-\beta_1(\mu)t}, e^{-\beta_L(\mu)t} w_L), e^{-\alpha_2(\mu)t}, e^{-\alpha_I(\mu)t}(x_I + a_I), w_L)\}$$

with $e^{-\beta_L(\mu)t} = \mathrm{diag}(e^{-\beta_2(\mu)t}, \ldots, e^{-\beta_{n_k}(\mu)t})$ and $e^{-\alpha_I(\mu)t} = \mathrm{diag}(e^{-\alpha_3(\mu)t}, \ldots, e^{-\alpha_{n_{k-1}}(\mu)t})$. Hence, the tangency between $W^u(\sigma_{k-1}(\mu))$ and $W^s(\sigma_{k+1}(\mu))$ in $S^c(\mu)$ is expressed by the system of equations:

$$\text{(E1)}\quad \begin{cases} e^{-\alpha_1 t}\, G(\xi) - F(\zeta) = 0 \\ \dfrac{\partial G}{\partial x_i}(\xi) - \dfrac{\partial F}{\partial v_i}(\zeta) . e^{(\alpha_1 - \alpha_i)t} = 0 \quad i = 3, \ldots, n_{k-1} \\ e^{-\alpha_1 t} \dfrac{\partial G}{\partial y_j}(\xi) . e^{-\beta_j t} - \dfrac{\partial F}{\partial w_j}(\zeta) = 0 \quad j = 2, \ldots, n_k \\ -\alpha_1 G(\xi) - \beta_1(\mu)\, e^{-\beta_1(\mu)t} \dfrac{\partial G}{\partial y_1}(\xi) - \Sigma_{j=2}^{n_k} \beta_j\, e^{-\beta_j(\mu)t} \dfrac{\partial G}{\partial y_i}(\xi) . w_j \\ \qquad + \alpha_2\, e^{(\alpha_1 - \alpha_2)t} \dfrac{\partial F}{\partial v_2}(\zeta) + \Sigma_{i=3}^{n_{k-1}} \alpha_i\, e^{(\alpha_1 - \alpha_i)t}(a_i + x_i) \dfrac{\partial F}{\partial v_i}(\zeta) = 0, \end{cases}$$

$\xi = (\mu, x_{\mathrm{I}}, e^{-\beta_1(\mu)t}, e^{-\beta_{\mathrm{L}}(\mu)t} w_{\mathrm{L}})$, $\zeta = (\mu, e^{-\alpha_2(\mu)t}, e^{-\alpha_{\mathrm{I}}(\mu)t}(x_{\mathrm{I}} + a_{\mathrm{I}}), w_{\mathrm{L}})$. Making $t \to +\infty$ we obtain $F(\mu, 0, 0, w_{\mathrm{L}}) = 0$, $\frac{\partial G}{\partial x_i}(\mu, x_{\mathrm{I}}, 0, 0) = 0$, $\frac{\partial F}{\partial w_j}(\mu, 0, 0, w_{\mathrm{L}}) = 0$ and $G(\mu, x_{\mathrm{I}}, 0, 0) = 0$ which is non-singular due to the generic and independent unfolding of the orbits of tangency.

Let $\alpha = \min\{\beta_1(0), \alpha_1(0), \alpha_2(0) - \alpha_1(0)\} > 0$ (so $\widetilde{\alpha}_j(\mu) = \alpha_j(\mu)/\alpha \geqslant 1$, $\widetilde{\beta}_j(\mu) = \beta_j(\mu)/\alpha \geqslant 1$, for μ near 0). By setting $e^{-\alpha t} = z$, we may extend the system to a neighbourhood of the origin in a C^1 fashion to apply the implicit function theorem and get a C^1 curve Γ_0 in the parameter space, tangent to the curve Γ at 0, such that $\mu \in \Gamma_0$ if and only if X_μ presents a tangency between $W^u(\sigma_{k-1}(\mu))$ and $W^s(\sigma_{k+1}(\mu))$. It also follows from the above equations that along Γ_0 the manifolds $W^u(\sigma_{k-1}(\mu))$ and $W^s(\sigma_{k+1}(\mu))$ present an orbit of quasi-transversal intersection.

For the singularities $\sigma_i(\mu)$ with $i < k - 1$, the transversality between $W^u(\sigma_i(0))$ and $W^{ss}(\sigma_k(0))$, and between $W^{cu}(\sigma_k(0))$ and $W^s(\sigma_{k+1}(0))$, guarantee the transversality between $W^u(\sigma_i(\mu))$ and $W^s(\sigma_{k+1}(\mu))$ for μ near 0. ∎

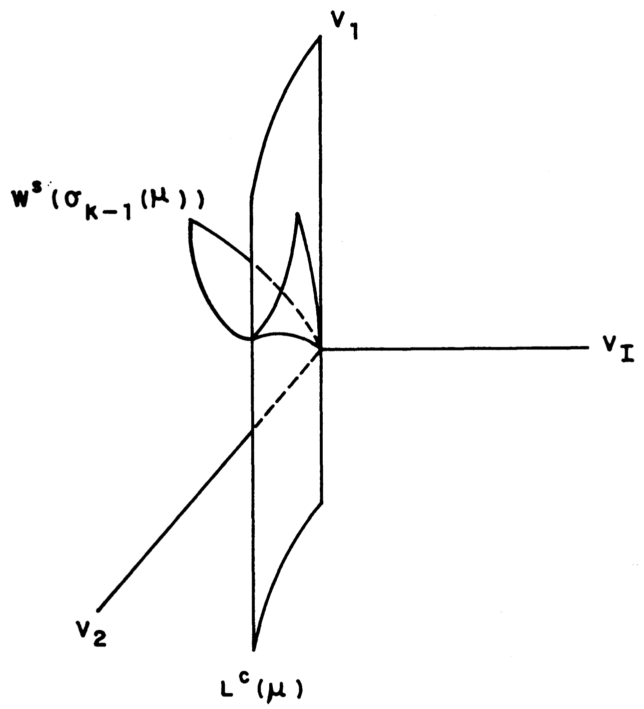

FIG. II

We now proceed to describe the above equations (E1) *in terms of tangencies between foliations: this geometric interpretation will be useful in the proof of the stability of the bifurcation of type I.* Let $F^{vs}_{k+1}(\mu)$ be the C^m foliation in $S^e(\mu)$ which is compatible with $W^s(\sigma_{k+1}(\mu))$ defined by $\pi^{vs}_{k+1}(\mu, v_1, v_2, v_{\mathrm{I}}, w_{\mathrm{L}}) = (v_1 - F(\mu, v_2, v_{\mathrm{I}}, w_{\mathrm{L}}) + F(\mu, v_2, 0, w_{\mathrm{L}}), v_2, w_{\mathrm{L}})$. It follows from the quasi-transversality between $W^u(\sigma_{k-1}(0))$ and $W^{ss}(\sigma_k(0))$ that the restriction of π^{vs}_{x+1} to $W^u(\sigma_{k-1}(\mu)) \cap S^c(\mu)$ is a submersion with fold along the set

$$\{ e^{-\alpha_1 t}\, G(\mu, x_I(\mu, e^{-\alpha t}, w_L), e^{-\beta_1 t}, e^{-\beta_2 t}, w_L), e^{-\alpha_2 t}, e^{-\alpha_1 t}\, x_1(\mu, e^{-\alpha t}, w_2), w_L \}$$

where $x_I(\mu, z, w_2)$ is a C^1 function which is of class C^{m-1} for $z > 0$. Hence, the submanifold $L^c(\mu) \subset S^c(\mu)$ defined by $(v_1, v_2, v_2^{\alpha_I/\alpha_2}\, x_I/(\mu, v_2^{\alpha/\alpha_2}, w_L), w_2)\ v_2 \geqslant 0$ is transversal to $F^{vs}_{k+1}(\mu)$ and, to $W^u(\sigma_{k-1}(\mu)) \cap S^c(\mu)$ and it contains the locus of tangency between $W^u(\sigma_{k-1}(\mu)) \cap S^c(\mu)$ and $F^{vs}_{k+1}(\mu)$; see Fig. II. Let $F^u(\mu)$ be a codimension-two C^1 foliation in $L^c(\mu)$ which is compatible with $W^u(\sigma_{k-1}(\mu)) \cap L^c(\mu)$ and with $W^u(\sigma_k(\mu)) \cap L^c(\mu)$. Then, from the quasi-transversality between $W^u(\sigma_k(0))$ and $W^s(\sigma_{k+1}(0))$, we get that $W^u(\sigma_{k+1}(\mu)) \cap L^c(\mu)$ is tangent to $F^u(\mu)$ along a two-dimensional C^1 manifold $S^c_0(\mu)$ which is the graph of a C^1 map $w_L = \Omega_L(\mu, v_2)$, $v_2 \geqslant 0$. In short, the second set of equations in (E1) above defines $L^c(\mu)$ and the second and third ones define $S^c_0(\mu)$.

In this way the bifurcation set is described as follows:

a) The *point* $W^u(\sigma_k(\mu)) \cap S^c_0(\mu)$ belongs to the *curve* $W^u(\sigma_{k+1}(\mu)) \cap S^c_0(\mu)$ if and only if there is a quasi-transversal orbit of tangency between $W^u(\sigma_k(\mu))$ and $W^s(\sigma_{k+1}(\mu))$,

b) The *curves* $T^s(\mu) = W^s(\sigma_{k+1}(\mu)) \cap S^c_0(\mu)$ and $T^u(\mu) = W^u(\sigma_{k-1}(\mu)) \cap S^c_0(\mu)$ are *tangent* in $S^c_0(\mu)$ if and only if there is a quasi-transversal orbit of tangency between $W^u(\sigma_{k-1}(\mu))$ and $W^s(\sigma_{k+1}(\mu))$. It corresponds to the curve Γ_0 in the parameter space while condition *a)* corresponds to Γ.

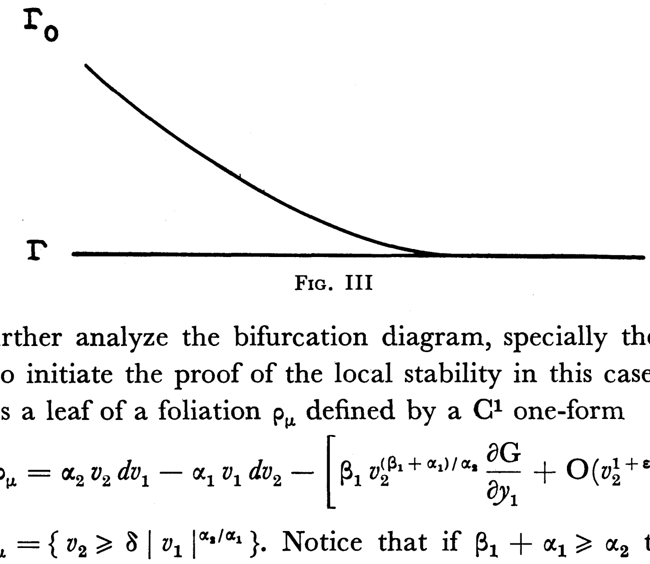

FIG. III

We now further analyze the bifurcation diagram, specially the second condition above, in order to initiate the proof of the local stability in this case. First, notice that the curve $T^u(\mu)$ is a leaf of a foliation ρ_μ defined by a C^1 one-form

$$\rho_\mu = \alpha_2 v_2\, dv_1 - \alpha_1 v_1\, dv_2 - \left[\beta_1 v_2^{(\beta_1+\alpha_1)/\alpha_2} \frac{\partial G}{\partial y_1} + O(v_2^{1+\varepsilon}) \right] dv_2, \quad \varepsilon > 0,$$

in the region $R_\mu = \{ v_2 \geqslant \delta \mid v_1 |^{\alpha_2/\alpha_1} \}$. Notice that if $\beta_1 + \alpha_1 \geqslant \alpha_2$ then 0 is a hyperbolic singularity of sink type with linear part equals to $\alpha_2 v_2\, dv_1 - \alpha_1 v_1\, dv_2$. If $\alpha_2 > \beta_1 + \alpha_1$, then by setting $w = v_2^{(\beta_1+\alpha_1)/\alpha_2}$ we obtain, after dividing by $\alpha_2 w^{-1+\alpha_2/(\alpha_1+\beta_1)}$, the expression of a C^1 one-form with a hyperbolic sink at the origin with linear part equals to $(\alpha_1 + \beta_1)\, w\, dw - \left(\alpha_1 v_1 + \beta_1 w \frac{\partial G}{\partial y_1}(0)\right) dv_2$. Hence, in both cases, the foliation $\rho_\mu = 0$

has a singularity of sink type at 0 and is such that all leaves, except one, are tangent to the axis $v_2 = 0$. Therefore, a tangency between ρ_μ and $T^s(\mu)$ occurs along a C^1 curve ℓ_μ defined by $\alpha_2 v_2 \frac{\partial F}{\partial v_2} - \alpha_1 v_1 - \beta_1 v_2^{(\beta_1+\alpha_1)/\alpha_2} \frac{\partial G}{\partial y_1} + 0(v_2^{1+\varepsilon}) = 0$, $v_2 \geqslant 0$. Using again the hypothesis (1.9), which means that $\frac{\partial F}{\partial v_2}(0) \neq 0$ and $\frac{\partial G}{\partial y_1}(0) \neq 0$, we obtain that ℓ_μ is transversal to $T^s(\mu)$ and to $\{ \rho_\mu = 0 \}$ and the contact along ℓ_μ is quadratic. Thus, the bifurcation set is characterized by the position of the points $p_{k+1}(\mu) = T^s(\mu) \cap \ell_\mu$, $p_{k-1}(\mu) = T^u(\mu) \cap \ell_\mu$ and $p_k(\mu) = W^u(\sigma_k(\mu)) \cap S_0^c(\mu)$. By taking μ in small neighbourhood V of $\mathbf{R}^2$ and shrinking $S(\mu)$, we can modify ρ_μ in a neighbourhood of the boundary of R_μ in order to include the curve $v_2 = \delta \mid v_1 \mid^{(\alpha_2/\alpha_1)(\mu)}$ as a leaf of ρ_μ and to extend it linearly by setting $\rho_\mu = \alpha_2 v_2 \, dv_1 - \alpha_1 v_1 \, dv_2$. The inverse image of ρ_μ by the Poincaré map between level sets $P_\mu : A_k(\mu) \to B_k(\mu)$ gives a continuous one-dimensional foliation with C^1 leaves which are topologically transversal to $A_k^s(\mu)$: this remark is easy to check by using the linearization of X_μ and it will be important in the proof of local stability at the end of the paragraph. Suppose now that $\hat{X}_\mu$ is a nearby family. Let $\varphi : V \to \mathbf{R}^2$ be a local homeomorphism which sends the bifurcation set of X_μ to the bifurcation set of $\hat{X}_\mu$. Then, we define a homeomorphism $h_\mu^c : S_0^c(\mu) \to \hat{S}_0^c(\varphi(\mu))$ which sends ρ_μ to $\hat{\rho}_{\varphi(\mu)}$ and the curve $T^s(\mu)$ to $\hat{T}^s(\varphi(\mu))$ as follows. We first define a homeomorphism *between the lines of tangency* ℓ_μ, $\hat{\ell}_{\varphi(\mu)}$ such that $p_{k-1}(\mu)$ is sent to $\hat{p}_{k-1}(\varphi(\mu))$ and $p_{k+1}(\mu)$ goes to $\hat{p}_{k+1}(\varphi(\mu))$. This is only necessary in the region $\mu_1 \geqslant 0$ or $\mu_2 \geqslant 0$ since, otherwise, the curves $T^s(\mu)$ and $T^u(\mu)$ do not intersect ℓ_μ. This gives a homeomorphism on part of the space of leaves of a foliation τ_μ defined by $dv_1 - \frac{\partial \tilde{F}}{\partial v_2}(\mu, v_2) \, dv_2 = 0$ and which has $T^s(\mu)$ as a distinguished leaf ($T^s(\mu)$ is defined by $v_1 = \tilde{F}(\mu, v_2)$). We extend such a homeomorphism to the axis $v_2 = 0$, $v_1 \leqslant 0$, also preserving its intersection with $T^s(\mu)$, to complete the definition of a homeomorphism on the space of leaves of δ_μ. This yields a homeomorphism also on the space of leaves of ρ_μ. By the reparametrization above, when $p_{k-1}(\mu)$ coincides with $p_{k+1}(\mu)$, the same occurs with $\hat{p}_{k-1}(\varphi(\mu))$ and $\hat{p}_{k+1}(\varphi(\mu))$. Therefore, since δ_μ and ρ_μ have quadratic contact along ℓ_μ, we define h_μ^c by sending leaves of ρ_μ to leaves of $\hat{\rho}_{\varphi(\mu)}$ and leaves of τ_μ to leaves of $\hat{\tau}_{\varphi(\mu)}$. In the sequel we will fully develop the proof of local stability.

(1.C) *The local stability of the bifurcation of type I.* — Let $\hat{X}_\mu$ be a family near X_μ so that all conditions described in (I.A) are satisfied with respect to a bifurcation value $\hat{\mu}$ near $\bar{\mu}$ where $\hat{X}_{\hat{\mu}}$ presents one orbit of quasi-transversality with a criticality.

The equivalence between X_μ and $\hat{X}_{\varphi(\mu)}$ (after choosing an appropriate reparametrization φ) will be a conjugacy outside a neighbourhood of the closure of $\gamma \cup \gamma'$ in M. This neighbourhood U is the union of two *distinguished neighbourdhoods* U_{k-1} and U_{k+1} of the orbits of tangency γ' and γ, respectively, which were constructed in the previous section (Theorem A). Inside U the equivalence will preserve the level sets of the potentials of X_μ and $\hat{X}_{\varphi(\mu)}$, which can be assumed to be the same for both families. We suppose that a compatible unstable system $F^u(\sigma_1(\mu)), \ldots, F^u(\sigma_{k-2}(\mu))$ together with a center-unstable foliation $F^{cu}(\sigma_{k-1}(\mu))$ have been already constructed and also a compatible stable system $F^s(\sigma_{k+2}(\mu)), \ldots, F^s(\sigma_\ell(\mu))$ with a center-stable foliation $F^{cs}(\sigma_{k+1}(\mu))$. We consider similar compatible systems for the nearby family $\{\hat{X}_\mu\}$ and assume that we already have homeomorphisms defined on the space of leaves of these foliations.

We first construct a center-unstable foliation $F^{cu}(\sigma_k(\mu))$ which is compatible with the unstable system and with $F^{cu}(\sigma_{k-1}(\mu))$. Besides the presence of criticality, *this last compatibility condition with a singular foliation is the novelty here.* As in the proof of Theorem A, we begin by describing the central foliation $F^c(\sigma_k(\mu))$ in a neighbourhood of the point of tangency $r_k(0) = \gamma' \cap A_k^s(0)$. Recall that $A_k^s(\mu)$ is the sphere $A_k(\mu) \cap W^s(\sigma_k(\mu))$, where $A_k(\mu)$ is the part of the boundary of $U_{k-1}(\mu) \cap U_{k+1}(\mu)$ which is contained in a non-critical level of the potential f_μ. The central foliation is constructed in $C_k^s(\mu)$, a tubular neighbourhood of $W^{ss}(\sigma_k(\mu)) \cap A_k^s(\mu)$ in $A_k^s(\mu)$ which is bounded by two spheres $\partial D_k^+(\mu)$ and $\partial D_k^-(\mu)$. We take μ-dependent coordinates (x_1, x_I, x_J) for the cylinder $C_k^s(\mu)$ centered at $r_k(0)$ such that

$$W^{ss}(\sigma_k(\mu)) \cap C_k^s(\mu) = \{ x_1 = 0 \},$$

$$W^{cu}(\sigma_{k-1}(\mu)) \cap C_k^s(\mu) = \{ (x_1, x_I, x_J = G^{cu}(\mu, x_1, x_I)) \}$$

and

$$W^u(\sigma_{k-1}(\mu)) \cap C_k^s(\mu) = \{ x_1 = G(\mu, x_I), x_J = G^{cu}(\mu, x_1, x_I) \}.$$

The leaves of $F^c(\sigma_k(\mu))$ inside $W^{cu}(\sigma_{k-1}(\mu)) \cap C_k^s(\mu)$ are the integral curves of the vector field Z_μ^c defined by

$$\dot{x}_1 = [x_1 - G(\mu, x_I)]^2 + \Sigma \left(\frac{\partial G}{\partial x_i} (\mu, x_I) \right)^2,$$

$$\dot{x}_i = \frac{\partial G}{\partial x_i} (\mu, x_I).$$

Hence, $F^c(\sigma_k(\mu))$ has a "saddle-node" singularity at $r_k(\mu)$, the point of tangency between $W^u(\sigma_{k-1}(\mu)) \cap C_k^s(\mu)$ and $F^{ss}(\sigma_k(\mu)) \cap C_k^s(\mu)$.

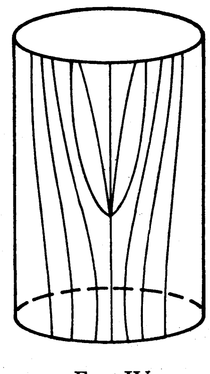

FIG. IV

Now, since $F^{cu}(\sigma_{k-1}(\mu)) \cap C^s_k(\mu)$ is transversal to $F^{ss}(\sigma_k(\mu)) \cap C^s_k(\mu)$ we are going to lift this foliation to each leaf of $F^{cu}(\sigma_{k-1}(\mu))$. Recall that there are two types of leaves of $F^{cu}(\sigma_{k-1}(\mu))$ which are denoted by $F^{cu}_1(\sigma_{k-1}(\mu))$ and $F^{cu}_2(\sigma_{k-1}(\mu))$ with dimensions equal to $(\dim W^u(\sigma_{k-1}(\mu)) + 1)$ and $(\dim W^u(\sigma_{k-1}(\mu)) + 2)$, respectively. Let $0 < \delta_1 < \delta_2$ be small numbers and consider the following two conic regions in a neighbourhood of $r_k(\mu)$ in $C^s_k(\mu)$: $C(\delta_1) = \{ | x_1 - G(\mu, x_I)|^2 \geqslant \delta_1 \| x_J - G^{cu}(\mu, x_1, x_I)\|^2 \}$ and $C(\delta_2) = \{ | x_1 - G(\mu, x_I)|^2 \leqslant \delta_2 \| x_J - G^{cu}(\mu, x_1, x_I)\|^2 \}$. The numbers δ_1, δ_2 are taken so that the boundaries of these regions are transversal to $F^{cu}_2(\sigma_{k-1}(\mu))$. Inside $C(\delta_1)$, we just lift the foliation $F^c(\sigma_k(\mu)) \subset W^{cu}(\sigma_{k-1}(\mu))$ to each leaf of $F^{cu}(\sigma_{k-1}(\mu))$ via the projection $(x_1, x_I, x_J) \mapsto (x_1, x_J)$. Hence, all leaves of type $F^{cu}_1(\sigma_{k-1}(\mu))$ are subfoliated by a one-dimensional foliation diffeomorphic to the one described above. In the region $C(\delta_2)$ we take the codimension-one foliation defined by the non-positive level sets of the map $(x_1, x_2, x_J) \overset{\pi_1}{\mapsto} | x_1 - G(\mu, x_I)|^2 - \delta_2 \| x_J - G^{cu}(\mu, x_1, x_I)\|^2$ and intersect it with $F^{cu}_2(\sigma_{k-1}(\mu))$ to obtain a codimension-one foliation transversal to $F^{ss}(\sigma_k(\mu))$. The central foliation, in the intersection of $F^{cu}_2(\sigma_{k-1}(\mu))$ with $C(\delta_2)$, is given by the one-parameter unfolding of Z^c_μ lifted to intersections of levels of π_1 with leaves of $F^{cu}_2(\sigma_{k-1}(\mu))$. That is, if F denotes a leaf, then in $F \cap \pi^{-1}(0)$ we just lift the vector field Z^c_μ and in $F \cap \pi_1^{-1}(-a)$ we lift the perturbed field $Z^c_{\mu,a} = Z^c_\mu + a \dfrac{\partial}{\partial x_1}$.

The region between $C(\delta_1)$ and $C(\delta_2)$ is used to match continuously the above foliations. To do that, we need to modify the intersections of $F^{cu}_1(\sigma_{k-1}(\mu))$ with the complement of $C(\delta_1)$ in order to include the boundary of $C(\delta_2)$ as a new leaf. In doing so we can glue a singular central foliation near the tangency point $r_k(0)$ with a non-singular foliation $F^c(\sigma_k(\mu))$ in $C^s_k(\mu)$ which is compatible with the unstable system $F^u(\sigma_1(\mu)), \ldots, F^u(\sigma_{k-2}(\mu)), F^{cu}(\sigma_{k-1}(\mu))$; see Theorem A. Before concluding the construction of $F^{cu}(\sigma_k(\mu))$, we indicate how to *extend* a homeomorphism h^c_μ, defined on a neighbourhood of the tangency point $r_k(0)$ in $W^{cu}(\sigma_{k-1}(\mu)) \cap C^s_k(\mu)$ and which preserves the central foliation $F^c(\sigma_k(\mu))$, to a full neighbourhood of $r_k(0)$ in $C^s_k(\mu)$. Inside the conic region $C(\delta_2)$, we just use the homeomorphism in the space of leaves of $F^{cu}(\sigma_{k-1}(\mu))$ and the projection $(x_1, x_I, x_J) \mapsto (x_1, x_I)$ to lift h^c_μ to each leaf of $F^{cu}(\sigma_{k-1}(\mu)) \cap C^s_k(\mu)$. Then, since $F^c(\sigma_k(\mu))$ is non-singular outside $C(\delta_2)$ we extend it to a neighbourhood of $C(\delta_2) \cap \partial D^+_k(\mu)$ in the sphere $\partial D^+_k(\mu)$. This neighbourhood is taken to be bounded by non-singular levels of π_1 and in this boundary the homeomorphism is actually a diffeomorphism C^1 close to the identity. Therefore, using the Isotopy Extension Theorem we can proceed as in Theorem A to extend it to all of $\partial D^+_k(\mu)$ in a way that it is compatible with the homeomorphism defined on the space of leaves of the unstable system. In the following lemma, *we show how to obtain this homeomorphism* h^c_μ.

Lemma 1. — Let $\hat{F}^c_k(\mu)$ *be a central foliation for the family* $\hat{X}_{\varphi(\mu)}$ *defined by a vector field* $\hat{Z}^c_\mu$ *which has a saddle-node at* $\hat{r}_k(\mu)$ *as above. Then, there is a homeomorphism* h^c_μ *defined* *on a neighbourhood of* $r_k(0)$ *and depending continuously on* μ, *which sends leaves of* $F_k^c(\mu)$ *to leaves of* $\hat{F}_k(\mu)$ *and* $W^{ss}(\sigma_k(\mu))$ *to* $W^{ss}(\hat{\sigma}_k(\varphi(\mu)))$.

