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THE DEFORMATION THEORY OF REPRESENTATIONS OF FUNDAMENTAL GROUPS OF COMPACT KÄHLER MANIFOLDS

by William M. GOLDMAN *and* John J. MILLSON*

Abstract. — Let Γ be the fundamental group of a compact Kähler manifold M and let G be a real algebraic Lie group. Let $\mathfrak{R}(\Gamma, G)$ denote the variety of representations $\Gamma \to G$. Under various conditions on $\rho \in \mathfrak{R}(\Gamma, G)$ it is shown that there exists a neighborhood of ρ in $\mathfrak{R}(\Gamma, G)$ which is analytically equivalent to a cone defined by homogeneous quadratic equations. Furthermore this cone may be identified with the quadratic cone in the space $Z^1(\Gamma, \mathfrak{g}_{\mathrm{Ad}\,\rho})$ of Lie algebra-valued 1-cocycles on Γ comprising cocycles u such that the cohomology class of the cup/Lie product square $[u, u]$ is zero in $H^2(\Gamma, \mathfrak{g}_{\mathrm{Ad}\,\rho})$. We prove that $\mathfrak{R}(\Gamma, G)$ is quadratic at ρ if either (i) G is compact, (ii) ρ is the monodromy of a variation of Hodge structure over M, or (iii) G is the group of automorphisms of a Hermitian symmetric space X and the associated flat X-bundle over M possesses a holomorphic section. Examples are given where singularities of $\mathfrak{R}(\Gamma, G)$ are not quadratic, and are quadratic but not reduced. These results can be applied to construct deformations of discrete subgroups of Lie groups.

The purpose of this paper is to investigate the local structure of the space o representations of a discrete group into a Lie group. If Γ is a finitely generated group and G is a linear algebraic Lie group the set $\mathrm{Hom}(\Gamma, G)$ of homomorphisms $\Gamma \to G$ has the natural structure of an affine algebraic variety $\mathfrak{R}(\Gamma, G)$, whose algebraic and geometric properties reflect the structure of Γ. In this paper we study one large class of groups—fundamental groups of compact Kähler manifolds—and deduce a general local property of their varieties of representations near representations satisfying fairly general conditions.

Let V be an algebraic variety and $x \in V$ be a point. We say that V *is quadratic at* x if there exists a neighborhood U of x in V and an analytic embedding of U into an affine space such that the image of U is a cone defined by a system of finitely many homogeneous quadratic equations. In particular the tangent cone to V at x will be defined by a system of homogeneous quadratic equations in the Zariski tangent space of V at x and V will be locally analytically equivalent to this quadratic cone.

Theorem **1.** — *Let* Γ *be the fundamental group of a compact Kähler manifold and* G *a real algebraic Lie group. Let* $\rho \in \mathrm{Hom}(\Gamma, G)$ *be a representation such that its image* $\rho(\Gamma)$ *lies in a compact subgroup of* G. *Then* $\mathfrak{R}(\Gamma, G)$ *is quadratic at* ρ.

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Suppose that $\{ H^{p,q}, Q, \sigma \}$ is a polarized Hodge structure of weight n (see § 8.1), that G is the group of real points of the isometry group of Q and $X = G/V$ is the classifying space for polarized Hodge structures of the above type (see Griffiths [Gr, p. 15]). Suppose further M is a complex manifold with fundamental group Γ. A representation $\rho : \Gamma \to G$ determines a flat principal G-bundle P_ρ over M and an associated X-bundle $P_\rho \times_G X$. A *horizontal holomorphic* V*-reduction* of P_ρ is a holomorphic section of $P_\rho \times_G X$ whose differential carries the holomorphic tangent bundle of M into the horizontal subbundle $T_h(X)$ defined in [Gr, p. 20].

Theorem **2.** — *Let* M *be a compact Kähler manifold with fundamental group* Γ *and* $X = G/V$ *a classifying space for polarized real Hodge structures. Suppose that* $\rho : \Gamma \to G$ *is a representation such that the associated principal bundle over* M *admits a horizontal holomorphic* V*-reduction. Then* $\mathfrak{R}(\Gamma, G)$ *is quadratic at* ρ.

Let $\pi : E \to M$ be a holomorphic family of smooth polarized projective varieties parametrized by M; then the period mapping which attaches to $x \in M$ the polarized Hodge structure on $H^n(\pi^{-1}(x))$ is a horizontal holomorphic V-reduction of the principal bundle associated to the monodromy representation. Thus we obtain the following:

Corollary. — *Suppose* $\rho : \Gamma \to G$ *is the monodromy of a variation of Hodge structure over* M. *Then* $\mathfrak{R}(\Gamma, G)$ *is quadratic at* ρ.

The idea behind the proof ot Theorem 2 can be applied to a number of other closely related situations. The next result appears to be one of the most useful of them.

Theorem **3.** — *Let* M *be a compact Kähler manifold with fundamental group* Γ *and* $X = G/K$ *a Hermitian symmetric space with automorphism group* G. *Suppose* $\rho : \Gamma \to G$ *is a representation such that the associated principal* G*-bundle over* M *admits a holomorphic* K*-reduction. Then* $\mathfrak{R}(\Gamma, G)$ *is quadratic at* ρ.

Recently C. Simpson has shown ([Si2, 5.3]) that for every reductive representation ρ the corresponding differential graded Lie algebra is formal, and by the techniques developed here, $\mathfrak{R}(\Gamma, G)$ is quadratic at ρ.

Perhaps the most important feature of such " quadratic singularity " theorems is that the quadratic functions are computable algebraic topological invariants. Thus we obtain a criterion for nonsingularity of $\mathfrak{R}(\Gamma, G)$ near a representation ρ. The Zariski tangent space to $\mathfrak{R}(\Gamma, G)$ near ρ equals the space $Z^1(\Gamma; \mathfrak{g}_{\mathrm{Ad}\,\rho})$ of Eilenberg-MacLane 1-cocycles of Γ with coefficients in the Γ-module $\mathfrak{g}_{\mathrm{Ad}\,\rho}$ (given the action defined by the composition of $\rho : \Gamma \to G$ with $\mathrm{Ad} : G \to \mathrm{Aut}(\mathfrak{g})$). The quadratic cone is defined by the cup-product where the Lie bracket $[,] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ is used as a coefficient pairing.

Corollary. — *Suppose that* M *is a compact Kähler manifold with fundamental group* Γ, *that* G *is a real algebraic group, and that* $\rho : \Gamma \to G$ *is a representation satisfying the hypotheses of Theorems 1, 2 or 3. Suppose that the cup-product*

$$H^1(\Gamma, \mathfrak{g}_{\mathrm{Ad}\,\rho}) \times H^1(\Gamma, \mathfrak{g}_{\mathrm{Ad}\,\rho}) \to H^2(\Gamma, \mathfrak{g}_{\mathrm{Ad}\,\rho})$$

is identically zero. Then $\mathfrak{R}(\Gamma, G)$ *is nonsingular at* ρ.

When G is compact, that the tangent cones to $\mathrm{Hom}(\Gamma, G)$ are quadratic was proved in Goldman-Millson [GM2]. When ρ is a reductive representation of the fundamental group of a closed surface into a reductive group, it was shown in Goldman [G1] that the tangent cone to $\mathrm{Hom}(\Gamma, G)$ is quadratic. It seems quite likely that there are further cases under which the above conclusion is true.

The proof given here uses a categorical language suggested to us by Deligne. We follow the philosophy that a deformation problem consists of a groupoid $\mathscr{C}$ whose objects are the items to be classified and whose morphisms are the allowable equivalences between them. As such the " moduli space " is the associated set Iso $\mathscr{C}$ of isomorphism classes of objects. Two deformation problems are regarded as equivalent if there is an equivalence of categories between the corresponding groupoids.

Although our principal aim is the space of representations, its deformation theory can be replaced by the equivalent deformation theory of flat connections on an associated principal bundle P. The groupoid here consists of gauge transformations acting on flat connections on P; the equivalence of flat connections with representations associates to a flat connection its holonomy homomorphism. In turn, this groupoid can be replaced by an isomorphic groupoid associated with a purely algebraic object: the differential graded Lie algebra of ad P-valued exterior differential forms on M associated with the flat connection on P. (Here ad P denotes the vector bundle associated with P by the adjoint representation of G on its Lie algebra.) Thus the deformation theory of representations of fundamental groups is equivalent to a deformation theory associated with differential graded Lie algebras.

The importance of differential graded Lie algebras in deformation theory was recognized early on in numerous contexts. In [NR2] Nijenhuis and Richardson detail an abstract approach to the Gerstenhaber deformation theory of algebras, the Kodaira-Spencer-Kuranishi deformation theory of complex manifolds, etc. If L denotes a differential graded Lie algebra, then the objects of the associated groupoid are those elements of L having degree 1 satisfying the *deformation equation*

$$d\alpha + \frac{1}{2}[\alpha, \alpha] = 0.$$

If $\mathscr{L}$ is a simply connected Lie group with Lie algebra L^0 then there is a natural action of $\mathscr{L}$ by affine transformations on L^1 given by

$$\exp(t\lambda) : \alpha \mapsto \exp(t\,\mathrm{ad}\,\lambda)\,(\alpha) + \frac{I - \exp(t\,\mathrm{ad}\,\lambda)}{\mathrm{ad}\,\lambda}\,(d\lambda)$$

for the one-parameter subgroup of $\mathscr{L}$ corresponding to $\lambda \in L^0$. This affine action preserves the solutions of the deformation equation in L^1. A morphism from α to β in this groupoid is an element $\eta \in \mathscr{L}$ with $\eta(\alpha) = \beta$.

For the local questions with which we are concerned we introduce formal infinitesimal parameters, given by elements of an Artin local $\mathbf{k}$-algebra A. For each such ring A we consider the set of morphisms from Spec(A) into the solution space of the deformation equation. In other words we are led to consider solutions of the deformation equation " parametrized " by Spec(A). This leads to a groupoid $\mathscr{C}(L; A)$ which depends both on the differential graded Lie algebra L and the Artin local ring A. The functor which associates to A the groupoid $\mathscr{C}(L; A)$ captures the local deformation theory associated with L.

The basic result concerning the groupoids $\mathscr{C}(L; A)$ is the following " equivalence theorem ", first observed and stated by Deligne, although an equivalent version can be found in the earlier work [SS] of Schlessinger-Stasheff:

Equivalence theorem. — Let $\mathbf{k}$ *be a field of characteristic zero and* $\varphi : L \to \bar{L}$ *be a homomorphism of differential graded Lie* $\mathbf{k}$*-algebras such that the induced maps* $H^i(\varphi) : H^i(L) \to H^i(\bar{L})$ *are isomorphisms for* $i = 0, 1$ *and injective for* $i = 2$. *Let* A *be an Artin local* $\mathbf{k}$*-algebra. Then the induced functor*

$$\varphi_* : \mathscr{C}(L; A) \to \mathscr{C}(\bar{L}; A)$$

is an equivalence of groupoids.

A differential graded algebra is *formal* if it is quasi-isomorphic to its cohomology algebra. In that case the deformation equation simplifies considerably, since the differential is identically zero. In particular the deformation equation is now a homogeneous quadratic equation whose set of solutions is a quadratic cone. The analogue of the fundamental observation of Deligne-Griffiths-Morgan-Sullivan [DGMS] that the de Rham algebra of a compact Kähler manifold is formal can then be applied (in various cases) to differential graded Lie algebras of exterior differential forms taking values in certain flat vector bundles. The key in all of these cases is the existence of a real variation of Hodge structure on ad P_C and the corresponding Hodge theory taking coefficients there. What seems to be crucial is that the covariant exterior differential and the covariant holomorphic (and anti-holomorphic) differential both give rise to the same harmonic spaces and satisfy the " principle of two types ". It follows from the Equivalence Theorem above that the deformation space is locally equivalent to the corresponding quadratic cone.

In [AMM], Arms, Marsden, and Moncrief prove that the inverse image of zero under the momentum map of an affine Hamiltonian action on a symplectic affine space is quadratic whenever the action preserves a positive complex structure. As noted in [GM2], this result implies Theorem 1 when M has complex dimension one. In [GM3] the result of Arms-Marsden-Moncrief is proved by applying the techniques of this paper.

In particular we associate to an affine Hamiltonian action a differential graded Lie algebra which under the assumptions of [AMM] we prove is formal. In another direction one can apply the Equivalence Theorem to deformations of pairs (V, ∇) where V is a holomorphic vector bundle over a Kähler manifold and ∇ is a compatible flat connection. One obtains then a quadratic cone of local deformations of (V, ∇) expressed as a family of quadratic cones over the quadratic cone which parametrizes local deformations of holomorphic structures on V.

This paper is organized as follows. The first section contains algebraic preliminaries concerning graded Lie algebras. In § 2 we describe the groupoid associated to a differential graded Lie algebra and an Artin local $\mathbf{k}$-algebra A. The equivalence theorem of Deligne-Schlessinger-Stasheff is stated. An obstruction theory is developed for extending objects and morphisms in the groupoids $\mathscr{C}(L; A)$ as the parameter ring A is enlarged. This obstruction theory relates the structure of the groupoid to cohomology classes in L and is used to prove the Equivalence Theorem. The third section of the paper discusses pro-representability of functors of Artin local $\mathbf{k}$-algebras by analytic germs. The definition of the quadratic cone " tangent " to an analytic germ is given. The fourth section discusses the algebraic structure of the variety $\mathfrak{R}(\Gamma, G)$. In particular the tangent space and the tangent quadratic cone to $\mathfrak{R}(\Gamma, G)$ at a representation $\rho : \Gamma \to G$ are computed in terms of the cohomology of Γ. The relation between representations and principal bundles is developed. In § 5, necessary background from differential geometry is summarized; in particular connections on principal bundles and the action of the group of gauge transformations on connections is discussed here. In § 6, the parallel deformation theories of flat connections parametrized by spectra of Artin local rings and representations of the fundamental group parametrized by spectra of Artin local rings, are discussed. This provides the bridge between infinitesimal deformations of representations and differential graded Lie algebras. By regarding (generalized) infinitesimal deformations of flat connections on a principal G-bundle as flat connections on principal G_A-bundles (where G_A is the Lie group consisting of A-points of G) we may apply the standard theory of connections to study connections parametrized by Spec A, for an Artin local $\mathbf{k}$-algebra A. The final result of this section, Theorem 6.9, is the basic result relating the local analytic structure of the variety of representations to a differential graded Lie algebra. Theorem 1 is proved in § 7. We have tried to present the proof in such a way that the modifications necessary for its generalizations are easily apparent. In § 8, Theorems 2 and 3 are proved by modifying the proof of Theorem 1. A basic point in the proof is the fundamental observation of Deligne that the complex of differential forms on a compact Kähler manifold with coefficients in a real variation of Hodge structure is formal; the formality follows in the usual way once the covariant differential is decomposed by total (base plus fiber) bidegree. This idea is expounded and exploited in Zucker [Z1] (see also Simpson [Si] and Corlette [C2]). Finally in § 9 various examples are given to illustrate the ideas and demonstrate further applications of these techniques. In 9.1 it is shown that if Γ is a lattice in the Heisenberg group then $\mathfrak{R}(\Gamma, G)$ generally does not have quadratic singularities; indeed the techniques developed here show that in many cases the germ of $\mathfrak{R}(\Gamma, G)$ at the trivial representation is analytically equivalent to a *cubic cone.* In 9.2 it is shown that the quadratic singularity theorems apply to a larger class of groups than fundamental groups of compact Kähler manifolds; in particular our techniques apply to Bieberbach groups and finite extensions of fundamental groups of compact Kähler manifolds arising from finite group actions on compact Kähler manifolds (fundamental groups of compact " Kähler orbifolds "). In particular we describe the example of Lubotzky-Magid [LM] where $\mathfrak{R}(\Gamma, G)$ is not reduced from our point of view. In 9.4 we briefly describe how these techniques apply to the deformation theory of holomorphic structures on vector bundles over Kähler manifolds. In 9.5 Theorem 3 is applied to discuss the existence of deformations of discrete groups acting on complex hyperbolic space.

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NOTATIONAL CONVENTIONS

Throughout this paper $\mathbf{k}$ will denote the field of real or complex numbers. By an Artin local $\mathbf{k}$-algebra we shall mean an Artinian local $\mathbf{k}$-algebra with unity such that the residue field $A/\mathfrak{m}$ is isomorphic to $\mathbf{k}$. All manifolds will be assumed to be C^∞, connected and paracompact and all tensor fields will also be assumed to be C^∞. By a $\mathbf{k}$-variety will be meant an affine scheme of finite type over $\mathbf{k}$ (not necessarily irreducible or reduced). If α, β are objects in a category (e.g. groups, algebras over $\mathbf{k}$), then $\mathrm{Hom}(\alpha, \beta)$ will denote the collection of morphisms $\alpha \to \beta$ and we denote the identity morphism $\alpha \to \alpha$ by I_α. If $\mathscr{C}$ is a small category, Obj $\mathscr{C}$ will denote its set of objects and

Iso $\mathscr{C}$ will denote its set of isomorphism classes of objects. All the tensor products we consider here are tensor products of **k**-vector spaces over **k**. We use the notation $\otimes$ for such tensor products.

1. Differential graded Lie algebras

1.1. In this section we review basic algebraic notions concerning graded Lie algebras and their derivations, referring to Nijenhuis-Richardson [NR2] for futher details. A *graded Lie algebra* over **k** will mean a **k**-vector space

$$L = \bigoplus_{i \geq 0} L^i$$

graded by the nonnegative integers, and a family of bilinear maps

$$[\ ,\ ] : L^i \times L^j \to L^{i+j}$$

satisfying (graded) skew-commutativity:

$$[\alpha, \beta] + (-1)^{ij} [\beta, \alpha] = 0$$

and the (graded) Jacobi identity:

$$(-1)^{ki} [\alpha, [\beta, \gamma]] + (-1)^{ij} [\beta, [\gamma, \alpha]] + (-1)^{jk} [\gamma, [\alpha, \beta]] = 0$$

where $\alpha \in L^i$, $\beta \in L^j$, $\gamma \in L^k$.

For each $\alpha \in L^i$, we shall denote the *adjoint transformation* by

$$\operatorname{ad} \alpha : L^j \to L^{j+i}$$
$$\beta \mapsto [\alpha, \beta].$$

Then L^0 is a Lie algebra and the adjoint representation of L^0 on L^i is a linear representation of the Lie algebra L^0.

A basic example of a graded Lie algebra arises as follows. Let $\mathfrak{g}$ denote a Lie algebra and $\mathscr{A}$ a graded commutative algebra, i.e. a graded vector space with associative multiplication $\mathscr{A}^i \times \mathscr{A}^j \to \mathscr{A}^{i+j}$ satisfying (graded) commutativity:

$$\alpha\beta = (-1)^{ij} \beta\alpha$$

where $\alpha \in \mathscr{A}^i$ and $\beta \in \mathscr{A}^j$. Examples include exterior algebras, cohomology algebras and the de Rham algebra of exterior differential forms on a manifold. Then $\mathscr{A} \otimes \mathfrak{g}$ is a graded Lie algebra under the operation

$$[\alpha \otimes u, \beta \otimes v] = \alpha\beta \otimes [u, v].$$

1.2. A *derivation* (of degree ℓ) consists of a family of linear maps $d : L^i \to L^{i+\ell}$ satisfying

$$d[\alpha, \beta] = [d\alpha, \beta] + (-1)^{i\ell} [\alpha, d\beta]$$

where $\alpha \in L^i$, $\beta \in L$. The Jacobi identity is equivalent to the assertion that for every $\alpha \in L^i$, $\operatorname{ad} \alpha$ is a derivation of degree i. It is easy to see that there is a graded Lie algebra

Der(L) (where $\mathrm{Der}(L)^{\ell}$ consists of derivations of degree ℓ) with operation the (graded) commutator

$$[d_1, d_2] = d_1 \circ d_2 - (-1)^{\ell_1 \ell_2} d_2 \circ d_1$$

where d_i is a derivation of degree ℓ_i for $i = 1, 2$.

A *differential graded Lie algebra* is a pair (L, d) where L is a graded Lie algebra and $d : L \to L$ is a derivation of degree 1 such that the composition $d \circ d = 0$. It follows that the space $Z^i(L) = \mathrm{Ker}\, d : L^i \to L^{i+1}$ of cocycles contains the space $B^i(L) = \mathrm{Image}\, d : L^{i-1} \to L^i$ of coboundaries. Thus the *cohomology* $H^i(L) = Z^i(L)/B^i(L)$ is defined and has the structure of a graded Lie algebra. Every graded Lie algebra becomes a differential graded Lie algebra by defining the differential d to be identically zero.

An *ideal* in a differential graded Lie algebra L is a graded subspace $L' \subset L$ such that $[L, L'] \subset L'$ and $d(L') \subset L'$. One checks easily that if $L' \subset L$ is an ideal, then the quotient L/L' is naturally a differential graded Lie algebra. If $\varphi : L \to \bar{L}$ is a homomorphism of differential graded Lie algebras, then the kernel $\mathrm{Ker}\,\varphi$ is an ideal. Similarly, if $D : L \to L$ is a derivation then its kernel Ker D is a differential graded subalgebra.

1.3. Let L denote a differential graded Lie algebra. We shall next define an action of the ordinary Lie algebra L^0 on the vector space L^1 by affine transformations and a corresponding quadratic mapping $Q : L^1 \to L^2$ which is equivariant respecting this affine action. Clearly the Lie algebra $\mathrm{ad}\, L^0 \subset \mathrm{Der}(L)^0$ acts linearly on the space of all derivations. We shall be particularly interested in the affine subspace $A \subset \mathrm{Der}(L)^1$ comprising derivations of degree 1 having the form $d + \mathrm{ad}\,\alpha$ where $\alpha \in L^1$. We claim that the subalgebra $\mathrm{ad}\, L^0$ preserves this affine subspace, i.e. the linear vector field on $\mathrm{Der}\,(L)^1$ determined by bracket with $\mathrm{ad}\,\lambda \in \mathrm{ad}\, L^0 \subset \mathrm{Der}(L)^0$ for $\lambda \in L^0$ is tangent to A. Let $\alpha \in L^1$ and $\beta \in L$. Then

$$\begin{aligned}[\mathrm{ad}\,\lambda, d + \mathrm{ad}\,\alpha]\,(\beta) &= [\mathrm{ad}\,\lambda, \mathrm{ad}\,\alpha]\,(\beta) + [\mathrm{ad}\,\lambda, d]\,(\beta) \\ &= \mathrm{ad}[\lambda, \alpha]\,(\beta) + [\lambda, d\beta] - d[\lambda, \beta] \\ &= \mathrm{ad}[\lambda, \alpha]\,(\beta) - \mathrm{ad}(d\lambda)\,(\beta) \\ &= \mathrm{ad}([\lambda, \alpha] - d\lambda)\,(\beta)\end{aligned}$$

so that the linear vector field on $\mathrm{Der}(L)^1$ has value $\mathrm{ad}([\lambda, \alpha] - d\lambda) \in \mathrm{ad}\, L^1$ at $d + \mathrm{ad}\,\alpha$. Since the tangent space to A equals $\mathrm{ad}\, L^1$, the claim is proved.

Moreover the correspondence which assigns to each $\lambda \in L^0$, the affine map $\rho(\lambda) : L^1 \to L^1$ defined by

$$\rho(\lambda) : \alpha \mapsto [\lambda, \alpha] - d\lambda$$

defines a homomorphism ρ of L^0 into the Lie algebra of affine vector fields on L^1. (This follows from the fact that the linear part of ρ is a Lie algebra homomorphism $\mathrm{ad} : L^0 \to \mathrm{End}(L^1)$ and the translational part of ρ is the derivation $-d : L^0 \to L^1$ with respect to the action given by the linear part.) If $\mathscr{L}$ is a simply-connected Lie group with Lie algebra L^0, then the corresponding affine action of the group $\mathscr{L}$ on L^1 is defined by the usual formula for one-parameter subgroups:

$$\exp(t\lambda) : \alpha \mapsto \exp(t \operatorname{ad} \lambda)\ (\alpha) + \frac{I - \exp(t \operatorname{ad} \lambda)}{\operatorname{ad} \lambda}\ (d\lambda) \tag{1-1}$$

in terms of power series, where $t \in \mathbf{k}$.

