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A FOLIATION OF \mathbf{R}^3 AND OTHER PUNCTURED 3-MANIFOLDS BY CIRCLES

by ELMAR VOGT

1. Introduction

In the proceedings of the 1976 Rio de Janeiro conference on Gelfand-Fuks Cohomology and Foliations P. Schweitzer published a list of problems which he had collected during the conference and the time prior to the publication of the proceedings [Sch]. The very last problem, Problem 36 of this list, is the following question of D. B. A. Epstein: *Can \mathbf{R}^3 be foliated by circles?*

A comment (attributed to L. Markus) is added, saying that decompositions of \mathbf{R}^3 into smooth circles are known but they fail to be foliations.

In this paper we construct a relatively simple foliation of \mathbf{R}^3 by circles and use this to prove that $M \setminus \{\text{point}\}$ admits a foliation by circles if M is a 3-manifold which admits one. At the end of the paper we give some examples. The foliations in this paper are not C^1 . They are only differentiable in the sense that coordinate changes of foliation charts are homeomorphisms which have a derivative at each point. Whether \mathbf{R}^3 admits a C^1 -foliation by circles remains an open question.

The problem of filling \mathbf{R}^3 by circles which form the leaves of a foliation should be seen, apart from its natural appeal, as part of a program to understand the geometric possibilities and complexities of foliations in which all leaves are compact or, for short, compact foliations.

In a remarkable paper [Ep1] D. B. A. Epstein showed that foliations by circles of compact 3-manifolds are quite well behaved: their leaves are the fibres of a Seifert fibre bundle (in a mildly more general sense than the usual one: passing to an orientable double cover gives a Seifert fibre bundle in the usual sense). Using Epstein's rather delicate arguments this result was later [EMS], [Vo1] generalized to arbitrary compact foliations of codimension 2 on compact manifolds. Here a foliation of arbitrary dimensions with all leaves compact is called a Seifert fibration if every leaf has a saturated neighborhood N (i.e. a union of leaves) such that the induced foliation on some *finite* regular cover of N is trivial (i.e. the leaves are the fibres of a product bundle).

A similar result is true in codimension 1 even on non-compact manifolds: after passage to a double covering the leaves of any compact foliation of codimension 1 are

the fibres of a locally trivial bundle over the circle or an interval. This statement is a direct consequence of the notion of holonomy and a proof is already in Reeb's thesis [Re]. There one also finds an example of a compact codimension 2 foliation on a non-compact manifold with the property that the volume-of-leaf function (with regard to any Riemannian metric) is not locally bounded. Thus it is not a Seifert fibration. In fact a compact foliation has a locally bounded volume-of-leaf function if and only if it is a Seifert fibration. This is also true for more general foliated sets. (A proof of this and a discussion of other equivalent conditions can be found in [Ep2].)

Thus given a compact foliation \mathcal{F} on a manifold B_0 and a volume-of-leaf function V the set B_1 of points where V is not locally bounded is bad in the sense that it is exactly at the leaves of B_1 where \mathcal{F} fails to be a Seifert fibration. B_1 is the first in a whole series of bad sets, one for each ordinal α . Given B_α one obtains $B_{\alpha+1}$ as the union of leaves where \mathcal{F} restricted to B_α fails to be a Seifert fibration, i.e. $B_{\alpha+1}$ is the set of points of B_α where V restricted to B_α is not locally bounded. If α is a limit ordinal one defines B_α to be the intersection of the preceding bad sets. Each B_α is clearly saturated. It is a closed set, in general not a submanifold, and $B_{\alpha+1}$ is nowhere dense in B_α . Thus B_0 is partitioned into sets $B_\alpha \setminus B_{\alpha+1}$ where the foliation is nice while something drastic happens when one moves from $B_\alpha \setminus B_{\alpha+1}$ to $B_{\alpha+1}$. The family $B_1 \supset B_2 \supset \dots$ of bad sets was introduced (in a slightly different form) by D. B. A. Epstein in [Ep1] and is called the Epstein hierarchy of \mathcal{F} . How many of the B_α actually appear is a first measure of complexity of the foliation. Because B_0 has a countable base the Epstein hierarchy always ends at a countable ordinal.

Epstein's proof ruled out the appearance of B_1 in the codimension 2 case on compact manifolds. That there was no replacement for Epstein's intricate arguments in higher codimensions was demonstrated by D. Sullivan [Su] who described a smooth circle foliation on a compact 5-manifold with B_1 a 3-sphere fibred by the Hopf fibration. Later an analytic example of a circle foliation with a non-trivial bad set was found on a compact 4-manifold [Ep - Vo].

R. Edwards, K. Millett, and D. Sullivan in their work on compact foliations [EMS] which dates back to 1974 gave a homological reinterpretation of Epstein's argument in [Ep1]. Their approach, based on deep geometrical insight and intuition, has the advantage to work in all dimensions and codimensions. Their main result says that under certain homological conditions the bad set of a compact foliation on a compact manifold must be empty, thus relating global algebraic properties of the manifold with the local behaviour of the foliation. This brought up the question of examples on homologically trivial or even contractible manifolds, and \mathbf{R}^3 is the first case that comes in mind (by Poincaré-Bendixson \mathbf{R}^2 does not even admit a non-singular vector field with one bounded semi-orbit, and circle foliations on \mathbf{R}^3 immediately give examples on \mathbf{R}^n for $n > 3$).

From the outset it was clear that a circle foliation of \mathbf{R}^3 has to be somewhat complicated. First of all it cannot be a fibration. More generally, A. Borel and J. P. Serre [Bo-Se] and A. Shapiro [Sh] proved that the only locally trivial fibration with an acyclic mani-

fold as total space and a compact fibre is the fibration by points. It also cannot be a Seifert fibration: no Euclidean space admits a 1-dimensional Seifert fibration because the leaves of a Seifert fibration on a simply connected n -manifold are the orbits of a fixed point free S^1 -action. By Smith theory, any S^1 -action on acyclic manifolds has fixed points.

Therefore, if there exists a circle foliation on \mathbf{R}^3 it has to have a non-empty bad set B_1 . Let W be a component of $\mathbf{R}^3 \setminus B_1$. Then W is the complement of a torus knot in S^3 and each leaf in W represents a non-trivial element of $H_1(W) \cong \mathbf{Z}$. Thus each component of $\mathbf{R}^3 \setminus B_1$ carries homology which has to disappear when various components of $\mathbf{R}^3 \setminus B_1$ are glued together along parts of B_1 . Exactly at these parts, where the closure of two (or more) components of the complement of B_1 meet, the circle foliation exhibits its most interesting features.

At the heart of our example is a circle foliation on a solid torus V from whose boundary ∂V the complement of two disjoint open annuli A_1 and A_2 has been removed. Restricted to the sets $V \setminus \partial V$, A_1 , and A_2 , the foliation is the product fibration with fiber S^1 . This is easy to achieve if A_1 and A_2 are tubular neighborhoods of longitudinal curves. The example of Reeb mentioned above is one in which A_1 and A_2 are neighborhoods of meridians. The novel feature of our example is that A_1 is the tubular neighborhood of a contractible curve in ∂V while A_2 is the neighborhood of a longitudinal curve. The bad set of our example is A_1 . A foliation of \mathbf{R}^3 is obtained by taking copies V_i , $i \in \mathbf{Z}$, of the above example and attaching V_i to V_{i+1} by identifying the annulus corresponding to A_1 in V_i to the one corresponding to A_2 in V_{i+1} .

