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C^k-CONJUGACY OF HOLOMORPHIC FLOWS NEAR A SINGULARITY

by MARC CHAPERON

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1. INTRODUCTION

(1.1) Notation and definitions

Let n denote a fixed positive integer, and let $\mathbf{d}$ be the Lie algebra of those germs at $0 \in \mathbf{C}^n$ of holomorphic vector fields which vanish at the origin. Call two elements X and Y of $\mathbf{d}$ *conjugate* if there exists a germ $h : (\mathbf{C}^n, 0) \circlearrowleft$ of a holomorphic diffeomorphism such that $h^* Y = X$. For each $X \in \mathbf{d}$, let X^1 stand for the linear part $dX(0) \in \mathfrak{gl}(n, \mathbf{C})$ of X.

Throughout the sequel, we denote by S a *diagonalisable* element of $\mathfrak{gl}(n, \mathbf{C})$. An S-*vector field* is an element X of $\mathbf{d}$ such that S is the semi-simple part of X^1 (thus, every $X \in \mathbf{d}$ is an S-vector field for a unique S). An S-*normal form* is an S-vector field of the form (¹) $S + N$, where N is *polynomial* and commutes whith S in $\mathbf{d}$ (since $S + N$ is an S-vector field, the linear part N^1 is nilpotent and commutes with S).

(1.2) Holomorphic classification: known results and obstacles

Define S to be in the *Poincaré domain* when the convex hull of its spectrum in $\mathbf{C}$ does not contain the origin.

Theorem (Poincaré-Dulac). — If S *is in the Poincaré domain, then*

(i) *the centraliser of* S *in* $\mathbf{d}$ *is finite dimensional and consists of polynomial vector fields—in particular, the degree of an* S-*normal form cannot be arbitrarily high*;

(ii) *every* S-*vector field is conjugate to an* S-*normal form.*

The reader is referred to (5.1) below for a proof of (i). The "preparation lemma" we shall state in section 2 is a natural generalisation of (ii) to *all* elements of $\mathbf{d}$.

The Poincaré-Dulac theorem provides very good models: if S is in the Poincaré domain, then (*see* (5.1))

a) every S-normal form generates a holomorphic $\mathbf{C}$-action on $\mathbf{C}^n$, given by an explicit formula;

b) if S lies outside an explicitly known closed subset of codimension one, then the only S-normal form is S itself (in this case, the above result is Poincaré's *linearisation* theorem).

(¹) We shall not distinguish between polynomial vector fields and their germs at 0.

The *Siegel domain* (i.e. the complementary subset of the Poincaré domain in $\mathfrak{gl}(n, \mathbf{C})$) *contains a full measure subset* $\mathscr{S}$ *with the following property*: *if* S *lies in* $\mathscr{S}$, *then every* S-*vector field is conjugate to* S (this is the *Siegel linearisation theorem*). Here are some reasons why this non-trivial, remarkable result is not quite as satisfactory as the previous one:

1) To show that a given S belongs to $\mathscr{S}$, one should check infinitely many inequalities—which might take some time in general.

2) The complementary subset $\mathscr{R}$ of $\mathscr{S}$ in the Siegel domain is dense, and even quite large. Moreover, the few known facts about the conjugacy classes of S-vector fields with $S \in \mathscr{R}$ show that there is little hope for simple results in that direction: for example, for a given S, the space of all *formal* conjugacy classes of S-vector fields may be really huge (and a given formal conjugacy class may contain an infinite dimensional space of conjugacy classes [MR]).

We shall prove that, most of the time (in a very simple sense), these pathological phenomena disappear if holomorphic conjugacy is replaced by C^k-conjugacy, $k \in \mathbf{N}$, defined in (1.3) below. This viewpoint will prove especially useful when dealing with *families* of vector fields—*see* our final remarks.

Our methods are based upon a rather thorough geometric understanding of the complex flows under study (sections 3 and 4): even in cases when the Siegel theorem holds true, this is an addition to our knowledge of the subject, allowing one to estimate the extent to which two different (holomorphic) conjugacy classes are geometrically different.

