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HARMONIC MAPPINGS OF KÄHLER MANIFOLDS TO LOCALLY SYMMETRIC SPACES

by JAMES A. CARLSON *and* DOMINGO TOLEDO*

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1. Introduction

A fundamental question in topology is to determine whether there exists a map of nonzero degree between given manifolds of the same dimension. In a lecture given in 1978, M. Gromov suggested looking at the existence of such mappings as defining an ordering on the set of homeomorphism classes of compact oriented manifolds of a given dimension, and formulated a number of stimulating conjectures as to which classes are or are not comparable in this ordering. This fascinating ordering is defined as follows: say that $M \geqslant N$, equivalently, that M dominates N, if there exists a continuous mapping from M to N of non-zero degree. Intuitively, $M \geqslant N$ means that M is more complicated than N. Thus, if $M \geqslant N$, then the Betti numbers of M are at least as large as those of N, since a map of non-zero degree is surjective in rational homology. For Riemann surfaces the ordering agrees with that given by the Betti numbers, i.e., by the genus: $M \geqslant N$ if and only if *genus* (M) $\geqslant$ *genus* (N). In general, however, the relation of domination is much more subtle. It is not reducible to an inequality of Betti numbers,

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and little more is known about it, with the exception of the case of locally symmetric spaces, which we review in part below. Note, however, that the sphere is an absolutely minimal element: $M \geqslant S^n$ for all n-dimensional manifolds M. The question of whether there is an easily described class of maximal elements, namely, a collection $\mathscr{C}$ of manifolds such that given any N there exists an $M \in \mathscr{C}$ so that $M \geqslant N$, is open, except for the case of surfaces. For these the class of hyperbolic manifolds is maximal.

We mention here only one of Gromov's original conjectures: $M \geqslant N$ is impossible if M is complex hyperbolic (i.e., has a Kähler metric of constant negative holomorphic sectional curvature) and N is real hyperbolic (i.e., has a metric of constant negative sectional curvature). This can now be seen in several ways, one of which is to apply a recent theorem of Sampson [27], which asserts that if M is Kähler and N is real hyperbolic, then any continuous map from M to N is trivial in homology of dimension larger than two. From this it follows that $M \geqslant N$ is impossible if $\dim N > 2$. Sampson's theorem on maps to real hyperbolic space was later independently deduced from Siu's rigidity theorem by Gromov.

The preceding conjecture and its solution via harmonic mappings form the principal motivation for the present paper, the goal of which is to find general properties of an extended Gromov ordering. For manifolds of possibly unequal dimension define $M \geqslant N$ to mean the existence of a continuous map $f: M \to N$ which is surjective in homology. Now take M to be a compact Kähler manifold and N to be a compact locally symmetric space of non-compact type, i.e., a space of the form $\Gamma \backslash G/K$, where G is semisimple Lie group without compact factors, where K is a maximal compact subgroup, and where Γ is a cocompact discrete subgroup. The main result (Theorem 3.1) then implies (via Corollary 3.3) that $M \geqslant N$ is impossible unless N is already Kähler in an obvious way, i.e., is locally Hermitian symmetric. Thus, a compact Kähler manifold cannot dominate a compact locally symmetric space of noncompact and non-Hermitian type.

If N has non-positive curvature, a natural tool for deciding whether $M \geqslant N$ for a given manifold M is the theory of harmonic mappings. A smooth mapping from M to N is called harmonic if it is an extreme value for the energy functional

$$E(f) = \frac{1}{2} \int_M \| df \|^2.$$

This integral makes sense if M and N are Riemannian manifolds, M is compact, and f is continuously differentiable. If in addition N is compact and has nonpositive Riemannian sectional curvatures, then the fundamental existence theorem of Eells and Sampson [12] asserts that each homotopy class of maps from M to N contains a harmonic representative. Thus $M \geqslant N$ is equivalent to the existence of a harmonic map from M to N which is surjective in rational homology.

Harmonic mappings became an effective tool for the study of geometric questions of this kind through the fundamental work of Siu [28]. In his generalization of Mostow's rigidity theorem for Hermitian symmetric spaces, Siu proved that a harmonic map of sufficiently high maximum rank of a compact Kähler manifold to a quotient of an irreducible bounded symmetric domain (other than the hyperbolic plane) must be holomorphic or anti-holomorphic. It follows that if a compact Kähler manifold dominates a compact quotient of such a domain, then the dominating map is homotopic to a holomorphic or anti-holomorphic map. Thus, if the original mapping is a homotopy equivalence, then it is homotopic to a biholomorphic or anti-biholomorphic map. Although this last statement is known as Siu's rigidity theorem, the actual statement on harmonic mappings is much stronger: the two manifolds need not be homotopy equivalent, in fact not even of the same dimension, and the map need not have any particular topological properties, but just sufficiently high rank at one point. The precise measure of " sufficiently high " is a function of the symmetric domain, given explicitly in [30]. Since appropriate homological conditions on a map force lower bounds on its rank, these methods are ideally suited to the domination question.

Technically, Siu's main accomplishment was *a)* the discovery of a Bochner-type identity for harmonic mappings which does not involve the Ricci tensor of the domain (this is where Kählerianity of the domain enters), and *b)* the algebraic study of the resulting vanishing theorem for targets which are symmetric domains. The vanishing theorem implies also that the harmonic mapping in question must satisfy further differential equations which, under the assumption of sufficiently high rank, can be satisfied only by holomorphic maps.

Sampson extended Siu's vanishing technique to treat harmonic mappings of compact Kähler manifolds to a class of real manifolds which includes the quotients of symmetric spaces of non-compact type [27]. His theorem is the following. Assume that M is compact Kähler and $N = \Gamma\backslash G/K$ is a non-positively curved locally symmetric space. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be a Cartan decomposition for the Lie algebra of G, and note that the real tangent space of N at any point can be identified (non-canonically) with $\mathfrak{p}$. Thus, if $f: M \to N$, then $df(T^{1,0})$ can be identified with a subspace W of $\mathfrak{p}^{\mathbf{C}}$. Sampson's theorem asserts that (under any such identification) the subspace W is abelian in the sense that $[X, Y] = 0$ for all X and Y in W. Furthermore, the harmonic map must satisfy further differential equations, to be discussed later.

Because of the relation rank $(df) \leqslant 2 \dim_{\mathbf{C}} W$, with equality when W has no real points, the purely algebraic conclusion of Sampson's theorem has strong geometric consequences. Indeed, it reduces the problem of estimating the maximum rank of a harmonic mapping to a problem in Lie algebra theory. The most elementary result of this kind is a matrix calculation which shows that when $G = SO(1, n)$, $\dim W \leqslant 1$, from which it follows that harmonic maps from a Kähler manifold to a compact quotient of hyperbolic n-space have real rank at most 2 [27]. Gromov has observed that this statement follows from Siu's rigidity theorem for mappings into the complex ball by considering the composition of the mapping with the inclusion of the real hyperbolic manifold into its complexification, which is a complex hyperbolic manifold.

Below we shall explore further the consequences of Sampson's theorem. Because the abelian spaces under consideration are not necessarily defined over **R**, they are not necessarily semisimple; consequently there is no obvious bound in terms of the rank of the group or of the symmetric space. However, the main result (Theorem 3.1) asserts that $\dim W \leqslant \frac{1}{2} \dim \mathfrak{p}^{\mathbf{C}}$, with equality possible only if $(\mathfrak{g}, \mathfrak{k})$ is a Hermitian-symmetric pair with $W = \mathfrak{p}^{1,0}$ or $W = \mathfrak{p}^{0,1}$. Here $\mathfrak{p}^{1,0}$ is the holomorphic tangent space of a G-invariant complex structure on G/K. Therefore rank $(df) < \dim N$ for f harmonic and N non-Hermitian (Corollary 3.2). If N is Hermitian and f has maximal rank at some point, then f is either holomorphic or antiholomorphic (Siu's rigidity theorem).

In the non-Hermitian case the bound on the rank, combined with the Eells-Sampson homotopy theorem, yields strong topological restrictions: the fundamental class of N is not in the image of the homology of a compact Kähler manifold under an arbitrary continuous map (Corollary 3.3). Consequently the fundamental class is not (rationally) representable by a homology class (fundamental or otherwise) on a compact Kähler manifold, and so $M \not\geqslant N$, as asserted above. The preceding obstruction to representability is quite different from those arising classically from cobordism theory. The latter are of finite order and measure representability of a homology class by a stably almost-complex manifold, while the ones presented here are of infinite order and measure representability by the smaller class of Kähler manifolds.

If both M and N are locally symmetric there is a good deal of information on the possible mappings from M to N, thanks to the rigidity theorems of Mostow and Margulis [21, 24], and Zimmer's use [32] of Kazhdan's property T [18]. For instance, if M has rank greater than 1, it follows from the theorem of Margulis that $M \geqslant N$ if and only if M and N have isometric universal covers and M is a covering space of N. Further information is given by a theorem of Zimmer [32, Cor. 20] which implies that if the fundamental group of M satisfies Kazhdan's property (M not necessarily locally symmetric), and if N is either real of complex hyperbolic (so that the group of isometries of the universal cover is not Kazhdan), then any mapping from M to N is homotopic to a constant. In particular $M \geqslant N$ is impossible. If M is covered by quaternionic hyperbolic space, then $\pi_1(M)$ is Kazhdan. Thus, if N is complex hyperbolic, then by Zimmer's theorem $M \geqslant N$ is impossible. By the main theorem $N \geqslant M$ is also impossible, and so these two classes of manifolds are not comparable.

The main result also gives topological restrictions on maps of a compact quotient M of the unit ball in $\mathbf{C}^n$ to a locally symmetric space N of non-Hermitian type which do not follow from the theories mentioned above. For example, if N is compact and of the same dimension as M, our results imply that there is no continuous mapping of M to N of non-zero degree, whereas Mostow's result gives only that there is no homotopy equivalence. The theorems of Margulis and Zimmer do not apply here.

The rank estimate of the main theorem is far from best possible, but we restrict ourselves to it (for the most part) because of its simplicity and generality. For specific symmetric spaces, special arguments lead to sharper inequalities, generally of the form $\mathrm{rank}(df) \lesssim \frac{1}{2}\dim \mathrm{N}$. Below (Corollary 3.7) we show that for maps to quaternionic hyperbolic space, $\mathrm{rank}(df) \leqslant \frac{1}{2}\dim \mathrm{N}$. Combining this result with the rank restriction of Sampson for real hyperbolic space and the rigidity theorem of Siu for complex hyperbolic space, one sees the complete picture of rank estimates for mappings into the hyperbolic spaces over the various fields (Theorem 3.5 and Corollary 3.7). A future paper, based on somewhat different algebraic methods, will treat the cases of $\mathrm{SL}(n, \mathbf{R})/\mathrm{SO}(n)$ and $\mathrm{SO}(p, q)/\mathrm{SO}(p) \times \mathrm{SO}(q)$. In the case of SL_n, for example, one has

$$\dim \mathrm{W} \leqslant \left[\frac{n^2 + 2n}{8}\right],$$

unless $n = 3$, in which case one has $\dim \mathrm{W} \leqslant 2$.