Our example is simple in the sense that the Epstein hierarchy is as short as possible: B_1 is a disjoint discrete union of countably many open annuli, and thus $B_2 = \emptyset$. But it is more complicated than Reeb's example in the way the bad set is approached by saturated 2-tori. To give this statement some content, assume that B_1 is a manifold, which it is in our example. If the circle foliation is defined on a connected 3-manifold M with $H_2(M) = 0$, then each component of $M \setminus B_1$ is of the form $C \cup T^2 \times [0, \infty)$, where C is a compact Seifert fibre space with boundary $\partial C = T^2 \times \{0\}$ and $T^2 \times [0, \infty)$ is trivially fibred with each $T^2 \times \{t\}$, $t \in [0, \infty)$, a union of leaves. Thus we have a family of saturated tori, $T^2 \times \{t\}$, going to infinity as t goes to infinity. In Reeb's example one can choose such a family and a tubular neighborhood $p: U \rightarrow B_1$ of B_1 such that p restricted to $T^2 \times \{t\} \cap U$ is locally a diffeomorphism. In our example, for any choice of family and tubular neighborhood, the 2-manifolds $T^2 \times \{t\} \cap U$ wobble terribly in the sense that this projection property no longer holds when they approach B_1 . (See fig. 7 in Section 2 and the accompanying text.) That such wobbling always occurs for circle foliations on acyclic 3-manifolds with $B_2 = \emptyset$ is one result of a forthcoming paper [Vo2] where the size of the homology groups of open 3-manifolds which admit circle foliations is investigated.

From a different point of view decompositions of \mathbf{R}^3 into compact sets have been intensively studied by geometric topologists (see for example [Bi]). Their motivation

comes from studying maps from \mathbf{R}^3 into a reasonable space, and thus they consider upper semicontinuous (u.s.c.) decompositions. The decomposition into leaves of a compact foliation is u.s.c. if and only if it is a Seifert fibration, and the space of leaves of a foliation which is not a Seifert fibration cannot be Hausdorff.

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2. A Circle Foliation of \mathbf{R}^3

We begin by replacing the standard Euclidean 3-space \mathbf{R}^3 by an open submanifold diffeomorphic to \mathbf{R}^3 . This will simplify the analytic description of the foliation considerably. Let $\lambda : [0, 1] \rightarrow [0, 1]$ be given by $\lambda(t) = 0$ for $0 \leq t \leq 1/4$, and $\lambda(t) = (4/3)t - 1/3$ for $1/4 \leq t \leq 1$, and fix $a \in (0, 1)$. Let N be the infinite open cylinder

$$N = \{(x, y, z) \in \mathbf{R}^3 : x^2 + y^2 < 1\}.$$

For $i \in \mathbf{Z}$ consider the piecewise linear half-open arc

$$L_i = \{(x, y, z) \in N : x = 0, i < z \leq i + 1, y = \lambda(i + 1 - z)\}$$

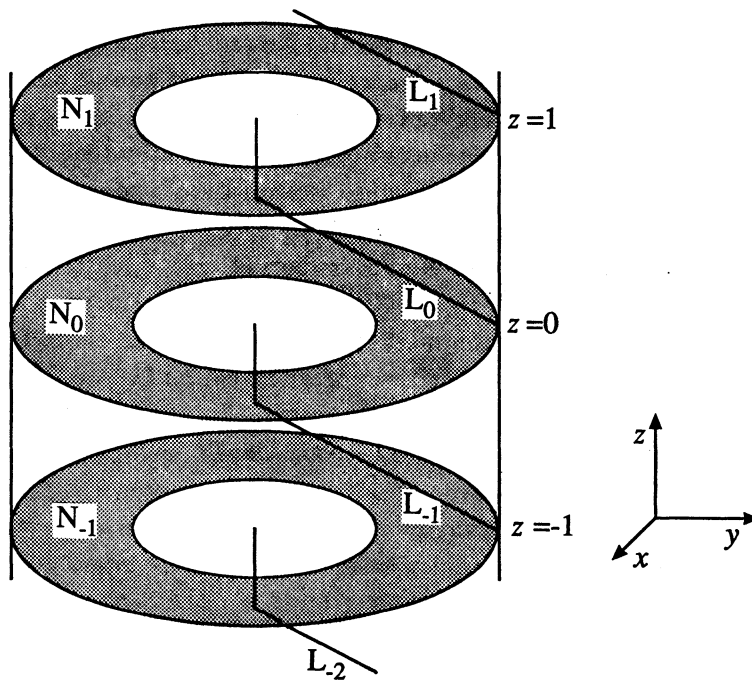


FIG. 1

in N , and denote by N_i the half-open annulus in N given by

$$N_i = \{(x, y, z) \in N : a^2 \leq x^2 + y^2 < 1, z = i\}.$$

The complement of $\bigcup_{i \in \mathbf{Z}} (N_i \cup L_i)$ in N is diffeomorphic to \mathbf{R}^3 and will be our model for \mathbf{R}^3 . We denote it by M . (See Fig. 1.)

Our next step will be a decomposition of M into smooth circles which just misses to be a foliation. For this, note that M intersects each horizontal plane of \mathbf{R}^3 in a punctured disk and thus M is smoothly foliated by open annuli

$$A_t = \{(x, y, z) \in M : z = t\}, \quad t \in \mathbf{R}.$$

Also note that M with this foliation is invariant under the \mathbf{Z} -action generated by the translation T defined by $T(x, y, z) = (x, y, z + 1)$. For $i \in \mathbf{Z}$ set

$$W_i = \{(x, y, z) \in M : i < z \leq i + 1\}.$$

Then W_i is diffeomorphic to a solid torus from whose boundary the complement of an open longitudinal annulus (corresponding to A_{i+1}) has been removed. We now fill each annulus A_t , $t \in \mathbf{R}$, by circles to obtain a decomposition \mathcal{D} of M which is required to have the following properties:

- (2.1) the decomposition \mathcal{D} is invariant under the action of \mathbf{Z} generated by T ,
- (2.2) the restriction of \mathcal{D} to each W_i , $i \in \mathbf{Z}$, defines a smooth (i.e. C^∞) foliation,
- (2.3) for each $i \in \mathbf{Z}$ and each $t \in [i - (1/8), i]$ the restriction of \mathcal{D} to A_t consists of round circles with center on the z -axis.

A decomposition with these properties is easy to construct. One may even assume that each element of \mathcal{D} is a true circle with center on the half plane $\{x = 0, y \geq 0\}$, and such a decomposition would be sufficient to carry out the next and final step. But to facilitate calculations we make one further assumption.

First orient the leaves of \mathcal{D} such that the projection to the x - y -plane induces the standard orientation there. Then we require in addition:

- (2.4) for each $i \in \mathbf{Z}$, $t \in (i, i + (1 - a)/2)$, the curves of \mathcal{D} cross the horizontal disk

$$D_i := \{(x, y, z) \in M : x^2 + y^2 \leq a^2, z = t\}$$

in lines parallel to the x -axis in positive x -direction. (See Fig. 2.)