Proof. — Using the above coordinates to describe $W^{cu}(\sigma_{k-1}(\mu)) \cap W^{cs}(\sigma_k(\mu))$, we observe that the projection $(x_1, x_I) \overset{\pi}{\mapsto} x_1$, along the leaves of $F^{ss}(\sigma_k(\mu))$, gives a Liapunov function for Z_μ^c and that Z_μ^c restricted to $W^u(\sigma_{k-1}(\mu)) \cap C_k^s(\mu)$ has a hyperbolic singularity at $r_k(\mu)$. Therefore, the proof consists in showing that there exists an equivalence between Z_μ^c and $\hat{Z}_{\varphi(\mu)}^c$ near saddle-nodes which preserves the level sets of a Liapunov function. To simplify the notation, we drop the parameter μ in the following arguments and denote by $W^j(Z^c)$, $j = u, s, cu, cs$, the invariant submanifolds of Z^c. For $\varepsilon > 0$, we let $D^- = \{ x_1 = -\varepsilon \}$ and $D^+ = \{ x_1 = \varepsilon \}$ be non-critical levels of π such that for small μ the singularities $r_k(\mu)$ and $\hat{r}_k(\varphi(\mu))$ are contained in $| x_1 | < \varepsilon$. We first obtain an equivalence on $W^{cs}(Z^c)$, starting by taking a C^1 diffeomorphism close to the identity from the closed disc $D^-(Z^c) = W^s(Z^c) \cap D^-$ to $W^s(\hat{Z}^c) \cap D^-$. Then, we take a tubular neighbourhood of $D^-(Z^c)$ in $W^{cs}(Z^c) \cap D^-$ with fibers forming a *radial* foliation δ^s. Each fiber of δ^s is a C^1 curve transversal and exterior to the boundary $\partial D^-(Z^c)$. Positive saturation of δ^s by the flow of Z^c, intersected with D^+, gives rise to a one-dimensional foliation in $W^{cs}(Z^c) \cap D^+$ which is singular at the point $W^c(Z^c) \cap D^+$. Hence, performing the same construction for $\hat{Z}^c$, we define a homeomorphism from $W^{cs}(Z^c) \cap D^+$ to $W^{cs}(\hat{Z}^c) \cap D^+$ which preserves this foliation and it is a diffeomorphism outside the point $W^c(\hat{Z}^c) \cap D^+$. By preserving the level sets of π and the trajectories of Z^c inside $W^{cs}(Z^c)$, we obtain an equivalence between $Z^c \mid W^{cs}(Z^c)$ and $\hat{Z}^c \mid W^{cs}(\hat{Z}^c)$. Proceeding dually, starting now at the level D^+, we get an equivalence between $Z^c \mid W^{cu}(Z^c)$ and $\hat{Z}^c \mid W^{cu}(\hat{Z}^c)$. The corresponding *radial* foliation in $W^{cu}(Z^c) \cap D^+$ is denoted by δ^u. We are now going to match these two equivalences. In D^-, we raise over each point of the disc $D^-(Z^c)$ a continuous foliation g^u with C^1 leaves, transversal to $W^{cs}(Z^c)$, which is compatible with $W^u(\sigma_{k-1}(\mu)) \cap D^-$. Each leaf of g^u has dimension equals to $d_u = \dim W^{uu}(Z^c)$, and $W^{cu}(Z^c) \cap D^-$ is taken as a distinguished leaf. In the complement of the component of $D^- \backslash W^u(\sigma_{k-1}(\mu))$ which contains $W^{cu}(Z^c)$, we take a $(d_u + 1)$-dimensional continuous foliation g_1^u, with C^1 leaves, which is transversal to $W^{cs}(Z^c) \cap D^-$ and such that the boundary of each leaf of g_1^u is a leaf of g^u in $W^u(\sigma_{k-1}(\mu)) \cap D^-$ and $g_1^u \cap W^{cs}(Z^c)$ is the radial foliation δ^u. Dually, we construct in D^+ the foliations g^s and g_1^s with dimension $d_s = \dim W^{ss}(Z^c)$ and $(d_s + 1)$, respectively. Still denoting by g^u the intersection of the positive saturation of g^u by the flow of Z^c with D^+, we observe that for each leaf $g_{1,b}^s$ of g_1^s the intersection $g^u \cap g_{1,b}^s$ is a one-dimensional foliation which is singular at the point

$$b = g_{1,b}^s \cap \partial D^+(Z^c) \subset W^u(\sigma_{k-1}(\mu)) \cap D^+.$$

Hence, as in the proof of Theorem A, we take two families of closed conic regions $E_1^b \subset E^b \subset g_{1,b}^s$ with vertices at b, such that $W^{cs}(Z^c) \cap g_{1,b}^s$ is contained in the interior of E_1^b. We then modify $g^u \cap g_{1,b}^s$ to get a new one-dimensional foliation $\mathscr{H}^b$ such that $\mathscr{H}^b \cap E_1^b = g^u \cap E_1^b$ and outside E^b it is non-singular and transversal to $g_b^s = g_{1,b}^s \cap W^u(\sigma_{k-1}(\mu))$. Clearly, this can be done depending continuously on b and μ. Once the same construction for $\hat{Z}^c$ is performed, we are prepared to define a homeomorphism h^+ on D^+ and conclude the proof of Lemma 1. The basic property of h^+ is that by preserving $\mathscr{H}^b$ it induces (via projection along the trajectories of the respective vector fields) a homeomorphism on D^- which is a continuous extension of the homeomorphisms already defined on $D^- \cap W^{cu}(Z^c)$ and $D^- \cap W^{cs}(Z^c)$. The definition of h^+ goes as follows. Let Δ^+ be the closure of the component of $D^+ \backslash W^u(\sigma_{k-1}(\mu))$ that contains the disc $D^+(Z^c)$ (i.e., the set $x_1 = \varepsilon$; $G(\mu, x_I) \geqslant \varepsilon$). In Δ^+, we take a continuous foliation by C^1 closed discs compatible with $\bar{g}_1^u$ (the positive saturation of g_1^u), which is transversal to $W^{cs}(Z^c) \cap D^+$ with complementary dimension, such that $D^+(Z^c)$ is a special leaf. The boundary of each of these discs is a sphere in $W^u(\sigma_{k-1}(\mu)) \cap D^+$ which is transversal to the foliation $g_1^s \cap W^u(\sigma_{k-1}(\mu))$, with complementary dimension. Thus, by preserving this family of discs and the foliation g^s and using the homeomorphisms already defined on $W^{cs}(Z^c) \cap D^+$ and on $W^{cu}(Z^c) \cap D^+ = D^+(Z^c)$ (space of leaves of these foliations) we obtain a homeomorphism on Δ^+. This gives a homeomorphism on the space of leaves of $\mathscr{H}^b$ which are outside the conic region E^b. Since in the space of the leaves of $\mathscr{H}^b$ which are inside E^b is the disc $D^-(Z^c)$ (where we also have defined a homeomorphism), we have obtained a homeomorphism in the total space of leaves of $\mathscr{H}^b$. To complete the definition of h^+, it is enough to preserve $\mathscr{H}^b$ and a codimension-one foliation whose space of leaves is $W^{cs}(Z^c) \cap g_{1,b}^s$, which can be defined by $G(\mu, x_I) = a$, $a \leqslant \varepsilon$. This concludes the construction of h^+. As observed above, the equivalence between Z^c and $\hat{Z}^c$ is obtained by preserving levels of π, using h^+ and preserving trajectories of the fields. ■

A center-unstable foliation $F^{cu}(\sigma_k(\mu))$ can now be defined as in the proof of Theorem A except in a neighbourhood of $r_k(0)$. In this neighbourhood, we want to distinguish a leaf that contains the tangency point $r_k(\mu)$ and contains the possible tangencies between $W^s(\sigma_{k+1}(\mu))$ and $W^u(\sigma_{k-1}(\mu))$. So, we let $\Sigma^c(\mu)$ be a neighbourhood of $r_k(0)$ in $A_k(\mu) \cap W^{cs}(\sigma_{k+1}(\mu)) \cap W^{cu}(\sigma_{k-1}(\mu))$ and consider coordinates (x_1, x_I, y_1, y_L), as in (1.B), to define a continuous family of vector fields Y_μ in $\Sigma^c(\mu)$ for $y_1 \geqslant 0$ by

$$\dot{x}_1 = [x_1 - G(\mu, x_I, y_1, y_1^{\beta_L/\beta_1} y_L)]^2 + \sum_{i=3} \frac{\partial G}{\partial x_i} (\mu, x_I, y_1, y_1^{\beta_L/\beta_1} y_L)\, \dot{x}_i,$$

$$\dot{x}_i = \frac{\partial G}{\partial x_i} (\mu, x_I, y_1, y_1^{\beta_L/\beta_1} y_L) - \frac{\partial F}{\partial v_i} (y_1^{\alpha_1/\beta_1}, y_1^{\alpha_I/\beta_1} x_I, y_L)\, y_1^{(\alpha_i - \alpha_1)/\beta_1},$$

$$\dot{y}_1 = 0, \quad \dot{y}_\ell = 0.$$

Y_μ is tangent to $W^u(\sigma_{k-1}(\mu)) \cap \Sigma^c(\mu)$ and its restriction to $A_k^s(\mu)$ is equal to Z_μ^c, the vector field which defines the central foliation $F^c(\sigma_k(\mu))$. For fixed (μ, y_1, y_2), Y_μ also has a singularity of saddle-node type and its singular set is a C^1 manifold contained in $W^u(\sigma_{k-1}(\mu)) \cap \Sigma^c(\mu)$. Moreover, the image of this singular set by the Poincaré map $P^c_\mu : \Sigma^c(\mu) \to S^c(\mu)$ coincides with the set of points of tangency between $W^u(\sigma_{k-1}(\mu)) \cap S^c(\mu)$ and the foliation $F^{vs}_{k+1}(\mu)$ defined in (1.B). In particular, this image contains the tangencies between $W^u(\sigma_{k-1}(\mu))$ and $W^s(\sigma_{k+1}(\mu))$ in the level set $B_k(\mu)$. Therefore, over each leaf of $F^c(\sigma_k(\mu))$ we can raise a $(u+1)$-dimensional singular foliation in $A_k(\mu) \cap W^{cu}(\sigma_{k-1}(\mu))$ such that its intersection with $\Sigma^c(\mu)$ is tangent to Y_μ. The positive saturation of this foliation gives part of the leaves of $F^{cu}(\sigma_k(\mu))$ which are contained in $W^{cu}(\sigma_{k-1}(\mu))$. The process to define $F^{cu}(\sigma_k(\mu))$ inside the other leaves of $F^{cu}(\sigma_{k-1}(\mu))$ is analogous to the one described above to obtain the central-foliation $F^c(\sigma_k(\mu))$, i.e. one uses conic regions and projections onto $W^{cu}(\sigma_k(\mu)) \cap A_k(\mu)$. Since outside a neighbourhood of the tangency point $r_k(0)$, $F^{cu}(\sigma_k(\mu))$ is exactly as in Theorem A, we have completed the definition of $F^{cu}(\sigma_k(\mu))$.

Now comes the main step in proving the stability of the bifurcation of type I: to define a homeomorphism in the level $B_k(\mu)$ which preserves the intersections of leaves of the stable system and the center-unstable foliation $F^{cu}(\sigma_k(\mu))$. Since the stable system is transversal to the singular set of $F^{cu}(\sigma_k(\mu))$, it is transversal to all leaves of $F^{cu}(\sigma_k(\mu))$ outside a neighbourhood of the tangency point $r_{k+1}(\mu)$. *Hence, it is enough to obtain a homeomorphism in a small neighbourhood* $S(\mu)$ *of* $r_{k+1}(\mu)$ *in* $B_k(\mu)$ *and proceed with a cone-like construction as in Theorem A outside* $S(\mu)$. The same is valid in the section $S^{cu}(\mu) = S(\mu) \cap W^{cu}(\sigma_k(\mu))$ since $W^{cu}(\sigma_k(\mu))$ is transversal to $W^s(\sigma_{k+1}(\mu))$. *The novelty here is to obtain a homeomorphism on the dual section* $S^{cs}(\mu) = S(\mu) \cap W^{cs}(\sigma_{k+1}(\mu))$. Following the methods of the non-critical case (Theorem A, Section II), we want to preserve the intersections of $F^{cu}(\sigma_k(\mu))$ with $S^{cs}(\mu)$ and a C^m foliation F^{ss}_μ, which is compatible with $W^s(\sigma_{k+1}(\mu))$ and transversal to $W^{cu}(\sigma_k(\mu)) \cap S^{cs}(\mu)$ with complementary dimension, $m \geq 3$. However, due to criticality, this process must be modified in a neighbourhood of $L^c(\mu)$. Let us recall the notation used at the end of (1.B). First, $S^c(\mu) = W^{cu}(\sigma_{k-1}(\mu)) \cap S^{cs}(\mu)$ and $F^{vs}_{k+1}(\mu)$ is a C^m foliation in $S^c(\mu)$ which is compatible with $W^s(\sigma_{k+1}(\mu))$ and has codimension equals to $\dim [W^u(\sigma_k(\mu)) \cap S^{cs}(\mu)] + 2$. Further, $L^c(\mu)$ is a submanifold of $S^c(\mu)$ which contains the set of tangencies between $F^{vs}_{k+1}(\mu)$ and $W^u(\sigma_{k-1}(\mu))$. In $L^c(\mu)$ there is a C^1 codimension-two foliation F^u_μ which is compatible with $W^u(\sigma_{k-1}(\mu)) \cap L^c(\mu)$ and with $W^u(\sigma_k(\mu)) \cap L^c(\mu)$. Finally, $S^c_0(\mu)$ is a two-dimensional manifold of $L^c(\mu)$ which is transversal to $W^u(\sigma_k(\mu))$ and contains the set of tangencies between $W^s(\sigma_{k+1}(\mu))$ and F^u_μ. The curve $T^u(\mu) = W^u(\sigma_{k-1}(\mu)) \cap S^c_0(\mu)$ is a leaf of a singular foliation defined by the one-form ρ_μ and the curve

$$T^s(\mu) = W^s(\sigma_{k+1}(\mu)) \cap S^c_0(\mu)$$

is a leaf of a foliation τ_μ. These foliations are defined in (1.B).

We now start constructing a homeomorphism $h^{cs}_\mu : S^{cs}(\mu) \to S^{cs}(\varphi(\mu))$. Using the reparametrization $\varphi(\mu)$, also obtained in (1.B), we define a homeomorphism from $S^c_0(\mu)$ to $S^c_0(\varphi(\mu))$ that preserves the singular foliation ρ_μ and the curves $T^s(\mu)$ and $T^u(\mu)$. *Next, we extend the homeomorphism to* $L^c(\mu)$. We take a two-dimensional foliation $(SN)_\mu$ in $L^c(\mu)$ which is compatible with $W^s(\sigma_{k+1}(\mu))$ and is singular along the curve $T^u(\mu)$.

The foliation $(SN)_\mu$ is tangent to a family of C^1 vector fields Y_μ with singularity of saddle-node type defined by

$$\dot v_1 = [v_1 - F(\mu, v_2, 0, w_L)]^2 + \Sigma \frac{\partial F}{\partial w_j}(\mu, v_2, 0, w_L)\, \dot w_j,$$

$$\dot w_j = \frac{\partial F}{\partial w_j}(\mu, v_2, 0, w_L) - \frac{\partial G}{\partial y_j}(\mu, x_I(\mu, v_2^{1/\alpha_1}, w_L), v_2^{\beta_1/\alpha_1}, v_2^{\beta_L/\alpha_1} w_L) \,.\, v_2^{(\alpha_1+\beta_j)/\alpha_1},$$

$$\dot v_2 = 0.$$

$S_0^c(\mu)$ is a distinguished leaf of $(SN)_\mu$ and F_μ^u is transversal to $(SN)_\mu$ except along the curve $T^u(\mu)$. Hence, the intersection of $(SN)_\mu$ with $(\pi_\mu^u)^{-1}(\tau_\mu)$ is a continuous one-dimensional foliation transversal to $W^{cu}(\sigma_k(\mu)) \cap S^{cs}(\mu)$, where π_μ^u is the projection into $S_0^c(\mu)$ along the leaves of F_μ^u. We can apply Lemma 1 to obtain a homeomorphism $W^{cu}(\sigma_k(\mu)) \cap S^{cs}(\mu)$ to $W^{cu}(\hat\sigma_k(\varphi(\mu)) \cap S^{cs}(\varphi(\mu))$ which sends trajectories of Y_μ to trajectories of $\hat Y_{\varphi(\mu)}$ and also preserves $W^u(\sigma_k(\mu)) \cap S^{cs}(\mu)$. This gives a homeomorphism in the space of leaves of $(SN)_\mu$. *We now have homeomorphisms defined on the space of leaves of two complementary foliations*: $S_\mu^u(\mu)$ (whose leaf space is $S_0^c(\mu)$) *and* $(SN)_\mu$. *Thus, we have a homeomorphism from* $L^c(\mu)$ *to* $\hat L^c(\varphi(\mu))$ *which preserves* $W^u(\sigma_k(\mu))$, $W^u(\sigma_{k-1}(\mu))$ *and* $W^s(\sigma_{k+1}(\mu))$. We extend this homeomorphism to all of $S^{cs}(\mu)$. We recall that there are two types of leaves of $F^{cu}(\sigma_k(\mu))$ which are denoted by $F_1^{cu}(\sigma_k(\mu))$ and $F_2^{cu}(\sigma_k(\mu))$ such that $\dim F_1^{cu}(\sigma_k(\mu)) = \dim W^u(\sigma_k(\mu)) + 1$ and $\dim F_2^{cu}(\sigma_k(\mu)) = \dim W^u(\sigma_k(\mu)) + 2$. The foliation $F_2^{cu}(\sigma_k(\mu))$ has a saddle-node type singularity along a u-dimensional submanifold, which contains the tangencies between $W^u(\sigma_{k-1}(\mu))$ and $W^s(\sigma_{k+1}(\mu))$. Outside this submanifold, $F_2^{cu}(\sigma_k(\mu))$ is transversal to $W^s(\sigma_{k+1}(\mu))$. Let us construct a foliation F_μ^{ss} in $S^{cs}(\mu)$ which is compatible with $W^s(\sigma_{k+1}(\mu))$ and transversal to $W^{cu}(\sigma_k(\mu)) \cap S^{cs}(\mu)$ with complementary dimension. We first take a C^m foliation F_μ^{vs}, $m \geqslant 3$, which is compatible with $W^s(\sigma_k(\mu))$ and transversal to $S^c(\mu)$ such that $F_\mu^{vs} \cap S^c(\mu) = F_{k+1}^{vs}(\mu)$; the foliation F_μ^{vs} is defined by a submersion π_μ^{vs}. We then define F_μ^{ss} by taking the pull back via π_μ^{vs} of the one-dimensional foliation in $L^c(\mu)$ defined by $(\pi_\mu^u)^{-1}(\tau_\mu) \cap (SN)_\mu$. By construction, $L^c(\mu)$ is a typical leaf of $F^{cu}(\sigma_k(\mu)) \cap S^{cs}(\mu)$, i.e. the restriction of π_μ^{vs} to each leaf of type $F_2^{cu}(\sigma_k(\mu)) \cap S^{cs}(\mu)$ near $L^c(\mu)$ is a homeomorphism onto $L^c(\mu)$. In particular, this is valid for all singular leaves of $F_2^{cu}(\sigma_k(\mu))$. Therefore, we can define a homeomorphism on a neighbourhood of $L^c(\mu)$ using the two complementary foliations: $F^{cu}(\sigma_k(\mu)) \cap S^{cs}(\mu)$ and F_μ^{vs}. Such a homeomorphism has one important property: it induces a homeomorphism in the level set $A_k(\mu)\backslash A_k^s(\mu)$ which extends continuously to the sphere $A_k^s(\mu) = A_k(\mu) \cap W^s(\sigma_k(\mu))$. Let us explain this point. Denote by $P_\mu^{cs} : S^{cs}(\mu) \to A_k(\mu)\backslash A_k^s(\mu)$ the restriction of the Poincaré-map. Since F_μ^{vs} is of class C^m, $m \geqslant 3$, and the singular foliation ρ_μ is preserved on its quotient space, the image by P_μ^{cs} of the singular foliation induced on each leaf of $F_2^{cu}(\sigma_k(\mu))$ by π_μ^{vs} is a continuous foliation with C^1 leaves which are topologically transversal to $A_k^s(\mu)$. Therefore, a homeomorphism which preserves $F^{cu}(\sigma_k(\mu))$ and F_μ^{vs} extends automatically to the sphere $A_k^s(\mu)$. As we pointed out above, the idea is to construct h_μ^{cs} by preserving F_μ^{ss}

and $F^{cu}(\sigma_k(\mu))$ outside a neighbourhood $\Delta_2(\mu)$ of $L^c(\mu)$. Inside $\Delta_2(\mu)$ it preserves F^{vs}_μ and $F^{cu}(\sigma_k(\mu))$. We detail this construction. Let $\Delta_1(\mu) \subset \Delta_2(\mu)$ be two wedge-shaped regions in $S^{cs}(\mu)$ that contain $L^c(\mu)$ in their interior and which are bounded by non-singular leaves of type $F^{cu}_2(\sigma_k(\mu))$. We also require that all singular leaves of $F^{cu}(\sigma_k(\mu))$ are contained in the interior of $\Delta_1(\mu)$. Each $\Delta_i(\mu)$, $i = 1, 2$, is the image by the Poincaré-map $A_k(\mu) \to S(\mu)$ of a solid cylinder transversal to $A^s_k(\mu)$. Inside the subset $(\pi^{vs}_\mu)^{-1}(R_\mu) \cap \Delta_1(\mu)$, R_μ as in (1.B), we preserve the two complementary foliation F^{vs}_μ and $F^{cu}_2(\sigma_k(\mu)) \cap S^{cs}(\mu)$. In the complement of this set in $\Delta_1(\mu)$ we preserve the complementary foliations $F^{cu}_1(\sigma_k(\mu)) \cap S^{cs}(\mu)$ and $F^{ss}(\mu)$. Since the intersection of the boundary of $(\pi^{vs}_\mu)^{-1}(R_\mu)$ with each leaf of $F^{cu}_2(\sigma_k(\mu)) \cap S^{cs}(\mu)$ is a leaf of type $F^{cu}_1(\sigma_k(\mu))$ and F^{vs}_μ is a codimension-one foliation in F^{ss}_μ, we obtain a homeomorphism on $\Delta_1(\mu)$. In the complement of $\Delta_2(\mu)$ in $S^{cs}(\mu)$ we proceed with the cone-like construction of Theorem A to define a homeomorphism preserving F^{ss}_μ and $F^{cu}(\sigma_k(\mu))$. The region $\Delta_2(\mu)\setminus\Delta_1(\mu)$ is now used to match these homeomorphisms. Notice that each non-singular leaf of type $F^{cu}_2(\sigma_k(\mu))$ is parametrized by a point in the sphere

$$A^{ss}_k(\mu) = A_k(\mu) \cap W^{ss}(\sigma_k(\mu)).$$

We assume that the boundaries of $\Delta_1(\mu)$ and $\Delta_2(\mu)$ correspond to codimension-one spheres S^μ_1 and S^μ_2 in $A^{ss}_k(\mu)$ centered at the point of tangency $r_{k-1}(0)$. Hence, the matching is done as we move radially from S^μ_1 to S^μ_2 preserving F^{vs}_μ in a subset R_x of R_μ. This subset is bounded by the pre-image of two leaves of ρ_μ whose distance gets smaller as we approximate the outer sphere S^μ_2. Finally, when we reach a point in S^μ_2, this region collapses into the unique leaf of ρ_μ which is transversal to the axis $v_2 = 0$. The picture illustrates this process in a section complementary to $W^u(\sigma_k(\mu))$ in $S^{cs}(\mu)$. It shows how the region R_μ, foliated by leaves of ρ_μ, shrinks to a curve.

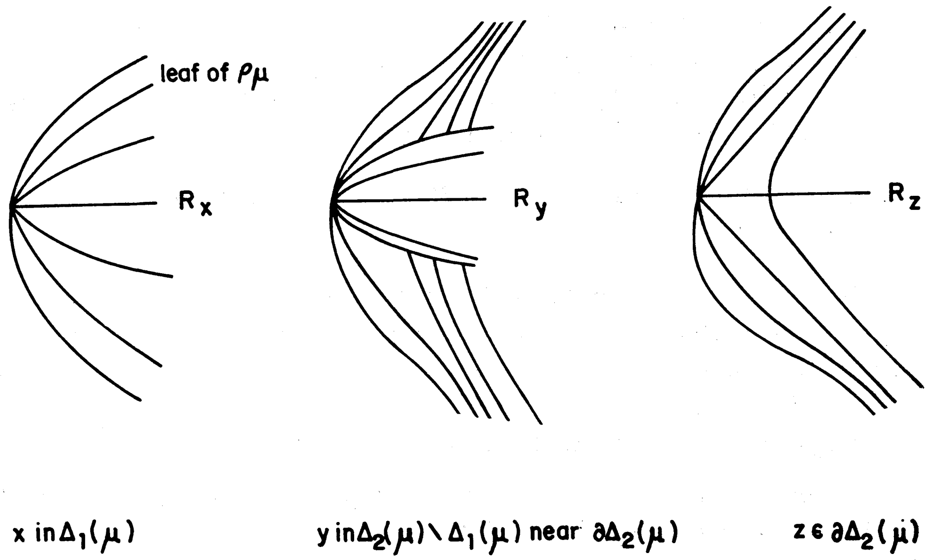

Fig. V

In this way we have obtained a continuous family of homeomorphisms

$$h_\mu^{cs} : S^{cs}(\mu) \to S^{cs}(\varphi(\mu)).$$

As pointed out above, this is enough to get a homeomorphism on the neighbourhood $S(\mu)$ of the tangent point $r_{k+1}(\mu)$ in the fence $B_k(\mu)$. We can now, using the methods in Theorem A, obtain an equivalence between X_μ and $\hat{X}_{\varphi(\mu)}$ on the neighbourhood $U(\mu) = U_{k-1}(\mu) \cup U_{k+1}(\mu)$ of the orbits of tangency γ' and γ preserving level sets of the potential f_μ: this is possible because we have preserved the center-unstable foliation $F^{cu}(\sigma_{k-1}(\mu))$ throughout the process. The extension of this equivalence to all of M is done as in Theorem A: outside the distinguished neighbourhood of the orbits of tangency we obtain a conjugacy between X_μ and $\hat{X}_{\varphi(\mu)}$.

§ 2. Bifurcations of type II: two orbits of quasi-transversality

(2.A) *Description of the bifurcation.* — This is a codimension-two bifurcation presented by families $\{ X_\mu \}$ in $\chi_2^g(M)$ such that, for $\bar{\mu} \in \mathbf{R}^2$, the vector field $X_{\bar{\mu}}$ presents exactly two orbits γ_1 and γ_2 of quasi-transversal intersection between stable and unstable manifolds of hyperbolic singularities:

$$\gamma_1 \subset W^u(p_1(\bar{\mu})) \cap W^s(q_1(\bar{\mu})) \quad \text{and} \quad \gamma_2 \subset W^u(p_2(\bar{\mu})) \cap W^s(q_2(\bar{\mu})).$$

In addition, we assume the following conditions:

(2.1) C^m linearizability of X_μ near each of these singularities with the eigenvalues of $dX_{\bar{\mu}}$ at these points having multiplicity one, m being sufficiently large as specified in Section I,

(2.2) Non-criticality of any other singularity with respect to the strong-stable or the strong-unstable manifolds: if $p \in M$ is a singularity of $X_{\bar{\mu}}$ different from $p_1(\bar{\mu})$, $p_2(\bar{\mu})$, $q_1(\bar{\mu})$ and $q_2(\bar{\mu})$, then $W^u(p)$ is transversal to $W^{ss}(p_1(\bar{\mu}))$ and to $W^{ss}(p_2(\bar{\mu}))$ and $W^s(p)$ is transversal to $W^{uu}(q_1(\bar{\mu}))$ and to $W^{uu}(q_2(\bar{\mu}))$,

(2.3) $W^{cu}(p_i(\bar{\mu}))$ is transversal to $W^s(q_i(\bar{\mu}))$ and $W^{cs}(q_i(\bar{\mu}))$ is transversal to $W^u(p_i(\bar{\mu}))$ for $i = 1, 2$,

(2.4) Generic and independent unfolding of the orbits of tangency of the family X_μ, so that there exist two C^1 curves Γ^1 and Γ^2 in the parameter space crossing each other transversally at the point $\bar{\mu}$ such that $\mu \in \Gamma^i$ if and only if $W^u(p_i(\mu))$ is not transversal to $W^s(q_i(\mu))$, for $i = 1, 2$.

We distinguish two possibilities, (II.a) and (II.b), that will be treated separately:

a) two of the above singularities coincide, namely $q_1(\bar{\mu}) = p_2(\bar{\mu})$ or $q_2(\bar{\mu}) = p_1(\bar{\mu})$ (which are dual) or the easier case $p_1(\bar{\mu}) = p_2(\bar{\mu})$,

b) all singularities above are distinct.

(2.B) *The bifurcation diagram of type* (II.a). — Let us first assume that $p_1(\bar{\mu}) = \sigma_{k-1}(\bar{\mu})$, $q_1(\bar{\mu}) = p_2(\bar{\mu}) = \sigma_k(\bar{\mu})$ and $q_2(\bar{\mu}) = \sigma_{k+1}(\bar{\mu})$ in the ordering $\sigma_1(\mu) \leqslant \ldots \leqslant \sigma_k(\mu)$ of the singularities of X_μ described in section I. We begin by analyzing the restriction X^c_μ of X_μ to the center manifold

$$W^c(\mu) = W^{cu}(\sigma_{k-1}(\mu)) \cap W^{cs}(\sigma_{k+1}(\mu))$$

near $\sigma_k(\mu)$. Let (x_1, x_I, y_1, y_L) be C^m linearizing coordinates for X^c_μ near $\sigma_k(\mu)$:

$$X^c_\mu = -\alpha_1(\mu)\, x_1 \frac{\partial}{\partial x_1} - \sum_{i=2} \alpha_i(\mu)\, x_i \frac{\partial}{\partial x_i} + \beta_1(\mu)\, y_1 \frac{\partial}{\partial y_1} + \sum_{j=2} \beta_j(\mu)\, y_j \frac{\partial}{\partial y_j}$$

with
$$0 < \alpha_1(\mu) < \ldots < \alpha_s(\mu), \qquad 0 < \beta_1(\mu) < \ldots < \beta_u(\mu),$$
$$x_I = (x_1, \ldots, x_s), \qquad y_L = (y_1, \ldots, y_u),$$
$$u = \dim[W^u(\sigma_k(\mu)) \cap W^{cs}(\sigma_{k+1}(\mu))]$$
and
$$s = \dim[W^s(\sigma_k(\mu)) \cap W^{cu}(\sigma_{k-1}(\mu))].$$

By the quasi-transversality assumption in a *cross section* $\Sigma^c(\mu) \subset \{ x_1 = 1 \}$, we have $W^u(\sigma_{k-1}(\mu)) \cap \Sigma^c(\mu) = \{(x_I, G(\mu, x_I, y_L), y_L)\}$, with G being a C^m function such that $x_I \to G(\bar{\mu}, x_I, 0)$ has a non-degenerate critical point at 0. Hence, we get from the generic unfolding of the orbit γ_1 that the map $\mu \mapsto G(\mu, x_I(\mu), 0)$ is a submersion at $\bar{\mu}$, where $x_I(\mu)$ is the solution of $\dfrac{\partial G}{\partial x_I}(\mu, x_I, 0) = 0$.

Also, by taking coordinates (v_1, v_I, w_L) in a *cross-section* $S^c(\mu) \subset \{y_1 = 1\}$ such that $W^u(\sigma_k(\mu)) \cap S^c(\mu) = \{(0, 0, w_L)\}$, we have

$$W^s(\sigma_{k+1}(\mu)) \cap S^c(\mu) = \{(F(\mu, v_I, w_L), v_I, w_L)\},$$

where F is a C^m function such that $w_L \mapsto F(\bar{\mu}, 0, w_L)$ has a non-degenerate critical point at 0. Hence, the conditions of generic and independent unfolding imply that the map

$$(\mu, x_I, w_L) \mapsto \left(G(\mu, x_I, 0), \frac{\partial G}{\partial x_I}(\mu, x_I, 0), F(\mu, x_I, 0), \frac{\partial F}{\partial w_L}(\mu, 0, w_L)\right)$$

is a local diffeomorphism at the point $(\bar{\mu}, 0, 0)$. Therefore, if $\mu \mapsto w_L(\mu)$ is the solution of $\dfrac{\partial F}{\partial w_L}(\mu, 0, w_L) = 0$, then the curves

$$\Gamma_1 = \{ G(\mu, x_I(\mu), 0) = 0 \} \quad \text{and} \quad \Gamma_2 = \{ F(\mu, 0, w_L(\mu)) = 0 \}$$

belong to the bifurcation diagram near $\bar{\mu}$. Also, $\mu \in \Gamma_1$ if and only if $W^u(\sigma_{k-1}(\mu))$ is not transversal to $W^s(\sigma_k(\mu))$ and $\mu \in \Gamma_2$ if and only if $W^u(\sigma_k(\mu))$ is not transversal to $W^s(\sigma_{k+1}(\mu))$. Furthermore, the intersection of $W^u(\sigma_{k-1}(\mu))$ with $S^c(\mu)$ is described by the equations $v_1 = e^{-\alpha_1(\mu)t}$, $v_I = e^{-\alpha_I(\mu)t} x_I$, $e^{-\beta_1(\mu)t} - G(\mu, x_I, e^{-\beta_L(\mu)t} w_L) = 0$. Using that the bifurcation unfolds generically and the implicit function theorem, we obtain a third C^1 curve Γ_3 in the parameter space tending to $\bar{\mu}$ (but disjoint from Γ_1

and Γ_2 outside this point), such that $\mu \in \Gamma_3 - \{\bar{\mu}\}$ if and only if $W^u(\sigma_{k-1}(\mu))$ is not transversal to $W^s(\sigma_{k+1}(\mu))$. Along the curve Γ_3 the family X_μ presents one orbit of quasi-transversality between $W^u(\sigma_{k-1}(\mu))$ and $W^s(\sigma_{k+1}(\mu))$.