The consequences of Sampson's theorem go beyond rank estimates. Study of the abelian subspaces for quaternionic hyperbolic space, for example, leads to the following result (Theorem 6.2). Suppose that N is covered by a quaternionic hyperbolic space. Then there is a naturally associated complex manifold $\tilde{\mathrm{N}}$ which fibers over N with projective lines as fibers and with a natural holomorphic horizontal distribution. If $f: \mathrm{M} \rightarrow \mathrm{N}$ is a harmonic map which has rank larger than two at some point, then f has a natural prolongation to a horizontal holomorphic map $\mathrm{F} : \mathrm{M} \rightarrow \tilde{\mathrm{N}}$. The universal cover of $\tilde{\mathrm{N}}$ is a special Griffiths period domain [13], and F satisfies the axioms of an abstract variation of Hodge structure. Thus a harmonic map of sufficiently high rank is the projection of a variation of Hodge structure. We believe that this is a fairly general fact about harmonic maps of sufficiently high rank of compact Kähler manifolds to non-Hermitian locally symmetric spaces, and that this constitutes the natural generalization of Siu's rigidity theorem to non-Hermitian targets. We note that the projection of a variation of Hodge structure to the canonically associated locally symmetric space is necessarily harmonic, a fact which follows easily from Theorem (2.3) below.

The connection with Hodge theory is not a fortuitous one. For both harmonic and period mappings the image $\mathrm{W} = df(\mathrm{T}^{1,0}_x)$ defines an abelian subspace (of $\mathfrak{p}^{\mathbf{C}}$ and of a suitable horizontal distribution, denoted $\mathfrak{g}^{-1,1}$, respectively). Indeed, it was this analogy at the algebraic level which led us to the rank restrictions of Corollary (3.2). In Hodge theory the non-integrability of the horizontal distribution implies that the dimension of W is strictly less than that of the horizontal distribution. By analogy, $\dim \mathrm{W} < \frac{1}{2}\dim \mathfrak{p}^{\mathbf{C}}$ in the non-Hermitian case. For similar reasons one expects $\mathrm{rank}(df) \lesssim \frac{1}{2}\dim \mathfrak{p}^{\mathbf{C}}$ for harmonic maps because of the inequality $\mathrm{rank}(df) \lesssim \frac{1}{2}\dim \mathfrak{g}^{-1,1}$ for variations of Hodge structure with values in a non-Hermitian period domain of weight two [5, 6].

A byproduct of our study is an emergent structure theory for harmonic maps of compact Kähler manifolds into compact locally symmetric spaces. We expect there to be a fairly complete classification of these in terms of special mappings, and hence, by the Eells-Sampson theorem, a structure theory for homotopy classes of mappings between such manifolds. To illustrate what we have in mind, we classify mappings into the hyperbolic spaces over the three fields. Using a theorem of Sampson [26], a straightforward application of a technique used by Jost and Yau [16], Mok [22] and Siu [31], and our results, one obtains the following. If $f: M \to N$ is harmonic and N is real hyperbolic, then either *a)* f maps to a closed geodesic or *b)* f factors as a product of a holomorphic map to a Riemann surface and a harmonic map of this surface into N. If N is complex hyperbolic, then f is either as in *a)* or *b)* above, or else *c)* f is either holomorphic or anti-holomorphic. Finally, if N is quaternionic hyperbolic, then f is as in *a)* or *b)* above or else *c)* f is the composition of a horizontal holomorphic map $F : M \to \tilde{N}$ and the projection $\pi : \tilde{N} \to N$, where $\tilde{N}$ is the complex manifold mentioned above and defined in section 6. J. Jost informs us that he and Yau have also obtained a factorization theorem similar to part *b)* above [17].

We close with two easy but remarkable applications of the theory sketched above. First, in Theorem (8.1) we show that a cocompact discrete subgroup $\Gamma \subset SO(1, n)$ cannot be the fundamental group of a compact Kähler manifold, provided that $n > 2$. We conjecture that the same holds for cocompact discrete subgroups of any semi-simple Lie group G without compact factors such that its symmetric space G/K is not Hermitian symmetric. Second, we show in Theorem (8.2) that a compact quotient of a Griffiths period domain of even weight and non-Hermitian type is not of the homotopy type of a compact Kähler manifold.

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2. Harmonic mappings and Sampson's theorem

Sampson's result, that $df(T^{1,0}_x)$ can be identified with an abelian subspace W of $\mathfrak{p}^{\mathbf{C}}$, is based on a Bochner-type formula obtained by integrating over M an iterated divergence of the symmetric (2, 0)-tensor

$$\varphi(X, Y) = \langle d'f(X), d'f(Y) \rangle.$$

We shall sketch the proof of this result, beginning with a few comments on the definition of φ. In the definition above the inner product is the complex linear extension of the Riemannian metric of N to the complexified tangent bundle $T^{\mathbf{C}} N$, and $d'f$ is the restriction of the complexification of df to the holomorphic tangent bundle $T^{1,0} M$. Since N is not assumed to have a complex structure, the map $d'f$ is not the one usual to complex manifold theory. Indeed, even if M were a complex manifold, $d'f(T^{1,0} M)$

would not in general be contained in $T^{1,0}N$, since $d'f(T^{1,0}M) \subset T^{1,0}N$ is the Cauchy-Riemann equation, asserting the holomorphicity of f.

Consider next the Euler-Lagrange equation for the energy, which asserts that df is a harmonic f^*TN-valued one-form on M, [12]. If the z^α denote local holomorphic coordinates on M and the y^i denote local C^∞ coordinates on N, then $d'f$ is locally represented by the matrix (y^i_α), where subscripts denote differentiation with respect to the coordinate with the indicated index. The Euler-Lagrange equation then takes the form

$$(*) \qquad g^{\alpha\bar\beta} y^i_{\alpha|\bar\beta} = 0.$$

Here the $y^i_{\alpha|\bar\beta}$ are the components of the tensor $D'' d'f$, defined as the (0, 1) part of the covariant differential of $d'f$ in the natural connection on $\mathrm{Hom}(T^{1,0}M, f^* T^{C}N)$, i.e., that determined by the Riemannian connections on TM and f^*TN. In local coordinates one has

$$(**) \qquad y^i_{\alpha|\bar\beta} = y^i_{\alpha\bar\beta} + \Gamma^i_{jk} y^j_\alpha y^k_{\bar\beta}.$$

Here Greek indices label all tensors on M, Latin indices label all tensors on N, and the $y^i_{\alpha\bar\beta}$ denote the ordinary mixed second partial derivatives of y^i rather than the components of a covariant derivative. It is a special feature of Kähler metrics that the covariant derivative on M does not enter explicitly in the formula for the Laplacian of a function.

Now construct a (1, 0)-form from φ by the divergence formula

$$\xi_\alpha = g^{\beta\bar\gamma} \varphi_{\alpha\beta,\bar\gamma},$$

and take the divergence a second time to obtain

$$\delta\xi = \| D'' d'f \|^2 - R_{ijkl} y^i_\beta y^j_\alpha y^k_{\bar\gamma} y^l_{\bar\mu} g^{\beta\bar\gamma} g^{\alpha\bar\mu},$$

where δ is the codifferential. This formula is remarkable in that it does not involve the curvature tensor of the domain, thus giving stronger restrictions on harmonic mappings than the original Bochner-type formula of Eells and Sampson [12]. The first relation of this kind was found by Siu [28] and was the basis of his proof of the complex analyticity of harmonic mappings.

Integrating this formula over the compact manifold M Sampson obtains the following theorem [27, Thm 1]:

Theorem **(2.1)**. — *Let* M *be a compact Kähler manifold, let* $f: M \to N$ *be a harmonic mapping, and suppose that*

$$\langle R(X, Y)\bar X, \bar Y \rangle \leqslant 0 \quad \textit{for all } X, Y \in T^{C}N.$$

Then $D'' d'f = 0$ *and for all* $x \in M$, $\langle R(X, Y)\bar X, \bar Y\rangle = 0$ *for all* $X, Y \in d_x f(T^{1,0}M)$.

Note that harmonicity, defined by the vanishing of a trace of the covariant differential $D'' d'f$ (see $(*)$), implies something stronger, namely, the vanishing of $D'' d'f$ itself. Because the Kähler metric on M does not appear explicitly in the expression $(**)$

for $D'' d'f$, harmonicity of f is independent of the metric on the source manifold. In addition, one observes that the restriction of a harmonic mapping to a complex submanifold is again harmonic. Thus, under the curvature assumption on N, harmonic mappings of a Kähler manifold into N are canonical objects.

When the target manifold is locally symmetric the vanishing of the curvature term in the preceding theorem also has strong consequences. Suppose, therefore, that N is a manifold of this kind with universal cover G/K, where G is noncompact and with associated Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ of the Lie algebra. For each point of N identify the complex tangent space to N with $\mathfrak{p}^{\mathbf{C}}$. This is unique up to the right action of K and the left action of Γ. Since these actions preserve all relevant structures, we may regard $df(T^{1,0}M)$ as a subspace of $\mathfrak{p}^{\mathbf{C}}$. Since

$$\langle R(X, Y)\,\overline{X}, \overline{Y} \rangle = - \langle [X, Y], [\overline{X}, \overline{Y}] \rangle \leqslant 0 \quad \text{for all } X, Y \text{ in } \mathfrak{p}^{\mathbf{C}},$$

the hypothesis of the theorem is satisfied and we obtain the main theorem that we need [27, Thm 3]:

Theorem **(2.2)**. — *If* M *is a compact Kähler manifold,* N *is a locally symmetric space of noncompact type, and* $f: M \to N$ *is harmonic, then* $D'' d'f = 0$ *and for each point of* M, df *maps* $T^{1,0}M$ *onto an abelian subspace* W *of* $\mathfrak{p}^{\mathbf{C}}$.

Theorem (2.2) has the following equivalent formulation, one that will be useful to us in section 7. Let ∇ denote the connection on $f^* T^{\mathbf{C}} N$ obtained by complexifying the pull-back of the Levi-Civita connection on $T^{\mathbf{C}} N$. The operator

$$\nabla : \Gamma(f^* T^{\mathbf{C}} N) \to \Gamma(T^* M \otimes f^* T^{\mathbf{C}} N),$$

where Γ denotes C^∞ sections, decomposes as $\nabla = \nabla' + \nabla''$, where

$$\nabla' : \Gamma(f^* T^{\mathbf{C}} N) \to \Gamma(T^{1,0} M \otimes f^* T^{\mathbf{C}} N),$$

$$\nabla'' : \Gamma(f^* T^{\mathbf{C}} N) \to \Gamma(T^{0,1} M \otimes f^* T^{\mathbf{C}} N).$$

Theorem **(2.3)**. — *Let* M *and* N *be as in Theorem* (2.2), *and let* $f: M \to N$ *be a smooth map. Then* f *is harmonic if and only if* ∇'' *is the* $\bar\partial$*-operator of a holomorphic structure on* $f^* T^{\mathbf{C}} N$ *and* $d'f$ *is a holomorphic section of the bundle* $\mathrm{Hom}(T^{1,0} M, f^* T^{\mathbf{C}} N)$.

Proof. — It is clear that ∇'' is the $\bar\partial$-operator of holomorphic structure on $f^* T^{\mathbf{C}} N$ if and only if it satisfies the integrability condition $(\nabla'')^2 = 0$, [9, Prop. 19.1]. But

$$(\nabla'')^2 (X, Y) = R(df(X), df(Y)),$$

where R is the complex-multilinear extension of the curvature tensor of N and $X, Y \in T^{0,1} M$. Since $R(X, Y) Z = - [[X, Y], Z]$, we see that $(\nabla'')^2 = 0$ is equivalent to

$$[df(X), df(Y)] = 0 \quad \text{for all } X, Y \in T^{0,1} M.$$

Since df and the Lie bracket are real operators, this is in turn equivalent to

$$[df(X), df(Y)] = 0 \quad \text{for all } X, Y \in T^{1,0} M.$$

If f is harmonic the last assertion holds by Theorem (2.2), so that $f^* T^{\mathbf{C}} N$ is a holomorphic bundle as asserted. The operator D'' appearing in the explicit form of the harmonic equation above as the (0, 1) part of the canonical connection on $\mathrm{Hom}(T^{1,0} M, f^* T^{\mathbf{C}} N)$ becomes the $\bar\partial$-operator on this holomorphic bundle. Therefore the equation $D'' d'f = 0$ is equivalent to the condition that $d'f$ be a holomorphic section of this bundle, establishing one of the implications in Theorem (2.3). The converse implication is clear from the above interpretation of D''.