Again it is clear that such a decomposition of M exists. It fails to be a foliation exactly in the points of A_i , $i \in \mathbf{Z}$.

In the final step of our construction we apply for each $i \in \mathbf{Z}$ a diffeomorphism h_i of W_i to the restriction $\mathcal{D}|_{W_i}$ of \mathcal{D} to W_i . The diffeomorphisms will be chosen so

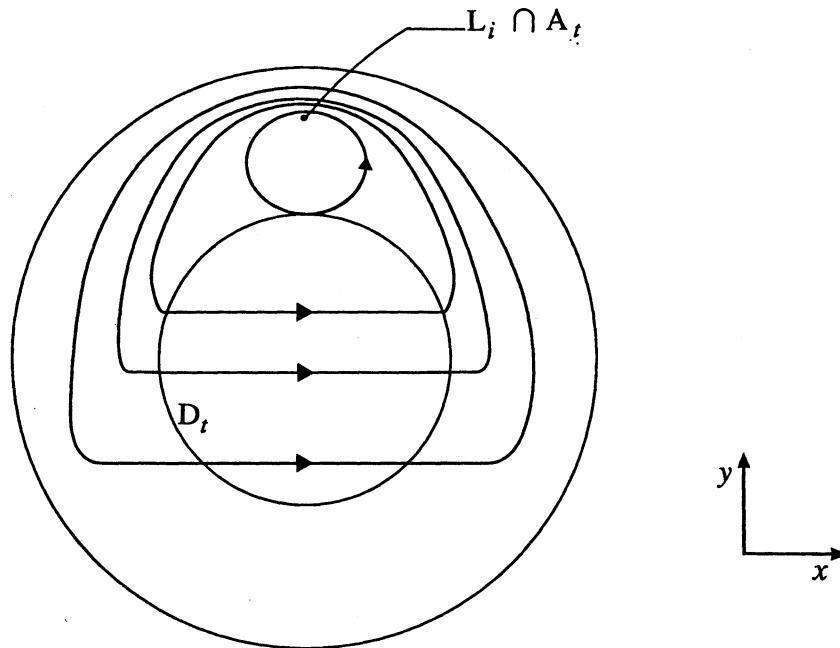


FIG. 2

that the decompositions $(h_i)_*(\mathcal{D} | W_i)$ fit together to form a foliation of M . Since everything we do will be invariant under the \mathbf{Z} -action (i.e. $h_i = T^i h_0 T^{-i}$) it suffices to consider $W := W_0$ and $h := h_0$. By abuse of language we call the restriction of \mathcal{D} to W_0 again \mathcal{D} . By (2.2) \mathcal{D} is a C^∞ -foliation on W_0 . Close to A_1 the leaves of \mathcal{D} are round circles and we do not want to change this. Therefore by \mathbf{Z} -invariance near points of A_0 we have to transform \mathcal{D} such that tangentially the curves of the new foliation approximate the circular flow around 0 which is given on A_0 .

The diffeomorphism of W which is going to achieve this is the composition $g \circ f$ of two diffeomorphisms. The first one, f , will introduce a vertical component into our hitherto strictly horizontal flow. The second one, g , will pick up this vertical component and force it to rotate so violently that the rotation will completely dominate the behaviour of the curves.

We begin with the analytic description of the diffeomorphisms. Fix b with $a < b < 1$. The support of f will be in

$$C_1 = \{(x, y, z) \in W : 0 < z \leq 3/4\},$$

the support of g in

$$C_2 = \{(x, y, z) \in W : x^2 + y^2 \leq b^2, 0 < z \leq (1 - b)/2\}.$$

Note that the closure of C_2 in N does not meet L_0 .

We require f to have the following properties:

- (2.5) $f(x, y, z) = (x, y, f_3(x, z))$,
- (2.6) f has support in C_1 ,
- (2.7) $f_3(0, z) = z$ if $(0, y, z) \in W$,
- (2.8) $f_3(x, z) = z \cdot e^{-x}$ for $0 < z \leq 1/4$.

One can construct f as the time-1-map of a flow given by a smooth vector field of the form $-x \cdot k(z) \frac{\partial}{\partial z}$, with $k(z) = 0$ for $z \geq 3/4$ and $k(z) = z$ for $0 < z \leq 1/4$. (See Fig. 3.)

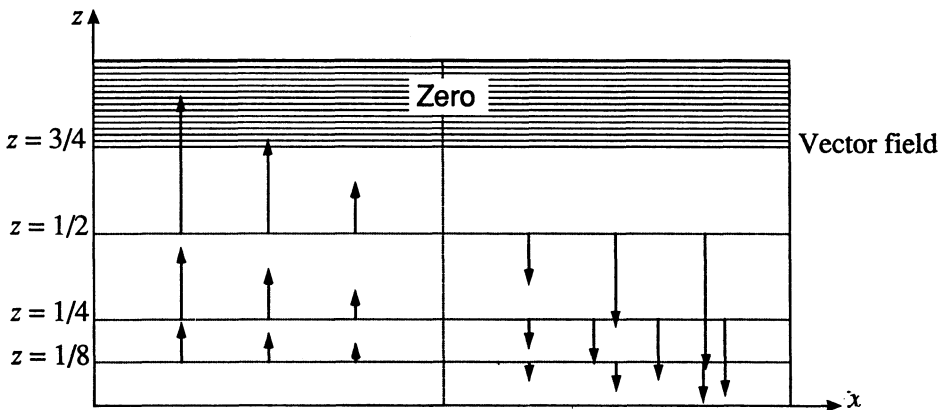


FIG. 3

Thus f tilts and bends every horizontal annulus A_t , $0 < t \leq 1/4$, a little. (See Fig. 4 for the projection of $f(A_t)$ to the x - z -plane.)

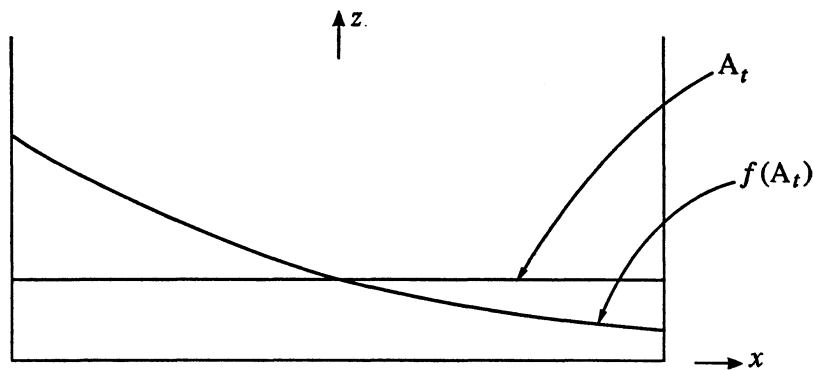


FIG. 4

Important for us is the fact that for $x^2 + y^2 \leq a^2$ and $0 < z < (1/4) e^{-a}$ the vector $\frac{\partial}{\partial x} - z \cdot \frac{\partial}{\partial z}$ is a positive tangent vector to $f_*(\mathcal{D})$ at the point (x, y, z) .