Let $\alpha \in L^1$. Then the square of the derivation $d + \operatorname{ad} \alpha$ is easily seen to be the derivation

$$(d + \operatorname{ad} \alpha) \circ (d + \operatorname{ad} \alpha) = \frac{1}{2}[d + \operatorname{ad} \alpha, d + \operatorname{ad} \alpha] = \operatorname{ad} Q(\alpha) \in \operatorname{Der}(L)^2$$

where

$$Q(\alpha) = d\alpha + \frac{1}{2}[\alpha, \alpha] \tag{1-2}$$

defines an (inhomogeneous) quadratic map $Q : L^1 \to L^2$. Clearly the action of L^0 by affine vector fields respects this quadratic map in the sense that the directional derivative of Q with respect to the tangent vector $\rho(\lambda)$ (where $\lambda \in L^0$) at $\alpha \in L^1$ equals

$$(\rho(\lambda)\, Q)\ (\alpha) = (\operatorname{ad} \lambda)\ (Q(\alpha)).$$

For the directional derivative $(\rho(\lambda)\, Q)\ (\alpha)$ equals

$$\begin{aligned} dQ_\alpha(\rho(\lambda)\ (\alpha)) &= (d + \operatorname{ad} \alpha)\ ([\lambda, \alpha] - d\lambda) \\ &= d[\lambda, \alpha] + [\alpha, [\lambda, \alpha]] - dd\lambda - [\alpha, d\lambda] \\ &= ([d\lambda, \alpha] + [\lambda, d\alpha]) + \frac{1}{2}[\lambda, [\alpha, \alpha]] - [d\lambda, \alpha] \\ &= [\lambda, d\alpha + \frac{1}{2}[\alpha, \alpha]] = [\lambda, Q(\alpha)] \end{aligned}$$

as claimed. In particular the affine action of $\mathscr{L}$ on L^1 preserves the subspace $Q^{-1}(0) \subset L^1$.

2. The groupoid associated to a differential graded Lie algebra and an Artin local ring

In this section we state the basic algebraic result on differential graded Lie algebras, and deformation theory. This result (Theorem 2.4) was first stated by Deligne, although it appeared earlier in a somewhat different formulation in Schlessinger-Stasheff [SS, Theorem 5.4]. It will be a basic tool for showing that two deformation theories are equivalent.

2.1. We begin by reviewing the relevant language from category theory (see Jacobson [J]) which will be needed to state the basic result.

Recall that a *groupoid* is a small category in which all morphisms are isomorphisms. Most of the groupoids we consider here arise from *transformation groupoids* as follows. Let G be a group which acts on a set X. The resulting groupoid (X, G) has for its set of objects $\mathrm{Obj}(X, G) = X$ and for $x, y \in X$ the morphisms $x \to y$ correspond to $g \in G$ such that $g(x) = y$. If G′ is a group acting on a set X′, then we say that a mapping $f: X \to X'$ is *equivariant with respect to* a homomorphism $\varphi: G \to G'$ if for each $g \in G$ the diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & X' \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle \varphi(g)} \\ X & \xrightarrow{f} & X' \end{array}$$

commutes. We shall also say that (f, φ) is a *transformation groupoid homomorphism* $(X, G) \to (X', G')$. A homomorphism of transformation groupoids determines a functor between the corresponding groupoids, although not every functor arises in this way.

A central notion is that of an *equivalence* of categories. (See Jacobson [J, 1.4] for the definition and discussion.) If $\mathscr{A}$ and $\mathscr{B}$ are categories, then a functor $F: \mathscr{A} \to \mathscr{B}$ is an equivalence if its satisfies three basic properties:

(i) F is *surjective on isomorphism classes*, i.e. the induced map $F_*: \mathrm{Iso}\,\mathscr{A} \to \mathrm{Iso}\,\mathscr{B}$ is surjective;

(ii) F is *full*, i.e. for any two objects $x, y \in \mathrm{Obj}\,\mathscr{A}$, the map

$$F(x, y): \mathrm{Hom}(x, y) \to \mathrm{Hom}(F(x), F(y))$$

is surjective;

(iii) F is *faithful*, i.e. for any two objects $x, y \in \mathrm{Obj}\,\mathscr{A}$, the map

$$F(x, y): \mathrm{Hom}(x, y) \to \mathrm{Hom}(F(x), F(y))$$

is injective.

An equivalence of categories $F: \mathscr{A} \to \mathscr{B}$ induces an bijection of sets

$$F_*: \mathrm{Iso}\,\mathscr{A} \to \mathrm{Iso}\,\mathscr{B}.$$

2.2. If A is a **k**-algebra and (L, d) is a differential graded Lie algebra, then the tensor product $L \otimes A$ is differential graded Lie algebra in the obvious way: for $\alpha, \beta \in L$ and $u, v \in A$, then

$$[\alpha \otimes u, \beta \otimes v] = [\alpha, \beta] \otimes uv$$
$$d(\alpha \otimes u) = d\alpha \otimes u$$

defines a bracket operation and a derivation giving $L \otimes A$ the structure of a differential graded Lie algebra.

Suppose that A is an Artin local **k**-algebra with maximal ideal $\mathfrak{m} \subset A$ and residue field **k** and consider the differential graded Lie algebra $L \otimes \mathfrak{m}$. Since $\mathfrak{m}^N = 0$ for

$N \geqslant 0$, the Lie algebra $L \otimes \mathfrak{m}$ is nilpotent and so is $(L \otimes \mathfrak{m})^0 = L^0 \otimes \mathfrak{m}$. The corresponding nilpotent Lie group $\exp(L^0 \otimes \mathfrak{m})$ has underlying space $L^0 \otimes \mathfrak{m}$ and is equipped with Campbell-Hausdorff multiplication

$$(X, Y) \mapsto \log(\exp(X)\exp(Y)).$$

As in 1.3 the group $\exp(L^0 \otimes \mathfrak{m})$ acts affinely on $L^1 \otimes \mathfrak{m}$ by formula (1-1).

Let $Q_A = Q : L^1 \otimes \mathfrak{m} \to L^2 \otimes \mathfrak{m}$ be the quadratic map $Q(\alpha) = d\alpha + \frac{1}{2}[\alpha, \alpha]$ defined in (1-1). The action of $\exp(L^0 \otimes \mathfrak{m})$ preserves the subspace $Q_A^{-1}(0) \subset L^1 \otimes \mathfrak{m}$. We define a groupoid $\mathscr{C}(L; A)$ as follows. The set of objects $\mathrm{Obj}\, \mathscr{C}(L; A)$ will be $Q_A^{-1}(0) \subset L^1 \otimes \mathfrak{m}$, and given $\alpha, \beta \in \mathrm{Obj}\, \mathscr{C}(L; A)$, morphisms $\alpha \to \beta$ correspond to elements λ of $L^0 \otimes \mathfrak{m}$ such that $\exp(\lambda)(\alpha) = \beta$.

2.3. Let $\varphi : L \to \bar{L}$ be a homomorphism of differential graded Lie algebras. There is an induced functor

$$\varphi_* : \mathscr{C}(L; A) \to \mathscr{C}(\bar{L}; A)$$

which arises from the corresponding homomorphism of transformation groupoids. Suppose further that $\psi : A \to A'$ is a homomorphism of Artin local **k**-algebras. (Such a homomorphism will map the maximal ideal of A to the maximal ideal of A'.) There are corresponding functors $\psi_* : \mathscr{C}(L; A) \to \mathscr{C}(L; A')$ such that the diagram

$$\begin{array}{ccc} \mathscr{C}(L; A) & \xrightarrow{\varphi_*} & \mathscr{C}(\bar{L}; A) \\ {\scriptstyle\psi_*}\downarrow & & \downarrow{\scriptstyle\psi_*} \\ C(L; A') & \xrightarrow[\varphi_*]{} & \mathscr{C}(\bar{L}; A') \end{array}$$

commutes, i.e. the functor $\varphi_* : \mathscr{C}(L; A) \to \mathscr{C}(\bar{L}; A)$ is *natural with respect to* A. All of our constructions will preserve this naturality.

We may now state the main algebraic result concerning differential graded Lie algebras.

2.4. *Theorem. — Suppose that* $\varphi : L \to \bar{L}$ *is a differential graded Lie algebra homomorphism inducing isomorphisms* $H^i(L) \to H^i(\bar{L})$ *for* $i = 0, 1$ *and a monomorphism* $H^2(L) \to H^2(\bar{L})$, *then for every Artin local* **k**-*algebra* A, *the induced functor* $\varphi_* : \mathscr{C}(L; A) \to \mathscr{C}(\bar{L}; A)$ *is an equivalence of groupoids.*

2.5. The proof of Theorem 2.4 proceeds by " Artinian induction " on the coefficient ring A. For every Artin local **k**-algebra A, there exists a sequence $A_0 = A, A_1, \ldots, A_{r-1}, A_r = \mathbf{k}$ of Artin local **k**-algebras and epimorphisms $\eta_i : A_i \to A_{i+1}$ such that $(\mathrm{Ker}\, \eta_i) . \mathfrak{m}_i = 0$ where $\mathfrak{m}_i \subset A_i$ is the maximal ideal. For example we might take for A_{i+1} the quotient ring A_i/K_i where $K_i \subset A_i$ is a minimal nonzero ideal and $\eta_i : A_i \to A_{i+1}$ the quotient map. We shall prove 2.4 by induction on the length r.

The initial case $r = 0$, i.e. $A = \mathbf{k}$, is completely trivial: for every L the category $\mathscr{C}(L; A)$ has but one object and one morphism. Thus to prove Theorem 2.4, it suffices to prove that *if* A *is an Artin local* $\mathbf{k}$*-algebra with maximal ideal* $\mathfrak{m} \subset A$ *and* $\mathfrak{J} \subset A$ *is an ideal such that* $\mathfrak{J}.\mathfrak{m} = 0$, *then* 2.4 *holds for* A *provided that it holds for* $A/\mathfrak{J}$.

2.6. To prove this induction we consider the relationship between the groupoids $\mathscr{C}(L; A)$ and $\mathscr{C}(L; A/\mathfrak{J})$ where $\mathfrak{J} \subset A$ is an ideal such that $\mathfrak{J}.\mathfrak{m} = 0$. Let $\pi : A \to A/\mathfrak{J}$ denote the ring epimorphism with kernel $\mathfrak{J}$ and consider the corresponding functor $\pi_* : \mathscr{C}(L; A) \to \mathscr{C}(L; A/\mathfrak{J})$. Theorem 2.4 will be proved by a detailed analysis of this functor. We state our results in terms of three " obstructions " o_i taking values in $H^i(L \otimes \mathfrak{J})$ for $i = 0, 1, 2$.

Proposition. — (1) There exists a map

$$o_2 : \mathrm{Obj}\, \mathscr{C}(L; A/\mathfrak{J}) \to H^2(L \otimes \mathfrak{J})$$

such that $\alpha \in \mathrm{Obj}\, \mathscr{C}(L; A/\mathfrak{J})$ *lies in the image of*

$$\pi_* : \mathrm{Obj}\, \mathscr{C}(L; A) \to \mathrm{Obj}\, \mathscr{C}(L; A/\mathfrak{J})$$

if and only if $o_2(\alpha) = 0$.

(2) Let $\xi \in \mathrm{Obj}\, \mathscr{C}(L; A/\mathfrak{J})$. *Let* $\pi_*^{-1}(\xi)$ *denote the category having for its set of objects the inverse image of* ξ *under* $\pi_* : \mathrm{Obj}\, \mathscr{C}(L; A) \to \mathrm{Obj}\, \mathscr{C}(L; A/\mathfrak{J})$ *and morphisms* γ *in* $\mathscr{C}(L; A)$ *such that* $\pi_*(\gamma) = I_\xi$. *There exists a simply transitive action of the group* $Z^1(L \otimes \mathfrak{J})$ *on the set* $\mathrm{Obj}(\pi_*^{-1}(\xi))$. *Moreover the composition of the difference map*

$$\mathrm{Obj}(\pi_*^{-1}(\xi)) \times \mathrm{Obj}(\pi_*^{-1}(\xi)) \to Z^1(L \otimes \mathfrak{J})$$

with the projection

$$Z^1(L \otimes \mathfrak{J}) \to H^1(L \otimes \mathfrak{J}),$$

which we denote by

$$o_1 : \mathrm{Obj}(\pi_*)^{-1}(\xi) \times \mathrm{Obj}(\pi_*)^{-1}(\xi) \to Z^1(L \otimes \mathfrak{J}),$$

has the following property: for $\alpha, \beta \in \mathrm{Obj}(\pi_*)^{-1}(\xi)$, *there exists a morphism* $\gamma : \alpha \to \beta$ *with* $\pi_*(\gamma) = I_\xi$ *if and only if* $o_1(\alpha, \beta) = 0$.

(3) Let $\widetilde{\alpha}, \widetilde{\beta} \in \mathrm{Obj}\, \mathscr{C}(L; A)$ *be isomorphic objects and* $f : \alpha \to \beta$ *a morphism in* $\mathscr{C}(L; A/\mathfrak{J})$ *from* $\alpha = \pi_*(\widetilde{\alpha})$ *to* $\beta = \pi_*(\widetilde{\beta})$. *Then there exists a simply transitive action of the group* $H^0(L \otimes \mathfrak{J})$ *on the set* $\pi_*^{-1}(f)$ *of morphisms* $\widetilde{f} : \widetilde{\alpha} \to \widetilde{\beta}$ *such that* $\pi_*(\widetilde{f}) = f$. *In particular the difference map*

$$o_0 : \pi_*^{-1}(f) \times \pi_*^{-1}(f) \to H^0(L \otimes \mathfrak{J})$$

has the following property: if $\widetilde{f}, \widetilde{f}' \in \pi_*^{-1}(f)$, *then* $\widetilde{f} = \widetilde{f}'$ *if and only if* $o_0(\widetilde{f}, \widetilde{f}') = 0$.

(If a group G acts simply transitively on a set X, the difference map

$$X \times X \to G$$

sends (x, y) to the unique $g \in G$ such that $g(x) = y$.) We shall denote the simply transitive actions of (2) and (3) by addition.

2.7. *Proof of 2.6 (1).* — Let $\omega \in \mathrm{Obj}\, \mathscr{C}(L; A/\mathfrak{J}) \subset L^1 \otimes \mathfrak{m}/\mathfrak{J}$. Then there exists $\tilde{\omega} \in L^1 \otimes \mathfrak{m}$ such that $\pi_*(\tilde{\omega}) = \omega$. Since ω is an object in $\mathscr{C}(L; A/\mathfrak{J})$,

$$Q(\tilde{\omega}) = d\tilde{\omega} + \frac{1}{2}[\tilde{\omega}, \tilde{\omega}] \in L^2 \otimes \mathfrak{J}.$$

Now
$$dQ(\tilde{\omega}) = d\left(\frac{1}{2}[\tilde{\omega}, \tilde{\omega}]\right) = [d\tilde{\omega}, \tilde{\omega}] \equiv -\frac{1}{2}[[\tilde{\omega}, \tilde{\omega}], \tilde{\omega}] \pmod{\mathfrak{J}}$$

since $Q(\tilde{\omega}) \in L^2 \otimes \mathfrak{J}$ and $[L^2 \otimes \mathfrak{J}, L^1 \otimes \mathfrak{m}] \subset L^3 \otimes \mathfrak{J}\mathfrak{m} = 0$. But $[[\tilde{\omega}, \tilde{\omega}], \tilde{\omega}] = 0$ by the Jacobi identity, whence $Q(\tilde{\omega})$ is a cocycle in $Z^2(L \otimes \mathfrak{J})$.

Furthermore suppose that $\tilde{\omega}' \in L^1 \otimes \mathfrak{m}$ is another lift of ω, i.e. $\tilde{\omega}' \equiv \tilde{\omega} \pmod{L^1 \otimes \mathfrak{J}}$. Then

$$Q(\tilde{\omega}') - Q(\omega) = d(\tilde{\omega}' - \tilde{\omega}) + [\tilde{\omega}, \tilde{\omega}' - \tilde{\omega}] + \frac{1}{2}[\tilde{\omega}' - \tilde{\omega}, \tilde{\omega}' - \tilde{\omega}] = d(\tilde{\omega}' - \tilde{\omega})$$

is exact since

$$[\tilde{\omega}, \tilde{\omega}' - \tilde{\omega}] \in [L^1 \otimes \mathfrak{m}, L^1 \otimes \mathfrak{J}] \subset L^2 \otimes \mathfrak{m}.\mathfrak{J} = 0$$

and
$$[\tilde{\omega}' - \tilde{\omega}, \tilde{\omega}' - \tilde{\omega}] \in [L^1 \otimes \mathfrak{J}, L^1 \otimes \mathfrak{J}] \subset L^2 \otimes \mathfrak{J}.\mathfrak{J} = 0.$$

Thus the cohomology class of the cocycle $Q(\tilde{\omega})$ is independent of the lift $\tilde{\omega}$. We define $o_2(\omega)$ to equal the cohomology class of $Q(\tilde{\omega})$ in $H^2(L \otimes \mathfrak{J})$.

Let $\tilde{\omega} \in \mathrm{Obj}\, \mathscr{C}(L; A)$ be an object which reduces (mod $\mathfrak{J}$) to $\omega \in \mathrm{Obj}\, \mathscr{C}(L; A/\mathfrak{J})$. Then $Q(\tilde{\omega}) = 0$ and

$$o_2(\omega) = [Q(\tilde{\omega})] = 0.$$

Conversely suppose that $\omega \in \mathrm{Obj}\, \mathscr{C}(L; A/\mathfrak{J})$ satisfies $o_2(\omega) = 0$. Let $\tilde{\omega}$ be as before. Then there exists $\psi \in L^1 \otimes \mathfrak{J}$ such that

$$Q(\tilde{\omega}) = d\tilde{\omega} + \frac{1}{2}[\tilde{\omega}, \tilde{\omega}] = d\psi.$$

Let $\tilde{\omega}' = \tilde{\omega} - \psi$. Then

$$Q(\tilde{\omega}') = Q(\tilde{\omega}) - d\psi - [\tilde{\omega}, \psi] = Q(\tilde{\omega}) - d\psi = 0$$

(since $[\tilde{\omega}, \psi] \in L^2 \otimes \mathfrak{m}.\mathfrak{J} = 0$) so there exists an object in $\pi_*^{-1}(\omega)$ if and only if $o_2(\omega) = 0$. Hence (1).

One can succinctly formulate 2.6 (1) as an " exact sequence ":

$$\mathrm{Obj}\, \mathscr{C}(L; A) \xrightarrow{\pi_*} \mathrm{Obj}\, \mathscr{C}(L; A/\mathfrak{J}) \xrightarrow{o_2} H^2(L \otimes \mathfrak{J}).$$

The proofs of (2) and (3) will be based on the following lemma.

2.8. *Lemma.* — *Let* $\alpha \in L^1 \otimes \mathfrak{m}$, $\eta \in L^0 \otimes \mathfrak{m}$, $u \in L^0 \otimes \mathfrak{J}$. *Then*

$$\exp(u + \eta)(\alpha) = \exp(\eta)(\alpha) - du.$$

Proof of Lemma 2.8. — If $n > 0$, then $(\operatorname{ad} u)^n (L \otimes \mathfrak{m}) \subset L \otimes \mathfrak{I}.\mathfrak{m} = 0$. It follows easily that

$$(\operatorname{ad}(u) + \operatorname{ad}\eta)^n (\alpha) = \operatorname{ad}(\eta)^n (\alpha),$$
$$(\operatorname{ad}(u) + \operatorname{ad}\eta)^n (d\eta) = \operatorname{ad}(\eta)^n (d\eta),$$
$$(\operatorname{ad}(u) + \operatorname{ad}\eta)^n (du) = 0.$$

Thus
$$\begin{aligned}\exp(u + \eta)(\alpha) &= \exp(\operatorname{ad}(u + \eta))(\alpha) + \frac{I - \exp(\operatorname{ad}(u + \eta))}{\operatorname{ad}(u + \eta)} (d(u + \eta)) \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} (\operatorname{ad}(u) + \operatorname{ad}(\eta))^n (\alpha) \\ &\qquad - \sum_{n=0}^{\infty} \frac{1}{(n+1)!} (\operatorname{ad}(u) + \operatorname{ad}(\eta))^n (du + d\eta) \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} (\operatorname{ad}(\eta))^n (\alpha) - du - \sum_{n=0}^{\infty} \frac{1}{(n+1)!} (\operatorname{ad}(\eta))^n (d\eta) \\ &= \exp(\eta)(\alpha) - du\end{aligned}$$

as desired.

2.9. *Proof of (2).* — Let $\xi \in \operatorname{Obj} \mathscr{C}(L; A/\mathfrak{I})$. We begin by defining a simply transitive action of $Z^1(L \otimes \mathfrak{I})$ on objects in $\pi_*^{-1}(\xi)$. Suppose that $\alpha \in \operatorname{Obj} \pi_*^{-1}(\xi) \subset L^1 \otimes \mathfrak{m}$ and let $\eta \in Z^1(L \otimes \mathfrak{I})$. Then

$$Q(\alpha + \eta) = Q(\alpha) + d\eta + [\alpha, \eta] + \frac{1}{2}[\eta, \eta] = 0$$

since $Q(\alpha) = d\eta = 0$, $[\alpha, \eta] \in [L^1 \otimes \mathfrak{m}, L^1 \otimes \mathfrak{I}] \subset L^2 \otimes \mathfrak{m}.\mathfrak{I} = 0$ and

$$[\eta, \eta] \in [L^1 \otimes \mathfrak{I}, L^1 \otimes \mathfrak{I}] \subset L^2 \otimes \mathfrak{I}.\mathfrak{I} = 0.$$

Thus $\alpha + \eta \in \operatorname{Obj} \mathscr{C}(L; A)$ satisfies $\pi_*(\alpha + \eta) = \xi$. This defines an action of $Z^1(L \otimes \mathfrak{I})$ on $\operatorname{Obj} \pi_*^{-1}(\xi)$.

Conversely let $\alpha, \beta \in \operatorname{Obj} \mathscr{C}(L; A)$ satisfy $\pi_*(\alpha) = \pi_*(\beta) = \xi$. Then $\alpha, \beta \in L^1 \otimes \mathfrak{m}$ and $\alpha - \beta \in L^1 \otimes \mathfrak{I}$. Now

$$[\beta, \alpha - \beta] = [\alpha - \beta, \alpha - \beta] = 0$$

since $[L^1 \otimes \mathfrak{m}, L^1 \otimes \mathfrak{I}] = [L^1 \otimes \mathfrak{I}, L^1 \otimes \mathfrak{I}] = 0$, so

$$d(\alpha - \beta) = d(\alpha - \beta) + [\beta, \alpha - \beta] + \frac{1}{2}[\alpha - \beta, \alpha - \beta] = Q(\alpha) - Q(\beta) = 0.$$

Thus $\alpha - \beta \in Z^1(L \otimes \mathfrak{I})$ as desired. Since $\alpha - \beta = 0$ if and only if $\alpha = \beta$, it follows that $Z^1(L \otimes \mathfrak{I})$ acts simply transitively on $\operatorname{Obj} \pi_*^{-1}(\xi)$.

We define the obstruction for the existence of an isomorphism covering I_ξ between two objects $\alpha, \beta \in \operatorname{Obj} \pi_*^{-1}(\xi)$. Let $o_1(\alpha, \beta)$ be the cohomology class of $\alpha - \beta$ in $H^1(L \otimes \mathfrak{I})$. There exists a morphism $\gamma : \alpha \to \beta$ with $\pi_*(\gamma) = I_\xi$ if and only if this obstruction vanishes.

Suppose that $\gamma : \alpha \to \beta$ is a morphism in $\mathscr{C}(L; A)$ such that $\pi_*(\gamma) = I_\xi$. Let $u \in L^0 \otimes \mathfrak{m}$ be the corresponding element such that $\exp(u) \in \exp(L^0 \otimes \mathfrak{m})$ sends α to β. Then $u \in L^0 \otimes \mathfrak{J}$ since $\pi_*(\gamma) = I_\xi$ and by Lemma 2.8

$$\beta = \exp(u)\ (\alpha) = \alpha - du.$$

Thus $\alpha - \beta$ is exact and $o_1(\alpha, \beta) = 0$.

Conversely suppose that $o_1(\alpha - \beta) = 0$. Then there exists $u \in L^0 \otimes \mathfrak{J}$ such that $\alpha - \beta = du$. As above, we see that

$$\beta = \exp(u)\ (\alpha)$$

as claimed. This completes the proof of (2).