3. The main theorems

Henceforth M will denote a compact Kähler manifold, N a locally symmetric space of noncompact type, and W the image under $d_x f$ of $T^{1,0}_x M$ for some $x \in M$, where $f: M \to N$ is a fixed harmonic mapping. The image under $d_x f$ of the real tangent space $T_x M$ is the subspace of real points of the space $W + \overline{W}$, so that $\dim_{\mathbf{R}} d_x f(T_x M) = \dim_{\mathbf{C}}(W + \overline{W}) \leqslant 2 \dim_{\mathbf{C}} W$. (In what follows we shall generally omit the subscript to *dim*: dim V shall mean $\dim_{\mathbf{R}} V$ for real vector spaces and $\dim_{\mathbf{C}} V$ for complex vector spaces. The same convention applies to the rank.) Combining the preceding inequality with Theorem (2.2), we obtain the following estimate:

$$\mathrm{rank}(df) \leqslant 2 \max\{ \dim W \mid W \subset \mathfrak{p}^{\mathbf{C}}, [W, W] = 0 \}.$$

Abelian subspaces of $\mathfrak{p}^{\mathbf{C}}$ which consist of semisimple elements lie in the noncompact part of a Cartan subalgebra, and so are bounded in dimension by the split rank, i.e., by the rank of G/K. At the opposite extreme are abelian subspaces which consist entirely of nilpotent elements. For an example, consider a Hermitian symmetric space G/K with invariant complex structure $J : \mathfrak{p} \to \mathfrak{p}$. (The number of such structures is 2^n, where n is the number of irreducible components of G/K.) Let $\mathfrak{p}^{1,0} = \mathfrak{p}_+$ denote the $+ i$-eigenspace of J, given explicitly by $\{ X - iJX \mid X \in \mathfrak{p} \}$. Then $\mathfrak{p}^{1,0}$ is an abelian subspace of $\mathfrak{p}^{\mathbf{C}}$, since the relation $[\mathfrak{p}^{1,0}, \mathfrak{p}^{1,0}] = 0$ is the integrability condition for the almost complex structure J. The dimension of this abelian space is one-half that of $\mathfrak{p}^{\mathbf{C}}$. Our main theorem asserts that the spaces $\mathfrak{p}^{1,0}$ are the largest possible:

Theorem **(3.1)**. — *Let* G/K *be a symmetric space of non-compact type, let* $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ *be the Cartan decomposition, and let* $W \subset \mathfrak{p}^{\mathbf{C}}$ *be an abelian subspace*: $[W, W] = 0$. *Then* $\dim W \leqslant \frac{1}{2} \dim \mathfrak{p}^{\mathbf{C}}$. *Moreover, in the case of equality the pair* $(\mathfrak{g}, \mathfrak{k})$ *must be Hermitian symmetric and the following holds*: *Let* $\mathfrak{g}_i$, $\mathfrak{k}_i$, $\mathfrak{p}_i$ *be the irreducible components of the pair* $(\mathfrak{g}, \mathfrak{k})$ *and let* $W_i = W \cap \mathfrak{p}_i$. *Then* $W = \bigoplus W_i$ *and for each i such that* $\mathfrak{g}_i$ *is not isomorphic to* $sl(2, \mathbf{R})$, $W_i = \mathfrak{p}_i^{1,0}$ *for one of the two invariant complex structures on* G_i/K_i.

As immediate consequences of this theorem one has the two results below:

Corollary **(3.2).** — *Let* $f: M \to N$ *be as above, and suppose that* N *is not locally Hermitian symmetric. Then for every* $x \in M$, *the rank of* $d_x f$ *is strictly smaller than the dimension of* N.

Proof. — Use Theorem (3.1) and the rank estimate above.

Corollary **(3.3).** — *Let* M *and* N *be as in Corollary* (3.2) *and suppose that* N *is compact. Let* $\varphi : M \to N$ *be a continuous mapping. Then* φ *is homotopic to a continuous mapping* $\psi : M \to N$ *whose image lies in a proper subskeleton of some cell subdivision of* N. *In particular,* φ *is not surjective in homology.*

Proof. — Since M and N are compact and N has non-positive sectional curvature, the existence theorem of Eells and Sampson [12] implies that φ is homotopic to a harmonic mapping $f: M \to N$. By Corollary (3.2) f is not surjective, hence by standard topology, can be deformed to a map ψ whose image lies in a proper subskeleton of some cell subdivision of N.

The above corollaries use only the case of strict inequality in Theorem (3.1). We have treated the case of equality in such detail in order to obtain the following part of Siu's rigidity theorem [28, 29, 30], which we now state:

Corollary **(3.4).** — *Let* $f: M \to N$ *be as above. Suppose that* N *is locally Hermitian symmetric, that its universal cover does not contain the hyperbolic plane as a factor, and that for some* $x \in M$ *the rank of* $d_x f$ *equals the dimension of* N. *Then* f *is holomorphic with respect to an invariant complex structure on* N.

Proof. — Since $df(T^{1,0} M)$ is an abelian subspace of half the dimension, it must be $\mathfrak{p}^{1,0}$ for an invariant complex structure on N. Thus $d_x f$ maps $T^{1,0}_x M$ into $\mathfrak{p}^{1,0}$, so that $d_x f$ is complex linear. By Theorem (2.3) $d'f$ is a holomorphic section of $\mathrm{Hom}(T^{1,0} M, f^* T^{\mathbf{C}} N)$, so that the rank of df must equal the dimension of N in the complement of some proper analytic subvariety of M. Consequently df is complex linear on this dense open set, hence is complex linear everywhere, i.e., f is holomorphic.

For each symmetric space G/K let $\alpha(G/K)$ denote the maximum complex dimension of an abelian subspace of $\mathfrak{p}^{\mathbf{C}}$. Theorem (3.1) states that if G/K is not Hermitian symmetric then $\alpha(G/K) < \frac{1}{2}\dim(G/K)$. For the hyperbolic spaces over the various fields, the following theorem gives more precise information on α:

Theorem **(3.5).** — *Let* $H^n_{\mathbf{K}}$ *denote the hyperbolic space of* **K**-*dimension* n *over the field* **K**, *where* $\mathbf{K} = \mathbf{R}$ *or* **C** *or the quaternions* **H**. *Let* W *be an abelian subspace of* $\mathfrak{p}^{\mathbf{C}}$ *where* $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ *is the Cartan decomposition of the group of isometries of* $H^n_{\mathbf{K}}$. *Then*:

a) *If* $\mathbf{K} = \mathbf{R}$, $\dim(W) \leqslant 1$.

b) *If* $\mathbf{K} = \mathbf{C}$, *and* $\dim(W) > 1$, *then* $W \subset \mathfrak{p}^{1,0}$ *for one of the two invariant complex structures on* $H^n_{\mathbf{C}}$.

c) *If* $\mathbf{K} = \mathbf{H}$, *and* $\dim(W) > 1$, *then there is a totally geodesic embedding of* $H^k_{\mathbf{C}}$ *in* $H^n_{\mathbf{H}}$ $(k \leqslant n)$ *such that* W *is the* $(1, 0)$-*space of the image of this embedding.*

Corollary **(3.6).** — *Let* $\alpha(G/K)$ *be as defined above. Then* $\alpha(H^n_{\mathbf{R}}) = 1$, $\alpha(H^n_{\mathbf{C}}) = n$, *and* $\alpha(H^n_{\mathbf{H}}) = n$.

The cases $K = \mathbf{R}$ or $\mathbf{C}$ of Theorem (3.5) were already known. For $\mathbf{R}$ this is a theorem of Sampson [27], and for $\mathbf{C}$ this is a reformulation of Siu's rigidity theorem for quotients of the unit ball [28]. The same remark applies to the following assertion.

Corollary **(3.7).** — *Let* M *be a compact Kähler manifold and* $f: M \to N$ *be a harmonic mapping, where* N *is a quotient of* $H^n_{\mathbf{K}}$. *Then:*

a) *If* $\mathbf{K} = \mathbf{R}$, f *has rank at most two.*
b) *If* $\mathbf{K} = \mathbf{C}$ *and the rank of* f *exceeds two at some point* $x \in M$, *then* f *is holomorphic with respect to one of the two invariant complex structures on* N.
c) *If* $\mathbf{K} = \mathbf{H}$, f *has rank at most* $2n = \frac{1}{2}\dim N$.

Proof. — The cases $\mathbf{K} = \mathbf{R}$ or $\mathbf{H}$ are an immediate application of the rank estimate above. The case $\mathbf{K} = \mathbf{C}$ is derived from Theorem (3.5) just as Corollary (3.4) was derived from Theorem (3.1).

Observe that the real dimension of $H^n_{\mathbf{H}}$ is $4n$, while $\alpha(H^n_{\mathbf{H}}) = n$. In this case $\alpha(G/K)$ is about one-quarter of the real dimension of G/K, so that the rank of a harmonic mapping is at most about one-half that value. As stated in the introduction, we believe that this is the typical situation. Observe also that for the hyperbolic spaces the complex abelian subspaces are totally classified by Theorem (3.5): they are either one-dimensional or subspaces of the $\mathfrak{p}^{1,0}$ space of a totally geodesic Hermitian symmetric subspace. For symmetric spaces of higher rank this will no longer be the case, and many examples can be deduced from the examples in Hodge theory presented in [5, 6]. We believe, however, that in many cases the abelian subspaces of sufficiently high dimension are contained in the complexification of the tangent space of a totally geodesic Hermitian subspace.

4. Proof of the main inequality

The proof of the main inequality given by Theorem (3.1) comes in two parts. First, we show that if W contains a semisimple element then strict inequality holds: $\dim W < \frac{1}{2}\dim \mathfrak{p}^{\mathbf{C}}$. Second, we show that if W contains no semisimple elements then it must be isotropic for the Killing form, so that $\dim W \leqslant \frac{1}{2}\dim \mathfrak{p}^{\mathbf{C}}$. If equality holds, then in addition $\mathfrak{p}^{\mathbf{C}} = W \oplus \overline{W}$, and in this case we show (using the isotropy condition) that $W = \mathfrak{p}^{1,0}$ for an invariant complex structure on G/K. We turn now to the first part of the argument:

Proposition **(4.1)**. — *Let* $\mathfrak{g}$ *be a real semisimple Lie algebra with no factors isomorphic to* $\mathfrak{sl}_2$, *let* $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ *be a Cartan decomposition, and let* W *be an abelian subspace of* $\mathfrak{p}^{\mathbf{C}}$ *with a nonzero semisimple element. Then*

$$\dim W < \frac{1}{2} \dim \mathfrak{p}^{\mathbf{C}}.$$

To begin the proof, consider a maximal abelian subspace W of $\mathfrak{p}^{\mathbf{C}}$, let X be an element of W, and let $X = X_s + X_n$ be its Jordan decomposition. By [19, Prop. 3], the semisimple and nilpotent parts of an element of $\mathfrak{p}^{\mathbf{C}}$ again lie in $\mathfrak{p}^{\mathbf{C}}$. Since X_s and X_n are polynomials in X, they commute with any element which commutes with X. This establishes the first part of

Lemma **(4.2)**. — *Let* W *be a maximal abelian subspace of* $\mathfrak{p}^{\mathbf{C}}$. *Then*

$$W = W_s \oplus W_n,$$

where the two summands consist entirely of semisimple and nilpotent elements, respectively. Moreover, any such decomposition is $K^{\mathbf{C}}$*-conjugate to one which is defined over* **R**.