(1.3) C^k-conjugacy: definitions and main results

Given $X \in \mathbf{d}$, recall that a *representative* of X is a holomorphic vector field $\widetilde{X}$ on some open neighbourhood dom $\widetilde{X}$ of 0 in $\mathbf{C}^n$, such that X is the germ of $\widetilde{X}$ at 0—in other words, germ X, viewed as a convergent power series, is the Taylor expansion of $\widetilde{X}$ at 0. The foliation defined by X is the foliation of $(\mathrm{dom}\, \widetilde{X}) \setminus \widetilde{X}^{-1}(0)$ by holomorphic curves everywhere tangent to $\widetilde{X}$. The germ of this foliation at 0 depends only on X, and will be denoted by $\mathscr{F}X$.

For each $k \in \mathbf{N}$, call two elements X and Y of $\mathbf{d}$

— C^k-*equivalent* if there exists a germ of a C^k-diffeomorphism $(\mathbf{C}^n, 0) \circlearrowleft$ (viewing $\mathbf{C}^n$ as $\mathbf{R}^{2n}$) sending $\mathscr{F}X$ onto $\mathscr{F}Y$;

— C^k-*conjugate* if they admit representatives $\widetilde{X}$ and $\widetilde{Y}$ respectively with the following property: there exists a C^k-diffeomorphism $h : \mathrm{dom}\, \widetilde{X} \to \mathrm{dom}\, \widetilde{Y}$ such that, for every $v \in \mathrm{dom}\, \widetilde{X}$, the image $h \circ c$ of the local integral curve $c : (\mathbf{C}, 0) \to (\mathbf{C}^n, v)$ of $\widetilde{X}$ is the local integral curve $(\mathbf{C}, 0) \to (\mathbf{C}^n, h(v))$ of $\widetilde{Y}$ (in the language of [Ch 86] (p. 68-70), the $\mathbf{C}$-action germs generated by X and Y are C^k-isomorphic).

Clearly, conjugacy implies C^k-conjugacy, which implies C^k-equivalence. Call S

— *hyperbolic* if its eigenvalues are simple and any two of them are **R**-independant;

— *weakly hyperbolic* if the closed line segment between two of its eigenvalues never contains 0 (thus, if S is hyperbolic, or in the Poincaré domain, it is weakly hyperbolic).

In section 5, we shall state and prove a more general version of the following result [Ch 80]:

Theorem 1. — *If* S *is hyperbolic, every* S-*vector field is* C^0-*conjugate to* S.

Now, Guckenheimer [G], Camacho, Kuiper, Palis [CKP] and Ladis [I] were able to determine the C^0-equivalence class of a hyperbolic S inside $\mathfrak{gl}(n, \mathbf{C})$, hence

Corollary 1. — *Let* S *and* T *be hyperbolic elements of* $\mathfrak{gl}(n, \mathbf{C})$, *and let* $\mathrm{Spec}^{-1}\,\mathrm{S}$ (*resp.* $\mathrm{Spec}^{-1}\,\mathrm{T}$) *denote the set of all inverses of eigenvalues of* S (*resp.* T).

(i) *If* S *and* T *are in the Poincaré domain, then any* S-*vector field is* C^0-*equivalent to any* T-*vector field.*

(ii) *In the remaining case, the following two conditions are equivalent*:

— *Any* S-*vector field is* C^0-*equivalent to any* T-*vector field.*

— *There exists* $A \in \mathrm{GL}(2, \mathbf{R})$ *such that* $A(\mathrm{Spec}^{-1}\,\mathrm{S}) = \mathrm{Spec}^{-1}\,\mathrm{T}$.

In the Siegel domain, this result—stated as a conjecture in [CKP]—exhibits a rigidity phenomenon which makes Theorem 1 much more surprising than the Grobman-Hartman linearisation theorem (*see* [A] or [Ch 86]), despite superficial analogy. For this very reason, the proof of Theorem 1 is hard and uses the following complex analogue of a theorem of Sternberg stated in (2.1) below:

Theorem 2. — *If* S *is weakly hyperbolic, then, for every positive integer* k, *each* S-*vector field is* C^k-*conjugate to an* S-*normal form* (1).

This statement is not quite as simple as Theorem 1—however, *see* Theorem 4 in the Conclusion—but its much nicer proof is very instructive.

(1.4) Plan of the article

In section 2, the real version of Theorem 2 is used as a good introduction to the general idea of our proofs (paragraph (2.3)), a good excuse for stating our main local lemmas (the Isolating Block Lemma and the Extension lemma in (2.2), the Preparation Lemma in (2.3)) and a good reason for introducing the basic notion of a strongly invariant manifold (paragraph (2.3)).

