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# VOLUMES OF S-ARITHMETIC QUOTIENTS OF SEMI-SIMPLE GROUPS

by GOPAL PRASAD\*

With an appendix by Moshe Jarden and Gopal Prasad

*Dedicated to the memory of Harish-Chandra.*

## Introduction

The purpose of this paper is twofold: The first is to give a computable formula for the volumes of the S-arithmetic quotients of  $G_S := \prod_{\mathfrak{v} \in S} G(k_{\mathfrak{v}})$ , in terms of a natural Haar measure on  $G_S$ , where  $G$  is an arbitrary absolutely quasi-simple, simply connected algebraic group defined over a global field  $k$  (i.e. a number field or the function field of a curve over a finite field) and  $S$  is a finite set of places of  $k$  containing all the archimedean ones; see § 3. The second is to use the results involved in the volume computation to provide a “good” lower (and also upper) bound for the class number of  $G$ ; this is done in § 4 of the paper.

Besides the results of C. L. Siegel for certain special classical groups, the only *general* results about the volumes of S-arithmetic quotients which were known until now were concerned with Chevalley groups (i.e. groups which split over  $k$ ); see Harder [12]. There is quite a bit of literature on the class number problem for classical groups. We would like to mention here the work of C. L. Siegel, T. Tamagawa, M. Kneser and his school. The bounds for general absolutely quasi-simple, simply connected groups given in § 4 include the bounds for the special classical groups obtained by earlier authors.

In this work we have made use of a considerable amount of Bruhat-Tits theory of reductive groups over local fields. This theory is needed here even in the case  $S$  consists only of archimedean places i.e. when  $G_S$  is a connected real semi-simple Lie group.

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In [4], the volume formula of Theorem 3.7 and Theorem 4.3 have been used to prove the following finiteness assertions:

(1) *Given a positive real number  $c$ , there are only finitely many triples  $(k, G, S)$  consisting of a number field  $k$ , an absolutely quasi-simple group  $G$  defined over  $k$  and of absolute rank  $> 1$ , and a finite set  $S$  of places of  $k$  containing all the archimedean places, such that  $G_S (= \prod_{v \in S} G(k_v))$  contains an  $S$ -arithmetic subgroup of covolume  $< c$ .*

(2) *Given a positive integer  $n$ , there are only finitely many pairs  $(k, G)$  consisting of a number field  $k$  and an absolutely quasi-simple, simply connected algebraic group  $G$  defined over  $k$  such that  $G_\infty := \prod_{v \in V_\infty} G(k_v)$  is compact and the class number of  $G$  (with respect to some coherent collection of parahoric subgroups  $P_v$  of  $G(k_v)$ ) is  $\leq n$ ; here  $V_\infty$  is the set of all archimedean places of  $k$ .*

The first of the above results answers a question of Jacques Tits in the affirmative.

It is a pleasure to thank Armand Borel for several helpful conversations and for his comments on the earlier drafts of this paper. I would also like to thank Pierre Deligne, Günter Harder and Robert Steinberg for useful conversations and encouragement. Finally I thank Jacques Tits for raising his question which led me to the present work.

## 0. Notation, conventions and preliminaries

In this section, we fix a number of notation and conventions to be used later, often without further reference.

**0.0.** As usual,  $\mathbf{Q}$ ,  $\mathbf{R}$  and  $\mathbf{C}$  will denote respectively, the fields of rational, real and complex numbers.  $\mathbf{Z}$  will denote the ring of rational integers.

For a finite set  $S$ ,  $\#S$  will denote its cardinality.

For a linear algebraic group  $H$ ,  $R_u(H)$  will denote its unipotent radical, i.e. its maximal connected normal unipotent subgroup.

**0.1.** In the sequel,  $k$  is a global field and  $A$  the  $k$ -algebra of adèles of  $k$  endowed with the usual locally compact topology. Let  $V$  be the set of places of  $k$ , and  $V_\infty$  (resp.  $V_f$ ) the subset of archimedean (resp. nonarchimedean) places. For  $v \in V$ ,  $k_v$  denotes the completion of  $k$  at  $v$  and  $|\cdot|_v$  the normalized absolute value on  $k_v$ . The absolute value  $|\cdot|_v$  has a unique extension to any algebraic extension of  $k_v$ , to be denoted in the same way. For  $v \in V_f$ ,  $\mathfrak{o}_v$  denotes the ring of integers of  $k_v$ ,  $v(x)$  the additive valuation of  $x \in k_v^\times$ ,  $\mathfrak{f}_v$  the (finite) residue field and  $q_v$  the order of  $\mathfrak{f}_v$ . We recall that for  $x \in k_v^\times$ ,

$$\begin{aligned} |x|_v &= [\mathfrak{o}_v : x\mathfrak{o}_v]^{-1} = q_v^{-v(x)} & \text{if } x \in \mathfrak{o}_v, \\ |x|_v &= [x\mathfrak{o}_v : \mathfrak{o}_v] = q_v^{-v(x)} & \text{if } x \notin \mathfrak{o}_v. \end{aligned}$$

For  $v \in V_\infty$ ,  $|x|_v = |x|$  if  $v$  is real, i.e.  $k_v = \mathbf{R}$ , and  $|x|_v = |x|^2$  if  $v$  is complex, i.e.  $k_v = \mathbf{C}$ .

We have the *product formula*: For all  $x \in k^\times$ ,  $\prod_{v \in V} |x|_v = 1$ .

For  $v \in V$ ,  $k_v$  is assumed to carry the Haar measure with respect to which the measure of  $\mathfrak{o}_v$  is 1 if  $v$  is nonarchimedean, the measure of the unit interval  $[0, 1]$  is 1 if  $v$  is real, and the measure of any square in  $k_v (\cong \mathbf{C})$ , with sides of length 1, is 2 if  $v$  is complex.

**0.2.** We shall denote by  $\mathcal{G}$  an absolutely quasi-simple, simply connected algebraic group defined and quasi-split over  $k$ . Let  $n = \dim \mathcal{G}$  and  $r$  be the absolute rank of  $\mathcal{G}$ .

If  $\mathcal{G}/k$  is not a triality form of type  ${}^6\mathbf{D}_4$ , let  $\ell$  be the smallest extension of  $k$  over which  $\mathcal{G}$  splits; then  $[\ell : k] \leq 3$ . If  $\mathcal{G}/k$  is a triality form of type  ${}^6\mathbf{D}_4$ , let  $\ell$  be a fixed extension of  $k$  of degree 3 contained in the Galois extension of  $k$ , of degree 6, over which  $\mathcal{G}$  splits; there are three such extensions, all isomorphic to each other over  $k$ .

If  $k$  is a number field, let  $D_k$  (resp.  $D_\ell$ ) be the absolute value of the discriminant of  $k/\mathbf{Q}$  (resp.  $\ell/\mathbf{Q}$ ). Let  $\mathfrak{d}(\ell/k)$  denote the *relative discriminant* of  $\ell$  over  $k$ ; it is an ideal in the ring of integers of  $k$ . It is well known that  $|\mathbf{N}_{k/\mathbf{Q}}(\mathfrak{d}(\ell/k))| \cdot D_k^{[\ell:k]} = D_\ell$ .

If  $k$  is the function field of a curve over a finite field, let  $q_k$  (resp.  $q_\ell$ ) be the cardinality of the finite field of the constant functions in  $k$  (resp.  $\ell$ ) and  $g_k$  (resp.  $g_\ell$ ) be the genus of  $k$  (resp.  $\ell$ ). Let  $D_k = q_k^{2g_k-2}$ ,  $D_\ell = q_\ell^{2g_\ell-2}$ .

**0.3.** Let  $v$  be a nonarchimedean place of  $k$  such that  $\ell_v := \ell \otimes_k k_v$  is a *ramified* field extension of  $k_v$  of degree 2. Let

$$v_v = \inf \{ |y|_v \mid y \in \ell_v, y + \bar{y} + 1 = 0 \},$$

where here, as well as in the sequel, for  $y \in \ell_v$ ,  $\bar{y}$  denotes its conjugate over  $k_v$ . Then  $v_v \geq 1$  and  $v_v = 1$  if and only if the characteristic of the residue field of  $k_v$  is odd. For later use, we fix a  $\lambda_v \in \ell_v$ , and a uniformizing element  $\pi_v$  of  $\ell_v$  such that  $\lambda_v + \bar{\lambda}_v + 1 = 0$ ,  $|\lambda_v|_v = v_v$  and  $\lambda_v \pi_v + \bar{\lambda}_v \bar{\pi}_v = 0$  (cf. Tits [33: 1.15]). Then

$$|\mathfrak{d}(\ell_v/k_v)|_v = |(\pi_v - \bar{\pi}_v)^2|_v = |\pi_v^2(1 + \lambda_v \bar{\lambda}_v^{-1})^2|_v = q_v^{-1} v_v^{-2},$$

where  $\mathfrak{d}(\ell_v/k_v)$  is the relative discriminant of  $\ell_v/k_v$ .

**0.4.** *The integer  $\mathfrak{s}(\mathcal{G})$ .* If  $\mathcal{G}$  splits over  $k$ , let  $\mathfrak{s}(\mathcal{G}) = 0$ . Now assume  $\mathcal{G}$  does not split over  $k$ . On the relative root system  ${}_k\Phi$  of  $\mathcal{G}$ , with respect to a maximal  $k$ -split torus  $\mathcal{E}$ , consider the ordering associated with a Borel  $k$ -subgroup containing  $\mathcal{E}$ . The integer  $\mathfrak{s}(\mathcal{G})$  is then defined as follows. If  ${}_k\Phi$  is reduced (which is the case if, and only if,  $\mathcal{G}$  is *not* a  $k$ -form of type  ${}^2\mathbf{A}_r$  with  $r$  even), then  $\mathfrak{s}(\mathcal{G})$  is equal to the sum of the number of short roots and of short simple roots. If  $\mathcal{G}$  is a  $k$ -form of type  ${}^2\mathbf{A}_r$  with  $r$  even, then  ${}_k\Phi$  is the non-reduced root system  $\mathbf{BC}_{r/2}$  and  $\mathfrak{s}(\mathcal{G}) = \frac{1}{2}r(r+3)$ , which is equal to the number of *all* roots in  ${}_k\Phi$  plus the number of simple roots.

Note that if  $\mathcal{G}$  is a  $k$ -form of type  ${}^2\mathbf{A}_r$  ( $r$  odd),  ${}^2\mathbf{D}_r$  ( $r$  arbitrary) or  ${}^2\mathbf{E}_6$ , then the root system  ${}_k\Phi$  is the reduced root system of type  $\mathbf{C}_{(r+1)/2}$ ,  $\mathbf{B}_{r-1}$ ,  $\mathbf{F}_4$  respectively and

$\mathfrak{s}(\mathcal{G})$  is  $\frac{1}{2}(r-1)(r+2)$ ,  $2r-1$ ,  $26$  respectively. If  $\mathcal{G}$  is a triality form of type  ${}^3\mathbf{D}_4$  or  ${}^6\mathbf{D}_4$ , then  ${}_x\Phi$  is of type  $\mathbf{G}_2$  and  $\mathfrak{s}(\mathcal{G}) = 7$ .

**0.5.** In this paper, we assume familiarity with the Bruhat-Tits theory [6] and recall just some notation and facts. All we need is stated in [33], and in most cases, proofs can be found in one of these references.

Let  $K$  be a nonarchimedean local field. In the sequel,  $K$  will always be a finite extension of  $k_v$ , for a nonarchimedean place  $v$ . Let  $G$  be an absolutely quasi-simple, simply connected group defined over  $K$ . Let  $\mathcal{B} = \mathcal{B}(G/K)$  be the associated Bruhat-Tits building. It is a contractible simplicial complex on which  $G(K)$  acts by simplicial automorphisms which are *special* (in particular, if  $g \in G(K)$  leaves a simplex of  $\mathcal{B}$  stable, then it fixes the simplex pointwise).

We recall that an *Iwahori subgroup* of  $G(K)$  can be defined as either the normalizer of a maximal pro- $p$  subgroup of  $G(K)$ , where  $p$  is the characteristic of the residue field of  $K$ , or as the subgroup of  $G(K)$  fixing a chamber (i.e. a maximal simplex) in  $\mathcal{B}$ . All Iwahori subgroups are conjugate in  $G(K)$ . A *parahoric subgroup*  $P$  of  $G(K)$  is the stabilizer of a simplex of  $\mathcal{B}$ . Every parahoric subgroup is compact, open and contains an Iwahori subgroup. The maximal ones are the maximal compact subgroups of  $G(K)$  and are the stabilizers of the vertices of  $\mathcal{B}$ . A (maximal) parahoric subgroup  $P$  is *special* if it fixes a *special vertex* of  $\mathcal{B}$ . A vertex  $x$  of  $\mathcal{B}$  is special if the *affine* Weyl group  $W$  is a semidirect product of the translation subgroup by the isotropy group  $W_x$  of  $x$  in  $W$ . If so, then  $W_x$  is canonically isomorphic to the Weyl group of the  $K$ -root system of  $G$ .

**0.6.** Let  $\hat{K}$  be the maximal unramified extension of  $K$  and  $\hat{\mathfrak{o}}$  be its ring of integers. Let  $\hat{\mathcal{B}}$  be the building of  $G(\hat{K})$  and  $\hat{A} \subset \hat{\mathcal{B}}$  be the apartment of a maximal  $\hat{K}$ -split torus of  $G$  which is defined over  $K$  and which contains a maximal  $K$ -split torus. There is an action of the Galois group of  $\hat{K}/K$  on  $\hat{\mathcal{B}}$  and  $\hat{A}$  is stable under this action; the fixed set in  $\hat{\mathcal{B}}$  may be identified with  $\mathcal{B}$  and the fixed set in  $\hat{A}$  with an apartment  $A$  of  $\mathcal{B}$  [33: 1.10]. A vertex of  $\hat{\mathcal{B}}$  lying in  $A$  which is special for  $\hat{\mathcal{B}}$  is also special for  $\mathcal{B}$  [33: 1.10.2]. If  $G$  splits over  $\hat{K}$ , such a point, viewed as a vertex of  $\mathcal{B}$  is called *hyperspecial* and its isotropy group in  $G(K)$  is a *hyperspecial* parahoric subgroup. If  $G$  is quasi-split over  $K$  and splits over an unramified extension of  $K$ , hyperspecial parahoric subgroups exist [33: 1.10.2]; these groups are the parahoric subgroups of  $G(K)$  of maximal volume [33: 3.8.2].

**0.7.** To any parahoric subgroup  $P$  of  $G(K)$ , the Bruhat-Tits theory associates a smooth affine group scheme defined over the ring  $\mathfrak{o}$  of integers of  $K$ , whose generic fiber is isomorphic to  $G/K$  and whose group of  $\mathfrak{o}$ -rational points is isomorphic to  $P$  (see [6: II] or [33: 3.4]). The coordinate ring of this group scheme is the  $\mathfrak{o}$ -algebra of those  $K$ -regular functions on  $G$  which on  $\hat{P}$  take values in  $\hat{\mathfrak{o}}$ , where  $\hat{P}$  is the parahoric subgroup of  $G(\hat{K})$  associated with  $P$ .