To complete the proof of the preceding lemma, consider a subspace $\mathfrak{a}'$ of $\mathfrak{p}^{\mathbf{C}}$ containing W_s which is abelian, consists entirely of semisimple elements, and is maximal with respect to these last two properties. By [19, Theorem 1] $\mathfrak{a}'$ is $K^{\mathbf{C}}$-conjugate to a space defined over **R**. Replacing $\mathfrak{a}'$ and W by suitable conjugates, we may assume that $\mathfrak{a}'$ is defined over **R**, hence is of the form $\mathfrak{a}' = \mathfrak{a}^{\mathbf{C}}$ with $\mathfrak{a} \subset \mathfrak{p}$ abelian. Then $\mathfrak{g}$ decomposes under the adjoint action of $\mathfrak{a}$ into restricted root spaces: there is a finite set Δ of linear forms α on $\mathfrak{a}$ (the restricted roots [15, p. 285]), and vectors $X_\alpha \in \mathfrak{p}$, $Y_\alpha \in \mathfrak{k}$ so that for all $H \in \mathfrak{a}$,

$$[H, X_\alpha] = \alpha(H)\, Y_\alpha,$$
$$[H, Y_\alpha] = \alpha(H)\, X_\alpha.$$

Let $\mathfrak{p}_\alpha = \mathbf{R}X_\alpha$, $\mathfrak{k}_\alpha = \mathbf{R}Y_\alpha$ and let $\mathfrak{g}_\alpha = \mathfrak{k}_\alpha \oplus \mathfrak{p}_\alpha$. Then there is a direct sum decomposition

(4.3) $$\mathfrak{g} = \mathfrak{g}_0 \oplus \sum_{\alpha \in \Delta^+} \mathfrak{g}_\alpha,$$

where $\mathfrak{g}_0$ denotes the centralizer of $\mathfrak{a}$ in $\mathfrak{g}$, and where the sum is interpreted to run over positive roots with appropriate multiplicity. We have similar direct sum decompositions

(4.4) $$\mathfrak{p} = \mathfrak{a} \oplus \sum_{\alpha \in \Delta^+} \mathfrak{p}_\alpha, \qquad \mathfrak{k} = \mathfrak{k}_0 \oplus \sum_{\alpha \in \Delta^+} \mathfrak{k}_\alpha,$$

where $\mathfrak{k}_0$ denotes the centralizer of $\mathfrak{a}$ in $\mathfrak{k}$. Define a subset of Δ by

$$\Delta(W) = \{\, \alpha \in \Delta \mid \alpha(W_s) = 0 \,\}$$

and a subspace of $\mathfrak{a}^{\mathbf{C}}$ by

$$W'_s = \bigcap_{\alpha \in \Delta(W)^+} \ker(\alpha),$$

where clearly $W_s \subset W'_s$. Since the roots α are defined over $\mathbf{R}$, so is W'_s. Consider now an element X of W, which we may decompose as

$$X = X_0 + \sum_{\alpha \in \Delta(W)^+} c_\alpha X_\alpha,$$

with $X_0 \in \mathfrak{g}_0$ and $X_\alpha \in \mathfrak{g}_\alpha$. If Y is an element of W'_s, then

$$[Y, X] = \sum_{\alpha \in \Delta(W)^+} c_\alpha \alpha(Y) X_\alpha = 0,$$

so that Y commutes with W. By the assumed maximality, Y lies in W, so that $W'_s \subset W_s$. Therefore $W_s = W'_s$, and so W_s is defined over $\mathbf{R}$, as required.

By the preceding lemma we may assume that W is normalized so that W_s is defined over $\mathbf{R}$. To complete the proof of proposition (4.1), consider the centralizer $\mathfrak{l}$ of $(W_s)_{\mathbf{R}}$ in $\mathfrak{g}$, a reductive Lie algebra which splits as

$$\text{(4.5)} \qquad \mathfrak{l} = \mathfrak{z} \oplus \mathfrak{l}'.$$

Here $\mathfrak{z}$ is the center of $\mathfrak{l}$ and $\mathfrak{l}' = [\mathfrak{l}, \mathfrak{l}]$ is the semisimple part. By construction, $W_s \subset \mathfrak{z}^{\mathbf{C}}$. Consider now an element X of W_n, which we may decompose as $X = X_1 + X_2$, with $X_1 \in \mathfrak{z}^{\mathbf{C}}$ and $X_2 \in \mathfrak{l}'^{\mathbf{C}}$. Let ad denote the adjoint representation of $\mathfrak{g}$. Then ad X is nilpotent, so that there is an n such that $(\operatorname{ad} X)^n = 0$. Therefore $(\operatorname{ad} X_1)^n = 0$ which, combined with the semisimplicity of X_1, yields $X_1 = 0$. Therefore $W_n \subset \mathfrak{l}'$, and so the decompositions (4.3) and (4.5) are compatible.

Next, observe that the given Cartan involution acts on $\mathfrak{l}$, $\mathfrak{z}$, and $\mathfrak{l}'$, so that $\mathfrak{l}' = \mathfrak{k}' \oplus \mathfrak{p}'$, where $\mathfrak{k}' = \mathfrak{k}' \cap \mathfrak{l}'$ and $\mathfrak{p}' = \mathfrak{p} \cap \mathfrak{l}'$. In particular, $W_n \subset \mathfrak{p}'^{\mathbf{C}}$. If W_n had nonzero real points, then these, as elements of $\mathfrak{p}$, would be semisimple, a contradiction. Therefore $W_n \oplus \overline{W}_n \subset \mathfrak{p}'^{\mathbf{C}}$, and so

$$\text{(4.6)} \qquad \dim W_n \leqslant \frac{1}{2} \dim \mathfrak{p}'^{\mathbf{C}}.$$

Because of this last inequality, the main step remaining in the proof of (4.1) is a bound on the dimension of W_s in terms of the codimension of $\mathfrak{p}'^{\mathbf{C}}$. Such a bound is given by the following result, whose proof (as well as the argument in the next section) was motivated by the study of centralizers in symmetric spaces by Bott and Samelson [4]:

Lemma (4.7). — *Let* $\mathfrak{g}$ *be a semisimple Lie algebra as above, and let* $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ *a Cartan decomposition. Then*

$$\dim W_s \leqslant \frac{1}{2} \operatorname{codim} \mathfrak{p}'^{\mathbf{C}},$$

with strict inequality unless $\mathfrak{g}$ *contains* $\mathfrak{sl}_2$ *as a factor.*

Here the codimension refers to the codimension in $\mathfrak{p}^{\mathbf{C}}$. Given the Lemma, we find that

$$\begin{aligned}\dim W &= \dim W_s + \dim W_n \\ &\leqslant \frac{1}{2}\operatorname{codim} \mathfrak{p}'^{\mathbf{C}} + \frac{1}{2}\dim \mathfrak{p}'^{\mathbf{C}} \\ &= \frac{1}{2}\dim \mathfrak{p}^{\mathbf{C}},\end{aligned}$$

which yields Proposition (4.1), as required.

For the proof of Lemma (4.7), let $m = \dim W_s$. Since the adjoint representation is faithful there are m positive roots $\alpha_1, \ldots, \alpha_m$ which are linearly independent on W_s. Since these m roots do not vanish on W_s, we see from the description (4.3) of $\mathfrak{p}$ that the m spaces $\mathfrak{p}_{\alpha_i}$ are complementary to $\mathfrak{p}'$ in $\mathfrak{p}$. Since the m-dimensional space $W_s \subset \mathfrak{z}^{\mathbf{C}}$ is also complementary to $\mathfrak{p}'^{\mathbf{C}}$ in $\mathfrak{p}^{\mathbf{C}}$, $\operatorname{codim} \mathfrak{p}'^{\mathbf{C}} \geqslant 2 \dim W_s$, as required. Suppose now that equality holds. Then the α_i are the only positive roots which do not vanish on W_s. Therefore if α is any positive root, $\alpha_i + \alpha$ is not a root. Since the Jacobi identity gives

$$[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta} \oplus \mathfrak{g}_{\pm(\alpha-\beta)},$$

where the sign in the second summand is determined so that it gives a positive root when non-zero, the $\mathfrak{g}_{\alpha_i}$ must commute with each other and with $\mathfrak{l}'$. From this it follows easily that each $\mathfrak{s}_i = [\mathfrak{g}_{\alpha_i}, \mathfrak{g}_{\alpha_i}] \oplus \mathfrak{g}_{\alpha_i}$ is a Lie algebra isomorphic to $\mathfrak{sl}_2$ and that

$$\mathfrak{g} = \sum_{i=1}^{m} \mathfrak{s}_i \oplus \mathfrak{l}'.$$

This relation establishes the Proposition and somewhat more: if equality holds in Lemma (4.7), the number of factors of type $\mathfrak{sl}_2$ equals the dimension of W_s, and all the semisimple elements lie in the direct sum of the $\mathfrak{sl}_2$ factors.

We turn now to the second part of the proof, namely the case in which W is maximal abelian and has no semisimple elements. Spaces of this kind consist entirely of nilpotent elements, and so are isotropic for the Killing form. Since the Killing form is nondegenerate on $\mathfrak{p}^{\mathbf{C}}$, this forces $\dim W \leqslant \frac{1}{2}\dim \mathfrak{p}^{\mathbf{C}}$. In addition W has no real points, since an element $X \in W \cap \overline{W}$ would be semisimple, contrary to hypothesis. Thus, if equality holds, $\mathfrak{p}^{\mathbf{C}} = W \oplus \overline{W}$. It remains to show that $W = \mathfrak{p}^{1,0}$ for an invariant complex structure on G/K. This is immediate from the lemma below:

Lemma **(4.8)**. — *Let* $\mathfrak{p}^{\mathbf{C}} = W \oplus \overline{W}$, *where* W *is abelian and isotropic for the Killing form. Then* $W = \mathfrak{p}^{1,0}$ *for an invariant complex structure on* G/K.

Proof. — Because the Killing form is invariant and W is abelian,

$$\langle [[W, \overline{W}], W], W \rangle = \langle [W, \overline{W}], [W, W] \rangle = 0,$$

so that $[[W, \overline{W}], W] \perp W$. Because W is a maximal isotropic subspace, $W^\perp = W$, so that $[[W, \overline{W}], W] \subset W$. But $[W, \overline{W}] = \mathfrak{k}^{\mathbf{C}}$, so that $[\mathfrak{k}, W] \subset W$. By [15, p. 373], this implies that $W = \mathfrak{p}^{1,0}$ for an invariant complex structure on G/K, as required.

5. Hyperbolic manifolds

We turn now to the proof of Theorem (3.5), which gives detailed information on the nature of harmonic maps with values in a hyperbolic space (real, complex, or quaternionic). These spaces, together with quotients of the Cayley plane, exhaust the class of rank one locally symmetric spaces of non-compact type. We begin with the lemma below, which asserts that the $K^{\mathbf{C}}$-orbit of an element of $X \in W$ tends to be small if the abelian subspace $W \subset \mathfrak{p}^{\mathbf{C}}$ is large. By orbit we mean that defined by the adjoint action, so that $[\mathfrak{k}^{\mathbf{C}}, X]$ is its tangent space at X. Codimension refers to codimension in $\mathfrak{p}^{\mathbf{C}}$, the ambient space of the orbit:

Lemma **(5.1).** — *Let* G/K *be a symmetric space, and let* W *be an abelian subspace of* $\mathfrak{p}^{\mathbf{C}}$. *Then*

$$\operatorname{codim}[\mathfrak{k}^{\mathbf{C}}, X] \geqslant \dim W$$

for all X *in* W.