In section 3, we explain the (global) structure of the complex flow generated by a weakly hyperbolic S, and discuss some further applications of our analysis. In

(1) The degree of which admits an upper bound depending only (and nicely) on S and k.

section 4, we show that, if S is weakly hyperbolic, an S-vector field of the form provided by the Preparation Lemma can really be considered a small perturbation of S near 0, and prove Theorem 2.

In Section 5, we state and prove the generalisation of Theorem 1 already mentioned.

In the conclusion, we discuss some related problems and results.

Aknowledgements.

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2. GENERAL IDEA AND MAIN LOCAL TOOLS OF OUR PROOFS

(2.1) Introduction: the real case

In this paragraph, we denote by R a diagonalisable (over $\mathbf{C}$) element of $\mathfrak{gl}(n, \mathbf{R})$. Define a *real* R*-vector field* and a *real* R*-normal form* as in (1.1), replacing $\mathbf{d}$ by the Lie algebra of those germs at $0 \in \mathbf{R}^n$ of C^∞-vector fields which vanish at 0.

The *stable subspace* E^+ of R and its *unstable subspace* E^- are defined as follows: E^+ is the unstable subspace of $-$ R, and E^- is the sum of those R-invariant subspaces of $\mathbf{R}^n$ corresponding to eigenvalues c with $\mathcal{R}e\, c > 0$, i.e. the set of those $v \in \mathbf{R}^n$ such that $\lim_{t \to -\infty} e^{t\mathrm{R}} v = 0$. For this last reason, a real R-normal form is tangent to E^- (and to E^+) at each of its points (*see* [Ch 86], p. 141, Lemme—an alternative, silly proof would follow from the calculations in (5.1) below). The following result is a (classical) particular case of [Ch 86 (4.4.2*b*), Théorème 1]:

Preparation Lemma. — For every $k \in \mathbf{N}$, *each real* R*-vector field is* C^∞*-conjugate (in the usual sense) to a real* R*-vector field* X *which has k-th order contact with a real* R*-normal form along* $E^+ \cup E^-$*—in particular, by the above remark,* X *is tangent to* E^- *(and to* E^+*) " at each of its points ".*

(In fact, the result proven in [Ch 86] is that one can take the same X for every k. The advantage of our weaker statement is that its proof can lead to effective computations.)

Call R *(real) hyperbolic* if none of its eigenvalues lies on the imaginary axis, i.e. $E^+ \oplus E^- = \mathbf{R}^n$ (thus, R need not be hyperbolic as an element of $\mathfrak{gl}(n, \mathbf{C})$). Here comes our real version of Theorem 2:

Theorem (Sternberg). — If R *is (real) hyperbolic, then, for each positive integer k, every real* R*-vector field is* C^k*-conjugate to a real* R*-normal form.*

(2.2) Proof of Sternberg's theorem

We shall first state two key lemmas, in the general form needed later on.

Notation. — Let Q denote the (riemannian) product $M \times E^+ \times E^-$, where M is a compact riemannian manifold and E^+, E^- denote two euclidean spaces with $E^+ \neq \{0\}$. Let W^+, W^- and Σ be the three submanifolds of Q defined by

$$W^+ = M \times E^+ \times \{0\}, \quad W^- = M \times \{0\} \times E^-,$$
$$\Sigma = W^+ \cap W^- = M \times \{0\} \text{ (1)}.$$

The canonical (orthogonal) projections of Q onto E^+ and E^- will be denoted by $x \mapsto x_+$ and $x \mapsto x_-$ respectively, and the euclidean norms will be written $v \mapsto |v|$. We let (*see* Fig. 1)

$$\begin{cases} B = \{x \in Q : |x_+| \leqslant 1 \text{ and } |x_-| \leqslant 1\} \\ \partial^+ B = \{x \in B : |x_+| = 1\}, \quad \partial^- B = \{x \in B : |x_-| = 1\} \\ W_1^+ = W^+ \cap B, \quad W_1^- = W^- \cap B, \end{cases}$$

hence in particular $\partial B = \partial^+ B \cup \partial^- B$. Denote the scalar products of E^+ and E^- by $(x,y) \mapsto (x \mid y)$, and the differential $TQ \to E^\pm$ of the projection $x \mapsto x_\pm$, by $v \mapsto v_\pm$.