So far we have not made much progress towards obtaining a foliation on M . The map f has changed the qualitative picture of \mathcal{D} very little. The really drastic move is caused by the second diffeomorphism, g , which is best described in partial polar coordinates (r, φ, z) , where (r, φ) are polar coordinates of the x - y -plane, and z is as before. We require g to be a diffeomorphism of W such that

(2.9) g has support in C_2 ,

(2.10) $g(r, \varphi, z) = (r, \varphi + G(r, z), z)$, and

(2.11) $G(r, z) = 1/z^2$ for $0 < z < (1 - b)/4$, $0 \leq r \leq a$.

Remark. — The function $1/z^2$ in (2.11) can be replaced by any function of the form $1/h(z)$ where $h : (-1, 1) \rightarrow \mathbf{R}$ is a C^1 -map with $h(0) = h'(0) = 0$ and $h'(z) > 0$ for $0 < z < 1$.

Under g every annulus A_t , $0 < t \leq 1$, is mapped to itself, and g rotates the disk $D_t \subset A_t$ by the angle $1/t^2$ if $0 < t < (1 - b)/4$. It fixes each point of $A_t \setminus D_t = \{(r, \varphi, t) \in A_t : r > b\}$ while it performs some twist map of the annulus $D_t \setminus D'_t = \{(r, \varphi, t) \in A_t : a < r \leq b\}$ (see Fig. 5).

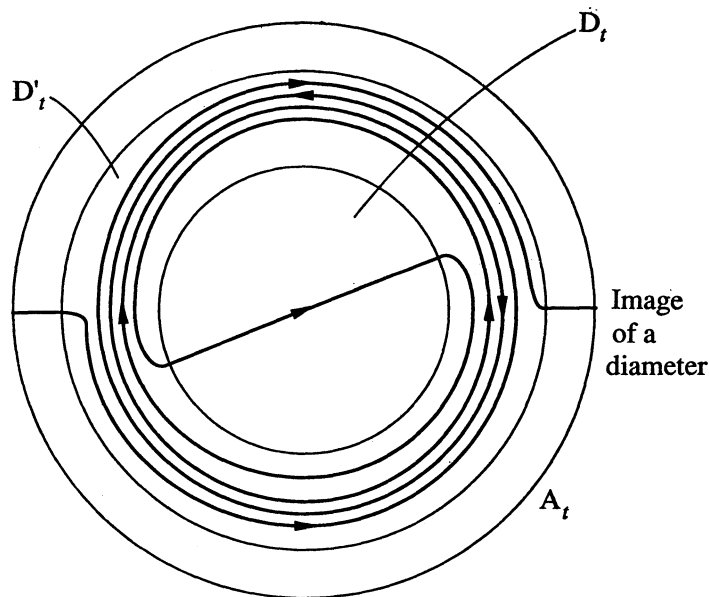


FIG. 5

The essential feature of g is the drastic increase of the angle by which D_t is rotated as t goes to 0. Thus the image of any curve in W which approaches a point of A_0 vertically gets whisked around the z -axis at an ever increasing rate as it approaches A_0 and gets stretched to become infinitely long. On the other hand, horizontal curves in each D_t are simply rotated and do not change in length.

To see what g does to the foliation $f_*(\mathcal{D})$ on W near A_0 we first investigate the

tangent field of $g_*f_*(\mathcal{D})$. At $g^{-1}(x, y, z)$ with $x^2 + y^2 \leq a$, $(1 - b)/4 > z > 0$, the vector $\frac{\partial}{\partial x} - z \frac{\partial}{\partial z}$ is a positive tangent vector to $f_*(\mathcal{D})$. Thus

$$\cos(1/z^2) \frac{\partial}{\partial x} + \sin(1/z^2) \frac{\partial}{\partial y} + \frac{2}{z^2} \left(-y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y} \right) - z \frac{\partial}{\partial z}$$

is a positive tangent vector to the foliation $g_*f_*(\mathcal{D})$ at (x, y, z) . Near A_1 , to be exact, at every point $(x, y, z) \in W$ with $7/8 \leq z \leq 1$, a positive tangent vector to $g_*f_*(\mathcal{D})$ is $\frac{\partial}{\partial \varphi} = -y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y}$. Therefore we can define $g_*f_*(\mathcal{D})$ on W by a nowhere vanishing C^∞ vector field V which is given by

$$\frac{\partial}{\partial \varphi} + \frac{z^2}{2} \left(\cos(1/z^2) \frac{\partial}{\partial x} + \sin(1/z^2) \frac{\partial}{\partial y} - z \frac{\partial}{\partial z} \right)$$

on $\{(x, y, z) \in W : x^2 + y^2 \leq a^2, 0 < z < (1 - b)/4\}$ and by $\frac{\partial}{\partial \varphi}$ on $\{(x, y, z) \in W : 7/8 \leq z \leq 1\}$.

Transporting $g_*f_*(\mathcal{D})$ and V with the translation T^t to $W_t = T^t(W)$ (remember: $W = W_0$), we obtain a new decomposition—which we call \mathcal{F} —of M by circles and a continuous non-singular vector field U tangent to \mathcal{F} . Clearly, the vector field U has unique solutions and therefore \mathcal{F} is a C^0 -foliation. We are done.

In the next section we will show that \mathcal{F} is differentiable. At the moment we only note that the vector field U is differentiable but not C^1 . The derivative of the horizontal components of U are not locally bounded at any point of A_i , $i \in \mathbf{Z}$.

To obtain a picture of the foliation \mathcal{F} as it approaches the annulus A_0 from above we stretch the cylinder

$$\{x, y, z : x^2 + y^2 < 1, 0 < z < 1\} \cup A_1$$

which contains W to the infinite cylinder

$$\{(x, y, z) : x^2 + y^2 < 1, 0 > z > -\infty\} \cup A_0$$

by $(x, y, z) \rightarrow (x, y, \log z)$.

The diffeomorphism f corresponds on this cylinder to a diffeomorphism which maps (x, y, z) to $(x, y, z - x)$ if $z < -\log 4$, while g corresponds to a diffeomorphism which transforms (r, φ, z) to $(r, \varphi + e^{-2z}, z)$ for $r \leq a$, $z < \log(1 - b)/4$. Thus for small z (z close to $-\infty$, this is the interesting part) the foliation corresponding to \mathcal{F} on the cylinder $Z = \{x, y, z : x^2 + y^2 \leq a^2, 0 > z\}$ is equal to the foliation $\beta_* \circ \alpha_*(\mathcal{H})$ on the doubly infinite cylinder $\{(x, y, z) : x^2 + y^2 \leq a^2, -\infty < z < \infty\}$, where \mathcal{H} is the foliation given by lines parallel to the x -axis and

$$\begin{aligned} \alpha(x, y, z) &= (x, y, z - x), & (x, y, z) \in Z \\ \beta(r, \varphi, z) &= (r, \varphi + e^{-2z}, z), & (r, \varphi, z) \in Z, \end{aligned}$$

with (r, φ) polar coordinates of \mathbf{R}^2 .