Proof. — Let $\mathfrak{p}_X$ and $\mathfrak{p}_W$ denote the centralizers of X and W, respectively, in $\mathfrak{p}^{\mathbf{C}}$, and note that $\mathfrak{p}_W \subset \mathfrak{p}_X$ if $X \in W$. Take orthogonal complements in $\mathfrak{p}^{\mathbf{C}}$ relative to the Killing form to obtain

$$\mathfrak{p}_X^{\perp} \subset \mathfrak{p}_W^{\perp}. \tag{5.2}$$

Apply the invariance relation $\langle [X, Y], Z \rangle = \langle Y, [Z, X] \rangle$ for arbitrary $Y \in \mathfrak{p}^{\mathbf{C}}$, $Z \in \mathfrak{k}^{\mathbf{C}}$, to get $\langle [X, Y], \mathfrak{k}^{\mathbf{C}} \rangle = \langle Y, [\mathfrak{k}^{\mathbf{C}}, X] \rangle$, which yields

$$[\mathfrak{k}^{\mathbf{C}}, X] = \mathfrak{p}_X^{\perp}. \tag{5.3}$$

Combining this with the preceding relation, we obtain $[\mathfrak{k}^{\mathbf{C}}, X] \subset \mathfrak{p}_W^{\perp}$, which yields the required assertion.

To gain information on the size of the orbits, we show first that when W is " large " it consists entirely of isotropic vectors:

Proposition **(5.4).** — *Let* G/K *be a rank one symmetric space, and let* W *be an abelian subspace of* $\mathfrak{p}^{\mathbf{C}}$. *If* $\dim W > 1$, *then* W *consists entirely of nilpotent elements, and so is isotropic for the Killing form.*

Proof. — Let W be an abelian subspace of $\mathfrak{p}^{\mathbf{C}}$. We must show that if W has nonzero semisimple part, then it has dimension one. Without loss of generality we may assume that W is maximal abelian in $\mathfrak{p}^{\mathbf{C}}$, so that by Lemma (4.2) there is a decomposition $W = W_s \oplus W_n$, with $W_s \neq 0$. Moreover, conjugating W by an element of $K^{\mathbf{C}}$, we may assume that W_s is defined over $\mathbf{R}$. Let X be a real element of W_s, and let $\mathfrak{p}_X$ denote its centralizer in $\mathfrak{p}$. Referring to the decomposition (4.4), we find that

$$\mathfrak{p}_X = \mathfrak{a} \oplus \sum_{\alpha(X)=0} \mathfrak{p}_\alpha.$$

Because G/K is of rank one, $\mathfrak{a}$ is the line generated by X, so that a restricted root vanishing on X is zero, which is absurd. Therefore $\mathfrak{p}_X = \mathfrak{a}$ is of dimension one. But $W \subset \mathfrak{p}_X^{\mathbf{C}}$, so W also has dimension one, as required.

The foregoing is already enough to prove the part of Theorem (3.5) which concerns real hyperbolic space. Suppose that $\dim W > 1$, and let X be any nonzero element of W. Identifying $\mathfrak{p}$ with $\mathbf{R}^n$, we identify the $K^{\mathbf{C}}$ action on $\mathfrak{p}^{\mathbf{C}}$ with that of $SO(n, \mathbf{C})$ on $\mathbf{C}^n$. By the proposition just proved, X is isotropic, and so lies on the homogeneous quadric

$$Q = \{ X \in \mathbf{C}^n \mid \langle X, X \rangle = 0 \}.$$

This set decomposes under the action of $K^{\mathbf{C}}$ into two orbits, the origin and the complement of the origin. Since X lies on the unique open orbit, $\operatorname{codim}[\mathfrak{k}^{\mathbf{C}}, X] = 1$. But Lemma (5.1), together with the hypothesis $\dim W > 1$ implies that $\operatorname{codim}[\mathfrak{k}^{\mathbf{C}}, X] > 1$, a contradiction. One can also give a completely elementary proof of the assertion $\dim W \leqslant 1$ for real hyperbolic space by computing the commutator [A, B] for two elements of $\mathfrak{p}^{\mathbf{C}}$: identifying these matrices with row vectors, the condition $[A, B] = 0$ becomes $A \wedge B = 0$. This is the proof of [27].

A more detailed analysis of the orbit structure of Q gives a complete picture of the abelian subspaces for the two remaining hyperbolic spaces. To give a uniform treatment, let $\mathbf{K}$ denote $\mathbf{R}$, $\mathbf{C}$, or $\mathbf{H}$, where the latter symbol stands for the quaternions, and note that $\mathfrak{p}$ can be identified with the right $\mathbf{K}$-vector space $\mathbf{K}^n$. Under this identification the Killing form corresponds to the usual real inner product, which we shall still be denote by $\langle\ ,\ \rangle$. Thus, if $X = (x_1, \ldots, x_n)$ and $Y = (y_1, \ldots, y_n)$, and if $X.Y = x_1.y_1 + \ldots + x_n.y_n$ is the dot product, then $\langle X, Y \rangle = \operatorname{Re}(\overline{X}.Y)$, where the conjugation is that defined by the conjugation of $\mathbf{C}$ or $\mathbf{H}$. The action of the isotropy group K corresponds to the usual action of $SO(n)$ if $\mathbf{K} = \mathbf{R}$, to that of $U(n)$ if $\mathbf{K} = \mathbf{C}$ and to an action of $Sp(n) \times Sp(1)$ if $\mathbf{K} = \mathbf{H}$, namely, that for which the pair (A, a) applied to $X \in \mathbf{H}^n$ yields $AX\bar{a}$. We say that $X, Y \in \mathfrak{p}$ are *$\mathbf{K}$-independent* if, under the above identification of $\mathfrak{p}$ with $\mathbf{K}^n$, X and Y are right linearly independent. In the complexified space $\mathfrak{p}^{\mathbf{C}} = (\mathbf{K}^n)^{\mathbf{C}}$ we shall always write the action of the complex scalars on the left and the induced action of the $\mathbf{K}$-scalars on the right. We then have the following:

Lemma **(5.5).** — *If* $X, Y \in \mathfrak{p}$ *are* $\mathbf{K}$*-independent and* $Z = X + iY$ *is isotropic, then* $\operatorname{codim}[\mathfrak{k}^{\mathbf{C}}, Z] = 1$.

Proof. — Let T denote the tangent space to Q at the point $Z = X + iY$ under consideration, and let T′ denote the tangent space at Z of the K-orbit of Z. Since T carries a complex structure, the smallest complex subspace $T'' \subset T$ containing the real subspace T′ is defined. Since the tangent space of the $K^{\mathbf{C}}$-orbit of Z is a complex subspace of T containing T′, it contains T″. We must therefore show that $T'' = T$. To this end we shall find the equations (1 — 4 below) satisfied by a general element $U + iV \in T'$, where $U, V \in \mathfrak{p}$. There are three cases *a)*, *b)* and *c)*, according to which of the equations (4) is applicable :

1) $\langle X, U \rangle - \langle Y, V \rangle = 0$,
2) $\langle X, V \rangle + \langle Y, U \rangle = 0$,
3) $\langle X, U \rangle + \langle Y, V \rangle = 0$,
4*a)* if $\mathbf{K} = \mathbf{C}$ then $\mathrm{Im}(\overline{U}.Y + \overline{X}.V) = 0$,
4*b)* if $\mathbf{K} = \mathbf{H}$ and $\overline{X}.Y \neq 0$ then $\overline{X}.Y \perp \mathrm{Im}(\overline{U}.Y + \overline{X}.V)$,
4*c)* if $\mathbf{K} = \mathbf{H}$ and $\overline{X}.Y = 0$ then $\mathrm{Im}(\overline{U}.Y + \overline{X}.V) = 0$.

To prove these relations, let $Z(t) = X(t) + iY(t)$ be a curve in the K-orbit of Z with $Z(0) = Z$ and $Z'(0) = U + iV$. The first two equations come from differentiating the relation $\langle Z(t), Z(t) \rangle = 0$, while the third comes from differentiating $\langle Z(t), \overline{Z}(t) \rangle =$ *constant*. To establish 4*a)*, let $Z(t) = A(t)\, Z(0)$, where $A(t)$ is a curve in $U(n)$ with $A(0) = 1$. The result follows from differentiation of the relation

$$\overline{X}(t).Y(t) = \overline{A(t)\, X}.A(t)\, Y = \overline{X}.Y = \textit{constant}.$$

For the remaining two assertions, let $Z(t) = A(t)\, Z\bar{a}(t)$ where $A(t)$ is a curve in $Sp(n)$ with $A(0) = 1$ and where $a(t)$ is a curve in the unit quaternions with $a(0) = 1$. Differentiate the relation

$$\overline{X}(t).Y(t) = \overline{A(t)\, X\bar{a}(t)}.A(t)\, Y\bar{a}(t) = a(t)\, \overline{X}.Y\bar{a}(t)$$

to obtain

$$\overline{U}.Y + \overline{X}.V = a'(0)\, \overline{X}.Y - \overline{X}.Ya'(0).$$

Because $Z(t)$ is isotropic, $\overline{X}(t).Y(t)$ is pure imaginary. Therefore the left-hand side (and so also the right-hand side) of the preceding relation is pure imaginary. Because $a(t)$ has unit length, $a'(0)$ is also pure imaginary, one obtains that the preceding relation can be rewritten as

$$\overline{U}.Y + \overline{X}.V = 2a'(0)\, (\overline{X}.Y).$$

Identifying purely imaginary quaternions with vectors in $\mathbf{R}^3$ and quaternion multiplication with the cross product, this becomes

$$\overline{U}.Y + \overline{X}.V = 2a'(0) \times (\overline{X}.Y).$$

The relations 4*b)* and 4*c)* are now clear (the " Im " is in fact superfluous).

It is easy to check that these equations actually define T', since in each case the solution space $\mathscr{E}$ has the same dimension as that of the K-orbit:

a) $\dim \mathscr{E} = 4n - 4 = \dim K.Z = \dim U(n)/U(n-2)$.
b) $\dim \mathscr{E} = 8n - 4 = \dim K.Z = \dim Sp(n) \times Sp(1)/Sp(n-2) \times S^1$.
c) $\dim \mathscr{E} = 8n - 6 = \dim K.Z = \dim Sp(n) \times Sp(1)/Sp(n-2) \times Sp(1)$.

We can now prove the equality $T = T''$. To this end let S be the subspace of T spanned by T' and $X + iY$. Since $X + iY$ is isotropic, $Y - iX$ satisfies all the defining equations of T′. Therefore $i(\mathrm{Y} - i\mathrm{X}) = \mathrm{X} + i\mathrm{Y} \in \mathrm{T}''$, and so $\mathrm{S} \subset \mathrm{T}''$. In cases *a)* and *b)* S has real codimension one in T, so that $\mathrm{T}'' = \mathrm{T}$. In case *c)* S has real codimension 3 in T. However, the three-dimensional space of vectors $\mathrm{X}q + i\mathrm{Y}q$, where q is a purely imaginary quaternion, satisfy all the equations of T′, so that the corresponding vectors $i(\mathrm{X}q + i\mathrm{Y}q)$ span a complement to S in T. Therefore $\mathrm{T}'' = \mathrm{T}$ in this case also. This concludes the proof of the lemma.