Isolating Block Lemma (Fig. 1). — Let ξ *be a smooth vector field on some open neighbourhood of* B *in* Q. *Assume that there exist positive constants* k_+ *and* k_- *such that every* $x \in B$ *satisfies*

$$(1) \qquad (\xi(x)_+ \mid x_+) \leqslant -k_+ |x_+|^2 \quad \textit{and} \quad (\xi(x)_- \mid x_-) \geqslant k_- |x_-|^2.$$

Then, the flow (Φ^t) *generated by* ξ *has the following properties:*

(i) *The two functions* $r_+, r_- : B \to [0, \infty]$ *defined by*

$$\begin{cases} r_+(x) = \sup\{t \geqslant 0 : \Phi^{-s}(x) \in B \textit{ for } 0 \leqslant s \leqslant t\} \\ r_-(x) = \sup\{t \geqslant 0 : \Phi^{s}(x) \in B \textit{ for } 0 \leqslant s \leqslant t\} \end{cases}$$

satisfy

$$\begin{cases} r_+^{-1}(\infty) = W_1^-, \quad r_-^{-1}(\infty) = W_1^+ \\ \Phi^{-r_+(x)}(x) \in \partial^+ B \textit{ for } x \in B \backslash W^-, \quad \Phi^{r_-(x)}(x) \in \partial^- B \textit{ for } x \in B \backslash W^+. \end{cases}$$

(ii) *The two smooth functions* $x \mapsto -r_+(x)/\mathrm{Log}\,|x_+|$ and $x \mapsto -r_-(x)/\mathrm{Log}\,|x_-|$, *defined on* $B \backslash (W^- \cup \partial^+ B)$ *and* $B \backslash (W^+ \cup \partial^- B)$ *respectively, are bounded and have positive lower bounds.*

(iii) *If* $\mathscr{B}$ *is a basis for the filter of neighbourhoods of* ∂W_1^+ *in* $\partial^+ B$, *then the sets* $\widetilde{V} = \{\Phi^t(x) : 0 \leqslant t \leqslant r_-(x) \textit{ and } x \in V\} \cup W_1^-$ *with* $V \in \mathscr{B}$ *form a basis* $\widetilde{\mathscr{B}}$ *for the filter of neighbourhoods of* $W_1^+ \cup W_1^-$ *in* B.

(1) Our results can be extended to the case when Q is the direct sum of two arbitrary riemannian vector bundles W^+ and W^- over M.

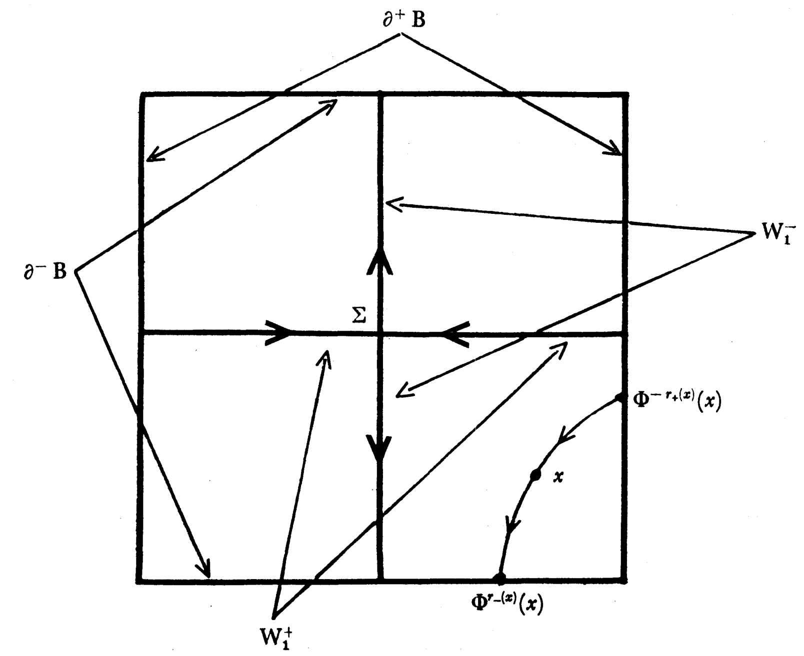

FIG. 1

Proof. — There exist positive constants K_+ and K_- such that, for every $x \in B$, the following estimates are satisfied:

$$\begin{cases} -K_+ \,|x_+|^2 \leqslant (\xi(x)_+ \mid x_+) \leqslant -k_+ \,|x_+|^2 \\ k_- \,|x_-|^2 \leqslant (\xi(x)_- \mid x_-) \leqslant K_- \,|x_-|^2. \end{cases}$$