**1. Tamagawa forms on quasi-split groups**

**1.1.** We fix a non-zero left-invariant exterior form  $\omega$  on  $\mathcal{G}$  of maximal degree and which is defined over  $k$ ; such a form is unique up to multiplication by an element of  $k^\times$  and is called a *Tamagawa form* on  $\mathcal{G}/k$ . As  $\mathcal{G}$  is a semi-simple group,  $\omega$  is bi-invariant.

**1.2.** For each  $v \in V_f$ , we fix, once and for all, a maximal parahoric subgroup  $\mathcal{P}_v$  of  $\mathcal{G}(k_v)$  with the following properties.

- (i) If  $\mathcal{G}$  splits over an unramified extension of  $k_v$ , then  $\mathcal{P}_v$  is a *hyperspecial* parahoric subgroup.
- (ii) If  $\mathcal{G}$  does not split over any unramified extension of  $k_v$  (then  $\mathcal{G} \times_k k_v$  is a residually split group over  $k_v$ ),  $\mathcal{P}_v$  is *special*. In case  $\mathcal{G}$  is an outer form of type  $A_r$ , with  $r$  even, we assume moreover that the gradient (i.e. the vector part) of the affine simple root corresponding to this special parahoric subgroup is a *divisible* root.
- (iii)  $\prod_{v \in V_\infty} \mathcal{G}(k_v) \cdot \prod_{v \in V_f} \mathcal{P}_v$  is an open subgroup of the adèle group  $\mathcal{G}(A)$ .

**1.3.** Let  $\mathcal{G}_v$  be the smooth affine  $\mathfrak{o}_v$ -group scheme associated with the parahoric subgroup  $\mathcal{P}_v$ , whose generic fiber ( $= \mathcal{G}_v \times_{\mathfrak{o}_v} k_v$ ) is isomorphic to  $\mathcal{G} \times_k k_v$  and whose group of  $\mathfrak{o}_v$ -rational points is isomorphic to  $\mathcal{P}_v$  (see 0.7).

Let  $c_v \in k_v^\times$  be such that  $c_v \omega$  induces an invariant exterior form on the  $\mathfrak{o}_v$ -group scheme  $\mathcal{G}_v$ , of maximal degree, which is defined over  $\mathfrak{o}_v$  and whose reduction to the group  $\mathcal{G}_v \times_{\mathfrak{o}_v} \mathfrak{f}_v$  over the residue field  $\mathfrak{f}_v$  is not zero. It is obvious that such a  $c_v$  exists and is unique up to multiplication by a unit. In particular,  $\gamma_v := |c_v|_v$  is a well-defined positive real number; it is equal to 1 for all but finitely many  $v$ 's.

**1.4.** If  $k$  is a number field, for an archimedean place  $v$  of  $k$ , let  $c_v$  be the positive real number such that with respect to the Haar measure determined by the form  $c_v \omega$ , the volume of any maximal compact subgroup of  $R_{k_v/\mathbb{R}}(\mathcal{G})(\mathbb{C})$  is 1, and let  $\gamma_v = |c_v|_v$ . We recall here that if  $v$  is real, then any maximal compact subgroup of  $R_{k_v/\mathbb{R}}(\mathcal{G})(\mathbb{C})$  is isomorphic to the unique (up to isomorphism) compact, simple, simply connected real-analytic Lie group of the same type as  $\mathcal{G}$  and if  $v$  is complex, then any maximal compact subgroup of  $R_{k_v/\mathbb{R}}(\mathcal{G})(\mathbb{C})$  is the direct product of two copies of this group.

**1.5.** Let  $r$  be the absolute rank of  $\mathcal{G}$  and let  $m_1, \dots, m_r$  ( $m_1 \leq \dots \leq m_r$ ) be the *exponents* of the simple, simply connected, compact real-analytic Lie group of the same type as  $\mathcal{G}$ . Note that  $\dim \mathcal{G} = r + 2\sum_{i=1}^r m_i$ .

We list below the exponents (see Bourbaki [5]).

Type	Exponents
$A_r$	$1, 2, \dots, r.$
$B_r$	$1, 3, 5, \dots, 2r - 1.$
$C_r$	$1, 3, 5, \dots, 2r - 1.$
$D_r$	$1, 3, 5, \dots, 2r - 5, 2r - 3, r - 1$ ( $r - 1$ has multiplicity 2 when $r$ is even).
$E_6$	$1, 4, 5, 7, 8, 11.$
$E_7$	$1, 5, 7, 9, 11, 13, 17.$
$E_8$	$1, 7, 11, 13, 17, 19, 23, 29.$
$F_4$	$1, 5, 7, 11.$
$G_2$	$1, 5.$

1.6. *Theorem.* — *We have*

$$\prod_{\mathfrak{v} \in V} \gamma_{\mathfrak{v}} = (D_{\mathfrak{t}}/D_k^{[\mathfrak{t}:k]})^{\frac{1}{2} \dim(\mathcal{G})} \prod_{\mathfrak{v} \in V_{\infty}} \left| \prod_{i=1}^r \frac{m_i!}{(2\pi)^{m_i+1}} \right|_{\mathfrak{v}}.$$

*Proof.* — Let  $\mathcal{Q}$  be any (not necessarily finite) Galois extension of  $k$  containing  $\mathfrak{t}$ , where  $\mathfrak{t}$  is as in 0.2. Then  $\mathcal{G}$  splits over  $\mathcal{Q}$ . Let  $L(\mathcal{G})$  be the Lie algebra of left-invariant vector fields on  $\mathcal{G}/k$ , and  $\mathfrak{g} = L(\mathcal{G}) \otimes_k \mathcal{Q}$ . Let  $\mathcal{E}$  be a maximal  $k$ -split torus of  $\mathcal{G}$  and  $\mathcal{Z}$  be its centralizer. Then  $\mathcal{Z}$  is defined over  $k$  and it is a torus since  $\mathcal{G}$  is quasi-split over  $k$ . Moreover, it splits over  $\mathcal{Q}$  since  $\mathcal{G}$  does. Let  $\Phi$  be the root system of  $\mathcal{G}$  with respect to  $\mathcal{Z}$ , and  $\Pi(\subset \Phi)$  be the set of simple roots with respect to the ordering on  $\Phi$  obtained by fixing a Borel  $k$ -subgroup containing  $\mathcal{Z}$ . Let  $\{H_a \mid a \in \Pi\} \cup \{X_b \mid b \in \Phi\}$  be a Chevalley basis of  $\mathfrak{g}$ , where the  $H_a$ 's constitute a basis of the Lie algebra  $L(\mathcal{Z}) \otimes_k \mathcal{Q}$  of  $\mathcal{Z}/\mathcal{Q}$  and for each  $b \in \Phi$ ,  $X_b$  is an element of the root space  $\mathfrak{g}_b$ . We fix an enumeration of this Chevalley basis, and for  $1 \leq i \leq n$  ( $= \dim \mathcal{G}$ ), let  $\mathfrak{X}_i$  be its  $i$ -th element. Let  $\mathfrak{X}^i$  be the dual basis of the dual  $\mathfrak{g}^*$  and let  $\omega^{\text{Ch}} = \mathfrak{X}^1 \wedge \dots \wedge \mathfrak{X}^n$ ;  $\omega^{\text{Ch}}$  is a  $\mathcal{G}$ -invariant exterior form on  $\mathcal{G}$  of maximal degree. The form  $\omega^{\text{Ch}}$  is defined over  $\mathcal{Q}$  and any other choice of Chevalley basis or its enumeration gives only  $\omega^{\text{Ch}}$  or  $-\omega^{\text{Ch}}$ .

Since the space of  $\mathcal{G}$ -invariant exterior forms on  $\mathcal{G}$  of maximal degree is 1-dimensional, there is an  $\alpha \in \mathcal{Q}^{\times}$  such that  $\omega = \alpha^{-1} \omega^{\text{Ch}}$ . As  $\omega$  is defined over  $k$ , for every  $\gamma \in \text{Gal}(\mathcal{Q}/k)$ ,  $\gamma(\omega) = \omega$ . Now since  $\gamma(\omega^{\text{Ch}}) = \pm \omega^{\text{Ch}}$ , we conclude that  $\gamma(\alpha)^2 = \alpha^2$  for all  $\gamma \in \text{Gal}(\mathcal{Q}/k)$  and hence  $\alpha^2 \in k^{\times}$ .

If  $k$  is a number field,  $\det(\langle \mathfrak{X}_i, \mathfrak{X}_j \rangle)$ , where  $\langle \mathfrak{X}_i, \mathfrak{X}_j \rangle = \text{Tr}(\text{ad } \mathfrak{X}_i \text{ ad } \mathfrak{X}_j)$  is the inner product of  $\mathfrak{X}_i$  with  $\mathfrak{X}_j$  with respect to the Killing form on  $\mathfrak{g}$ , is an integer. Let  $m$  be its absolute value. Then  $m$  is uniquely determined by the absolute root system of  $\mathcal{G}$ ; it does not depend on the choice of the Chevalley basis of  $\mathfrak{g}$ .

We fix a  $k$ -basis  $X_1, \dots, X_n$  of the Lie algebra  $L(\mathcal{G})$  so that if  $X^1, \dots, X^n$  is the dual basis,  $\omega = X^1 \wedge \dots \wedge X^n$ . If  $k$  is a number field, for every archimedean place  $\mathfrak{v}$

of  $k$ , we fix a basis  $Y_1^v, \dots, Y_n^v$  of  $L(\mathcal{G}) \otimes_k k_v$  such that with respect to the Killing form  $\langle \cdot, \cdot \rangle_v$  on  $L(\mathcal{G}) \otimes_k k_v$ ,  $Y_i^v$  is orthogonal to  $Y_j^v$  for all  $1 \leq i \neq j \leq n$ , and moreover, if  $v$  is real, then  $|\langle Y_i^v, Y_i^v \rangle_v| = 1$ , whereas, if  $v$  is complex, then  $\langle Y_i^v, Y_i^v \rangle_v = 1$  for all  $i \leq n$ . Now let  $Y_1^v, \dots, Y_n^v$  be the dual basis and  $\omega_v^K = Y_1^v \wedge \dots \wedge Y_n^v$ . Then  $\omega_v^K$  is an invariant exterior form on  $\mathcal{G} \times_k k_v$ , of maximal degree, defined over  $k_v$ ; it determines a Haar measure on  $\mathcal{G}(k_v)$  as well as on every maximal compact subgroup of  $R_{k_v/\mathbf{R}}(\mathcal{G})(\mathbf{C})$ . The volume of each of the latter subgroups is equal to

$$m^{\frac{1}{2}} \prod_{i=1}^r \frac{(2\pi)^{m_i+1}}{m_i!} \quad \text{if } v \text{ is real,}$$

$$\text{and} \quad \left( m^{\frac{1}{2}} \prod_{i=1}^r \frac{(2\pi)^{m_i+1}}{m_i!} \right)^2 = \left| m^{\frac{1}{2}} \prod_{i=1}^r \frac{(2\pi)^{m_i+1}}{m_i!} \right|_v \quad \text{if } v \text{ is complex;}$$

see, for example, [25: § 3] or [20].

Let  $\mathfrak{d} = \det(\langle X_i, X_j \rangle)$ , where  $\langle X_i, X_j \rangle = \text{Tr}(\text{ad } X_i \text{ ad } X_j)$  is the inner product of  $X_i$  with  $X_j$  under the Killing form on  $L(\mathcal{G})$ . Then it is obvious that if  $v$  is a complex place,  $\omega_v^K \otimes \omega_v^K$  equals  $\mathfrak{d}\omega \otimes \omega$ , and if  $v$  is real, then  $\omega_v^K \otimes \omega_v^K$  equals either  $\mathfrak{d}\omega \otimes \omega$  or  $-\mathfrak{d}\omega \otimes \omega$ . Now as the volume of any maximal compact subgroup of  $R_{k_v/\mathbf{R}}(\mathcal{G})(\mathbf{C})$  with respect to the Haar measure determined by  $\omega_v^K$  is

$$\left| m^{\frac{1}{2}} \prod_{i=1}^r \frac{(2\pi)^{m_i+1}}{m_i!} \right|_v,$$

we conclude that, for all archimedean  $v$ ,

$$\gamma_v = |\mathfrak{d}|_v^{\frac{1}{2}} \left| m^{-\frac{1}{2}} \prod_{i=1}^r \frac{m_i!}{(2\pi)^{m_i+1}} \right|_v.$$

But since  $\omega^{\text{Ch}} = \alpha\omega$ , we find that  $\alpha^4 m^2 = \mathfrak{d}^2$ , which implies that  $|\mathfrak{d}m^{-1}|_v = |\alpha^2|_v$ , and hence for all archimedean  $v$ ,

$$\gamma_v = |\alpha^2|_v^{\frac{1}{2}} \left| \prod_{i=1}^r \frac{m_i!}{(2\pi)^{m_i+1}} \right|_v.$$

Therefore,

$$\begin{aligned} \prod_{v \in \mathbf{V}} \gamma_v^2 &= \prod_{v \in \mathbf{V}_f} \gamma_v^2 \cdot \prod_{v \in \mathbf{V}_\infty} \gamma_v^2 \\ &= \prod_{v \in \mathbf{V}_f} \gamma_v^2 \cdot \prod_{v \in \mathbf{V}_\infty} \left( |\alpha^2|_v \left| \prod_{i=1}^r \frac{m_i!}{(2\pi)^{m_i+1}} \right|_v^2 \right) \\ &= \prod_{v \in \mathbf{V}_f} \gamma_v^2 \cdot \prod_{v \in \mathbf{V}_f} |\alpha^2|_v^{-1} \cdot \prod_{v \in \mathbf{V}_\infty} \left| \prod_{i=1}^r \frac{m_i!}{(2\pi)^{m_i+1}} \right|_v^2 \\ &\quad \text{(by the product formula (0.1); recall that } \alpha^2 \in k^\times) \\ &= \prod_{v \in \mathbf{V}_f} (|\alpha^2|_v^{-1} \gamma_v^2) \cdot \prod_{v \in \mathbf{V}_\infty} \left| \prod_{i=1}^r \frac{m_i!}{(2\pi)^{m_i+1}} \right|_v^2. \end{aligned}$$



Next we shall prove that

$$\prod_{\mathfrak{v} \in \mathfrak{V}_f} (|\alpha^2|_{\mathfrak{v}}^{-1} \gamma_{\mathfrak{v}}^2) = (D_{\ell}/D_k^{[\ell:k]})^{\mathfrak{q}(\mathcal{G})}.$$

This will establish the theorem.