We can now conclude the proof of Theorem (3.5), beginning with the case of $\mathbf{K} = \mathbf{C}$. Let W be an abelian subspace of dimension greater than one. By Lemma (5.1), $\operatorname{codim}[\mathfrak{k}^{\mathbf{C}}, \mathrm{Z}] > 1$ for any nonzero element of Z of W. Write $\mathrm{Z} = \mathrm{X} + i\mathrm{Y}$ with $\mathrm{X}, \mathrm{Y} \in \mathfrak{p}$ and apply Lemma (5.5) to conclude that X and Y are **C**-dependent, i.e.,

$$(*) \qquad \mathrm{Y} = a\mathrm{X} + b\mathrm{J}\mathrm{X}$$

for some real a and b. By Proposition (5.4), Z is isotropic for the Killing form, or, equivalently, X and Y are perpendicular vectors of equal length. From these last two facts we conclude that in (∗) the coefficients satisfy $a = 0$ and $b = \pm 1$. Since b is a continuous function of $\mathrm{Z} \in \mathrm{W}$, just one of the signs appears. Therefore W is in either the $+i$ or the $-i$ eigenspace of J, as required by the theorem: W lies in $\mathfrak{p}^{1,0}$ for one of the two invariant complex structures.

For the case $\mathbf{K} = \mathbf{H}$ we must consider an abelian subspace $\mathrm{W} \subset \mathfrak{p}^{\mathbf{C}}$ of dimension greater than one and a general element $\mathrm{X} + i\mathrm{Y} \in \mathrm{W}$, with $\mathrm{X}, \mathrm{Y} \in \mathfrak{p}$. By Lemma (5.5) the vectors X and Y must be right **H**-dependent: there is a quaternion q such that $\mathrm{Y} = \mathrm{X}q$. We show first that q does not vary as $\mathrm{X} + i\mathrm{Y}$ varies in W. To this end consider an element $\mathrm{X}' + i\mathrm{Y}' \in \mathrm{W}$ which is (left **C**-)independent from $\mathrm{X} + i\mathrm{Y}$. Then $\mathrm{Y}' = \mathrm{X}' q'$ for some $q' \in \mathbf{H}$, and we must show that $q = q'$. If X and X′ are **H**-dependent, then the four vectors X, Y, X′, Y′ lie on the same right **H**-line. This line is the tangent space of a totally geodesic $\mathrm{H}^1_{\mathbf{H}}$ in $\mathrm{H}^n_{\mathbf{H}}$, so that the subspace of W spanned by $\mathrm{X} + i\mathrm{Y}$ and $\mathrm{X}' + i\mathrm{Y}'$ is a two-dimensional abelian subspace of the complexification of the tangent space of a quaternionic hyperbolic line; since the latter is the same as real hyperbolic 4-space, this contradicts the first part of Theorem (3.5). We conclude that X and X′ are **H**-independent. Now consider the vector $(\mathrm{X} + \mathrm{X}') + i(\mathrm{Y} + \mathrm{Y}')$. There is a quaternion q'' such that $\mathrm{Y} + \mathrm{Y}' = (\mathrm{X} + \mathrm{X}')\, q''$, so that

$$\mathrm{X}q + \mathrm{X}'\, q' = (\mathrm{X} + \mathrm{X}')\, q''.$$

By the **H**-independence of X and X′, this implies $q = q' = q''$, as required.

To complete the proof of (3.5) we take a basis $\mathrm{X}_k + i\mathrm{X}_k\, q$ for which the X_k form an orthonormal set. Then W is of the form

$$\mathrm{W} = \{\, \mathrm{X} + i\mathrm{X}q \mid \mathrm{X} \in \mathrm{V}, \mathrm{V} \subset \mathfrak{p}, \mathrm{V} \text{ totally real}, q \in \mathbf{H}, q^2 = -1, q + \bar{q} = 0 \,\},$$

where the restriction on q follows from the isotropy of W, and V totally real means that V is equivalent under the action of K to a space of the form $\{(x_1, \ldots, x_k, 0, \ldots, 0) \in \mathbf{H}^n \mid x_1, \ldots, x_k \in \mathbf{R}\}$ for some $k \leqslant n$, namely $k = \dim_{\mathbf{C}}(\mathrm{W})$. The space $V \oplus Vq \subset \mathfrak{p}$ is the real tangent space of a totally geodesic embedding of $H^k_{\mathbf{C}}$ in $H^n_{\mathbf{H}}$, as described in [2, Thm. 3.25], and W is the (1, 0)-tangent space of the complex structure on $V' \oplus V' q$ given by right multiplication by $-q$. Thus W has the required form.

6. Connections with Hodge theory

The purpose of this section is to reformulate the last part of the proof of Theorem (3.5) in a more invariant way, and to show that this reformulation leads directly to an equivalence between harmonic mappings to quotients of quaternionic hyperbolic space and horizontal holomorphic mappings to an associated space. This space can be viewed either as a Griffiths period domain or as a twistor space. The latter point of view has been used by Eells and Salamon to study harmonic mappings from Riemann surfaces to quaternionic manifolds [11, 25]. We will use the first point of view because the main tool that we will need is a removable singularities theorem proved by Griffiths in [13].

We present this equivalence as a test case for a general conjecture that harmonic mappings of sufficiently high rank from a Kähler manifold to a locally symmetric space of non-compact type are in one-to-one correspondence with horizontal holomorphic mappings to an associated period domain. The converse, that the composition of a period mapping F with the canonical projection to the associated locally symmetric space is harmonic, is easily established using Theorem (2.3) and the fact that $dF(T^{1,0}_x)$ is abelian. More details on this general principle will be presented in a future article.

To begin, we consider the quaternionic hyperbolic space $D = H^n_{\mathbf{H}}$ studied in the previous section. Thus, if $V = \mathbf{H}^{n+1}$ is a right-quaternionic vector space endowed with the inner product

$$b(X, Y) = \mathrm{Re}(-x_1 y_1 - \dots - x_n y_n + x_{n+1} y_{n+1}),$$

then D is the open subset of $\mathbf{HP}^n$ consisting of all right-quaternionic lines L on which the Hermitian form

$$h(X, Y) = b(X, \overline{Y})$$

is positive-definite. Note that the real tangent space to D at L is the vector space $\mathrm{Hom}_{\mathbf{H}}(L, L^{\perp})$ consisting of right-quaternionic-linear maps of L to $L^{\perp}$.

Let D′ be the set of all pairs (L, J) with L in D, where J is an orthogonal complex structure on L. By this we mean that

$$J^2 = -1,$$

$$J(Xq) = J(X)\, q \quad \text{for all } q \in \mathbf{H},$$

and $$b(JX, JY) = b(X, Y) \quad \text{for all } X, Y \in L.$$

Such structures are given by left multiplication by a purely imaginary unit quaternion; consequently the set of possible J's constitutes an S^2. There is an obvious projection

$\pi : D' \to D$, and this map is equivariant with respect to the natural group actions relative to which the source and target are homogeneous. Thus,

$$D = Sp(n, 1)/Sp(n) \times Sp(1),$$
$$D' = Sp(n, 1)/Sp(n) \times U(1),$$

and the fiber of π is $Sp(1)/U(1) \cong \mathbf{CP}^1 \cong S^2$, as noted above.

The manifold D' admits both a natural complex structure and a natural " horizontal bundle ". To describe the first, let $L^{1,0}$ and $L^{0,1}$ be the $+i$ and $-i$ eigenspaces of J on $L^{\mathbf{C}}$. Because J satisfies the identity

$$b(Jx, y) + b(x, Jy) = 0,$$

these spaces are b-isotropic. Thus, to the datum (L, J) one may associate the datum $L^{1,0}$, where this latter is 1) stable under right multiplication by quaternions, 2) b-isotropic, and 3) h-positive. Conversely, the second datum yields the first. Now the first two conditions define a complex submanifold $\hat{D}'$ of the Grassmannian of complex 2-planes in $V^{\mathbf{C}}$, and the third condition defines D' as an open subset of $\hat{D}'$. This gives the required complex structure.

The real tangent space to $x = L^{1,0}$ in D' is

$$T^{\mathbf{R}}_x = {}_{\mathbf{C}}\mathrm{Hom}_{\mathbf{H}}(L^{1,0}, V^{\mathbf{C}}/L^{1,0}) \cong {}_{\mathbf{C}}\mathrm{Hom}_{\mathbf{H}}(L^{1,0}, L^{\perp\mathbf{C}} \oplus L^{0,1})$$

viewed as a real vector space. The subscripts signal the fact that the homomorphisms are linear with respect to the right action of quaternions and the left action of complex numbers. Clearly one has a splitting into horizontal and vertical subspaces defined by

$$T^{\mathbf{R}}_{h,x} = {}_{\mathbf{C}}\mathrm{Hom}_{\mathbf{H}}(L^{1,0}, L^{\perp\mathbf{C}})$$
$$T^{\mathbf{R}}_{v,x} = {}_{\mathbf{C}}\mathrm{Hom}_{\mathbf{H}}(L^{1,0}, L^{0,1}),$$

with the latter tangent to the fibers of π. A map with values in D' is called *horizontal* if it is tangent to the bundle $T^{\mathbf{R}}_h$ formed by the $T^{\mathbf{R}}_{h,x}$.

Given a discrete group $\Gamma \subset Sp(n, 1)$, the holomorphic structure and horizontal bundle pass to the quotient $N' = \Gamma \backslash D'$, and the projection passes to a map $\pi : N' \to N$, where $N = \Gamma \backslash D$. We can now state the main result:

Theorem **(6.1)**. — *Let* M *be a compact Kähler manifold and let* N *and* N' *be as above. Let* $f : M \to N$ *be a harmonic mapping such that, for some* $x \in M$, *the rank of* $d_x f$ *is larger than two. Then there exists a horizontal holomorphic mapping* $F : M \to N'$ *that lifts* f, i.e., $f = \pi \circ F$.

Before taking up the proof, we note the connection with Hodge theory. First, the decomposition

$$V^{\mathbf{C}} = L^{1,0} \oplus L^{\perp\mathbf{C}} \oplus L^{0,1} \underset{\text{def}}{=} V^{2,0} \oplus V^{1,1} \oplus V^{0,2}$$

defines a b-polarized Hodge structure of weight two on the real vector space V with Hodge numbers $h^{2,0} = \dim V^{2,0} = 2$ and $h^{1,1} = \dim V^{1,1} = 2n$. As such it defines a point in the Griffiths period domain

$$D'' \cong SO(4, 4n)/U(2) \times SO(4n).$$

In fact, D' sits in D'' as a homogeneous complex submanifold, namely that consisting of all Hodge structures stable under right multiplication by quaternions. Moreover, the horizontal bundle as defined above coincides with that that induced by the usual Hodge-theoretic horizontal bundle on D''.

To begin the proof, let m be the maximum rank of df, let U be the open subset of M where the rank of df is m, and note that $m > 2$. Because of Theorem (3.5 c) $W = df'(T^{1,0}_x M)$ can be identified with the image of the $(1, 0)$-tangent space at $f(x)$ of a geodesically imbedded copy of complex hyperbolic space. Therefore $m = 2k$ is even, and the map $df: T_x M \to T_{f(x)} H^k_{\mathbf{C}}$ is complex-linear.

The totally geodesic embeddings of $H^k_{\mathbf{C}}$ in D are given as follows. Let E' be a right **H**-subspace of V of **H**-dimension $k + 1$ such that $b \mid E'$ has signature $(4k, 4)$ and let $J : E' \to E'$ be a b-orthogonal and right **H**-linear complex structure. Then the set of all right **H**-lines in E' on which b is positive definite and which are invariant under J forms a totally geodesic submanifold of D isomorphic to $H^k_{\mathbf{C}}$. Moreover, all such submanifolds are obtained in this way for suitable E' and J. The real tangent space of such a submanifold at a line L is ${}_J\mathrm{Hom}_{\mathbf{H}}(L, E)$, where $E = E' \cap L^{\perp}$ and ${}_J\mathrm{Hom}_{\mathbf{H}}(L, E)$ denotes the left-J-linear homomorphisms which are right-**H**-linear. This is the invariant description of the space denoted by $V \oplus Vq$ at the end of the proof of Theorem (3.5), described now as a subspace of $\mathrm{Hom}_{\mathbf{H}}(L, L^{\perp}) = T^{\mathbf{R}}_L D$ without reference to a particular basis.