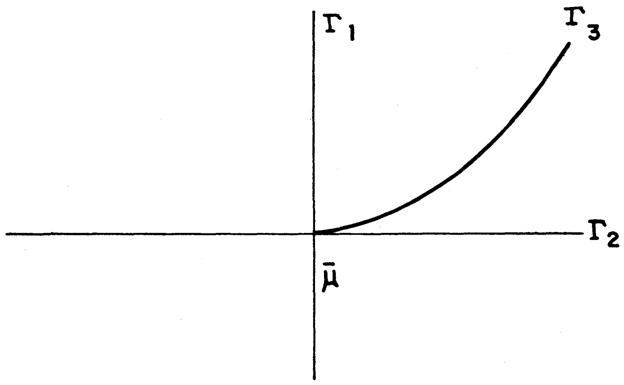

FIG. VI

(2.C) *The stability of the bifurcation of type* (II.a). — As in Theorem A, we focus our attention to a neighbourhood $U(\mu)$ of the closure of the orbits of tangency γ_1 and γ_2 in M which is constructed by glueing together *distinguished neighbourhoods* of these orbits. We construct in $U(\mu)$ flow equivalences that preserve compatible systems of foliations, so that they can be extended to flow equivalences on all of M.

Suppose we have already constructed a compatible unstable system $F^u(\sigma_1(\mu)), \ldots, F^u(\sigma_{k-2}(\mu)), F^{cu}(\sigma_{k-1}(\mu))$ and a compatible stable system $F^{cs}(\sigma_{k+1}(\mu)), \ldots, F^s(\sigma_{k+2}(\mu)), \ldots, F^s(\sigma_k(\mu))$, together with a homeomorphism in the space of leaves of these foliations. We start by constructing a center-unstable foliations $F^{cu}(\sigma_k(\mu))$ compatible with the unstable system whose main leaf, $W^{cu}(\sigma_k(\mu))$, is a C^1 invariant manifold contained in $W^{cu}(\sigma_{k-1}(\mu))$ and transversal to $W^u(\sigma_{k-1}(\mu))$ and which contains all possible tangencies between $W^u(\sigma_{k-1}(\mu))$ and $W^s(\sigma_{k+1}(\mu))$. This construction resembles very much the one done in § 1 for the orbit of tangency with criticality. In the cross section $\Sigma^c(\mu) \subset \{x_1 = 1\}$ consider coordinates (x_I, y_1, y_L) centered at $r_k(0) = \gamma_1 \cap \Sigma^c(0)$, as in (2.B) above. Let the vector field Z_μ, tangent to $W^u(\sigma_{k-1}(\mu))$, be defined by

$$\dot{y}_1 = (y_1 - G(\mu, x_I, y_1^{\beta_L/\beta_1} y_L))^2 + \Sigma \frac{\partial G}{\partial x_i}(\mu, x_I, y_1^{\beta_L/\beta_1} y_L)\, \dot{x}_i,$$

$$\dot{x}_i = \frac{\partial G}{\partial x_i}(\mu, x_I, y_1^{\beta_L/\beta_1} y_L) - \left(\frac{-\beta_1 y_1 + \dfrac{\partial G}{\partial y_L} \cdot y_1^{\beta_L/\beta_1} \beta_L \cdot y_L}{\alpha_1 - \dfrac{\partial F}{\partial v_I}(y_1^{\alpha_I/\beta_1} x_I, y_L)\, y_1^{(\alpha_I - \alpha_1)/\beta_1} \alpha_I x_I} \right) \cdot \frac{\partial F}{\partial v_i} \cdot y_1^{(\alpha_i - \alpha_1)/\beta_1},$$

$$\dot{y}_L = 0,$$

$$\dot{\mu} = 0,$$

$i = 1, \ldots, s$, for $y_1 > 0$. Since $\beta_j(\mu) > \beta_1(\mu)$ and $\alpha_i(\mu) > \alpha_1(\mu)$ for $i \geqslant 2$ and $j \geqslant 2$, this extends to a C^1 vector field in $\Sigma^c(\mu)$ which has for each (μ, y_L) a singularity of saddle-node type. The singular set of Z_μ, $\mathrm{Sing}(Z_\mu)$, is a submanifold of dimension u of $W^u(\sigma_{k-1}(\mu)) \cap \Sigma^c(\mu)$ which is topologically transversal to $W^s(\sigma_k(\mu)) \cap \Sigma^c(\mu)$. Its image by the Poincaré-map $P^c_\mu : \Sigma^c(\mu) \to S^c(\mu)$ contains the tangencies between $W^u(\sigma_{k-1}(\mu))$ and $W^s(\sigma_{k+1}(\mu))$ in $S^c(\mu)$, a cross-section in $\{y_1 = 1\}$. We consider a foliation $\widetilde{F}^u_k(\mu)$ in $\Sigma^c(\mu)$ which is tangent to the vector field Z_μ and singular along $\mathrm{Sing}(Z_\mu)$ having C^1 leaves of dimension $(u+1)$. We distinguish a leaf $M^u_k(\mu)$ which is transversal to $W^u(\sigma_{k-1}(\mu))$ and such that $M^u_k(\mu) \cap W^u(\sigma_{k-1}(\mu)) = \mathrm{Sing}(Z_\mu)$. Let $F^u_k(\mu)$ be a u_k-dimensional ($u_k = \dim W^u(\sigma_k(\mu))$ foliation in $W^{cu}(\sigma_{k-1}(\mu)) \cap \Sigma(\mu)$ which is compatible with $W^u(\sigma_{k-1}(\mu)) \cap \Sigma(\mu)$ and such that $F^u_k(\mu) \cap \Sigma^c(\mu) = \widetilde{F}^u_k(\mu)$. Positive saturation of $F^u_k(\mu)$ gives part of the center-unstable foliation $F^{cu}(\sigma_k(\mu))$ inside $W^{cu}(\sigma_{k-1}(\mu))$, which has a distinguished leaf denoted by $W^{cu}_k(\mu)$. In the next figure we see these leaves in a slice complementary to $W^u(\sigma_k(\mu))$. The construction of the other leaves of $F^{cu}(\sigma_k(\mu))$ corresponding to points near the singular set $\mathrm{Sing}(Z_\mu)$ follows as in § 1 of the present section. Dually, we obtain an s_k-dimensional singular foliation $F^s_k(\mu)$, ($s_k = \dim W^s(\sigma_k(\mu))$) in the level set $B_k(\mu) = [f_\mu^{-1}[f_\mu(\sigma_k(\mu)) + \varepsilon]]$ which is compatible with the stable-system $F^{cs}(\sigma_{k+1}(\mu))$, $F^s(\sigma_{k+2}(\mu))$, $\ldots$, $F^s(\sigma_\ell(\mu))$. We denote by $M^s_k(\mu)$ the distinguished leaf of $F^s_k(\mu)$ that contains the point $p_{k+1}(\mu) = \gamma_2(\mu) \cap B_k(\mu)$.

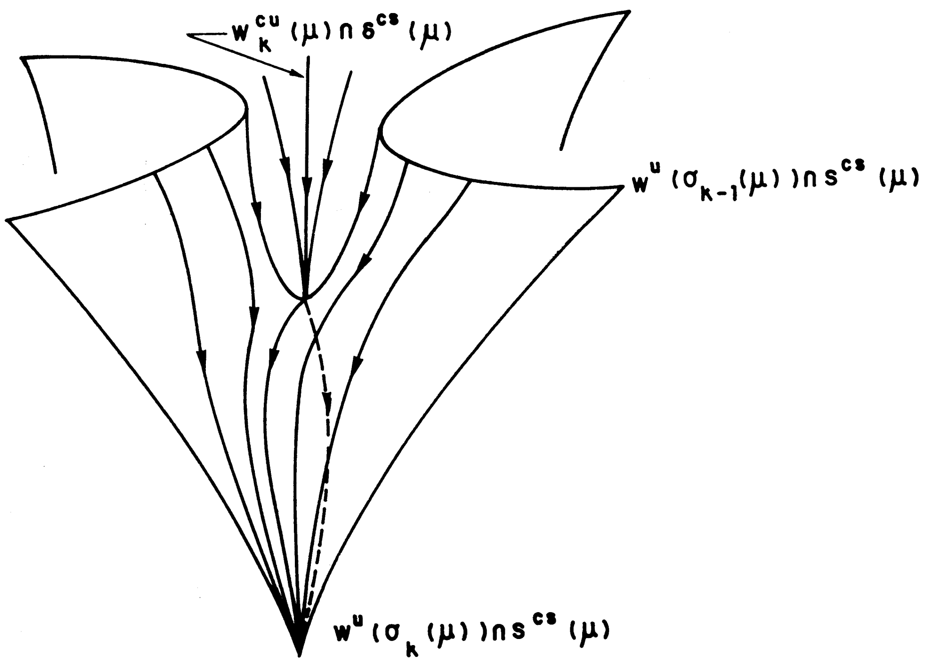

Fig. VII

Let $\lambda^c(\mu)$ be the C^1 curve defined by $\lambda^c(\mu) = W^{cu}_k(\mu) \cap M^s_k(\mu) \cap S^c(\mu)$ and consider the points $p_{k-1}(\mu) = W^u(\sigma_{k-1}(\mu)) \cap \lambda^c(\mu)$, $W^u(\sigma_k(\mu)) \cap \lambda^c(\mu) = p_k(\mu)$ $(= \{0\})$ and $p_{k+1}(\mu) = W^s(\sigma_{k+1}(\mu)) \cap \lambda^c(\mu)$. Notice that $p_{k-1}(\mu)$ is only defined for μ on a connected component of $V - \Gamma_1$, where V is a neighbourhood of $\bar{\mu}$ in $\mathbf{R}^2$. The curves Γ_2 and Γ_3 of the bifurcation set correspond to $\{p_{k+1}(\mu) = p_k(\mu)\}$ and $\{p_{k+1}(\mu) = p_{k-1}(\mu)\}$ respectively.

We start the construction of an equivalence between X_μ *and a nearby family* $\hat{X}_\mu$. Consider a reparametrization $\varphi : (V, \bar{\mu}) \to (\mathbf{R}^2, \hat{\mu})$, such that $\varphi(\Gamma_i) = \hat{\Gamma}_i$ for $i = 1, 2, 3$ and which sends the regions A_{ij} between the curves onto corresponding regions $\hat{A}_{ij}$ as in the picture (Fig. VIII).

To obtain an equivalence between X_μ and $\hat{X}_{\varphi(\mu)}$, we first want to define a homeomorphism on the level set $B_k(\mu)$ which preserves the foliations $F_k^s(\mu)$ and $F_k^{cu}(\mu) \cap B_k(\mu)$. The main step is the construction of a homeomorphism on $S(\mu)$, a neighbourhood of $p_{k+1}(\mu)$ in $B_k(\mu)$. Let $M^{cu}(\mu) = W_k^{cu}(\mu) \cap W^{cs}(\sigma_{k+1}(\mu)) \cap S(\mu)$ and let (v_1, w_L) be a system of coordinates for $M^{cu}(\mu)$ such that

$$\{ v_1 - \widetilde{F}(\mu, w_L) = 0 \} = W^s(\sigma_{k+1}(\mu)) \cap M^{cu}(\mu),$$

$$\{ v_1 = 0 \} = W^u(\sigma_k(\mu)) \cap M^{cu}(\mu)$$

and

$$W^u(\sigma_{k-1}(\mu)) \cap M^{cu}(\mu) = \{ v_1^{\beta_1/\alpha_1} - \widetilde{G}(\mu, v_1^{\beta_L/\alpha_1}.w_L) = 0, v_1 > 0 \},$$

where $\widetilde{F}$, $\widetilde{G}$ are of class C^2. Hence, by construction the foliation $\widetilde{F}_k^s(\mu) \cap M^{cu}(\mu)$ is tangent to the vector field Y_μ defined by

$$\dot{v}_1 = (v_1 - \widetilde{F}(\mu, w_L))^2 + \frac{\partial \widetilde{F}}{\partial w_L}(\mu, w_L).\dot{w}_L,$$

$$\dot{w}_L = \frac{\partial \widetilde{F}}{\partial w_L}(\mu, w_L) - \left[\frac{\alpha_1 v_1}{-\beta_1 v_1^{\beta_1/\alpha_1} + \dfrac{\partial \widetilde{G}}{\partial y_L}(\mu, v_1^{\beta_L/\alpha_1}.w_L).\beta_L, w_L}\right] \frac{\partial \widetilde{G}}{\partial y_L}(\mu, v_1^{\beta_L/\alpha_1})\, v_1^{\beta_L/\alpha_1}$$

for $v_1 > 0$. We extend it to $v_1 \leqslant 0$ by setting

$$\dot{v}_1 = (v_1 - \widetilde{F}(\mu, w_L))^2 + \frac{\partial \widetilde{F}}{\partial w}(\mu, w_L)\,\dot{w}_L,$$

$$\dot{w}_L = \frac{\partial \widetilde{F}}{\partial w_L}(\mu, w_L).$$

Let $L^u(\mu, v_1, w_L) = [v_1^{\beta_1/\alpha_1(\mu)} - \widetilde{G}(\mu, v_1^{\beta_L/\alpha_1(\mu)} w_L) + \widetilde{G}(\mu, 0)]^{\alpha_1(\mu)/\beta_1(\mu)}$ be a C^1 submersion defined for $v_1 \geqslant 0$; observe that $(L^u)^{-1}(\widetilde{G}(\mu, 0)^{\alpha_1/\beta_1(\mu)}) = W^u(\sigma_{k-1}(\mu))$ and $(L^u)^{-1}(0) = W^u(\sigma_k(\mu))$. For $v_1 < 0$, we extend it as $L^u(\mu, v_1, w_L) = v_1$. It is easy to check that $-L^u(\mu, v_1, w_L)$ is a Liapunov function for the vector field Y_μ. We apply Lemma 1 of § 1 to get a homeomorphism $M^{cu}(\mu) \to \widehat{M}^{cu}(\varphi(\mu))$ which is a topological equivalence between Y_μ and $\hat{Y}_{\varphi(\mu)}$ preserving the level sets of the respective functions L_μ^u and $\hat{L}_{\varphi(\mu)}^u$. The same procedure is used in order to get a homeomorphism on the cross section $M^{cs}(\mu) = W_k^{cs}(\mu) \cap W^{cu}(\sigma_{k-1}(\mu)) \cap \Sigma(\mu)$, where $\Sigma(\mu)$ is a neighbourhood of $r_k(\mu)$ in $A_k(\mu)$.

Now, to complete the definition of the homeomorphism on the cross-section

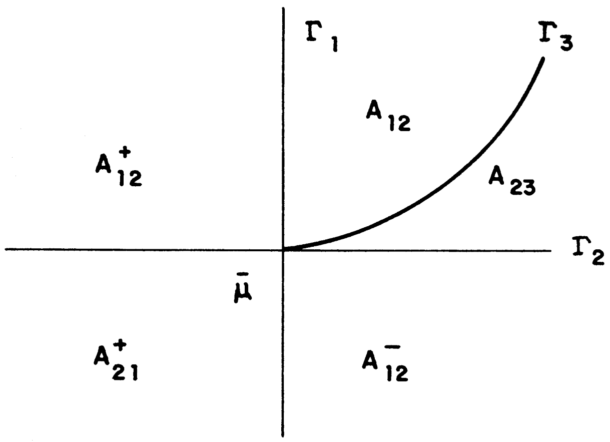

FIG. VIII

$S(\mu) \cap W^u(\sigma_k(\mu))$, which corresponds to part of the space of leaves of the foliation $F^s_k(\mu)$, we take a C^1 foliation on $W^u(\sigma_k(\mu)) \cap S(\mu)$ transversal to

$$W^{cs}(\sigma_{k+1}(\mu)) \cap W^u(\sigma_k(\mu)) \cap S(\mu)$$

and of complementary dimension. We make the same construction for

$$W^u(\hat{\sigma}_k(\varphi(\mu))) \cap S(\varphi(\mu))$$

and obtain the extension of the homeomorphism by requiring that this foliation be preserved and also the intersections of the center-stable foliation

$$F^{cs}(\sigma_{k+1}(\mu)) \cap W^u(\sigma_k(\mu)) \cap S(\mu).$$

This homeomorphism is further extended to the whole sphere (fundamental domain) $W^u(\sigma_k(\mu)) \cap B_k(\mu)$ by preserving the stable system $F^s(\sigma_{k+2}(\mu)), \ldots, F^s(\sigma_\ell(\mu))$ in a compatible way with the homeomorphisms on the space of leaves of these foliations. Similarly we obtain a homeomorphism on the space of leaves of the center-unstable foliation $F^{cu}(\sigma_k(\mu))$, which is compatible with the unstable system $F^u(\sigma_1(\mu)), \ldots, F^{cu}(\sigma_{k-1}(\mu))$.

The homeomorphism on the level set $B_k(\mu)$ is then well-defined since we want it to preserve the complementary foliations $F^s_k(\mu)$ and $F^{cu}_k(\mu) \cap B_k(\mu)$. As we saw in Theorem A, Section II, this is enough to obtain the equivalence in the neighbourhood of the singularity $\sigma_k(\mu)$. The arguments to define the equivalence in $B_{k-1}(\mu)$ and, afterwards, its extension to all of M are now very similar to those in Theorem A. (The corresponding facts in case (II.b) are somewhat more delicate and will be treated in more details in the sequel.) ■

(2.D) *The case of two orbits of quasi-transversality corresponding to disjoint pairs of singularities (type* II.b*).* — This case goes much in parallel with the previous one: the main difference consists in a more careful construction of an equivalence for the two nearby families. This is due to the existence of intermediate singularities between the ones corresponding to the orbits of tangency.

Let X_μ be one of these families and let us order the singularities at the bifurcation point $\bar{\mu}$, so that γ_1 is an orbit of quasi-transversality between $W^u(\sigma_i(\bar{\mu}))$ and $W^s(\sigma_{i+1}(\bar{\mu}))$ and γ_2 is a similar orbit between $W^u(\sigma_{i+k}(\bar{\mu}))$ and $W^s(\sigma_{i+k+1}(\bar{\mu}))$ for some $k \geqslant 2$. We first observe that the construction for μ near $\bar{\mu}$ of compatible unstable and stable systems, which are now denoted by $F_1^u(\mu), \ldots, F_i^{cu}(\mu), F_{i+1}^{cu}(\mu), F_{i+2}^u(\mu), \ldots, F_{i+k}^{cu}(\mu)$ and $F_{i+1}^{cs}(\mu), F_{i+2}^s(\mu), \ldots, F_{i+k}^{cs}(\mu), F_{i+k+1}^{cs}(\mu), \ldots, F_\ell^s(\mu)$, respectively, is very similar to the previous case. The difference is that now we have to construct $F_{i+j}^u(\mu)$ for $2 \leqslant j \leqslant k-1$. To do this, we just note that although like before $F_{i+1}^{cu}(\mu)$ is a singular foliation, its singular set, $\mathrm{Sing}(F_{i+1}^{cu}(\mu))$, is the union of two manifolds which are transversal to $W^s(\sigma_{i+j}(\mu))$ for $j \geqslant 2$. Moreover, since each leaf of $F_{i+1}^{cu}(\mu)$ accumulates in a C^1 fashion on $\mathrm{Sing}(F_{i+1}^{cu}(\mu))$, any foliation on this singular set can be extend to the leaves of $F_{i+1}^{cu}(\mu)$ in a continuous way. Thus, when constructing $F_{i+2}^u(\mu)$, in a compatible way with $F_{i+1}^{cu}(\mu)$, it is enough to do so in $\mathrm{Sing}(F_{i+1}^{cu}(\mu))$, and then extend it to each leaf of $F_{i+1}^{cu}(\mu)$. The same reasoning applies to $F_{i+j}^u(\mu)$, $2 \leqslant j \leqslant i+k-1$ and to $F_{i+k}^{cu}(\mu)$.

In the construction of the above foliations, we can also require $W^u(\sigma_i(\mu))$ to be foliated by leaves of $F_{i+k}^{cu}(\mu)$. In particular, since $W^s(\sigma_{i+k+1}(\bar{\mu}))$ is transversal to $W^{cu}(\sigma_{i+k}(\bar{\mu}))$, we conclude that $W^s(\sigma_{i+k+1}(\mu))$ is transversal to $W^u(\sigma_i(\mu))$, μ near $\bar{\mu}$. Thus, the bifurcation set of X_μ near $\bar{\mu}$ consists exactly of two C^1 curves Γ_1 and Γ_2 that intersect transversally at $\bar{\mu}$: $\mu \in \Gamma_1$ if and only if $W^u(\sigma_i(\mu))$ is quasi-transversal to $W^s(\sigma_{i+1}(\mu))$ and $\mu \in \Gamma_2$ if and only if $W^u(\sigma_{i+k}(\mu))$ is quasi-transversal to $W^s(\sigma_{i+k+1}(\mu))$.

(2.E) *Local stability.* — Let X_μ be a family of type (II.b) and let $\hat{X}_\mu$ be a nearby family with main bifurcation value $\hat{\mu}$ near $\bar{\mu}$. Let $(F, G) : (V, \bar{\mu}) \to (\mathbf{R}^2, 0)$ be a C^1 map defined in a neighbourhood U of $\bar{\mu}$ in $\mathbf{R}^2$ such that $F^{-1}(0) = \{ \mu \in V; W^u(\sigma_{i+k}(\mu))$ is quasi-transversal to $W^s(\sigma_{i+k+1}(\mu)\}$ and $G^{-1} = \{ \mu \in V; W^u(\sigma_i(\mu))$ is quasi-transversal to $W^s(\sigma_{i+1}(\mu)\}$. By the hypothesis of independent unfolding, (F, G) is a local diffeomorphism. Therefore, if $(\hat{F}, \hat{G})$ is the corresponding map associated with the family $\{ \hat{X}_\mu \}$, we can define the reparametrization $\varphi = (\hat{F}, \hat{G})^{-1} \circ (F, G)$.

To prove that X_μ is equivalent to $\hat{X}_{\varphi(\mu)}$ we take two distinguished neighbourhoods $U_i(\mu)$ and $U_{i+k}(\mu)$ of the closure of the two orbits of tangency, γ_1 and γ_2, as constructed in Theorem A. *Inside these neighbourhoods the equivalence h_μ will preserve the level sets of the potential function f_μ and outside them it will be a conjugacy.* The idea of the proof is to first define a continuous family of homeomorphisms on the space of leaves of the unstable system from $F_1^u(\mu)$ up to $F_{i+k}^{cu}(\mu)$. The important point here is to preserve the leaves of the stable system which are contained in the stable manifolds. Dually we define a family of homeomorphisms on the space of leaves of the stable system from $F_\ell^s(\mu)$ to $F_{i+k}^{cs}(\mu)$. We then obtain a homeomorphism on the fence $B_{i+k}(\mu) \subset f_\mu^{-1}(f_\mu(\sigma_{i+k}(\mu)) + \varepsilon)$, preserving $F_{i+k}^{cu}(\mu) \cap B_{i+k}(\mu)$ and the stable system. At this point we obtain an equivalence on a full neighbourhood of the singularity σ_{i+k}: we use the cone-like construction in Theorem A, Section II, and preserve the level sets of the potential f_μ. The equivalence is extended to the distinguished neighbourhood U_{i+k} by preserving level sets of the potential and repeating the cone-like construction near σ_{i+k+1}. With this we define homeomorphisms on the space of leaves of $F^{cs}_{i+1}(\mu)$ and complete the definition on the fence $B_{i+1}(\mu) \subset f_\mu^{-1}(f_\mu(\sigma_{i+1}(\mu)) + \varepsilon)$ also preserving $F^{cu}_{i+1}(\mu)$. Since the foliation $F^{cu}_i(\mu)$ is preserved in this process, the equivalence can be extended to the second distinguished neighbourhood $U_i(\mu)$ again by the methods explained in Theorem A.

Let us give more detail on this construction. We assume that we already have homeomorphisms on the space of leaves of $F^u_1(\mu), \ldots, F^u_{i-1}(\mu)$ and $F^{cu}_i(\mu)$ as well as on the space of leaves of $F^{cs}_{i+k+1}(\mu), \ldots, F^s_\ell(\mu)$. The homeomorphism on the space of leaves of the foliation $F^{cu}_{i+1}(\mu)$ is obtained as in (II.a) using Lemma 1. Next, we obtain homeomorphisms $h^s_{i+j,\mu}$ on the space of leaves of the foliation $F^u_{i+j}(\mu)$ for $j = 2, \ldots, k-1$, and, also, of $F^{cu}_{i+k}(\mu)$. We will perform the construction for $j = 2$, since the general case can be done by induction in a similar way.

Construction of $h^s_{i+2,\mu}$. — Let us suppose that $W^u(\sigma_{i+1}(\mu)) \cap W^s(\sigma_{i+2}(\mu)) \neq \emptyset$. We denote by $F^{cs}_{i+2}(\sigma_{i+1}(\mu))$ the set of leaves of $F^{cs}(\sigma_{i+1}(\mu))$ which are contained in $W^s(\sigma_{i+2}(\mu))$. We recall that $A_{i+1}(\mu)$ and $B_{i+1}(\mu)$ are two small fences contained in the non-critical levels $f_\mu^{-1}(f_\mu(\sigma_{i+1}(\mu)) \pm \varepsilon)$ for $\varepsilon > 0$ small, respectively. We are going to define a homeomorphism on $B_{i+1}(\mu) \cap W^s(\sigma_{i+2}(\mu))$ which preserves $F^{cs}_{i+2}(\sigma_{i+1}(\mu))$. So, we first construct a homeomorphism on the space of leaves of this foliation: this is done leaf by leaf using the Isotopy Extension Theorem, as in the previous cases. Since we already have a homeomorphism on the space of leaves of the foliation $F^{cu}_{i+1}(\mu)$, we obtain a homeomorphism on $A_{i+1}(\mu) \cap W^s(\sigma_{i+2}(\mu))$ which preserves $F^{cs}_{i+2}(\sigma_{i+1}(\mu))$, $F^{cu}_{i+1}(\mu)$ and a complementary foliation F^{su}_μ: this is exactly like in the proof of Theorem A when we restrict ourselves to $W^s(\sigma_{i+2}(\mu))$. Therefore, through the Poincaré map $P_{i+1,\mu} : A_{i+1}(\mu) \backslash W^s(\sigma_{i+1}(\mu)) \to B_{i+1}(\mu) \backslash W^u(\sigma_{i+1}(\mu))$, we get the required homeomorphism on $W^s(\sigma_{i+2}(\mu)) \cap B_{i+1}(\mu)$. Let $D^s(i+2, \mu)$ be a fundamental domain for $W^s(\sigma_{i+2}(\mu))$ which is contained in the non-critical level set $f_\mu^{-1}(f_\mu(\sigma_{i+1}(\mu)) + \varepsilon)$. Using the Isotopy Extension Theorem and the compatibility of the homeomorphisms on the space of leaves of the foliations $F^u_1(\mu), \ldots, F^{cu}_i(\mu)$, we obtain the extension of the homeomorphism $W^s(\sigma_{i+2}(\mu)) \cap B_{i+1}(\mu) \to W^s(\hat{\sigma}_{i+2}(\varphi(\mu))) \cap \hat{B}_{i+1}(\varphi(\mu))$ to $D^s(i+2, \mu)$, finishing the construction of $h_{i+2,\mu}$. ■

As mentioned before the construction of the other homeomorphisms $h^s_{i+j,\mu}$, for $3 \leqslant j \leqslant k$, is analogous to the one described above: we proceed by induction, using the leaves of the stable system $F^{cs}_{i+1}(\mu), F^s_{i+2}(\mu), \ldots, F^s_{i+j-1}(\mu)$ which are contained in $W^s(\sigma_{i+j}(\mu))$.

We are now prepared to define an equivalence on the distinguished neighbourhood $U_{i+k}(\mu)$ of the orbit of tangency γ_2. To do that we again apply Lemma 1 to obtain a homeomorphism on the space of leaves of $F^{cs}_{i+k}(\mu)$. The construction is dual to the one used to obtain a homeomorphism on the space of leaves of $F^{cu}_{i+1}(\mu)$. This homeomorphism, together with the homeomorphism $h^s_{i+k,\mu}$ constructed above on the space of leaves of $F^{cu}_{i+k}(\mu)$, yields the definition of an equivalence on the neighbourdhood $U_{i+k}(\mu)$ according to the methods in Theorem A. We conclude our arguments with the construction of an equivalence in the distinguished neighbourhood $U_i(\mu)$. We have already defined a homeomorphism on the set $\bigcup_{2\leqslant j\leqslant k+1} W^s(\sigma_{i+j}(\mu)) \cap B_{i+1}(\mu)$ which preserves the foliation $F^{cs}_{i+1}(\mu)$. We can then extend this homeomorphism to the remaining part of the space of leaves of $F^{cs}_{i+1}(\mu)$ corresponding to the leaves contained in $W^s_{i+k+2}(\mu), \ldots, W^s_l(\mu)$. This extension, which is by now standard, is compatible with the homeomorphisms already defined on the space of leaves of the corresponding stable foliation. With this, since we also have preserved the foliation $F^{cu}_i(\mu)$ throughout the process, we can define the equivalence on the neighbourhood $U_i(\mu)$ again by the methods in Theorem A. To obtain the globalization of the equivalence to all of M, we just choose the non-critical level $F_c = f_{\bar\mu}^{-1}(c)$ where $f_{\bar\mu}(\sigma_{i+k}(\bar\mu)) < c < f_{\bar\mu}(\sigma_{i+k+1}(\bar\mu))$ and proceed as it was done at the end of Theorem A. ∎

§ 3. Bifurcations of type III: saddle-node with criticality

In this paragraph, which is similar to § 1, we treat the case of a saddle-node with criticality. Let X_μ be a family in $\chi^g_2(M)$ such that for a value $\bar\mu \in \mathbf{R}^2$, the vector field $X_{\bar\mu}$ presents a unique nonhyperbolic singularity $p(\bar\mu)$ which is a saddle-node unfolding generically, as defined in Section I. Suppose that there is one hyperbolic singularity $q(\bar\mu)$ such that the unstable manifold of $q(\bar\mu)$ is transversal to the stable manifold of $p(\bar\mu)$, but there is one orbit γ of quasi-transversal intersection between $W^u(q(\bar\mu))$ and $W^{ss}(p(\bar\mu))$, the strong stable manifold of $p(\bar\mu)$. In addition we assume the following conditions to hold for the family X_μ.