Orienting leaves of \mathcal{H} in positive x -direction, $\frac{\partial}{\partial x} - \frac{\partial}{\partial z}$ is the tangent vector field of $\alpha_*(\mathcal{H})$. If we follow the image of a leaf $L_{y,z} = \{(t, y, z - t) : -\sqrt{a^2 - y^2} \leq t \leq \sqrt{a^2 - y^2}\}$ of $\alpha_*(\mathcal{H})$ under β we see the following. If we move from $(t, y, z - t)$ to $(t + \varepsilon, y, z - (t + \varepsilon))$ for some small positive ε and z close to $-\infty$, the image curve slowly loses height by ε while horizontally it moves from $R(e^{2t-2z})(t, y)$ to $R(e^{2(t+\varepsilon)-2z})(t + \varepsilon, y)$, where $R(\tau)$ is the rotation of \mathbf{R}^2 by the angle τ around the origin. Thus the image point while moving radially very slowly from $\sqrt{t^2 + y^2}$ to $\sqrt{(t + \varepsilon)^2 + y^2}$ has a rapid angular movement from

$$\varphi_0 = e^{2t-2z} \quad \text{to} \quad \varphi_1 = \varphi_0 + e^{2(t-\varepsilon)} \cdot (e^{2\varepsilon} - 1) \geq \varphi_0 + e^{|\varepsilon|} \cdot 2\varepsilon/e^2.$$

Thus, if we follow $\beta(L_{y,z})$ from beginning to end it will, while slowly losing height, rapidly spiral many times on almost circular trajectories around the z -axis, first spiraling in towards the z -axis until it touches the circle of radius y and then spiraling outwards again (Fig. 6).

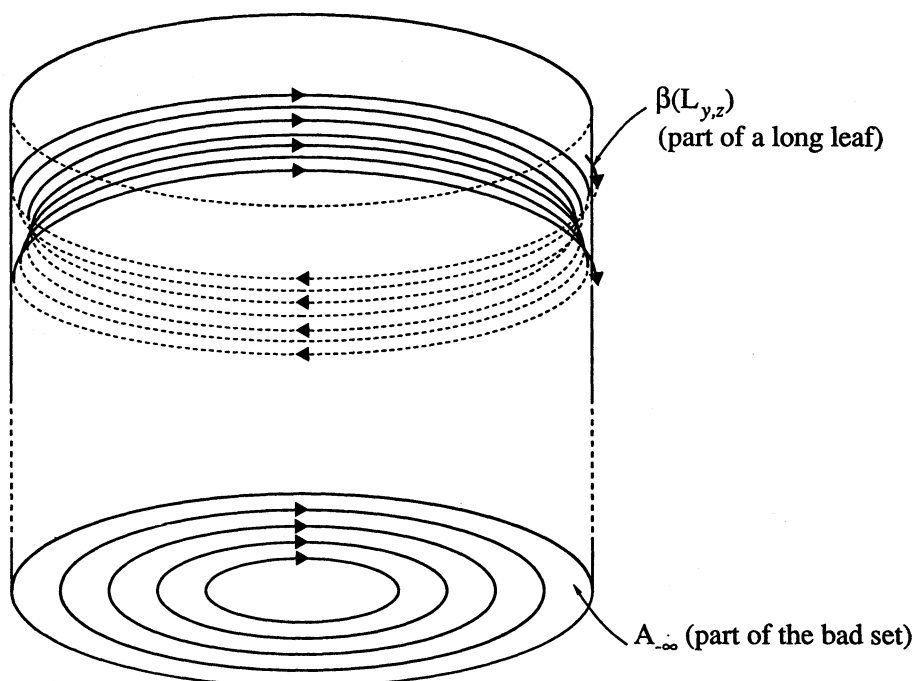


FIG. 6

If one fixes y_0 and lets z go to $-\infty$, then, in the limit, $\beta(L_{y_0,z})$, or rather its counterpart in W , will be smeared over the whole subannulus $y_0^2 \leq x^2 + y^2 < a^2$ of A_0 for $y_0 \neq 0$, and all over A_0 for $y_0 = 0$. In the language of [EMS] each $0 \leq y_0 < a$ gives rise to a moving leaf approaching B_1 , namely the family $\{L_z; 0 \geq z > -\infty\}$, where L_z is the leaf containing $\beta(L_{y_0,z})$. It is interesting to note that for different values of y_0 the limits of the corresponding moving leaves are different subsets of B_1 .

One obtains a somewhat more global view of the behaviour of \mathcal{F} near the bad set by considering for decreasing t the image of all of D_t under $\beta \circ \alpha$. (To the family $\beta \circ \alpha(D_t)$, $t \rightarrow -\infty$, corresponds in W a family of disks, $g \circ f(D_s)$, $s \rightarrow 0$, ($s = e^t$). Each $g \circ f(D_s)$ lies in an invariant torus T_s of $\mathcal{F} | W$, which bounds as solid torus V_s , and $W = \bigcup_{s>0} V_s$. As s goes to 0, the really interesting part of T_s is $g \circ f(\text{Int}(D_s \setminus \{0\}))$. These annuli “converge” to the bad set. They are not invariant, but the length of the intersection of a leaf of \mathcal{F} with $g \circ f(\text{Int}(D_s \setminus \{0\}))$ goes to infinity as s goes to 0.) One way to visualize $\beta \circ \alpha(D_t)$ is to consider the image of the circle $C_{r,t}$ of radius r around the center of D_t as r grows from 0 to a , i.e. contemplate the curves

$$C'(r, t) = \{(r, \varphi + e^{2(r \cos \varphi - t)}, t - r \cos \varphi) : 0 \leq \varphi \leq 2\pi\}$$

for $0 \leq r \leq a$, and for t negative of large absolute value. For very small r the dominant part of $\varphi + e^{2(r \cos \varphi - t)}$ will be $\varphi + e^{-2t}$ and $C'(r, t)$ is very close to the ellipse $\alpha(C_{r,t})$ rotated by e^{-2t} . But as r grows the oscillating factor $e^{2r \cos \varphi}$ will dictate the behaviour of the angular component $\varphi + e^{2r \cos \varphi} \cdot e^{-2t}$ more and more, and the projection of $\beta \circ \alpha(D_t)$ to the x - y -plane will look like a cusp singularity whose fold lines spiral a large number of times around the origin as they move outwards away from the cusp point. In Fig. 7 we indicate this phenomenon for $|t|$ quite small. As t decreases further, one should imagine the outward ends of the folds to rotate away from their current positions in opposite direction (if one ignores the rotation by e^{-2t}), the lower one in positive, the upper one in negative direction. We also depict two flow lines of \mathcal{F} in Fig. 7. One, called front, is $\beta(L_{y_0, t})$ with $y_0 < 0$, the other, called back, is $\beta(L_{y_0, t})$ with y_0 rather close to a . One notices that the folding over of a large part of the back half $\{(x, y, t) \in D_t : y > 0\}$ of D_t has two effects. It enables us to stretch D_t in angular direction and at the same time it flips the direction of the flow on the back half of D_t around, so that it also points in positive φ -direction.