2.10. *Proof of (3).* — Suppose $\tilde{\alpha}, \tilde{\beta} \in \operatorname{Obj} \mathscr{C}(L; A)$, and $\alpha = \pi_*(\tilde{\alpha})$, $\beta = \pi_*(\tilde{\beta})$ and consider a morphism $\gamma : \alpha \to \beta$ in $\mathscr{C}(L; A/\mathfrak{J})$. We define a simply transitive action of $H^0(L \otimes \mathfrak{J})$ on the set $\pi_*^{-1}(\gamma)$ of morphisms $\tilde{\gamma} : \tilde{\alpha} \to \tilde{\beta}$ such that $\pi_*(\tilde{\gamma}) = \gamma$. We identify $H^0(L \otimes \mathfrak{J})$ with the subset $Z^0(L \otimes \mathfrak{J})$ consisting of $u \in L^0 \otimes \mathfrak{J}$ with $du = 0$. There exists $v \in L^0 \otimes \mathfrak{m}$ such that $\tilde{\gamma}$ is defined by $\exp(v) : \tilde{\alpha} \to \tilde{\beta}$. Now if $u \in L^0 \otimes \mathfrak{J}$, then it follows from Lemma 2.8 that

$$\exp(v + u)\ (\tilde{\alpha}) = \exp(v)\ (\tilde{\alpha}) - du = \tilde{\beta} - du$$

so that if $u \in H^0(L \otimes \mathfrak{J})$, then $\tilde{\gamma} + u = \exp(v + u)$ is a morphism $\tilde{\alpha} \to \tilde{\beta}$ with $\pi_*(\tilde{\gamma} + u) = \gamma$. This defines the action of $H^0(L \otimes \mathfrak{J})$ on $\pi_*^{-1}(\gamma)$.

To show this action is simply transitive, consider another morphism $\tilde{\gamma}' : \tilde{\alpha} \to \tilde{\beta}$ in $\pi_*^{-1}(\gamma)$. Then $\tilde{\gamma}, \tilde{\gamma}' \in L^0 \otimes \mathfrak{m}$ satisfy $\tilde{\gamma} \equiv \tilde{\gamma}' \pmod{\mathfrak{J}}$. We define the obstruction $o_0(\tilde{\gamma}, \tilde{\gamma}')$ to equal $u = \tilde{\gamma} - \tilde{\gamma}' \in L^0 \otimes \mathfrak{J}$. Then

$$du = \tilde{\gamma}'(\tilde{\alpha}) - \tilde{\gamma}(\tilde{\alpha}) = \tilde{\beta} - \tilde{\beta} = 0$$

i.e. $u \in H^0(L \otimes \mathfrak{J})$. Furthermore u is the unique element of $H^0(L \otimes \mathfrak{J})$ sending $\tilde{\gamma}$ to $\tilde{\gamma}'$. Thus the action is simply transitive. This concludes the proof of (3), and also the proof of 2.6.

2.11. We now prove the Equivalence Theorem 2.4. Assuming that $\varphi : L \to \bar{L}$ induces homology isomorphisms in degrees 0 and 1 and a monomorphism in degree 2, we show that for any Artin local **k**-algebra the corresponding functor $\varphi_* : \mathscr{C}(L; A) \to \mathscr{C}(\bar{L}; A)$ is an equivalence. By induction we assume that $\mathfrak{J} \subset A$ is an ideal with $\mathfrak{m}.\mathfrak{J} = 0$ and that φ induces an equivalence $\mathscr{C}(L; A/\mathfrak{J}) \to \mathscr{C}(\bar{L}; A/\mathfrak{J})$. We prove that φ induces an equivalence $\varphi_* : \mathscr{C}(L; A) \to \mathscr{C}(\bar{L}; A)$ by checking that φ_* satisfies the three basic properties 2.1 (i)-(iii).

Surjective on isomorphism classes. — Let $\bar{\omega} \in \operatorname{Obj} \mathscr{C}(\bar{L}; A)$. Then

$$\pi_*\, \bar{\omega} \in \operatorname{Obj} \mathscr{C}(\bar{L}; A/\mathfrak{J})$$

and by the induction hypothesis there exists $\omega' \in \operatorname{Obj} \mathscr{C}(L; A/\mathfrak{J})$ and an isomorphism $g : \varphi_*\, \omega' \to \pi_*\, \bar{\omega}$. Now

$$H^2(\varphi)\ o_2(\omega') = o_2(\varphi_*\, \omega') = o_2(g^{-1}\, \pi_*\, \bar{\omega}) = o_2(\pi_*\, \bar{\omega}) = 0$$

since $\bar{\omega}$ is an object in $\mathscr{C}(\bar{L}; A)$ covering $\pi_*(\bar{\omega})$. Since $H^2(\varphi)$ is injective, $o_2(\omega') = 0$ and by 2.6 (1) there exists $\tilde{\omega} \in \operatorname{Obj} \mathscr{C}(L; A)$ such that $\pi_* \tilde{\omega} = \omega'$. The obstruction to the existence of an isomorphism $\varphi_* \tilde{\omega} \to \bar{\omega}$ covering $I_{\omega'}$ is an element $o_1(\varphi_* \tilde{\omega}, \bar{\omega}) \in H^1(L \otimes \mathfrak{J})$. Since $H^1(\varphi)$ is surjective it follows that there exists a cocycle $u \in Z^1(L \otimes \mathfrak{J})$ such that $H^1(\varphi)\,[u] = o_1(\varphi_* \tilde{\omega}, \bar{\omega})$. Now let $\omega = \tilde{\omega} - u$. Then $\omega \in \operatorname{Obj} \mathscr{C}(L; A)$ and

$$\begin{aligned} o_1(\varphi_* \omega, \bar{\omega}) &= o_1(\varphi_* \omega, \varphi_* \tilde{\omega}) + o_1(\varphi_* \tilde{\omega}, \bar{\omega}) \\ &= H^1(\varphi)\, o_1(\omega, \tilde{\omega}) + o_1(\varphi_* \tilde{\omega}, \bar{\omega}) \\ &= -\, H^1(\varphi)\,[u] + o_1(\varphi_* \tilde{\omega}, \bar{\omega}) = 0 \end{aligned}$$

proving that $\varphi_* : \operatorname{Iso} \mathscr{C}(L; A) \to \mathscr{C}(\bar{L}; A)$ is surjective.

Full. — Let $\bar{\gamma} : \varphi_* \omega_1 \to \varphi_* \omega_2$ be a morphism in $\mathscr{C}(\bar{L}; A)$. Then

$$\pi_* \bar{\gamma} : \varphi_* \pi_* \omega_1 = \pi_* \varphi_* \omega_1 \to \pi_* \varphi_* \omega_2 = \varphi_* \pi_* \omega_2$$

is a morphism in $\mathscr{C}(\bar{L}; A/\mathfrak{J})$. By the induction hypothesis

$$\varphi_* : \operatorname{Hom}(\pi_* \omega_1, \pi_* \omega_2) \to \operatorname{Hom}(\varphi_* \pi_* \omega_1, \varphi_* \pi_* \omega_2)$$

is surjective, so there exists $\gamma_1 : \pi_* \omega_1 \to \pi_* \omega_2$ such that $\varphi_* \gamma_1 = \pi_*(\bar{\gamma})$. The obstruction to the existence of a morphism $\omega_1 \to \omega_2$ is an element $o_1(\omega_1, \omega_2) \in H^1(L \otimes \mathfrak{J})$. Now

$$H^1(\varphi)\, o_1(\omega_1, \omega_2) = o_1(\varphi_* \omega_1, \varphi_* \omega_2) = 0$$

since $\bar{\gamma}$ is a morphism $\varphi_* \omega_1 \to \varphi_* \omega_2$. Since $H^1(\varphi)$ is injective it follows that there exists a morphism $\gamma' : \omega_1 \to \omega_2$. Now γ_1 and $\pi_* \gamma'$ are both morphisms $\pi_* \omega_1 \to \pi_* \omega_2$ so there exists an automorphism $g : \pi_* \omega_1 \to \pi_* \omega_1$ such that $\gamma_1 = \pi_* \gamma' \circ g$. Now g is given by an element of $\exp(L^0 \otimes \mathfrak{m}/\mathfrak{J})$ and there exists a lift $\tilde{g} \in \exp(L^0 \otimes \mathfrak{m})$. Now $o_1(\tilde{g}\omega_1, \omega_1) \in H^1(L \otimes \mathfrak{J})$ is represented by a cocycle $u \in Z^1(L \otimes \mathfrak{J})$ and $\tilde{g}' = \exp(-u) \circ \tilde{g}$ defines an automorphism $\omega_1 \to \omega_1$ covering g. Then

$$\gamma'' = \gamma' \circ \tilde{g}' : \omega_1 \to \omega_2$$

covers $\pi_* \gamma' \circ g = (\varphi_*)^{-1} \pi_* \bar{\gamma}$. Now $\varphi_* \gamma''$ and $\bar{\gamma}$ are both morphisms covering $\pi_* \bar{\gamma}$ so their difference $o_0(\varphi_* \gamma'', \bar{\gamma}) \in H^0(L \otimes \mathfrak{J})$ is defined. Since $H^0(\varphi)$ is surjective, there exists $v \in L^0 \otimes \mathfrak{J}$ such that $\varphi_*(v) = o_0(\varphi_* \gamma'', \bar{\gamma}) \in H^0(L \otimes \mathfrak{J})$. It follows that $\gamma = \gamma'' - v$ defines a morphism $\omega_1 \to \omega_2$ satisfying $\varphi_* \gamma = \bar{\gamma}$.

Faithful. — Let $\gamma_1, \gamma_2 : \omega_1 \to \omega_2$ be morphisms in $\mathscr{C}(L; A)$ with

$$\varphi_*(\gamma_1) = \varphi_*(\gamma_2).$$

Then $\pi_* \gamma_1, \pi_* \gamma_2 : \pi_* \omega_1 \to \pi_* \omega_2$ are morphisms in $\mathscr{C}(L; A/\mathfrak{J})$ satisfying

$$\varphi_* \pi_* \gamma_1 = \pi_* \varphi_* \gamma_1 = \pi_* \varphi_* \gamma_2 = \varphi_* \pi_* \gamma_2.$$

By the induction hypothesis $\pi_* \gamma_1 = \pi_* \gamma_2$ so the obstruction

$$o_0(\gamma_1, \gamma_2) \in H^0(L \otimes \mathfrak{J})$$

is defined. Now $H^0(\varphi)\, o_0(\gamma_1, \gamma_2) = o_0(\varphi_* \gamma_1, \varphi_* \gamma_2) = 0$. Since $H^0(\varphi)$ is injective it follows that $\gamma_1 = \gamma_2$ as desired.

This completes the proof of Theorem 2.4.

2.12. Let (L, d) and $(\bar{L}, \bar{d})$ be differential graded Lie algebras. We say that (L, d) and $(\bar{L}, \bar{d})$ are *quasi-isomorphic* if there exists a sequence of differential graded Lie algebra homomorphisms

$$L = L_0 \to L_1 \leftarrow L_2 \to \ldots \leftarrow L_{m-1} \to L_m = \bar{L}$$

such that each homomorphism induces a cohomology isomorphism. A differential graded Lie algebra is *formal* if it is quasi-isomorphic to one with zero differential—thus a differential graded Lie algebra (L, d) is formal if and only if it is quasi-isomorphic to its cohomology algebra $(H(L), 0)$. A repeated application of 2.4 yields the following.

Corollary. — Suppose (L, d) *and* $(\bar{L}, \bar{d})$ *are quasi-isomorphic as above. Then for each Artin local* **k***-algebra* A *the groupoids* $\mathscr{C}(L; A)$ *and* $\mathscr{C}(\bar{L}; A)$ *are equivalent. Furthermore the induced sequence of equivalences*

$$\mathscr{C}(L; A) \to \mathscr{C}(L_1; A) \leftarrow \mathscr{C}(L_2; A) \to \ldots \leftarrow \mathscr{C}(L_{m-1}; A) \to \mathscr{C}(\bar{L}; A)$$

depends naturally on A.

3. Pro-representability of functors by analytic germs

Our ultimate goal is to prove that two analytic spaces are locally equivalent. To this end we shall replace germs of analytic spaces by more algebraic objects—analytic local rings, complete local rings, and functors from Artin local rings into sets. Therefore it will be crucial for us to know that two analytic germs are equivalent if the corresponding functors they represent are naturally isomorphic. This is the primary goal of the present section. In particular we desire criteria for an analytic germ to be equivalent to a quadratic cone (see below). (For a description of tangent cones, the reader is referred to Kunz [Kz] and Mumford [M].)

3.1. If X is a **k**-variety and $x \in X$ then we denote the germ of X at x by (X, x) (for a complete discussion of analytic germs and analytic equivalence thereof, the reader is referred to Gunning [Gu, pp. 62-68]). We denote by $\mathcal{O}_{(X,x)}$ the corresponding *analytic local ring* consisting of germs of functions on X which are analytic at x. If A is a local ring then the completion of A with respect to its maximal ideal will be denoted by $\hat{A}$; if (X, x) represents an analytic germ, the corresponding complete local ring is $\hat{\mathcal{O}}_{(X,x)}$.

If R is a $\mathbf{k}$-algebra, there is a corresponding functor from the category of Artin local $\mathbf{k}$-algebras to the category of sets defined by

$$A \mapsto \mathrm{Hom}(R, A)$$

(where Hom denotes the set of local $\mathbf{k}$-algebra homomorphisms) which we denote by F_R. Let F be a functor from Artin local $\mathbf{k}$-algebras to sets. We shall say that the analytic germ (X, x) *pro-represents* F if the functors F and $F_{\hat{\mathcal{O}}_{(X,x)}}$ are naturally isomorphic.

The basic results we need in the sequel are summarized in the following.

Theorem. — *Let* (X, x) *and* (Y, y) *represent germs of* $\mathbf{k}$*-varieties. Then the following conditions are equivalent:*

(1) *The analytic germs of* (X, x) *and* (Y, y) *are analytically equivalent;*
(2) *The analytic local rings* $\mathcal{O}_{(X,x)}$ *and* $\mathcal{O}_{(Y,y)}$ *are isomorphic;*
(3) *The complete local rings* $\hat{\mathcal{O}}_{(X,x)}$ *and* $\hat{\mathcal{O}}_{(Y,y)}$ *are isomorphic;*
(4) *The functors* $F_{\hat{\mathcal{O}}_{(X,x)}}$ *and* $F_{\hat{\mathcal{O}}_{(Y,y)}}$ *are naturally isomorphic.*

Proof. — For the equivalence (1) $\Leftrightarrow$ (2), the reader is referred to Gunning [Gu, pp. 67-68]. The equivalence (2) $\Leftrightarrow$ (3) is proved in Artin [A, p. 282]. The equivalence (3) $\Leftrightarrow$ (4) follows from the next lemma. (Compare Schlessinger [Sc, 2.9].)

3.2. *Lemma.* — *Let* R *and* S *be complete local* $\mathbf{k}$*-algebras and let* $\eta : F_R \Rightarrow F_S$ *be a natural transformation. Then there exists a unique* $f \in \mathrm{Hom}(S, R)$ *such that* $\eta = f^*$.

Proof. — Let $\mathfrak{m}$ denote the maximal ideal in R and let $\pi_n : R \to R/\mathfrak{m}^n$ denote projection. Let $f_n = \eta(\pi_n) : S \to R/\mathfrak{m}^n$. We claim that the f_n form a compatible family, defining a homomorphism $f : S \to R$ such that $f \circ \pi_n = f_n$. Indeed let $p : R/\mathfrak{m}^{n+1} \to R/\mathfrak{m}^n$ be the projection. Since η is natural, there is a commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}(R, R/\mathfrak{m}^{n+1}) & \xrightarrow{\eta} & \mathrm{Hom}(S, R/\mathfrak{m}^{n+1}) \\ {\scriptstyle p_*}\downarrow & & \downarrow{\scriptstyle p_*} \\ \mathrm{Hom}(R, R/\mathfrak{m}^{n}) & \xrightarrow[\eta]{} & \mathrm{Hom}(S, R/\mathfrak{m}^{n}) \end{array}$$

Thus $$p_* f_{n+1} = p_*\, \eta(\pi_{n+1}) = \eta p_*(\pi_{n+1}) = \eta(\pi_n) = f_n$$

establishing the claim.

Let $f \in \mathrm{Hom}(S, R)$ be the corresponding homomorphism. We now prove that $\eta = f^*$. Let A be an Artin local $\mathbf{k}$-algebra and let $\chi \in \mathrm{Hom}(R, A)$. Since A is Artinian, there exists $n \geqslant 0$ and a homomorphism $h : R/\mathfrak{m}^n \to A$ such that $\chi = h \circ \pi_n$. It follows from the commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}(R, R/\mathfrak{m}^{n}) & \xrightarrow{\eta} & \mathrm{Hom}(S, R/\mathfrak{m}^{n}) \\ {\scriptstyle h_*}\downarrow & & \downarrow{\scriptstyle h_*} \\ \mathrm{Hom}(R, A) & \xrightarrow[\eta]{} & \mathrm{Hom}(S, A) \end{array}$$

that $h \circ f_n = h_* \eta(\pi_n) = \eta(h_* \pi_n) = \eta(\chi)$. Since $f_n = \pi_n \circ f$, it follows that

$$\eta(\chi) = h \circ \pi_n \circ f = \chi \circ f = f^* \chi$$

as desired.

We have established the existence of f with $\eta = f^*$. It remains to prove that f is unique. Suppose that $f \in \mathrm{Hom}(S, R)$ satisfies $f^* = \eta$. Then f is the limit of the induced maps $f_n = \pi_n \circ f$. But since $\eta = f^*$ we have $\pi_n \circ f = f^* \pi_n = \eta(\pi_n)$. Hence the maps f_n are determined by η and therefore f is determined by η. This concludes the proof of the lemma.

Quadratic cones and analytic germs

3.3. Let E be a finite-dimensional vector space over $\mathbf{k}$. A *quadratic cone* in E is an algebraic variety $\mathcal{Q} \subset E$ which can be defined by a (finite, possibly empty) family of equations

$$B(u, u) = 0$$

where $u \in E$ and $B : E \times E \to F$ is a $\mathbf{k}$-bilinear mapping to a $\mathbf{k}$-vector space F. Clearly any affine space is a quadratic cone and the Cartesian product of two quadratic cones is a quadratic cone. A variety $\mathcal{Q} \subset E$ is a quadratic cone if and only if its complexification $\mathcal{Q}(\mathbf{C})$ is a quadratic cone in $E_C = E \otimes \mathbf{C}$. (Suppose that $\mathcal{Q}(\mathbf{C})$ is defined by equations $f_i(z_1, \ldots, z_n) = 0$ where the f_i are homogeneous quadratic functions with complex coefficients; writing $z_j = x_j + iy_j$, we see that $\mathcal{Q}$ is defined by the homogeneous quadratic equations with real coefficients $\mathrm{Re} f_i(x_1 + iy_1, \ldots, x_n + iy_n) = 0$, $\mathrm{Im} f_i(x_1 + iy_1, \ldots, x_n + iy_n) = 0$ inside the vector space defined by $y_1 = \ldots = y_n = 0$.)

Let X be a $\mathbf{k}$-variety and let $x \in X$. We say that X is *quadratic at* x if there exists a quadratic cone $\mathcal{Q} \subset E$ and neighborhoods U_x of x in X and U_0 of 0 in $\mathcal{Q}$ which are analytically isomorphic. Equivalently, X is quadratic at x if the complete local $\mathbf{k}$-algebra $\hat{\mathcal{O}}$ of x in X admits a presentation of the form $\mathbf{k}[[x_1, \ldots, x_m]]_0/\mathfrak{I}$ where $\mathbf{k}[[x_1, \ldots, x_m]]_0$ is the ring of formal power series in m variables, i.e. the complete local ring of $\mathbf{k}^m$ at 0 and $\mathfrak{I}$ is an ideal generated by homogeneous quadratic polynomials in $(x_1, \ldots, x_m)$. In particular if X is quadratic at x then the tangent cone to X at x is a quadratic cone and X is locally analytically isomorphic to its tangent cone.

Consider an arbitrary analytic germ (X, x) and let $\hat{\mathcal{O}} = \hat{\mathcal{O}}_{(X, x)}$ be its complete local $\mathbf{k}$-algebra. The Zariski tangent space $T_x X$ is naturally identified with the dual of the vector space $\mathfrak{m}/\mathfrak{m}^2$ where $\mathfrak{m} \subset \hat{\mathcal{O}}$ is the maximal ideal. Every analytic germ admits a canonical embedding in its Zariski tangent space ([GR], p. 153, Corollary 14). Fixing an isomorphism $\mathfrak{m}/\mathfrak{m}^2 \cong \mathbf{k}^m$, we consider such an embedding $X \subset \mathbf{k}^m$. The tangent cone $\mathcal{C}$ to X at x is then the algebraic cone corresponding to the associated graded $\mathbf{k}$-algebra

$$\bigoplus_{n \geq 0} \mathfrak{m}^n/\mathfrak{m}^{n+1}.$$

The defining ideal $\mathfrak{I}_C \subset \mathbf{k}[[x_1, \ldots, x_m]]$ is then a homogeneous ideal which contains no homogeneous polynomials of degree 0 or 1. Then the ideal $\mathfrak{I}_C^{(2)} \subset \mathfrak{I}_C$ generated by the elements of $\mathfrak{I}_C$ of degree 2 defines a quadratic cone $C^{(2)}$ canonically associated to the germ (X, x). We call $C^{(2)}$ the *tangent quadratic cone* to X at x. Let

$$\widehat{\mathcal{O}}^{(2)} = \widehat{\mathcal{O}}^{(2)}_{X,x} = \mathbf{k}[[x_1, \ldots, x_m]]/\mathfrak{I}_C^{(2)}$$

denote the complete local $\mathbf{k}$-algebra corresponding to $C^{(2)}$. The following assertion is then immediate.

Proposition. — X *is quadratic at* x *if and only if* $\widehat{\mathcal{O}} \cong \widehat{\mathcal{O}}^{(2)}$.

In terms of Artin rings, the tangent quadratic cone can be described as follows. Let A_n denote the truncated polynomial ring $\mathbf{k}[t]/(t^{n+1})$, i.e. $A_0 = \mathbf{k}$ and A_1 is the ring of dual numbers. Then the Zariski tangent space $T_x X$ can be identified with the set $\mathrm{Hom}(\widehat{\mathcal{O}}, A_1)$. The tangent cone of X at x is the image

$$\mathrm{Image}(\mathrm{Hom}(\widehat{\mathcal{O}}, \mathbf{k}[[t]]) \to \mathrm{Hom}(\widehat{\mathcal{O}}, A_1))$$

(since it consists of elements of the Zariski tangent space $T_x X$ which are tangent to analytic paths in X, see [Wh]) and the tangent quadratic cone $C_x^{(2)} X$ equals the image

$$\mathrm{Image}(\mathrm{Hom}(\widehat{\mathcal{O}}, A_2) \to \mathrm{Hom}(\widehat{\mathcal{O}}, A_1)) \subset T_x X.$$

In practice one proceeds as follows to compute the quadratic cone associated to an analytic germ. Consider an ideal $\mathfrak{I} \subset \mathbf{k}[x_1, \ldots, x_n]$ of functions which vanish at the origin $0 \in \mathbf{k}^n$ and consider the variety X defined by $\mathfrak{I}$. Choose a finite set of generators $f_1, \ldots, f_l$ for $\mathfrak{I}$. The Zariski tangent space is defined by the differentials $df_i(0)$; in particular, for each f_i which has nonzero differential at 0 one obtains a linear functional vanishing on the tangent space. The tangent quadratic cone $C^{(2)}$ is similarly defined (inside the Zariski tangent space) by the quadratic terms $d^2 f(0)$ of functions $f \in \mathfrak{I}$ such that $df_i(0) = 0$ (although in general one may need more f than those in the original generating set $\{f_1, \ldots, f_l\}$).

The quadratic cone associated to a differential graded Lie algebra

3.4. Let $\mathfrak{g}$ be a Lie algebra. A $\mathfrak{g}$-*augmented differential graded Lie algebra* is a triple (L, d, ε) where (L, d) is a differential graded Lie algebra and $\varepsilon : L^0 \to \mathfrak{g}$ is a homomorphism of Lie algebras. A *homomorphism of* $\mathfrak{g}$-*augmented differential graded Lie algebras* $(L, d, \varepsilon) \to (\bar{L}, \bar{d}, \bar{\varepsilon})$ is a differential graded Lie algebra homomorphism $\varphi : (L, d) \to (\bar{L}, \bar{d})$ such that $\varepsilon \circ \varphi = \bar{\varepsilon}$. If (L, d, ε) is a $\mathfrak{g}$-augmented differential graded Lie algebra, then the augmentation extends trivially to a differential graded Lie algebra homomorphism $\varepsilon : L \to \mathfrak{g}$ where $\mathfrak{g}$ is given the differential graded Lie algebra structure with no nonzero elements of positive degree. The *augmentation ideal* $L' = \mathrm{Ker}(\varepsilon : L \to \mathfrak{g})$ is then an ideal in (L, d) and thus also a differential graded Lie algebra.