(3.A) *Other generic conditions.*

(3.1) The pair $(p(\mu), \gamma(\mu))$ unfolds generically at $\mu = \bar\mu$. This means that, provided that the saddle-node unfolds generically, there is a C^1 curve Γ_{SN} in the parameter space such that $\mu \in \Gamma_{SN}$ if and only if the vector field X_μ exhibits a saddle-node singularity $p(\mu)$, and an orbit of tangency between $W^u(q(\mu))$ and $W^{ss}(p(\mu))$ occurs only for the isolated value $\bar\mu$ in Γ_{SN}. This is equivalent to say that, if $\mu \in \Gamma_{SN} \mapsto \sigma^u(\mu)$ and $\mu \in \Gamma_{SN} \mapsto \sigma^{ss}(\mu)$ are two C^1 curves in M such that $\sigma^u(\mu) \in W^u(q(\mu))$, $\sigma^{ss}(\mu) \in W^{ss}(p(\mu))$ and $\sigma^u(\bar\mu) = \sigma^{ss}(\bar\mu) = r \in \gamma$, then the projection of $\dot\sigma^u(\bar\mu) - \dot\sigma^{ss}(\bar\mu)$ onto $T_r M / T_r W^u(q(\bar\mu)) + T_r W^{ss}(p(\bar\mu))$ is not zero,

(3.2) X_μ is C^m linearizable near $q(\bar\mu)$ and partially linearizable near the saddle-node $p(\bar\mu)$ as described in Section I (c.4), its linear part having distinct eigenvalues at these points and $m \geqslant 3$,

(3.3) $W^{cu}(q(\bar\mu))$ is transversal to $W^{ss}(p(\bar\mu))$,

(3.4) Let $W^{vs}(p(\bar\mu))$ be the invariant manifold of codimension one in $W^{ss}(p(\bar\mu))$ whose tangent space at $p(\bar\mu)$ is complementary to the eigenspace corresponding to the weakest contraction for $X_{\bar\mu} \mid W^{ss}(p(\bar\mu))$. Then, $W^u(q(\bar\mu))$ is transversal to $W^{vs}(p(\bar\mu))$,

(3.5) There are no other criticalities: for any singularity σ different from $p(\bar\mu)$ and $q(\bar\mu)$, $W^u(\sigma)$ is transversal to $W^{ss}(q(\bar\mu))$ and to $W^{ss}(p(\bar\mu))$ and $W^s(\sigma)$ is transversal to $W^{uu}(q(\bar\mu))$ and to $W^{uu}(p(\bar\mu))$. All other invariant manifolds intersect transversally.

(3.B) *The bifurcation set.* — The hypothesis of generic unfolding of the saddle-node implies that there exists a C^1 curve Γ_{SN} near $\bar\mu$ in the parameter space, such that along Γ_{SN} the family X_μ presents a saddle-node bifurcation. Γ_{SN} is the image of the singular set of the restriction of the projection $(x, \mu) \mapsto \mu$ to the manifold $\left\{ \frac{\partial f_\mu}{\partial x}(x) = 0 \right\}$, where (x, μ) are C^∞ coordinates in a neighbourhood of $(p(\bar\mu), \bar\mu)$ in $M \times \mathbf{R}^2$ and f_μ is the potential function associated to the family X_μ.

Let $W^{cs}(p(\mu))$ be a C^3 center-stable manifold. From the linearizing assumptions, we can write

$$X_\mu^{cs} = X_\mu \mid W^{cs}(p(\mu)) = B(\mu, x)\frac{\partial}{\partial x} + \sum_{j=1}^{s} A_{ij}(\mu, x)\, y_i \frac{\partial}{\partial y_j}$$

in a neighbourhood of $p(\mu)$ in M, where $s = \dim W^{ss}(p(0))$ and $(x, y_1, \ldots, y_s)$ are μ-dependent C^m coordinates, $m \geqslant 2$, such that the eigenvalues of the matrix $A(\mu, x) = (A_{ij}(\mu, x))_{s \times s}$ are distinct and negative. The ordering $(y_1, \ldots, y_s)$ corresponds to the ordering $\alpha_1(\mu) < \ldots < \alpha_s(\mu)$ of the absolute values of the eigenvalues of $A(\mu, x)$. The generic unfolding of the saddle-node implies that $B(0,0) = \frac{\partial B}{\partial x}(0,0) = 0$, $\frac{\partial^2 B}{\partial x^2}(0,0) \neq 0$ (say positive) and $\frac{\partial B}{\partial \mu_1}(0,0) \neq 0$. Therefore, there is a diffeomorphism $\varphi(\mu, x) = (\varphi_1(\mu), \varphi_2(\mu, x))$ such that $B \circ \varphi(\mu, x) = x^2 + \mu_1$. Using the change of coordinates $x = \varphi_2(\mu, \bar x)$, $y_j = \bar y_j$, $\bar\mu = \varphi_1(\mu)$, we have

$$\dot{\bar x} = \left[\frac{\partial \varphi_2}{\partial \bar x}(\mu, \bar x)\right]^{-1} . B(\varphi(\mu, \bar x)) = \left[\frac{\partial \varphi_2}{\partial \bar x}(\mu, \bar x)\right]^{-1} (\bar x^2 + \bar\mu_1),$$

$$\dot{\bar y}_j = \dot y_j.$$

Multiplying by the nonvanishing function $\frac{\partial \varphi_2}{\partial \bar x}(\mu, \bar x)$, we obtain a family $\bar X_\mu$ equivalent to X_μ^{cs} near $p(\mu)$ such that $\bar X_\mu = (\bar x^2 + \bar\mu_1)\frac{\partial}{\partial \bar x} + \Sigma \bar A_{ij}(\mu, \bar x)\, \bar y_i \frac{\partial}{\partial \bar y_j}$. From now on we drop the bars to simplify the notation. Let $\Sigma_\mu^{cs} \subset \{y_1 = 1\}$ be a cross section such that

$W^{ss}(p(0, \mu_2)) \cap \Sigma_\mu^{cs} = \{(0, 1, y_2, \ldots, y_s)\}$. Since $W^{cu}(q(0))$ is transversal to $W^{ss}(p(0))$, we may write

$$W^{cu}(q(\mu)) \cap \Sigma_\mu^{cs} = \{(x, y_L, y_K = y_K(\mu, x, y_L))\}$$

and

$$W^{u}(q(\mu)) \cap \Sigma_\mu^{cs} = \{(F(\mu, y_L), y_L, y_K(\mu, F(\mu, y_L), y_L))\},$$

with $y_L = (y_2, \ldots, y_{s_q})$, $y_K = (y_{s_q+1}, \ldots, y_s)$, $1 + s_q + \dim W^{uu}(p(0)) = \dim W^u(q(0))$, $F(0,0) = 0$, $\dfrac{\partial F}{\partial y_L}(0,0) = 0$ and $\left(\dfrac{\partial^2 F}{\partial y_j\, \partial y_i}(0,0)\right)_{2 \leqslant j, i \leqslant s_q}$ nondegenerate (we assume $\dim W^u(q(0)) + \dim W^{ss}(p(0)) \geqslant n + 1$).

For $\mu_1 < 0$, we have two distinguished hyperplanes in Σ_μ^{cs}, namely $x = \pm\sqrt{-\mu_1}$, which correspond to $W^{ss}(p_1(\mu)) \cap \Sigma_\mu^{cs}$ and to $W^s(p_2(\mu)) \cap \Sigma_\mu^{cs}$, where $p_1(\mu)$ and $p_2(\mu)$ are the two hyperbolic singularities that collapse to form the saddle-node. Therefore, $W^u(q(\mu))$ is nontransversal to $W^s(p_2(\mu))$ if and only if $\dfrac{\partial F}{\partial y_L}(\mu, y_L) = 0$ and $\sqrt{-\mu_1} = F(\mu, y_L)$. From the hypothesis of quasi-transversality and the implicit function theorem, we obtain a C^1 curve Γ in the parameter space defined by $\sqrt{-\mu_1} = F(\mu, \Omega_L(\mu))$, where $y_L = \Omega_L(\mu)$ is a C^1 solution of $\dfrac{\partial F}{\partial y_L}(\mu, y_L) = 0$. Since $\dfrac{\partial F}{\partial \mu_2}(0,0) \neq 0$ (by the independent unfolding hypothesis), we obtain that Γ is a C^1 curve tangent to Γ_{SN} at 0. There are no other criticalities and $W^u(q(\bar\mu))$ is transversal to $W^s(p(\bar\mu))$, and, thus, the bifurcation diagram for the family X_μ for μ near 0 is exactly $\Gamma \cup \Gamma_{SN}$.

Remark. — Along Γ the field X_μ presents one orbit of quasi-transversality between $W^u(q(\mu))$ and $W^s(p_2(\mu))$. If $\dim W^u(q(0)) + \dim W^{ss}(p(0)) = n$, then the above equations simplify to $x = F(\mu)$ and Γ is given by $\sqrt{-\mu_1} = F(\mu)$.

(3.C) *Stability.* — Let X_μ be in $\chi_2^g(M)$ such that $X_{\bar\mu}$ presents a saddle-node with criticality and the family satisfies all the conditions described in (3.A). If $\hat X_\mu$ is close to X_μ so that it also has a bifurcation of type III for $\hat\mu$ near $\bar\mu$, then we will show that $\{X_\mu\}_{\mu \in U}$ is equivalent to $\{\hat X_\mu\}_{\mu \in U'}$, where U and U' are open neighbourhoods of $\bar\mu$ and $\hat\mu$ in $\mathbf{R}^2$. We may assume in the usual ordering of the singularities of X_μ, $\sigma_1(\mu) \leqslant \sigma_2(\mu) \leqslant \ldots \leqslant \sigma_\ell(\mu)$, that $\sigma_k(\mu) = \sigma_{k+1}(\mu) = p(\mu)$ for $\mu \in \Gamma_{SN}$ and $q(\mu) = \sigma_{k-1}(\mu)$. We will see at the end of this paragraph that there is no loss of generality in doing so. We consider a distinguished neighbourhood $U_{k-1}(\mu)$ of $\sigma_{k-1}(\mu)$ as constructed in Theorem A and connect it along the orbit of tangency γ to a neighbourhood $V(\mu)$ of the saddle-node. As in previous cases, we construct an equivalence h_μ that preserves the level sets of f_μ inside $U_{k-1}(\mu)$. In $V(\mu)$ it preserves two continuous invariant foliations with C^1 leaves and depending continuously on μ; these foliations, denoted by F_μ^{uu} and F_μ^{cs}, have complementary dimensions. The leaves of F_μ^{uu} have dimension equal to $\dim W^{uu}(\sigma_k(\bar\mu))$ and its space of leaves is the center-stable manifold $W^{cs}(\sigma_k(\mu))$.

We start constructing the equivalence between X_μ and $\hat{X}_\mu$ on the neighbourhood $V(\mu)$ by obtaining an equivalence between $X_\mu^{cs} = X_\mu \mid W^{cs}(\sigma_k(\mu))$ and $\hat{X}_\mu^{cs} = \hat{X}_\mu \mid W^{cs}(\hat{\sigma}_k(\mu))$. To do that, let us consider a compatible unstable system $F_1(\mu), \ldots, F_{k-2}^u(\mu), F_{k-1}^{cu}(\mu)$ as before. We take a continuous family of C^2 cylinders $C(\mu)$ in $W^{cs}(\sigma_k(\mu))$ and a continuous family of C^1 closed discs $D(\mu)$ contained in some leaf F of the strong-stable foliation F_μ^{ss} so that, for $\mu \in \Gamma_{SN}$, $C(\mu)$ is transversal to $W^{ss}(\sigma_k(\mu))$ and $C(\mu) \cup D(\mu)$ contains a fundamental domain for $W^s(\sigma_k(\mu))$. On $C(\mu)$, we construct a C^1 foliation $F_\mu^c(\mu)$ of dimension one, which is compatible with the induced system $F_1^u(\mu) \cap C(\mu), \ldots, F_{k-2}^u(\mu) \cap C(\mu), F_{k-1}^{cu}(\mu) \cap C(\mu)$. Also, $F_\mu^c(\mu)$ is compatible with $W^u(\sigma_{k-1}(\mu)) \cap C(\mu)$ and has a unique singularity of saddle-node type which is the point of tangency between $W^u(\sigma_{k-1}(\mu)) \cap C(\mu)$ and $F_\mu^{ss} \cap C(\mu)$; outside this point, $F^c(\mu)$ is transversal to $F_\mu^{ss} \cap C(\mu)$. The construction of $F^c(\mu)$ is exactly like in the previous paragraph. Let $M^c(\mu)$ be a distinguished leaf of $F^c(\mu)$, namely the curve in $W^{cu}(\sigma_{k-1}(\mu)) \cap C(\mu)$ defined by $y_L = \Omega_L(\mu)$, where $\mu \in U \mapsto \Omega_L(\mu)$ is the C^{m-1} solution of $\frac{\partial F}{\partial y_L}(\mu, y_L) = 0$ and, as above, (x, y_L, y_K) are C^m coordinates for $C(\mu)$ near the point of tangency $\gamma(\mu) \cap C(\mu)$, $m \geqslant 3$. For $\mu_1 \leqslant 0$ in this curve, there are three distinguished points $p_{k-1}(\mu) = W^u(\sigma_{k-1}(\mu)) \cap M^c(\mu)$, $p_k(\mu) = W^s(\sigma_k(\mu)) \cap M^c(\mu)$ and $p_{k+1}(\mu) = W^s(\sigma_{k+1}(\mu)) \cap M^c(\mu)$, so that the curve $p_{k-1}(\mu) = p_{k+1}(\mu)$ represents the values of the parameter such that $W^u(\sigma_{k-1}(\mu))$ is quasi-transversal to $W^s(\sigma_{k+1}(\mu))$. Therefore, in the three-dimensional manifold $M^c = \bigcup_{p \in U} M^c(\mu)$, we have two C^{m-1} surfaces intersecting transversally at 0 defined by $M_1 = \{ x = F(\mu, \Omega_L(\mu))\}$ and $M_2 = \{ B(\mu, x) = 0 \}$. So, let $\bar{\varphi} : M^c \to M^c$ be a diffeomorphism of the form $\bar{\varphi}(\mu, x) = (\varphi_1(\mu), h^c(\mu, x))$ such that $\bar{\varphi}(M_1) = \{ x - \mu_2 = 0 \}$, $B \circ \bar{\varphi}(\mu, x) = x^2 + \mu_1$. Then, it is clear that X_μ^{cs} is topologically equivalent to

$$\widetilde{X}_\mu^{cs} = (x^2 + \mu_1)\frac{\partial}{\partial x} + \Sigma \widetilde{A}_{ij}(\mu, x)\, y_i \frac{\partial}{\partial y_j}$$

and the manifold $\widetilde{M}_1(\mu) = \widetilde{M}^c(\mu) \cap W^u(\sigma_{k-1}(\mu))$ is represented by $\{ x - \mu_2 = 0 \}$. If we repeat the construction for the nearby family $\hat{X}_\mu$, we obtain $\widetilde{X}_\mu^{cs}$ equivalent to a family with the same normal form along the central manifold (still denoted $W^c(\mu)$) and with the same expression for the manifold $\widetilde{M}_1(\mu)$. Hence, $\hat{X}_\mu^{cs} \mid W^c(\mu)$ is conjugate to $\widetilde{X}_\mu^{cs} \mid W^c(\mu)$ and the conjugacy preserves the distinguished point $x_{k-1}(\mu)$, which is the projection via the strong-stable foliation F_μ^{ss} of the point $p_{k-1}(\mu) = W^u(\sigma_{k-1}(\mu)) \cap M^c(\mu)$. Thus, $X_\mu^{cs} \mid W^c(\mu)$ is equivalent to $\hat{X}_{\varphi(\mu)}^{cs} \mid W^c(\varphi(\mu))$ with $\varphi : (U, 0) \to (\mathbf{R}^2, 0)$ being a homeomorphism that sends the region A_i onto $\hat{A}_i$ as in the picture (Figure IX). This gives a homeomorphism in the space of leaves of the strong-stable foliation F_μ^{ss}. We now define a homeomorphism $h_\mu^{cs} : W^{cs}(\sigma_k(\mu)) \to W^{cs}(\hat{\sigma}_k(\varphi(\mu)))$. Let us consider, as in previous paragraphs, a continuous family of compatible homeomorphisms h_μ^i, for $i = 1, \ldots, k-1$, defined on the space of leaves of the foliations $F_1(\mu), \ldots, F_{k-2}^u(\mu)$, $F_{k-1}^{cu}(\mu)$. We define h_μ^{cs} in the same way as in Theorem A, Chapter III of [15], the only difference arising from the singularity of the central foliation F^c_μ. Hence, we begin by applying Lemma 1 to get a homeomorphism between $W^{cu}(\sigma_{k-1}(\mu)) \cap C(\mu)$ and $W^{cu}(\hat{\sigma}_{k-1}(\varphi(\mu))) \cap \hat{C}(\varphi(\mu))$ preserving the central foliation. We then proceed as in § 1 to extend this to a homeomorphism on $C(\mu)$ which is compatible with h^i_μ, $i = 1, \ldots, k-1$ and sends $F^c(\mu)$ to $F^c(\varphi(\mu))$. This induces a homeomorphism on the boundary of the disc $D(\mu)$ which is extended to its interior, the extension being compatible with the homeomorphisms h^i_μ. Finally, we define h^{cs}_μ by sending F^{ss}_μ to $\hat{F}^{ss}_{\varphi(\mu)}$.

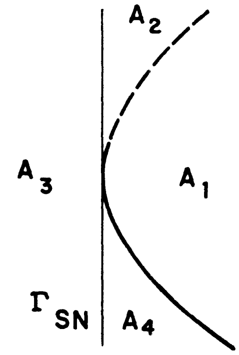

FIG. IX

Now, over each point of $C(\mu) \cup D(\mu)$ we raise a u-dimensional ($u = \dim W^{uu}(\sigma_k(\mu))$) continuous foliation F^{uu}_μ, with C^1 leaves compatible with the unstable system $F^u_1(\mu), \ldots, F^u_{k-2}(\mu), F^{cu}_{k-1}(\mu)$ and with $W^u(\sigma_{k-1}(\mu))$. Positively saturating it by the flow $X_{\mu,t}$ and adding the strong-unstable foliation restricted to $W^u(\sigma_k(\mu))$ for $\mu_1 \leqslant 0$, we obtain a strong-unstable foliation F^{uu}_μ whose space of leaves is $W^{cs}(\sigma_k(\mu))$. We then construct a complementary foliation denoted by F^{cs}_μ compatible with a stable system $F^s_i(\mu)$ for $i = k+2, \ldots, \ell$. We start by constructing a compatible stable system $F^s_{k+2}(\mu), \ldots, F^s_\ell(\mu)$, together with a homeomorphism in the space of leaves of each of these foliations. Let $L_+(\mu)$ be a leaf of F^{ss}_μ in $W^{cs}(\mu)$ such that $F^{ss}_{\mu,\varepsilon} \cap W^c(\mu)$ consists of a point x_ε with coordinate $\varepsilon > 0$ small. Over each point x of $L_+(\mu)$ we take $F^{uu}_{\mu,x}$ the part of the leaf of the strong-unstable foliation that contains x and is contained in the neighbourhood $V(\mu)$. If we let $D_{\mu,\varepsilon} = \bigcup_{x \in L_+(\mu)} F^{uu}_{\mu,x}$, then $D_{\mu,\varepsilon}$ is a C^0 disc of codimension one which is C^m outside $L_+(\mu)$. We also take a continuous family of C^2 cylinders $C^1(\mu)$ in $W^{cu}(\mu)$ transversal to $W^{uu}(\mu)$, so that $C^1(\mu) \cup D^{cu}_{\mu,\varepsilon}$ contains a fundamental domain for $W^u(\mu)$, where $D^{cu}_{\mu,\varepsilon} = D_{\mu,\varepsilon} \cap W^{cu}(\sigma_k(\mu))$, and the vector field X_μ is tangent to $C^1(\mu) \cap W^{cu}_{\mu,\varepsilon}$. In $C^1(\mu)$, we let $F^d(\mu)$ be a one-dimensional central foliation compatible with the stable system $F^s_{k+2}(\mu), \ldots, F^s_\ell(\mu)$. Over each leaf of $F^d(\mu)$ we raise an $(s+1)$-dimensional foliation compatible with the stable system. Over each point of $D_{\mu,\varepsilon}$ we raise an s-dimensional continuous foliation compatible with the induced system $F^s(\sigma_i(\mu)) \cap D_{\mu,\varepsilon}$, $k+2 \leqslant i \leqslant \ell$. The center-stable foliation F^{cs}_μ is the $(s+1)$-foliation obtained by saturating negatively the foliation and adding to it the center-stable manifold $W^{cs}(\mu)$ for $\mu_1 \leqslant 0$. We repeat the same constructions for $\hat{X}_{\varphi(\mu)}$.

We can now get a homeomorphism $h_\mu^{cu} : D_{\mu,\varepsilon}^{cu} \to \hat{D}_{\varphi(\mu),\varepsilon}^{cu}$ compatible with the homeomorphism on the space of leaves of the stable system, by first constructing it on $\partial D_{\mu,\varepsilon}^{cu}$ and then extending it to the interior of the disc. The equivalence between X_μ and $\hat{X}_{\varphi(\mu)}$ in the neighbourhood $V(\mu)$ is finally obtained by preserving the complementary foliations F_μ^{cs} and F_μ^{uu}. Since we are preserving the center-unstable foliation $F_{k-1}^{cu}(\mu)$, we may extend it to a neighbourhood $U_{k-1}(\mu)$ of $\sigma_{k-1}(\mu)$ by preserving the level sets of the function f_μ. The globalization of the equivalence to all M follows exactly like in § 1 of this section or in Theorem A, Section II.

Finally, if in the ordering of the singularities of X_μ,

$$\sigma_1(\mu) \leqslant \ldots \leqslant \sigma_{k-1}(\mu) \leqslant \sigma_k(\mu) \leqslant \sigma_{k+1}(\mu) \leqslant \ldots \leqslant \sigma_\ell(\mu),$$

with $\sigma_k(\mu) = \sigma_{k+1}(\mu) = p(\mu)$ along the curve of saddle-nodes Γ_{SN}, the orbit of quasi-transversality occurs in the unstable manifold of a singularity $\sigma_j(\mu)$, with $j \leqslant k-2$, we proceed as in § 2, case II.b. That is, we construct a compatible system of unstable foliations $F^u(\sigma_1(\mu)), \ldots, F^{cu}(\sigma_j(\mu)), F^u(\sigma_{j+1}(\mu)), \ldots, F^u(\sigma_{k-1}(\mu))$ and follow the same steps as above. Again, we connect the distinguished neighbourhood $U_j(\mu)$ of $\sigma_j(\mu)$ to the neighbourhood $V(\mu)$ along the orbit of tangency.

§ 4. Bifurcations of type V: saddle-node with an orbit of tangency

So far we have treated the cases which present at most one secondary bifurcation: in a neighbourhood of the bifurcation value $\bar{\mu}$, the family X_μ presents for $\mu \neq \bar{\mu}$ at most one new bifurcation. Contrary to this, the bifurcations corresponding to types V, VI and VII of the list in Section I may present several secondary bifurcations. This lead us to analyze orbits of tangency between several invariant manifolds and a certain invariant foliation. For this reason, to prove stability, a globalization of Lemma 1 in § 1 (Lemma 2 below) will be necessary.

In this paragraph we study the case where $X_{\bar{\mu}}$ presents a saddle-node $p(\bar{\mu})$ and an orbit γ of quasi-transversality. We assume that γ belongs to the unstable manifold $W^u(q(\bar{\mu}))$ of a hyperbolic singularity and the stable manifold $W^s(p(\bar{\mu}))$ of the saddle-node. The case where the quasi-transversal orbit occurs between invariant manifolds of hyperbolic singularities, will be discussed at the end of this paragraph. Besides the assumptions that we have already used in previous cases, like linearizability and partial linearizability for X_μ near $q(\bar{\mu})$ and $p(\bar{\mu})$, generic and independent unfolding of the saddle-node and the orbit of quasi-transversality, and transversality between $W^{cu}(q(\bar{\mu}))$ and $W^s(p(\bar{\mu}))$, several others are required here. They are satisfied by generic families $X_\mu \in \chi_2^g(M)$ which present a bifurcation of type V.

(4.A) *Other generic assumptions.*

(4.1) Let $W^{vu}(p(\bar{\mu}))$ be the codimension-one invariant submanifold of $W^{uu}(p(\bar{\mu}))$ such that $T_{p(\bar{\mu})} W^{vu}(p(\bar{\mu}))$ is complementary to the eigenspace corresponding to the smallest nonzero eigenvalue of $dX_{\bar{\mu}}(p(\bar{\mu}))$ (weakest expansion). Then, for $r \in \gamma$, there exists a linear subspace $E_r \subset T_r W^u(q(\bar{\mu}))$ with $\dim E_r = \dim W^{vu}(p(\bar{\mu}))$ such that $\lim_{t\to\infty} dX_{\bar{\mu},t}(r) . E_r = T_{p(\bar{\mu})} W^{vu}(p(\bar{\mu}))$. Moreover, if σ is a singularity of $X_{\bar{\mu}}$ different from $p(\bar{\mu})$ and $q(\bar{\mu})$, then $W^s(\sigma)$ is transversal to $W^u(p(\bar{\mu}))$, $W^{uu}(p(\bar{\mu}))$ and $W^{vu}(p(\bar{\mu}))$, and $W^u(\sigma)$ is transversal to $W^{ss}(q(\bar{\mu}))$, $W^s(p(\bar{\mu}))$ and $W^{ss}(p(\bar{\mu}))$.

(4.2) Let $F^{vu}_{\bar{\mu}}$ be the unique codimension-two invariant foliation in $W^u(p(\bar{\mu}))$ which has $W^{vu}(p(\bar{\mu}))$ as a distinguished leaf. $F^{vu}_{\bar{\mu}}$ is compatible with $F^{uu}_{\bar{\mu}}$, each leaf L of $F^{uu}_{\bar{\mu}}$ is subfoliated by leaves of $F^{vu}_{\bar{\mu}}$. Suppose $L \neq W^{uu}(p(\bar{\mu}))$ and that $\pi^{vu} : L \to \mathbf{R}$ is a submersion that defines $F^{vu}_{\bar{\mu}}$ in L. Then, the restriction of π^{vu} to each stable manifold $W^s(\sigma(\bar{\mu})) \cap L$ is a Morse function with distinct critical values. For any stable manifold such that $W^s(\sigma(\bar{\mu})) \cap L$ is tangent to $F^{vu}_{\bar{\mu}}$, the eigenvalues of $dX_\mu(\sigma(\bar{\mu}))$ are distinct. In this case the center-stable manifold $W^{cs}(\sigma(\bar{\mu})) \cap L$ is transversal to $F^{vu}_{\bar{\mu}}$.

Comments. — Clearly, these conditions do not depend on the leaf L. Also, if $W^s(\sigma(\bar{\mu})) \cap L$ is compact, it is easy to perturb $X_{\bar{\mu}}$ so that $\pi^{vu} \mid W^s(\sigma(\bar{\mu})) \cap L$ is a Morse function with distinct critical values and $W^{cs}(\sigma(\bar{\mu})) \cap L$ is transversal to $F^{vu}_{\bar{\mu}}$. To get the genericity of these hypotheses, we use the ordering

$$\sigma_1(\bar{\mu}) \leqslant \dots \leqslant \sigma_k(\bar{\mu}) \leqslant \sigma_{k+2}(\bar{\mu}) \leqslant \dots \leqslant \sigma_\ell(\bar{\mu})$$

of the singularities of $X_{\bar{\mu}}$ such that $p(\bar{\mu}) = \sigma_k(\bar{\mu})$, assuming that $W^{uu}(\sigma_i(\bar{\mu}))$ is transversal to $W^s(\sigma_j(\bar{\mu}))$ for $k + 2 \leqslant i \leqslant \ell - 1$; $i + 1 \leqslant j \leqslant \ell$ and proceed by induction using transversality arguments, in particular, transversality between $W^{cs}(\sigma_j(\bar{\mu}))$ and $F^{vu}_{\bar{\mu}}$.