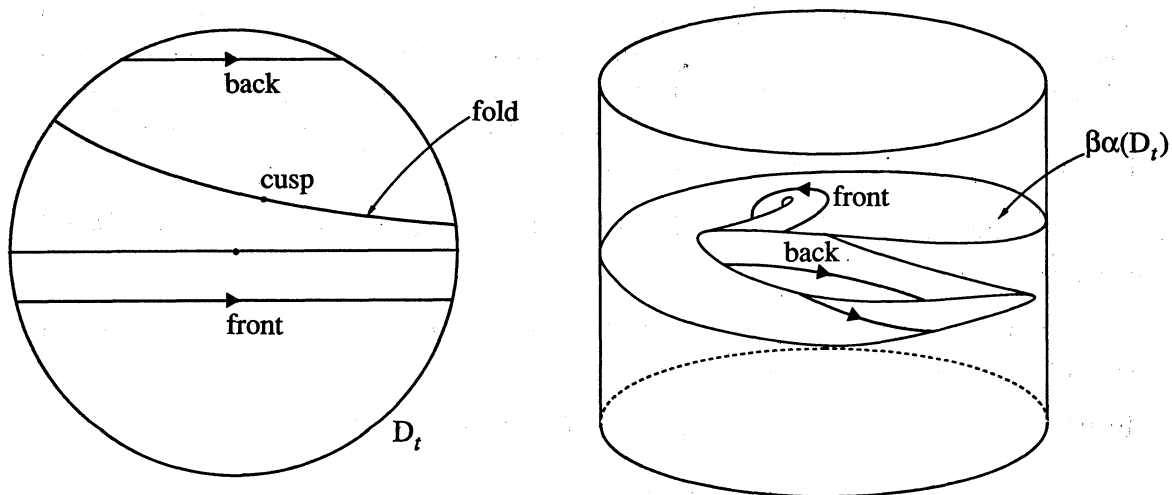


FIG. 7

3. The Question of Differentiability

We return to the foliation \mathcal{F} on $\mathbf{M} \subset \mathbf{R}^3$ with tangent vector field U of the previous section. By construction U is C^∞ in the complement of

$$B_1 = \bigcup_{i \in \mathbf{Z}} A_i = \{(x, y, z) \in \mathbf{M} : z \in \mathbf{Z}\}.$$

Therefore, \mathcal{F} is C^∞ in $\mathbf{M} \setminus B_1$. Since U and \mathcal{F} are equivariant with regard to the translation $T : \mathbf{M} \rightarrow \mathbf{M}$, $T(x, y, z) = (x, y, z + 1)$, it remains to investigate \mathcal{F} near $A_0 = \{(x, y, 0) : 0 < x^2 + y^2 < a^2\}$. In a neighborhood of A_0 the vector field U equals

$$U(x, y, z) = \begin{cases} -y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y} + \frac{z^2}{2} \left(\cos(1/z^2) \frac{\partial}{\partial x} + \sin(1/z^2) \frac{\partial}{\partial y} - z \frac{\partial}{\partial z} \right), & \text{for } z > 0, \\ -y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y}, & \text{for } z \leq 0, \end{cases}$$

or, in partial polar coordinates (r, φ, z) ,

$$(3.1) \quad U(r, \varphi, z) = \begin{cases} \frac{\partial}{\partial \varphi} + \frac{z^2}{2} \left(A(r, \varphi, z) \frac{\partial}{\partial r} + B(r, \varphi, z) \frac{\partial}{\partial \varphi} - z \frac{\partial}{\partial z} \right), & z > 0, \\ \frac{\partial}{\partial \varphi}, & z \leq 0, \end{cases}$$

where A, B are smooth in $A_0 \times (0, \varepsilon)$, with

$$(3.2) \quad A(r, \varphi, z)^2 + r^2 \cdot B(r, \varphi, z)^2 = 1.$$

We already mentioned that U has unique solutions which are pointwise periodic. Thus U defines a global flow $F : \mathbf{R} \times \mathbf{M} \rightarrow \mathbf{M}$. If U is C^r with $r \geq 1$, then the flow F as well as the foliation \mathcal{F} are C^r . In our situation where U is only differentiable a general theorem which guarantees the differentiability of F and \mathcal{F} does not seem to be available. We give a direct proof covering our example, but the method applies to other situations where a smooth vector field U_+ on $N \times (0, \varepsilon)$ and a smooth vector field U_- on $N \times (-\varepsilon, 0]$ form together a continuous vector field on $N \times (-\varepsilon, \varepsilon)$.

Proposition 3.1. — *Let $F : \mathbf{R} \times \mathbf{M} \rightarrow \mathbf{M}$ be the flow defined by the vector field U . Then F is differentiable.*

Proof. — Let $0 < r_0 < a$, $\varphi_0 \in \mathbf{R} \bmod 2\pi$, $t_0 \in \mathbf{R}$. We have to show that F has a derivative at $(t_0, (r_0, \varphi_0, 0))$.

$$(3.3) \quad \begin{aligned} F(t, (r, \varphi, z)) &= \int_0^t \frac{\partial}{\partial \tau} F(\tau, r, \varphi, z) \, d\tau + (r, \varphi, z) \\ &= \int_0^t U(F(\tau, r, \varphi, z)) \, d\tau + (r, \varphi, z). \end{aligned}$$

If we denote the r -, φ - and z -components of F by F_r , F_φ , and F_z and assume that $(t, (r, \varphi, z))$ is close to $(t_0, (r_0, \varphi_0, 0))$ the last expression is equal to

$$(r, \varphi + t, z), \quad \text{if } z \leq 0,$$

and to

$$\begin{aligned} (r, \varphi + t, z) + \frac{1}{2} \int_0^t F_z(\tau, r, \varphi, z)^2 (A \circ F(\tau, r, \varphi, t), \\ B \circ F(\tau, r, \varphi, z), -F_z(\tau, r, \varphi, z)) d\tau \\ =: (r, \varphi + t, z) + \tilde{F}(t, r, \varphi, z), \quad \text{if } z > 0. \end{aligned}$$

We claim that for a small neighborhood V_1 of $(t_0, r_0, \varphi_0, 0)$ an equation of the form

$$(3.4) \quad \tilde{F}(t, r, \varphi, z) = z^2 \cdot C(t, r, \varphi, z)$$

holds in $V = V_1 \cap \{z > 0\}$ for some \mathbb{R}^3 -valued function C which is continuous and bounded on V . Therefore, the derivative of F at $(t_0, r_0, \varphi_0, 0)$ is given by $(t, r, \varphi, z) \mapsto (r, \varphi + t, z)$.

The equation (3.4) holds for the following reasons:

- (i) F_z depends only on τ and z and is the solution of $\dot{F}_z = -\frac{1}{2} F_z^3$ with initial value z , i.e.

$$F_z(\tau, z) = \frac{z}{\sqrt{1 + \tau \cdot z^2}}.$$

If V_1 is small enough, we therefore may assume that $F_z(\tau, r, \varphi, z)^2 < 2 \cdot z^2$ for $(t, r, \varphi, z) \in V$ and $-|t| \leq \tau \leq |t|$.