" Integrating " these inequations (*see* [Ch 86], p. 362, Lemme), we get

$$(2) \qquad \begin{cases} e^{k_+ t}\,|x_+| \leqslant |\Phi^{-t}(x)_+| \leqslant e^{K_+ t}\,|x_+| & \text{for } 0 \leqslant t \leqslant r_+(x) \\ e^{k_- t}\,|x_-| \leqslant |\Phi^{t}(x)_-| \leqslant e^{K_- t}\,|x_-| & \text{for } 0 \leqslant t \leqslant r_-(x). \end{cases}$$

From this and the (therefore) obvious fact that each function $t \mapsto |\Phi^{-t}(x)_+|$, $0 \leqslant t \leqslant r_+(x)$ and $t \mapsto |\Phi^{t}(x)_-|$, $0 \leqslant t \leqslant r_-(x)$, is either zero, or strictly increasing, assertions (i) and (ii) follow at once.

Assertion (iii) is obtained as follows: for each $x \in \partial^+ B$, the greatest distance Δ_x between $\Phi^t(x)$ and $W_1^+ \cup W_1^-$ for $0 \leqslant t \leqslant r_-(x)$ can be estimated quite easily: either $x \in W_1^+$, in which case $\Delta_x = 0$, or, by the growth properties of $t \mapsto |\Phi^{-t}(x)_-|$ and $t \mapsto |\Phi^t(x)_+|$,

$$\Delta_x = |\Phi^{t_x}(x)_+| = |\Phi^{t_x}(x)_-|,$$

where $t_x \in [0, r_-(x)]$ is defined by this equality. Therefore, (2) yields

$$(3) \qquad \Delta_x \leqslant |x_-|^{k_+/(K_+ + K_-)}.$$

Similarly, the least distance δ_x between $\Phi^t(x)$ and $W_1^+ \cup W_1^-$ for $0 \leqslant t \leqslant r_-(x)$ is either zero, or $\min\{|x_-|, |\Phi^{r_-(x)}(x)_+|\}$, hence, by (2),

$$(4) \qquad \delta_x \geqslant \min\{|x_-|, |x_-|^{K_+/k_-}\}.$$

Assertion (iii) follows at once from (3) and (4).

Note. — The name of our lemma comes from the fact that B is an isolating block for (Φ^t) in the sense of Conley (*see* for example [CZ]).

Notation and definition. — For each smooth vector field X on a manifold (with corners) C, we let f_X denote the flow of X (defined on a subset $\mathrm{dom} f_X$ of $\mathbf{R} \times C$), and sometimes write $f_X^t(x)$ instead of $f_X(t, x)$ for $(t, x) \in \mathrm{dom} f_X$. A subset U of C is called X-*saturated* if $\{t : f_X^t(x) \in U\}$ is an interval for every $x \in U$.

Extension lemma. — For each ξ as in the Isolating Block Lemma and each positive integer k, there exist an integer $m \geqslant k$ and a C^1-neighbourhood $\mathcal{N}$ of $\xi|_B$ in the space of smooth vector fields on B such that the following hold true:

(i) *If $\mathcal{N}_0$ denotes the set of those $X \in \mathcal{N}$ which are tangent to W_1^+ and to W_1^- at each of its points, then, Hypothesis* (1) *of the Isolating Block Lemma is satisfied (with different positive constants) if $\xi|_B$ is replaced by any $X \in \mathcal{N}_0$.*

(ii) *Let X and Y be two arbitrary elements of $\mathcal{N}_0$, having m-th order contact along $W_1^+ \cup W_1^-$, and let $h : \omega \to \omega'$ be a C^m-diffeomorphism with the following properties*:

a) *ω is the X-saturated intersection of an open neighbourhood of Σ in Q and an open neighbourhood of $W_1^+ \backslash \Sigma$ in $B \backslash (W_1^- \cup \partial^- B)$;*

b) *ω' is Y-saturated, open in B, and $h_*(X|_\omega) = Y|_{\omega'}$;*

c) *h has m-th order contact with the identity along $W^+ \cap \omega$.*

Then, the set

$$\Omega = \{f_X^t(x) : x \in \omega,\ t \geqslant 0,\ (t, x) \in \mathrm{dom} f_X \text{ and } (t, h(x)) \in \mathrm{dom} f_Y \backslash (f_Y)^{-1}(\partial^- B)\} \cup W_1^- \backslash \partial^- B$$