Let  $\mathcal{G}_{\mathfrak{v}}$  be the smooth affine  $\mathfrak{o}_{\mathfrak{v}}$ -group scheme associated with the parahoric subgroup  $\mathcal{P}_{\mathfrak{v}}$  of  $\mathcal{G}(k_{\mathfrak{v}})$  (see 1.3). Let  $L(\mathcal{G}_{\mathfrak{v}})$  be the Lie algebra of  $\mathcal{G}_{\mathfrak{v}}$ ; it is an  $\mathfrak{o}_{\mathfrak{v}}$ -Lie algebra. Since the generic fiber  $\mathcal{G}_{\mathfrak{v}} \times_{\mathfrak{o}_{\mathfrak{v}}} k_{\mathfrak{v}}$  of  $\mathcal{G}_{\mathfrak{v}}$  is  $\mathcal{G} \times_k k_{\mathfrak{v}}$ , it follows that

$$L(\mathcal{G}_{\mathfrak{v}}) \otimes_{\mathfrak{o}_{\mathfrak{v}}} k_{\mathfrak{v}} \cong L(\mathcal{G}) \otimes_k k_{\mathfrak{v}}.$$

We use this isomorphism to identify  $L(\mathcal{G}_{\mathfrak{v}})$  with an  $\mathfrak{o}_{\mathfrak{v}}$ -subalgebra of the  $k_{\mathfrak{v}}$ -Lie algebra  $\mathfrak{g}_{\mathfrak{v}} := L(\mathcal{G}) \otimes_k k_{\mathfrak{v}}$ . The  $k_{\mathfrak{v}}$ -span of  $L(\mathcal{G}_{\mathfrak{v}})$  is clearly all of  $\mathfrak{g}_{\mathfrak{v}}$ . Let  $\{Y_i^{\mathfrak{v}}\}$  be an  $\mathfrak{o}_{\mathfrak{v}}$ -basis of  $L(\mathcal{G}_{\mathfrak{v}})$  ( $\subset \mathfrak{g}_{\mathfrak{v}}$ ), and let  $a_{\mathfrak{v}} \in k_{\mathfrak{v}}^{\times}$  be such that  $X_1 \wedge \dots \wedge X_n = a_{\mathfrak{v}} Y_1^{\mathfrak{v}} \wedge \dots \wedge Y_n^{\mathfrak{v}}$ . Then it is obvious that  $a_{\mathfrak{v}} \omega$  induces an invariant exterior form on the  $\mathfrak{o}_{\mathfrak{v}}$ -group scheme  $\mathcal{G}_{\mathfrak{v}}$ , which is defined over  $\mathfrak{o}_{\mathfrak{v}}$  and whose reduction to the group  $\mathcal{G}_{\mathfrak{v}} \times_{\mathfrak{o}_{\mathfrak{v}}} \mathfrak{f}_{\mathfrak{v}}$  is not zero. Hence,  $|a_{\mathfrak{v}}|_{\mathfrak{v}} = \gamma_{\mathfrak{v}}$  (see 1.3).

Now let  $v$  be a nonarchimedean place of  $k$  such that  $\mathcal{G}$  splits over the maximal unramified extension  $\hat{k}_{\mathfrak{v}}$  of  $k_{\mathfrak{v}}$ . (Then  $\ell \otimes_k k_{\mathfrak{v}}$  is a direct sum of certain *unramified* extensions of  $k_{\mathfrak{v}}$ ; we note here, for future use, that for any unramified extension  $K$  of  $k_{\mathfrak{v}}$ ,  $|\mathfrak{d}(K/k_{\mathfrak{v}})|_{\mathfrak{v}} = 1$ .) Let  $\hat{\mathfrak{o}}_{\mathfrak{v}}$  be the ring of integers of  $\hat{k}_{\mathfrak{v}}$ . Then it is clear that if  $\{Z_i^{\mathfrak{v}}\}$  is any  $\hat{\mathfrak{o}}_{\mathfrak{v}}$ -basis of  $L(\mathcal{G}_{\mathfrak{v}}) \otimes_{\mathfrak{o}_{\mathfrak{v}}} \hat{\mathfrak{o}}_{\mathfrak{v}}$  and  $b_{\mathfrak{v}} (\in \hat{k}_{\mathfrak{v}}^{\times})$  is such that  $X_1 \wedge \dots \wedge X_n = b_{\mathfrak{v}} Z_1^{\mathfrak{v}} \wedge \dots \wedge Z_n^{\mathfrak{v}}$ , then  $b_{\mathfrak{v}} a_{\mathfrak{v}}^{-1}$  is a unit and hence,  $|b_{\mathfrak{v}}|_{\mathfrak{v}} = |a_{\mathfrak{v}}|_{\mathfrak{v}} = \gamma_{\mathfrak{v}}$ . We observe now that since  $\mathcal{P}_{\mathfrak{v}}$  is a hyperspecial parahoric subgroup of  $\mathcal{G}(k_{\mathfrak{v}})$ , and  $\mathcal{G}_{\mathfrak{v}}$  is the associated  $\mathfrak{o}_{\mathfrak{v}}$ -group scheme,  $\mathcal{G}_{\mathfrak{v}}(\hat{\mathfrak{o}}_{\mathfrak{v}})$  is a hyperspecial parahoric subgroup of  $\mathcal{G}(\hat{k}_{\mathfrak{v}})$  ([33: 2.6.1 and 3.4.1]). But as  $\mathcal{G}$  splits over  $\hat{k}_{\mathfrak{v}}$ , this implies that there is an  $\hat{\mathfrak{o}}_{\mathfrak{v}}$ -basis  $Z_1^{\mathfrak{v}}, \dots, Z_n^{\mathfrak{v}}$  of  $L(\mathcal{G}_{\mathfrak{v}}) \otimes_{\mathfrak{o}_{\mathfrak{v}}} \hat{\mathfrak{o}}_{\mathfrak{v}}$  which is a Chevalley basis of the split Lie algebra  $\mathfrak{g}_{\mathfrak{v}} \otimes_{k_{\mathfrak{v}}} \hat{k}_{\mathfrak{v}}$  ([33: 3.4.2 and 3.4.3]). Now as  $\omega^{\text{ch}} = \alpha \omega$ , and, up to sign,  $\omega^{\text{ch}}$  is independent of the choice of the Chevalley basis and its enumeration, we conclude from this that  $|\alpha^2|_{\mathfrak{v}}^{-1} \gamma_{\mathfrak{v}}^2 = 1$  for every nonarchimedean place  $v$  such that  $\mathcal{G}$  splits over the maximal unramified extension  $\hat{k}_{\mathfrak{v}}$  of  $k_{\mathfrak{v}}$ .

Let now  $\mathcal{R}$  be the set of all nonarchimedean places  $v$  of  $k$  such that  $\mathcal{G}$  does not split over any unramified extension of  $k_{\mathfrak{v}}$ , or equivalently,  $\ell \otimes_k k_{\mathfrak{v}}$  contains a nontrivial ramified field extension of  $k_{\mathfrak{v}}$ . Then  $\mathcal{R}$  is finite. Let  $v \in \mathcal{R}$ ; then there are two possibilities:

1.  $\ell \otimes_k k_{\mathfrak{v}}$  is a field, we shall denote it by  $\ell_{\mathfrak{v}}$ , it is a ramified extension and  $[\ell_{\mathfrak{v}} : k_{\mathfrak{v}}] = [\ell : k]$ .
2.  $\ell \otimes_k k_{\mathfrak{v}}$  is a direct sum of  $k_{\mathfrak{v}}$  and a ramified field extension  $\ell_{\mathfrak{v}}$  of  $k_{\mathfrak{v}}$  of degree 2. This is the case if  $\mathcal{G}/k$  is a form of type  ${}^6\mathbf{D}_4$  of  $k$ -rank 2 and  $\mathcal{G}/k_{\mathfrak{v}}$  is a form of type  ${}^2\mathbf{D}_4$  of  $k_{\mathfrak{v}}$ -rank 3. In this case, the  $k$ -root system of  $\mathcal{G}$  is of type  $\mathbf{G}_2$  and  $\mathfrak{s}(\mathcal{G}) = 6 + 1 = 7$ ; its  $k_{\mathfrak{v}}$ -root system is of type  $\mathbf{B}_3$  which has 6 short roots and one short simple root.

To compute  $|\alpha^2|_{\mathfrak{v}}^{-1} \gamma_{\mathfrak{v}}^2$ , we shall construct a suitable  $\mathfrak{o}_{\mathfrak{v}}$ -basis of the Lie algebra  $L(\mathcal{G}_{\mathfrak{v}})$ . For this purpose, we fix a maximal  $k_{\mathfrak{v}}$ -split torus  $\mathcal{E}_{\mathfrak{v}}$  such that in the Bruhat-Tits building of  $\mathcal{G}/k_{\mathfrak{v}}$  the vertex fixed by  $\mathcal{P}_{\mathfrak{v}}$  lies on the apartment determined

by  $\mathcal{E}_v$ . Let  $\mathcal{Z}_v$  be the centralizer of  $\mathcal{E}_v$  in  $\mathcal{G}$ . Then as  $\mathcal{G}$  is quasi-split over  $k$  (and so also over  $k_v$ ),  $\mathcal{Z}_v$  is a torus, and it is clearly defined over  $k_v$ . Let  $\Phi(\mathcal{Z}_v)$  (resp.  $\Phi(\mathcal{E}_v)$ ) be the root system of  $\mathcal{G}$  with respect to  $\mathcal{Z}_v$  (resp.  $\mathcal{E}_v$ ). We fix a Borel subgroup of  $\mathcal{G}$  which contains  $\mathcal{Z}_v$  and is defined over  $k_v$ . This gives compatible orderings on  $\Phi(\mathcal{E}_v)$  and  $\Phi(\mathcal{Z}_v)$ . Let  $\Phi(\mathcal{Z}_v)^+$  (resp.  $\Phi(\mathcal{E}_v)^+$ ) be the set of roots in  $\Phi(\mathcal{Z}_v)$  (resp.  $\Phi(\mathcal{E}_v)$ ) positive with respect to this ordering and let  $\Pi(\mathcal{Z}_v)$  (resp.  $\Pi(\mathcal{E}_v)$ ) be the set of simple roots.

We fix a minimal Galois extension  $L_v$  of  $k_v$  containing  $\ell_v$  and denote by  $\Gamma$  the Galois group of  $L_v/k_v$ . Then  $\mathcal{G}$ , and so also the torus  $\mathcal{Z}_v$ , splits over  $L_v$ . This implies that  $\Gamma$  operates on the character group  $X^*(\mathcal{Z}_v)$  of  $\mathcal{Z}_v$ ; under this action of  $\Gamma$ ,  $\Phi(\mathcal{Z}_v)$ ,  $\Phi(\mathcal{Z}_v)^+$  and  $\Pi(\mathcal{Z}_v)$  are stable.

The restriction of roots in  $\Phi(\mathcal{Z}_v)$  to  $\mathcal{E}_v$  gives a bijective correspondence between the set of  $\Gamma$ -orbits in  $\Phi(\mathcal{Z}_v)$  and the set  $\Phi(\mathcal{E}_v)$ ; under this correspondence, the orbits in  $\Pi(\mathcal{Z}_v)$  correspond to the roots in  $\Pi(\mathcal{E}_v)$ , [32: 2.5]. Also, it is easy to see that the restriction to  $\mathcal{E}_v$  of a root  $\mathfrak{b}$  in  $\Phi(\mathcal{Z}_v)$  is a long root of the root system  $\Phi(\mathcal{E}_v)$  if and only if  $\mathfrak{b}$  is  $\Gamma$ -invariant.

For  $\mathfrak{b} \in \Phi(\mathcal{Z}_v)$ , let  $\Gamma_{\mathfrak{b}}$  be the isotropy group at  $\mathfrak{b}$  in  $\Gamma$ , and let  $\ell_v^{\mathfrak{b}}$  be the subfield of  $L_v$  fixed by  $\Gamma_{\mathfrak{b}}$ . Then for all  $\mathfrak{b} \in \Phi(\mathcal{Z}_v)$ ,  $\ell_v^{\mathfrak{b}}$  is a ramified extension of  $k_v$  of degree  $\leq 3$ .

For every  $b \in \Phi(\mathcal{E}_v)$ , we fix a root  $\mathfrak{b}$  in  $\Phi(\mathcal{Z}_v)$  such that (1) the restriction of  $\mathfrak{b}$  to  $\mathcal{E}_v$  is  $b$ , (2) if  $b$  is short,  $\ell_v^{\mathfrak{b}} = \ell_v$ , and (3) the root associated with  $-b$  is the negative of the root associated with  $b$ .

Let  $\pi_v$  be a uniformizing element of  $\ell_v$ . In case  $\mathcal{G}/k_v$  is an outer form of type  $A_r$  with  $r$  even, we let  $\lambda_v$  be as in 0.3 and assume  $\pi_v$  so chosen that  $\lambda_v \pi_v + \bar{\lambda}_v \bar{\pi}_v = 0$ . The ring of integers of  $\ell_v$  equals the direct sum of the  $\pi_v^i \mathfrak{o}_v$ ,  $0 \leq i < [\ell_v : k_v]$ .

The Lie algebra  $\mathfrak{g}_v = L(\mathcal{G}) \otimes_k k_v$  splits over  $L_v$  and the action of the Galois group  $\Gamma$  on  $L_v$  induces an action on  $\mathfrak{g}_v \otimes_{k_v} L_v$ .

The following assertion can be proved using the considerations in §§ 1, 2 and 7.1 of [28] (see also [6: II, §§ 4.3, 4.4]).

There exists a Chevalley basis  $\{X_{\mathfrak{b}} \mid \mathfrak{b} \in \Phi(\mathcal{Z}_v)\} \cup \{H_{\mathfrak{a}} \mid \mathfrak{a} \in \Pi(\mathcal{Z}_v)\}$  of the Lie algebra  $\mathfrak{g}_v \otimes_{k_v} L_v$  such that:

(i)  $\gamma(X_{\mathfrak{b}}) = X_{\gamma(\mathfrak{b})}$  for all  $\gamma \in \Gamma$  and  $\mathfrak{b} \in \Phi(\mathcal{Z}_v)$  whose restriction to  $\mathcal{E}_v$  is a non-divisible root in  $\Phi(\mathcal{E}_v)$ .

(ii)  $\gamma(X_{\mathfrak{b}}) = -X_{\mathfrak{b}}$  for  $\gamma \in \Gamma$ ,  $\gamma \neq 1$ , and any  $\mathfrak{b} \in \Phi(\mathcal{Z}_v)$  whose restriction to  $\mathcal{E}_v$  is a divisible root.