(4.B) *The bifurcation set.* — Assume that in the ordering of the singularities

$$\sigma_1(\mu) \leqslant \dots \leqslant \sigma_{k-1}(\mu) \leqslant \sigma_k(\mu) \leqslant \sigma_{k+1}(\mu) \leqslant \dots \leqslant \sigma_\ell(\mu) \text{ of } X_\mu,$$

we have $\sigma_k(\mu) = \sigma_{k+1}(\mu) = p(\mu)$ for $\mu \in \Gamma_{SN}$, the curve of saddle-nodes, and that $\sigma_{k-1}(\mu) = q(\mu)$; also assume $\bar{\mu} = 0$. Using the transversality between $W^{cu}(\sigma_{k-1}(\mu))$ and $W^s(\sigma_k(\mu))$ and the partial linearizability of X_μ near $\sigma_k(\mu)$, we extend $W^{cu}(\sigma_{k-1}(\mu))$ to a neighbourhood of the closure of the orbit of tangency γ so that it contains the saddle-node. We may suppose that we have a normal form for $X_\mu \mid W^{cu}(\sigma_{k-1}(\mu))$ near $\sigma_k(\mu)$ and, as in § 3, we can write

$$X^{cu}_\mu = X_\mu \mid W^{cu}(\sigma_{k-1}(\mu)) = (\pm x_1^2 + \mu_1)\frac{\partial}{\partial x_1} + \Sigma\alpha_{ij}(x_1, \mu)\, y_i \frac{\partial}{\partial y_j} + \Sigma\beta_{ij}(x_1, \mu)\, z_i \frac{\partial}{\partial z_j}$$

in a neighbourhood of $p(0)$, with all eigenvalues of $A_I(x_1, \mu) = (\alpha_{ij}(x_1, \mu))$ being negative and of $B_L(x_1, \mu) = (\beta_{ij}(x_1, \mu))$ being positive. In these coordinates we assume that the z_1-axis corresponds to the direction of the weakest expansion, $u = \dim W^{uu}(p(\bar{\mu}))$; we choose the positive sign in the above expression. Let $\Sigma^{cu}_-(\mu)$ be a cross-section intersecting the orbit of tangency γ. Then, the intersection of $W^u(\sigma_{k-1}(\mu))$ with $\Sigma^{cu}_-(\mu)$ is

$\{(y_{\mathrm{I}}, z_1, z_{\mathrm{L}}) \mid z_1 = \mathrm{F}(\mu, y_{\mathrm{I}}, z_{\mathrm{L}})\}$ with $y_{\mathrm{I}} \mapsto \mathrm{F}(\mu, y_{\mathrm{I}}, 0)$ being a deformation of a Morse function. Moreover, the generic and independent unfolding of the orbit of quasi-transversality implies that the map $(\mu_1, \mu_2) \mapsto (\mu_1, \mathrm{F}(\mu, \mathrm{Y}_{\mathrm{I}}(\mu), 0))$ is a local diffeomorphism, where $\mu \mapsto \mathrm{Y}_{\mathrm{I}}(\mu)$ is the solution of $\dfrac{\partial \mathrm{F}}{\partial y_{\mathrm{I}}}(\mu, y_{\mathrm{I}}, 0) = 0$. Therefore, the curve $\Gamma_{k-1,k}$ of quasi-transversality between $\mathrm{W}^u(\sigma_{k-1}(\mu))$ and $\mathrm{W}^s(\sigma_k(\mu))$ is locally defined by $\{\mu_1 < 0\} \cap \{\mathrm{F}(\mu, \mathrm{Y}_{\mathrm{I}}(\mu), 0) = 0\}$. By changing coordinates, we get $\Gamma_{k-1,k} = \{\mu_2 = 0, \mu_1 < 0\}$.

Since there are no criticalities, other bifurcations may occur only in the region $\mu_1 > 0$, where the corresponding vector field does not present singularities near the point $p(0)$. To analyze these possibilities, we let $\Sigma^{cu}_{+}(\mu)$ be a small closed disc contained in the section $\{x_1 = \varepsilon_1\}$ such that $\Sigma^{cu}_{+}(\mu) \cap \mathrm{W}^u(p(\mu))$ is contained in a leaf of the strong unstable foliation. The positive number ε_1 is taken so that if $\mathrm{W}^s(\sigma_j(\mu)) \cap \partial\Sigma^{cu}_{+}(\mu) \neq \emptyset$, then $\mathrm{W}^s(\sigma_j(0)) \cap \mathrm{W}^{uu}(p(0)) \neq \emptyset$. Hence, if there is an orbit of tangency between $\mathrm{W}^u(\sigma_{k-1}(\mu))$ and $\mathrm{W}^s(\sigma_j(\mu))$, then it necessarily intersects the interior of $\Sigma^{cu}_{+}(\mu)$. Moreover, from (4.1) and (4.2), these tangencies may occur only near the points of tangency between $\mathrm{W}^s(\sigma_j(0)) \cap \mathrm{W}^u(p(0)) \cap \Sigma^{cu}_{+}(0)$ and the foliation F^{vu}_0. For each $j \geqslant k+2$ we denote by $p_{j,1}, \ldots, p_{j,n(j)}$ these points. Let $(v_{\mathrm{I}}, w_1, w_{\mathrm{L}})$ be C^m coordinates in $\Sigma^{cu}_{+}(\mu)$ such that $\mathrm{W}^u(\sigma_k(\mu)) \cap \Sigma^{cu}_{+}(\mu) = (0, w_1, w_{\mathrm{L}})$, $m \geqslant 3$. We may assume that $\mathrm{W}^s(\sigma_j(0))$ has codimension one in $\Sigma^{cu}_{+}(0)$; if not, we just restrict ourselves to $\mathrm{W}^{cs}(\sigma_j(0))$. Then, from (4.2), near each point p_{ji} we may write

$$\mathrm{W}^s(\sigma_j(\mu)) \cap \Sigma^{cu}_{+}(\mu) = \{w_1 = \mathrm{G}_{ji}(\mu, v_{\mathrm{I}}, w_{\mathrm{L}})\}$$

with $\mathrm{G}_{ji}(0, v_{\mathrm{I}}, 0)$ having a nondegenerate critical point at $v_{\mathrm{I}}(p_{ji})$.

Let us extend F^{vu}_0, previously only defined on $\mathrm{W}^u(p(0))$, see (4.2). Let

$$\Sigma^c_{-}(\mu) = \{(y_{\mathrm{I}}, z_1, z_{\mathrm{L}}) \in \Sigma^{cu}_{-}(\mu) \mid z_{\mathrm{L}} = 0\} \quad \text{and} \quad \pi^{vu}_{\mu} : \Sigma^{cu}_{-}(\mu) \to \Sigma^c_{-}(\mu)$$

$$\pi^{vu}_{\mu}(y_{\mathrm{I}}, z_1, z_{\mathrm{L}}) = (y_{\mathrm{I}}, z_1 - \mathrm{F}(\mu, y_{\mathrm{I}}, z_{\mathrm{L}}) + \mathrm{F}(\mu, y_{\mathrm{I}}, 0), 0)$$

be a submersion that defines a C^m foliation F^{vu}_{μ} compatible with $\mathrm{W}^u(\sigma_{k-1}(\mu)) \cap \Sigma^{cu}_{-}(\mu)$. *For latter purpose, the flow saturation of this foliation will still be denoted by* F^{vu}_{μ}. Using the normal form for X^{cu}_{μ} near $p(0)$ to get a linear expression for the Poincaré map $\mathrm{P}^{cu}_{\mu} : \Sigma^{cu}_{-}(\mu) \to \Sigma^{cu}_{+}(\mu)$ for $\mu_1 > 0$, we obtain that the restriction of π^{vu}_{μ} to

$$\mathrm{W}^s(\sigma_j(\mu)) \cap \Sigma^{cu}_{-}(\mu)$$

is singular along disjoint C^{m-1} manifolds $\widetilde{\mathrm{M}}_{ji}(\mu)$ for $i = 1, \ldots, n(j)$, with dimension equal to $|\mathrm{I}| = \dim \mathrm{W}^{ss}(p(0)) \cap \mathrm{W}^{cu}(q(0))$ and which depend differentiably on μ. As $\mu \to 0$, all these manifolds become C^1 close to the y_{I}-plane and for $\mu_1 = 0$ they collapse into this set. Since the points $\{p_{ji}\}$ belong to distinct leaves of the foliation F^{vu}_0, the images $\mathrm{M}_{ji}(\mu) = \tilde{\pi}^{vu}_{\mu}(\widetilde{\mathrm{M}}_{ji}(\mu))$ are *disjoint* submanifolds of codimension one in $\Sigma^c_{-}(\mu)$. If $\mathrm{W}^s(\sigma_j(\mu))$ has minimal dimension (equal to $\dim \mathrm{W}^{ss}(p(0))$), then

$$\widetilde{\mathrm{M}}_{ji}(\mu) = \mathrm{W}^s(\sigma_j(\mu)) \cap \Sigma^{cu}_{-}(\mu).$$

From this construction we conclude that $W^u(\sigma_{k-1}(\mu))$ is tangent to $W^s(\sigma_j(\mu))$ in $\Sigma_-^{cu}(\mu)$ if and only if $M_{ji}(\mu)$ is tangent to $W^u(\sigma_{k-1}(\mu)) \cap \Sigma_-^c(\mu)$ for some $i = 1, \ldots, n(j)$. Hence, for each (j, i), we consider possible tangencies between the manifold $M_{ji}(\mu)$ and the foliation defined by $(y_I, z_1) \overset{\pi}{\mapsto} z_1 - F(\mu, y_I, 0) + F(\mu, 0, 0)$. Using now the hypothesis of quasi-transversality between $W^s(p(0))$ and $W^u(q(0))$, we obtain for each μ in a neighbourhood of 0 in $\{\mu_1 > 0\}$ a unique point of tangency $q_{ji}(\mu) \in M_{ji}(\mu)$. The map $\mu \mapsto q_{ji}(\mu)$ is of class C^1 in a neighbourhood of 0 in $\{\mu_1 \geqslant 0\}$ and $q_{ji}(0, \mu_2) = 0$. Therefore, X_μ presents a quasi-transversal orbit of tangency between $W^u(\sigma_{k-1}(\mu))$ and $W^s(\sigma_j(\mu))$ if and only if $q_{ji}(\mu)$ belongs to $W^u(\sigma_{k-1}(\mu)) \cap \Sigma_-^c(\mu)$. These values of μ correspond to a finite number of disjoint C^1 curves $\Gamma_{k-1,j}^i$ tangent to the μ_1-axis at 0. The bifurcation diagram is as in the figure.

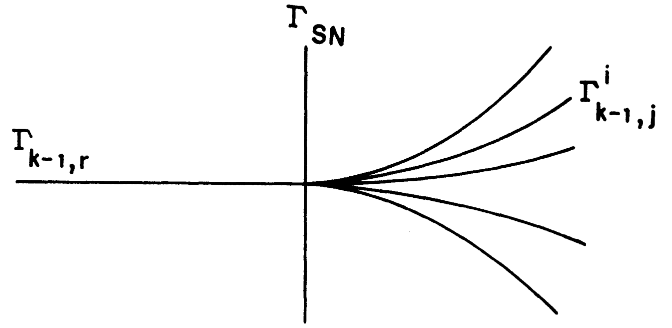

FIG. X

(4.C) *Stability.* — Let X_μ be a family in $\chi_2^g(M)$ which presents a bifurcation of type V at $\bar{\mu}$ and satisfies all the assumptions described in (4.A). If $\hat{X}_\mu$ is a nearby family, with $\hat{\mu}$ as the corresponding bifurcation value, then we show that there are neighbourhoods U and $\hat{U}$ of $\bar{\mu}$ and $\hat{\mu}$ in $\mathbf{R}^2$ such that $\{X_\mu\}_{\mu \in U}$ is equivalent to $\{\hat{X}_\mu\}_{\mu \in \hat{U}}$. We assume that $\bar{\mu} = \hat{\mu} = 0$.

We start by taking a compatible unstable system $F_1^u(\mu), \ldots, F_{k-2}^u(\mu), F_{k-1}^{cu}(\mu)$ and neighbourhoods $U_{k-1}(\mu)$ of $\sigma_{k-1}(\mu)$ and $V(\mu)$ of $p(\mu)$ in M which are connected along the orbit of tangency γ. From the description of the bifurcation set, each point of tangency between F_0^{vu} and $W^s(\sigma_j(0))$ yields a quasi-transversal orbit between $W^u(\sigma_{k-1}(\mu))$ and $W^s(\sigma_j(\mu))$. So, we consider distinguished neighbourhoods $U_j(\mu)$ of each such singularity and connect them to $V(\mu)$ with tubes along each orbit of tangency γ_{ji}. The equivalence will preserve the level sets of f_μ inside the neighbourhood $U_j(\mu)$. Using the transversality between $W^s(\sigma_i(0))$ and $W^{uu}(\sigma_j(0))$ for $i > j \geqslant k+2$, and proceeding as in § 2 of the present section, we construct a compatible center-stable system $F_{k+2}^{cs}(\mu), F_{k+3}^{cs}(\mu), \ldots, F_l^{cs}(\mu)$. It may happen that for some $i \geqslant k+2$, the stable manifold $W^s(\sigma_i(0))$ is transversal to F_0^{vu} (for instance, when $\sigma_i(0)$ is a sink). In this case we take, as in § 2, the stable foliation $F_i^u(\mu)$.

To obtain the equivalence between X_μ and $\tilde{X}_\mu$ on a neighbourhood V of the saddle-node $p(\bar{\mu})$ in $M \times \mathbf{R}^2$, we construct a center-unstable foliation $F^{cu}(\mu)$ compatible with the unstable system and the unstable manifold $W^u(\sigma_{k-1}(\mu))$. The method to construct $F_k^{cu}(\mu)$ is similar to the one already used in § 1 and § 2. The main difference here is that we want the singular set of $F_k^{cu}(\mu)$ to contain the points of tangency between $W^u(\sigma_{k-1}(\mu))$ and the manifolds $W^s(\sigma_j(\mu))$ for $j = k+2, \ldots, \ell$. Let $\Sigma_-^c(\mu)$ be the leaf space of the foliation F_μ^{vu} constructed in (4.B). We recall that to describe the bifurcation set we have obtained codimension-one submanifolds $M_{ji}(\mu) \subset \Sigma_-^c(\mu)$ such that $W^u(\sigma_{k-1}(\mu))$ is tangent to $W^s(\sigma_j(\mu))$ if and only if $W^u(\sigma_{k-1}(\mu)) \cap \Sigma_-^c(\mu)$ is tangent to $M_{ji}(\mu)$ for some $i = 1, \ldots, n(j)$. For each pair (j, i) and $\mu_1 > 0$, we let

$$Y_{ji}(\mu, y_I) = \frac{\partial F}{\partial y_I}(\mu, y_I, 0) - \frac{\partial \Lambda_{ji}}{\partial y_I}(\mu, y_I),$$

where graph $(\Lambda_{ji}) = M_{ji}(\mu)$. Since $\lim_{\mu_1 \to 0} Y_{ji}(\mu, y_I) = \frac{\partial F}{\partial y_I}(\mu, y_I, 0)$, we may extend this family to $Y_{ji}(\mu, y_I) = \frac{\partial F}{\partial y_I}(\mu, y_I, 0)$ for $\mu_1 \leqslant 0$. For $0 < \varepsilon_1 < \varepsilon_2$ small, we let $\Delta_{ji}(\varepsilon_1) \subset \Delta_{ji}(\varepsilon_2)$ be open neighbourhoods of $M_{ji}(\mu) \cap W^u(\sigma_{k-1}(\mu))$ such that $\Delta_{ji}(\varepsilon_2) \cap \Delta_{j'i'}(\varepsilon_2) = \emptyset$ for $(j', i') \neq (j, i)$. We define a family of vector fields $Y(\mu, y_I)$ such that $Y \mid \Delta_{ji}(\varepsilon_1) = Y_{ji}$ and Y_μ in the complement of $\cup \Delta_{ji}(\varepsilon_2)$ is equal to $\frac{\partial F}{\partial y_I}(\mu, y_I)$. As in § 1 and § 2, the central foliation which gives rise to the leaves of $F_k^{cu}(\mu)$ inside $W^{cu}(\sigma_{k-1}(\mu))$ is tangent to a vector field Z_μ with a saddle-node singularity such that Z_μ restricted to each Δ_{ji} is equal to Y_μ. Associated to a central manifold of Z_μ we have a special leaf denoted by $W_k^{cu}(\mu)$. This invariant manifold is completed for $\mu_1 \leqslant 0$ by adding part of a center-unstable manifold which is linear in the above normal form coordinates. By construction, $W_k^{cu}(\mu)$ contains all tangencies between $W^u(\sigma_{k-1}(\mu))$ and $W^s(\sigma_j(\mu))$, $j \geqslant k+2$. The other leaves of $F_k^{cu}(\mu)$ are obtained exactly as in § 1.

Complementary to $F_k^{cu}(\mu)$, we define a strong-stable foliation F_μ^{ss}. Since all stable manifolds $W^s(\sigma_j(0))$ are transversal to $W^u(\sigma_k(0))$, the method described in § 3 can also be applied here. However, since $F_k^{cu}(\mu)$ is a singular foliation, in order to have transversality between F_μ^{ss} and $F_k^{cu}(\mu)$ outside $W_k^{cu}(\mu)$ we modify F_μ^{ss} for $\mu_1 > 0$ near the points of tangency $p_{ji}(0)$. Let $\Sigma_\pm(\mu)$ be two cross-sections such that $\Sigma_\pm(\mu) \cap W^{cu}(\sigma_{k-1}(\mu)) = \Sigma_\pm^{cu}(\mu)$ and suppose that $F_{\mu,j}^{ss}$ is a strong-stable foliation in $W^s(\sigma_j(\mu)) \cap \Sigma_+(\mu)$, as constructed in § 3, which is transversal to $W_k^{cu}(\mu)$. We can also assume that $F_{\mu,j}^{ss}$ is transversal to $F_k^{cu}(\mu)$ outside a neighbourhood of each point $p_{ji}(0)$. Let $P_\mu : \Sigma_-(\mu) \to \Sigma_+(\mu)$ be the Poincaré map for $\mu_1 > 0$. We modify $F_{\mu,j}^{ss}$ in a neighbourhood of $p_{ji}(0)$ in such way that each leaf of the induced foliation $P_\mu^{-1}(F_{\mu,j}^s) \cap \Sigma_-^{cu}(\mu)$ projects by π_μ^{vu} onto a level set of a Liapounov function of the vector field Z_μ in $\Sigma_-^c(\mu)$. Proceeding in this way for all stable manifolds $W^s(\sigma_j(\mu))$ and extending this modifyied foliation to each leaf of $F_j^{cs}(\mu)$ as in § 1, we get the required strong stable foliation F_μ^{ss}. By preserving F_μ^{ss} and F_k^{cu}, we can obtain an equivalence between the two families X_μ and $\hat{X}_\mu$ on a neighbourhood of the saddle-node singularity similarly to § 3. *Hence, to prove local stability of* X_μ *we have now to obtain homeomorphisms on the space of leaves of these foliations.*

Let us first construct a suitable reparametrization φ. Consider X_μ restricted to $W_k^{cu}(\mu)$ (the space of leaves of F_μ^{ss}). Since $W_k^{cu}(\mu)$ depends differentiably on μ, it is transversal to $W^{ss}(\sigma_k(\mu))$ for $\mu \in \Gamma_{SN}$ and admits a C^r smoothing structure, $r \geqslant 3$ (see [15]), we conclude that $X_\mu \mid W_k^{cu}(\mu)$ has a μ-dependent normal form near the saddle-node as in (4.B). In $W_k^{cu}(\mu)$ we consider a codimension-two invariant foliation compatible with F_μ^{vu} such that for $\mu \in \Gamma_{SN}$ it has as special leaf $W^{vu}(\sigma_k(\mu))$, the codimension-two strong unstable manifold (see (4.1)). For $\mu_1 > 0$, F_μ^{vs} is obtained by saturating the foliation used at the end of (4.B) and intersecting with the leaves of F_μ^{uu}. This foliation is extended to a neighbourhood of $p(0)$ for $\mu_1 \leqslant 0$ by adding to it a codimension-two linear foliation. In particular for $\mu = 0$ this gives the foliation defined in (4.A). For each μ the leaf space of F_μ^{vu} is an invariant surface $W(\mu)$ that contains a center manifold, and it is defined in the above coordinates by $z_L = 0$. In $W(\mu)$ we take a fundamental domain $C(\mu) \cup E^c$, where $E^c = \{ x_1 = \varepsilon, \mid z_1 \mid \leqslant \delta \}$ and $C(\mu) = C_+ \cup C_- = \{ \mid z_1 \mid = \delta, \mid x_1 \mid \leqslant \varepsilon \}$. Let $C = \bigcup_{\mu \in U} C(\mu)$ and define $I_X : U \setminus \Gamma_{k-1,k} \to C \cup E^c$, the map that associates to each μ the point of intersection of $W^u(\sigma_{k-1}(\mu))$ with $C(\mu) \cup E_\mu^c$. If $c_{ji}(\mu) \in E^c$ represents the leaf of F_μ^{vu} which contains the tangency point $p_{ji}(\mu)$, then the curve $\Gamma_{k-1,j}^i$ obtained at the end of (4.B) is defined by $I_X^{-1}(c_{ji}(\mu))$. Moreover, from the hypothesis of generic unfolding of the orbit of tangency $\gamma \left(\frac{\partial F}{\partial \mu_2}(0) \neq 0\right)$ we obtain that $I_X^{-1}(E^c)$ is a wedged shape region $\Delta \subset \{ \mu_1 \geqslant 0 \}$ with vertice at 0, which is bounded by two curves $I_X^{-1}(\pm \delta)$. We also have in Δ a singular foliation Γ defined by $I_X^{-1}(x)$ for $x \in E^c$ with special leaves $\Gamma_{k-1,j}^i$. We define a reparametrization $\varphi : (\Delta, 0) \to (\hat{\Delta}, 0)$ of the form $(\varphi_1(\mu_1), \varphi_2(\mu_1, \mu_2))$ which sends Γ to $\hat{\Gamma}$. Since a conjugacy on a center manifold induces via the strong unstable foliation a homeomorphism $h^c : C \to \hat{C}$, we choose the reparametrization on $U \setminus \Delta$ in such way that $I_{\hat{X}} \circ \varphi = h^c \circ I_X$. This gives a reparametrization on a full neighbourhood of 0.

We now prove that X_μ and $\hat{X}_{\varphi(\mu)}$ are equivalent. We begin by taking a continuous family of diffeomorphisms $\psi_\mu : E^c(\mu) \to \hat{E}^c(\varphi(\mu))$ sending $c_{ji}(\mu)$ to $\hat{c}_{ji}(\varphi(\mu))$. Using a conjugacy we define a homeomorphism on the space of leaves of F_μ^{vu}. To define an equivalence between $X_\mu \mid W_k^{cu}(\mu)$ and $\hat{X}_{\varphi(\mu)} \mid W_k^{cu}(\varphi(\mu))$ we use a conjugacy which preserves F_μ^{vu} inside each leaf of F_μ^{uu}. Therefore, for $\mu_1 > 0$, it is enough to obtain a continuous family of homeomorphisms on a leaf $\Sigma_+^{uu}(\mu)$, preserving F_μ^{vu} and the center-stable system, in order to get an equivalence on a neighbourhood of the saddle-node $p(0)$ in $W_k^{cu}(\mu)$. Contrary to this, for $\mu_1 \leqslant 0$, the negative flow saturation of $\Sigma_+^{uu}(\mu)$ just fills a conic region $\Lambda(\mu)$ with vertex at the singularity $\sigma_{k+1}(\mu)$. Therefore, to get an equivalence on a full neighbourhood of $p(0)$ in $W_k^{cu}(\mu)$, we construct a two-dimensional foliation $F_2^c(\mu)$ in the complement of $\Lambda(\mu)$ which is compatible with the center-stable system and transversal to F_μ^{vu}. Thus, the equivalence is defined by preserving F_μ^{vu} and

$F_2^c(\mu)$. Let us construct a foliation $F_2^c(\mu)$: this construction resembles very much the one of a central foliation in [15]; the main difference here is that we want it to be transversal to the codimension-two foliation F_μ^{vu}. Let $K(\mu) = \bigcup_{x \in C(\mu)} D_x^{vu}(\mu)$, where $D_x^{vu}(\mu)$ is a closed disc centered at x and contained in the leaf of F_μ^{vu} over x. As above, $C(\mu)$ is the intersection of the fundamental domain C with the plane $\mu =$ constant. Let $E^{uu}(\mu)$ be a closed solid cylinder in the leaf $\Sigma_+^{uu}(\mu)$ which is bounded by two closed discs $K_i^{uu}(\mu) = K(\mu) \cap \Sigma_+^{uu}(\mu)$, $i = 1, 2$, and by a cylinder $S(\mu)$. Over each disc $K_i^{uu}(\mu)$ we raise a one-dimensional continuous foliation $\lambda^c(\mu)$ in $K(\mu)$ which is compatible with the center-stable system. We can assume that $C(\mu)$ is a leaf of $\lambda^c(\mu)$. We construct $\lambda^c(\mu)$ in such way that the union of the leaves of $\lambda^c(\mu)$ which are over the spheres $\partial K_1(\mu)$ and $\partial K_2^{uu}(\mu)$ is the closed cylinder $\bigcup_{x \in C(\mu)} \partial D_x^{vu}(\mu)$. Since the tangencies between $W^s(\sigma_j(\mu))$ and F_μ^{vu} occur in the interior of $E^{uu}(\mu)$, the cylinder $S(\mu) \subset \partial E^{uu}(\mu)$ can be foliated by a one-dimensional foliation $\delta^c(\mu)$, which is compatible with the center-stable system, and whose leaves are C^1 and transversal to $F_\mu^{vu} \cap S(\mu)$. Over each leaf of $\delta^c(\mu)$ we raise a two-dimensional foliation $\lambda_2^c(\mu)$ also compatible with the center-stable system, with each leaf of $\delta^c(\mu)$ being bounded by two leaves of $\lambda^c(\mu)$. Thus, $F_2^c(\mu)$ is obtained by taking the negative saturate of $\lambda^c(\mu)$ and of $\lambda_2^c(\mu)$ by the flow of X_μ. This finishes the construction of $F_2^c(\mu)$ which has as space of leaves the boundary of $E^{uu}(\mu)$. Since we already have defined a homeomorphism on the space of leaves of F_μ^{vu}, in order to conclude the construction of the equivalence between $X_\mu \mid W_k^{cu}(\mu)$ and $\hat{W}_{\varphi(\mu)} \mid W_k^{cu}(\varphi(\mu))$ it is *enough to obtain a continuous family of homeomorphisms* $h_\mu^{uu} : E^{uu}(\mu) \to E^{uu}(\varphi(\mu))$ *which preserves* F_μ^{vu} *and the center-stable system.* The idea to obtain h_μ^{uu} is to " project " $E^{uu}(\mu)$ onto $E^{uu}(0)$ along the leaves of F_μ^{vu} and to construct a homeomorphism from $E^{uu}(0)$ to $\hat{E}^{uu}(0)$ which satisfies the above requirements. We then pull back this homeomorphism to $E^{uu}(\mu)$ to get h_μ^{uu}. This process is achieved by constructing a continuous foliation $\mathscr{H}$ on $E^{uu} = \bigcup_{\mu \in U} E^{uu}(\mu)$, with C^1 leaves of dimension two, which is transversal to $E^{uu}(0)$ and compatible with both the center-stable system and with the foliation F_μ^{vu}. The construction of $\mathscr{H}$ is easy except at neighbourhoods of the tangency points $p_{ji}(0)$. Near each point $p_{ji}(0)$, $\mathscr{H}$ restricted to $W^{cu}(\sigma_j(\mu))$ is defined by intersecting F_μ^{vu} with a three-dimensional foliation given by a continuous family of vector fields, parametrized by μ, which has a saddle-node type singularity at $p_{ji}(\mu)$. Therefore the surfaces of tangency $(\mu, p_{ji}(\mu))$ are special leaves of $\mathscr{H}$. The extension of $\mathscr{H}$ to the leaves of the system F_j^{cs} near $p_{ji}(0)$ is done as in § 1. The foliation $\mathscr{H}$ was conceived so that it may be used to trivialize the foliation F_μ^{vu} along the center-stable system. Suppose that $h_0^{uu} : E^{uu}(0) \to \hat{E}^{uu}(0)$ is a homeomorphism preserving F_0^{vu} and the center-stable system. Then, we define $h_\mu^{uu} : E^{uu}(\mu) \to E^{uu}(\varphi(\mu))$ by sending $\mathscr{H} \cap E^{uu}(\mu)$ to $\hat{\mathscr{H}} \cap E^{uu}(\varphi(\mu))$. The reparametrization φ obtained above guarantees that the point $p_{ji}(\mu)$ is sent to the corresponding one $\hat{p}_{ji}(\varphi(\mu))$. Thus, to finish the construction of an equivalence between $X_\mu \mid W_u^{cu}(\mu)$ and $\hat{X}_{\varphi(\mu)} \mid W_u^{cu}(\varphi(\mu))$ it remains to prove the existence of h_0^{uu}. This is the content of the following key lemma.

Lemma 2. — *There is a homeomorphism* $h_0^{uu} : \mathrm{E}^{uu}(0) \to \hat{\mathrm{E}}^{uu}(0)$ *that preserves the foliation* F_0^{vu}, *the center-stable system* $\mathrm{F}^{cs}(\sigma_j(0)) \cap \mathrm{E}^{uu}(0)$ *and the stable manifolds* $\mathrm{W}^s(\sigma_j(0)) \cap \mathrm{E}^{uu}(0)$ *for* $j = k+2, \ldots, \ell$.

