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Appendix on return-time sequences

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APPENDIX

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Return Times of Dynamical Systems

Let $(X, \mathscr{B}, \mu, T)$ be an ergodic system and let $A \in \mathscr{B}$ be of positive measure $\mu(A) > 0$. For $x \in X$, consider the return time sequence $\Lambda_x = \{ n \in \mathbf{Z}_+ \mid T^n x \in A \}$. By Birkhoff's pointwise ergodic theorem, the sequence Λ_x has positive density for μ-almost all $x \in X$. This fact refines the classical Poincaré recurrence principle (cf. [Fu]). An even stronger statement is given by the Wiener-Wintner theorem: there is a set X' of X of full measure such that the sums

$$\frac{1}{N} \sum_{1 \leqslant n \leqslant N} \chi_A(T^n x)\, z^n$$

converge for all z in the unit circle $\mathbf{C}_1 = \{ z \in \mathbf{C} \mid |z| = 1 \}$ and $x \in X'$. Thus from general theory of unitary operators, this fact may be reinterpreted by saying that almost all sequences Λ_x satisfy the L^2, hence the mean ergodic theorem. Our purpose here is to prove the following fact, answering a question open for some time.

Theorem. — With the notation above, Λ_x satisfies almost surely the pointwise ergodic theorem, i.e., the averages

$$\frac{1}{N} \sum_{\substack{1 \leqslant n \leqslant N \\ n \in \Lambda_x}} S^n g$$

converge almost surely for any measure preserving system $(Y, \mathscr{D}, \nu, S)$ and $g \in L^1(Y)$.

The argument given next actually yields a more precise condition on the point x.

Let $f \in L^\infty(X)$ be obtained by projecting χ_A on the orthogonal complement of the eigenfunctions of T. It clearly suffices to prove that for almost all $x \in X$, $\{f(T^n x)\}$ is a " summing sequence ", i.e.,

$$(*) \qquad \frac{1}{N} \sum_{1 \leqslant n \leqslant N} f(T^n x)\, g(S^n y) \to 0 \quad \text{a.e. } y \in Y$$

for any measure preserving system $(Y, \mathscr{D}, \nu, S)$ and $g \in L^\infty(Y)$. (The contribution of the eigenfunctions is taken care of by Birkhoff's theorem.)

Observe the equivalence of the following statements:

(i) f has continuous spectral measure,

(ii) $\langle T^n f, f \rangle = \hat{\sigma}_f(n)$, σ a continuous measure,

(iii) $(1/N) \sum_1^N f(T^n x) f(T^n \xi) \to 0$ a.e. in (x, ξ) as $N \to \infty$.

Proof of (ii) $\Rightarrow$ (iii). — Write $F = \lim (1/N) \sum_1^N f(T^n x) f(T^n \xi)$, a limit which exists by the ergodic theorem, and $\| F \|^2 = \lim (1/N^2) \sum_{m,n=1}^N (\hat{\sigma}_f(n-m))^2 = 0$.

Proposition. — *Assume x generic for f and* $(1/N) \sum f(T^n x) f(T^n \xi) \to 0$, *a.e. in* ξ *(!). Then* $\{f(T^n x)\}$ *is a summing sequence.*

Proof. — I) Assume that for some $(Y, \mathscr{D}, \nu, S)$ and $g \in L^\infty(Y)$ there is a set B^* of positive measure for which the limsup of $(*)$ is positive. Then there exists $a > 0$, $B \subset B^*$, $\nu(B) > 0$ and a sequence of intervals $R_j = (L_j, M_j)$ (called " ranges ") such that for every $y \in B$ and every j there exists $n_j \in \mathbf{R}_j$ $(n_j = n_j(y))$ such that

$$(**) \qquad \sum_{n=1}^{n_j} f(T^n x)\, g(S^n y) > a n_j.$$

II) Given $\delta > 0$, there exists $K = K(N, \delta)$ such that

$$\nu\Big(\bigcup_1^K S^j B\Big) > 1 - \delta.$$

III) Write φ for the indicator function of $\bigcup_{j=1}^K S^j B$. If M_0 is large enough, and if we denote by G the set $G = \{y : | (1/n) \sum_1^n \varphi(S^j y) - 1 | < 2\delta \text{ for all } n > M_0\}$, then $\nu(G) > 1 - \delta$.

IV) For notational convenience we assume that f has finite range, and we denote by B_n the set of all n-blocks for f, i.e., the set of words $w_k^{(n)} = (f(T^{k+1} x), \ldots, f(T^{k+n} x))$; $w_k^{(n)}$ appears with density $p(w_k^{(n)})$.