(iii) If  $\mathcal{G}/k_v$  is *not* an outer form of type  $A_r$ , with  $r$  even, then the union of the following sets is an  $\mathfrak{o}_v$ -basis of the Lie algebra  $L(\mathcal{G}_v) (\subset \mathfrak{g}_v)$ :

$$\begin{aligned} & \{H_{\mathfrak{a}} \mid \mathfrak{a} \in \Pi(\mathcal{E}_v) \text{ long}\}, \quad \left\{ \sum_{\gamma \in \Gamma/\Gamma_{\mathfrak{a}}} \gamma(\pi^i) H_{\gamma(\mathfrak{a})} \mid \mathfrak{a} \in \Pi(\mathcal{E}_v) \text{ short}, 0 \leq i < [\ell_v : k_v] \right\}, \\ & \{X_{\mathfrak{b}} \mid \mathfrak{b} \in \Phi(\mathcal{E}_v) \text{ long}\}, \quad \left\{ \sum_{\gamma \in \Gamma/\Gamma_{\mathfrak{b}}} \gamma(\pi^i) X_{\gamma(\mathfrak{b})} \mid \mathfrak{b} \in \Phi(\mathcal{E}_v) \text{ short}, 0 \leq i < [\ell_v : k_v] \right\}. \end{aligned}$$

(iv) If  $\mathcal{G}/k_v$  is an outer form of type  $A_r$  with  $r$  even, let  $\mathfrak{v}$  be the  $\mathbf{R}$ -valued homomorphism of the character group  $X^*(\mathcal{Z}_v)$  which takes the value  $+1$  at the unique root in  $\Pi(\mathcal{Z}_v)$  whose restriction to  $\mathcal{E}_v$  is a multipliable root and takes the value zero at all the other elements of  $\Pi(\mathcal{Z}_v)$ ; note that  $\mathfrak{v}(\mathfrak{b}) = 0$  or  $\pm 2$  if  $\mathfrak{b}$  is a root whose restriction to  $\mathcal{E}_v$  is neither multipliable nor divisible. Then the union of the following sets is an  $\mathfrak{o}_v$ -basis of the Lie algebra  $L(\mathcal{G}_v)$ :

$$\begin{aligned} & \{ \pi_v^i H_a + \bar{\pi}_v^i H_{\bar{a}} \mid a \in \Pi(\mathcal{E}_v), i = 0, 1 \}, \quad \{ \lambda_v \pi_v X_{\mathfrak{b}}, \lambda_v^{-1} \bar{\pi}_v X_{-\bar{\mathfrak{b}}} \mid \mathfrak{b} (\in \Phi(\mathcal{E}_v)^+) \text{ divisible} \}, \\ & \{ \pi_v^i X_{\mathfrak{b}} + \bar{\pi}_v^i X_{\bar{\mathfrak{b}}}; \lambda_v^{-1} \pi_v^i X_{-\mathfrak{b}} + \bar{\lambda}_v^{-1} \bar{\pi}_v^i X_{-\bar{\mathfrak{b}}} \mid \mathfrak{b} (\in \Phi(\mathcal{E}_v)^+) \text{ multipliable}, i = 0, 1 \}, \\ & \{ \lambda_v^{\frac{1}{2}\mathfrak{v}(\mathfrak{b})} \pi_v^i X_{\mathfrak{b}} + \bar{\lambda}_v^{\frac{1}{2}\mathfrak{v}(\mathfrak{b})} \bar{\pi}_v^i X_{\bar{\mathfrak{b}}} \mid \mathfrak{b} \in \Phi(\mathcal{E}_v), \mathfrak{b} \text{ nonmultipliable and nondivisible}, i = 0, 1 \}; \end{aligned}$$

where for  $\mathfrak{b} \in \Phi(\mathcal{Z}_v)$ ,  $\bar{\mathfrak{b}}$  denotes its conjugate under the nontrivial element of the Galois group of  $\ell_v/k_v$ .

Now using the above basis of  $L(\mathcal{G}_v)$ , and the fact that  $\omega^{\text{Ch}} = \alpha\omega$ , it is not difficult to see that  $|\alpha^2|_v^{-1} \gamma_v^2 = |\mathfrak{d}(\ell_v/k_v)|_v^{-s(\mathcal{G})}$ . Therefore,

$$\prod_{v \in V_f} (|\alpha^2|_v^{-1} \gamma_v^2) = \prod_{v \in \mathcal{R}} |\mathfrak{d}(\ell_v/k_v)|_v^{-s(\mathcal{G})} = (D_\ell/D_k^{(\ell:k)})^{s(\mathcal{G})};$$

see the appendix at the end of this paper. (Recall that for  $v \in V_f - \mathcal{R}$ ,  $\ell \otimes_k k_v$  is a direct sum of certain unramified extensions of  $k_v$ , and for any unramified extension  $K$  of  $k_v$ ,  $|\mathfrak{d}(K/k_v)|_v = 1$ .) This proves the theorem.

## 2. Volumes of parahoric subgroups

We begin with the following general lemma.

**2.0. Lemma.** — *Let  $F$  be an arbitrary field.  $G$  and  $G'$  be connected semi-simple  $F$ -groups. Assume that  $G$  is an inner  $F$ -form of  $G'$  and  $G'$  is quasi-split over  $F$ . Let  $F'$  be a separable extension of  $F$  such that  $G$  is quasi-split over  $F'$ . Then  $G$  and  $G'$  are isomorphic over  $F'$ . Moreover, if  $F'$  is a Galois extension of  $F$ , we can find an  $F'$ -isomorphism  $\varphi: G \rightarrow G'$  such that for all  $\gamma$  in the Galois group of  $F'/F$ ,  $\varphi^{-1} \cdot \gamma \varphi \in \text{Int}(G)$ .*

*Proof.* — By assumption, there exists an isomorphism  $f: G \rightarrow G'$  defined over a (fixed) separable closure  $F'_s$  of  $F'$  such that for all  $\gamma$  in the Galois group  $\Gamma(F'_s/F)$  of  $F'_s/F$ , we have  $a_\gamma := f^{-1} \cdot \gamma f \in \text{Int}(G)$ . Choose a Borel subgroup  $B$  of  $G$  (resp.  $B'$  of  $G'$ ) and a maximal torus  $T$  (resp.  $T'$ ) of  $B$  (resp.  $B'$ ), all defined over  $F'$ . Then we can arrange that  $f$  maps  $B$  and  $T$  onto  $B'$  and  $T'$  respectively. Then so does  $\gamma f$  for all  $\gamma$  in the Galois group  $\Gamma(F'_s/F')$  ( $\subset \Gamma(F'_s/F)$ ) of  $F'_s/F'$ . Hence for  $\gamma \in \Gamma(F'_s/F')$ ,  $a_\gamma$  preserves  $B, T$  and so it is of the form  $\text{Int } t_\gamma$  ( $t_\gamma \in T$ ).

Assume now that  $G$  is adjoint. Then  $t_\gamma$  is uniquely determined and it follows that  $\gamma \mapsto t_\gamma$  is a 1-cocycle on  $\Gamma(F'_s/F')$  with values in  $T$ . The Galois group  $\Gamma(F'_s/F')$  acts on  $X^*(T)$  by permuting the simple roots. These form a basis of  $X^*(T)$  since  $G$  is adjoint. Therefore  $\Gamma(F'_s/F')$  acts as a permutation representation and this implies that  $T$  is a

direct product of certain tori of the form  $R_{L/F'}(\mathrm{GL}_1)$ . Therefore it is cohomologically trivial over  $F'$  and so  $(a_\gamma)$  is a coboundary: there exists a  $t \in T$  such that  $t_\gamma = t \cdot \gamma t^{-1}$ . Then for  $\gamma \in \Gamma(F'_s/F')$ ,  $(f \cdot \mathrm{Int} t)^{-1} \cdot \gamma (f \cdot \mathrm{Int} t) = \mathrm{Int} t^{-1} \cdot a_\gamma \cdot \mathrm{Int} \gamma t = \mathrm{Id}$ , hence the isomorphism  $\varphi := f \cdot \mathrm{Int} t$  is defined over  $F'$ .

If  $G$  is not adjoint, let  $G \rightarrow \mathrm{Ad} G$  be the canonical central isogeny. If  $\hat{t}$  is in the inverse image of the previous  $t$ , then again  $\varphi := f \cdot \mathrm{Int} \hat{t}$  is defined over  $F'$ . Moreover, since  $f^{-1} \cdot \gamma f \in \mathrm{Int}(G)$  for all  $\gamma \in \Gamma(F'_s/F)$ , it is obvious that if  $F'$  is a Galois extension of  $F$ , then for all  $\gamma$  in the Galois group of  $F'/F$ ,  $\varphi^{-1} \cdot \gamma \varphi \in \mathrm{Int}(G)$ .

**2.1.** Let  $G$  be an absolutely quasi-simple, simply connected algebraic  $k$ -group which is an inner form of  $\mathcal{G}$ . It is known that for all but finitely many places  $v$ ,  $G$  is quasi-split over  $k_v$  ([30: 4.9 (ii)]) and so it is isomorphic to  $\mathcal{G}$  over  $k_v$  (Lemma 2.0).

We shall use the notation introduced in the previous section. Thus  $\omega$  is the  $\mathcal{G}$ -invariant exterior form on  $\mathcal{G}$  of maximal degree (and defined over  $k$ ) fixed in 1.1.

Let  $\varphi : G \rightarrow \mathcal{G}$  be an isomorphism defined over a (not necessarily finite) Galois extension  $K$  of  $k$  such that for every  $\gamma$  in the Galois group  $\Gamma(K/k)$  of  $K/k$ ,  $\varphi^{-1} \cdot \gamma \varphi$  is an inner automorphism of  $G$ . Then  $\omega^* := \varphi^*(\omega)$  is an invariant exterior form on  $G$  of maximal degree; moreover it is defined over  $k$  (see [15: pp. 475-476]). If  $\psi : G \rightarrow \mathcal{G}$  is some other isomorphism defined over an extension of  $k$ , then as any commutative quotient of  $\mathrm{Aut}(\mathcal{G})/\mathrm{Int}(\mathcal{G})$  is of order  $\leq 3$ , it is clear that  $\psi^*(\omega) = u(\psi) \omega^*$ , where  $u(\psi)$  is a root of unity of order  $\leq 3$ .

For each  $v \in V$ ,  $\omega$  (resp.  $\omega^*$ ), together with the normalized absolute value  $|\cdot|_v$  on  $k_v$  (see 0.1), determines a Haar measure on  $\mathcal{G}(k_v)$  (resp.  $G(k_v)$ ) which we shall denote by  $\omega_v$  (resp.  $\omega_v^*$ ). The Haar measure  $\omega_v^*$  on  $G(k_v)$  is *independent* of the choice of the isomorphism  $\varphi : G \rightarrow \mathcal{G}$ .

**2.2.** A collection  $P = (P_v)_{v \in V_f}$  of parahoric subgroups  $P_v$  of  $G(k_v)$  is said to be *coherent* if  $\prod_{v \in V_\infty} G(k_v) \prod_{v \in V_f} P_v$  is an open subgroup of the adèle group  $G(A)$ .

Let a coherent collection  $P = (P_v)_{v \in V_f}$  of parahoric subgroups be given. For  $v \in V_f$ , let  $G_v$  be the smooth affine  $\mathfrak{o}_v$ -group scheme associated with the parahoric subgroup  $P_v$  of  $G(k_v)$  (0.7). Its generic fiber ( $= G_v \times_{\mathfrak{o}_v} k_v$ ) is isomorphic to  $G \times_k k_v$  and its group of integral points is isomorphic to  $P_v$ .

Let the parahoric subgroups  $\mathcal{P}_v$  and the smooth affine  $\mathfrak{o}_v$ -group scheme  $\mathcal{G}_v$  associated with  $\mathcal{P}_v$  be as in 1.2 and 1.3 respectively. We shall denote by  $\overline{\mathcal{G}}_v$  (resp.  $\overline{G}_v$ ) the group  $\mathcal{G}_v \times_{\mathfrak{o}_v} \mathfrak{f}_v$  (resp.  $G_v \times_{\mathfrak{o}_v} \mathfrak{f}_v$ ) over the (finite) residue field  $\mathfrak{f}_v$  of  $k_v$ . It is known (see [33: 3.5.2]) that since  $\mathcal{G}$  and  $G$  are simply connected, the groups  $\overline{\mathcal{G}}_v$  and  $\overline{G}_v$  are connected; also the ‘‘reduction mod  $\mathfrak{p}_v$ ’’ homomorphisms  $\mathcal{P}_v = \mathcal{G}_v(\mathfrak{o}_v) \rightarrow \overline{\mathcal{G}}_v(\mathfrak{f}_v)$  and  $P_v = G_v(\mathfrak{o}_v) \rightarrow \overline{G}_v(\mathfrak{f}_v)$  are surjective [33: 3.4.4]. Both  $\overline{\mathcal{G}}_v$  and  $\overline{G}_v$  admit a Levi decomposition over  $\mathfrak{f}_v$  [33: 3.5]. Let  $\overline{\mathcal{M}}_v$  (resp.  $\overline{M}_v$ ) be a fixed maximal connected reductive  $\mathfrak{f}_v$ -subgroup such that

$$\overline{\mathcal{G}}_v = \overline{\mathcal{M}}_v \cdot R_u(\overline{\mathcal{G}}_v) \quad (\text{resp. } \overline{G}_v = \overline{M}_v \cdot R_u(\overline{G}_v)),$$

where  $R_u(\overline{\mathcal{G}}_v)$  (resp.  $R_u(\overline{G}_v)$ ) is the unipotent radical of  $\overline{\mathcal{G}}_v$  (resp.  $\overline{G}_v$ ). As  $\mathfrak{f}_v$  is a finite field, both  $\overline{\mathcal{M}}_v$  and  $\overline{M}_v$  are quasi-split over  $\mathfrak{f}_v$  (see, for example, [2: Proposition 16.6]). We fix a Borel  $\mathfrak{f}_v$ -subgroup  $\overline{\mathcal{B}}_v$  of  $\overline{\mathcal{M}}_v$ ,  $\overline{B}_v$  of  $\overline{M}_v$  and a maximal  $\mathfrak{f}_v$ -torus  $\overline{\mathcal{E}}_v$  of  $\overline{\mathcal{B}}_v$ ,  $\overline{T}_v$  of  $\overline{B}_v$ . Let  $\overline{\mathcal{U}}_v$  (resp.  $\overline{U}_v$ ) be the unipotent radical of  $\overline{\mathcal{B}}_v$  (resp.  $\overline{B}_v$ ).

Let  $\mathcal{I}_v$  (resp.  $I_v$ ) be the inverse image in  $\mathcal{P}_v$  (resp.  $P_v$ ) of  $\overline{\mathcal{B}}_v(\mathfrak{f}_v) \cdot R_u(\overline{\mathcal{G}}_v)(\mathfrak{f}_v)$  (resp.  $\overline{B}_v(\mathfrak{f}_v) \cdot R_u(\overline{G}_v)(\mathfrak{f}_v)$ ) under the reduction map  $\mathcal{P}_v \rightarrow \overline{\mathcal{G}}_v(\mathfrak{f}_v)$  (resp.  $P_v \rightarrow \overline{G}_v(\mathfrak{f}_v)$ ). Then  $\mathcal{I}_v$  (resp.  $I_v$ ) is an Iwahori subgroup of  $\mathcal{G}(k_v)$  (resp.  $G(k_v)$ ). Obviously,  $[\mathcal{P}_v : \mathcal{I}_v] = [\overline{\mathcal{M}}_v(\mathfrak{f}_v) : \overline{\mathcal{B}}_v(\mathfrak{f}_v)]$  and  $[P_v : I_v] = [\overline{M}_v(\mathfrak{f}_v) : \overline{B}_v(\mathfrak{f}_v)]$ .