Proof. — By using a diffeomorphism which preserves F_0^{vu} we may assume that $\mathrm{E}^{uu}(0) = \hat{\mathrm{E}}^{uu}(0)$ and $\mathrm{F}_0^{vu} = \hat{\mathrm{F}}_0^{vu}$. We may also assume that $\pi^{vu} \mid \mathrm{W}^s(\sigma_j(0))$ and $\pi^{vu} \mid \mathrm{W}^s(\hat{\sigma}_j(0))$ have the *same* critical values for $j = k+2, ., \ell$. Let us assume that $\mathrm{W}^s_{k+2} = \mathrm{W}^s(\sigma_{k+2}(0)) \cap \mathrm{E}^{uu}(0)$ is compact and disjoint from the boundary of $\mathrm{E}^{uu}(0)$. If $\pi_0^{vu}(w_1, w_\mathrm{L}) = w_1$ is the projection along the leaves of F_0^{vu} then $\pi_{k+2}^{vu} = \pi_0^{vu} \mid \mathrm{W}^s_{k+2}$ is a Morse function with distinct critical values. Analogously, for $\hat{\pi}_{k+2}^{vu} = \hat{\pi}_0^{vu} \mid \hat{\mathrm{W}}^s_{k+2}$. Let $\varphi_{k+2} : \mathrm{W}^s_{k+2} \to \hat{\mathrm{W}}^s_{k+2}$ be a diffeomorphism C^2 close to the inclusion map. Then, $\hat{\pi}_{k+2}^{vu} \circ \varphi_{k+2}$ and π_{k+2}^{vu} are C^2 close Morse functions with the same critical values. Therefore, there exists a C^2 diffeomorphism $h_2 : \mathrm{W}^s_{k+2} \to \hat{\mathrm{W}}^s_{k+2}$, close to the identity, such that $\pi_{k+2}^{vu} \circ h_{k+2} = \hat{\pi}_{k+2}^{vu} \circ \varphi_{k+2}$. We define the restriction of h_0^{uu} to W^s_{k+2} by $h_0^{uu} = \varphi_{k+2} \circ h_{k+2}^{-1}$. The same is done, in a continuous way, for all leaves of $\mathrm{F}^s_{k+2} = \mathrm{F}^s(\sigma_{k+2}(0)) \cap \mathrm{E}^{uu}(0)$ which are contained in the center-stable manifold $\mathrm{W}^{cs}_{k+2} = \mathrm{W}^{cs}(\sigma_{k+2}(0)) \cap \mathrm{E}^{uu}(0)$. Since W^{cs}_j is transversal to $\mathrm{F}_+^{vu}(0)$, we let F_{k+2}^{vu} be a C^1 foliation in a neighbourhood of W^{cs}_{k+2} which is transversal to W^{cs}_{k+2} and compatible with F_0^{vu} such that $\dim \mathrm{F}_{k+2}^{vu} = \mathrm{codim}_{\mathrm{E}^{uu}(0)} \mathrm{W}^{cs}_{k+2}$. The foliation F_{k+2}^{vu} is defined on a tube τ_{k+2} along the stable manifold W^s_{k+2}. The intersection of τ_{k+2} with each leaf F of F_0^{vu} is a closed box $\mathrm{B}^\mathrm{F}_{k+2}$ bounded by a cylinder transversal to F_{k+2}^{vu} together with two closed discs $\partial_1^\mathrm{F} \cup \partial_2^\mathrm{F}$ contained in leaves of F_{k+2}^{vu}, such that any leaf of the center-stable foliation F^{cs}_{k+2}, whose dimension is equal to the dimension of W^{cs}_{k+2}, intersects transversally $\partial_1^\mathrm{F} \cup \partial_2^\mathrm{F}$. To obtain τ_{k+2}, we first define local tubes $\tau_{k+2,i}$ in a neighbourhood of each critical point $p_{k+2,i}$ between two non-critical levels $\partial^-_{k+2,i}$ and $\partial^+_{k+2,i}$. Let $\mathrm{Z}_{k+2,i}$ be a C^2 vector field (as constructed in § 1) tangent to each leaf of $\mathrm{F}^{cs}_{k+2} = \mathrm{F}^{cs}(\sigma_{k+2}(0)) \cap \mathrm{E}^{uu}(0)$, whose restriction to W^{cs}_{k+2} has a saddle-node singularity at $p_{k+2,i}$ and whose restriction to W^s_{k+2} is the gradient of π_{k+2}^{vu}. In the leaf $\partial^-_{k+2,i}$ we take a closed box $\mathrm{B}^-_{k+2,i}$ as above and positive saturate it by the flow of $\mathrm{Z}_{k+2,i}$ in the strip between $\partial^-_{k+2,i}$ and $\partial^+_{k+2,i}$. We add to this set the stable manifold $\mathrm{W}^s(\mathrm{Z}_{k+2,i})$ in order to obtain the local tube $\tau_{k+2,i}$. The local tubes $\tau_{k+2,i}$ for $i = 1, \ldots, n(j)$ are then connected along W^s_{k+2} by using the integral curves of Z_{k+2}, a C^1 extension of $\mathrm{Z}_{k+2,i}$ along the leaves of F^{cs}_{k+2} in a neighbourhood of W^s_j such that Z_j restricted to W^s_j is a Morse-Smale vector field. Therefore, by preserving the two complementary foliations F^{cs}_{k+2} and F^{vu}_{k+2}, we obtain a homeomorphism h_0^{uu} on the tube τ_{k+2}. Observe also that $\mathrm{E}^{uu}(0)$ is bounded by a cylinder $\mathrm{S}(0)$ transversal to F_0^{vu} and two closed discs, each one contained in a leaf of F_0^{vu}. Since $\mathrm{W}^s(\sigma_j)$ is transversal to F_0^{vu} at the boundary of $\mathrm{E}^{uu}(0)$, the cylinder S_0 can be foliated by one-dimensional leaves $\delta^c(0)$ compatible with the stable system and transversal to F_0^{vu}. Hence, if $\mathrm{W}^s_{k+2} \cap \partial \mathrm{E}^{uu}(0) \neq \emptyset$, we take the diffeomorphism described above also preserving the foliation δ^c_{k+2} (leaves of δ^c which are contained in $\mathrm{W}^s_{k+2} \cap \partial \mathrm{E}^{uu}(0)$). Next, suppose that $\mathrm{W}^s_{k+3} = \mathrm{W}^s(\sigma_{k+3}(0)) \cap \mathrm{E}^{uu}(0)$

is nonempty and does not intersect the boundary of $E^{uu}(0)$. If W^s_{k+3} is compact, we repeat the above argument. If not, then $W^u(\sigma_{k+2}(0)) \cap W^s(\sigma_{k+3}(0)) \neq \emptyset$ and we consider the foliation $F^{cs}_{k+2,k+3}$, consisting of the leaves of the center-stable foliation F^{cs}_{k+2} which are contained in $W^s(\sigma_{k+3}(0))$. We also take a diffeomorphism on the space of leaves of this foliation. This is possible because the intersection of $W^s(\sigma_{k+3}(0))$ with a fundamental domain of $W^s(\sigma_{k+2}(0))$ is compact. By preserving $F^{cs}_{k+2,k+3}$ and F^{vu}_{k+2}, we get a homeomorphism from $W^s_{k+3} \cap \tau_{k+2}$ to $\hat{W}^s_{k+3} \cap \tau_{k+2}$. Using the Isotopy Extension Theorem, we get a homeomorphism $\varphi_{k+3} : W^s_{k+3} \to \hat{W}^s_{k+3}$, which is a C^2 diffeomorphism on $W^s_{k+3} \backslash \tau_{k+2}$. The functions π^{vu}_{k+3} and $\hat{\pi}^{vu}_{k+3} \circ \varphi_{k+3}$ have the same critical values and coincide in $W^s_{k+3} \cap \tau_{k+2}$. Let π^t_{k+3} be the homotopy $(1-t)\,\pi^{vu}_{k+3} + t\hat{\pi}^{vu}_{k+3} \circ \varphi_{k+3}$ for $t \in [0, 1]$. Then $\pi^t_{k+3} \mid \tau_{k+2} \cap W^s_{k+3} = \pi^{vu}_{k+3}$. By defining a family of vector fields ξ^{k+3}_t on W^s_{k+3} with supp $\xi^{k+3}_t \subset W^s_{k+3} \backslash \tau_{k+2}$, such that $\dfrac{\partial \pi^t_{k+3}}{\partial t} = d\pi^t_{k+3} . \xi^{k+3}_t$, we obtain that π^t_{k+3} is topologically trivial. That is, there exists a continuous family of homeomorphisms $h^t_{k+3} : W^s_{k+3} \to W^s_{k+3}$ such that $\pi^t_{k+3} \circ h^t_{k+3} = \pi^0_{k+3} = \pi^{vu}_{k+3}$. Hence we define h^{uu}_0 restricted to W^s_{k+3} by $h^{uu}_0 = \varphi_{k+3} \circ h^1_{k+3}$. We do the same on each leaf of F^s_{k+3} contained in W^{cs}_{k+3}, to extend h^{uu}_0 to a neighbourhood of W^s_{k+3} in W^{cs}_{k+3}. Again, since W^{cs}_{k+3} is transversal to F^{vu}_0, we take a C^1 foliation F^{vu}_{k+3} in a neighbourhood of W^s_{k+2} which is transversal to W^{cs}_{k+2} and compatible with F^{vu}_0 and with F^{vu}_{k+2} such that $\dim F^{vu}_{k+3} = \operatorname{codim}_{E^{uu}(0)} W^{cs}_{k+3}$. This foliation is constructed in a tube τ_{k+3} along W^s_{k+3} exactly as in the previous step of this induction. If $W^s_{k+3} \cap \partial E^{uu}(0) \neq \emptyset$, we take the homeomorphism φ_{k+3} also preserving $\delta^c(0)$ in $W^s_{k+3} \cap \partial E^{uu}(0)$. Proceeding by induction on the ordering of the singularities, we obtain the homeomorphism h^{uu}_0 as wished. ■

Thus, we have obtained a homeomorphism on $W^{cu}_k(\mu)$, the space of leaves of F^{ss}_μ. By applying Lemma 1 and the methods described in § 1, we obtain a homeomorphism on the space of leaves of $F^{cu}_k(\mu)$. These homeomorphisms define an equivalence on a neighbourhood $V(\mu)$ of the saddle-node in M as in § 3, by imposing that the two complementary foliations F^{ss}_μ and $F^{cu}_k(\mu)$ must be preserved.

To extend this equivalence to a distinguished neighbourhood $U_{k-1}(\mu)$ of $\sigma_{k-1}(\mu)$, we connect it to $V(\mu)$ with an invariant tube $T_{k-1}(\mu)$ along the orbit of tangency. In the fence $B_{k-1}(\mu) \subset \partial U_{k-1}(\mu)$, we let $D_{k-1}(\mu)$ be the intersection of $\bigcup_{t \leqslant 0} X_{\mu,t}(V_\mu)$ with $B_{k-1}(\mu)$. We can assume, after a reparametrization of time, that $D_{k-1}(\mu)$ is contained in $X_{\mu,-T}(\Sigma_-(\mu))$ for some T. Hence, we have defined a homeomorphism on $D_{k-1}(\mu)$ which preserves the center-unstable foliation $F^{cu}(\sigma_{k-1}(\mu))$. The same arguments as for Theorem A are now applied to extend this homeomorphism to the fence $B_{k-1}(\mu)$ preserving $F^{cu}(\sigma_{k-1}(\mu))$ and the center-stable system. We define a homeomorphism on $U_{k-1}(\mu)$ by preserving level sets and trajectories. Inside the tube $T_{k-1}(\mu)$ the equivalence is a conjugacy. Analogously, we get an equivalence between X_μ and $\hat{X}_{\varphi(\mu)}$ on a distinguished neighbourhood $U_{k+2}(\mu)$ of $\sigma_{k+2}(\mu)$. Proceeding inductively and using the compatibility of the center-stable system, we construct equivalences on distinguished neighbourhood $U_j(\mu)$ of $\sigma_j(\mu)$, $j \geqslant k+3$. Finally, as in § 2, we extend the equivalence to all of M as a conjugacy outside these neighbourhoods. ∎

It remains to deal with the case where the vector field $X_{\bar{\mu}}$ presents a saddle-node $p(\bar{\mu})$ and one orbit γ of quasi-transversality between $W^u(q(\bar{\mu}))$ and $W^s(q'(\bar{\mu}))$, $q(\bar{\mu})$ and $q'(\bar{\mu})$ being hyperbolic singularities. We assume the linearizability conditions and the non-criticality condition with respect to the strong-stable and strong-unstable manifolds of $p(\bar{\mu})$, $q(\bar{\mu})$ and $q'(\bar{\mu})$ and also the generic and independent unfoldings of the saddle-node and the orbit of quasi-transversality. Similarly to the case (II.b) of § 2, since there are no criticalities, we conclude that the bifurcation set near $\bar{\mu}$ is the union of two C^1 curves $\Gamma_{QT} \cup \Gamma_{SN}$ intersecting transversally at $\bar{\mu}$, such that for $\mu \in \Gamma_{QT}$ the field X_μ presents one orbit of quasi-transversality between $W^u(q(\mu))$ and $W^s(q'(\mu))$ and for $\mu \in \Gamma_{QT}$ a saddle-node $p(\mu)$. The equivalence between X_μ and a nearby family $\hat{X}_\mu$ is obtained without much difficulty using a combination of the methods developed in (II.*b*) of § 2 and § 3. ∎

§ 5. Quasi-transversal orbit with tengency between center-unstable and stable manifolds

In this paragraph we consider a family $X_\mu \in \chi_2^g(M)$ such that for a value $\bar{\mu} \in \mathbf{R}^2$ the vector field $X_{\bar{\mu}}$ presents a bifurcation of type VI: there is an orbit of quasi-transversality between $W^u(p(\bar{\mu}))$ and $W^s(q(\bar{\mu}))$, $p(\bar{\mu})$ and $q(\bar{\mu})$ hyperbolic singularities, satisfying all the generic conditions described in Section I except (*c*.3); i.e. the center-unstable manifold $W^{cu}(p(\bar{\mu}))$ is not transversal to $W^s(q(\bar{\mu}))$. To have a codimension-two bifurcation, we assume that $W^u(p(\bar{\mu}))$ is transversal to $W^{cs}(q(\bar{\mu}))$. Since we also assume that $X_{\bar{\mu}}$ is C^m linearizable near $p(\bar{\mu})$, choosing a C^m center-unstable manifold $W^{cu}(p(\bar{\mu}))$ which is linear in these coordinates, we suppose that along the orbit of tangency γ the stable manifold $W^s(q(\bar{\mu}))$ is quasi-transversal to $W^{cu}(p(\bar{\mu}))$. Here we take

$$m \geqslant \max\left\{3, \left[\frac{\alpha_2(\bar{\mu})}{\alpha_1(\bar{\mu})}\right]+1, \left[\frac{2\alpha_1(\bar{\mu})-\alpha_2(\bar{\mu})}{\beta_1(\bar{\mu})}\right]+1\right\}.$$

Comment. — Although center-unstable manifolds are not unique, this condition does not depend on the choice of a C^m center-unstable manifold, if m is sufficiently high. In fact, if N is a $(u+1)$-dimensional invariant manifold of class C^m for m as above such that $T_{p(0)} N = E_0 \oplus T_{p(0)} W^u(p(\bar{\mu}))$, E_0 being the eigenspace corresponding to the weakest contraction, then the contact between N and $W^{cu}(p(\bar{\mu}))$ along γ is of order at least two. That is, for each point $r \in \gamma$ there is a local diffeomorphism ψ in a neighbourhood U of r in M, $j^2\psi(r) =$ 2-jet of the identity map, such that

$$\psi(U \cap N) = W^u(p(\bar{\mu})) \cap U.$$

Proof. — Let (x_1, x_I, y_1, y_L) be C^m linearizing coordinates for $X_{\bar{\mu}}$ near $p(0)$ such that $W^s(p(0)) = (x_1, x_I, 0, 0)$ and $W^u(p(0)) = (0, 0, y, y_L)$. We suppose that γ is tangent to the y_1-axis (weakest expansion). Then $N \cap \{y_1 = 1\} = \{x_1, N_I(x_1, y_L), y_L)\}$ with N_I of class C^m. Let $P : \{y_1 = 1\} \to \{x_1 = 1\}$ be the Poincaré map

$$P(x_1, x_I, y_L) = (x_1^{-\alpha_I/\alpha_1} x_I, x_1^{\beta_1/\alpha_1}, x_1^{\beta_L/\alpha_1}, y_L);$$

then $$P(N \cap \{y_1 = 1\})$$

is a C^m manifold parametrized by $(y_1^{-\alpha_I/\beta_1} N_I(y_1^{\alpha_1/\beta_1}, y_L), y_1, y_1^{\beta_L/\beta_1} y_L)$. Since each component $y_1^{-\alpha_i/\beta_1} N_j(y_1^{\alpha_1/\beta_1}, y_L)$ is of class C^m with $m \geqslant \max\left\{3, \left[\frac{2\alpha_1 - \alpha_2}{\beta_1}\right] + 1, \left[\frac{\alpha_2}{\alpha_1}\right] + 1\right\}$, and there are no resonances between the eigenvalues, we must have $dN_j(0) = 0$ and $d^2 N_j(0) = 0$. Thus, $j^2 N_I(0) = 0$, proving our statement. ■

We also suppose the generic unfolding of the orbit γ, so there is in the parameter space a curve Γ_{QT} containing $\bar{\mu}$ along which X_μ exhibits a quasi-transversality between $W^u(p(\mu))$ and $W^s(q(\mu))$. We require the tangency between $W^{cu}(p(\mu))$ and $W^s(q(\mu))$ to unfold generically, so we also get a curve Γ_0 containing $\bar{\mu}$, along which X_μ presents a quasi-transversality between $W^{cu}(p(\mu))$ and $W^s(q(\mu))$. It is easy to see that Γ_{QT} and Γ_0 are always tangent at the point $\bar{\mu}$. Therefore we require that Γ_{QT} and Γ_0 have a *quadratic contact at* $\bar{\mu}$.

(5.A) *Other generic assumptions.* — In addition, we assume that the family X_μ satisfies the following generic conditions. First, let $W_1^{cu}(p(\bar{\mu}))$ be a $(u + 2)$-dimensional center-unstable manifold, $u = \dim W^u(p(\bar{\mu}))$, which we assume linear in the linearizing coordinates. Then, $W_1^{cu}(p(\bar{\mu}))$ is transversal to $W^s(q(\bar{\mu}))$. Now, let $W^{ss}(p(\bar{\mu}))$ and $W^{vs}(p(\bar{\mu}))$ be the invariant submanifolds of $W^s(p(\bar{\mu}))$ of codimension one and two, respectively, which corresponds to the eigenspaces of strongest contractions. For any singularity $\sigma(\bar{\mu})$ of $X_{\bar{\mu}}$, we assume that $W^u(\sigma(\bar{\mu}))$ is transversal to $W^{ss}(p(\bar{\mu}))$ and to $W^{vs}(p(\bar{\mu}))$. Moreover, let F^{vs} be the codimension-two foliation in $W^s(p(\bar{\mu}))$ having $W^{vs}(p(\bar{\mu}))$ as a distinguished leaf. If $W^u(\sigma(\bar{\mu}))$ is not transversal to F^{vs} and $\dim W^u(\sigma(\bar{\mu})) \cap W^s(p(\bar{\mu})) \geqslant 2$, we require that the restriction of π^{vs} (projection along F^{vs}) to $W^u(\sigma(\bar{\mu})) \cap W^s(p(\bar{\mu}))$ has a fold singularity along one orbit. This last hypothesis is similar to the one used in § 4. If L is a leaf of the strong stable foliation and π_L^{vs} is the projection along the leaves of F^{vs} contained in L, then we assume that π_L^{vs} restricted to $W^s(\sigma(\bar{\mu})) \cap L$ is a Morse function. It is easy to show the genericity of this hypothesis and that it does not depend on the leaf L. We also require that the points of tangency between $W^u(\sigma(\bar{\mu})) \cap L$ and F^{vs} belong to distinct leaves. *For each $j < k$, we denote by $p_{ji}(\mu)$ the distinguished points of tangency between* $W^u(\sigma_j(\mu)) \cap L$ *and* F^{vs}. Since we are going to use a compatible center-unstable system, we assume that for each $\sigma(\bar{\mu}) \leqslant p(\bar{\mu})$ there is the smallest contraction and that $W^{cu}(\sigma(\bar{\mu}))$ is transversal to F^{vs} in $W^s(p(\bar{\mu}))$. We also require $W^u(\sigma'(\bar{\mu}))$ to be transversal to $W^{ss}(\sigma(\bar{\mu}))$ for all singularities $\sigma'(\bar{\mu}) \leqslant \sigma(\bar{\mu})$ and $W^{uu}(q(\bar{\mu}))$ to be transversal to $W^s(\sigma^*(\bar{\mu}))$ if $q(\bar{\mu}) \leqslant \sigma^*(\bar{\mu})$. The genericity of these conditions follows exactly as in (4.A).

(5.B) *The bifurcation set.* — Let X_μ be a family satisfying the conditions described above at a bifurcation value $\bar{\mu}$. Let $X_\mu^{cs} = X_\mu \mid W^{cs}(\sigma_{k+1}(\mu))$. We assume that $\bar{\mu} = 0$ and that $p(\mu) = \sigma_k(\mu)$ and $q(\mu) = \sigma_{k+1}(\mu)$ in the usual ordering of the singularities of X_μ. Let us take μ-dependent C^m linearizing coordinates near $\sigma_k(\mu)$, such that

$$X_\mu^{cs} = -\sum_{i=1}^{n-u} \alpha_i(\mu)\, x_i \frac{\partial}{\partial x_i} + \sum_{j=1}^{r} \beta_j(\mu)\, y_j \frac{\partial}{\partial y_j},$$

with $\quad u = \dim W^u(\sigma_k(\mu)), \qquad r = \dim[W^u(\sigma_k(\mu)) \cap W^{cs}(\sigma_{k+1}(\mu)].$

Considering a cross-section $S^{cs}(\mu) \subset \{ y_1 = 1 \}$ with coordinates (v_1, v_2, v_I, w_L), we get $W^u(\sigma_k(\mu)) \cap S^{cs}(\mu) = \{(0, \ldots, 0, w_L)\}$, $W^{cu}(\sigma_k(\mu)) = \{(v_1, 0, \ldots, 0, w_L)\}$ and $W_1^{cu}(\sigma_k(\mu)) = \{(v_1, v_2, 0, \ldots, 0, w_L)\}$. The generic assumptions of (5.A) imply that $W^s(\sigma_{k+1}(\mu)) \cap S^{cs}(\mu) = \{ v_2 = F(\mu, v_1, v_I, w_L)\}$, F being a C^m function such that $F(0) = 0$ and $F(0, v_1, 0, w_L)$ is a Morse function with critical point at the origin. The condition of generic unfolding imply that

$$\operatorname{rank} \begin{bmatrix} \dfrac{\partial F}{\partial \mu_1}(0) & \dfrac{\partial F}{\partial \mu_2}(0) \\[2ex] \dfrac{\partial^2 F}{\partial v_1\, \partial \mu_1}(0) & \dfrac{\partial^2 F}{\partial v_1\, \partial \mu_2}(0) \end{bmatrix} = 2.$$

From this we obtain the curve $\Gamma_{k,k+1}$ of quasi-transversality between $W^u(\sigma_k(\mu))$ and $W^s(\sigma_{k+1}(\mu))$ by solving the system of equations $\left\{ F(\mu, 0, 0, w_L) = 0, \dfrac{\partial F}{\partial w_L}(\mu, 0, 0\, w_L) = 0 \right\}$.
The curve of tangency Γ_0 between $W^{cu}(\sigma_k(\mu))$ and $W^s(\sigma_{k+1}(\mu))$ is given by

$$\left\{ F(\mu, v_1, 0, w_L) = 0,\ \frac{\partial F}{\partial v_1}(\mu, v_1, 0, w_L) = 0,\ \frac{\partial F}{\partial w_L}(\mu, v_1, 0, w_L) = 0 \right\}.$$

We may write $\Gamma_{k,k+1} = \{ \mu_2 = 0 \}$ and $\Gamma_0 = \left\{ \mu_2 = \frac{1}{4}\mu_1^2 \right\}$. (We are assuming that $\dfrac{\partial^2 F}{\partial v_1^2}(0)$, $\dfrac{\partial^2 F}{\partial v_1\, \partial \mu_1}(0)$ and $\dfrac{\partial F}{\partial \mu_2}(0)$ are all positive.) Although Γ_0 does not belong to the bifurcation set, it serves as a guide to obtain the other curves along which the family presents a quasi-transversality between $W^u(\sigma_j(\mu))$ and $W^s(\sigma_{k+1}(\mu))$, $j < k$. Let us assume that $W^u(\sigma_j(\mu))$ has codimension one (if not, just restrict X_μ^{cs} to $W^{cu}(\sigma_j(\mu)) \cap W^{cs}(\sigma_k(\mu))$). As is § 4 it is easy to see that if $W^u(\sigma_j(0))$ is transversal to the foliation F^{vs} in $W^s(\sigma_k(0))$, then, for μ near 0, $W^u(\sigma_j(\mu))$ is transversal to $W^s(\sigma_{k+1}(\mu))$. Hence, possible tangencies between these manifolds occur near the tangency points $p_{ji}(\mu)$ between the unstable manifolds and the foliation F^{vs}, see (5.A). We write the intersection of $W^u(\sigma_j(\mu))$ with a cross-section $S^{cs}(\mu) \subset \{ x_1 = 1 \}$ near $p_{ji}(0)$ as a graph $x_2 = G_{ji}(\mu, x_I, y_1, y_L)$ with $x_I \mapsto G_{ji}(0, x_I, 0, 0)$ being a C^m Morse function with critical point $x_I(p_{ji}(0))$. Using this expression and the generic unfolding of the quasi-transversality, we obtain as in § 1 a C^1 curve Γ_{ji} such that $\mu \in \Gamma_{ji}$ if and only if X_μ presents one orbit of quasi-transversal tangency between $W^u(\sigma_j(\mu))$ and $W^s(\sigma_{k+1}(\mu))$. This curve Γ_{ji} is tangent to Γ_0 at 0. It follows from the fact that the points of tangency $p_{j'i'}$ and p_{ji} belong to distinct leaves of F^{vs} if $(j, i) \neq (j', i')$, that $\Gamma_{ji} \cap \Gamma_{j'i'} = \emptyset$. We stress the similarity between the present bifurcation and the one treated in § 1.

In order to analyze all the secondary bifurcations simultaneously, we let $F^{vs}_{k+1}(\mu)$ be a C^m foliation in $S^{cs}(\mu)$, compatible with $W^s(\sigma_{k+1}(\mu))$ and defined by $\pi^{vs}_{k+1,\mu}(v_1, v_2, v_I, w_L) = (v_1, v_2 - F(\mu, v_1, v_I, w_L) + F(\mu, v_1, 0, w_L), w_L)$. Negative saturation of $F^{vs}_{k+1}(\mu)$ by the flow of X^{cs}_μ converges to the foliation F^{vs}. Since the tangencies between $W^u(\sigma_j(\mu))$ and $W^s(\sigma_{k+1}(\mu))$ in $S^{cs}(\mu)$ occur in the set of tangencies between $W^u(\sigma_j(\mu))$ and $F^{vs}_{k+1}(\mu)$, we associate to each distinguished point $p_{ji}(0)$ a submanifold $\widetilde{T}_{ji}(\mu)$ in $W^u(\sigma_j(\mu)) \cap S^{cs}(\mu)$ such that $\bigcup_{i=1,\ldots,n(j)} \widetilde{T}_{ji}(\mu)$ contains all those tangencies. Using the above notation, the submanifold is obtained by solving the system $\frac{\partial G_{ji}}{\partial x_I}(\mu, x_I, e^{-\beta_1 t}, e^{-\beta_L t} w_L) - \frac{\partial F}{\partial v_I}(\mu, e^{-\alpha_1 t}, e^{-\alpha_I t} x_I, w_L) \cdot e^{-(\alpha_I - \alpha_1)t} = 0$. Since the points $\{p_{ji}(0)\}$ belong to distinct leaves of F^{vs}, *the images* $T_{ji}(\mu) = \pi^{vs}_{k+1,\mu}(\widetilde{T}_{ji}(\mu))$ *are disjoint submanifolds of codimension one in* $L^c(\mu)$, the leaf space of $F^{vs}_{k+1}(\mu)$ with coordinates (v_1, v_2, w_L). All these manifolds are contained in a wedged shape region of the form $|v_2| \leqslant \delta_2 |v_1|^{\alpha_2/\alpha_1}$ in $L^c(\mu)$. The tangencies between $W^s(\sigma_{k+1}(\mu))$ and $W^u(\sigma_j(\mu))$ in $S^{cs}(\mu)$ correspond to tangencies between $W^s(\sigma_{k+1}(\mu))$ and $T_{ji}(\mu)$ in $L^c(\mu)$ for some $i \in \{1, \ldots, n(j)\}$. Proceeding as in § 1, we let F^u_μ be a C^1 foliation of codimension two in $L^c(\mu)$ which is compatible with $W^u(\sigma_k(\mu))$ and with all submanifolds $T_{ji}(\mu)$, for $j \geqslant k-1$ and $i = 1, \ldots, n(j)$. Since $W^u(\sigma_k(\mu)) \cap L^c(\mu) = \{v_1 = v_2 = 0\}$ we may also choose F^u_μ compatible with the "horizontal" foliation $v_1 =$ constant. As in § 1, we obtain a two-dimensional C^1 manifold $S^c_0(\mu)$ of class C^2 outside the origin, which is transversal to $W^u(\sigma_k(\mu)) \cap L^c(\mu)$ and to $W^s(\sigma_{k+1}(\mu)) \cap L^c(\mu)$. With this process we reduce the analysis of the bifurcation of type VI to the corresponding one for two-parameter families of gradients in a three-dimensional manifold. Thus, the bifurcation set is obtained by analyzing the following situations:

a) the point $p_k(\mu) = W^u(\sigma_k(\mu)) \cap S^c_0(\mu)$ belongs to the curve $T^s(\mu) = W^s(\sigma_{k+1}(\mu)) \cap S^c_0(\mu)$, i.e., $W^u(\sigma_k(\mu))$ is tangent to $W^s(\sigma_{k+1}(\mu))$,

b) the curves $T^u_{ji}(\bar{\mu}) = T_{ji}(\mu) \cap S^c_0(\mu)$ and $T^s(\mu)$ are tangent, i.e. the manifold $W^u(\sigma_j(\mu))$ is tangent to $W^s(\sigma_{k+1}(\mu))$.