Consider now a point x in the open set U introduced above, and let L_x be an **H**-line whose Γ-orbit is $f(x)$. According to the preceding discussion, there is a space E'_x containing L_x and a complex structure J_x on E'_x such that $d_x f: T_x M \to \mathrm{Hom}_{J_x}(L_x, E_x) \subset T_{L_x} D$ is a complex-linear map with respect to the complex structure given by precomposing with $J_x \mid L_x$. But then $F(x) = (L_x, J_x) \bmod \Gamma$ defines a lifting of f from N to N'.

It remains to show that F is holomorphic and horizontal. We shall treat horizontality first, beginning with the following result, which asserts that the lift is horizontal if and only if the tensor J is parallel.

Lemma **(6.2)**. — *A smooth map* $F : M \to D'$ *is horizontal if and only if* $\nabla J = 0$, *where* J *is the complex structure on* $F^* \pi^* L$ *induced from the tautological complex structure on* $\pi^* L$ *and* ∇ *is the connection induced by* $F^* \pi^*$ *from the canonical connection on* L.

Proof. — By the definition of $T_h D'$, F is horizontal on an open set U' if the following condition holds: let s be a $V^{\mathbf{C}}$-valued function on U' with $s(p) \in L^{1,0}_p$; then $d_X s(p) \in L^{1,0}_p \oplus L^{\perp\mathbf{C}}_p$ for any real tangent vector X at $p \in U'$. Functions s of the required form are given by $s(p) = (1 - iJ_p)\, t(p)$, where $t(p) \in L_p$. Let ∇ denote the connection operator on L, defined by the relation $\nabla t = dt - dt^\perp$, where orthogonal complement is taken relative to L. Then

$$d_X[(1-iJ)\,t] = (1-iJ)\,\nabla_X t + \{\,d_X[(1-iJ)\,t]\}^\perp + i(\nabla_X J)\,t.$$

The first two terms on the right-hand side lie in $L^{1,0} \oplus L^{\perp C}$. The third term is a purely imaginary vector in L^C, hence is zero if and only if its component in $L^{0,1}$ is zero. Therefore $d_X s$ lies in the required subspace for all t if and only if $\nabla_X J$ vanishes. The proof is therefore complete.

We must now show that J is indeed parallel. To this end we show (Lemma (6.3)), that for all $X, Y \in T_x M$, $df(X)\,(\nabla_Y J) = df(Y)\,(\nabla_X J)$, where T_x is the real tangent space, $df(X) \in \mathrm{Hom}(f^* L, f^* L^\perp)$, and $J \in \mathrm{End}(f^* L)$. Moreover, we observe that *a)* if $df(X) \neq 0$, then $df(X)$ is injective, and *b)* if $df(X)$ and $df(Y)$ are independent, then $df(X)\,L \cap df(Y)\,L = 0$. Both assertions follow from the fact that $df(X)$ is a right-quaternionic-linear map whose domain is a quaternionic line. If rank $df > 1$, we are therefore in the situation of the Lemma (6.4) below, with $A(X) = df(X)$ and $B(Y) = \nabla_Y J$. Applying the lemma, we conclude that $\nabla J = 0$, as required.

Lemma **(6.3).** — *The tensor* $df(Y)\,(\nabla_X J)$ *is symmetric in* X *and* Y.

Proof. — Let $\beta(X, Y) = (\nabla_X\, df)\,(Y)$. The three assertions below imply the required symmetry condition:

a) $\beta(X, JY) - \beta(X, Y)\,J = df(Y)\,(\nabla_X J)$,

b) $\beta(X, Y)$ is symmetric in X and Y,

c) $\beta(X, JY)$ is symmetric in X and Y.

Here we view $df(Y)$ in $f^* T^C N$, so that $\nabla_X\, df(Y) \underset{\text{def}}{=} \nabla_{df(X)}\, df(Y)$ makes sense.

To begin, we note that the harmonic equation implies $\nabla'' d' f = 0$, or, equivalently, that $(\nabla\, df)^{1,1} = 0$, so that

$$(*) \qquad \beta(JX, JY) = -\,\beta(X, Y).$$

The symmetry of $\beta(X, JY)$ then follows from the symmetry of $\beta(X, Y)$:

$$\beta(X, JY) = -\,\beta(JX, J^2 Y) = \beta(JX, Y) = \beta(Y, JX).$$

The symmetry of β itself is an immediate consequence of the fact that the connections on N and M are torsion-free. It therefore remains to establish *a)*, which we do by noting the validity of the following sequence of identities:

$$\begin{aligned}
\beta(X, JY) &= (\nabla_X\, df)\,(JY) \\
&= \nabla_X(df(JY)) - df(\nabla_X(JY)) \\
&= \nabla_X(df(Y)\,J) - df(J(\nabla_X Y)) \\
&= \nabla_X(df(Y))\,J + df(Y)\,\nabla_X J - df(\nabla_X Y)\,J \\
&= [(\nabla_X\, df)\,(Y)]\,J + df(Y)\,\nabla_X J \\
&= \beta(X, Y)\,J + df(Y)\,\nabla_X J.
\end{aligned}$$

We have used the fact that df is J-linear and that J is parallel on M.

Lemma **(6.4).** — *Let* $A : V \to \mathrm{Hom}(E, F)$, $B : V \to \mathrm{End}(E)$ *be linear maps such that*

a) *if* $A(X) \neq 0$ *then* $A(X)$ *is injective,*

b) *if* $A(X)$ *and* $A(Y)$ *are independent, then* $A(X)\,E \cap A(Y)\,E = 0$,

c) $A(X)\,B(Y) = A(Y)\,B(X)$ *for all* $X, Y \in V$.

d) $\dim A(V) > 1$.

Then $B = 0$.

Proof. — Suppose first that $A(X) \neq 0$, $A(Y) = 0$. The commutation relation *c)* gives $A(X)\,B(Y)\,E = A(Y)\,B(X)\,E = 0$. Since $A(X)$ is injective, $B(Y) = 0$, so that B factors through the kernel of A. It therefore suffices to prove the lemma in the case that A is injective. To this end, observe that *d)* is now equivalent to the condition $\dim V > 1$. Consequently for any $X \neq 0$ there is a $Y \in V$ independent of X. Since $A(X)$ and $A(Y)$ are independent, $A(X)\,E \cap A(Y)\,E = 0$. The commutation relation $A(X)\,B(Y)\,E = A(Y)\,B(X)\,E$ then implies that $A(Y)\,B(X)\,E = 0$, and the injectivity of $A(Y)$ yields $B(X) = 0$. Since $X \in V$ was arbitrary, the proof is complete.

To prove that F is holomorphic on U we must show that dF_p is complex-linear as a map from the real tangent space of M at p to the real tangent space at N′ at $F(p)$. Now the map df_p takes values in the real tangent space of N at $f(p)$, namely $\mathrm{Hom}_H(L, L^\perp)$. According to the discussion above, its values lie in the real tangent space to a geodesically imbedded complex hyperbolic space, given explicitly by ${}_J\mathrm{Hom}_H(L, E)$. Moreover, df_p is complex linear as map from $T_p^R M$ to ${}_J\mathrm{Hom}_H(L, E)$. Now df_p is given explicitly as

$$df_p(X)\, t = \{ d_X\, t \}^\perp,$$

while dF_p is given by

$$dF_p(X)\, [(1 - iJ)\, t] = \{ d_X[(1 - iJ)\, t] \}^\perp + \{ d_X[(1 - iJ)\, t] \}^{0,1}.$$

However, for horizontal maps J is parallel, so the previous formula reduces to

$$dF_p(X)\, [(1 - iJ)\, t] = \{ d_X[(1 - iJ)\, t] \}^\perp,$$

i.e., to

$$dF_p(X)\, s = \{ d_X\, s \}^\perp$$

for all sections s of $L^{1,0}$. But now the complex-linearity of dF_p follows from that of df_p.

Returning to the global situation, we have constructed a horizontal holomorphic lift $F : U \to N'$ of f. Since $d'f$ is holomorphic by Theorem (2.3), $M - U$ is a complex analytic subvariety of M. But the removable singularities theorem for horizontal holomorphic mappings defined on $\Delta^d \times \Delta^*$, where Δ is a disk in **C**, [13, Prop. 9.10] (applied, say, to the period domain $SO(4, 4n)/U(2) \times SO(4n)$ which contains D′ as a complex submanifold), immediately implies that a horizontal holomorphic mapping defined in the complement of a proper analytic subvariety must extend holomorphically across this subvariety. Thus F extends to a horizontal holomorphic mapping $F : M \to N'$, thereby concluding the proof of the theorem.

7. Maps of low rank

The purpose of this section is to prove the following structure theorem for harmonic maps $f: \mathrm{M} \to \mathrm{N}$ of rank at most two, where (throughout) M is a compact Kähler manifold and N is a hyperbolic space form, i.e., $\mathrm{N} = \Gamma\backslash\mathbf{H}^n_{\mathbf{K}}$.

Theorem (**7.1**). — *Let* M *and* N *be as above and let* $f: \mathrm{M} \to \mathrm{N}$ *be a non-constant harmonic map such that rank* $d_x f \leqslant 2$ *for all* $x \in \mathrm{M}$. *Then either*

a) $f(\mathrm{M})$ *is a closed geodesic in* N, *or*

b) *There exists a compact Riemann surface* S, *a holomorphic map* $\varphi: \mathrm{M} \to \mathrm{S}$, *and a harmonic map* $\psi: \mathrm{S} \to \mathrm{N}$ *so that* $f = \psi\varphi$.

Combining this result with the classification of harmonic maps of rank larger than two described in the last two sections, we obtain at once the following complete structure theorem for harmonic mappings into the hyperbolic space forms over the three fields:

Theorem (**7.2**). — *Let* M *and* N *be as above and let* $f: \mathrm{M} \to \mathrm{N}$ *be a non-constant harmonic map. Then*

a) *If* $\mathbf{K} = \mathbf{R}$, *then* f *is as in Theorem* (7.1).

b) *If* $\mathbf{K} = \mathbf{C}$, *then* f *is either as in Theorem* (7.1) *or* f *is holomorphic or anti-holomorphic.*

c) *If* $\mathbf{K} = \mathbf{H}$, *then* f *is either as in Theorem* (7.1) *or* f *lifts to a variation of Hodge structure as in Theorem* (6.1).

We now proceed to the proof of Theorem (7.1). Observe that if rank $d_x f \leqslant 1$ for all $x \in \mathrm{M}$, then by a theorem of Sampson [26] $f(\mathrm{M})$ is a closed geodesic in N, which is part *a)* of the theorem. Thus we need only consider maps $f: \mathrm{M} \to \mathrm{N}$ satisfying rank $d_x f \leqslant 2$ for all $x \in \mathrm{M}$ with equality for some $x \in \mathrm{M}$.

Observe next that rank $d_x f \leqslant 2$ is equivalent to $\dim d_x f(\mathrm{T}^{1,0}\,\mathrm{M}) \leqslant 1$, since, by the discussion in section 5, $\dim d_x f(\mathrm{T}^{1,0}\,\mathrm{M}) \geqslant 2$ implies rank $d_x f \geqslant 4$, the latter possible only if $\mathbf{K} = \mathbf{C}$ or $\mathbf{H}$. Moreover, rank $d_x f = 2$ is equivalent to $\dim d_x f(\mathrm{T}^{1,0}\,\mathrm{M}) = 1$ with no real vectors in $d_x f(\mathrm{T}^{1,0}\,\mathrm{M})$. Thus, to prove Theorem (7.1), we only need consider maps $f: \mathrm{M} \to \mathrm{N}$ satisfying

(i) $\dim d_x f(\mathrm{T}^{1,0}\,\mathrm{M}) \leqslant 1$ for all $x \in \mathrm{M}$, and

(ii) for some $x \in \mathrm{M}$ this space is one-dimensional and contains no real vectors.