For all but finitely many  $v$ ,  $P_v$  is a hyperspecial parahoric subgroup of  $G(k_v)$  and hence there exists an isomorphism of  $G(k_v)$  onto  $\mathcal{G}(k_v)$  which carries  $P_v$  onto  $\mathcal{P}_v$  (see [33: 2.5]); this isomorphism induces an isomorphism of the  $\mathfrak{o}_v$ -group scheme  $G_v$  onto the  $\mathfrak{o}_v$ -group scheme  $\mathcal{G}_v$ . Therefore, for all but finitely many  $v$ ,  $\overline{M}_v$  is isomorphic to  $\overline{\mathcal{M}}_v$  over  $\mathfrak{f}_v$ .

**2.3. Proposition.** — For  $v \in V_f$ ,

$$\omega_v^*(I_v) = \frac{\#\overline{T}_v(\mathfrak{f}_v)}{\#\overline{\mathcal{E}}_v(\mathfrak{f}_v)} \omega_v(\mathcal{I}_v)$$

and

$$\omega_v^*(P_v) = \frac{[\overline{M}_v(\mathfrak{f}_v) : \overline{B}_v(\mathfrak{f}_v)]}{[\overline{\mathcal{M}}_v(\mathfrak{f}_v) : \overline{\mathcal{B}}_v(\mathfrak{f}_v)]} \cdot \frac{\#\overline{T}_v(\mathfrak{f}_v)}{\#\overline{\mathcal{E}}_v(\mathfrak{f}_v)} \cdot \omega_v(\mathcal{P}_v).$$

*Proof.* — According to the Bruhat-Tits theory, the Iwahori subgroup  $\mathcal{I}_v$  (resp.  $I_v$ ) determines a smooth affine  $\mathfrak{o}_v$ -group scheme  $\mathcal{G}_{\mathcal{I}_v}$  (resp.  $G_{I_v}$ ) whose generic fiber is  $\mathcal{G} \times_k k_v$  (resp.  $G \times_k k_v$ ) and  $\mathcal{G}_{\mathcal{I}_v}(\mathfrak{o}_v) \cong \mathcal{I}_v$  (resp.  $G_{I_v}(\mathfrak{o}_v) \cong I_v$ ); see 0.7.

It is a well known consequence of a theorem of Steinberg [31] that since the residue field  $\hat{\mathfrak{f}}_v$  of the maximal unramified extension  $\hat{k}_v$  of  $k_v$  is algebraically closed,  $G$  is quasi-split over  $\hat{k}_v$ . Now since  $\mathcal{G}$  is a quasi-split inner  $k$ -form of  $G$ , we conclude that  $G$  is isomorphic to  $\mathcal{G}$  over  $\hat{k}_v$  and there is an isomorphism  $\varphi_v : G \times_k \hat{k}_v \rightarrow \mathcal{G} \times_k \hat{k}_v$  such that  $\varphi_v^{-1} \cdot \gamma \varphi_v$  is an inner automorphism of  $G$  for all  $\gamma \in \text{Gal}(\hat{k}_v/k_v)$ ; see Lemma 2.0. The exterior form  $\varphi_v^*(\omega)$  is then defined over  $k_v$  and the Haar measure on  $G(k_v)$  determined by it (and the absolute value  $|\cdot|_v$  on  $k_v$ ) is  $\omega_v^*$  (2.1). Now let  $\hat{\mathfrak{o}}_v$  be the ring of integers of  $\hat{k}_v$ . Then  $\hat{I}_v := G_{I_v}(\hat{\mathfrak{o}}_v)$  (resp.  $\hat{\mathcal{I}}_v := \mathcal{G}_{\mathcal{I}_v}(\hat{\mathfrak{o}}_v)$ ) is an Iwahori subgroup of  $G(\hat{k}_v)$  (resp.  $\mathcal{G}(\hat{k}_v)$ ) and in view of the conjugacy of Iwahori subgroups, we may (and we will) assume that  $\varphi_v(\hat{I}_v) = \hat{\mathcal{I}}_v$ . Then the isomorphism  $\varphi_v$  is induced from a unique isomorphism

$$G_{I_v} \times_{\mathfrak{o}_v} \hat{\mathfrak{o}}_v \rightarrow \mathcal{G}_{\mathcal{I}_v} \times_{\mathfrak{o}_v} \hat{\mathfrak{o}}_v,$$

defined over  $\hat{\mathfrak{o}}_v$ , which we denote again by  $\varphi_v$ , by base change  $\hat{\mathfrak{o}}_v \hookrightarrow \hat{k}_v$ ; this is seen at once using the description of the coordinate rings of the group schemes  $G_{I_v}$  and  $\mathcal{G}_{\mathcal{I}_v}$  (given in 0.7).

It is obvious that there is an  $a_v \in k_v^\times$  such that the exterior form  $a_v \omega$  induces an invariant exterior form on the group scheme  $\mathcal{G}_{\mathcal{J}_v}$ , which is defined over  $\mathfrak{o}_v$  and whose reduction to the group scheme  $\bar{\mathcal{G}}_{\mathcal{J}_v} := \mathcal{G}_{\mathcal{J}_v} \times_{\mathfrak{o}_v} \hat{\mathfrak{f}}_v$  is not zero. Then as  $\varphi_v$  is an isomorphism defined over  $\hat{\mathfrak{o}}_v$ , the exterior form  $\varphi_v^*(a_v \omega) = a_v \varphi_v^*(\omega)$  on the  $\mathfrak{o}_v$ -group scheme  $G_{I_v}$  is defined over  $\mathfrak{o}_v$  and its reduction to  $G_{I_v} \times_{\mathfrak{o}_v} \hat{\mathfrak{f}}_v$ , and hence also to  $\bar{G}_{I_v} := G_{I_v} \times_{\mathfrak{o}_v} \hat{\mathfrak{f}}_v$ , is not zero.

The inclusion  $I_v \subset P_v$  (resp.  $\mathcal{J}_v \subset \mathcal{P}_v$ ) induces a homomorphism  $\bar{G}_{I_v} \rightarrow \bar{G}_v$  (resp.  $\bar{\mathcal{G}}_{I_v} \rightarrow \bar{\mathcal{G}}_v$ ), where  $\bar{G}_v$  (resp.  $\bar{\mathcal{G}}_v$ ) is as in 2.2. Also, there is a (unique) maximal  $\hat{\mathfrak{f}}_v$ -torus of  $\bar{G}_{I_v}$  (resp.  $\bar{\mathcal{G}}_{\mathcal{J}_v}$ ) which is mapped isomorphically onto  $\bar{T}_v$  (resp.  $\bar{\mathcal{E}}_v$ ) under this homomorphism. Since no confusion is likely, we denote this torus of  $\bar{G}_{I_v}$  (resp.  $\bar{\mathcal{G}}_{\mathcal{J}_v}$ ) again by  $\bar{T}_v$  (resp.  $\bar{\mathcal{E}}_v$ ). Then  $\bar{G}_{I_v} = \bar{T}_v \cdot R_u(\bar{G}_{I_v})$  and  $\bar{\mathcal{G}}_{\mathcal{J}_v} = \bar{\mathcal{E}}_v \cdot R_u(\bar{\mathcal{G}}_{\mathcal{J}_v})$ .

Now as the Haar measure on  $\mathcal{G}(k_v)$  given by  $a_v \omega$  is  $|a_v|_v \omega_v$ , we conclude (cf. [24: I, 2.5]) that

$$\begin{aligned} |a_v|_v \omega_v(\mathcal{J}_v) &= \# \bar{\mathcal{G}}_{\mathcal{J}_v}(\hat{\mathfrak{f}}_v) \cdot q_v^{-\dim \mathcal{G}} \\ &= \# \bar{\mathcal{E}}_v(\hat{\mathfrak{f}}_v) \cdot q_v^{-\dim \bar{\mathcal{E}}_v}. \end{aligned}$$

Similarly, as the Haar measure on  $G(k_v)$  given by  $\varphi_v^*(a_v \omega) = a_v \varphi_v^*(\omega)$  is  $|a_v|_v \omega_v^*$ , we have

$$|a_v|_v \omega_v^*(I_v) = \# \bar{T}_v(\hat{\mathfrak{f}}_v) \cdot q_v^{-\dim \bar{T}_v}.$$

Now note that

$$\dim \bar{\mathcal{E}}_v = \hat{k}_v\text{-rank } \mathcal{G} = \hat{k}_v\text{-rank } G = \dim \bar{T}_v,$$

and so we deduce from the above that

$$\omega_v^*(I_v) = \frac{\# \bar{T}_v(\hat{\mathfrak{f}}_v)}{\# \bar{\mathcal{E}}_v(\hat{\mathfrak{f}}_v)} \cdot \omega_v(\mathcal{J}_v).$$

Then

$$\begin{aligned} \omega_v^*(P_v) &= [P_v : I_v] \omega_v^*(I_v) \\ &= \frac{[P_v : I_v]}{[\mathcal{P}_v : \mathcal{J}_v]} \cdot \frac{\omega_v^*(I_v)}{\omega_v(\mathcal{J}_v)} \cdot \omega_v(\mathcal{P}_v) \\ &= \frac{[\bar{M}_v(\hat{\mathfrak{f}}_v) : \bar{B}_v(\hat{\mathfrak{f}}_v)]}{[\bar{\mathcal{M}}_v(\hat{\mathfrak{f}}_v) : \bar{\mathcal{B}}_v(\hat{\mathfrak{f}}_v)]} \cdot \frac{\# \bar{T}_v(\hat{\mathfrak{f}}_v)}{\# \bar{\mathcal{E}}_v(\hat{\mathfrak{f}}_v)} \cdot \omega_v(\mathcal{P}_v). \end{aligned}$$

This proves the proposition.

**2.4.** Let  $\Delta_v$  be the basis of the *absolute* affine root system of  $G$  at  $v$  (i.e. the affine root system of  $G$  over the maximal unramified extension  $\hat{k}_v$  of  $k_v$ ) determined by the Iwahori subgroup  $I_v$ .

Let  $\Theta_v$  be the subset of  $\Delta_v$  corresponding to the parahoric subgroup  $P_v$ . The Galois group of  $\hat{k}_v/k_v$  operates on  $\Delta_v$ , leaving  $\Theta_v$  stable. The Tits index [32] of the reductive group  $\bar{M}_v/\hat{\mathfrak{f}}_v$  is obtained from the Dynkin diagram of  $\Delta_v$  together with the

action of the Galois group of  $\hat{k}_v/k_v$  (i.e. “the local index” of  $G/k_v$ ) by *deleting* the vertices corresponding to the roots in  $\Theta_v$  and all the edges containing such vertices (see [33: 3.5.2]). Note that there is a canonical identification of the Galois group of  $\hat{k}_v/k_v$  with the Galois group of  $\hat{f}_v/f_v$ , where  $\hat{f}_v$  is the residue field of  $\hat{k}_v$ —it is an algebraic closure of  $f_v$ .

**2.5.** Fixing a  $\hat{k}_v$ -isomorphism of  $G$  onto  $\mathcal{G}$ , we identify the root system as well as the affine root system of  $\mathcal{G}/\hat{k}_v$  with those of  $G/\hat{k}_v$ .

Let  $d_v \in \Delta_v$  be the affine simple root corresponding to the parahoric subgroup  $\mathcal{P}_v$ . Then  $\Delta_v - \{d_v\}$  can be identified with a basis of the absolute root system  $\Psi_v$  of the reductive group  $\bar{\mathcal{M}}_v$ ; see [33: 3.5.2]. Since  $\mathcal{P}_v$  is a hyperspecial parahoric subgroup if  $\mathcal{G}$  splits over an unramified extension of  $k_v$ , otherwise ( $\mathcal{G}$  is residually split over  $k_v$  and)  $\mathcal{P}_v$  is a special parahoric subgroup,  $d_v$  is “special” ([28: 7.1]).  $\Psi_v$  is then a reduced and irreducible root system of rank  $r_v := \hat{k}_v$ -rank  $\mathcal{G} = \hat{k}_v$ -rank  $G$ . Hence the reductive group  $\bar{\mathcal{M}}_v$  is in fact absolutely quasi-simple and its absolute rank is  $r_v$ .

**2.6.** We note here, for future use, the following empirical fact about connected semi-simple groups defined over a finite field: *If  $H$  is a connected semi-simple group defined over a finite field  $\mathfrak{f}$  of cardinality  $q$ , then  $\#H(\mathfrak{f}) < q^{\dim H}$ .* This assertion can be checked by looking at the table of orders of finite groups of Lie type given in [25: Table 1]. Note that connected isogeneous groups over a finite field have an equal number of rational points [2: Proposition 16.8]; note also that it suffices to check the assertion for absolutely simple groups since every nonabelian simple group over  $\mathfrak{f}$  is obtained by restriction of scalars from an absolutely simple group defined over a finite extension of  $\mathfrak{f}$ .

The two lemmas that follow (2.7 and 2.8) are needed for the proof of Proposition 2.10.

Let  $\Theta_v$  be as in 2.4 and  $t_v = \#\Theta_v - 1$ .

**2.7. Lemma.** — (i)  $\dim \bar{\mathcal{M}}_v \geq r_v(r_v + 2)$ .

(ii) *If either  $G$  is not quasi-split over  $k_v$ , or  $\mathcal{P}_v$  is not special, or  $G$  splits over  $\hat{k}_v$  but  $\mathcal{P}_v$  is not hyperspecial, then*

$$\dim \bar{\mathcal{M}}_v - \dim \bar{M}_v \geq 2r_v.$$

*If, moreover,  $t_v \geq 1$ ,*

$$\dim \bar{\mathcal{M}}_v - \dim \bar{M}_v \geq 2(r_v + t_v - 1).$$

*Proof.* — The absolute root system of  $\bar{M}_v$  is the root system  $\Phi_v$  with the basis  $\Delta_v - \Theta_v$  (see [33: 3.5.2]). Hence,  $\dim \bar{M}_v = r_v + \#\Phi_v$ .

Let  $d_v$  and  $\Psi_v$  be as in 2.5. Since  $\Psi_v$  is the absolute root system of  $\bar{\mathcal{M}}_v$  (2.5),  $\dim \bar{\mathcal{M}}_v = r_v + \#\Psi_v$ . Now to prove assertion (i), we just need to note that among the reduced and irreducible root systems of a given rank  $s$ , one with the smallest cardinality is of type  $A_s$ , which has  $s(s+1)$  roots.