We now define two $\mathfrak{g}$-augmented differential graded Lie algebras (L, d, ε)

and $(\bar{L}, \bar{d}, \bar{\varepsilon})$ to be *quasi-isomorphic* if there exists a sequence of $\mathfrak{g}$-augmented differential graded Lie algebra homomorphisms

$$L = L_0 \to L_1 \leftarrow L_2 \to \dots \leftarrow L_{m-1} \to L_m = \bar{L}$$

such that each homomorphism induces a cohomology isomorphism. A $\mathfrak{g}$-augmented differential graded Lie algebra is *formal* if it is quasi-isomorphic to its cohomology.

For any differential graded Lie algebra L let $\mathcal{Q}_L$ denote the quadratic cone consisting of all $u \in L^1$ such that $[u, u] = 0$. The main abstract result relating formal differential graded Lie algebras to quadratic cones is the following.

3.5. *Theorem. — Suppose* (L, d, ε) *is a formal* $\mathfrak{g}$*-augmented differential graded Lie algebra. Suppose that the augmentation* $\varepsilon : L^0 \to \mathfrak{g}$ *is surjective and its restriction to* $H^0(L) \subset L^0$ *is injective. Let* $L' = \operatorname{Ker} \varepsilon$ *be the augmentation ideal. Then the analytic germ of the quadratic cone* $\mathcal{Q}_{H(L)} \times \mathfrak{g}/\varepsilon(H^0(L))$ *pro-represents the functor*

$$A \mapsto \operatorname{Iso} \mathscr{C}(L'; A).$$

3.6. *Corollary. — Suppose that* (L, d) *is a formal differential graded Lie algebra with* $H^0(L) = 0$. *Then the analytic germ of* $\mathcal{Q}_{H(L)}$ *pro-represents the functor*

$$A \mapsto \operatorname{Iso} \mathscr{C}(L; A).$$

Proof of 3.6. — Apply 3.5 with $\mathfrak{g} = 0$.

3.7. The proof of 3.5 involves a simple general construction with transformation groupoids. Let (X, G) be a transformation groupoid and let Y be a set upon which G acts. Then the transformation groupoid $(X \times Y, G)$, where G acts on $X \times Y$ by the diagonal action is a new groupoid which we denote by $(X, G) \bowtie Y$. If $\varphi : (X', G') \to (X, G)$ is a morphism of transformation groupoids and Y is a G-set, then there is a corresponding morphism of transformation groupoids

$$\varphi \bowtie Y : (X', G') \bowtie Y \to (X, G) \bowtie Y$$

where the G'-action on Y is induced from the G-action on Y by the homomorphism $\varphi : G' \to G$.

3.8. *Lemma. — If* $\varphi : (X', G') \to (X, G)$ *is an equivalence of groupoids, then* $\varphi \bowtie Y : (X', G') \bowtie Y \to (X, G) \bowtie Y$ *is also an equivalence of groupoids.*

Proof. — We show that $\varphi \bowtie Y$ satisfies the three basic properties of an equivalence of categories.

Surjective on isomorphism classes. — Let $(x, y) \in X \times Y = \operatorname{Obj}(X, G) \bowtie Y$. Since $\varphi_* : \operatorname{Iso}(X', G') \to \operatorname{Iso}(X, G)$ is surjective, there exists $x' \in X'$, $g \in G$ such that $g\varphi(x') = x$. Thus $g.(\varphi \bowtie Y)\,(x', g^{-1}y) = (x, y)$ as desired.

Full. — Suppose that $g : (\varphi(x'_1), y_1) \to (\varphi(x'_2), y_2)$. We show that there exists $g' : (x'_1, y_1) \to (x'_2, y_2)$ with $\varphi(g') = g$. As $g\varphi(x'_1) = \varphi(x'_2)$ and

$$\varphi_* : \mathrm{Hom}(x'_1, x'_2) \to \mathrm{Hom}(\varphi(x'_1), \varphi(x'_2))$$

is surjective, it follows that there exists $g' \in \mathrm{G}'$ with $\varphi(g') = g$. Since $gv_1 = y_2$, it follows that $g' : (x'_1, y_1) \to (x'_2, y_2)$ as claimed.

Faithful. — If $g'_1, g'_2 : (x'_1, y_1) \to (x'_2, y_2)$ and $\varphi(g'_1) = \varphi(g'_2)$, then $g'_1 = g'_2$ since $\varphi : \mathrm{Hom}(x'_1, x'_2) \to \mathrm{Hom}(\varphi(x'_1), \varphi(x'_2))$ is injective. This completes the proof of Lemma 3.8.

3.9. We apply this construction to the groupoid $\mathscr{C}(\mathrm{L}; \mathrm{A})$ as follows. The augmentation $\varepsilon : \mathrm{L}^0 \to \mathfrak{g}$ determines a group homomorphism $\varepsilon : \exp(\mathrm{L}^0 \otimes \mathfrak{m}) \to \exp(\mathfrak{g} \otimes \mathfrak{m}) = \mathrm{G}^0_{\mathrm{A}}$ and hence an action of $\exp(\mathrm{L}^0 \otimes \mathfrak{m})$ on $\mathrm{G}^0_{\mathrm{A}}$ by left-multiplication. The construction of 3.7 defines a new transformation groupoid $\mathscr{C}(\mathrm{L}; \mathrm{A}) \bowtie \mathrm{G}^0_{\mathrm{A}}$, depending naturally on A. There is a transformation groupoid homomorphism

$$\varphi : \mathscr{C}(\mathrm{L}'; \mathrm{A}) \to \mathrm{C}(\mathrm{L}; \mathrm{A}) \bowtie \mathrm{G}^0_{\mathrm{A}}$$

defined by the inclusion

$$\mathrm{Obj}\, \mathscr{C}(\mathrm{L}'; \mathrm{A}) \to \mathrm{Obj}\, \mathscr{C}(\mathrm{L}; \mathrm{A}) \times \mathrm{G}^0_{\mathrm{A}}$$
$$\omega \to (\omega, 1)$$

on objects and the inclusion $\exp(\mathrm{L}'^0 \otimes \mathfrak{m}) \hookrightarrow \exp(\mathrm{L}^0 \otimes \mathfrak{m})$ on morphisms.

Lemma. — Suppose that $\varepsilon : \mathrm{L}^0 \to \mathfrak{g}$ *is surjective. Then*

$$\varphi : \mathscr{C}(\mathrm{L}'; \mathrm{A}) \to \mathscr{C}(\mathrm{L}; \mathrm{A}) \bowtie \mathrm{G}^0_{\mathrm{A}}$$

is an equivalence of groupoids.

Proof. — We check that φ satisfies the three basic properties of an equivalence.

Surjective on isomorphism classes. — Let $(\omega, \exp \mathrm{X}) \in \mathrm{Obj}\, \mathscr{C}(\mathrm{L}; \mathrm{A}) \bowtie \mathrm{G}^0_{\mathrm{A}}$. Since ε is surjective, there exists $\widetilde{\mathrm{X}} \in \mathrm{L}^0 \otimes \mathfrak{m}$ with $\varepsilon(\widetilde{\mathrm{X}}) = \mathrm{X}$. Then

$$\exp(\widetilde{\mathrm{X}}) : \varphi(\exp(-\widetilde{\mathrm{X}}) . \omega) \mapsto (\omega, \exp \mathrm{X})$$

as desired.

Full. — Let $\omega_1, \omega_2 \in \mathrm{Obj}\, \mathscr{C}(\mathrm{L}'; \mathrm{A})$. Suppose that $\lambda \in \mathrm{L}^0 \otimes \mathfrak{m}$ defines a morphism $\exp(\lambda) : \varphi(\omega_1) \to \varphi(\omega_2)$. Since $\exp(\lambda) . (\omega_1, 1) = (\omega_2, 1)$ it follows that $\varepsilon(\lambda) = 0$, i.e. $\lambda \in (\mathrm{L}')^0$. Thus $\exp(\lambda)$ defines a morphism $\omega_1 \to \omega_2$ in $\mathscr{C}(\mathrm{L}'; \mathrm{A})$ which maps under φ to $\exp(\lambda) : \varphi(\omega_1) \to \varphi(\omega_2)$ as claimed.

Faithful. — That φ maps morphisms injectively is immediate from the definition. The proof of 3.9 is complete.

3.10. *Lemma.* — *Suppose that* (L, d, ε) *is a* $\mathfrak{g}$*-augmented differential graded Lie algebra with* $d = 0$ *and that* $\varepsilon : L^0 \to \mathfrak{g}$ *is injective. Then the analytic germ of* $\mathcal{Q}_L \times \mathfrak{g}/\varepsilon(L^0)$ *pro-represents the functor*

$$A \mapsto \mathrm{Iso}(\mathscr{C}(L; A) \bowtie G^0_A).$$

Proof. — The analytic germ of the vector space $\mathfrak{g}/\varepsilon(L^0)$ at 0 pro-represents the functor

$$A \mapsto \frac{\exp(\mathfrak{g} \otimes \mathfrak{m})}{\exp(\varepsilon(L^0) \otimes \mathfrak{m})}.$$

Since $\exp(L^0 \otimes \mathfrak{m})$ acts freely on G^0_A by left-multiplication it follows that there is a natural isomorphism between the sets

$$\mathrm{Iso}(\mathscr{C}(L; A) \bowtie G^0_A) = \frac{\mathrm{Obj}\,\mathscr{C}(L; A) \times \exp(\mathfrak{g} \otimes \mathfrak{m})}{\exp(L^0 \otimes \mathfrak{m})}$$

and

$$\mathrm{Obj}\,\mathscr{C}(L; A) \times \frac{\exp(\mathfrak{g} \otimes \mathfrak{m})}{\exp(\varepsilon(L^0) \otimes \mathfrak{m})}.$$

Now $\mathrm{Obj}\,\mathscr{C}(L; A) = \{ u \in L^1 \otimes \mathfrak{m} \mid [u, u] = 0 \}$ is the set of A-points of $\mathcal{Q}_L$ over the origin $0 \in L^1 \otimes \mathfrak{m}$ and therefore the analytic germ of $\mathcal{Q}_L$ at 0 pro-represents the functor

$$A \mapsto \mathrm{Obj}\,\mathscr{C}(L; A).$$

Thus the analytic germ of $\mathcal{Q}_L \times \mathfrak{g}/\varepsilon(L^0)$ pro-represents

$$A \mapsto \mathrm{Obj}\,\mathscr{C}(L; A) \times \frac{\exp(\mathfrak{g} \otimes \mathfrak{m})}{\exp(\varepsilon(L^0) \otimes \mathfrak{m})}$$

and hence also

$$A \mapsto \mathrm{Iso}(\mathscr{C}(L; A) \bowtie G^0_A)$$

as desired.

Proof of 3.5. — Let (L, d, ε) be a $\mathfrak{g}$-augmented differential graded Lie algebra satisfying the hypotheses of 3.5. Since (L, d, ε) is formal, there exists a quasi-isomorphism of $\mathfrak{g}$-augmented differential graded Lie algebras from L to its cohomology H(L)

$$L = L_0 \to L_1 \leftarrow L_2 \to \dots \leftarrow L_{m-1} \to L_m = H(L) \tag{3-1}$$

and a corresponding sequence of groupoid homomorphisms

$$\mathscr{C}(L; A) \to \mathscr{C}(L_1; A) \leftarrow \mathscr{C}(L_2; A) \to \dots \leftarrow \mathscr{C}(L_{m-1}; A) \to \mathscr{C}(H(L); A).$$

By 2.4 each of these groupoid homomorphisms is an equivalence and depends naturally on A. Since (3-1) consists of homomorphisms of $\mathfrak{g}$-*augmented* differential graded Lie algebra homomorphisms, one can form a new sequence of groupoid homomorphisms

$$\begin{aligned}\mathscr{C}(L; A) \bowtie G^0_A \to \mathscr{C}(L_1; A) \bowtie G^0_A \leftarrow \mathscr{C}(L_2; A) \bowtie G^0_A \\ \to \dots \leftarrow \mathscr{C}(L_{m-1}; A) \bowtie G^0_A \\ \to \mathscr{C}(H(L); A) \bowtie G^0_A\end{aligned}$$

which according to Lemma 3.8 consists of equivalences. Combining this sequence with the groupoid homomorphism φ we obtain by Lemma 3.9 a sequence of equivalences

$$\mathscr{C}(L'; A) \xrightarrow{\varphi} \mathscr{C}(L; A) \bowtie G^0_A \to \dots \to \mathscr{C}(H(L); A) \bowtie G^0_A$$

which induces a natural isomorphism of sets

$$\mathrm{Iso}\, \mathscr{C}(L'; A) \leftrightarrow \mathrm{Iso}(\mathscr{C}(H(L); A) \bowtie G^0_A).$$

Applying Lemma 3.10 the functor

$$A \mapsto \mathrm{Iso}\, \mathscr{C}(L'; A)$$

is pro-represented by the analytic germ of the quadratic cone $\mathscr{Q}_{H(L)} \times \mathfrak{g}/\varepsilon(H^0(L))$ at 0, as claimed. The proof of Theorem 3.5 is complete.

4. Representations of the fundamental group

4.1. In this section we focus on the primary object of interest in this paper: the variety of representations of a finitely generated group in an algebraic Lie group. Let $\overline{G}$ be an algebraic group defined over $\mathbf{k}$ and let $G = \overline{G}(\mathbf{k})$ denote its group of $\mathbf{k}$-points, with its natural structure as a Lie group. Then the set $\mathrm{Hom}(\Gamma, G)$ of homomorphisms $\Gamma \to G$ has the natural structure of the set of $\mathbf{k}$-points of an algebraic variety $\mathfrak{R}(\Gamma, G)$ defined over $\mathbf{k}$. Composition of a homomorphism $\Gamma \to G$ with an inner automorphism $G \to G$ defines an algebraic action of G on $\mathfrak{R}(\Gamma, G)$. We shall denote the corresponding transformation groupoid $(\mathrm{Hom}(\Gamma, G), G)$ by $\mathscr{R}(\Gamma, G)$. For more details concerning the variety $\mathfrak{R}(\Gamma, G)$ the reader is referred to [G], [JM], [LM], [MS].

4.2. We shall be interested in the local structure of $\mathfrak{R}(\Gamma, G)$ near a representation ρ_0. To this end we consider " infinitesimal deformations " of ρ_0, i.e. representations " parametrized " by Spec(A) where A is a fixed Artin local $\mathbf{k}$-algebra A. As always we assume that A has unity and its residue field is isomorphic to $\mathbf{k}$. Let

$$\mathbf{k} \xrightarrow{i} A \xrightarrow{q} \mathbf{k} \tag{4-1}$$

denote the corresponding $\mathbf{k}$-algebra homomorphisms. For a $\mathbf{k}$-algebraic group $\overline{G}$, there is an associated group $\overline{G}(A)$ of A-points, for any $\mathbf{k}$-algebra A. When A is a finite dimensional $\mathbf{k}$-algebra, then $\overline{G}(A)$ itself has the structure of the group of $\mathbf{k}$-points of another $\mathbf{k}$-algebraic group $\overline{G}_A$; in particular the group $\overline{G}(A) = \overline{G}_A(\mathbf{k})$ is a Lie group which we denote by G_A.

The structure of the group G_A can be understood as follows. The Lie algebra $\mathfrak{g}_A$ of G_A in easily seen to be the tensor product $\mathfrak{g} \otimes A$, where the Lie bracket is defined by the formula

$$[X \otimes a, Y \otimes b] = [X, Y] \otimes ab$$

and tensor product with $\mathfrak{g}$ gives a sequence of homomorphisms of Lie algebras

$$\mathfrak{g} \xrightarrow{i} \mathfrak{g}_A \xrightarrow{q} \mathfrak{g}.$$

The kernel of the homomorphism $q : \mathfrak{g}_A \to \mathfrak{g}$ equals $\mathfrak{g} \otimes \mathfrak{m}$ and one sees that $\mathfrak{g}_A$ is the semidirect product of $\mathfrak{g}$ with the ideal $\mathfrak{g} \otimes \mathfrak{m}$. Since A is Artinian, $\mathfrak{m}^N = 0$ for $N \gg 0$ and $\mathfrak{g} \otimes \mathfrak{m}$ is a nilpotent Lie algebra.

There is a corresponding sequence of Lie groups

$$G \xrightarrow{i} G_A \xrightarrow{q} G \tag{4-2}$$

whose composition is the identity map $G \to G$. The kernel of $q : G_A \to G$ is the unipotent group $G_A^0 = \exp(\mathfrak{g} \otimes \mathfrak{m})$. Moreover G_A is the semidirect product of G with the nilpotent normal subgroup $G_A^0 \lhd G_A$.

Let $\mathfrak{R}(\Gamma, G)$ be as above. Then its set of A-points

$$\mathfrak{R}(\Gamma, G)\,(A)$$

equals the set $\mathrm{Hom}(\Gamma, G_A)$ of all homomorphisms $\Gamma \to G_A$. (Compare [LM], § 1.) Composition with the homomorphism $q : G_A \to G$ defines a map $q_* : \mathrm{Hom}(\Gamma, G_A) \to \mathrm{Hom}(\Gamma, G)$ which is equivariant with respect to the actions by inner automorphisms via the homomorphism $q : G_A \to G$. The pair (q_*, q) thus defines a functor between the corresponding groupoids

$$\mathscr{R}(\Gamma, G_A) \to \mathscr{R}(\Gamma, G).$$

Let $\rho_0 \in \mathrm{Hom}(\Gamma, G)$ and A an Artin local $\mathbf{k}$-algebra. We define a transformation groupoid $\mathscr{R}_A(\rho_0)$ which captures the infinitesimal deformations of ρ_0. The objects of $\mathscr{R}_A(\rho_0)$ will be $\rho \in \mathrm{Hom}(\Gamma, G_A)$ such that $q_*\, \rho = \rho_0$; clearly the group $G_A^0 = \exp(\mathfrak{g} \otimes \mathfrak{m})$ preserves $\mathrm{Obj}\, \mathscr{R}_A(\rho_0)$. We define $\mathscr{R}_A(\rho_0)$ to be the corresponding groupoid.

Applying 3.1 to the analytic local $\mathbf{k}$-algebra of $\mathfrak{R}(\Gamma, G)$ at ρ_0 we obtain the following:

4.3. *Theorem. — Let* (X, x) *be a germ of an analytic variety. Then* (X, x) *pro-represents the functor*

$$A \mapsto \mathrm{Obj}\, \mathscr{R}_A(\rho_0) = \{\, \rho \in \mathrm{Hom}(\Gamma, G_A) \mid q_*(\rho) = \rho_0 \,\}$$

if and only if the analytic germ of $\mathfrak{R}(\Gamma, G)$ *at* ρ_0 *is analytically equivalent to* (X, x).

4.4. We shall describe the Zariski tangent space and the tangent quadratic cone to $\mathfrak{R}(\Gamma, G)$. Let A_n denote the truncated polynomial ring $\mathbf{k}[t]/(t^{n+1})$ so that $A_0 = \mathbf{k}$ and A_1 equals the ring of dual numbers. For $i \geqslant j$, let $q^i_j : A_i \to A_j$ denote the quotient projection. If V is an $\mathbf{k}$-variety we denote the corresponding map of A-points by $q^i_j : V(A_i) \to V(A_j)$. Let $\rho_0 : \Gamma \to G$ be a homomorphism. Then the Zariski tangent space at ρ_0 equals the set of A_1-points of $\mathfrak{R}(\Gamma, G)$, which map under q^1_0 to ρ_0, i.e. the fiber over $\rho_0 \in \mathrm{Hom}(\Gamma, G)$ of the map $q^1_0 : \mathrm{Hom}(\Gamma, G_{A_1}) \to \mathrm{Hom}(\Gamma, G_{A_0})$. For any Artin local $\mathbf{k}$-algebra A, let $\mathfrak{m} \subset A$ denote its maximal ideal; then we have a semidirect product decomposition of the group $G_A \cong G(A)$ of A-points of G as

$$G_A = \exp(\mathfrak{g} \otimes \mathfrak{m})\,.\,G.$$

Let $A = A_1$ with maximal ideal $\mathfrak{m}$. If $\rho \in \operatorname{Hom}(\Gamma, G_A)$ satisfies $q \circ \rho = \rho_0$, then we shall write (uniquely)

$$\rho(\gamma) = \exp(X(\gamma)) \cdot \rho_0(\gamma)$$

where $X : \Gamma \to \mathfrak{g} \otimes \mathfrak{m}$ is given by

$$X(\gamma) = u(\gamma) \otimes t$$

where $u \in Z^1(\Gamma, \mathfrak{g}_{\operatorname{Ad}\rho_0})$ is a 1-cocycle. (The condition that ρ is a homomorphism o groups translates into the cocycle condition on $u : \Gamma \to \mathfrak{g}$

$$\delta u(\alpha, \beta) = u(\alpha) - u(\alpha\beta) + \operatorname{Ad}\rho_0(\alpha)\, u(\beta) = 0$$

for any $\alpha, \beta \in \Gamma$.) In this way the Zariski tangent space $T_{\rho_0} \mathfrak{R}(\Gamma, G) \subset \mathfrak{R}(\Gamma, G)(A_1)$ is identified as the vector space $Z^1(\Gamma, \mathfrak{g}_{\operatorname{Ad}\rho_0})$. (Compare Lubotzky-Magid [LM, Proposition 2.2].)

In a similar way we determine the tangent quadratic cone to $\mathfrak{R}(\Gamma, G)$ at ρ_0. Let $A = A_2$ with maximal ideal $\mathfrak{m}$. If $X \in \mathfrak{g} \otimes \mathfrak{m}$, we write $X = X_1 \otimes t + X_2 \otimes t^2$. We have the Campbell-Hausdorff formula for the group $\exp(\mathfrak{g} \otimes \mathfrak{m})$:

$$\begin{aligned}\log(\exp X \exp Y) &= X + Y + \frac{1}{2}[X, Y] \\ &= (X_1 + Y_1) \otimes t + \left(X_2 + Y_2 + \frac{1}{2}[X_1, Y_1]\right) \otimes t^2\end{aligned}$$

valid for $X, Y \in \mathfrak{g} \otimes \mathfrak{m}$. Suppose that ρ is an A-point of $\mathfrak{R}(\Gamma, G)$ which maps to ρ_0. Writing

$$\rho(\gamma) = \exp(u_1(\gamma) \otimes t + u_2(\gamma) \otimes t^2) \cdot \rho_0(\gamma)$$

the condition that ρ is a homomorphism translates into two conditions:

(4-3) $$\delta u_1 = 0$$

(4-4) $$\delta u_2 = -\frac{1}{2}[u_1, u_1]$$

where the bracket pairing

$$[\ ,\ ] : Z^1(\Gamma, \mathfrak{g}_{\operatorname{Ad}\rho_0}) \times Z^1(\Gamma, \mathfrak{g}_{\operatorname{Ad}\rho_0}) \to Z^2(\Gamma, \mathfrak{g}_{\operatorname{Ad}\rho_0})$$

is defined by

$$[u, v](\alpha, \beta) = [u(\alpha), \operatorname{Ad}\rho_0(\alpha)\, v(\beta)].$$

Condition (4-3) asserts that $q_1^2(X) = u_1 \otimes t$ corresponds to a cocycle $u_1 \in Z^1(\Gamma, \mathfrak{g}_{\operatorname{Ad}\rho_0})$ and condition (4-4) asserts that the cohomology class of the bracket square $[u_1, u_1]$ is zero in $H^2(\Gamma, \mathfrak{g}_{\operatorname{Ad}\rho_0})$. Thus we obtain the following:

Proposition. — The tangent quadratic cone to $\mathfrak{R}(\Gamma, G)$ *at* ρ_0 *equals the set* $\mathcal{Q}$ *of all* $u \in Z^1(\Gamma, \mathfrak{g}_{\operatorname{Ad}\rho_0})$ *such that* $[u, u]$ *is zero in* $H^2(\Gamma, \mathfrak{g}_{\operatorname{Ad}\rho_0})$. *In particular* $\mathfrak{R}(\Gamma, G)$ *is quadratic at* ρ_0 *if and only if the analytic germ of* $\mathfrak{R}(\Gamma, G)$ *at* ρ_0 *is equivalent to the analytic germ of* $\mathcal{Q}$ *at* 0.