In [31, § 3, 4] Siu studies the same situation as in Theorem (7.1) with the further assumptions that N is a Riemann surface (where (i) is automatic) and that (ii) holds, and he obtains conclusion *b)* in the theorem. To complete the proof of Theorem (7.1) one merely has to check that Siu's method (which is motivated by [16] and [22]) extends to the present situation, which we proceed to do; cf. also [17].

The mapping f is real analytic; thus it is known that $f(M)$ is a two dimensional polyhedron K in N. Let $y_0 \in K$ be a regular value of f regarded as a map from M to K, and let U be a small contractible neighborhood of y_0 in K such that rank $d_x f = 2$ for all $x \in f^{-1}(U)$. Since $f \mid f^{-1}(U)$ is a proper map, it follows that $f^{-1}(U)$ is diffeomorphic, as a fibration over U, to $U \times f^{-1}(y_0)$. Let C_{y_0} be a connected component of $f^{-1}(y_0)$, let V denote the component of $f^{-1}(U)$ containing C_{y_0}, and let $C_y = V \cap f^{-1}(y)$, which is a connected component of $f^{-1}(y)$.

Lemma (**7.3**). — *For each $y \in U$, C_y is a complex analytic submanifold of* M *of complex codimension one, and for each $x \in C_y$, $T_x^{1,0} C_y = \ker d'_x f$. Moreover* U *has a unique complex structure so that $f \mid V$ is holomorphic.*

Proof. — Let $x \in C_y$. Since $d_x f(T^{1,0} M)$ contains no real vectors, it contains no purely imaginary vectors. Thus $d_x f(X) = 0$ holds if and only if $d_x f(X - iJX) = 0$. It follows that $T_x C_y = \ker d_x f$ is invariant under J, hence that C_y is a complex submanifold, and that the corresponding space of $(1, 0)$ tangent vectors is as asserted in the lemma. Finally, for each $x \in C_y$ let $L(x) = d_x f(T^{1,0} M)$. Then $L(x)$ is a complex line in the two dimensional complex vector space $T_y^{\mathbf{C}} U$ which varies holomorphically with x and contains no real vectors. Since the collection of such lines is parametrized by the complement of the equator in the Riemann sphere, it follows from the connectedness and compactness of C_y that $L(x)$ is constant, say L. Give U the complex structure determined by $T_y^{1,0} U = L$. This is clearly the unique complex structure on U that makes $f \mid V$ holomorphic.

Theorem (7.1) is proved by showing that $f \mid V$ has an analytic continuation to a holomorphic mapping $\varphi : M \to S$, where S is some compact Riemann surface, and that the original map f is constant on the fibres of φ, hence factors through S. First we give the general statement on analytic continuation.

Lemma (**7.4**). — *Let* M *be a compact Kähler manifold, let* U *and* V *be open sets in* **C** *and* M *respectively, and let $f : V \to U$ be a holomorphic map with compact connected fibres. Then there exists a compact Riemann surface* S *containing* U *and a holomorphic map $\varphi : M \to S$ such that $\varphi \mid V = f$.*

Proof. — The proof uses the Chow scheme C(M) constructed by Barlet in [1], the compactness of its components, as proved by Liebermann in [20] as a consequence of a theorem of Bishop [3], and the assumption that M is Kähler. All the information we need concerning C(M) is contained in the first section of [20]. We only mention that the points of C(M) are in one to one correspondence with the analytic cycles in M, that holomorphic families of cycles are suitably induced by universal families, and we refer the reader to [1, 20] for the unexplained details.

The cycles $f^{-1}(y)$, $y \in U$, form a holomorphic family of cycles in M parametrized by U. Let S be the irreducible component of C(M) containing this family.

Then S is compact, may be assumed reduced and normal, and there is a subvariety $Z \subset S \times M$ (the universal cycle) so that $Z_s \subset M$, defined by $\{s\} \times Z_s = Z \cap \{s\} \times M$, is the cycle in M corresponding to $s \in S$. Let $p : Z \to S$ and $q : Z \to M$ denote the restrictions to Z of the projections of $S \times M$ to S and M respectively. Then $q(Z) = M$, since it contains the open set V; since the cycles Z_s are disjoint for $s \in U$, S has complex dimension one, hence is a compact Riemann surface.

The cycles Z_s belong to a single homology class in M, and since $Z_s \cap Z_t = \emptyset$ for $s, t \in U$, $s \neq t$, this class has zero self-intersection. Since M is Kähler, it follows that if $s, t \in S$ and $s \neq t$, then $Z_s \cap Z_t$ is either empty or Z_s and Z_t are both reducible and their intersection consists of a union of common components. Since Z_s is reducible for only finitely many values of s, it follows that $q : Z \to M$ is finite-to-one. Since q is generically one-to-one and M is a manifold, q is one-to-one, hence bijective, hence biholomorphic. Since Z maps to S by p, we get the desired extension of f by defining $\varphi = pq^{-1}$, and the proof is complete.

Returning to the proof of the theorem, we apply the lemma to $f \mid V$ to obtain the desired holomorphic map φ, so that it remains only to show that f is constant on the fibres of φ. This would follow if we knew that for each non-singular point x of $\varphi^{-1}(s) = Z_s$, $d'f$ vanishes on $T^{1,0}_x Z_s$, for then df would vanish on a dense subset of the connected variety Z_s. To this end let W be the open subset of Z defined by $W = \{(s, x) \in Z \mid x \text{ is a non-singular point of } Z_s\}$. If $z = (s, x) \in W$, let $E_z = T^{1,0}_z \cap \ker dp$. Then E_z is canonically isomorphic to $T^{1,0}_x Z_s$, and the spaces E_z, being of constant dimension, form a holomorphic vector bundle E over W. Now let $\Phi : T^{1,0}(S \times M) \to (qf)^* TN^{\mathbf{C}}$ be the holomorphic bundle map obtained by composing $d'q$ and $d'f$. Tracing through the definitions, one sees that $\Phi \mid E_z = 0$ if and only if $d'_x f(T^{1,0}_x(\varphi^{-1}(s))) = 0$. Since the latter condition holds for all $s \in U$, we see that $\Phi \mid E$ vanishes over a non empty open subset of W, hence vanishes identically on W, hence the desired conclusion.

From the foregoing we see that there is a continuous map $\psi : S \to N$ such that $f = \psi\varphi$; it remains only to show that ψ is harmonic. But, as in [31], this follows immediately from existence and uniqueness theorems for harmonic mappings. First, ψ is homotopic to a harmonic mapping ψ': If N is compact this follows from the existence theorem of Eells and Sampson. If N is not compact, it is an easy consequence of Corlette's criterion [8] that a harmonic map exists if and only if the Zariski closure of the image of the fundamental group of the domain is a reductive subgroup of the group G of isometries of the universal cover of N. An easy application of Theorem (2.3) shows that $\psi' \varphi$ is harmonic: since it is homotopic to f and harmonic mappings of rank larger than one into manifolds of strictly negative curvature are unique in their homotopy class, one must have $\psi = \psi'$. The proof of the theorem is complete.

8. Applications

We close with two applications:

Theorem **(8.1)**. — *Let* Γ *be a cocompact discrete subgroup of* $SO(1, n)$ *with* $n > 2$. *Then* Γ *is not the fundamental group of a compact Kähler manifold.*

Proof. — By passing to a subgroup of finite index, we may assume that Γ is torsionfree. An Eilenberg-MacLane space $K(\Gamma, 1)$ can be constructed as $\Gamma\backslash D$, where $D = SO(1, n)/SO(n)$ is hyperbolic n-space. Let M be a compact Kähler manifold, assume $\pi_1(M) \cong \Gamma$. This isomorphism is induced by a map $f : M \to \Gamma\backslash D$ which classifies the universal cover of M. By the theorem of Eells and Sampson, we may assume that f is harmonic. By Theorem (7.1) above, f factors as $\psi\varphi$, where $\varphi : M \to S$ and S is either a circle or a compact Riemann surface. Now $f_* : \pi_1(M) \to \Gamma$ is an isomorphism, so $\varphi_* : \pi_1(M) \to \pi_1(S)$ is injective. Therefore, Γ, identified as a subgroup of $\pi_1(S)$ acts freely on a contractible complex of dimension at most 2, namely, the universal cover of S. Consequently the cohomological dimension of Γ is at most 2. However, the cohomological dimension of Γ is in fact n, since $\Gamma\backslash D$ is a $K(\Gamma, 1)$. If $n > 2$, we are in the presence of a contradiction.

Our last theorem concerns the locally homogeneous complex manifolds studied in [14]. Let $D = G/V$ be a manifold of the following form: G is a semisimple Lie group without compact factors with maximal compact subgroup K such that there is a Cartan subgroup of G contained in K (equivalently, G and K have the same rank), and $V \subset K$ is the centralizer of a torus in K. Then $D = G/V$ has a finite number of homogeneous complex structures (called duals of Kähler C-spaces in [14]). Fix one of these complex structures on G/V. If $\Gamma \subset G$ is a torsion-free co-compact subgroup, then $M = \Gamma\backslash D$ is a locally homogeneous complex manifold which fibres over the locally symmetric space $\Gamma\backslash G/K$ with fibres compact homogeneous Kähler manifolds. Moreover, by [14, 4.23, p. 277], M has a pseudo-Kähler form which is positive on the fibres and negative in the directions orthogonal to the fibres.

Theorem **(8.2)**. — *Let* $D = G/V$ *and* $\Gamma \subset G$ *be as above. Suppose that the associated symmetric space* G/K *is not Hermitian symmetric. Then the complex manifold* $M = \Gamma\backslash D$ *is not of the homotopy type of a compact Kähler manifold.*

Proof. — Let $\pi : D \to X = G/K$ be the canonical map to the associated symmetric space, and let $\pi : M \to N$ be the associated map of discrete quotients, where $N = \Gamma\backslash X$. Assume M to be Kähler, and apply Corollary (3.3) to conclude that the fundamental class of N is not in the image of π_*. Next, let κ be the curvature form of the canonical bundle of M. According to [14, 4.23, p. 277], κ is positive on the fibers of π. Now let Ω be a volume form for N, and note (by Fubini's theorem) that $\pi^* \Omega \wedge \kappa^r$ has nonzero integral on M, where r is the dimension of the fibers of π. Therefore the cohomology class of $\pi^*\Omega$ is nonzero. By Poincaré duality there is a cycle c on M such that $\pi^*(\Omega)(c) \neq 0$. But then $\Omega(\pi_*(c)) \neq 0$. Therefore the fundamental class of N is in the image of π_*, and we are once again in the presence of a contradiction.

A special case of the manifolds considered in Theorem (8.2) are the compact quotients of the Griffiths period domains D of even weight, defined in [13], for which D is not already a Hermitian symmetric space. The simplest examples are the (non-Hermitian) period domains of weight two, namely $D = SO(2p, q)/U(p) \times SO(q)$, $p \neq 1$ and $q \neq 2$.

We note that the groups of Theorem (8.1) and the manifolds of Theorem (8.2) resist the relevant non-Kählerianity tests of rational homotopy theory [10, 23]. The Betti numbers of M behave in a similarly refractory manner: for M a quotient of a Griffiths period domain it is known that in Borel's stable range the rational cohomology ring is a polynomial algebra on generators of even degree [7], and outside this range the Betti numbers are difficult to compute.

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