We shall now prove (ii). We begin by observing that the root system  $\Psi_v$  can be identified with the root system consisting of the non-divisible roots of the root system of  $\mathcal{G}/\hat{k}_v$ , and with this identification, the gradient of  $d_v$  is the negative of the dominant (i.e. the highest) root of the root system of  $\mathcal{G}/\hat{k}_v$ , with respect to the basis determined by  $\Delta_v - \{d_v\}$ , if  $\mathcal{G}$  splits over  $\hat{k}_v$ , and is the negative of the dominant short root if  $\mathcal{G}$  does not split over  $\hat{k}_v$  but its  $\hat{k}_v$ -root system is reduced ([28: 2.10]).

We now first take-up the case where the  $\hat{k}_v$ -root system of  $G$  is not reduced.  $G/k_v$  is then an outer form of type  $A_r$ , with  $r$  even, say  $r = 2n$ , which does not split over  $\hat{k}_v$ . In this case  $G$  is quasi-split over  $k_v$ ,  $\mathcal{M}_v$  is an absolutely quasi-simple group of type  $B_n$  and  $\bar{M}_v$  is isogeneous to a  $\mathfrak{f}_v$ -group which is the direct product of a  $t_v$ -dimensional torus and a semi-simple group whose absolute Dynkin diagram is obtained from the Dynkin diagram of  $\Delta_v$  by deleting the vertices corresponding to the roots in  $\Theta_v$  and all the edges containing such vertices (2.4). From this description of  $\mathcal{M}_v$  and  $\bar{M}_v$ , (ii) follows at once. Note that since  $P_v$  is not special, if  $\Theta_v$  contains a special root, then it contains at least two roots (so  $t_v \geq 1$ ).

We assume now that the  $\hat{k}_v$ -root system of  $G$  is reduced. If  $G$  is not quasi-split over  $k_v$ , then the Galois group of  $\hat{k}_v/k_v$  does not fix any special (affine) root ([28: 7.2]), so if  $\Theta_v$  contains a special root, being stable under the Galois group, it contains at least two special roots. It is easy to see that whenever  $\Theta_v$  contains two or more special roots,  $\Phi_v$  can be realized as an integrally closed proper subroot system of  $\Psi_v$ . This is also the case if  $G$  splits over  $\hat{k}_v$  and  $\Theta_v$  contains a non-special root. On the other hand, if  $G$  does not split over  $\hat{k}_v$ , and  $\Theta_v$  contains a non-special root, then the dual root system  $\Phi_v^\vee$  can be realized as an integrally closed proper subroot system of the dual  $\Psi_v^\vee$ .

It is well-known that for any integrally closed proper subroot system  $\Phi$  of an irreducible and reduced root system  $\Psi$  of rank  $s$ ,  $\#\Psi - \#\Phi \geq 2s$ ; see [3: Corollaire on p. 210 and Théorème 4]. From this we conclude that

$$\dim \mathcal{M}_v - \dim \bar{M}_v = (\#\Psi_v - \#\Phi_v) \geq 2r_v,$$

and if  $t_v \geq 1$ , then in fact  $\#\Psi_v - \#\Phi_v \geq 2r_v + 2(t_v - 1)$ .

**2.8. Lemma.** — *Let  $\mathfrak{f}$  be a finite field with  $q$  elements and  $T$  an  $s$ -dimensional  $\mathfrak{f}$ -torus. Then  $\#T(\mathfrak{f}) \leq (q + 1)^s$ .*

*Proof.* — Let  $\mathfrak{f}'$  be the smallest extension of  $\mathfrak{f}$  over which  $T$  splits;  $\mathfrak{f}'$  is necessarily a finite cyclic extension of  $\mathfrak{f}$ . Then  $T$  is isogeneous to a direct product of tori  $T_i$  such that the canonical representation of the Galois group  $\Gamma(\mathfrak{f}'/\mathfrak{f})$  on  $X^*(T_i) \otimes_{\mathbf{Z}} \mathbf{Q}$  is irreducible. Since connected isogeneous groups over a finite field have the same number of rational elements ([2: Proposition 16.8]), this reduces us to the case where the representation of  $\Gamma(\mathfrak{f}'/\mathfrak{f})$  on  $X^*(T) \otimes_{\mathbf{Z}} \mathbf{Q}$  is irreducible. Now let  $[\mathfrak{f}' : \mathfrak{f}] = n$ . Then  $(s =) \dim T$  equals the value of Euler's  $\varphi$ -function at  $n$ . It is not difficult to see, using, for example, Möbius inversion in the multiplicative form, that  $\#T(\mathfrak{f}) = P_n(q)$ , where  $P_n(x) \in \mathbf{Z}[x]$  is the



$n$ -th cyclotomic polynomial. We recall that  $P_n(x)$  is a monic polynomial of degree  $s$  whose roots are precisely the primitive  $n$ -th roots of unity. From this it follows at once that  $\#T(\mathfrak{f}) = P_n(q) \leq (q+1)^s$ .

**2.9.** Let  $\gamma_v$  be as in 1.3. Then (cf. [24: I, 2.5])

$$\gamma_v \omega_v(\mathcal{P}_v) = \#\bar{\mathcal{G}}_v(\mathfrak{f}_v) \cdot q_v^{-\dim \mathcal{G}} = \#\bar{\mathcal{M}}_v(\mathfrak{f}_v) \cdot q_v^{-\dim \bar{\mathcal{M}}_v};$$

where  $\bar{\mathcal{G}}_v$  and  $\bar{\mathcal{M}}_v$  are as in 2.2.

**2.10. Proposition.** — For  $v \in V_f$ ,

- (i) 
$$\gamma_v \omega_v^*(\mathbf{I}_v) = \frac{\#\bar{T}_v(\mathfrak{f}_v)}{q_v^{(r_v + \dim \bar{\mathcal{M}}_v)/2}}.$$
- (ii) 
$$\frac{\#\bar{T}_v(\mathfrak{f}_v)}{q_v^{(r_v + \dim \bar{\mathcal{M}}_v)/2}} \leq (q_v + 1)^{r_v} q_v^{-r_v(r_v+3)/2}.$$
- (iii) 
$$\gamma_v \omega_v^*(\mathbf{P}_v) = \frac{\#\bar{M}_v(\mathfrak{f}_v)}{q_v^{(\dim \bar{M}_v + \dim \bar{\mathcal{M}}_v)/2}}.$$

(iv) For all  $v \in V_f$ ,

$$\frac{\#\bar{M}_v(\mathfrak{f}_v)}{q_v^{(\dim \bar{M}_v + \dim \bar{\mathcal{M}}_v)/2}} < 1.$$

Moreover if either  $G$  is not quasi-split over  $k_v$ , or  $\mathbf{P}_v$  is not special, or  $G$  splits over  $\hat{k}_v$  but  $\mathbf{P}_v$  is not hyperspecial, then

$$\frac{\#\bar{M}_v(\mathfrak{f}_v)}{q_v^{(\dim \bar{M}_v + \dim \bar{\mathcal{M}}_v)/2}} \leq (q_v + 1) q_v^{-r_v-1}.$$

*Proof.* — (1) According to Proposition 2.3,

$$\omega_v^*(\mathbf{I}_v) = \frac{\#\bar{T}_v(\mathfrak{f}_v)}{\#\bar{\mathcal{E}}_v(\mathfrak{f}_v)} \cdot \omega_v(\mathcal{I}_v),$$

and hence,

$$\begin{aligned} \gamma_v \omega_v^*(\mathbf{I}_v) &= \frac{\#\bar{T}_v(\mathfrak{f}_v)}{\#\bar{\mathcal{E}}_v(\mathfrak{f}_v)} \cdot \frac{\gamma_v \omega_v(\mathcal{P}_v)}{[\mathcal{P}_v : \mathcal{I}_v]} \\ &= \frac{\#\bar{T}_v(\mathfrak{f}_v)}{\#\bar{\mathcal{E}}_v(\mathfrak{f}_v)} \cdot \frac{\#\bar{\mathcal{M}}_v(\mathfrak{f}_v) \cdot q_v^{-\dim \bar{\mathcal{M}}_v}}{[\bar{\mathcal{M}}_v(\mathfrak{f}_v) : \bar{\mathcal{B}}_v(\mathfrak{f}_v)]} \quad (\text{see 2.2 and 2.9}) \\ &= \frac{\#\bar{T}_v(\mathfrak{f}_v)}{\#\bar{\mathcal{E}}_v(\mathfrak{f}_v)} \cdot \frac{\#\bar{\mathcal{B}}_v(\mathfrak{f}_v)}{q_v^{\dim \bar{\mathcal{M}}_v}}. \end{aligned}$$

Now recall that  $\bar{\mathcal{U}}_v$  (resp.  $\bar{U}_v$ ) is the unipotent radical of  $\bar{\mathcal{B}}_v$  (resp.  $\bar{B}_v$ ). So

$$\bar{\mathcal{B}}_v(\mathfrak{f}_v) = \bar{\mathcal{E}}_v(\mathfrak{f}_v) \cdot \bar{\mathcal{U}}_v(\mathfrak{f}_v), \quad \bar{B}_v(\mathfrak{f}_v) = \bar{T}_v(\mathfrak{f}_v) \cdot \bar{U}_v(\mathfrak{f}_v),$$

$$\#\bar{\mathcal{U}}_v(\mathfrak{f}_v) = q_v^{\dim \bar{\mathcal{U}}_v}, \quad \#\bar{U}_v(\mathfrak{f}_v) = q_v^{\dim \bar{U}_v},$$

and  $\dim \bar{\mathcal{M}}_v = \dim \bar{\mathcal{E}}_v + 2 \dim \bar{\mathcal{U}}_v; \quad \dim \bar{M}_v = \dim \bar{T}_v + 2 \dim \bar{U}_v.$

Moreover, as we have observed before,

$$\dim \bar{\mathcal{E}}_v = r_v = \dim \bar{T}_v.$$

So

$$\begin{aligned} \gamma_v \omega_v^*(\mathbf{I}_v) &= \frac{\#\bar{T}_v(\mathfrak{f}_v)}{\#\bar{\mathcal{E}}_v(\mathfrak{f}_v)} \cdot \frac{\#\bar{\mathcal{B}}_v(\mathfrak{f}_v)}{q_v^{\dim \bar{\mathcal{M}}_v}} \\ &= \frac{\#\bar{T}_v(\mathfrak{f}_v)}{q_v^{r_v + \dim \bar{\mathcal{U}}_v}} = \frac{\#\bar{T}_v(\mathfrak{f}_v)}{q_v^{(r_v + \dim \bar{\mathcal{M}}_v)/2}}. \end{aligned}$$

Since  $\#\bar{T}_v(\mathfrak{f}_v) \leq (q_v + 1)^{r_v}$  by 2.8, the second assertion of the proposition follows from Lemma 2.7 (i).

$$\begin{aligned} (2) \quad \gamma_v \omega_v^*(\mathbf{P}_v) &= \frac{[\bar{M}_v(\mathfrak{f}_v) : \bar{B}_v(\mathfrak{f}_v)]}{[\bar{\mathcal{M}}_v(\mathfrak{f}_v) : \bar{\mathcal{B}}_v(\mathfrak{f}_v)]} \cdot \frac{\#\bar{T}_v(\mathfrak{f}_v)}{\#\bar{\mathcal{E}}_v(\mathfrak{f}_v)} \cdot \gamma_v \omega_v(\mathcal{P}_v) \quad (\text{by 2.3}) \\ &= \frac{[\bar{M}_v(\mathfrak{f}_v) : \bar{B}_v(\mathfrak{f}_v)]}{[\bar{\mathcal{M}}_v(\mathfrak{f}_v) : \bar{\mathcal{B}}_v(\mathfrak{f}_v)]} \cdot \frac{\#\bar{T}_v(\mathfrak{f}_v)}{\#\bar{\mathcal{E}}_v(\mathfrak{f}_v)} \cdot \frac{\#\bar{\mathcal{M}}_v(\mathfrak{f}_v)}{q_v^{\dim \bar{\mathcal{M}}_v}} \quad (\text{by 2.9}) \\ &= \frac{\#\bar{M}_v(\mathfrak{f}_v)}{q_v^{\dim \bar{\mathcal{M}}_v}} \cdot \frac{\#\bar{\mathcal{B}}_v(\mathfrak{f}_v)}{\#\bar{B}_v(\mathfrak{f}_v)} \cdot \frac{\#\bar{T}_v(\mathfrak{f}_v)}{\#\bar{\mathcal{E}}_v(\mathfrak{f}_v)} \\ &= \frac{\#\bar{M}_v(\mathfrak{f}_v)}{q_v^{\dim \bar{\mathcal{M}}_v}} \cdot \frac{\#\bar{\mathcal{U}}_v(\mathfrak{f}_v)}{\#\bar{U}_v(\mathfrak{f}_v)} \\ &= \frac{\#\bar{M}_v(\mathfrak{f}_v)}{q_v^{(\dim \bar{M}_v + \dim \bar{\mathcal{M}}_v)/2}}. \end{aligned}$$

(3) Let  $\bar{Z}_v$  be the connected component of the identity in the center of  $\bar{M}_v$  and  $[\bar{M}_v, \bar{M}_v]$  be the derived group. Then the product map  $\bar{Z}_v \times [\bar{M}_v, \bar{M}_v] \rightarrow \bar{M}_v$  is an isogeny defined over  $\mathfrak{f}_v$  and, hence,  $\#\bar{M}_v(\mathfrak{f}_v) = \#\bar{Z}_v(\mathfrak{f}_v) \cdot \#[\bar{M}_v, \bar{M}_v](\mathfrak{f}_v)$ . So

$$\begin{aligned} \frac{\#\bar{M}_v(\mathfrak{f}_v)}{q_v^{(\dim \bar{M}_v + \dim \bar{\mathcal{M}}_v)/2}} &= \frac{\#\bar{Z}_v(\mathfrak{f}_v) \cdot \#[\bar{M}_v, \bar{M}_v](\mathfrak{f}_v)}{q_v^{\dim \bar{Z}_v} \cdot q_v^{\dim [\bar{M}_v, \bar{M}_v]} \cdot q_v^{(\dim \bar{\mathcal{M}}_v - \dim \bar{M}_v)/2}} \\ &< \#\bar{Z}_v(\mathfrak{f}_v) q_v^{-(\dim \bar{\mathcal{M}}_v - \dim \bar{M}_v)/2 - \dim \bar{Z}_v} \\ &\leq (q_v + 1)^{\dim \bar{Z}_v} q_v^{-(\dim \bar{\mathcal{M}}_v - \dim \bar{M}_v)/2 - \dim \bar{Z}_v}. \end{aligned}$$

We have used here the facts that  $\#[\bar{M}_v, \bar{M}_v](\mathfrak{f}_v) < q_v^{\dim[\bar{M}_v, \bar{M}_v]}$  and  $\#\bar{Z}_v(\mathfrak{f}_v) \leq (q_v + 1)^{\dim \bar{Z}_v}$ , which follow from the observation in 2.6 and Lemma 2.8. Finally we note that  $\dim \bar{Z}_v = \#\mathcal{O}_v - 1 = t_v$ . Lemma 2.7 (ii) now implies assertion (iv) of the proposition if either  $G$  is not quasi-split over  $k_v$ , or  $P_v$  is not special, or  $G$  splits over  $\hat{k}_v$  but  $P_v$  is not hyperspecial. Note that if  $t_v \geq 1$ ,

$$(q_v + 1)^{t_v} q_v^{-(r_v + t_v - 1) - t_v} \leq (q_v + 1) q_v^{-r_v - 1}.$$

Let us assume now that  $G$  is quasi-split over  $k_v$  (hence is isomorphic to  $\mathcal{G}$  over  $k_v$ , see Lemma 2.0). Then  $\#\bar{M}_v(\mathfrak{f}_v) q_v^{-(\dim \bar{M}_v + \dim \bar{\mathcal{N}}_v)/2}$  ( $= \gamma_v \omega_v^*(P_v)$ ) is maximal when  $P_v$  is isomorphic to  $\mathcal{P}_v$ , in which case it is equal to  $\#\bar{\mathcal{N}}_v(\mathfrak{f}_v) q_v^{-\dim \bar{\mathcal{N}}_v}$ , and, according to the observation in 2.6, this number is less than 1; recall that  $\bar{\mathcal{N}}_v$  is (absolutely) quasi-simple (2.5).