4.5. We shall recast the deformation theory of homomorphisms in the context of differential graded Lie algebras by interpreting a representation $\Gamma \to G$ geometrically as the holonomy of a flat connection on a principal G-bundle over a manifold M with fundamental group Γ. We realize Γ explicitly as the group of covering transformations of a fixed universal covering space $\widetilde{M} \to M$. Let $\rho : \Gamma \to G$ be a representation. Then one may form a (right) principal G-bundle over M as follows. Consider the trivial principal G-bundle $\widetilde{M} \times G$ over $\widetilde{M}$ with G acting by right multiplications on the fibers of $M \times G \to M$. Then

$$\gamma : (\tilde{x}, g) \mapsto (\gamma\tilde{x}, \rho(\gamma)\, g) \tag{4-5}$$

defines an action of Γ on $\widetilde{M} \times G$ over $\widetilde{M}$ by principal bundle automorphisms. Since Γ acts properly discontinuously and freely on $\widetilde{M}$ the action defined by (4-5) is also properly discontinuous and free; we denote the quotient $(\widetilde{M} \times G)/\Gamma$ by P_ρ. Then P_ρ is the total space of a principal G-bundle over M associated to ρ.

Proposition. — Let $\mathbf{P}(M; G)$ *denote the set of isomorphism classes of principal* G*-bundles over* M *and let* $\mathrm{Hom}(\Gamma, G)$ *be the set of homomorphisms* $\Gamma \to G$ *given the classical topology as a* $\mathbf{k}$*-algebraic set. Then the map*

$$\mathscr{P} : \mathrm{Hom}(\Gamma, G) \to \mathbf{P}(M; G)$$

which associates to $\rho \in \mathrm{Hom}(\Gamma, G)$ *the isomorphism class of* P_ρ *is continuous, where* $\mathbf{P}(M; G)$ *is given the discrete topology.*

Remark. — This map is not continuous in the Zariski topology, however. In [G2] it is shown that if M is a closed surface with $\chi(M) < 0$ and $G = SL(2, \mathbf{R})$, then $\mathrm{Hom}(\Gamma, G)$ is connected in the Zariski topology although many different isomorphism classes of principal G-bundles may arise.

Proof. — Let $\rho_0 \in \mathrm{Hom}(\Gamma, G)$. Since $\mathrm{Hom}(\Gamma, G)$ is a real algebraic set, it is locally contractible. Choose a contractible neighborhood U of ρ_0 in $\mathrm{Hom}(\Gamma, G)$ and form a principal G-bundle P_U over $U \times M$ as follows. Let P_U be the quotient of $U \times M \times G$ by the action of Γ defined by

$$\gamma : (\rho, \tilde{x}, g) \mapsto (\rho, \gamma\tilde{x}, \rho(\gamma)\, g).$$

The projection $U \times \widetilde{M} \times G \to U \times \widetilde{M}$ defines a principal bundle $\pi : P_U \to U \times M$ such that the pullback $\iota_\rho^* P_U$ equals P_ρ where $\iota_\rho : M \to U \times M$ denotes the inclusion $x \mapsto (\rho, x)$. Since U is contractible the covering homotopy property ([St, p. 53]) implies that $P_\rho \cong P_{\rho_0}$ for any $\rho \in U$. It follows that $\mathscr{P}$ is locally constant, i.e. continuous, hence the Proposition.

4.6. In § 5.9 homomorphisms $\Gamma \to G$ will be related to flat connections on P via the *holonomy correspondence*, which is inverse to the construction in 4.5. Proposition 4.4 implies that the local structure of $\mathfrak{R}(\Gamma, G)$ near ρ_0 can be understood in terms of flat connections on a *fixed* principal bundle P. Strictly speaking, however, we do not need 4.5 at all (since we never explicitly use the classical topology on $\mathrm{Hom}(\Gamma, G)$). In its place we may use the " infinitesimal analogue " which is based on the following elementary fact:

Lemma. — *For every Artin local* $\mathbf{k}$*-algebra* A, *the maps* $i_* : \mathbf{P}(M; G) \to \mathbf{P}(M; G_A)$ *and* $q_* : \mathbf{P}(M; G_A) \to \mathbf{P}(M; G)$ *induced by the group homomorphisms* $i : G \to G_A$ *and* $q : G_A \to G$ *are isomorphisms of sets.*

Proof. — Since $\mathrm{Coker}\, i = \mathrm{Ker}\, q = G_A^0$ is nilpotent and hence contractible, the homomorphisms $G \xrightarrow{i} G_A \xrightarrow{q} G$ are homotopy equivalences; the result follows by a standard argument [St, § 12.6-7].

5. Connections on principal bundles

5.1. In this section we summarize the basic facts we need concerning connections, bundle automorphisms, curvature and holonomy. We shall mainly follow Kobayashi-Nomizu [KN] for notational conventions, etc. The reader is also referred to Atiyah-Bott [AB], Greub-Halperin-Vanstone [GHV] and Chern [Ch] although the reader is warned that there are several different conventions which are commonly used. If M is a smooth manifold, then the graded algebra of all exterior differential forms on M will be denoted $\Omega^*(M)$. If $f : M \to V$ is a smooth map where V is a vector space, we shall identify the derivative of f as a V-valued 1-form on M, i.e. a linear map $TM \to V$ using the translations in V to identify each tangent space $T_v V$ with V itself.

Now let G be a (nonabelian) Lie group. The Lie algebra $\mathfrak{g}$ of G consists of the left-invariant vector fields on G. If M is a smooth manifold and $f : M \to G$ is a smooth map, we shall use the following (nonstandard) notation to denote its derivative. For each $x \in M$ and $\xi \in T_x(M)$ the derivative $df(\xi) \in T_{f(x)} G$ extends to a unique left-invariant vector field, which we denote $\mathscr{D}f(x)\,(\xi)$. As such $\mathscr{D}f$ is a linear map $TM \to \mathfrak{g}$ which we regard as a $\mathfrak{g}$-valued 1-form on M, i.e. an element of $\Omega^1(M) \otimes \mathfrak{g}$.

Suppose that $G = GL(n; \mathbf{k})$ and let $f : M \to G$ be a smooth function. We may conveniently confuse f with its composition with the inclusion of $GL(n; \mathbf{k})$ in the vector space $M_n(\mathbf{k})$ of all $n \times n$ matrices with entries from $\mathbf{k}$. For each $x \in M$, the derivative $df(x) \in \Omega^1(M) \otimes M_n(\mathbf{k})$ and $\mathscr{D}f$ is given by matrix multiplication

$$\mathscr{D}f(x) = f(x)^{-1}\, df(x).$$

Thus the notation $\mathscr{D}f = f^{-1}\, df$ is commonly in use for matrix groups, but our present notation is chosen to avoid reference to a specific linear embedding.

Let M denote a connected smooth manifold and $\pi : P \to M$ a fixed principal G-bundle over M. We shall let G act on P on the *right*; the action of $g \in G$ will be denoted $R_g : p \mapsto p.g$. The infinitesimal generators of this right action are the *fundamental vector* *fields* on P. If $p \in P$, then we denote by $\sigma_p : \mathfrak{g} \to T_p(P)$ the map which associates to an element $X \in \mathfrak{g}$ the value of the corresponding fundamental vector field $\sigma(X)$ at p, i.e.

$$\sigma_p(X) = \frac{d}{dt}\bigg|_{t=0} p.\exp(tX).$$

If $p \in P$ there is an exact sequence of vector spaces

$$0 \longrightarrow \mathfrak{g} \xrightarrow{\sigma_p} T_p(P) \xrightarrow{d\pi} T_{\pi(p)} M \longrightarrow 0$$

which defines a trivialization of the *vertical subbundle*

$$\operatorname{Ker} d\pi : TP \to \pi^* TM$$

as a G-invariant subbundle of TP with fiber $\mathfrak{g}$.

5.2. We shall let ad P denote the vector bundle over M with fiber $\mathfrak{g}$ associated to the principal bundle P, i.e. $\operatorname{ad} P = P \times_G \mathfrak{g}$ where G acts on $\mathfrak{g}$ by the adjoint representation. That is, the fiber of ad P over $x \in M$ equals the space of all $\xi : \pi^{-1}(x) \to \mathfrak{g}$ satisfying the identity

$$\xi \circ R_g = \operatorname{Ad}(g^{-1}) \circ \xi.$$

It is easy to see that if $p \in \pi^{-1}(x)$ then the correspondence $(\xi, p) \mapsto \xi(p)$ defines a trivialization $\pi^* \operatorname{ad} P \to \mathfrak{g}$ of the pullback $\pi^* \operatorname{ad} P$ over P. Thus the vertical subbundle of TP is the pullback of the $\mathfrak{g}$-bundle associated to P.

There is a natural graded Lie algebra of exterior differential forms on M which take values in the vector bundle ad P. We shall describe this algebra in terms of the principal bundle P thereby avoiding all reference to local coordinates. As in § 1.1 the space $\Omega^*(P) \otimes \mathfrak{g}$ of $\mathfrak{g}$-valued exterior differential forms on P is a graded Lie algebra. Let $\omega \in \Omega^q(P) \otimes \mathfrak{g}$ be a $\mathfrak{g}$-valued exterior differential form on P. We shall say that ω is *horizontal* is and only if for each $X \in \mathfrak{g}$, the interior product

$$\iota_{\sigma(X)}\, \omega = 0$$

i.e. $\omega(\xi_1, \ldots, \xi_q) = 0$ whenever one of $\xi_1, \ldots, \xi_q$ is vertical. We shall say that ω is *equivariant* if and only if

(5-1) $$R_g^* \omega = \operatorname{Ad}(g^{-1}) \circ \omega.$$

We shall thus *define* an ad P-valued exterior q-form on M to be a $\mathfrak{g}$-valued exterior q-form on P which is both horizontal and equivariant. (Compare [KN, p. 75].) We denote the space of such exterior forms by $\Omega^q(M; \operatorname{ad} P)$; clearly both equivariance and horizontality are preserved under the bracket operation, whence $\Omega^*(M; \operatorname{ad} P)$ is a graded Lie subalgebra of $\Omega^*(P) \otimes \mathfrak{g}$. (Although the exterior differential $d : \Omega^q(P) \otimes \mathfrak{g} \to \Omega^{q+1}(P) \otimes \mathfrak{g}$ preserves the equivariant forms, it will not preserve horizontality. Thus one will need a flat connection in P to make $\Omega^*(M; \operatorname{Ad} P)$ into a differential graded Lie algebra.)

5.3. A *gauge transformation* of P will be a bundle automorphism $F : P \to P$ which covers the identity $I_M : M \to M$. That is, for each $g \in G$, $p \in P$,

(5-2) $$F(p.g) = F(p).g$$

and $\pi \circ F = \pi$. Thus there exists a map $f : P \to G$ such that

(5-3) $$F(p) = p.f(p).$$

Condition (5-2) implies that f satisfies

(5-4) $$f(p.g) = g^{-1} f(p)\, g$$

from which it follows that the derivative $\mathscr{D}f \in \Omega^1(P) \otimes \mathfrak{g}$ satisfies

$$R_g^* \mathscr{D}f = \mathrm{Ad}(g^{-1}) \circ \mathscr{D}f$$

i.e. is an equivariant $\mathfrak{g}$-valued 1-form on P. In general, however, $\mathscr{D}f$ is not horizontal: its interior product on a fundamental vector field is given by the formula

$$\mathscr{D}f \circ \sigma_p(X) = X - \mathrm{Ad}(f(p)^{-1})\,(X)$$

which is easily established by taking the derivative of (5-4) with $g = \exp(tX)$.

5.4. We wish to compute the derivative of the gauge transformation $F : P \to P$ in terms of the equivariant map $f : P \to G$. To this end we let $R : P \times G \to P$ denote the action of G on P and rewrite (5-2) as $F = R \circ (I_P \times f)$. Then the differential of $F : P \to P$ at $p \in P$ is the composition of the linear maps

$$T_p\, P \xrightarrow{I \oplus df} T_p(P) \oplus T_{f(p)}(P) \xrightarrow{dR_{f(p)}} T_{F(p)}(P).$$

Now the differential of $R : P \times G \to P$ at $(p, f(p))$ equals the sum of the differential $dR_{f(p)} : T_p(P) \to T_{F(p)}(P)$ and the linear map $T_{f(p)}(G) \to T_{F(p)}(P)$ which associates to a tangent vector $\xi \in T_{f(p)}(G)$ the vector $\sigma_{F(p)}(X)$ where the left-invariant vector field X equals ξ at $f(p)$. It follows that the differential of $F : P \to P$ is given by

$$(dF)_p = dR_{f(p)} + \sigma_{F(p)} \circ \mathscr{D}f.$$

5.5. A *connection* on P is a $\mathfrak{g}$-valued 1-form $\omega \in \Omega^1(P) \otimes \mathfrak{g}$ on P satisfying the vertical condition

(5-5) $$\omega(\sigma(X)) = X$$

for each $X \in \mathfrak{g}$ and the equivariance condition (5-1). It is easy to show that this notion of a connection is equivalent to any of the other standard definitions of a connection, see e.g. Atiyah-Bott [AB, p. 547]. We denote the space of all connections on P by $\mathbf{A}(P)$.

It is easy to see from the above definition that the space of connections $\mathbf{A}(P)$ is an affine space with underlying vector space of translations $\Omega^1(M; \mathrm{ad}\, P)$: the difference between two connections is a horizontal equivariant $\mathfrak{g}$-valued 1-form on P, and hence an element of $\Omega^1(M; \mathrm{ad}\, P)$; adding to a connection a horizontal equivariant $\mathfrak{g}$-valued 1-form on P gives an equivariant $\mathfrak{g}$-valued 1-form on P which satisfies the vertical condition and hence is a connection.

We shall need the covariant differential operator d_ω associated with a connection ω, which is a derivation of $\Omega^*(M; \operatorname{ad} P)$ having degree 1. As observed above, the exterior derivative is a derivation of $\Omega^*(P) \otimes \mathfrak{g}$ which does not preserve horizontal forms: indeed it is easy to see that if $\eta \in \Omega^*(P) \otimes \mathfrak{g}$ is an equivariant horizontal form and $X \in \mathfrak{g}$ then

$$\iota_{\sigma(X)}\, d\eta = \mathscr{L}_{\sigma(X)}(\eta) = -\operatorname{ad}(X) \circ \eta.$$

Similarly the vertical condition (5-5) implies that the derivation $\operatorname{ad}\omega$ of $\Omega^*(P) \otimes \mathfrak{g}$ satisfies

$$\iota_{\sigma(X)}[\omega, \eta] = \operatorname{ad}(X) \circ \eta$$

whence the derivation $d + \operatorname{ad}\omega$ of $\Omega^*(P) \otimes \mathfrak{g}$ preserves the subspace of horizontal forms. Since both d and $\operatorname{ad}\omega$ preserve equivariant forms, $d + \operatorname{ad}\omega$ preserves equivariant forms and therefore defines a derivation $d_\omega : \Omega^q(M; \operatorname{ad} P) \to \Omega^{q+1}(M; \operatorname{ad} P)$ and hence a connection on the vector bundle ad P. (Compare Greub-Halperin-Vanstone [GHV, Vol. II, 6.13].)

5.6. Now we shall compute the action of gauge transformations on connections. Clearly the pullback of a connection by a gauge transformation is a connection. To this end, fix a connection $\omega \in \mathbf{A}(P)$; then an arbitrary connection on P can be uniquely written as $\omega + \eta$ where $\eta \in \Omega^1(M; \operatorname{ad} P)$. We shall show that the action of a gauge transformation $F : P \to P$ on $\mathbf{A}(P)$ is given by the formula

$$F^*(\omega + \eta) = \omega + \operatorname{Ad}(f^{-1}) \circ \eta + ((\operatorname{Ad}(f^{-1}) - I) \circ \omega + \mathscr{D}f). \tag{5-6}$$

Let $p \in P$. Denoting the value of a tensor field ω at p by ω_p we have the following:

$$\begin{aligned}
(F^*\omega)_p &= \omega_{F(p)} \circ (dF)_p \\
&= (\operatorname{Ad} f(p)^{-1} \circ \omega_p \circ (dR_{f(p)})^{-1}) \circ ((dR_{f(p)}) + \sigma_{F(p)} \circ \mathscr{D}f) \\
&= \operatorname{Ad} f(p)^{-1} \circ \{ \omega_p + \omega_p \circ \sigma_p \circ \operatorname{Ad} f(p) \circ \mathscr{D}f \} \\
&\qquad\qquad (\text{since } dR_g \circ \sigma = \sigma \circ \operatorname{Ad}(g^{-1})) \\
&= \operatorname{Ad} f(p)^{-1} \circ \omega_p + \operatorname{Ad} f(p)^{-1} \circ \omega_p \circ \sigma \circ \operatorname{Ad} f(p) \circ \mathscr{D}f \\
&= \operatorname{Ad} f(p)^{-1} \circ \omega_p + (\mathscr{D}f)_p.
\end{aligned}$$

Applying this calculation to the connection $\omega + \eta$ and subtracting off ω we obtain the following:

$$(F^*(\omega + \eta))_p = \omega_p + \operatorname{Ad} f(p)^{-1} \circ \eta_p + ((\operatorname{Ad} f(p)^{-1} - I) \circ \omega_p + \mathscr{D}f).$$

Thus the group of gauge transformations acts affinely on the space of connections; the linear part of the action is given by

$$\eta_p \mapsto \operatorname{Ad} f(p)^{-1} \circ \eta_p$$

and the translational part is equal to

$$(\operatorname{Ad} f(p)^{-1} - I) \circ \omega_p + \mathscr{D}f.$$

5.7. An *infinitesimal gauge transformation* is a vector field on P which infinitesimally generates a one-parameter group of gauge transformations. Such a one-parameter group $F_t : P \to P$ is given by maps $f_t : P \to G$ of the form $f_t(p) = \exp - t\lambda(p)$ where $\lambda : P \to \mathfrak{g}$ is equivariant. Thus we identify infinitesimal gauge transformations with the Lie algebra $\Omega^0(M; \mathrm{ad}\, P)$ of sections of ad P.

We now apply (5-6) to such a one-parameter group of gauge transformations and differentiate in order to see the action of a infinitesimal gauge transformation on $\mathbf{A}(P)$:

$$\begin{aligned}\frac{d}{dt}\bigg|_{t=0} F_t^*(\omega + \eta)_p &= [\lambda, \eta] - d\lambda - [\omega, \lambda] \\ &= - [\eta, \lambda] - d_\omega(\lambda) \\ &= - (d_\omega + \mathrm{ad}\, \eta)\, (\lambda).\end{aligned}$$

Thus an infinitesimal gauge transformation $\lambda \in \Omega^0(M; \mathrm{ad}\, P)$ determines an affine vector field $\rho(\lambda)$ on $\mathbf{A}(P)$ whose value at the connection $\omega + \eta \in \mathbf{A}(P)$ equals

$$- (d_\omega + \mathrm{ad}\, \eta)\, (\lambda) \in \Omega^1(M; \mathrm{ad}\, P) = T_{\omega+\eta}(\mathbf{A}(P)).$$

It follows that the action of the one-parameter group F_t corresponding to $\exp(- t\lambda)$ on the space of connections is given by

$$F_t^*(\omega + \eta) = \omega + \mathrm{Ad}\exp(t\lambda) \circ \eta + \frac{I - \exp(t\, \mathrm{ad}\, \lambda)}{\mathrm{ad}\, \lambda}\, (d_\omega\, \lambda). \tag{5-7}$$

5.8. Let $\omega \in \mathbf{A}(P)$ be a connection. The $\mathfrak{g}$-valued exterior 2-form on P defined by

$$K(\omega) = d\omega + \frac{1}{2}\, [\omega, \omega]$$

is clearly equivariant and because of the vertical condition (5-5) the Maurer-Cartan equations imply that $K(\omega)$ is horizontal. Thus $K(\omega)$, the *curvature* of ω, is an ad P-valued exterior 2-form on M. One checks that the derivation $d + \mathrm{ad}\, \omega$ of $\Omega^*(P) \otimes \mathfrak{g}$ satisfies $(d + \mathrm{ad}\, \omega) \circ (d + \mathrm{ad}\, \omega) = 0$ if and only if $\mathrm{ad}\, K(\omega) = 0$. The connection ω is said to be *flat* if and only if $K(\omega) = 0$; in that case $(\Omega^*(M; \mathrm{ad}\, P), d_\omega)$ is a differential graded Lie algebra.

We denote the space of all flat connections on P by $\mathbf{F}(P)$. If P is a principal G-bundle and $\omega \in \mathbf{F}(P)$, we refer to the pair (P, ω) as a *flat principal* G-*bundle.* If ω is a fixed flat connection, then an arbitrary connection $\omega + \eta \in \mathbf{A}(P)$ is flat precisely when

$$Q(\eta) = d_\omega(\eta) + \frac{1}{2}\, [\eta, \eta] = K(\omega + \eta) = 0.$$

As in § 1.3, the space $\mathbf{F}(P)$ is invariant under the affine action of the group $\mathbf{G}(P)$ of gauge transformations of P.

5.9. Choose a base-point $x \in M$. Suppose that ω is a connection in P. For any smooth path $\sigma : [0, 1] \to M$ starting at $\sigma(0) = x$ and ending at $\sigma(1) = y \in M$, there is a parallel transport mapping $T_\sigma : \pi^{-1}(x) \to \pi^{-1}(y)$ defined as follows. (Compare [KN, § II.3].) If $p \in \pi^{-1}(x)$, there exists a unique path $\tilde{\sigma}_p : [0, 1] \to P$ such that $\pi \circ \tilde{\sigma}_p = \sigma$, $\tilde{\sigma}_p(0) = p$ and $\tilde{\sigma}_p^* \omega = 0$. Define $T_\sigma(p) = \tilde{\sigma}_p(1)$. If ω is a flat connection, then T_σ depends only on the relative homotopy class of σ. Parallel transport has the following basic properties:

$$T_\sigma = I \qquad \text{when } \sigma \text{ is the constant path } x,$$
$$T_\sigma(p.g) = T_\sigma(p).g,$$
$$T_{\tau*\sigma} = T_\tau \circ T_\sigma \qquad \text{where } \tau \text{ is a path from } y \text{ to } z$$
$$\text{and } \tau * \sigma \text{ denotes the composite path from } x \text{ to } z.$$

Suppose that $\omega \in \mathbf{F}(P)$ is a flat connection on P. Let $\gamma : [0, 1] \to M$ be a loop in M based at x. Then the parallel transport operator $T_\gamma : \pi^{-1}(x) \to \pi^{-1}(x)$ depends only on the homotopy class of γ in the fundamental group $\Gamma = \pi_1(M, x)$. Thus there exists $\rho(\gamma) \in G$ such that $T_\gamma(p) = p.\rho(\gamma)$; it is easily verified that $\rho : \Gamma \to G$ is a homomorphism of groups, the ***holonomy representation*** of the flat connection ω at p. We shall write $\rho = \mathrm{hol}_p(\omega)$, whereby there results a map

$$\mathrm{hol}_p : \mathbf{F}(P) \to \mathrm{Hom}(\Gamma, G).$$

This map is equivariant with respect to certain natural group actions. Namely, the group $\mathbf{G}(P)$ of gauge transformations of P has a homomorphism $\varepsilon_p : \mathbf{G}(P) \to G$ defined by

$$F(p) = p.(\varepsilon_p(F))$$

and G acts on $\mathrm{Hom}(\Gamma, G)$ by composition with inner automorphisms of G. One can easily see that the holonomy map hol_p is equivariant respecting ε_p, i.e. for each $F \in \mathbf{G}(P)$ the diagram

$$\begin{array}{ccc} \mathbf{F}(P) & \xrightarrow{\mathrm{hol}_p} & \mathrm{Hom}(\Gamma, G) \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle \varepsilon_p(F)} \\ \mathbf{F}(P) & \xrightarrow{\mathrm{hol}_p} & \mathrm{Hom}(\Gamma, G) \end{array} \tag{5-8}$$

commutes.

5.10. We define a groupoid $\mathscr{F}(M, G)$ as follows. For each isomorphism class of principal G-bundle over M choose a representative principal G-bundle $\pi : P \to M$ over M and a base-point $p \in \pi^{-1}(x)$. We shall let $\mathscr{F}(M, G)$ be the disjoint union, over the set $\mathbf{P}(M; G)$ of isomorphism classes of principal G-bundles, of the groupoids $\mathscr{F}(P)$ arising from the action of the group $\mathbf{G}(P)$ of gauge transformations on the set of flat connections $\mathbf{F}(P) = \mathrm{Obj}\,\mathscr{F}(P)$. It follows from (5-8) that the disjoint union over the set $\mathbf{P}(M; G)$ of the $(\mathrm{hol}_p, \varepsilon_p) : \mathscr{F}(P) \to \mathscr{R}(\Gamma, G)$ defines a functor

$\mathrm{hol}: \mathscr{F}(\mathrm{M}; \mathrm{G}) \to \mathscr{R}(\Gamma, \mathrm{G})$ depending on the choices (P, p). The basic results concerning this functor are the following:

Theorem. — *The functor* $\mathrm{hol}: \mathscr{F}(\mathrm{M}; \mathrm{G}) \to \mathscr{R}(\Gamma, \mathrm{G})$ *is an equivalence of groupoids. In particular for any principal* G*-bundle* P *over* M *the holonomy correspondence* hol_p *defines a bijection between gauge equivalence classes of flat connections on* P *and conjugacy classes of representations of the fundamental group which induce* P.