- (ii) Using (i), (3.1), and (3.3) one obtains $|F_r(\tau, r, \varphi, z) - r| < |\tau| \cdot z^2$ for $(t, r, \varphi, z) \in V$ and $-|t| \leq \tau \leq |t|$. This in turn together with (3.2) implies that $B \circ F(\tau, r, \varphi, z)$ has a global bound for $(t, r, \varphi, z) \in V$ and $-|t| \leq \tau \leq |t|$.
- (iii) The absolute value of A is bounded by 1. \square

It is an easy consequence of Proposition (3.1) that \mathcal{F} is differentiable. We cover M by flow boxes of F which either lie in the complement of the bad set $\bigcup_{i \in \mathbb{Z}} A_i$ and are smooth or are of the form

$$T^i \circ F|_{(-\delta, \delta) \times S(\psi, \varepsilon)},$$

where $T^i: M \rightarrow M$ is the translation $(r, \varphi, z) \rightarrow (r, \varphi, z + i)$, and $S(\psi, \varepsilon)$ is the open rectangle

$$S(\psi, \varepsilon) = \{(r, \varphi, z) \in M : 0 < r < a, \varphi = \psi, |z| < \varepsilon\}$$

which is transverse to \mathcal{F} if ε is sufficiently small. These flow boxes form an atlas of foliation charts for \mathcal{F} . In the complement of $\bigcup_{i \in \mathbb{Z}} A_i$ coordinate changes are C^∞ . Therefore it suffices to show that

$$F_{12} = (F|_{(-\delta_1, \delta_1) \times S(\psi_1, \varepsilon_1)})^{-1} \circ F|_{(-\delta_2, \delta_2) \times S(\psi_2, \varepsilon_2)}$$

is differentiable in $(t, r, \psi_2, 0)$ if F_{12} is defined in $(t, r, \psi_2, 0)$. By Proposition (3.1) it suffices to prove that $(F|_{(-\delta, \delta) \times S(\psi, \varepsilon)})^{-1}$ is differentiable in every point of the form $((r, \varphi), 0)$

which lies in $F((-\delta, \delta) \times S(\psi, \varepsilon))$. By Proposition (3.1) the linear approximation of $F|_{(-\delta, \delta) \times S(\psi, \varepsilon)}$ at $(t_0, (r_0, \psi, 0))$ is given by $(t, r, \psi, z) \rightarrow (r_0, \psi + t_0, 0) + (r - r_0, t - t_0, z)$ and thus the derivative of $F|_{(-\delta, \delta) \times S(\psi, \varepsilon)}$ at $(t_0, (r_0, \psi, 0))$ can be identified with the identity. The differentiability of $(F|_{(-\delta, \delta) \times S(\psi, \varepsilon)})^{-1}$ follows then from the next lemma whose straightforward proof we omit (it probably is well known).

Lemma 3.2. — *Let $h: (\mathbf{R}^n, 0) \rightarrow (\mathbf{R}^n, 0)$ be a local homeomorphism at 0. Assume that h is differentiable at 0 with derivative A . If A is non-singular, then the local inverse of h at 0 is differentiable at 0 with derivative A^{-1} . \square*

Summarizing Sections 2 and 3 we obtain

Theorem 3.3. — *There exists a foliation \mathcal{F} of \mathbf{R}^3 which has the following properties:*

- (i) *all leaves of \mathcal{F} are smooth simple closed curves,*
- (ii) *\mathcal{F} is defined by a differentiable, but not continuously differentiable, non-singular vector field,*
- (iii) *\mathcal{F} is differentiable,*
- (iv) *the bad set $B_1(\mathcal{F})$ of \mathcal{F} is a countable union of properly embedded open annuli.*

Using arguments similar to those in the proof of Proposition (3.1) one can show that neither the flow F nor the foliation \mathcal{F} constructed in section 2 are C^1 . Thus we have the following

Problem 3.4. — *For which r , with $1 \leq r \leq \omega$, does there exist a C^r -foliation of \mathbf{R}^3 by circles?*

4. More Circle Foliations

With the help of the foliation \mathcal{F} of \mathbf{R}^3 by circles described in Section 2 many "new" circle foliations can be constructed. We will illustrate this by giving some examples.

All of these examples are obtained by taking 3-manifolds which carry a circle foliation and glueing them together along parts of their boundaries. Since the foliation \mathcal{F} or variants of \mathcal{F} will always be involved our examples will at best be as smooth as \mathcal{F} . Following a suggestion of N. Kuiper we will call a 1-dimensional foliation \mathcal{G} of a C^∞ 3-manifold M a C^1 -foliation if

(4.1) all leaves of \mathcal{G} are C^∞ ,

(4.2) the tangent distribution of \mathcal{G} is continuous,

(4.3) \mathcal{G} is differentiable (in the sense of the preceding section).

To prevent a loss of smoothness in the glueing process along parts of ∂M , let $\partial M \times [0, \infty)$ be a smooth outward collar of ∂M , and extend \mathcal{G} to a foliation \mathcal{G}' of

$M \cup \partial M \times [0, \infty)$ by setting $\mathcal{G}' \mid \partial M \times \{t\} = (\mathcal{G} \mid \partial M) \times \{t\}, t \in [0, \infty)$. We then require in addition:

(4.4) The extension \mathcal{G}' of \mathcal{G} to $M \cup \partial M \times [0, \infty)$ is C^1 -.

All examples considered below will satisfy (4.1-4). This is easy to check and we leave that task to the reader.

For our first example we need a circle foliation which is essentially due to G. Reeb ([Re], A, III, c, 2).

Example 4.1 (Reeb). — Let S^1 be the unit circle in \mathbf{R}^2 and $J \subset S^1$ the set of points with $x_2 \leq 0$. Then there exists a smooth circle foliation \mathcal{F}_N on

$$N = [0, 1] \times S^1 \times S^1 \setminus \{0\} \times J \times S^1$$

with the following properties:

(4.5a) on $\{1\} \times S^1 \times S^1$ the leaves are

$$\{1\} \times S^1 \times \{\varphi\}, \quad \varphi \in S^1;$$

(4.5b) on $\{0\} \times (S^1 \setminus J) \times S^1$ the leaves are

$$\{0\} \times \{\psi\} \times S^1, \quad \psi \in S^1 \setminus J.$$

Proof. — We modify slightly the example of [Ep1], pp. 66, 67. Let $g: \mathbf{R} \rightarrow [0, 1]$ be a C^∞ -function such that $g^{-1}(0) = [3/4, \infty)$ and $g^{-1}(1) = (-\infty, 0]$. For $t \in [0, 1]$ let $H_t \subset \mathbf{R}^3$ be the image of $\mathbf{R}^2 = \mathbf{R}^2 \times \{0\}$ under the rotation around the y -axis of angle $(\pi/2) \cdot g(t)$. Then the translates of H_t in z -direction for $t > 0$, and in x -direction for $t \leq 0$ define a foliation of $\{t\} \times \mathbf{R}^3$ by planes, and altogether a foliation of $[0, 1] \times \mathbf{R}^3$ by planes, where the leaves in $\{1\} \times \mathbf{R}^3$ are parallel to the x - y -plane and the leaves in $\{0\} \times \mathbf{R}^3$ are parallel to the y - z -plane. This foliation induces a foliation on $[0, 1] \times S^1 \times \mathbf{R} \subset [0, 1] \times \mathbf{R}^2 \times \mathbf{R} = [0, 1] \times \mathbf{R}^3$, except in the lines $\{0\} \times \{(\pm 1, 0)\} \times \mathbf{R}$ where the foliation of $\{0\} \times \mathbf{R}^2 \times \mathbf{R}$ is not transverse to $\{0\} \times S^1 \times \mathbf{R}$. Since $(\pm 1, 0) \subset J$ we obtain a 1-dimensional foliation on

$$\tilde{N} = (0, 1] \times S^1 \times \mathbf{R} \cup \{0\} \times (S^1 \setminus J) \times \mathbf{R}$$

and the leaves on $\{t\} \times S^1 \times \mathbf{R}$ are ellipses of slope $(\pi/2) \cdot g(t)$ for $0 < t \leq 1$, and vertical lines on $\{0\} \times (S^1 \setminus J) \times \mathbf{R}$. Thus after factoring out the \mathbf{Z} -action $(x, y, z) \rightarrow (x, y, z + n), n \in \mathbf{Z}$, we obtain a circle foliation \mathcal{F}_N on $N = \tilde{N}/\mathbf{Z}$ with the required properties. \square

Proposition 4.2. — Let $V = D^2 \times S^1$ be the solid torus and let x be a point in the interior of V . Then there exists a C^1 -foliation \mathcal{F}_a by circles on $V \setminus \{x\}$ such that \mathcal{F}_a restricted to ∂V is the foliation by parallels $\{z\} \times S^1, z \in \partial D^2$.