Given $\delta > 0$ (δ can be chosen once and for all as a function of a and $\nu(B)$ in I)) let N_δ be such that for each set $A_\delta \subset X$, $\mu(A_\delta) > 1 - \delta$, $|(1/N) \sum f(T^n x) f(T^n \xi)| < \delta$ for all $\xi \in A_\delta$ and $N > N_\delta$ (cf. assumption (!)).

Given a range (L, M) with $L > N_\delta$, set $N = N(M)$ so that in any interval on the integers of length $\geqslant N$ the statistics of the n-blocks (for f) with $n \leqslant M$ is correct. Denote by B_n^* the n-blocks that have the form $(f(T\xi), \ldots, f(T^n \xi))$ with $\xi \in A_\delta$ (we are interested in $n \in (L, M)$). For $L < n < M$ the total probability (= density) of the blocks in B_n^* exceeds $1 - \delta$ (in any interval of length $\geqslant N(M)$). Notice also that heads of M-blocks which are in B_M^* are in the appropriate B_n^*.

V) A sequence of ranges $\{(L_j, M_j)\}$ is ***properly spaced*** if $L_{j+1} > N(M_j)$. (We also assume $L_1 > N_\delta$. Another assumption on L_1 is that it is $> M_0$ (recall the definition of G in III) and assume that K (II)) is $\ll L_1$.) *Going back to* I), we select a properly spaced sequence of ranges $\{(L_j, M_j)\}_{j=1}^J$ (J depending on a) and N large enough so that $N \gg N(M_J)$.

Recall B from I) and G from III).

For any $y \in B \cap G$ we define a sequence $\{c_n(y)\}_{n=1}^{N}$ which is a sum of J sequences (layers) $\{c_n^j(y)\}$ having the following properties:

(α) For all j, n and y, $c_n^j(y)$ is in the range of f (in particular uniformly bounded)

(β) For $j_1 \neq j_2$, $|(1/N) \sum_{n=1}^{N} c_n^{j_1}(y)\, c_n^{j_2}(y)| < \delta$

(γ) $(1/N) \sum_{n=1}^{N} c_n^j(y)\, g(S^n y) > a - \delta, j = 1, \ldots, J$

(α) and (β) together imply $[(1/N) \Sigma(c_n(y))^2]^{1/2} = O(\sqrt{J} + \delta J)$, and ($\gamma$) implies $(1/N) \sum_1^N c_n(y)\, g(S^n y) > J(a - \delta)$. Contradiction.

We construct $\{c_n^j\}$ in reverse order on j. The number $c_n^J(y)$ is defined as follows: $\ell_1(y)$ is the first index $k > 0$ such that $S^k y \in B$; on the interval $(\ell_1(y), \ell_1(y) + n_J(S^{\ell_1(y)} y))$ we set

$$c_n^J(y) = f(T^{n - \ell_1(y)}\, x),$$

$\ell_2(y)$ is the index of the first point in the S-orbit of y after $\ell_1(y) + n_J(S^{\ell_1(y)} y)$ which is in B, and on the interval $(\ell_2(y), \ell_2(y) + n_J(S^{\ell_2(y)} y)$ we copy again $\{f(T^k x)\}_{k=1}^{n_J(S^{\ell_2(y)} y)}$ etc. The intervals on which we copy those starting n_J blocks fill most of [1, N]. We refer to these as the basic intervals of the J-layer. Outside of these, set $c_n^J(y)$ arbitrarily.

We now define $c_n^{J-1}(y)$ in a similar manner within every basic interval of the J-layer, with the additional restriction on the starting place of the new basic blocks that (in addition to the fact that the corresponding point in the orbit of y is in B) the matching piece of the basic J-layer block in is B^*, i.e., more or less orthogonal to the " new " basic block; see IV). Since the " orthogonal " blocks have density $> 1 - \delta$, the new basic blocks cover more than $1 - 3\delta$ of [1, N]. We continue with $c_n^{J-2}(y), \ldots, c_n^1(y)$, working each time within the basic blocks of the previous level and introducing blocks which are " orthogonal " to all previous levels.

Remarks.

(i) The condition that $(1/N) \sum_1^N f(T^n x) f(T^n \xi) \to 0$ a.e. in ξ(!) is a special case of (*) and hence necessary. One can construct examples showing that it is not a consequence of the genericity of x.

(ii) One may construct a sequence $\Lambda = \{k_n\}$, $k_n = o(n)$, and a weakly mixing system (Y, S) such that $(1/N) \sum_1^N g(S^{k_n} y)$ does not converge a.e., for some $g \in L^\infty(Y)$. (This question was considered in [Fu], p. 96.)

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