Proof. — It suffices to show that every representation $\rho \in \mathrm{Hom}(\Gamma, \mathrm{G})$ arises as the holonomy of a flat connection on some principal bundle P, and for two flat principal bundles (P_1, ω_1) and (P_2, ω_2) the set of bundle isomorphisms $\mathrm{F}: (\mathrm{P}_1, \omega_1) \to (\mathrm{P}_2, \omega_2)$ is mapped bijectively under ε_p to the set of elements $g \in \mathrm{G}$ conjugating $\mathrm{hol}_p(\omega_1)$ to $\mathrm{hol}_p(\omega_2)$.

Surjective on isomorphism classes. — Let $\rho \in \mathrm{Hom}(\Gamma, \mathrm{G})$; we shall construct a flat connection ω_ρ on the principal bundle P_ρ constructed in 4.5. Let $\widetilde{\mathrm{M}}_x$ denote the set of relative homotopy classes of paths $\sigma: [0, 1] \to \mathrm{M}$ with $\sigma(0) = x$; the map $\widetilde{\mathrm{M}}_x \to \mathrm{M}$ defined by $\sigma \mapsto \sigma(1)$ is a universal covering space of M. If γ is a loop in M based at x, then the corresponding covering transformation $\gamma: \widetilde{\mathrm{M}} \to \widetilde{\mathrm{M}}$ is given by the composition of paths $\sigma \mapsto \sigma * \gamma^{-1}$. Let $\Pi: \widetilde{\mathrm{M}} \times \mathrm{G} \to \mathrm{G}$ denote projection. Then $\widetilde{\omega} = \mathscr{D}(\Pi)$ defines a flat connection on the trivial principal G-bundle $\widetilde{\mathrm{M}} \times \mathrm{G}$ over $\widetilde{\mathrm{M}}$, which is invariant under the action of G on $\widetilde{\mathrm{M}} \times \mathrm{G}$ by left multiplication. It follows that the Γ-action defined by (4-5) preserves the connection $\widetilde{\omega}$ and thus there is an induced flat connection ω_ρ on the principal G-bundle P_ρ defined in 4.5. Since parallel transport of $p = (x, g) \in \widetilde{\mathrm{M}} \times \mathrm{G}$ with respect to $\widetilde{\omega}$ along a path σ in $\widetilde{\mathrm{M}}$ with $\sigma(0) = x$ is the path $t \mapsto (\sigma(t), g)$ it follows that $\mathrm{hol}_p(\omega_\rho) = \rho$ as desired.

Full. — For $i = 1, 2$ let (P_i, ω_i) be flat principal G-bundles over M. Let $p_i \in \mathrm{P}_i$ be base-points covering the base-point $x \in \mathrm{M}$. Let $\rho_i = \mathrm{hol}_{p_i}(\omega_i) \in \mathrm{Hom}(\Gamma, \mathrm{G})$. We must show that if $\rho_2 = g_0 . \rho_1$ then there exists an isomorphism $\mathrm{F}: \mathrm{P}_1 \to \mathrm{P}_2$ of principal bundles such that $\mathrm{F}^* \omega_2 = \omega_1$ and $\mathrm{F}(p_1) . g_0 = p_2$.

To construct such a bundle isomorphism we proceed as follows. For any path σ from x to y let $\mathrm{T}^i_\sigma: \pi^{-1}(x) \to \pi^{-1}(y)$ be the parallel transport operator associated with ω_i. The map $\mu_i: \widetilde{\mathrm{M}}_x \times \mathrm{G} \to \mathrm{P}_i$ defined by

$$\mu_i(\sigma, g) = \mathrm{T}^i_\sigma(p) . g$$

expresses P_i as a quotient of $\widetilde{\mathrm{M}}_x \times \mathrm{G}$ by the action of Γ defined by

$$\gamma: (\sigma * \gamma, g) \mapsto (\sigma, \rho(\gamma)\, g).$$

Indeed, $\mu_i(\sigma, g) = \mu_i(\sigma', g')$ if and only if there exists a loop γ based at x such that $\sigma' = \sigma * \gamma$ and $g = \rho_i(\gamma)\, g'$. We define a map $\mathrm{F}: \mathrm{P}_1 \to \mathrm{P}_2$ covering the identity map $\mathrm{M} \to \mathrm{M}$ by the formula

$$\mathrm{F}(\mu_1(\sigma, g)) = \mu_2(\sigma, g_0\, g). \tag{5-9}$$

To show that F is well-defined, it suffices to show that if $\mu_1(\sigma, g) = \mu_1(\sigma', g')$ then $\mu_2(\sigma, g_0 g) = \mu_2(\sigma', g_0 g')$. So suppose that $\sigma' \simeq \sigma * \gamma$ and $g = \rho_1(\gamma) g'$. Then

$$\begin{aligned}\mu_2(\sigma, g_0 g) &= T^2_\sigma(p_2) . g_0 \rho_1(\gamma) g' = T^2_\sigma(p_2) . \rho_2(\gamma) g_0 g' \\ &= T^2_{\sigma * \gamma}(p_2) . g_0 g' = \mu_2(\sigma * \gamma, g_0 g') \\ &= \mu_2(\sigma', g_0 g')\end{aligned}$$

and we see that F is a well-defined map $P_1 \to P_2$.

We now check that F is a bundle isomorphism, i.e. that it satisfies (5-2). Let $g \in G$ and let $p \in P$ be arbitrary; there exists a path σ from x to $\pi_1(p_1)$ and $h \in G$ such that $p = \mu_1(\sigma, h)$. Now

$$\begin{aligned}F(p.g) &= F(\mu_1(\sigma, hg)) = \mu_2(\sigma, g_0 hg) \\ &= \mu_2(\sigma, g_0 h) . g = F(p) . g\end{aligned}$$

and F is an isomorphism of principal bundles. Clearly $F(p_1) = p_2 . g_0$.

It remains to check that $F^* \omega_2 = \omega_1$. Since the field $\operatorname{Ker} \omega_i \subset TP$ of horizontal subspaces defining the connection ω_i is determined by the parallel transport operators T^i, it suffices to show that F maps T^1 to T^2, i.e. for any path τ starting at an arbitrary point $y \in M$ that

$$F \circ T^1_\tau = T^2_\tau \circ F.$$

Let $p \in \pi_1^{-1}(y)$ and write $p = \mu_1(\sigma_1, g)$ as above. Then

$$\begin{aligned}F \circ T^1_\tau(p) &= F \circ T^1_\tau(T^1_\sigma(p_1) . g) = F \circ \mu_1(\tau * \sigma, g) \\ &= \mu_2(\tau * \sigma, g_0 g) = T^2_\tau \mu_2(\sigma, g_0 g) = T^2_\tau \circ F(p)\end{aligned}$$

and the result follows.

Faithful. — Suppose that $g_1, g_2 \in \operatorname{Hom}(\omega_1, \omega_2)$ are gauge transformations satisfying

$$\varepsilon_p(g_1) = \varepsilon_p(g_2).$$

Then $g_1^{-1} \circ g_2$ is an automorphism $\omega_1 \to \omega_1$ and $\varepsilon_p(g_1^{-1} \circ g_2) = 1$. By a standard argument (e.g. [GM1, Lemma 1.2]) this implies that $g_1^{-1} \circ g_2$ is parallel with respect to ω_1 and since M is connected $g_1 = g_2$.

This concludes the proof of Theorem 5.10.

5.11. Since our main results concern the space of homomorphisms $\operatorname{Hom}(\Gamma, G)$ and not equivalence classes, we do not consider the groupoid $\mathscr{F}(M; G)$ but rather a small modification of it. Namely let $\mathscr{F}'(M; G)$ denote the groupoid with the same set of objects as $\mathscr{F}(M; G)$ but whose morphisms are gauge transformations in the kernel of $\varepsilon_p : \mathbf{G}(P) \to G$. Let $\mathscr{R}'(\Gamma, G)$ be the transformation groupoid $(\operatorname{Hom}(\Gamma, G), 1)$ (only the identity morphisms). Proposition 5.10 immediately implies the following:

Corollary. — *The functor*

$$\mathrm{hol} : \mathscr{F}(M; G) \to \mathscr{R}(\Gamma, G)$$

induces an equivalence of groupoids

$$\mathrm{hol}: \mathscr{F}'(\mathrm{M}; \mathrm{G}) \to \mathscr{R}'(\Gamma, \mathrm{G}).$$

In other words, hol *induces an isomorphism of sets*

$$\mathrm{Iso}\, \mathscr{F}'(\mathrm{M}; \mathrm{G}) \to \mathrm{Hom}(\Gamma, \mathrm{G}).$$

5.12. For later use we record the following simple fact:

Lemma. — *The map* $\varepsilon_p : \Omega^0(\mathrm{M}; \mathrm{ad}\,\mathrm{P}) \to \mathfrak{g}$ *is surjective.*

Proof. — Let $\mathrm{X} \in \mathfrak{g}$. Choose a smooth function $\psi : \mathrm{M} \to \mathbf{R}$ such that $\psi(x) = 1$ and $\psi \equiv 0$ outside a coordinate neighborhood of $x \in \mathrm{M}$. Let $p' \in \mathrm{P}$ be an arbitrary point; there exists a path σ from x to $\pi(p')$ in M and $g \in \mathrm{G}$ such that $p' = \mathrm{T}_\sigma(p).g$. Then

$$\lambda(p') = \psi(\pi(p'))\, \mathrm{Ad}\, g^{-1}(\mathrm{X})$$

defines an element $\lambda \in \Omega^0(\mathrm{M}; \mathrm{ad}\,\mathrm{P})$ such that $\varepsilon_p(\lambda) = \mathrm{X}$. Hence the claim.

6. Infinitesimal deformations of flat connections

In this section we shall develop the theory of flat connections and their holonomy representations "parametrized" by the spectrum of a fixed Artin local **k**-algebra A.

6.1. Let M be a smooth manifold and $\pi : \mathrm{P} \to \mathrm{M}$ a principal G-bundle over M. Let $i : \mathrm{G} \to \mathrm{G}_\mathrm{A}$ be as in § 4 and $\mathrm{P}_\mathrm{A} = \mathrm{P} \times_\mathrm{G} \mathrm{G}_\mathrm{A}$ be the associated fiber product, i.e. the fiber of P_A over $x \in \mathrm{M}$ equals the set of all maps $\xi : \pi^{-1}(x) \to \mathrm{G}_\mathrm{A}$ such that

$$\xi(p.g) = i(g^{-1})\, \xi(p)$$

for $p \in \mathrm{P}$, $g \in \mathrm{G}$; then P_A has the natural structure as a principal G_A-bundle with right G_A-action defined by

$$\xi.g(p) = \xi(p)\, g$$

where $p \in \mathrm{P}$, $g \in \mathrm{G}_\mathrm{A}$. We shall denote the bundle projection by $\pi_\mathrm{A} : \mathrm{P}_\mathrm{A} \to \mathrm{M}$. Corresponding to the sequence (4-2) there is a sequence of maps

$$\mathrm{P} \xrightarrow{i} \mathrm{P}_\mathrm{A} \xrightarrow{q} \mathrm{P}$$

where $i : \mathrm{P} \to \mathrm{P}_\mathrm{A}$ is a G-equivariant embedding and $q : \mathrm{P}_\mathrm{A} \to \mathrm{P}$ is a principal fibration with structure group G^0_A.

The vector bundle $\mathrm{ad}\,\mathrm{P}_\mathrm{A}$ over M is simply the tensor product $\mathrm{ad}\,\mathrm{P} \otimes \mathrm{A}$ and the tensor product of (4-1) with the graded Lie algebra $\Omega^*(\mathrm{M}; \mathrm{ad}\,\mathrm{P})$ defines a sequence of homomorphisms of graded Lie algebras

$$\Omega^*(\mathrm{M}; \mathrm{ad}\,\mathrm{P}) \xrightarrow{i} \Omega^*(\mathrm{M}, \mathrm{ad}\,\mathrm{P}_\mathrm{A}) = \Omega^*(\mathrm{M}; \mathrm{ad}\,\mathrm{P}) \otimes \mathrm{A} \xrightarrow{q} \Omega(\mathrm{M}; \mathrm{ad}\,\mathrm{P}).$$

Let $\mathbf{G}(P)$ (resp. $\mathbf{G}(P_A)$) denote the group of all gauge transformations of P (resp. P_A). Corresponding to the homomorphisms (4-2) there are homomorphisms of the gauge groups

$$\text{(6-1)}\qquad \begin{array}{ccccc} \mathbf{G}(P) & \xrightarrow{i} & \mathbf{G}(P_A) & \xrightarrow{q} & \mathbf{G}(P) \\ {\scriptstyle \varepsilon_p}\downarrow & & {\scriptstyle \varepsilon_p}\downarrow & & {\scriptstyle \varepsilon_p}\downarrow \\ G & \xrightarrow[i]{} & G_A & \xrightarrow[q]{} & G \end{array}$$

The kernel of the homomorphism $q : \mathbf{G}(P_A) \to \mathbf{G}(P)$ equals the (infinite-dimensional) nilpotent Lie group $\exp(\Omega^0(M; \operatorname{ad} P) \otimes \mathfrak{m})$ and will be denoted $\mathbf{G}^0(P_A)$. In particular $\mathbf{G}(P_A)$ equals the semidirect product of $i(\mathbf{G}(P)) \cong \mathbf{G}(P)$ with the nilpotent normal subgroup $\mathbf{G}^0(P_A)$.

6.2. By 4.6 the homomorphisms $G \xrightarrow{i} G_A \xrightarrow{q} G$ induce bijections between isomorphism classes of principal G-bundles over M and principal G_A-bundles over M. Thus for each isomorphism class we choose a representative principal G-bundle P and a base-point $p \in P$ as well as a corresponding base-point $p \in P_A = i_*(P)$. Lemma 4.6 implies that every principal G_A-bundle arises (up to isomorphism) as such a P_A.

Suppose that $\tilde{\omega}$ is a connection on P_A. Consider the pullback $i^* \tilde{\omega} \in \Omega^1(P) \otimes \mathfrak{g}_A$ to P; its composition with $q : \mathfrak{g}_A \to \mathfrak{g}$

$$q \circ i^* \tilde{\omega} \in \Omega^1(P) \otimes \mathfrak{g}$$

is easily seen to be a connection in P, which we call the *restriction* of $\tilde{\omega}$ to P. Thus every connection on P_A determines a connection on P. (See [KN, § II.6.4] for a general discussion of restriction of connections to subbundles.) The curvature of this connection is easily seen to satisfy

$$K(q \circ i^* \tilde{\omega}) = qK(\tilde{\omega})$$

and it follows that if $\tilde{\omega}$ is a flat connection on P_A then its restriction $q \circ i^* \tilde{\omega}$ to P is also flat.

Composition with $q : G_A \to G$ defines a map $q_* : \operatorname{Hom}(\Gamma, G_A) \to \operatorname{Hom}(\Gamma, G)$. The following diagram

$$\begin{array}{ccc} \mathbf{F}(P_A) & \xrightarrow{q \circ i^*} & \mathbf{F}(P) \\ {\scriptstyle \mathrm{hol}_p}\downarrow & & \downarrow{\scriptstyle \mathrm{hol}_p} \\ \operatorname{Hom}(\Gamma, G_A) & \xrightarrow[q_*]{} & \operatorname{Hom}(\Gamma, G) \end{array}$$

commutes. Furthermore this commutative square is equivariant with respect to the right-hand square in (6-1).

Suppose that $\omega_0 \in \mathbf{F}(P)$ is a flat connection. We shall define a groupoid $\mathscr{F}_A(\omega_0)$ corresponding to the deformation theory of flat connections with formal infinitesimal parameters taken from A. Let

$$\mathbf{F}_A(\omega_0) = \{ \tilde{\omega} \in \mathbf{F}(P_A) \mid q \circ i^*(\tilde{\omega}) = \omega_0 \}$$

denote the set of flat connections on P_{A} which restrict to ω_0 on P; then clearly the kernel $\mathrm{G}^0(\mathrm{P}_{\mathrm{A}})$ of $q : \mathbf{G}(\mathrm{P}_{\mathrm{A}}) \to \mathbf{G}(\mathrm{P})$ acts on $\mathbf{F}_{\mathrm{A}}(\omega_0)$. We denote the corresponding groupoid by $\mathscr{F}_{\mathrm{A}}(\omega_0)$.

Let $\rho_0 \in \mathrm{Hom}(\Gamma, \mathrm{G})$. Let $\mathscr{R}_{\mathrm{A}}(\rho_0)$ be the groupoid defined in § 4.2. Let $g \in \mathbf{G}^0(\mathrm{P}_{\mathrm{A}})$ be a gauge transformation; then $\varepsilon_p(g) \in \mathrm{G}^0_{\mathrm{A}}$. The diagram

$$\begin{array}{ccc} \mathbf{F}_{\mathrm{A}}(\omega_0) & \xrightarrow{g} & \mathbf{F}_{\mathrm{A}}(\omega_0) \\ {\scriptstyle \mathrm{hol}_p}\downarrow & & \downarrow{\scriptstyle \mathrm{hol}_p} \\ \mathrm{Obj}\,\mathscr{R}_{\mathrm{A}}(\rho_0) & \xrightarrow[\varepsilon_p(g)]{} & \mathrm{Obj}\,\mathscr{R}_{\mathrm{A}}(\rho_0) \end{array}$$

commutes, whence

$$(\mathrm{hol}_p, \varepsilon_p) : \mathscr{F}_{\mathrm{A}}(\omega_0) \to \mathscr{R}_{\mathrm{A}}(\rho_0)$$

defines a functor.

6.3. *Proposition. — The functor* $(\mathrm{hol}_p, \varepsilon_p) : \mathscr{F}_{\mathrm{A}}(\omega_0) \to \mathscr{R}_{\mathrm{A}}(\rho_0)$ *is an equivalence of groupoids.*

Proof. — There is a commutative diagram of groupoids:

$$\begin{array}{ccccc} \mathscr{F}_{\mathrm{A}}(\omega_0) & \longrightarrow & \mathscr{F}(\mathrm{P}_{\mathrm{A}}) & \xrightarrow{q \circ i^*} & \mathscr{F}(\mathrm{P}) \\ {\scriptstyle (\mathrm{hol}_p,\, \varepsilon_p)}\downarrow & & \downarrow{\scriptstyle (\mathrm{hol}_p,\, \varepsilon_p)} & & \downarrow{\scriptstyle (\mathrm{hol}_p,\, \varepsilon_p)} \\ \mathscr{R}_{\mathrm{A}}(\rho_0) & \longrightarrow & \mathscr{R}(\Gamma, \mathrm{G}_{\mathrm{A}}) & \xrightarrow[q]{} & \mathscr{R}(\Gamma, \mathrm{G}) \end{array}$$

where the last two vertical arrows are equivalences. We shall prove that the first vertical arrow is an equivalence by showing that it induces a surjection on isomorphism classes on objects and that it is fully faithful:

Surjective on isomorphism classes. — Let $\rho \in q_*^{-1}(\rho_0)$ be an object in $\mathscr{R}_{\mathrm{A}}(\rho_0)$. By 5.10 applied to P_{A} there exists $\omega_1 \in \mathbf{F}(\mathrm{P}_{\mathrm{A}})$ such that $\mathrm{hol}_p(\omega_1) = \rho$. Now $\omega_2 = q \circ i^*(\omega_1) \in \mathbf{F}(\mathrm{P})$ satisfies $\mathrm{hol}_p(\omega_2) = \rho_0$ whence by 5.10 (applied to P) there exists $g_1 \in \mathbf{G}(\mathrm{P})$ such that $g_1 : \omega_2 \to \omega_0$ and $\varepsilon_p(g_1) = 1$. Let $\omega = i(g_1)^* \omega_1$. Then $\mathrm{hol}_p(\omega) = \varepsilon_p(i(g_1)) . \mathrm{hol}_p(\omega_1) = \rho$ and $q \circ i^*(\omega) = q \circ i^*(i(g_1)^* \omega_1) = g_1^* \omega_2 = \omega_0$. Thus $\mathrm{hol}_p : \mathbf{F}_{\mathrm{A}}(\omega_0) \to \mathrm{Obj}\,\mathscr{R}_{\mathrm{A}}(\rho_0)$ is surjective whence $(\mathrm{hol}_p, \varepsilon_p)$ induces a surjection $\mathrm{Iso}\,\mathscr{F}_{\mathrm{A}}(\omega_0) \to \mathrm{Iso}\,\mathscr{R}_{\mathrm{A}}(\rho_0)$ as desired.

Full. — Suppose that $\omega_1, \omega_2 \in \mathbf{F}_{\mathrm{A}}(\omega_0)$ satisfy $\mathrm{hol}_p(\omega_i) = \rho_i$ for $i = 1, 2$ and that there exists $g \in \mathrm{G}^0_{\mathrm{A}}$ such that $g : \rho_1 \to \rho_2$. By 5.10 applied to P_{A} there exists $\tilde{g}_1 \in \mathbf{G}(\mathrm{P}_{\mathrm{A}})$ such that $\tilde{g}_1 : \omega_1 \to \omega_2$ and $\varepsilon_p(\tilde{g}_1) = g$. Let $\tilde{g} = \tilde{g}_1 \circ (iq(\tilde{g}_1))^{-1}$. Then $\tilde{g} : \omega_1 \to \omega_2$ and $\varepsilon_p(\tilde{g}) = g$. Furthermore $q(\tilde{g}) = q(\tilde{g}_1)\, q(\tilde{g}_1)^{-1} = 1$ so $\tilde{g} \in \mathbf{G}_0(\mathrm{P}_{\mathrm{A}})$ as desired.

Faithful. — Since $\mathscr{F}(\omega_0)$ and $\mathscr{R}(\rho_0)$ are subcategories of $\mathscr{F}(\mathrm{P}_{\mathrm{A}})$ and $\mathscr{R}(\Gamma, \mathrm{G}_{\mathrm{A}})$ respectively, this follows from the corresponding assertion in 5.10 applied to P_{A}.

This concludes the proof of 6.3.

6.4. There are analogous results for the groupoids $\mathscr{F}'(M; G)$ and $\mathscr{R}'(\Gamma, G)$. Namely we define $\mathscr{F}'_A(\omega_0)$ to be the transformation groupoid

$$(\mathbf{F}_A(\omega_0), \operatorname{Ker}(\varepsilon_p : \mathbf{G}(P_A) \to G_A))$$

and we define $\mathscr{R}'_A(\rho^0)$ to be the groupoid with $\operatorname{Obj} \mathscr{R}'_A(\rho_0) = \operatorname{Obj} \mathscr{R}_A(\rho_0)$ but only the identity morphisms. Then Corollary 5.11 and Proposition 6.3 imply:

Corollary. — The map $\operatorname{hol} : \mathscr{F}'_A(\omega_0) \to \mathscr{R}'_A(\rho_0)$ *is an equivalence of groupoids. In other words the holonomy correspondence induces an isomorphism of sets*

$$\operatorname{Iso} \mathscr{F}'_A(\omega_0) \to \operatorname{Iso} \mathscr{R}'_A(\rho_0)$$

which depends functorially on A.

6.5. Similar to the restriction operation $q \circ i^* : \mathbf{F}(P_A) \to \mathbf{F}(P)$ one can extend connections on P to connections on P_A. To this end observe that there is an isomorphism $j : i^* T(P_A) \to TP \otimes A$ of $\mathfrak{g}_A$-bundles over P. Let $\omega_0 : TP \to \mathfrak{g}$ be a connection on P. Then $(\omega \otimes A) \circ j$ is a section of $\operatorname{Hom}(i^* TP_A, \mathfrak{g}_A)$ satisfying the vertical condition (5-5) and is equivariant with respect to the action of G. It follows that there is a unique G_A-equivariant $\mathfrak{g}_A$-valued 1-form $\tilde{\omega}_0$ on P_A extending $(\omega \otimes A) \circ j$. Thus $\tilde{\omega}_0$ is a connection on P_A. We shall call $\tilde{\omega}_0$ the *extension* of ω_0 to P_A. It is easy to see that $q \circ i_*(\tilde{\omega}_0) = \omega_0$. The curvature of the extension $\tilde{\omega}_0$ satisfies $K(\tilde{\omega}_0) = i(K(\omega_0))$ whereby if ω_0 is a flat connection, then so is its extension to P_A. In that case its holonomy is easily seen to be $\operatorname{hol}_p(\tilde{\omega}_0) = i_*(\operatorname{hol}_p(\omega_0))$ where $i_* : \operatorname{Hom}(\Gamma, G) \to \operatorname{Hom}(\Gamma, G_A)$ denotes composition with $i : G \to G_A$.

