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MOSHE JARDEN
GOPAL PRASAD

Appendix on the discriminant quotient formula for global field

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APPENDIX

The discriminant quotient formula for global fields

by Moshe JARDEN *and* Gopal PRASAD

We shall use the notation introduced in § 0, however, in the following ℓ will be an arbitrary finite separable extension of k and if k is a number field, we will now let A denote its ring of integers and B that of ℓ. If k is a global function field, let $\mathfrak{k}$ be its field of constants and $\mathfrak{l}$ be that of ℓ; q_k (resp. q_ℓ) is then the cardinality of $\mathfrak{k}$ (resp. $\mathfrak{l}$). For a place v of k (resp. w of ℓ), k_v (resp. ℓ_w) will denote the completion of k (resp. ℓ) at v (resp. w). If v is nonarchimedean and k is a number field, then A_v (resp. B_w) will denote the closure of A (resp. B) in k_v (resp. ℓ_w); A_v is the same as the ring denoted by $\mathfrak{o}_v$ earlier.

$|\ |_\infty$ will denote the usual absolute value on $\mathbf{Q}$, and for each rational prime p, $|\ |_p$ the p-adic absolute value.

For $v \in V_f$, the absolute value $|\ |_v$ extends to the fractional ideals of k if k is a number field and to the divisors of k if k is a function field.

A.1. In case k is a number field, let $\mathfrak{d}(A/\mathbf{Z})$, $\mathfrak{d}(B/\mathbf{Z})$ be the discriminants of $A/\mathbf{Z}$, $B/\mathbf{Z}$ respectively ([10: § 4]), and $D_k = |\mathfrak{d}(A/\mathbf{Z})|_\infty$, $D_\ell = |\mathfrak{d}(B/\mathbf{Z})|_\infty$. The *relative discriminant* $\mathfrak{d}(\ell/k)$ of ℓ/k is by definition the discriminant $\mathfrak{d}(B/A)$ of B/A ([10: § 4]), it is an ideal in A.

A.2. The group of divisors of function fields will be written multiplicatively.

Let K be a global function field. If $\mathfrak{a} = \prod \mathfrak{a}_v$ is the prime factorization of a divisor $\mathfrak{a}$ of K, then its degree, to be denoted $\deg_K(\mathfrak{a})$, is defined by

$$q_K^{\deg_K(\mathfrak{a})} = \prod_v |\mathfrak{a}_v|_v^{-1}, \tag{1}$$

where q_K is the cardinality of the field of constants of K. The discriminant D_K of K is by definition equal to $q_K^{2g_K-2}$, where g_K is the genus of K.

If L is a finite separable extension of K, then $\mathfrak{D}(L/K)$ will denote the *different* of L/K (see [8: Chapter IV, § 8] for the definition of the different). The *relative discriminant* $\mathfrak{d}(L/K)$ is by definition the divisor $N_{L/K}(\mathfrak{D}(L/K))$ of K.

A.3. For a place w of ℓ lying over a nonarchimedean place v of k, let $\mathfrak{d}(\ell_w/k_v)$ be the relative discriminant of ℓ_w/k_v. Then ℓ_w/k_v is unramified if and only if $\mathfrak{d}(\ell_w/k_v)$ is trivial. The v-component of the discriminant $\mathfrak{d}(\ell/k)$ is $\prod_{w|v} \mathfrak{d}(\ell_w/k_v)$ and $|\mathfrak{d}(\ell/k)|_v = \prod_{w|v} |\mathfrak{d}(\ell_w/k_v)|_v$; see [10: § 4, Proposition 5], [14: p. 463].

Theorem **A.** — *Let ℓ be a finite separable extension of k. Then*

$$D_\ell/D_k^{[\ell:k]} = \prod_{v \in V_f} \prod_{w|v} |\mathfrak{d}(\ell_w/k_v)|_v^{-1}. \tag{2}$$

Proof. — Number fields and function fields will be treated separately.

(i) *k is a number field.* We use the following relation for the relative discriminants of the ring of integers ([10: § 4, Proposition 7 (ii)])

$$\mathfrak{d}(B/\mathbf{Z})/\mathfrak{d}(A/\mathbf{Z})^{[\ell:k]} = N_{k/\mathbf{Q}}(\mathfrak{d}(B/A)). \tag{3}$$

Taking the absolute value of both sides of the above, we obtain

$$\begin{aligned} D_\ell/D_k^{[\ell:k]} &= |N_{k/\mathbf{Q}}(\mathfrak{d}(\ell/k))|_\infty \\ &= \prod_p |N_{k/\mathbf{Q}}(\mathfrak{d}(\ell/k))|_p^{-1} \quad \text{by the product formula (0.1)} \\ &= \prod_p \prod_{v|p} |\mathfrak{d}(\ell/k)|_v^{-1} \quad \text{(by [7: Theorem in § 11])} \\ &= \prod_{v \in V_f} \prod_{w|v} |\mathfrak{d}(\ell_w/k_v)|_v^{-1} \quad \text{(cf. A.3).} \end{aligned}$$

(ii) *k is a function field**. Let $k' = \mathfrak{l}k$. Then k' and ℓ have the same field of constants, the genus of k' equals that of k ([9: p. 132, Theorem 2]) and the different $\mathfrak{D}(k'/k)$ is trivial. Theorem 8 of [8: Chapter IV] implies then that $\mathfrak{D}(\ell/k') = \mathfrak{D}(\ell/k)$.

The Riemann-Hurwitz formula for ℓ/k' ([8: p. 106, Corollary 2]) gives

$$2g_\ell - 2 = [\ell : k'] (2g_{k'} - 2) + \deg_\ell(\mathfrak{D}(\ell/k')). \tag{4}$$

By a result on p. 110 of [9], we have

$$[\mathfrak{l} : \mathfrak{k}] \deg_\ell(\mathfrak{D}(\ell/k)) = \deg_k(\mathfrak{d}(\ell/k)),$$

since $\mathfrak{d}(\ell/k) = N_{\ell/k}(\mathfrak{D}(\ell/k))$. Now multiplying (4) by $[\mathfrak{l} : \mathfrak{k}]$ we obtain

$$[\mathfrak{l} : \mathfrak{k}] (2g_\ell - 2) = [\ell : k] (2g_k - 2) + \deg_k(\mathfrak{d}(\ell/k)).$$

As $q_\ell = q_k^{[\mathfrak{l}:\mathfrak{k}]}$, this leads to

$$q_\ell^{2g_\ell - 2} = q_k^{(2g_k - 2)[\ell:k]} q_k^{\deg_k(\mathfrak{d}(\ell/k))}. \tag{5}$$

By (1) and the last result of A.3, $q_k^{\deg_k(\mathfrak{d}(\ell/k))} = \prod_v \prod_{w|v} |\mathfrak{d}(\ell_w/k_v)|_v^{-1}$, formula (2) follows therefore from (5).

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* We are indebeted to W.-D. Geyer for a simplification of an earlier version of the proof in this case.

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Tata Institute of Fundamental Research
Homi Bhabha Road
Bombay 400 005
India

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