**2.11. Remark.** — For particular groups, one can give bounds which are better than those provided by Proposition 2.10 (ii), (iv). If, for example,  $G$  is an inner or outer form of type **A** and  $v$  is such that  $G(k_v)$  is isomorphic to  $\mathrm{SL}_{n_v}(\mathcal{D}_v)$ , where  $\mathcal{D}_v$  is a central division algebra over  $k_v$  of degree  $d_v$ , then

$$\begin{aligned} \frac{\#\bar{T}_v(\mathfrak{f}_v)}{q_v^{(r_v + \dim \bar{\mathcal{N}}_v)/2}} &= (q_v - 1)^{-1} (q_v^{d_v} - 1)^{(r+1)/d_v} q_v^{-r(r+3)/2} \\ &< (q_v - 1)^{-1} q_v^{-r(r+1)/2 + 1} \end{aligned}$$

and

$$\frac{\#\bar{M}_v(\mathfrak{f}_v)}{q_v^{(\dim \bar{M}_v + \dim \bar{\mathcal{N}}_v)/2}} < (q_v - 1)^{-1} q_v^{-(r^2 + 2r - (r+1)^2 d_v^{-1} - 1)/2}.$$

To establish the latter, it is sufficient to consider the case where  $P_v$  is a maximal parahoric subgroup of  $G(k_v)$ ; we assume now this to be the case. Let  $F_v$  be the unique extension of  $\mathfrak{f}_v$  of degree  $d_v$ . Then the dimension of  $\bar{M}_v$  is  $d_v n_v^2 - 1$  and  $\bar{M}_v(\mathfrak{f}_v)$  is isomorphic to the subgroup of  $\mathrm{GL}_{n_v}(F_v)$  consisting of matrices whose determinant is of norm 1 over  $\mathfrak{f}_v$ , so its order is less than  $(q_v - 1)^{-1} q_v^{d_v n_v^2}$ ;  $\bar{T}_v(\mathfrak{f}_v)$  is isomorphic to the diagonal subgroup of this group, its order is  $(q_v - 1)^{-1} (q_v^{d_v} - 1)^{n_v}$ . Now as  $d_v n_v - 1 = r$  and  $\dim \bar{\mathcal{N}}_v = r^2 + 2r$ , the above bounds are obvious.

On the other hand, if  $r$  is odd, say  $r = 2n + 1$ , and  $G/k_v$  is an outer form of type **A**, of  $k_v$ -rank  $n$ , which does not split over  $\hat{k}_v$ , then, as can be seen,

$$\frac{\#\bar{M}_v(\mathfrak{f}_v)}{q_v^{(\dim \bar{M}_v + \dim \bar{\mathcal{N}}_v)/2}} \leq \frac{(q_v^{n+1} + 1) \prod_{i=1}^n (q_v^{2i} - 1)}{q_v^{(n+1)(n+2)}} \leq q_v^{-(n+1)}.$$

### 3. Covolumes of the principal $S$ -arithmetic subgroups

As in § 2,  $G$  is an inner  $k$ -form of  $\mathcal{G}$  and  $\varphi: G \rightarrow \mathcal{G}$  is an isomorphism defined over some Galois extension  $K$  of  $k$  such that for every  $\gamma$  in the Galois group of  $K/k$ ,  $\varphi^{-1} \cdot \gamma \varphi$  is an inner automorphism of  $G$ .

Let  $\omega$  be an invariant exterior form on  $\mathcal{G}$  defined over  $k$  and of maximal degree. Then  $\omega^* = \varphi^*(\omega)$  is an invariant exterior form on  $G$  of maximal degree; it is defined over  $k$  (2.1).

We shall use the notation introduced in the preceding sections. Thus, for  $v \in V_f$ , the parahoric subgroups  $P_v$ 's are as in 2.2.

**3.1.** The natural embedding of  $k$  in the  $k$ -algebra  $A$  of adèles, gives an embedding of  $G(k)$  in  $G(A)$ ; we identify  $G(k)$  with a subgroup of  $G(A)$  in terms of this embedding. Then  $G(k)$  is a discrete subgroup of the locally compact group  $G(A)$ , and it is well-known ([1], [11]) that in the measure on  $G(A)/G(k)$  induced by any Haar measure on  $G(A)$ , the volume of  $G(A)/G(k)$  is finite.

**3.2.**  $\omega^*$  determines a Haar measure  $\omega_A^*$  on  $G(A)$ , which coincides with the product measure  $\prod_{v \in v_\infty} \omega_v^* \cdot \prod_{v \in v_f} (\omega_v^* |_{P_v})$  on the open subgroup  $\prod_{v \in v_\infty} G(k_v) \cdot \prod_{v \in v_f} P_v$ ; note that since  $G$  is semi-simple, the product  $\prod_{v \in v_f} \omega_v^*(P_v)$  is absolutely convergent and hence the product measure  $\prod_{v \in v_\infty} \omega_v^* \cdot \prod_{v \in v_f} (\omega_v^* |_{P_v})$  is a Haar measure on  $\prod_{v \in v_\infty} G(k_v) \cdot \prod_{v \in v_f} P_v$  (cf. [26: § 1] where this is proved over number fields; a similar proof applies in the case of global function fields).

In the sequel, we shall let  $\omega_A^*$  also denote the finite invariant measure on  $G(A)/G(k)$  induced by the Haar measure  $\omega_A^*$  on  $G(A)$ .

**3.3.** Let  $D_k$  be as in 0.2. The *Tamagawa number*  $\tau_k(G)$  of  $G/k$  is by definition the positive real number  $D_k^{-\frac{1}{2} \dim G} \omega_A^*(G(A)/G(k))$ ; in view of the product formula (see 0.1), it depends only on  $G/k$  and not on the choice of the invariant exterior  $k$ -form  $\omega$ .

It was conjectured by André Weil that  $\tau_k(G) = 1$ . He and T. Tamagawa proved this for all *inner* forms of type **A**, and in case  $k$  is of characteristic different from two, for all  $k$ -forms of type **B**, **C** and all forms of type **D** except the triality forms of type **D**<sub>4</sub>; M. Demazure verified the conjecture for the forms of type **G**<sub>2</sub> (see [34]). J. G. M. Mars then proved the conjecture for *outer* forms of type **A** ([23]), all forms of type **F**<sub>4</sub> and certain *inner* forms of type **E**<sub>8</sub> ([22]) over number fields. For split groups over number fields, the conjecture was proved by R. P. Langlands ([19]). Using some of his ideas, G. Harder ([13]) proved the conjecture for all split groups over global function fields, and K. F. Lai proved it for quasi-split groups over number fields ([18]).

R. Kottwitz ([17]), following a proposal of Jacquet-Langlands [15], has recently proved the conjecture for groups over number fields, without any case-by-case considerations, modulo the Hasse principle for the Galois cohomology of simply connected semi-simple groups ([16]). The Hasse principle has been known to hold for all groups of type other than **E**<sub>8</sub>. V. I. Chernousov has just announced its verification also for the groups of type **E**<sub>8</sub>. Hence, the work of Kottwitz ([17]) implies that  $\tau_k(G) = 1$  if  $k$  is a number field.

**3.4.** Let  $S$  be a finite set of places of  $k$ , containing all the archimedean places, such that for some  $v \in S$ ,  $G(k_v)$  is noncompact, or, equivalently,  $G$  is isotropic over  $k_v$ . Let  $G_S = \prod_{v \in S} G(k_v)$ . Then the strong approximation property ([27], [21]) implies that

$$G_S \cdot \prod_{v \notin S} P_v \cdot G(k) = G(A).$$

Let  $\Lambda$  be the image of  $G(k) \cap (G_S \cdot \prod_{v \notin S} P_v)$  under the natural projection

$$G_S \cdot \prod_{v \notin S} P_v \rightarrow G_S.$$

Then  $\Lambda$  is a *lattice* in  $G_S$  i.e., it is a discrete subgroup of  $G_S$  of finite covolume; we will say that it is the *principal  $S$ -arithmetic subgroup determined by the parahoric subgroups  $P_v$*  ( $v \notin S$ ). The object of this section is to compute the volume  $\mu_S(G_S/\Lambda)$  with respect to a natural measure  $\mu_S$  (see 3.6 below).

Let  $\omega_S^*$  denote the measure on  $G_S/\Lambda$  induced by the product measure  $\prod_{v \in S} \omega_v^*$  on  $G_S (= \prod_{v \in S} G(k_v))$ . As

$$G(A) = G_S \cdot \prod_{v \notin S} P_v \cdot G(k),$$

$G(A)/G(k)$  has a natural identification with  $G_S \prod_{v \notin S} P_v / (G(k) \cap G_S \prod_{v \notin S} P_v)$ , and so there is a (principal) fibration  $G(A)/G(k) \rightarrow G_S/\Lambda$  with fiber  $\prod_{v \notin S} P_v$ . Hence,

$$D_k^{\frac{1}{2} \dim G} \tau_k(G) = \omega_S^*(G(A)/G(k)) = \omega_S^*(G_S/\Lambda) \cdot \prod_{v \notin S} \omega_v^*(P_v).$$

Therefore,

$$\omega_S^*(G_S/\Lambda) = D_k^{\frac{1}{2} \dim G} \tau_k(G) \left( \prod_{v \notin S} \omega_v^*(P_v) \right)^{-1}.$$

**3.5.** Let  $v$  be an archimedean place of  $k$ . Let  $c_v (\in \mathbf{R}^\times)$  be as in 1.4 and  $\gamma_v = |c_v|_v$ . We recall that  $c_v$  is such that under the Haar measure induced by the invariant exterior form  $c_v \omega$ , any maximal compact subgroup of  $R_{k_v/\mathbf{R}}(\mathcal{G})(\mathbf{C})$  has volume 1. We claim that the volume of any maximal compact subgroup of  $R_{k_v/\mathbf{R}}(G)(\mathbf{C})$  in the Haar measure induced by the invariant form  $c_v \omega^*$  is also 1. To prove this, we fix a basis  $\mathcal{Y}_1^v, \dots, \mathcal{Y}_n^v$  (resp.  $Y_1^v, \dots, Y_n^v$ ) of the Lie algebra  $L(\mathcal{G}) \otimes_k k_v$  (resp.  $L(G) \otimes_k k_v$ ) such that with respect to the Killing form  $\langle \cdot, \cdot \rangle_v$  on  $L(\mathcal{G}) \otimes_k k_v$  (resp.  $L(G) \otimes_k k_v$ )  $\mathcal{Y}_i^v$  is orthogonal to  $\mathcal{Y}_j^v$  (resp.  $Y_i^v$  is orthogonal to  $Y_j^v$ ) for all  $1 \leq i \neq j \leq n$ , and moreover, if  $v$  is real,

$$|\langle \mathcal{Y}_i^v, \mathcal{Y}_i^v \rangle_v|_v = 1 = |\langle Y_i^v, Y_i^v \rangle_v|_v \quad \text{for all } i \leq n,$$

and if  $v$  is complex, then

$$\langle \mathcal{Y}_i^v, \mathcal{Y}_i^v \rangle_v = 1 = \langle Y_i^v, Y_i^v \rangle_v \quad \text{for all } i \leq n.$$

Let  $\mathcal{Y}_v^1, \dots, \mathcal{Y}_v^n$  (resp.  $Y_v^1, \dots, Y_v^n$ ) be the dual basis and  $\omega_{v, \mathcal{G}}^K = \mathcal{Y}_v^1 \wedge \dots \wedge \mathcal{Y}_v^n$  (resp.  $\omega_{v, G}^K = Y_v^1 \wedge \dots \wedge Y_v^n$ ). Let  $\theta_v : G \rightarrow \mathcal{G}$  be an isomorphism defined over the algebraic closure  $\bar{k}_v (\cong \mathbf{C})$  of  $k_v$  such that for all  $\gamma$  in the Galois group of  $\bar{k}_v/k_v$ ,  $\theta_v^{-1} \cdot \gamma \theta_v$  is an inner automorphism of  $G$  (2.0). Then  $\theta_v^*(\omega_{v, \mathcal{G}}^K)$  is defined over  $k_v$  ([15: pp. 475-476]),

$\theta_v$  induces a Lie algebra isomorphism  $L(G) \otimes_k \bar{k}_v \rightarrow L(\mathcal{G}) \otimes_k \bar{k}_v$ , and as any isomorphism of Lie algebras preserves the Killing form, it follows that  $\theta_v^*(\omega_{v,\mathcal{G}}^K) = \pm \omega_{v,G}^K$ . Now we note that since  $G$  is a form of  $\mathcal{G}$ , the maximal compact subgroup of  $R_{k_v/\mathbf{R}}(\mathcal{G})(\mathbf{C})$  and  $R_{k_v/\mathbf{R}}(G)(\mathbf{C})$  are isomorphic and hence have equal volume ( $= |a|_v$ ; where  $a = m^{\frac{1}{2}} \prod_{i=1}^r \frac{(2\pi)^{m_i+1}}{m_i!}$ ,  $m$  is as in the proof of Theorem 1.6 and  $m_1 \leq \dots \leq m_r$  are the exponents (1.5)) with respect to the Haar measures determined by  $\omega_{v,\mathcal{G}}^K$  and  $\omega_{v,G}^K$  respectively. Now since  $\omega$  is a multiple of  $\omega_{v,\mathcal{G}}^K$ , and  $\theta_v^*(\omega) = \zeta_v \omega^*$ , where  $\zeta_v$  is a root of unity (cf. 2.1), our claim is obvious.