Let $\omega_0 \in \mathbf{F}(P)$ be a flat connection in P. Its extension $\tilde{\omega}_0$ is a distinguished object in $\mathscr{F}_A(\omega_0)$. If $\tilde{\omega} \in \operatorname{Obj} \mathscr{F}_A(\omega_0)$ is an arbitrary object, then the difference

$$\eta = \tilde{\omega} - \tilde{\omega}_0$$

belongs to $\Omega^1(M; \operatorname{ad} P) \otimes \mathfrak{m} \subset \Omega^1(M; \operatorname{ad} P_A)$ and satisfies the deformation equation

$$Q(\eta) = d\eta + \frac{1}{2}[\eta, \eta] = 0.$$

Furthermore the action of $\mathbf{G}^0(P_A) = \exp(\Omega^0(M; \operatorname{ad} P) \otimes \mathfrak{m})$ is given by (5-6) and we immediately obtain the following:

6.6. *Proposition. — Let* $\omega_0, \tilde{\omega}_0$ *be as above. Let* L *be the differential graded Lie algebra* $\Omega^*(M; \operatorname{ad} P)$ *with differential* d_{ω_0}. *Then the correspondence*

$$\begin{aligned} \mathbf{F}_A(\omega_0) &\to L^1 \otimes \mathfrak{m} \\ \tilde{\omega} &\mapsto \tilde{\omega} - \tilde{\omega}_0 \end{aligned} \qquad (6\text{-}2)$$

defines an isomorphism of groupoids

$$\mathscr{F}_A(\omega_0) \to \mathscr{C}(L; A)$$

depending naturally on A.

6.7. We now modify the above result by considering ad P-valued differential forms which vanish at the base-point. An infinitesimal gauge transformation $\eta \in \Omega^0(M; \text{ad } P)$ is an equivariant map $P \to \mathfrak{g}$. The evaluation map $\varepsilon_p : \Omega^0(M; \text{ad } P) \to \mathfrak{g}$ at p is a Lie algebra homomorphism and is thus a $\mathfrak{g}$-augmentation (in the sense of 3.4) for (L, d_ω). We denote its kernel by $\Omega^*(M; \text{ad } P)_p$. Clearly

$$\exp(\Omega^0(M; \text{ad } P)_p \otimes \mathfrak{m} = \text{Ker}(\varepsilon_p : \mathbf{G}^0_A(P) \to G^0_A)$$

and we have:

Corollary. — *Let* $L' = \Omega^*(M; \text{ad } P)_p$ *and let* $\omega_0 \in \mathbf{F}(P)$. *Then the correspondence* (6-2) *defines an isomorphism of groupoids*

$$\mathscr{F}'_A(\omega_0) \to \mathscr{C}(L'; A).$$

Combining Corollary 6.4, Corollary 6.7, and Theorem 4.3 we obtain the following:

6.8. *Theorem.* — *Let* $\rho_0 \in \text{Hom}(\Gamma, G)$ *and let* (P, ω_0) *be a flat principal* G-*bundle. Suppose that* $p \in P$ *and that* $\text{hol}_p(\omega_0) = \rho_0$. *Let* L′ *denote the augmentation ideal* $\text{Ker } \varepsilon_p = \Omega^*(M; \text{ad } P)_p$. *Then the analytic germ of* $\mathfrak{R}(\Gamma, G)$ *at* ρ_0 *pro-represents the functor*

$$A \mapsto \text{Iso } \mathscr{C}(L'; A).$$

7. Proof of Theorem 1

7.1. Let M be a compact Kähler manifold with fundamental group Γ and let G be a real algebraic Lie group and $K \subset G$ a compact subgroup. Let (P, ω) be a flat principal G-bundle over M whose holonomy homomorphism $\rho : \Gamma \to G$ takes values in K; let ad P be the associated Lie algebra bundle. Let $\mathfrak{g}_{\mathbf{C}} = \mathfrak{g} \otimes \mathbf{C}$ denote the complexified Lie algebra of G and let ad $P_{\mathbf{C}}$ be the complexification of ad P. Since K is compact, there exists an Ad(K)-invariant positive definite symmetric bilinear form $\mathbf{B} : \mathfrak{g} \times \mathfrak{g} \to \mathbf{R}$; we fix such a bilinear form once and for all. The complexification of **B** is a nondegenerate symmetric **C**-bilinear form $\mathfrak{g}_{\mathbf{C}} \times \mathfrak{g}_{\mathbf{C}} \to \mathbf{C}$, also denoted **B**, and the corresponding Hermitian form $\mathfrak{g}_{\mathbf{C}} \times \mathfrak{g}_{\mathbf{C}} \to \mathbf{C}$ defined by

$$\langle X, Y \rangle = \mathbf{B}(X, \overline{Y})$$

is positive definite and Ad(K)-invariant. By invariance it follows that $\langle\, , \rangle$ defines a parallel Hermitian form on the vector bundle ad $P_{\mathbf{C}}$ which is positive definite.

7.2. The connection ω in P determines a connection d_ω in the vector bundle ad P. The complexification D of d_ω then defines a connection in the complex vector bundle ad $P_{\mathbf{C}}$. Let $L = \Omega^*(M; \text{ad } P_{\mathbf{C}})$ be the complex graded Lie algebra; then (L, D) is a differential graded Lie algebra. We may decompose D as $D = D' + D''$ where D′ has type (1, 0) and D″ has type (0, 1); then

$$D'^2 = D''^2 = D' D'' + D'' D' = 0.$$

Using the Kähler metric on M and the parallel Hermitian metric $\langle\,,\,\rangle$ on the fibers of ad P_G, one may define adjoints D^*, D'^*, D''^* and Laplacians $\Delta_D = DD^* + D^* D$, etc. One then has the standard identities which are proved in the usual way (see e.g. [W, 4.7-4.10], [DGMS, § 5]):

(7-1) $$\Delta_D = 2\,\Delta_{D'} = 2\,\Delta_{D''},$$

(7-2) $$[\Lambda, D'] = i\,D''^*, \quad [\Lambda, D''] = -\,i\,D'^*.$$

7.3. *Proposition.* — (1) *The covariant exterior differential decomposes as* $D = D' + D''$ *where*

$$(D')^2 = (D'')^2 = D'\,D'' + D''\,D' = 0.$$

(2) *(Principle of two types)*

$$(\mathrm{Ker}(D') \cap \mathrm{Ker}(D'')) \cap (\mathrm{Image}(D') \cup \mathrm{Image}(D'')) = \mathrm{Image}(D'\,D'').$$

7.4. The subspace $\mathrm{Ker}\,D' \subset L$ is a graded subalgebra upon which the restriction of the differential D of L equals the derivation D''. By 7.3 (1), the pair $(\mathrm{Ker}\,D', D'')$ is a differential graded Lie subalgebra of (L, d).

Lemma. — *The inclusion* $i : (\mathrm{Ker}\,D', D'') \to (L, D)$ *induces an isomorphism on cohomology.*

Proof. — $H(i)$ *is surjective*: Let $\alpha \in \mathrm{Ker}(D)$; then

$$D''\,\alpha \in \mathrm{Ker}(D') \cap \mathrm{Image}(D'') = \mathrm{Image}(D'\,D'')$$

and there exists β such that

$$D''\,\alpha = D'\,D''\,\beta.$$

Similarly, there exists γ such that

$$D'\,\alpha = D'\,D''\,\gamma.$$

Now $$\alpha + D(\beta - \gamma) = \alpha + D'\,\beta - D''\,\gamma \in \mathrm{Ker}(D') \cap \mathrm{Ker}(D'')$$

is a cycle in $(\mathrm{Ker}(D'), D'')$ whose cohomology class maps under $H(i)$ to $[\alpha] \in H^*(L)$.

$H(i)$ *is injective*: Suppose that $\alpha \in \mathrm{Ker}\,D' \cap \mathrm{Ker}\,D''$ is a cycle in $(\mathrm{Ker}\,D', D'')$ such that $H(i)\,[\alpha] = 0 \in H(L)$. Then there exists $\beta \in L$ such that $\alpha = D\beta = D'\,\beta + D''\,\beta$. Now $D'\,\beta \in \mathrm{Ker}\,D'' \cap \mathrm{Image}\,D' = \mathrm{Image}\,D''\,D'$ so there exists γ such that $D'\,\beta = D''\,D'\gamma$. Then $D'(\beta + D\gamma) = D'\,\beta + D'\,D''\,\gamma = 0$ and $D''(\beta + D\gamma) = D''\,\beta + D''\,D'\,\gamma = D\beta = \alpha$ whence $\alpha \in D''(\mathrm{Ker}\,D')$ is a boundary in $(\mathrm{Ker}\,D', D'')$ as desired. This completes the proof of Lemma 7.3.

7.5. Let $H_{D'}$ denote the graded vector space $\mathrm{Ker}\,D'/\mathrm{Image}\,D'$ and let $\Pi : \mathrm{Ker}\,D' \to H_{D'}$ denote the quotient projection. Since $D' : L \to L$ is a derivation, $H_{D'}$ inherits from L the structure of a graded Lie algebra. We give $H_{D'}$ the zero differential.

Lemma. — *The map* $\Pi : (\operatorname{Ker} D', D'') \to (H_{D'}, 0)$ *is a differential graded Lie algebra homomorphism which induces an isomorphism of cohomology.*

Proof. — To see that Π is a differential graded Lie algebra homomorphism it suffices to show

$$D''(\operatorname{Ker} D') \subset \operatorname{Image} D'$$

which follows from

$$D''(\operatorname{Ker} D') \subset \operatorname{Image} D'' \cap \operatorname{Ker} D' = \operatorname{Image} D'' D' \subset \operatorname{Image} D'.$$

We next show that $H(\Pi)$ is an isomorphism. Surjectivity follows as in 7.4: let $\alpha \in \operatorname{Ker} D'$; then $D''\alpha \in \operatorname{Im} D'' \cap \operatorname{Ker} D' = \operatorname{Im} D'' D'$ and there exists β such that $D''\alpha = D'' D' \beta$. Now $D'(\alpha - D'\beta) = D'\alpha = 0$ and $D''(\alpha - D'\beta) = 0$ whence $\alpha - D'\beta$ is a cycle in $(\operatorname{Ker} D', D'')$ whose cohomology class maps to $[\alpha] \in H_{D'}$ under Π. To establish injectivity, let α be a cycle in $(\operatorname{Ker} D', D'')$ which satisfies $\Pi[\alpha] = 0$. Then $\alpha \in \operatorname{Ker} D' \cap \operatorname{Ker} D'' \cap \operatorname{Image} D'$ and $\alpha \in \operatorname{Image} D'' D' \subset D''(\operatorname{Ker} D')$ is a boundary in $(\operatorname{Ker} D', D'')$ as claimed. This completes the proof of 7.5.

7.6. We thus have homomorphisms of differential graded Lie algebras:

$$(L, D) \overset{i}{\leftarrow} (\operatorname{Ker} D', D'') \overset{\Pi}{\rightarrow} (H_{D'}, 0) \tag{7-3}$$

which induce isomorphisms on cohomology. Furthermore the Lie algebra homomorphisms $\varepsilon_p : L \to \mathfrak{g}_{\mathbf{C}}$ and $H^0(L) \hookrightarrow \mathfrak{g}_{\mathbf{C}}$ define $\mathfrak{g}_{\mathbf{C}}$-augmentations respected by the above homomorphisms. Thus (7-3) defines a quasi-isomorphism of $\mathfrak{g}_{\mathbf{C}}$-augmented differential graded Lie algebras and we obtain the following:

Corollary. — *The* $\mathfrak{g}_{\mathbf{C}}$*-augmented differential graded Lie algebra* $(\Omega^*(M; \operatorname{ad} P_{\mathbf{C}}), D, \varepsilon_p)$ *is formal.*

Now we are in position to prove Theorem 1. Let (L, D, ε_p) and L' be as above. By Theorem 6.9, the analytic germ of $\mathfrak{R}(\Gamma, G)$ at ρ pro-represents the functor

$$A \mapsto \operatorname{Iso} \mathscr{C}(L'; A). \tag{7-4}$$

By Lemma 5.12, $\varepsilon_p : \Omega^0(M; \operatorname{ad} P_{\mathbf{C}}) \to \mathfrak{g}_{\mathbf{C}}$ is surjective. It follows from the faithful assertion in 5.10 that the restriction of ε_p to $H^0(M; \operatorname{ad} P_{\mathbf{C}})$ is injective. Together with Corollary 7.6, all the hypotheses of Theorem 3.5 are satisfied, and we conclude that the functor (7-4) is also pro-represented by the analytic germ of a quadratic cone $\mathscr{Q}$. By Theorem 3.1 it follows that the analytic germ of $\mathfrak{R}(\Gamma, G)$ at ρ is equivalent to the analytic germ of $\mathscr{Q}$ at 0, i.e. that $\mathfrak{R}(\Gamma, G)$ is quadratic at ρ. The proof of Theorem 1 is complete.

8. Proof of theorem 2 and theorem 3

8.1. In this section we prove Theorems 2 and 3 of the introduction. We begin by reviewing the notion of real variation of Hodge structure.

In what follows we adopt the notation of Griffiths [Gr, § 1], although for us a *polarized Hodge structure of weight n* on a complex vector space H will mean a triple $(\{H^{p,q}\}_{p+q=n}, \sigma, Q)$ satisfying the following axioms:

(i) $H = \bigoplus_{p+q=n} H^{p,q}$ where each $H^{p,q}$ is a finite-dimensional complex vector space;
(ii) $\sigma : H \to H$ is a real structure on H (i.e. conjugation with respect to a real form on H) such that $\sigma(H^{p,q}) = H^{q,p}$;
(iii) $Q : H \times H \to \mathbf{C}$ is a bilinear form which is symmetric for n even and skew-symmetric for n odd and satisfies the Hodge-Riemann bilinear relations:
(iv) $Q(H^{p,q}, H^{p',q'}) = 0$ unless $p + p' = q + q' = n$;
(v) $i^{p-q} Q(\psi, \bar{\psi}) > 0$ for any nonzero $\psi \in H^{p,q}$.

The bilinear form Q is called the *polarization* and the collection of integers $\{h^{p,q}\}_{p+q=n}$ is called the *type* of the Hodge structure. The collection of polarized Hodge structures of a given type on the vector space H forms a homogeneous space G/V, the *classifying space of polarized Hodge structures of type* $\{h^{p,q}\}$, where G is the automorphism group of the triple (H, σ, Q) and V is the subgroup of G stabilizing a fixed polarized Hodge structure. The classifying space G/V admits a G-homogeneous complex structure. For each $y \in G/V$, we denote by $T_y^{\text{horiz}}(G/V)$ the horizontal subspace of $T_y^{(1,0)}(G/V)$ as defined in [Gr, p. 21].

8.2. Next suppose that M is a complex manifold. A *real variation of polarized Hodge structure over* M consists of $(E, D, \{\mathscr{H}^{p,q}\}_{p+q=n}, \sigma, Q)$ satisfying the following axioms:

(i) E is a holomorphic complex vector bundle over M and $D : \Omega^j(M; E) \to \Omega^{j+1}(M; E)$ is a flat connection on E with $D'' = \bar{\partial}_E$;
(ii) $\mathscr{H}^{p,q} \subset E$ is a C^∞ complex vector subbundle;
(iii) $\sigma : E \to E$ is a parallel (with respect to D) real structure on E;
(iv) $Q : E \times E \to \mathbf{C}$ is a parallel bilinear form such that for each $x \in M$, the triple $(\{\mathscr{H}^{p,q}\}_{p+q=n}, \sigma_x, Q_x)$ is a polarized Hodge structure of weight n on E_x;
(v) The subbundle $\mathscr{F}^r \subset E$ defined by $\mathscr{F}^r = \bigoplus_{p \geqslant r} \mathscr{H}^{p,q}$ for $0 \leqslant r \leqslant n$ is holomorphic with respect to the holomorphic structure on E defined by the flat connection D;
(vi) (Transversality) For every holomorphic tangent vector v and smooth section s of $\mathscr{F}^r$, the covariant derivative $D_v s \in F^{r-1}$.

We may interpret the above notion in terms of principal bundles as follows. Since D is flat and σ and Q are parallel, there is a flat principal G-bundle $\pi : P \to M$ to which E is associated by the representation $G \to GL(H)$. Let Γ be the fundamental group of M and let $\rho : \Gamma \to G$ be the holonomy representation of P; ρ is the *monodromy representation* of the variation of Hodge structure over M. The associated G/V-bundle $S = P \times_G G/V$ then parametrizes Hodge structures on E in the following sense: a point of the fiber S_x is a polarized Hodge structure of the given type on the vector space E_x. A collection $(E, D, \{\mathscr{H}^{p,q}\}, \sigma, Q)$ satisfying conditions (i)-(iv) above thus defines a section of S, i.e. a reduction of the structure group of P to V. Condition (v) implies that the V-reduction is *holomorphic* in the following sense. Let $\omega \in \Omega^1(P) \otimes \mathfrak{g}$ denote the connection form on P; then the restriction $(d\pi)_p : \operatorname{Ker} \omega_p \to T_{\pi(p)} M$ is an isomorphism for each $p \in P$ and we denote by $\operatorname{Ker} \omega^{(1,0)}$ the inverse image $(d\pi)^{-1} T^{(1,0)} M$. A V-reduction of P is given by an equivariant map $f: P \to G/V$; such an equivariant map defines a *holomorphic* V-reduction if and only if $df(\operatorname{Ker} \omega^{(1,0)}) \subset T^{(1,0)} G/V$. Moreover a V-reduction f is said to be *horizontal* if $df(\operatorname{Ker} \omega^{(1,0)}) \subset T^{\text{horiz}} G/V$. The transversality condition (vi) is equivalent to the horizontality of the corresponding V-reduction.

We may also interpret this in terms of a universal covering $M \to M$. Write $P = P_\rho = M \times_\Gamma G$ as in 4.5 where ρ is the monodromy representation of a real variation of Hodge structure over M. Then a V-reduction of P is given by a map $\tilde{f}: \tilde{M} \to G/V$ such that

$$\tilde{f} \circ \gamma = \rho(\gamma) \circ \tilde{f} \tag{8-1}$$

for each $\gamma \in \Gamma$. The V-reduction is holomorphic if and only if $\tilde{f}$ is a holomorphic map; the V-reduction is horizontal if and only if $\tilde{f}$ is horizontal in the sense that $d\tilde{f}(T^{(1,0)} \tilde{M}) \subset T^{\text{horiz}} G/V$. We may summarize the equivalence of these three points of view as follows:

8.3. *Lemma. — Let* M *be a complex manifold with fundamental group* Γ *and universal covering* M. *Let* G/V *be the classifying space for polarized Hodge structures of weight n and type* $\{ h^{p,q} \}$. *Then the following categories are equivalent*:

(1) *Real variations of polarized Hodge structures of weight n and type* $\{ h^{p,q} \}$ *over* M;
(2) *Flat principal* G*-bundles over* M *with horizontal holomorphic* V*-reduction*;
(3) *Pairs* $(\rho, \tilde{f})$ *where* $\rho \in \operatorname{Hom}(\Gamma, G)$ *and* $\tilde{f}: \tilde{M} \to G/V$ *is a horizontal holomorphic map satisfying* (8-1).

8.4. We now prove Theorem 2. Suppose that M is a compact Kähler manifold with fundamental group Γ. Let H, n, G/V be as above. Let $\pi : P \to M$ be a principal G-bundle over M with holonomy representation $\rho : \Gamma \to G$ and let $f: P \to G/V$ be a horizontal holomorphic V-reduction. Let $(\operatorname{ad} P_{\mathbf{C}}, D)$ denote the flat complex Lie algebra bundle associated to P and let $p \in P$. We shall show that the $\mathfrak{g}$-augmented differential graded Lie algebra $(\Omega^*(M; \operatorname{ad} P_{\mathbf{C}}), D, \varepsilon_p)$ is formal by the same method as 7.6. We shall first prove the following:

Lemma. — The flat vector bundle $(\operatorname{ad} P_{\mathbf{C}}, D)$ *has a polarized Hodge structure of weight* 0.

Proof. — A polarized Hodge structure of weight n on a vector space H induces a polarized Hodge structure of weight 0 on the vector space End(H) of linear maps $H \to H$, [Gr, p. 16]. We claim this latter structure induces a polarized Hodge structure of weight 0 on $\mathfrak{g}_{\mathbf{C}} \subset \operatorname{End}(H)$ by

$$\mathfrak{g}_{\mathbf{C}}^{(r,-r)} = \mathfrak{g}_{\mathbf{C}} \cap \operatorname{End}(H)^{(r,-r)}.$$

Clearly it suffices to show that

$$\mathfrak{g}_{\mathbf{C}} = \bigoplus_{r \in \mathbf{Z}} \mathfrak{g}_{\mathbf{C}}^{(r, -r)}.$$

The action of $S^1 \subset \mathbf{C}^*$ on H defined by

$$\zeta : v \mapsto \zeta^{p-q} v$$

for $v \in H^{(p,q)}$ is real and preserves Q and thus factors through the action of G on H. The Hodge decomposition of End(H) is then the weight decomposition for this action of S^1; since $\mathfrak{g}_{\mathbf{C}}$ is stable for the adjoint representation of $G \supset S^1$, the polarized Hodge structure of End(H) induces one on $\mathfrak{g}_{\mathbf{C}}$. If $h \in G/V$ is a polarized Hodge structure on H, we denote by $h^* \otimes h$ the corresponding polarized Hodge structure of weight 0 on $\mathfrak{g}_{\mathbf{C}}$.

Suppose that $\tilde{f} : \tilde{M} \to G/V$ is a ρ-equivariant horizontal holomorphic map corresponding to the real variation of polarized Hodge structure over M. To each point $x \in \tilde{M}$, $F(x) = \tilde{f}(x)^* \otimes \tilde{f}(x)$ is a polarized Hodge structure of weight zero on $\mathfrak{g}_{\mathbf{C}}$. Let U denote the classifying space for polarized Hodge structures of weight zero and the corresponding type for $\mathfrak{g}_{\mathbf{C}}$; then it is clear from the definition of the complex structure on U and the definition of the filtration on $F(x)$ that $F : \tilde{M} \to U$ is a horizontal holomorphic map. It follows that F defines a real variation of polarized Hodge structure of weight 0 on ad $P_{\mathbf{C}}$ as desired.

8.5. This lemma can be applied, thanks to the following fundamental observation of Deligne:

Theorem (Deligne). — Suppose M *is a compact Kähler manifold and* E *is a real variation of polarized Hodge structure over* M. *Then the complex* $\Omega^*(M; E)$ *is formal.*

Since the proof of this result has not been published, we briefly explain how it follows from Zucker [Z1, § 2]. We decompose the covariant differential D on $\Omega^*(M; E)$ as $D = D' + D''$, according to total (i.e. base plus fiber) Hodge degree. By the transversality axiom D has total degree 1 and we let D' be its (1, 0)-part and D'' its (0, 1)-part. The identities (7-1) and (7-2) remain true (with the new interpretations of D' and D'') where adjoints are taken with respect to the positive definite metric on E associated to the polarizing bilinear form. (See Zucker [Z1, 2.7] and Simpson [Si1, Si2]). Formality follows as in § 7.

Theorem 2 now follows immediately. The real variation of Hodge structure ad $P_{\mathbf{C}}$ has two elements of extra structure we must consider, namely its multiplicative structure as a Lie algebra and the $\mathfrak{g}_{\mathbf{C}}$-augmentation $\varepsilon_p : \Omega^*(M; \text{ad } P_{\mathbf{C}}) \to \mathfrak{g}_{\mathbf{C}}$. But the homomorphisms of complexes given in (7-3) are quasi-isomorphisms of $\mathfrak{g}_{\mathbf{C}}$-augmented differential graded Lie algebras so the above theorem of Deligne implies:

8.6. *Proposition. — The* $\mathfrak{g}_{\mathbf{C}}$*-augmented differential graded Lie algebra*

$$(\Omega^*(M; \mathrm{ad}\, P_{\mathbf{C}}), D, \varepsilon_p)$$

is formal.

Theorem 2 nows follows from the same formal argument (see § 7.6) as the proof of Theorem 1 in § 7.