is an X-saturated open subset of B, and h extends to a unique C^k-diffeomorphism H of Ω onto the Y-saturated open subset

$$\Omega' = \{f_Y^t(y) : y \in \omega',\ t \geqslant 0,\ (t, y) \in \mathrm{dom} f_Y \text{ and } (t, h^{-1}(y)) \in \mathrm{dom} f_X \backslash (f_X)^{-1}(\partial^- B)\} \cup W_1^- \backslash \partial^- B$$

such that $H_*(X|_\Omega) = Y|_{\Omega'}$. Moreover, H has k-th order contact with the identity along $W^+ \cup W^-$.

Proof. — (i) is clear, and so is the fact that $\Omega \backslash W^-$ (resp. $\Omega' \backslash W^-$) is an X-saturated (resp. Y-saturated) open subset of B—whatever m may be. Moreover, still assuming $m \geqslant k$ arbitrary, $H|_{\Omega \backslash W^-}$ is uniquely determined by the conjugacy condition, which implies that it has to be the C^m-diffeomorphism onto $\Omega' \backslash W^-$ defined by $H(f_X^t(x)) = f_Y^t(h(x))$ for $t \geqslant 0$, $x \in \omega$ and $f_X^t(x) \in \Omega$. Therefore, uniqueness comes from the following fact: if Ω *is* a neighbourhood of $W_1^-\backslash\partial^- B$ in B, then, as E^+ is not trivial, $H|_{\Omega\backslash W^-}$ has at most one continuous extension to Ω. Thus, we just have to prove

Lemma 1. — If m is large enough and $\mathcal{N}$ small enough, then, under the hypotheses of (ii), *there exists an open subset* U *of* B *with* $\Sigma \subset U \subset \Omega$ *such that the mapping*

$$U \ni v \mapsto \begin{cases} H(v) & \text{if } v \notin W^- \\ v & \text{if } v \in W^- \end{cases}$$

is of class C^k *and has k-th order contact with* Id *along* W^-.

Indeed, Lemma 1 implies our result: if its conclusion is true, then, by the analogue of (2) for X and Y (which implies that W_1^- is their unstable manifold at Σ) and the fact that they coincide up to order k along W_1^-, we shall have $H(f_X^t(v)) = f_Y^t(H(v))$ for $t \geqslant 0$, $v \in U$ and $f_X^t(v) \in B\backslash\partial^- B$ and, by the flow-box theorem, Ω will be a neighbourhood of $W_1^-\backslash\partial^- B$ and H will have k-th order contact with Id along $W_1^-\backslash\partial^- B$.

We shall give a rather tricky proof of Lemma 1, again adapted from [Ch 86] ((4.2.3), théorème 2). Of course, we exclude the case when E^-—and, therefore, Lemma 1—is trivial. Let $u \in C^\infty(\mathbf{R}, [0, 1])$ satisfy $u^{-1}(1) = (-\infty, 1/3]$ and $u^{-1}(0) = [2/3, \infty)$, and let X_0, $\tilde{X}$, $\tilde{Y}$ be the smooth vector fields on Q defined by

$$\begin{cases} X_0(x) = (0, -x_+, x_-) \\ \tilde{X}(x) = \begin{cases} X_0(x) + u(|x_+|)\, u(|x_-|)\,(X(x) - X_0(x)) & \text{if } x \in B \\ X_0(x) = \tilde{Y}(x) & \text{if } x \notin B \end{cases} \\ \tilde{Y}(x) = X_0(x) + u(|x_+|)\, u(|x_-|)\,(Y(x) - X_0(x)) & \text{if } x \in B. \end{cases}$$

As X_0 generates an **R**-action on Q, so do $\tilde{X}$ and $\tilde{Y}$.

Step 1. — If $\mathcal{N}$ has been chosen small enough, then

(i) $\tilde{X}$ *and* $\tilde{Y}$ *satisfy inequations of type* (2) *in the whole of* Q, *and have m-th order contact along* $W^+ \cup W^-$;

(ii) *let* $B' = \{x \in Q : |x_-| < 1/3$ *and* $|x_+| < 1/3\}$ *and* $\omega'' = B' \cap H^{-1}(B')\backslash W^-$; *then,* $\omega^+ = \{f_{\tilde{X}}^t(x) : x \in \omega''$ *and* $t \leqslant 0\}$ *is an* $\tilde{X}$*-saturated open neighbourhood of* $W^+\backslash\Sigma$ *in* $Q\backslash W^-$, *and* $H|_{\omega''}$ *extends to a unique* C^m*-embedding* $\tilde{H} : \omega^+ \to Q$ *such that* $\tilde{H}_*(\tilde{X}|_{\omega^+}) = \tilde{Y}|_{\tilde{H}(\omega^+)}$, *having m-th order contact with* Id *along* $W^+\backslash\Sigma$.