**3.6.** For any archimedean place  $v$  of  $k$ , let  $\mu_v$  be the Haar measure on  $G(k_v)$  determined by the invariant exterior form  $c_v \omega^*$  ( $c_v$  as in 1.4), and for  $v$  nonarchimedean, let  $\mu_v$  be the *Tits measure* on  $G(k_v)$ , i.e., the Haar measure with respect to which every Iwahori subgroup of  $G(k_v)$  has volume 1. Of course,  $\mu_v = \omega_v^*(I_v)^{-1} \omega_v^*$  for all  $v \in V_f$ ; where  $I_v$  is an Iwahori subgroup of  $G(k_v)$ . Let  $\mu_S = \prod_{v \in S} \mu_v$  be the product measure on  $G_S (= \prod_{v \in S} G(k_v))$ ; we shall denote the  $G_S$ -invariant induced measure on  $G_S/\Lambda$  also by  $\mu_S$ .

Let  $\ell$ ,  $D_\ell$  and  $s(\mathcal{G})$  be as in 0.2 and 0.4 respectively.

**3.7. Theorem.** — *We have the following*

$$\mu_S(G_S/\Lambda) = D_k^{\frac{1}{2} \dim G} (D_\ell/D_k^{[\ell:k]})^{\frac{1}{2} s(\mathcal{G})} \left( \prod_{v \in V_\infty} \left| \prod_{i=1}^r \frac{m_i!}{(2\pi)^{m_i+1}} \right|_v \right) \tau_k(G) \mathcal{E};$$

where 
$$\mathcal{E} = \prod_{v \in S_f} \frac{q_v^{(r_v + \dim \bar{\mathcal{M}}_v)/2}}{\# \bar{T}_v(\bar{f}_v)} \cdot \prod_{v \notin S} \frac{q_v^{(\dim \bar{M}_v + \dim \bar{\mathcal{M}}_v)/2}}{\# \bar{M}_v(\bar{f}_v)},$$

and  $S_f = S \cap V_f$ .

*Proof.* — Clearly

$$\begin{aligned} \mu_S(G_S/\Lambda) &= \left( \prod_{v \in V_\infty} |c_v|_v \right) \left( \prod_{v \in S_f} \omega_v^*(I_v) \right)^{-1} \omega_S^*(G_S/\Lambda) \\ &= \left( \prod_{v \in V_\infty} \gamma_v \right) \left( \prod_{v \in S_f} \omega_v^*(I_v) \right)^{-1} D_k^{\frac{1}{2} \dim G} \tau_k(G) \left( \prod_{v \notin S} \omega_v^*(P_v) \right)^{-1} \\ &\hspace{20em} \text{(cf. 3.4)} \\ &= D_k^{\frac{1}{2} \dim G} \prod_{v \in V} \gamma_v \left( \prod_{v \in S_f} \gamma_v \omega_v^*(I_v) \prod_{v \notin S} \gamma_v \omega_v^*(P_v) \right)^{-1} \tau_k(G) \\ &= D_k^{\frac{1}{2} \dim G} (D_\ell/D_k^{[\ell:k]})^{\frac{1}{2} s(\mathcal{G})} \left( \prod_{v \in V_\infty} \left| \prod_{i=1}^r \frac{m_i!}{(2\pi)^{m_i+1}} \right|_v \right) \tau_k(G) \mathcal{E} \\ &\hspace{10em} \text{(by Theorem 1.6);} \end{aligned}$$

where,

$$\begin{aligned} \mathcal{E} &= \left( \prod_{v \in S_f} \gamma_v \omega_v^*(I_v) \cdot \prod_{v \notin S} \gamma_v \omega_v^*(P_v) \right)^{-1} \\ &= \prod_{v \in S_f} \frac{q_v^{(r_v + \dim \bar{\mathcal{M}}_v)/2}}{\#\bar{T}_v(\mathfrak{f}_v)} \cdot \prod_{v \notin S} \frac{q_v^{(\dim \bar{M}_v + \dim \bar{\mathcal{M}}_v)/2}}{\#\bar{M}_v(\mathfrak{f}_v)} \quad (\text{by Proposition 2.10}). \end{aligned}$$

This proves the theorem.

**3.8. Remark.** — If  $V_\infty \neq \emptyset$ , i.e. if  $k$  is a number field, then

$$\prod_{v \in V_\infty} \left| \prod_{i=1}^r \frac{m_i!}{(2\pi)^{m_i+1}} \right|_v = \left( \prod_{i=1}^r \frac{m_i!}{(2\pi)^{m_i+1}} \right)^{[k:\mathbb{Q}]}$$

**3.9. Remark.** — The reductive groups  $\bar{\mathcal{M}}_v$ ,  $\bar{M}_v$  and the tori  $\bar{T}_v(\mathbb{C} \bar{M}_v)$  can be described in terms of the local index of  $\mathcal{G}/k_v$ , of  $G/k_v$  and the subset  $\Theta_v$  of 2.4 (see [33: 3.5]). Thus, in principle,  $\mu_s(G_S/\Lambda)$  can be computed, using the formula given by the above theorem, when  $\tau_k(G)$  is known, for example, if  $k$  is a number field (see 3.3).

**3.10. Remark.** — The factors  $\frac{q_v^{(r_v + \dim \bar{\mathcal{M}}_v)/2}}{\#\bar{T}_v(\mathfrak{f}_v)}$  ( $v \in S_f$ ) and  $\frac{q_v^{(\dim \bar{M}_v + \dim \bar{\mathcal{M}}_v)/2}}{\#\bar{M}_v(\mathfrak{f}_v)}$  ( $v \notin S$ ), of the Euler product  $\mathcal{E}$  in the statement of Theorem 3.7, are all greater than 1. Moreover the former is at least  $(q_v + 1)^{-r_v} q_v^{\frac{1}{2} r_v (r_v + s)}$ . For  $v \notin S$ , if  $G$  is not quasi-split over  $k_v$ , then the later factor is at least  $(q_v + 1)^{-1} q_v^{r_v + 1}$  (and if  $G$  is anisotropic over  $k_v$ , then this factor is  $\geq (q_v + 1)^{-r_v} q_v^{\frac{1}{2} r_v (r_v + s)}$ ; see Proposition 2.10 (ii), (iv). These observations are crucial for the proof of the finiteness assertions in [4].

**3.11. Remark.** — Let  $k$  be a number field and  $G$  be such that for some archimedean place  $v$  of  $k$ ,  $G(k_v)$  is noncompact and for every nonarchimedean  $v$ ,  $G$  is quasi-split over  $k_v$ . We assume that for every nonarchimedean  $v$ ,  $P_v$  is special and whenever  $G$  splits over the maximal unramified extension of  $k_v$ , it is hyperspecial. Then for all nonarchimedean  $v$ ,  $\bar{M}_v$  is isomorphic to  $\bar{\mathcal{M}}_v$  over  $\mathfrak{f}_v$ , and hence,

$$\frac{q_v^{(\dim \bar{M}_v + \dim \bar{\mathcal{M}}_v)/2}}{\#\bar{M}_v(\mathfrak{f}_v)} = \frac{q_v^{\dim \bar{M}_v}}{\#\bar{M}_v(\mathfrak{f}_v)}.$$

Now let  $\Lambda_\infty$  be the projection of  $G(k) \cap (\prod_{v \in V_\infty} G(k_v) \cdot \prod_{v \in V_f} P_v)$  into

$$G_\infty := \prod_{v \in V_\infty} G(k_v).$$

Then

$$\mu_\infty(G_\infty/\Lambda_\infty) = D_k^{\frac{1}{2} \dim G} (D_\ell/D_k^{[\ell:k]})^{\frac{1}{2} s(\mathcal{G})} \left( \prod_{i=1}^r \frac{m_i!}{(2\pi)^{m_i+1}} \right)^{[k:\mathbb{Q}]} \tau_k(G) \mathcal{E};$$

where  $\mu_\infty = \mu_{v_\infty}$ , and  $\mathcal{E} = \prod_{v \in v_f} \frac{q_v^{\dim \bar{M}_v}}{\# \bar{M}_v(\mathfrak{f}_v)}$ . Using the orders of finite groups of Lie type, given, for example, in [25: Table 1], we easily see that  $\mathcal{E}$  is a product of the values of the Dedekind zeta function of  $k$  and certain Dirichlet L-functions at the integers  $m_i + 1, i \leq r$ . If, moreover, the absolute rank of  $G_\infty$  equals that of any maximal compact subgroup, then ( $k$  is totally real and) using the functional equations of the Dedekind zeta function and Dirichlet L-functions, we get a very concise formula for the volume of  $G_\infty/\Lambda_\infty$ .

**4. Class numbers of absolutely quasi-simple, simply connected groups**

**4.1.** We shall assume in this section that  $G$  is *anisotropic* over  $k$ . If  $k$  is a number field, we assume moreover that  $\prod_{v \in v_\infty} G(k_v)$  is *compact*;  $k$  is then totally real.

As in 2.2, let  $P = (P_v)_{v \in v_f}$  be a coherent collection of parahoric subgroups. It is known that the set  $(\prod_{v \in v_\infty} G(k_v) \cdot \prod_{v \in v_f} P_v) \backslash G(A)/G(k)$  of double cosets is finite ([1: Theorem 5.1], [11: 2.2.7 (iii)]); the cardinality of this set is called the *class number of  $G$  relative to  $P$*  and will be denoted by  $c(P)$ .

We shall denote the compact-open subgroup  $\prod_{v \in v_\infty} G(k_v) \cdot \prod_{v \in v_f} P_v$ , of  $G(A)$ , by  $C$ . We shall use the notation introduced in the preceding sections. In particular,  $\omega_A^*$  is the Haar measure on  $G(A)$  defined in 3.2.

**4.2.** We fix representatives  $g_i \in G(A), 1 \leq i \leq c(P)$ , of the double cosets in  $C \backslash G(A)/G(k)$ . Then

$$G(A) = \bigcup_{i=1}^{c(P)} Cg_i G(k)$$

and so 
$$\omega_A^*(G(A)/G(k)) = \sum_{i=1}^{c(P)} \frac{\omega_A^*(C)}{\# F_i}; \tag{*}$$

where  $F_i = g_i^{-1} Cg_i \cap G(k)$  is a finite subgroup of  $G(k)$  since  $G(k)$  is a discrete subgroup of  $G(A)$  and  $C$ , and hence also  $g_i^{-1} Cg_i$ , is a compact subgroup. If there is a finite upper bound for the orders of finite subgroups of  $G(k)$  (which is the case if  $k$  is a number field—this follows, for example, from [29: LG, Chapter IV, Appendix 3, Theorem 1]), let  $f = f(G/k)$  be the smallest integer such that the order of any finite subgroup of  $G(k)$  is at most  $f$ , otherwise let  $f = \infty$ . Then as

$$\omega_A^*(G(A)/G(k)) = D_k^{\frac{1}{2} \dim G} \tau_k(G) \quad (\text{see 3.3}),$$

we conclude from (\*) that

$$c(P) \omega_A^*(C) \geq D_k^{\frac{1}{2} \dim G} \tau_k(G) \geq f^{-1} c(P) \omega_A^*(C).$$

So,

$$f D_k^{\frac{1}{2} \dim G} \tau_k(G) (\omega_A^*(C))^{-1} \geq c(P) \geq D_k^{\frac{1}{2} \dim G} \tau_k(G) (\omega_A^*(C))^{-1}.$$



Now we shall determine  $(\omega_A^*(C))^{-1}$  using the results proved in §§ 1-3. Obviously,

$$\begin{aligned} (\omega_A^*(C))^{-1} &= \left( \prod_{v \in V_\infty} \omega_v^*(G(k_v)) \cdot \prod_{v \in V_f} \omega_v^*(P_v) \right)^{-1} \\ &= \prod_{v \in V} \gamma_v \cdot \left( \prod_{v \in V_\infty} \gamma_v \omega_v^*(G(k_v)) \cdot \prod_{v \in V_f} \gamma_v \omega_v^*(P_v) \right)^{-1} \\ &= (D_l/D_k)^{\frac{1}{2} s(\mathcal{G})} \left( \prod_{v \in V_\infty} \left| \prod_{i=1}^r \frac{m_i!}{(2\pi)^{m_i+1}} \right|_v \right) \left( \prod_{v \in V_f} \gamma_v \omega_v^*(P_v) \right)^{-1} \end{aligned}$$

since according to Theorem 1.6,

$$\prod_{v \in V} \gamma_v = (D_l/D_k)^{\frac{1}{2} s(\mathcal{G})} \left( \prod_{v \in V_\infty} \left| \prod_{i=1}^r \frac{m_i!}{(2\pi)^{m_i+1}} \right|_v \right),$$

and it is clear from the definition of  $\gamma_v$  (see 3.5) that as  $G(k_v)$  is compact for all  $v \in V_\infty$ ,  $\gamma_v \omega_v^*(G(k_v)) = 1$  for  $\forall v \in V_\infty$ .

Let

$$\begin{aligned} \zeta(P) &= \left( \prod_{v \in V_f} \gamma_v \omega_v^*(P_v) \right)^{-1} \\ &= \prod_{v \in V_f} \frac{q_v^{(\dim \bar{M}_v + \dim \bar{\mathcal{N}}_v)/2}}{\# \bar{M}_v(\bar{f}_v)} \quad (\text{by Proposition 2.10 (ii)}). \end{aligned}$$

Then

$$\omega_A^*(C)^{-1} = (D_l/D_k)^{\frac{1}{2} s(\mathcal{G})} \left( \prod_{v \in V_\infty} \left| \prod_{i=1}^r \frac{m_i!}{(2\pi)^{m_i+1}} \right|_v \right) \zeta(P).$$

Therefore, we conclude the following:

**4.3. Theorem.** — *Let*

$$c(P) = \# \left( \left( \prod_{v \in V_\infty} G(k_v) \cdot \prod_{v \in V_f} P_v \right) \backslash G(A) / G(k) \right).$$

*Then*

$$c(P) \geq D_k^{\frac{1}{2} \dim G} (D_l/D_k)^{\frac{1}{2} s(\mathcal{G})} \left( \prod_{v \in V_\infty} \left| \prod_{i=1}^r \frac{m_i!}{(2\pi)^{m_i+1}} \right|_v \right) \tau_k(G) \zeta(P),$$

*and*

$$c(P) \leq f D_k^{\frac{1}{2} \dim G} (D_l/D_k)^{\frac{1}{2} s(\mathcal{G})} \left( \prod_{v \in V_\infty} \left| \prod_{i=1}^r \frac{m_i!}{(2\pi)^{m_i+1}} \right|_v \right) \tau_k(G) \zeta(P);$$

*where*

$$\zeta(P) = \prod_{v \in V_f} \frac{q_v^{(\dim \bar{M}_v + \dim \bar{\mathcal{N}}_v)/2}}{\# \bar{M}_v(\bar{f}_v)}.$$