Proof. — By (2.3) the foliation \mathcal{F} on $M \cong \mathbf{R}^3$ of Section 2 restricted to $\{(x, y, z) : 0 < x^2 + y^2 < 1, 7/8 < z < 1\}$ consists of round, horizontal circles with centers on the z -axis. Thus \mathcal{F} can be extended to $M \cup C$ where the open annulus

$$C = \{(x, y, z) : x^2 + y^2 = 1, 7/8 < z < 1\}$$

is also foliated by round circles. Now attach $M \cup C$ to N of Reeb's example (4.1) by identifying the annulus C with the annulus $\{0\} \times (S^1 \setminus J) \times S^1$ of N via a foliation preserving diffeomorphism. The resulting manifold is diffeomorphic to a solid torus with an interior point removed and the boundary of the solid torus is $\{1\} \times S^1 \times S^1 \subset N$, with $\{1\} \times S^1 \times \{\varphi\}$, $\varphi \in S^1$, corresponding to parallels, and $\{1\} \times \{\varphi\} \times S^1$ corresponding to meridians. By (4.5a) the leaves of $\{1\} \times S^1 \times S^1$ are parallels. \square

Corollary 4.3. — *Let M be a 3-manifold which admits a C^1 -circle foliation and let y be an interior point of M . Then $M \setminus \{y\}$ admits a C^1 -foliation.*

Proof. — Let \mathcal{H} be a C^1 -circle foliation of M and let L be a typical leaf of the Seifert fibration $\mathcal{H} | (M \setminus B_1(\mathcal{H}))$. Also let L and y be in the same component of M . The leaf L has an invariant neighborhood which is isomorphic in the category of C^1 -foliations to $V = D^2 \times S^1$ with the product circle foliation. Replace V by $V \setminus \{x\}$ with the foliation \mathcal{F}_a of (4.2). The resulting manifold is diffeomorphic to $M \setminus \{y\}$. \square

$N.$ Kuiper pointed out how one could use the foliation \mathcal{F} of $M \cong \mathbf{R}^3$ to obtain

Example 4.4. — *The foliation by meridians $\partial D^2 \times \{\varphi\}$, $\varphi \in S^1$, of the boundary $\partial D^2 \times S^1$ of the solid torus $V = D^2 \times S^1$ can be extended to a C^1 -foliation of $V \setminus \{\text{interior point}\}$ by circles.*

To see this just note that all leaves of the foliation \mathcal{F} of $M \cong \mathbf{R}^3$ are unknotted circles in M . Let L be a leaf in $M \setminus B_1$, and let X be a small invariant neighborhood of L isomorphic to $D^2 \times S^1$ with the product foliation. Then $M \setminus \text{int } X$ is a solid torus minus a point, and the curves $\{z\} \times S^1$, $z \in \partial D^2$, are meridians of this solid torus. \square

Remark. — The foliation by meridians of $\partial D^2 \times S^1$ cannot be extended to a circle foliation of $D^2 \times S^1$ because, by [Ep1], the resulting foliation is a Seifert fibration of a solid torus and these are known to be standard local models of Seifert fibrations [Sei].

Our final example uses a modification of the construction in Section 2. From this one gets a good idea how one can adjust the construction of Section 2 in order to obtain "unexpected" circle foliations on many 3-manifolds.

Example 4.5. — *Let V_h be the interior of a handlebody of genus h . Let $E \subset V$ be a finite set with at least $h - 1$ points. Then $V_h \setminus E$ admits a C^1 -foliation by circles.*

Proof. — Consider again the circle foliation $f_*(\mathcal{D})$ of Section 2 on the solid torus $W = \{(x, y, z) \in M : 0 < z \leq 1\}$. The diffeomorphism g deforms $f_*(\mathcal{D})$ into a circle

foliation $g_*f_*(\mathcal{D})$ which admits a C^1 -extension to $W \cup A_0$, where the foliation on A_0 is given by round circles around the origin. The diffeomorphism g has support in the cylinder

$$C_2 = \{(x, y, z) \in W : x^2 + y^2 \leq b^2, 0 < z \leq (b - 1)/2\}.$$

Replace g by a diffeomorphism g' which has support in a disjoint union of $h - 1$ cylinders Z_1, \dots, Z_{h-1} in C_2 , where Z_i has the form

$$Z_i = d_i \times (0, (b - 1)/2] \subset W$$

with $d_i \subset \{(x, y) \in \mathbf{R}^2 : x^2 + y^2 \leq b^2\}$ a round disk: $d_i = \{(x, y) : (x - x_i)^2 + (y - y_i)^2 \leq b_i^2\}$.

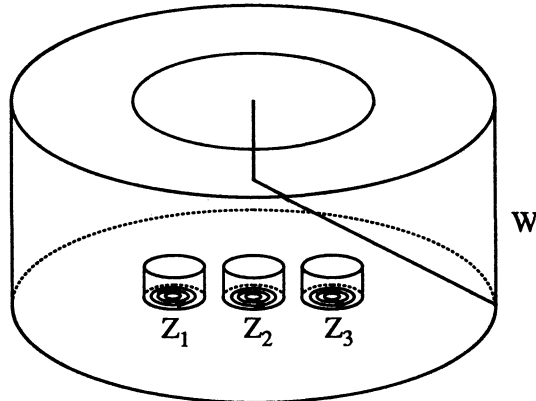


FIG. 8

Let g' act on each cylinder Z_i the way g acts on C_2 . Then $g'_*f_*(\mathcal{D})$ is a C^∞ -circle foliation on W which extends to a C^1 -foliation on $W \cup \bigcup_{i=1}^{h-1} a_i$, where the a_i are round open disks punctured in their center and lying in

$$\{(x, y, z) : x^2 + y^2 < b^2, z = 0\} \subset \mathbf{R}^3, \quad Cl(a_i) \subset d_i,$$

and the a_i are foliated by round circles around the center of a_i .

If we take two copies of $W \cup \bigcup_{i=1}^{h-1} a_i$ and identify for $i = 1, \dots, h - 1$ the a_i of the first copy with the a_i of the second copy via the "identity" we obtain V_h minus $h - 1$ points with a C^1 -foliation by circles. \square

Remark. — If $h > 2$ and one a_i -pair is identified via a diffeomorphism which changes the orientation of the circles, we obtain an open solid non-orientable handlebody with $h - 1$ points removed together with a C^1 -foliation by circles.

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