The first situation yields the curve $\Gamma_{k,k+1}$ obtained above. In the second one we have to consider two non-equivalent cases: $\alpha_2(0) < 2\alpha_1(0)$ and $\alpha_2(0) > 2\alpha_1(0)$ as in the three-dimensional case analyzed in [22]; if $\alpha_2(0) = 2\alpha_1(0)$, the family is not stable in general. By parametrizing the curve $T^s(\mu)$ by $(v_1, F(\mu, v_1, 0, \Omega_L(\mu, v_1)))$ and each curve $T^u_{ji}(\mu)$ by $(v_1, M_{ji}(\mu, v_1))$ for $v_1 \geqslant 0$ (or for $v_1 \leqslant 0$), and letting $S^c_0 = \bigcup_\mu S^c_0(\mu)$, the hypothesis $\frac{\partial F}{\partial \mu_2}(0) \neq 0$ implies that the set $M_{ji} = \{F(\mu, v_1, \Omega_L(\mu, v_1)) = M_{ji}(\mu, v_1)\}$

is a two-dimensional submanifold of S_0^c. Hence, the bifurcation set Γ_{ji} which corresponds to tangencies between $W^s(\sigma_{k+1}(\mu))$ and $W^u(\sigma_j(\mu))$ is the image of the singular set of the map π_{ji}, restriction of the projection $\pi(\mu, v_1) = \mu$ to M_{ji}. Each Γ_{ji} is a branch of a C^1 curve which is tangent to the curve Γ_0 defined by

$$F(\mu, v_1, 0, \Omega_L(\mu, v_1, 0)) = 0 = \frac{\partial F}{\partial v_1}(\mu, v_1, 0, \Omega_L(\mu, v_1, 0))$$

and for $(j', i') \neq (j, i)$ the branches are disjoint in a neighbourhood of 0. If $\alpha_2(0) > 2\alpha_1(0)$, then all branches of Γ_{ji} are on the same side of $\Gamma_{k, k+1}$; otherwise one may find branches in both sides. See Figure XI.

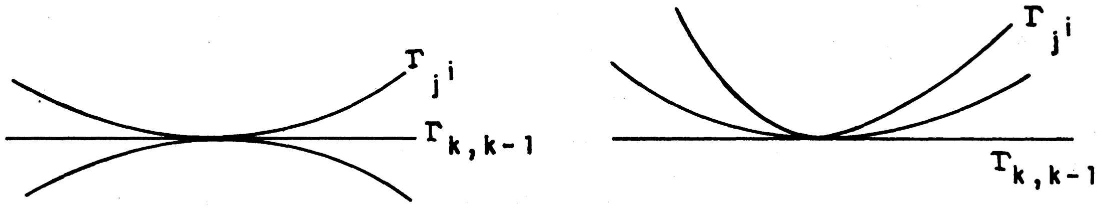

FIG. XI

(5.C) *Local stability.* — Let us construct an equivalence between X_μ and a nearby family $\hat{X}_\mu$. We take a compatible center-unstable system $F_1^{cu}(\mu), \ldots, F_{k-1}^{cu}(\mu)$ and a stable system $F_{k+1}^{cs}(\mu), F_{k+2}^s(\mu), \ldots, F_\ell^s(\mu)$ for X_μ. In the discussion of the bifurcation set, we have already observed the similarity between this case and the one in § 1. As in that case the main point to prove stability is to obtain a homeomorphism h_μ^{cs} on the cross-section $S^{cs}(\mu) = S(\mu) \cap W^{cs}(\sigma_{k+1}(\mu))$ where $S(\mu)$ is a small neighbourhood of the tangency point $p_k(0) = \gamma \cap B_k(\mu)$ in a fence $B_k(\mu)$. We now describe this homeomorphism, beginning with a reparametrization φ together with a homeomorphism $h_\mu^c : S_0^c(\mu) \to \hat{S}_0^c(\varphi(\mu))$, $S_0^c(\mu)$ as defined in (5.B) above. Each curve $T_{ji}^u(\mu) = (v_1, M_{ji}(\mu, v_1))$ obtained at the end of (5.B) is a leaf of a singular foliation defined by a one-form on $S_0^c(\mu)$, $w_{ji}(\mu) = -\alpha_1(\mu)\, v_1\, dv_2 + [\alpha_2(\mu)\, v_2 + o(v_1^{1+\varepsilon})]\, dv_1$. Using a partition of unity, we may define a C^1 one-form $w(\mu)$ such that restricted to a sector of the form $| v_2 - M_{ji}(\mu, v_1)| < \delta\, | v_1 |^{\alpha_2/\alpha_1}$ it coincides with $w_{ji}(\mu)$; outside the origin, w may be taken of class C^2. We can also take $w(\mu) = -\alpha_1(\mu)\, v_1\, dv_2 + \alpha_2(\mu)\, v_2\, dv_1$ for $\delta\, | v_1 |^{\alpha_2/\alpha_1} \leqslant | v_2 |$ and assume that the curve $T_0(\mu) = W^{cu}(\sigma_k(\mu)) \cap S_0^c(\mu)$ is a leaf of $w(\mu) = 0$. The set of tangency points between the curve $T^s(\mu) = W^s(\sigma_{k+1}(\mu)) \cap S_0^c(\mu)$ and the leaves of $w(\mu) = 0$ is described by an equation

$$-\alpha_1(\mu)\, v_1 \frac{\partial F}{\partial v_1}(\mu, v_1) + \alpha_2(\mu)\, F(\mu, v_1) + \Gamma(\mu, v_1, F(\mu, v_1)),$$

where F is as in (5.B), Γ is C^1 and $T^s(\mu) = \operatorname{graph} F(\mu, \cdot)$. Since $\frac{\partial F}{\partial \mu_2}(0) \neq 0$, this is the graph G of a C^1 function $\mu_2 = \mu_2(\mu_1, v_1)$ (which is even C^2 outside the origin), whose projection $\pi : G \to \mathbf{R}^2$ has a fold singularity along a C^1 curve τ. Thus, there exists a homeomorphism $\sigma : G \to G$ with fixed point set τ, such that $\pi(\mu_1, v_1) = \pi(\mu_1, \bar{v}_1)$ if and only if $\bar{v}_1 = \sigma(\mu_1, v_1)$. The image of τ by π is a C^1 curve tangent to $\Gamma_{k,k+1}$ at the origin. Moreover, if $M_0 = \{ \widetilde{F}(\mu, v_1) = 0 \}$ represents the intersection of the curve $T^s(\mu)$ with $T_0(\mu)$, then τ is transversal to $M_0 \cap T$ at the origin. Since all curves $T^u_{ji}(\mu)$ are tangent to $T_0(\mu)$ at the origin, the curves $\Lambda_{ji} = M_{ji} \cap G$ are also transversal to τ at the origin, where $M_{ji} = \{ F(\mu, v_1) - M_{ji}(\mu, v_1) = 0 \} = T^s(\mu) \cap T^u_{ji}(\mu)$. Let Λ^c be the foliation in G defined by the pull-back of ω by the map

$$(\mu_1, v_1) \mapsto (\mu_1, \mu_2(\mu_1, v_1), v_1, F(\mu_1, \mu_2, v_1)).$$

The leaves of Λ^c, except two of them, are tangent to $\Lambda^c_0 = M_0 \cap G$. We can take a homeomorphism from the space of leaves of Λ^c to the space of leaves of $\widehat{\Lambda}^c$, sending Λ_{ji} to $\widehat{\Lambda}_{ji}$ and Λ^c_0 to $\widehat{\Lambda}^c_0$. Let $\Delta \subset G$ be a closed conic region, with vertex at the origin, which contains all the distinguished leaves Λ_{ji} and Λ^c_0, intersects τ at the origin and whose boundary is transversal to Λ^c. This region Δ is taken so that $\tau \cap \Delta \neq \emptyset$ and also $\pi^{-1}(\Gamma_{k,k-1}) \cap \Delta = \emptyset$. We let $\varphi(\mu_1, \mu_2) = (\varphi_1(\mu_1), \varphi_2(\mu_1, \mu_2))$ be a reparametrization that sends the π-image of the curves Λ^c contained in Δ to the $\hat{\pi}$-image of $\widehat{\Lambda}^c$, $\pi(\tau)$ to $\hat{\pi}(\hat{\tau})$ and $\Gamma_{k,k+1}$ to $\widehat{\Gamma}_{k,k+1}$. This induces a homeomorphism $\xi : G \to \widehat{G}$ by sending Δ to $\widehat{\Delta}$, preserving the foliations Λ^c, $\widehat{\Lambda}^c$ and $\mu_1 =$ constant, and, by using the involutions σ, $\hat{\sigma}$, in such way that $\varphi \circ \hat{\pi} = \pi \circ \xi$. By preserving the surfaces M_c that represent the intersection of $T^c(\mu)$ with $T^s(\mu)$, we already have a continuous family of homeomorphisms $v_1 \mapsto \eta_\mu(v_1)$ in the set $| F(\mu, v_1) | \leqslant \delta \, | v_1 |$. They are extended continuously outside this region by performing an extension on each fiber $\mu =$ constant. This gives a homeomorphism on the space of leaves of the foliation $dv_1 = 0$ in $S^c_0(\mu)$. The homeomorphism $h^c_\mu : S^c_0(\mu) \to \widehat{S}^c_0(\varphi(\mu))$ in the conic region $| v_2 | \leqslant \delta_3 \, | v_1 |$ preserves the foliations $dv_1 = 0$ and $T^c(\mu)$. Also h^c_μ automatically sends $T^s(\mu)$ to $\widehat{T}^s(\varphi(\mu))$. We extend h^c_μ arbitrarily outside the conic region but preserving $T^s(\mu)$.

We now extend h^c_μ to the tangency submanifold $L^c(\mu) \subset S^{cs}(\mu)$ (see (5.B)). This is analogous to the construction used in § 1; the difference, due to the tangency between $W^s(\sigma_{k+1}(\mu))$ and $W^{cu}(\sigma_k(\mu))$, is that we need a new process to define a two-dimensional foliation $(SN)_\mu$: like in (1.C), this foliation has a saddle-node singularity along the curve $T^s(\mu)$. Since we preserve the foliation given by $dv_1 = 0$, we may define $(SN)_\mu$ using once more a family of vector fields Y_{μ, v_1}, now also parametrized by v_1, which is compatible with $W^s(\sigma_{k+1}(\mu))$. For fixed (μ, v_1), Y_{μ, v_1} presents a unique singularity of saddle-node type at $\{ T^s(\mu) \cap (v_1 = \text{constant}) \}$. Outside this point, the trajectories of Y_{μ, v_1} are transversal to F^u_μ (5.B). Hence, by applying a parametrized version of Lemma 1, we obtain a homeomorphism from $L^c(\mu)$ to $\widehat{L}^c(\varphi(\mu))$. We can now define a homeomorphism on a neighbourhood $S(\mu)$ of the tangency point $p_k(0) = \gamma \cap B_{k-1}(\mu)$ in the fence $B_{k-1}(\mu)$. Since $W^{cs}(\sigma_{k+1}(\mu))$ is transversal to $W^u(\sigma_k(\mu))$, the cone-like method of Theorem A is applied to obtain a homeomorphism on

$$S^{cu}(\mu) = S(\mu) \cap W^{cu}(\sigma_k(\mu))$$

which preserves the center-stable foliation $F^{vs}(\sigma_{k+1}(\mu))$. Hence, as in § 1, it is enough to construct a homeomorphism h^{cs}_μ on $S^{cs}(\mu) = S(\mu) \cap W^{cs}(\sigma_{k+1}(\mu))$. We already have a homeomorphism on the space of leaves of $F^{vs}_{k+1}(\mu)$, a foliation of dimension $(s-2)$, $s = \dim W^s(\sigma_k(\mu))$, which is compatible with $W^s(\sigma_{k+1}(\mu))$. It remains to construct a suitable center-unstable foliation $F^{cu}(\sigma_k(\mu))$ and to adapt the cone-like construction to this case. (We recall that in the previous applications of this method (Theorem A) the foliation F^{ss}_μ, dual to $F^{cu}(\sigma_k(\mu))$ in $S^{cs}(\mu)$, had dimension equal to $(s-1)$.) We describe the $(u+1)$-dimensional leaves of type $F^{cu}_1(\sigma_k(\mu))$ whose space of leaves correspond to closed discs $D^s_\pm(\mu)$ in the fundamental domain $A^s_k(\mu) = A_k(\mu) \cap W^s(\sigma_k(\mu))$, $A_k(\mu)$ being a fence in a level set. Let $\Omega(\mu) = (\pi^{vs}_{k+1,\mu})^{-1}(L^c(\mu))$, where $\pi^{vs}_{k+1,\mu}$ is the projection along $F^{vs}_{k+1}(\mu)$, and consider $P^{cs}_\mu : \Omega(\mu) \to A_k(\mu)$ the restriction of the Poincaré map to $\Omega(\mu)$. Using the linearizing coordinates and the fact that F^{vs} is of class C^m, it is easy to see that the image of this foliation is a codimension-one foliation in $P^{cs}_\mu(\Omega(\mu))$, which extends continuously to the strong stable foliation $F^{vs}_k(\mu)$ in the discs $D^s_\pm(\mu)$. We raise over each point of $D^s_\pm(\mu)$ a one-dimensional foliation $F_1(\mu)$ in $P^{cs}_\mu(\Omega(\mu))$, which is compatible with the center-unstable system and also has its inverse image by P^{cs}_μ compatible with $(\pi^{vs}_{k+1,\mu})^{-1}(T^c(\mu))$. We then raise over $F_1(\mu)$ a u-dimensional continuous foliation $F^u_1(\mu)$ also compatible with the stable system and transversal to $D^s_\pm(\mu)$. We define $F^{cu}_1(\sigma_k(\mu))$ by taking the positive saturate of $F^u_1(\mu)$ by the flow of X_μ. The $(u+2)$-dimensional leaves of type $F^{cu}_2(\sigma_k(\mu))$ are obtained as in Theorem A, its leaf space is a sphere $A^{ss}_k(\mu) = W^{ss}(\sigma_k(\mu)) \cap A_k(\mu)$. However, to avoid tangencies between $F^{vs}_{k+1}(\mu)$ and $F^{cu}_2(\sigma_k(\mu))$ in $S^{cs}(\mu)$ we go one step further and distinguish a new type of leaves, denoted by $F^{cu}_3(\sigma_k(\mu))$, which are $(u+3)$-dimensional. Let $C^{vs}_k(\mu)$ be a small tubular neighbourhood of the sphere $A^{vs}_k(\mu) = W^{vs}(\sigma_k(\mu)) \cap A_k(\mu)$ in $A^{ss}_k(\mu)$, which is bounded by two leaves of $F^{vs}_k(\mu) \cap A^{ss}_k(\mu)$. Using the transversality between $W^u(\sigma_j(\mu))$ and $W^{vs}(\sigma_k(\mu))$, we can construct a one-dimensional foliation $F^c_1(\mu)$ on $C^{vs}_k(\mu)$ which is compatible with the center-unstable system. We let $F^{cu}_3(\sigma_k(\mu))$ be the foliation whose leaves are of the form $\bigcup_{x \in \ell^c_1(\mu)} F^{cu}_{2,x}(\sigma_k(\mu))$, where $\ell^c_1(\mu)$ is the leaf of $F^c_1(\mu)$ containing $x \in A^{vs}_k(\mu)$. We construct homeomorphisms on the space of leaves of $F^{cu}_2(\sigma_k(\mu))$ and of $F^{cu}_3(\sigma_k(\mu))$ and apply Lemma 2 to get a homeomorphism on the space of leaves of $F^{cu}_1(\sigma_k(\mu))$. In this case we need Lemma 2 in order to preserve $F^{vs}_k(\mu)$. With these homeomorphisms together with the foliation $F^{vs}_{k+1}(\mu)$ we obtain h^{cs}_μ as follows. We divide $S^{cs}(\mu)$ into three conic regions:

$$A(\mu) = \{ v_1^2 \geqslant \delta[v_2^2 + |v_I|^2]\}, \quad B(\mu) = \{ v_1^2 \leqslant \delta[v_2^2 + |v_I|^2]\} \cap \{ v_2^2 \geqslant \delta |v_I|^2 \}$$

and $$C(\mu) = \{ v_1^2 \leqslant \delta[v_2^2 + |v_I|^2]\} \cap \{ \delta |v_I|^2 \geqslant |v_2|^2 \},$$

with $\delta > 0$ small. On $A(\mu)$ we preserve $F^{cu}_1(\sigma_k(\mu))$ and $F^{vs}_{k+1}(\mu)$; in each leaf of $(\pi^{vs}_{k+1,\mu})^{-1}(T^c(\mu))$ these foliations are complementary. On $B(\mu)$ it is defined by preserving the complementary foliations $F^{cu}_2(\sigma_k(\mu))$ and F^{vs}_{k+1}. On $C(\mu)$ we preserve $F^{cu}_3(\sigma_k(\mu))$ and $F^{vs}_{k+1}(\mu)$. Let $F^{cu}_{3,d}(\sigma_k(\mu))$ be a leaf of $F^{cu}_3(\sigma_k(\mu))$. The intersection of $F^{cu}_{3,d}(\sigma_k(\mu))$ with $\partial C(\mu)$ projects homeomorphically, via $\pi^{vs}_{k+2}(\mu)$, to $L^c(\mu)$. Hence, it defines a homeomorphism on the leaf space of the leaves of type $F_1^{cu}(\sigma_k(\mu))$ which are contained in $F_{3,d}^{cu}(\sigma_k(\mu))$. So, h_μ^{cs} restricted to $F_{3,d}^{cu}(\sigma_k(\mu))\setminus \mathrm{Int}(C(\mu))$ preserves $F_1^{cu}(\sigma_k(\mu))$ and $F_{k+1}^{vu}(\mu)$. We then extend it to $\mathrm{Int}(C(\mu)) \cap F_{3,d}^{cu}(\sigma_k(\mu))$ arbitrarily but preserving $F_{k+1}^{vs}(\mu)$. The definition of h_μ^{cs} is now complete. As observed above, this is enough to obtain a homeomorphism on $S(\mu)$. We extend this homeomorphism to all of $B_k(\mu)$ preserving the center-stable system and $F^{cu}(\sigma_k(\mu))$. By reasoning as in Theorem A we obtain an equivalence on a neighbourhood of the closure of the orbit of tangency γ preserving level sets of f_μ. Proceeding by induction, we construct the equivalence on distinguished neighbourhoods of the singularities, $i \geqslant 1$, also preserving level sets of f_μ, as it is done at the end of § 4. We conclude the result by extending these equivalences to all of M as in Theorem A. ∎

§ 6. Orbit of tangency of codimension two

(6.A) *Generic assumptions.* — We consider in this paragraph a family X_μ, such that for a value $\bar{\mu}$ there is a unique orbit γ contained in the intersection of an unstable manifold $W^u(p(\bar{\mu}))$ and a stable manifold $W^s(q(\bar{\mu}))$ of two hyperbolic singularities of $X_{\bar{\mu}}$ such that $\dim[T_r W^u(p(\bar{\mu})) + T_r W^s(q(\bar{\mu}))] = n - 2$ for $r \in \gamma$ ($\dim M = n$). We assume X_μ to be C^m linearizable, $m \geqslant 3$, near $p(\bar{\mu})$ and $q(\bar{\mu})$ and that the eigenvalues of the linear part of X_μ have multiplicity one at these points. We also assume that for any hyperbolic singularity $\sigma(\bar{\mu}) \neq p(\bar{\mu})$ the unstable manifold $W^u(\sigma(\bar{\mu}))$ is transversal to $W^{ss}(p(\bar{\mu}))$ and to $W^{vs}(p(\bar{\mu}))$, the strong-stable manifolds of $p(\bar{\mu})$ of codimension one and two. We suppose that $W^u(\sigma(\bar{\mu}))$ has at most a quadratic contact with the very strong-stable foliation $F_L^{vs}(p(\bar{\mu}))$ in a leaf L of the strong stable foliation $F^{ss}(p(\bar{\mu}))$ (see § 5). Dually, we require transversality between $W^s(\sigma'(\mu))$ and $W^{uu}(q(\bar{\mu}))$ and between $W^s(\sigma'(\bar{\mu}))$ and $W^{vu}(q(\bar{\mu}))$ and quadratic contact between $W^s(\sigma'(\bar{\mu}))$ and $F_{L'}^{vu}(q(\bar{\mu}))$, the very strong-unstable foliation, in a leaf L' of the strong-unstable foliation $F^{uu}(q(\bar{\mu}))$. Let $W^{cu}(p(\bar{\mu}))$ and $W_1^{cu}(p(\bar{\mu}))$ be $(u+1)$- and $(u+2)$-dimensional C^m center-unstable manifolds of $p(\bar{\mu})$ ($u = \dim W^u(p(\bar{\mu}))$); then, $W^s(q(\bar{\mu}))$ is transversal to $W_1^{cu}(p(\bar{\mu}))$ and dually $W^u(p(\bar{\mu}))$ is transversal to $W_1^{cs}(q(\bar{\mu}))$, a C^m center-stable manifold of dimension $(s+2)$. We also suppose that $W^{cu}(p(\bar{\mu}))$ is transversal to $W^{cs}(q(\bar{\mu}))$ and the generic unfolding of the orbit of tangency γ. This means that if $\sigma^s, \sigma^u : \mathbf{R}^2 \to M$ are immersions with $\sigma^s(\bar{\mu}) = \sigma^u(\bar{\mu}) = r \in \gamma$ and $\sigma^s(\mu) \subset W^s(q(\mu))$, $\sigma^u(\mu) \subset W^u(p(\mu))$, then the restriction of the projection $T_r M \to T_r M / T_r W^u(p(\bar{\mu})) + T_r W^s(q(\bar{\mu}))$ to $\mathrm{Im}[d\sigma^s(\bar{\mu}) - d\sigma^u(\bar{\mu})]$ is an isomorphism. Actually, a generic family X_μ presents an orbit of tangency of codimension two when there is *lack of dimensions*, that is $u + s = n - 1$.

Proposition **5**. — *There is an open and dense subset $\mathscr{G}' \subset \chi_2^g(M)$ such that if $X_\mu \in \mathscr{G}'$ and for some value $\bar{\mu}$ the vector field presents an orbit of tangency $\gamma \subset W^u(p(\bar{\mu})) \cap W^s(q(\bar{\mu}))$ with*

$$\dim[T_r W^u(p(\bar{\mu})) + T_r W^s(q(\bar{\mu}))] = n - 2 \quad \textit{for } r \in \gamma,$$

then

$$\dim W^u(p(\bar{\mu})) + \dim W^s(q(\bar{\mu})) = n - 1.$$

Proof. — Let $u = \dim W^u(p(\bar\mu))$ and $s = \dim W^s(q(\bar\mu))$. We take μ-dependent C^m coordinates $(x_1, \ldots, x_{n-u}, y_1, \ldots, y_{u-1}, z)$ $(m \geq 3)$ in a neighbourhood U of r in M such that

$$X_\mu \mid U = \frac{\partial}{\partial z} \quad \text{and} \quad W^u(p(\mu)) \cap U = \{ x_1 = \ldots = x_{n-u} = 0 \},$$

$W^s(q(\mu)) \cap U = \{ x_1 = F_1(\mu, y_1, \ldots, y_\ell), x_2 = F_2(\mu, y_1, \ldots, y_\ell), y_{\ell+1} = \ldots = y_{u-1} = 0 \}$. We are assuming $\ell \geq 1$, where $\ell = \dim[T_r W^u(p(\bar\mu)) \cap T_r W^s(q(\bar\mu))] - 1$. Hence, we can associate to X_μ a two-parameter family of C^m maps $F : \mathbf{R}^2 \times \mathbf{R}^\ell \to \mathbf{R}^2$, $F(\mu, y_1, \ldots, y_\ell) = (F_1(\mu, y_1, \ldots, y_\ell), F_2(\mu, y_1, \ldots, y_\ell))$. If $j^1_2 F(\mu, \bar y)$ denotes the one-jet with respect to the variables $(y_1, \ldots, y_\ell) = \bar y$, then $\dim[T_r W^u(p(\bar\mu)) + T_r W^s(q(\bar\mu))] = n - 2$ is equivalent to $j^1_2 F(\bar\mu, 0) = (0, 0) \in \mathbf{R}^2 \times L(\mathbf{R}^\ell, \mathbf{R}^2) \approx J^1(\mathbf{R}^\ell, \mathbf{R}^2)_{(\bar\mu, 0)}$. But, since $\ell \geq 1$, we have $\dim(\mathbf{R}^2 \times \mathbf{R}^\ell) = 2 + \ell < 2 + 2\ell = \operatorname{codim}_{J^1(\mathbf{R}^\ell, \mathbf{R}^2)}(0, 0)$ and the transversality theorem implies that $(0, 0)$ is generically avoided. That is, with a small perturbation of F we get $j^1_2 F(\bar\mu, 0) \neq (0, 0)$. This proves the proposition. ■

(6.B) *The bifurcation set.* — Assume that $p(\mu) = \sigma_k(\mu)$ and $q(\mu) = \sigma_{k+1}(\mu)$ in the usual ordering of the singularities of X_μ. Let us describe the bifurcation set associated to tangencies between $W^u(\sigma_j(\mu))$ and $W^s(\sigma_{k+1}(\mu))$ for $j \leq k - 1$ and between $W^s(\sigma_{j'}(\mu))$ and $W^u(\sigma_k(\mu))$ for $j' \geq k + 2$. It is easy to see, as in § 5, that these tangencies correspond to criticalities of $W^u(\sigma_j(\mu))$ (resp. $W^s(\sigma_{j'}(\mu))$) with respect to $F^{vs}(\sigma_k(\mu))$ (resp. $F^{vu}(\sigma_{k+1}(\mu))$). Let $W^{cs}_1(\sigma_{k+1}(\mu))$ be a $(s + 2)$-dimensional center-stable manifold of $\sigma_{k+1}(\mu)$ extended as in § 1 to a neighbourhood of $\sigma_k(\mu)$, and consider the restriction $X^{cs}_\mu = X_\mu \mid W^{cs}_1(\sigma_{k+1}(\mu))$. Assume that there are C^m linearizing coordinates (x_1, x_2, x_I, y_1) for $W^{cs}_1(\sigma_{k+1}(\mu))$ near $\sigma_k(\mu)$ such that

$$X^{cs}_\mu = - \sum_{i=1}^{n-u} \alpha_i(\mu)\, x_i \frac{\partial}{\partial x_i} + \beta_1(\mu)\, y_1 \frac{\partial}{\partial y_1}$$
$$(u = \dim W^u(\sigma_k(\mu)),\ 0 < \alpha_i(\mu) < \alpha_{i+1}(\mu),\ \beta_1(\mu) > 0).$$

In a cross-section Σ^{cs}_μ contained in $\{y_1 = 1\}$ and with coordinates (v_1, v_2, v_I) such that $W^u(\sigma_k(\mu)) \cap \Sigma^{cs}_\mu = \{(0, 0, 0)\}$, $W^{cu}_1(\sigma_k(\mu)) \cap \Sigma^{cs}_\mu = \{(v_1, v_2, 0)\}$, we have

$$W^s(\sigma_{k+1}(\mu)) \cap \Sigma^{cs}_\mu = \{ v_1 = F^1(\mu, v_I), v_2 = F^2(\mu, v_I)\}$$

with $F^1(\bar\mu, 0) = F^2(\bar\mu, 0) = 0$. The hypothesis of generic unfolding of the family X_μ implies that the map $\mu \mapsto (F^1(\mu, 0), F^2(\mu, 0))$ is a local diffeomorphism near $\bar\mu$ and, hence, after a change of coordinates in the parameter space, we may assume $\bar\mu = 0$, $F^1(\mu, 0) = \mu_1$ and $F^2(\mu, 0) = \mu_2$. To " reduce dimensions ", we consider the C^m foliation $F^{vs}_{k+1}(\mu)$ in Σ^{cs}_μ whose main leaf is $W^s(\sigma_{k+1}(\mu)) \cap \Sigma^{cs}_\mu$, and which is defined by $\pi^{vs}_{k+1}(\mu, v_1, v_2, v_I) = [v_1 - F^1(\mu, v_I) + F^1(\mu, 0), v_2 - F^2(\mu, v_I) + F^2(\mu, 0)]$. Let $\bigcup_{1 \leq i \leq n(j)} \lambda^u_{ji}(\mu)$ be the image in $W^{cu}_1(\sigma_k(\mu)) \cap \Sigma^{cs}_\mu$ of the set of points of tangency between $F^{vs}_{k+1}(\mu)$ and $W^u(\sigma_j(\mu))$. Each $\lambda^u_{ji}(\mu)$ is a branch of a C^1 curve tangent to the v_1-axis and corresponds to the distinguished point $p_{ji}(\mu)$ in the cross sections

$S^{cs}_{\pm}(\mu) \subset \{ | x_1 | = 1 \}$. As in § 5, $p_{ji}(\mu)$ is a point of tangency between $W^u(\sigma_j(\mu))$ and $F^{vs}(\mu)$ in $W^u(\sigma_j(k)) \cap W^s(\sigma_k(\mu))$. Since these critical points belong to distinct leaves of $F^{vs}(\mu)$, we obtain $\lambda^u_{ji}(\mu) \cap \lambda^u_{j'i'}(\mu) = \emptyset$ for $(i, j) \neq (i', j')$ and for μ in a neighbourhood of $(0, 0)$. These curves are contained in the region $\Delta_1(\mu) = \{ | v_2 | \leqslant \delta | v_1 |^{\alpha_2/\alpha_1} \}$ and are tangent to a vector field Z^k_μ which has a hyperbolic singularity at the origin and is equal to $-\alpha_1(\mu)\, v_1 \dfrac{\partial}{\partial v_1} - \alpha_2(\mu)\, v_2 \dfrac{\partial}{\partial v_2}$ outside $\Delta_1(\mu)$. It is clear from this construction that an orbit of quasi-transversality between $W^u(\sigma_j(\mu))$ and $W^s(\sigma_{k+1}(\mu))$ will occur in Σ^{cs}_μ if and only if the point $r_{k+1}(\mu) = W^s(\sigma_{k+1}(\mu)) \cap W^{cu}_1(\sigma_k(\mu)) \cap \Sigma^{cs}_\mu$ belongs to the curve $\lambda^u_{ji}(\mu)$, for some $1 \leqslant i \leqslant n(j)$.