8.7. We now prove Theorem 3. (Compare [Z2].) Suppose that G is the automorphism group of a Hermitian symmetric space $X = G/K$ where $K \subset G$ is a maximal compact subgroup. To each point $z \in X$ we associate a real polarized Hodge structure of weight 0 on $\mathfrak{g}_{\mathbf{C}}$ as follows. Let K_z be the maximal compact subgroup of G fixing z and let $J_z \in K_z$ be the element of the center of K which acts by multiplication by i on the holomorphic tangent space to X at z. Let $\mathfrak{p}_z^{\pm}$ to the $\pm$ 1-eigenspace of $\mathrm{Ad}\, J_z$ on $\mathfrak{g}_{\mathbf{C}}$. Let σ denote conjugation with respect to the real form $\mathfrak{g} \subset \mathfrak{g}_{\mathbf{C}}$ and let Q denote the Killing form. Then

$$\begin{aligned} H^{(1,-1)} &= \mathfrak{p}_z^- \\ H^{(0,0)} &= \mathfrak{k}_z \otimes \mathbf{C} \\ H^{(-1,1)} &= \mathfrak{p}_z^+ \end{aligned}$$

together with σ, Q is a polarized Hodge structure of weight 0 on $\mathfrak{g}_{\mathbf{C}}$. Thus points of X parametrize Hodge structures on $\mathfrak{g}_{\mathbf{C}}$ and any holomorphic map $\tilde{f}: \tilde{M} \to X$ is automatically horizontal. Thus if P admits a holomorphic K-reduction, then $\mathrm{ad}\, P_{\mathbf{C}}$ has the structure of a real variation of polarized Hodge structure. Theorem 3 now follows as before.

8.8. Recall (see [C1] or [GM1]) that if M has fundamental group Γ, G is a semisimple Lie group, and $\rho : \Gamma \to G$ is a homomorphism, then every G-invariant q-form ω on the symmetric space G/K defines a characteristic class $\omega(\rho) \in H^q(M)$.

Corollary. — Suppose that M *is a compact Kähler manifold with fundamental group* Γ *and* G *is the group of automorphisms of a Hermitian symmetric space* G/K *and let* $\rho \in \mathrm{Hom}(\Gamma, G)$. *Let* ω^j *be the j-th power of the Kähler form on* G/K. *Suppose that the corresponding characteristic class* $\omega^j(\rho) \in H^{2j}(M)$ *is nonzero for* $j > 1$. *Then* $\mathfrak{R}(\Gamma, G)$ *and* $\mathfrak{R}(\Gamma, G_{\mathbf{C}})$ *are quadratic at* ρ.

Proof. — By Corlette [C1, 5.3], the associated principal G-bundle admits a holomorphic K-reduction. Now apply Theorem 3.

9. Examples and applications

The Heisenberg group

9.1. In general the singularities of $\mathfrak{R}(\Gamma, G)$ will not be quadratic, as the following simple example demonstrates. Let H be the three-dimensional real Heisenberg group and let $\Gamma \subset H$ be a lattice. The quotient $M = H/\Gamma$ is a closed 3-manifold which is the total space of an oriented circle bundle over the 2-torus with nonzero Euler class. Let G be an algebraic Lie group which is not two-step nilpotent and ρ the trivial representation. Then $\mathfrak{R}(\Gamma, G)$ is not quadratic at ρ.

In fact the analytic germ of $\mathfrak{R}(\Gamma, G)$ at ρ is isomorphic to the *cubic cone*

$$\mathscr{C} = \{ (X, Y) \in \mathfrak{g} \times \mathfrak{g} \mid [[X, Y], X] = [[X, Y], Y] = 0 \}.$$

To see this one needs only apply the chain of groupoid equivalences between $\mathscr{R}_A(\rho)$ and $\mathscr{C}(L; A) \bowtie G^0_A$ where L is the differential graded Lie algebra $\Omega^*(M; \mathrm{ad}\, P)$. Since ρ is trivial, the associated principal bundle P as well as the Lie algebra bundle ad P is a trivial bundle so that $L = \Omega^*(M) \otimes \mathfrak{g}$. By a well-known theorem of Nomizu [N] the inclusion of the subcomplex L^H of H-invariant $\mathfrak{g}$-valued forms induces an isomorphism of homology and we are reduced to studying the groupoids associated to the differential graded Lie algebra L^H. Now the algebra of H-invariant $\mathbf{R}$-valued differential forms on M is an exterior algebra generated by three invariant one-forms α, β, γ subject to the relations

$$d\alpha = d\beta = 0, \qquad d\gamma = \alpha \wedge \beta.$$

It follows that an object in $\mathscr{C}(L^H; A)$ is given by

$$\omega = \alpha \otimes X + \beta \otimes Y + \gamma \otimes Z$$

$(X, Y, Z \in \mathfrak{g} \otimes \mathfrak{m})$ satisfying

$$0 = d\omega + \frac{1}{2}[\omega, \omega] = (\alpha \wedge \beta) \otimes (Z + [X, Y]) + (\beta \wedge \gamma) \otimes [Y, Z] + (\gamma \wedge \alpha) \otimes [Z, X].$$

Then the set of A-points over the origin can be identified with

$$\{ (X, Y, Z) \in (\mathfrak{g} \otimes \mathfrak{m})^3 \mid Z = -[X, Y], [Y, Z] = [Z, X] = 0 \}$$

which by eliminating Z is isomorphic to

$$\{ (X, Y) \in (\mathfrak{g} \otimes \mathfrak{m})^2 \mid [[X, Y], X] = [[X, Y], Y] = 0 \}$$

the set of A-points of $\mathscr{C}$ over the origin. Thus the analytic germ of $\mathfrak{R}(\Gamma, G)$ at ρ is isomorphic to the cubic cone $\mathscr{C}$. It follows easily that the Zariski tangent space and the tangent quadratic cone are both isomorphic to the vector space $\mathfrak{g}^2$. In particular every infinitesimal deformation is unobstructed at the first level, although possibly not at the second level.

Bieberbach groups

9.2. The methods of this paper yield quadratic singularity theorems for a class of groups much larger than fundamental groups of compact Kähler manifolds; the Kähler condition is only used to derive the fact that the $\mathfrak{g}$-augmented differential graded

Lie algebra $\Omega^*(M; \mathrm{ad}\, P)$ is formal. Such a conclusion can be drawn in several other cases, for example for certain representations of *Bieberbach groups*, i.e. lattices in the group of isometries of Euclidean space. (Bieberbach proved that such a group is an extension of a lattice Λ of translations by a finite group H such that the corresponding representation $H \to \mathrm{Aut}(\Lambda)$ is faithful. Torsion free Bieberbach groups are the fundamental groups of compact flat Riemannian manifolds.) Suppose that M is a compact flat Riemannian manifold with fundamental group Γ and $\rho \in \mathrm{Hom}(\Gamma, G)$ a homomorphism such that $\rho(\Gamma)$ lies in a compact subgroup of G. Since harmonic forms on a flat Riemannian manifold are parallel and hence closed under exterior multiplication, the inclusion

$$\mathscr{H}^*(M; \mathrm{ad}\, P) \hookrightarrow \Omega^*(M; \mathrm{ad}\, P)$$

is a quasi-isomorphism and hence $\Omega^*(M; \mathrm{ad}\, P)$ is a formal $\mathfrak{g}$-augmented differential graded Lie algebra. It follows that, if Γ is a torsion free Bieberbach group and ρ is as above, $\mathfrak{R}(\Gamma, G)$ is quadratic at ρ.

9.3. By a further argument one can obtain the same conclusion for an arbitrary Bieberbach group Γ. Essentially this is just an extension of these techniques from manifolds to their quotients by finite groups considered as orbifolds. Indeed, suppose that M is a manifold and H is a finite group of diffeomorphisms of M. Let (P, ω) be a flat principal G-bundle over M upon which the H-action lifts to connection-preserving automorphisms. The quotient is then a Euclidean orbifold (crystallographic quotient) whose orbifold fundamental group Γ is an extension of the fundamental group $\pi_1(M)$ of M by the finite group H. The holonomy representation $\rho : \pi_1(M) \to G$ then extends to a representation $\rho_\dagger : \Gamma \to G$. Conversely every representation $\Gamma \to G$ determines (by restriction to $\pi_1(M)$) a flat principal G-bundle over M with an H-action.

If the $\mathfrak{g}$-augmented differential graded Lie algebra $\Omega^*(M; \mathrm{ad}\, P)$ is formal, then it follows from 3.5 that $\mathfrak{R}(\pi_1(M), G)$ is quadratic at ρ. Suppose that $\Omega^*(M; \mathrm{ad}\, P)$ is formal as a $\mathfrak{g}$-augmented differential graded Lie algebra with H-action—i.e. $\Omega^*(M; \mathrm{ad}\, P)$ is quasi-isomorphic to its cohomology by H-equivariant differential graded Lie algebra homomorphisms between differential graded Lie algebras with H-actions by differential graded Lie algebra automorphisms. We claim that $\mathfrak{R}(\Gamma, G)$ is quadratic at $\rho_\dagger$.

To this end we replace M by another space with fundamental group Γ. Let K be a 1-connected compact Lie group into which H embeds (such as $SU(|H|)$) and consider the diagonal action of H on $M \times K$ where H acts on K by right-multiplication. Since this action is free the quotient $M_\dagger = (M \times K)/H$ is a manifold with fundamental group Γ. The action of K on $M \times K$ by left-multiplication determines a K-action on $M_\dagger$ whose quotient is M. Let $\Pi_\dagger : M_\dagger \to M$ denote the quotient projection. The flat principal G-bundle (P, ω) determines a flat principal G-bundle $(P_\dagger, \omega_\dagger)$ on $M_\dagger$ whose holonomy representation is $\rho_\dagger$ and such that the K-action lifts to a K-action on P preserving ω.

To show that $\mathfrak{R}(\Gamma, G)$ is quadratic at $\rho_\dagger$ consider the groupoid equivalence

$\mathscr{R}_A(\rho_\dagger) \to \mathscr{C}(L_\dagger; A) \bowtie G^0_A$ where L is the g-augmented differential graded Lie algebra $\Omega^*(M_\dagger; \mathrm{ad}\, P_\dagger)$. By a standard argument, the inclusion $(L_\dagger)^K \hookrightarrow L_\dagger$ of K-invariant differential forms is a quasi-isomorphism of augmented differential graded Lie algebras. Furthermore the operation

$$L \to L_\dagger$$
$$\alpha \mapsto H^*_\dagger \alpha$$

defines a differential graded Lie algebra isomorphism $L^H \to (L_\dagger)^K$. Thus the functor $A \mapsto \mathscr{C}(L^H; A) \bowtie G^0_A$ is pro-represented by the analytic germ of $\mathfrak{R}(\Gamma, G)$ at $\rho_\dagger$. Since H acts by isometries, the quasi-isomorphism of L to its cohomology is H-equivariant and thus there is a corresponding quasi-isomorphism of the subalgebras of invariants $L^H \to H(L)^H$ and thus L^H is formal and $\mathfrak{R}(\Gamma, G)$ is quadratic at $\rho_\dagger$.

The preceding argument works equally well when H is a finite group acting on a compact manifold M preserving a Kähler structure on M. We obtain the following:

Theorem. — *Let* M *be a compact manifold with a universal covering* $\widetilde{M}$ *and covering group* $\pi_1(M)$. *Let* H *be a finite group acting on* M. *Let* Γ *be the group of automorphisms of* M *generated by* $\pi_1(M)$ *and lifts of elements of* H. *Suppose that* H *preserves either a flat Riemannian metric or a Kähler structure on* M. *If* G *is a real algebraic Lie group and* $\rho : \Gamma \to G$ *is a homomorphism whose image lies in a compact subgroup of* G, *then* $\mathfrak{R}(\Gamma, G)$ *is quadratic at* ρ.

We may use this theorem to interpret the example of Lubotzky-Magid [LM] of a Bieberbach group Γ for which $\mathfrak{R}(\Gamma, G)$ is not reduced. In their example $G = GL(2, \mathbf{C})$, Γ is the (3, 3, 3)-triangle group (an extension of a free abelian group Δ of rank two by the symmetric group S_3 on three letters) and ρ is a representation with kernel precisely Δ. The cohomology of this representation is given by:

$$H^0(\Gamma, \mathfrak{g}_{\mathrm{Ad}\,\rho}) = \mathbf{C}$$
$$H^1(\Gamma, \mathfrak{g}_{\mathrm{Ad}\,\rho}) = \mathbf{C}$$
$$H^2(\Gamma, \mathfrak{g}_{\mathrm{Ad}\,\rho}) = \mathbf{C}$$

and the cup-product defines an isomorphism

$$H^1(\Gamma, \mathfrak{g}_{\mathrm{Ad}\,\rho}) \otimes H^1(\Gamma, \mathfrak{g}_{\mathrm{Ad}\,\rho}) \to H^2(\Gamma, \mathfrak{g}_{\mathrm{Ad}\,\rho}).$$

It follows that the analytic germ of $\mathfrak{R}(\Gamma, G)$ at ρ is isomorphic to the non-reduced scheme $PGL(2, \mathbf{C}) \times \mathrm{Spec}(\mathbf{C}[t]/(t^2))$.

Deformations of holomorphic vector bundles

9.4. These ideas may be applied to deformation theories other than flat bundles. For example let M be a compact Kähler manifold and let $\rho : \pi_1(M) \to GL(n, \mathbf{C})$ be a representation defining a flat $\mathbf{C}^n$-bundle V over M. Then there is a unique holomorphic structure on V compatible with the flat structure on V and the complex structure on M (compare Kobayashi [K]). The deformation theory of holomorphic structures on V is controlled by the transformation groupoid $\mathscr{E}(V)$ with objects

$$\mathrm{Obj}\ \mathscr{E}(V) = \{\eta \in \Omega^{(0,1)}(M; \mathrm{End}(V)) \mid \bar\partial\eta + \frac{1}{2}[\eta, \eta] = 0\}$$

and morphisms the C^∞ sections of Aut(V). Here $\bar\partial$ denotes the covariant $\bar\partial$-operator on V defined by the flat connection, extended to End(V). Thus, $\mathscr{E}(V)$ is the transformation groupoid associated to the differential graded Lie algebra L with

$$L^q = \Omega^{(0,q)}(M; \mathrm{End}(V))$$

and differential $\bar\partial$.

Suppose now that V is stable, or equivalently (by Uhlenbeck-Yau [UY]), that ρ is conjugate to an irreducible unitary representation. Then $H^0(L) = 0$ and by [GM3], Theorem 4.10, there is a complex analytic space $\mathscr{K}(L)$, the Kuranishi space, which pro-represents the functor

$$A \mapsto \mathrm{Iso}\ \mathscr{C}(L; A)$$

Moreover if V is stable it is well-known ([GM3], Theorem 4.7), that $\mathscr{K}(L)$ is a neighborhood of V in the space of isomorphism classes of holomorphic structures on V. Moreover L is formal: On a Kähler manifold the harmonic $(0, q)$-forms are exactly the anti-holomorphic ones; this extends to forms taking values in a flat vector bundle which are harmonic with respect to a parallel Hermitian metric (such a metric exists by Uhlenbeck-Yau [UY]). Thus the subcomplex of harmonic $(0, q)$-forms is closed under multiplication and the inclusion

$$\mathscr{H}^{(0,*)}(M; \mathrm{End}(V)) \hookrightarrow \Omega^{(0,*)}(M; \mathrm{End}(V))$$

is a quasi-isomorphism. Now the moduli space $\mathfrak{M}(V)$ of isomorphism classes of stable holomorphic structures on V has the structure of a complex analytic space. The following is then an immediate consequence of Theorem 3.5.

Theorem. — $\mathfrak{M}(V)$ *has quadratic singularities.*

This theorem has been proved independently by A. M. Nadel [N].

Deformations of lattices acting on complex hyperbolic spaces

9.5. Let $G_0 = U(n, 1)$ and let $\Gamma \subset G_0$ be a cocompact lattice. Let $M = \Gamma \backslash G_0/K_0$ be the corresponding locally symmetric space. We consider deformations of the inclusions ρ of Γ in $G = SU(n+1, 1)$ and $G_{\mathbf{C}} = SL(n+2, \mathbf{C})$. Let $\mathfrak{g}_0 \subset \mathfrak{g} \subset \mathfrak{g}_{\mathbf{C}}$ be the Lie algebras of $G_0 \subset G \subset G_{\mathbf{C}}$; then the adjoint representation of G restricted to G_0 decomposes as a sum of G_0-modules

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$$

where $\mathfrak{g}_0$ is the adjoint representation of G_0 and $\mathfrak{g}_1$ is the G_0-module C^{n+1} corresponding to the standard representation of $G_0 = U(n, 1)$ twisted by the determinant character. Let $\langle\ ,\ \rangle$ denote the G_0-invariant Hermitian form on $\mathfrak{g}_1$ and

$$\Lambda : \mathfrak{g}_1 \times \mathfrak{g}_1 \to \mathbf{R}$$
$$(u, v) \mapsto \operatorname{Im} \langle u, v \rangle$$

the corresponding alternating form. In terms of the Lie bracket on $\mathfrak{g}$, one may verify that $[\mathfrak{g}_1, \mathfrak{g}_1] = \mathfrak{g}_0$ and

$$\Lambda(u, v) = \frac{i}{2} \operatorname{tr}[u, v]$$

where $\operatorname{tr} : \mathfrak{g}_0 = u(n, 1) \to \mathbf{C}$ denotes trace.

Let $\mathscr{S}$ denote the saturation $G.\mathfrak{R}(\Gamma, G_0)$; then it follows by Weil rigidity that $\mathscr{S}$ is nonsingular at ρ. The Zariski normal space to $\mathscr{S}$ in $\mathfrak{R}(\Gamma, G)$ is the cohomology group $H^1(\Gamma, \mathfrak{g}_1)$; the normal quadratic cone is defined by the quadratic form associated to the pairing

$$[\ ,\ ]_* : H^1(\Gamma, \mathfrak{g}_1) \otimes H^1(\Gamma, \mathfrak{g}_1) \to H^2(\Gamma, \mathfrak{g}_0).$$

Let $\Pi : H^2(\Gamma, \mathbf{R}) \to \mathbf{R}$ denote the projection on the class of the Kähler form in $H^2(\Gamma) = H^2(M)$. In Goldman-Millson [GM1] it is proved that the composition

$$H^1(\Gamma, \mathfrak{g}_1) \otimes H^1(\Gamma, \mathfrak{g}_1) \longrightarrow H^2(\Gamma, \mathfrak{g}_0) \xrightarrow{\frac{i}{2}\operatorname{tr}_*} H^2(\Gamma) \xrightarrow{\Pi} \mathbf{R}$$

is positive definite and it follows that $\mathscr{S}$ is an open neighborhood of ρ in $\mathfrak{R}(\Gamma, G)$ in the classical topology. It follows that every representation $\Gamma \to G$ near ρ must be conjugate to one of the form

$$\gamma \mapsto \rho(\gamma)\, \zeta(\gamma)$$

where ζ is a homomorphism of Γ into the center $U(1) \subset U(n, 1)$. This is a " local rigidity " result, for which the corresponding global rigidity result (involving characteristic numbers as in 8.8) has been recently proved by Corlette [C1] for $n > 1$ and Toledo [T] for $n = 1$. (Compare also [C2] and [C3].)

Thus there are no interesting deformations of ρ in the *real* group G. However Theorem 3 can be used to predict the existence of deformations of ρ in the complex group $G_{\mathbf{C}}$. Let P denote the principal G-bundle over M with holonomy ρ; then the inclusion of locally symmetric spaces $M = \Gamma\backslash G_0/K_0 \hookrightarrow \Gamma\backslash G/K$ defines a holomorphic K-reduction of P. Thus Theorem 3 applies and we see that the analytic germ of $\mathfrak{R}(\Gamma, G_{\mathbf{C}})$ is isomorphic to the product $\mathscr{S}_{\mathbf{C}} \times \mathscr{Q}$ where $\mathscr{S}_{\mathbf{C}}$ is the saturation $G_{\mathbf{C}}.\mathfrak{R}(\Gamma, (G_0)_{\mathbf{C}})$ and $\mathscr{Q}$ is the quadratic cone defined by the cup-product

$$\text{(9-1)} \qquad H^1(\Gamma, \mathfrak{g}_1 \otimes \mathbf{C}) \times H^1(\Gamma, \mathfrak{g}_1 \otimes \mathbf{C}) \to H^2(\Gamma, \mathfrak{g}_0 \otimes \mathbf{C}).$$

In general $H^1(\Gamma, \mathfrak{g}_1) \neq 0$ and there will be nontrivial deformations corresponding to points in the cone $\mathscr{Q}$; some of these deformations form a union of two vector spaces isomorphic to $H^1(\Gamma, \mathfrak{g}_1)$ with a common origin. This family is constructed as follows. Since $\mathfrak{g}_1$ has a G_0-invariant complex structure, its complexification decomposes

$$\mathfrak{g}_1 \otimes \mathbf{C} = \mathfrak{g}_1^+ \oplus \mathfrak{g}_1^-$$

as a sum of two G_0-modules each isomorphic to $\mathfrak{g}_1$. (As matrices, the elements of $\mathfrak{g}_1^+$ have last column $(z_1, \ldots, z_{n+1}, 0)$ and all other entries are zero; the elements of $\mathfrak{g}_1^-$ are the transposes of the elements of $\mathfrak{g}_1^+$.) Furthermore each $\mathfrak{g}_1^\pm$ is an abelian subalgebra of $\mathfrak{g}_\mathbf{C}$ and hence $H^1(\Gamma, \mathfrak{g}_1^\pm)$ is isotropic with respect to (9-1). Hence the union $H^1(\Gamma, \mathfrak{g}_1^+) \cup H^1(\Gamma, \mathfrak{g}_1^-)$ embeds in $\mathcal{Q}$. One may regard the deformations corresponding to points in this union as "affine deformations" of ρ, having ρ as linear part and a representative cocycle $\Gamma \to \mathfrak{g}_1^\pm$ as translational part. We do not know if these deformations are the only ones possible, or equivalently, if ρ can be approximated by representations with image Zariski-dense in $G_\mathbf{C}$, for $n > 1$.

Consider the case $n = 1$. Suppose that $\Gamma \subset U(1, 1) = G_0$ is the fundamental group of a closed oriented surface M of genus $g > 1$. We consider deformations of the inclusion $\rho : \Gamma \hookrightarrow U(1, 1) \hookrightarrow SL(3, \mathbf{C}) = G_\mathbf{C}$. Now the complexification $\mathfrak{R}(\Gamma, GL(2, \mathbf{C}))$ of $\mathfrak{R}(\Gamma, U(1, 1))$ is nonsingular at ρ and has dimension $8g - 3$; its saturation $\mathcal{S}_\mathbf{C} = SL(3, \mathbf{C}) . \mathfrak{R}(\Gamma, GL(2, \mathbf{C}))$ has dimension $8g + 1$. (See Goldman [G1, G2] for calculations.) Now the Zariski normal space to $\mathcal{S}_\mathbf{C}$ in $\mathfrak{R}(\Gamma, SL(3, \mathbf{C}))$ is the cohomology group $H^1(\Gamma, \mathfrak{g}_1 \otimes \mathbf{C}) \cong \mathbf{C}^{8(g-1)}$. The map given by trace $\mathrm{tr} : \mathfrak{g}_0 \otimes \mathbf{C} \to \mathbf{C}$ induces an isomorphism $\mathrm{tr}_* : H^2(\Gamma, \mathfrak{g}_0 \otimes \mathbf{C}) \to H^2(\Gamma, \mathbf{C})$ and it follows from Poincaré duality that the bilinear form

$$H^1(\Gamma, \mathfrak{g}_1 \otimes \mathbf{C}) \otimes H^1(\Gamma, \mathfrak{g}_1 \otimes \mathbf{C}) \longrightarrow H^2(\Gamma, \mathfrak{g}_0 \otimes \mathbf{C}) \xrightarrow{\mathrm{tr}_*} \mathbf{C}$$

is nondegenerate. Thus Theorem 3 implies that the analytic germ of $\mathfrak{R}(\Gamma, SL(3, \mathbf{C}))$ is isomorphic to the quadratic cone $\mathbf{C}^{8g-1} \times \mathcal{Q}$ where $\mathcal{Q}$ is the cone in $\mathbf{C}^{8(g-1)}$ on a quadric hypersurface. The isotropic subspaces $H^1(\Gamma, \mathfrak{g}_0^\pm)$ have dimension $4(g - 1)$ and thus are properly contained in $\mathcal{Q}$. It follows that ρ can be approximated by representations $\Gamma \to SL(3, \mathbf{C})$ having Zariski-dense image.

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