We now apply the same reasoning to $X^{cu}_\mu = X_\mu \mid W^{cu}_1(\sigma_k(\mu))$, by taking a codimension-two foliation $F^{vu}_k(\mu)$ in a cross-section Σ^{cu}_μ, near the singularity $\sigma_{k+1}(\mu)$, having $W^u(\sigma_k(\mu)) \cap \Sigma^{cu}_\mu$ as a distinguished leaf. In this way we get a vector field $\widetilde{Z}^{k+1}_\mu$ on $\Sigma^{cu}_\mu \cap W^{cs}_1(\sigma_{k+1}(\mu))$ with distinguished trajectories $\lambda^s_{ji}(\mu)$, $j \geqslant k+2$, such that $W^u(\sigma_k(\mu))$ is quasi-transversal to $W^s(\sigma_j(\mu))$ in Σ^{cu}_μ if and only if

$$r_k(\mu) = W^u(\sigma_k(\mu)) \cap \Sigma^{cu}_\mu \cap W^{cs}_1(\sigma_{k+1}(\mu)) \text{ belongs to } \lambda^s_{ji}(\mu) \quad \text{for some } 1 \leqslant i \leqslant n(j).$$

Therefore, by taking $X^c_\mu = X_\mu \mid X^{cu}_1(\sigma_k(\mu)) \cap W^{cs}_1(\sigma_{k+1}(\mu))$, we are reduced to consider the three-dimensional case with the corresponding singular foliations in the cross sections $\Sigma^c_1(\mu) = \Sigma^{cs}(\mu) \cap W^{cu}_1(\sigma_k(\mu))$ and $\Sigma^c_2(\mu) = \Sigma^{cu}(\mu) \cap W^{cs}_1(\sigma_{k+1}(\mu))$. Let $P^c_\mu : \Sigma^c_1(\mu) \to \Sigma^c_2(\mu)$ be the Poincaré map, $P^c_\mu(v_1, v_2) = (P^c_1(\mu, v_1, v_2), P^c_2(\mu, v_1, v_2))$, and consider the induced field $(P^c_\mu)^{-1}_* \widetilde{Z}^{k+1}_\mu = Z^{k+1}_\mu$. Since the integral curves of Z^{k+1}_μ (except for two of them) are tangent to $W^{cs}(\sigma_{k+1}(\mu)) \cap \Sigma^{cu}_\mu$, using the transversality between $W^{cu}(\sigma_k(\mu))$ and $W^{cs}(\sigma_{k+1}(\mu))$ and restricting Z^k_μ and $\widetilde{Z}^{k+1}_\mu$ to the regions

$$A_1(\mu) = \{ | v_2 | \leqslant \delta | v_1 |, | v_1 | \leqslant \varepsilon \}$$

and

$$A_2(\mu) = \{ | P^c_2(\mu, v_1, v_2)| \leqslant \delta | P^c_1(\mu, v_1, v_2)|, | P^c_1(\mu, v_1, v_2)| \leqslant \varepsilon \}$$

for $0 < \delta < 1$ small, the trajectories of Z^k_μ and of $\widetilde{Z}^{k+1}_\mu$ are transversal to each other for μ close to 0. In the parameter space we obtain the corresponding regions

$$B_1 = \{ | F_2(\mu, 0)| \leqslant \delta | F_1(\mu, 0)|, | F_1(\mu, 0)| \leqslant \varepsilon \}$$

and

$$B_2 = \{ | P^c_2(\mu, 0)| \leqslant \delta | P^c_1(\mu, 0)|, | P^c_1(\mu, 0)| \leqslant \varepsilon \},$$

which contain the bifurcation set.

Note that $B_1 \cap B_2 = \{(0, 0)\}$ and the bifurcations are characterized by the fact that $(0, 0)$ belongs to the integral curve $\lambda^s_{ji}(\mu)$ of Z^{k+1}_μ, or $r_{k+1}(\mu)$ belongs to the integral curve $\lambda^u_{k\ell}(\mu)$ of Z^k_μ. Since the map $\mu \mapsto r_{k+1}(\mu)$ is a local diffeomorphism, we get a finite number of integral curves $\Gamma^u_{k\ell}$ of $(r^{-1}_{k+1})_* Z^k_\mu$ in the region B_1. Similarly, we obtain finitely many trajectories Γ^s_{ji} of $(P^c_*)^{-1} \widetilde{Z}^{k+1}_\mu$ in the region B_2. Thus, the bifurcation set is as in the picture.

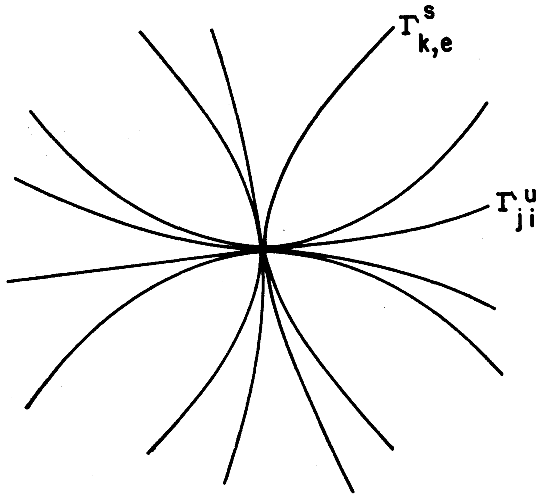

FIG. XII

(6.C) *Stability.* — The construction of an equivalence between X_μ and a nearby family $\hat{X}_\mu$ is similar to the one in § 5 and we will describe only its main steps. The important point is to obtain a homeomorphism on a cross-section Σ contained in a distinguished neighbourhood of the orbits of tangency γ and $\hat{\gamma}$. For that, we first obtain a homeomorphism $h^c_\mu : \Sigma^c_1(\mu) \to \hat{\Sigma}^c_1(\varphi(\mu))$ between cross-sections in the center manifold $W^{cu}_1(\sigma_k(\mu)) \cap W^{cs}_1(\sigma_{k+1}(\mu))$ and $W^{cu}_1(\hat{\sigma}_k(\varphi(\mu)) \cap W^{cs}_1(\hat{\sigma}_{k+1}(\varphi(\mu)))$. We start by taking a homeomorphism from $\partial A_1(0)$ to $\partial\hat{A}_1(0)$ which sends $\lambda^u_{ji}(0) \cap \partial A_1(0)$ to $\hat{\lambda}^u_{ji}(0) \cap \partial\hat{A}_1(0)$; this induces (via r_{k+1} defined above) a homeomorphism from ∂B_1 to $\partial\hat{B}_1$. We also consider a homeomorphism from $\partial A_2(0)$ to $\partial\hat{A}_2(0)$ sending $\lambda^s_{ji}(0) \cap \partial A_2(0)$ to $\hat{\lambda}^s_{ji}(0) \cap \partial\hat{A}_2(0)$, which induces a homeomorphism from ∂B_2 to $\partial\hat{B}_2$ (via P^c defined above).

Let $\varphi : (u, 0) \mapsto (\varphi(u), 0)$ be a homeomorphism sending B_i to $\hat{B}_i$, $i = 1, 2$, that extends the above homeomorphisms and preserves the trajectories of the fields $(r^{-1}_{k+1})_* Z^k_\mu$ and $(P^c_*)^{-1} \tilde{Z}^{k+1}_\mu$. The homeomorphism $h^c_\mu : \Sigma^c(\mu) \to \hat{\Sigma}^c(\varphi(\mu))$ is defined in such way that it sends trajectories of Z^k_μ to trajectories of $\hat{Z}^k_{\varphi(\mu)}$ inside $A_1(\mu)$ and $\hat{A}_1(\varphi(\mu))$, respectively, and trajectories of Z^{k+1}_μ to trajectories of $\hat{Z}^{k+1}_{\varphi(\mu)}$ inside $B_1(\mu)$ and $\hat{B}_1(\varphi(\mu))$. We choose h^c_μ to send $\lambda^s_{ji}(\mu)$ to $\hat{\lambda}^s_{ji}(\varphi(\mu))$ and $\lambda^u_{ji}(\mu)$ to $\hat{\lambda}^u_{ji}(\varphi(\mu))$. Since $\Sigma^c(\mu)$ is the space of leaves of $F^{vs}_{k+1}(\mu)$, in order to define a homeomorphism $h^{cs}_\mu : \Sigma^{cs}(\mu) \to \hat{\Sigma}^{cs}(\varphi(\mu))$ it is enough to define a center-unstable foliation $F^{cu}(\sigma_k(\mu))$ as in § 5 and proceed exactly like in that case. It is important to observe that, by construction, the pull-back of the trajectories of Z^k_μ via the projection $\pi^{vs}_{k+1}(\mu)$ gives a codimension-one foliation, singular along $[\pi^{vs}_{k+1}(\mu)]^{-1} [W^u(\sigma_k(\mu)) \cap \Sigma^c(\mu)]$, such that the foliations in the cross-sections $S^{cs}_\pm(\mu) \subset \{ | x_1 | = 1 \}$ induced by the Poincaré map, extends continuously to the very strong-stable foliation $F^{vs}(\sigma_k(\mu))$. We apply Lemma 2 again, to obtain a homeomorphism in the space of leaves of the center-unstable foliation. The same procedure also works to define a homeomorphism $h^{cu}_\mu : \Sigma^{cu}(\mu) \to \hat{\Sigma}^{cu}(\varphi(\mu))$ preserving the foliation $F^{vu}_k(\mu)$

and a center-stable foliation $F^{cs}(\sigma_{k+1}(\mu))$ as above. Finally, to get a homeomorphism h_μ from $\Sigma(\mu)$ to $\hat{\Sigma}(\varphi(\mu))$, we take a u-dimensional continuous foliation G_μ^{vs} in $\Sigma(\mu)$ with C^1 leaves transversal to $W_1^{cu}(\sigma_k(\mu)) \cap \Sigma(\mu)$, which extends $F_{k+1}^{vs}(\mu)$ and is compatible with $F^{vs}(\sigma_{k+1}(\mu)) \cap \Sigma(\mu)$. Dually, we take an s-dimensional foliation G_μ^{vu} in $\Sigma(\mu)$ which extends $F_k^{vu}(\mu)$ and is compatible with $F^{cu}(\sigma_k(\mu)) \cap \Sigma(\mu)$. Similarly, we define $\hat{G}_\mu^{vs}$ and $\hat{G}_\mu^{vs}$ in the section $\hat{\Sigma}(\mu)$. Since $u + s = n - 1$ and we already have defined the homeomorphisms h_μ^{cs} and h_μ^{cu} in the space of leaves of these foliations, the homeomorphism $h_\mu : \Sigma(\mu) \to \hat{\Sigma}(\varphi(\mu))$ is defined by sending G_μ^{vs} to $\hat{G}_{\varphi(\mu)}^{vs}$ and G_μ^{vu} to $\hat{G}_{\varphi(\mu)}^{vu}$. The extension to the distinguished neighbourhood of the closure of the orbit of tangency γ and to the whole manifold M is exactly like in § 5.

§ 7. Remaining cases: two saddle-nodes and codimension-one and two singularity

In this paragraph we finish the proof of local stability for the codimension-two bifurcations by analyzing the two remaining cases.

(7.A) *Two saddle-nodes.* — We assume that the family X_μ has a bifurcation value $\bar{\mu}$ where the vector field $X_{\bar{\mu}}$ presents two saddle-nodes $p(\bar{\mu})$ and $q(\bar{\mu})$. We assume the existence of C^m, $m \geq 3$, linearizations transversally to center-manifolds of $p(\bar{\mu})$ and $q(\bar{\mu})$ and transversality between all unstable manifolds and the strong-stable manifolds $W^{ss}(p(\bar{\mu}))$ and $W^{ss}(q(\bar{\mu}))$ and between all stable manifolds and the strong-unstable manifolds $W^{uu}(p(\mu))$ and $W^{uu}(q(\bar{\mu}))$. The saddle-nodes unfold generically and do so independently. Hence, $\bar{\mu}$ belongs to the transversal intersection of two C^1 curves Γ_1 and Γ_2 with $\mu \in \Gamma_1$ if and only if X_μ presents one saddle-node near $p(\bar{\mu})$ and μ belongs to Γ_2 if and only if there is a saddle-node for X_μ near $q(\bar{\mu})$. In a neighbourhood U of $\bar{\mu}$ the bifurcation set is the union of Γ_1 with Γ_2.

Let us prove the local stability of X_μ. Suppose, in the usual ordering of the singularities of X_μ, that $p(\mu) = \sigma_i(\mu) = \sigma_{i+1}(\mu)$ for $\mu \in \Gamma_1$ and $q(\mu) = \sigma_k(\mu) = \sigma_{k+1}(\mu)$ if $\mu \in \Gamma_2$. We have two possibilities: there is an intermediate singularity

$$\sigma_{k+1}(\bar{\mu}) < \sigma_j(\bar{\mu}) < \sigma_i(\bar{\mu})$$

or not. We will construct an equivalence between X_μ and a nearby family $\hat{X}_\mu$ for the first case; the second case is simpler and can be derived from the first. We begin by considering a reparametrization $\varphi : (U, \bar{\mu}) \to (\mathbf{R}^2, \varphi(\bar{\mu}))$ that sends Γ_1 to $\hat{\Gamma}_1$, Γ_2 to $\hat{\Gamma}_2$ and it is defined so that there are conjugacies between X_μ restricted to the center-manifolds $W^c(p(\mu))$ and $W^c(q(\mu))$ and $\hat{X}_{\varphi(\mu)}$ restricted to $W^c(p(\varphi(\mu)))$ and $W^c(q(\varphi(\mu)))$. We then consider a compatible unstable system $F_1^u(\mu), \ldots, F_{k-1}^u(\mu)$ and construct a center-unstable foliation $F_k^{cu}(\mu)$ which is compatible with this system and has a center-unstable manifold $W^{cu}(\sigma_k(\mu))$ as its main leaf (see § 3). Since the singularities of $F_k^{cu}(\mu)$ occur along C^1 manifolds which are transversal to all intermediate manifolds $W^s(\sigma_j(\mu))$, we can proceed as in § 2 to get an unstable foliation $F^u_j(\mu)$ which is compatible with the system $F^u_1(\mu), \ldots, F^u_{k-1}(\mu), F^{cu}_k(\mu), \ldots, F^u_{j-1}(\mu)$ for $k+2 \leqslant j \leqslant i-1$. Now we construct a compatible strong-unstable foliation $F^{uu}_i(\mu)$ whose space of leaves is a center-stable manifold $W^{cs}(\sigma_i(\mu))$. Dually, we let $F^s_{i+2}(\mu), \ldots, F^s_\ell(\mu)$ be a compatible stable system, and construct a center-stable foliation $F^{cs}_i(\mu)$. Actually, the procedure here is not quite dual since we are going to use the strong-unstable foliation $F^{uu}_i(\mu)$ as part of a system of coordinates near $\sigma_i(\bar{\mu})$. To do that, let $K^+_i(\mu)$ be a closed disc contained in a leaf of the strong-stable foliation $F^{ss}_i(\mu)$ inside $W^{cs}(\sigma_i(\mu))$ and let $V^u_i(\mu)$ be a cross-section of the form $V^u_i(\mu) = \bigcup_{x \in K^+_j(\mu)} F^{uu}_{i,x}(\mu)$, where $F^{uu}_{i,x}(\mu)$ is the leaf of F^{uu}_i through x. Part of the leaves of $F^{cs}_i(\mu)$ is obtained by negative saturation by the flow $X_{\mu,t}$ of an s_i-dimensional continuous foliation $\widetilde{F}^{ss}_j(\mu)$ in $V^u_i(\mu)$ ($s_i = \dim W^{ss}(\sigma_i(\mu))$), topologically transversal to $W^{cu}(\sigma_i(\mu)) \cap V^u_i(\mu)$ and compatible with the stable system. The other leaves of $F^{cs}_i(\mu)$ are obtained exactly as in § 3. The process to construct an equivalence is now clear by previous arguments, but we briefly describe it as follows.

We begin with a (compatible) family of homeomorphisms

$$h^s_i(\mu) : W^s(\sigma_i(\mu)) \to W^s(\hat{\sigma}_i(\varphi(\mu))), \quad j = 1, \ldots, k-1.$$

It induces a homeomorphism in part of the space of leaves of $F^{cu}_k(\mu)$, which can be extended to all of $W^{cs}(\sigma_k(\mu))$, by first extending it to a fundamental domain $D^s_k(\mu) \cup C^s_k(\mu)$ and then to all of $W^{cs}(\sigma_k(\mu))$ by preserving the strong stable foliation and the intersections of $F^{cu}_k(\mu)$ with $W^{cs}(\sigma_k(\mu))$. Next, we consider successively fundamental domains $D^s_j(\mu)$ of $W^s(\sigma_j(\mu))$ for $k+2 \leqslant j \leqslant i-1$ to get (compatible) homeomorphisms in the space of leaves of the unstable foliations $F^u(\sigma_j(\mu))$. We finally reach the domain $D^s_i(\mu) \cup C^s_i(\mu)$, corresponding to the space of leaves of the strong-unstable foliation $F^{uu}_i(\mu)$. Here, again, the equivalence restricted to the center-stable manifold $W^{cs}(\sigma_j(\mu))$ is a conjugacy preserving the strong-stable foliation $F^{ss}_i(\mu)$. Proceeding dually, a family of homeomorphisms $h^u_j(\mu) : W^u(\sigma_j(\mu)) \to W^u(\hat{\sigma}_j(\varphi(\mu)))$, for $j = i+2, \ldots, \ell$, gives rise to a homeomorphism in the space of leaves of the center-stable foliation $F^{cs}_i(\mu)$ and the equivalence near $\sigma_i(\bar{\mu})$ is obtained as in § 3 by preserving the complementary foliations $F^{uu}_i(\mu)$ and $F^{cs}_i(\mu)$. We now extend this equivalence to a neighbourhood of each singularity $\sigma_j(\mu)$, for $j = k+2, \ldots, i-1$, by using the procedure say of § 2 to construct compatible stable foliations $F^s_{k+2}(\mu), \ldots, F^s_{i-1}(\mu)$ and homeomorphisms in the space of leaves of these foliations. The equivalence in these neighbourhoods is a conjugacy preserving stable and unstable foliations. Proceeding by induction we reach the saddle-node $\sigma_k(\mu)$. We then construct a strong-stable foliation $F^{ss}_k(\mu)$ compatible with the stable system and extend the equivalence to a neighbourhood of $\sigma_k(\mu)$ by preserving $F^{cu}_k(\mu)$ and $F^{ss}_k(\mu)$. For the extension of the equivalence to all of M, we proceed as in previous paragraphs, concluding the proof of the local stability of this case. ■

(7.B) *Codimension-one or two singularity.* — We consider here a family of gradients X_μ such that the vector field $X_{\bar{\mu}}$ presents exactly one non-hyperbolic singularity $\sigma(\bar{\mu})$. We suppose that 0 is an eigenvalue of $dX_{\bar{\mu}}(\sigma(\bar{\mu}))$ of multiplicity one. Therefore, there exists a center-manifold $W^c(\sigma(\bar{\mu}))$ containing $\sigma(\bar{\mu})$, which is of class C^m, m large (see [10]). It is well known that transversal to $W^c(\sigma(\bar{\mu}))$ there are unique strong-stable and strong-unstable manifolds $W^{ss}(\sigma(\bar{\mu}))$ and $W^{uu}(\sigma(\bar{\mu}))$. We assume that $X_{\bar{\mu}} \mid W^c(\sigma(\bar{\mu})) = [(x^k + 0(|x|^{k+1})] \frac{\partial}{\partial x}$ (i.e. the germ of X_μ has finite codimension) and that the family X_μ unfolds generically the singularity $\sigma(\bar{\mu})$. This means that the potential f_μ is a versal unfolding of $f_{\bar{\mu}}$. In addition, we require that all stable and unstable manifolds are transversal, and for each singularity $\sigma'(\bar{\mu})$ its stable and unstable manifolds are transversal to $W^{uu}(\sigma(\bar{\mu}))$ and $W^{ss}(\sigma(\bar{\mu}))$. These assumptions imply that there are no secondary bifurcations and, hence, the bifurcation set of X_μ near $\bar{\mu}$ coincides with the catastrophe set of f_μ. That is, it coincides with the set of values μ such that f_μ presents a degenerate critical point. In particular, let us consider $\mu \in \mathbf{R}^2$, $\bar{\mu} = 0$ and $k = 3$. We then obtain the cusp-family which is equivalent to $f(\mu, x, y) = x^4 + \mu_1 x^2 + \mu_2 x + Q(y)$ in a neighbourhood of the bifurcation of type IX described in Section I, and the bifurcation value $\bar{\mu} = 0$ represents two collapsing saddle-nodes.

Theorem. — Let X_μ *be a family in* $\mathfrak{X}^0_2(M)$ *which unfolds generically a non-hyperbolic singularity of type IX as above. Then* X_μ *is stable at* $\bar{\mu}$.

Proof. — We will actually proof that if X_μ is a *d-parameter family of gradients* which unfolds generically a $(k-1)$*-codimension singularity* such that 0 is an eigenvalue of multiplicity one of $dX_{\bar{\mu}}(\sigma(\bar{\mu}))$ and $d \geqslant k-1$, then X_μ is stable at $\bar{\mu}$. For simplicity, we suppose $\bar{\mu} = 0$. From the theory of singularity of functions [7], if X_μ is a nearby family with associated potential $\hat{f}_\mu$, there is a local diffeomorphism of the form $[\varphi(\mu), \hat{\varphi}(\mu, z)]$ defined in a neighbourhood of $(0, \sigma(0))$ in $\mathbf{R}^d \times M$ such that $\hat{f}_\mu \circ \hat{\varphi}(\mu, z) = f_{\varphi(\mu)}(z)$. Moreover, if $f^c(\mu, x)$ is the restriction of f_μ to the central manifold $W^c(\sigma(\mu))$, then there exists a C^{m-2} diffeomorphism of the form $[\psi(\mu), \hat{\psi}(\mu, x)]$ such that

$$f^c_\mu \circ \hat{\psi}(\mu, z) = x^{k+1} + \sum_{i=1}^{k-1} \mu_i x^i.$$

Hence, since $W^c(\sigma(\mu))$ is one-dimensional, in this new μ-dependent coordinate we can write $X_\mu \mid W^c(\sigma(\mu)) = \tau(\mu, x)\,[(k+1)\,x^k + \sum_{i=1}^{k-1} \mu_i x^{i-1}]$, where $\tau(\mu, x)$ is a positive C^{m-2} function defined in a neighbourhood of $(0, \sigma(0))$ in $\mathbf{R}^d \times M$. Now, extending τ to all of $\mathbf{R}^d \times M$ so that $\tau = 1$ outside a neigbourhood of $(0, \sigma(0))$, we define a new family of vector fields $Y_\mu = \frac{1}{r} X_\mu$ which is equivalent to X_μ. By performing the same construction for $\hat{X}_\mu$, we define a family $\hat{Y}_\mu$ which is equivalent to $\hat{X}_\mu$ and is such that $\hat{Y}_\mu \mid W^c(\sigma(\mu))$ and $Y_\mu \mid W^c(\sigma(\mu))$ have exactly the *same expressions* in the respective coordinates $\bar{x}$ and x. Therefore, by taking $h^c(\mu, x) = \bar{x}$ we obtain a conjugacy between

$Y_\mu \mid W^c(\sigma(\mu))$ and $\hat{Y}_{\varphi(\mu)} \mid W^c(\varphi(\mu))$. We can now proceed exactly like in Theorem A of Chapter III of [15] to extend h^c_μ to a conjugacy $h_\mu : M \to M$ between Y_μ and $\hat{Y}_{\varphi(\mu)}$. In this way we obtain an equivalence between X_μ and $\hat{X}_{\varphi(\mu)}$, concluding the proof of the stability of X_μ. ■

Section IV. — Globalization

In Sections II and III we have obtained a finite number of open and dense subsets of $\chi^g_2(M)$, each one corresponding to the cases described in Section I, with the property that every family X_μ contained in their intersection $\mathscr{Y}_1$ is locally stable at every value of the parameter. Suppose now that μ varies on a fixed closed disc D in $\mathbf{R}^2$. From our analysis of the bifurcation sets in previous sections, it follows that there exists a subset $\mathscr{Y} \subset \mathscr{Y}_1$, also open and dense, such that there are no codimension-two bifurcations on ∂ D and the curves that represent the codimension-one bifurcations are transversal to ∂ D. Hence, for $X_\mu \in \mathscr{Y}_1$, the codimension-one bifurcations occur on isolated points in ∂ D and there is a finite number, say r, of codimension-two bifurcation values in the interior of D. For each $1 \leqslant i \leqslant r$, let D_i be a small closed disc transversal to all branches of codimension-one bifurcations and containing a unique codimension-two bifurcation value in its interior. Let $D^* = D - \bigcup_i \operatorname{Int} D_i$. For $X_\mu \in \mathscr{Y}_1$, the intersection of the bifurcation set with D^* consists of the union of a finite number of closed C^1 simple curves or intervals, $\Gamma_1, \ldots, \Gamma_m$, each one corresponding either to a saddle-node or to an orbit of quasi-transversality. We denote by $\bar{\mu}_1, \ldots, \bar{\mu}_r$ the codimension-two bifurcation values of X_μ inside D. From the local stability of X_μ at $\bar{\mu}_i$, there exist open neighbourhoods V of X_μ in $\mathscr{Y}_1$ and U_i of $\bar{\mu}_i$ in D such that any family $\hat{X}_\mu$ in V is equivalent to X_μ for $\mu \in U_i$. We are now going to piece together these equivalences. To do this, we take for each $i = 1, \ldots, r$ a smooth function $\rho_i : \mathbf{R}^2 \to \mathbf{R}$ such that $0 \leqslant \rho_i \leqslant 1$, $\operatorname{supp}(\rho_i) \subset U_i$ and $\rho_i = 1$ in a closed disc D_i centered at $\bar{\mu}_i$ and define perturbations $\hat{X}^i_\mu = \operatorname{grad}_{\hat{g}^i_\mu} \hat{f}^i_\mu$ where the metrics $\hat{g}^i_\mu$ and the potentials $\hat{f}^i_\mu$ are defined inductively as follows:

$$\hat{f}^1_\mu = f_\mu + \rho_1(\mu)\,[\hat{f}_\mu - f_\mu], \qquad \hat{g}^1_\mu = g_\mu + \rho_1(\mu)\,[\hat{g}_\mu - g\mu]$$

and

$$\hat{f}^i_\mu = \hat{f}^{i-1}_\mu + \rho_i(\mu)\,[\hat{f}_\mu - \hat{f}^{i-1}_\mu], \qquad \hat{g}^i_\mu = \hat{g}^{i-1}_\mu + \rho_i(\mu)\,[\hat{g}_\mu - \hat{g}^{i-1}_\mu].$$

Hence, $\hat{X}^i_\mu = \hat{X}^{i-1}_\mu$ for $\mu \notin U_i$, $\hat{X}^i_\mu = X_\mu$ for $\mu \notin \bigcup^i_{j=1} U_j$ and $\hat{X}^i_\mu = \hat{X}_\mu$ for $\mu \in \bigcup^i_{j=1} D_j$. Using the remark concerning local stability made after the proof of Theorem A, and which applies to all bifurcation cases in Sections II and III, we obtain that $\hat{X}^i_\mu$ is equivalent to $\hat{X}^{i-1}_{\varphi_i(\mu)}$ with the reparametrization φ_i satisfying $\varphi_i(\mu) = \mu$ for $\mu \notin U_i$ and the equivalence $h^i_\mu =$ identity for $\mu \notin U_i$. Therefore, by transitivity, X_μ is equivalent to $\hat{X}^r_\mu$. Now, let Γ_1 be the first curve of codimension-one bifurcation in $D \setminus \bigcup^r_{j=1} D_j$. We cover Γ_1 by a finite number of domains of reparametrizations, $U^1_1, \ldots, U^1_{\ell_1}$, and starting with $\hat{X}^r_\mu$, define perturbations $\hat{X}^{1,k}_\mu$, $k = 1, \ldots, \ell_1$, along Γ_1, as above, such that $\hat{X}^{1,k}_\mu = \hat{X}^r_\mu$

for $\mu \notin \bigcup_{j=1}^{k} U_i^1$ and $\hat{X}_\mu^{1,k} = \hat{X}_\mu$ for $\mu \in \bigcup_{j=1}^{k} D_j^1$, where D_j^1 is a closed disc inside U_j^1 and $\bigcup_{j=1}^{\ell_1} D_j^1 \supset \Gamma_1$.

Performing again the modifications referred to above of the equivalences inside each domain U_k^1, we show that the family $\hat{X}_\mu^{1,k-1}$ is equivalent to the family $\hat{X}_\mu^{1,k}$ with an equivalence which is the identity outside U_k^1. In this way, starting with the equivalence between $\hat{X}_\mu^r$ and $\hat{X}_\mu^{1,1}$ and proceeding by induction, we construct an equivalence between $\hat{X}_\mu^r$ and $\hat{X}_\mu^{1,\ell_1}$ in a *neighbourhood* of Γ_1. It is now clear that by covering each curve Γ_j, $j = 1, \ldots, m$, with domains of reparametrizations $U_1^j, \ldots, U_{\ell_j}^j$, we obtain inductively an equivalence between $\hat{X}_\mu^r$ and $X_\mu^{m,\ell m}$ (and therefore between X_μ and $\hat{X}_\mu$) in a *neighbourhood* W of the entire bifurcation set in D. It is important to observe that all the reparametrizations that we perform preserve ∂ D. Finally, we repeat the same procedure in each component of D\W thus achieving a global equivalence between X_μ and $\hat{X}_\mu$. The proof of the main theorem in the paper is complete. ∎

Remark. — In the arguments presented above, the closed disc D can be replaced by any compact surface as the parameter space.

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Departamento de Matemática - I.C.E.X.
Universidade Federal de Minas Gerais
Belo Horizonte, M.G.
Brésil

Instituto de Matemática Pura e Aplicada (I.M.P.A.)
Estrada Dona Castorina, 110
Jardim Botánico
Rio de Janeiro, R.J.
Brésil

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