Step 2. — Let $W_r = \{x \in Q : |x_-| < r\}$, $r > 0$, *and let* $x \mapsto x_0$ *denote the projection* $Q \to M$. *If r is small enough, then*

(i) *the mapping* $H_0 : W_r \cap (\omega^+ \cup B) \to Q$ *defined by*

$$H_0(v) = \begin{cases} (\tilde{H}(v_0, v_+, u(1 - |v_+|)\, v_-)_0,\ \tilde{H}(v)_+ + u(|v_+|)\,(v_+ - \tilde{H}(v)_+), \\ \qquad \tilde{H}(v)_- + u(|v_+|)\,(v_- - \tilde{H}(v)_-)) & \text{for } 1/3 \leqslant |v_+| \leqslant 2/3 \\ v & \text{for } |v_+| \leqslant 1/3 \\ \tilde{H}(v) & \text{for } |v_+| \geqslant 2/3 \end{cases}$$

is a C^m*-embedding, having m-th order contact with* Id *along* $W^+ \cup W^-$;

(ii) *for r small enough, the vector field* Z *on* W_r *given by*

$$Z(v) = H_0^* \widetilde{Y}(v) \quad \textit{if } v \in B \quad \textit{and} \quad Z(v) = X_0(v) \quad \textit{if } v \notin B$$

is of class C^{m-1} *and has* $(m-1)$*-th order contact with* $\widetilde{X}$ *along* $W^+ \cup W^-$.

Outside B, we have that $H_0^{-1} \circ \widetilde{H} = \mathrm{Id}$ and $\widetilde{X} = Z$; therefore, replacing $(\widetilde{X}, \widetilde{Y}, \widetilde{H})$ by $(\widetilde{X}, Z, H_0^{-1} \circ \widetilde{H})$, we get a more or less standard extension problem, which we shall now solve.

Step 3 ([Ch 86], *théorème A6-5, p. 361*). *Define* $\widetilde{X}_- : Q \to E^-$ *by* $\widetilde{X}_-(x) = \widetilde{X}(x)_-$, *and let*

$$\begin{cases} \lambda_X = \inf\left\{ \left(\dfrac{\partial \widetilde{X}_-}{\partial x_-}(x).y \,|\, y \right) \Big/ |y|^2 \;: x \in W^+,\, y \in E^- \backslash \{0\} \right\} \\ \mu_X^+ = \sup\{ L_{\widetilde{X}}\, G(x)\,(y,y)/2\,|y|^2 : y \in T_x Q \backslash \{0\},\ x \in W^+ \} \\ \mu_X^- = \inf\ \{ L_{\widetilde{X}}\, G(x)\,(y,y)/2\,|y|^2 : y \in T_x Q \backslash \{0\},\ x \in W^+ \}, \end{cases}$$

where $L_{\widetilde{X}}\, G$ *is the Lie derivative of the riemannian metric of* Q *with respect to* $\widetilde{X}$. *If we have* $\lambda_X > 0$ *(hence* $\mu_X^+ > 0$*),* $\mu_X^- < 0$ *and* $m > k + 1 + (\mu_X^+ - k\mu_X^-)/\lambda_X$, *then, for r small enough, the family* $(f_Z^t \circ f_{\widetilde{X}}^{-t}|_{W_r})_{t \geqslant 0}$ *is well-defined and converges in the* C^k *sense to an embedding* H_r *which has k-th order contact with* Id *along* $W^+ \cup W^-$.

We can now explain *how to choose m and $\mathcal{N}$*: m is the least integer greater than $k + 1 + (\mu_\xi^+ - k\mu_\xi^-)/\lambda_\xi$ (we have $\mu_\xi^- < 0 < \lambda_\xi$ because E^+ and E^- are non-trivial), and $\mathcal{N}$ is such that each $X \in \mathcal{N}_0$ satisfies (i), $\mu_X^- < 0 < \lambda_X$ and $m > k + 1 + (\mu_X^+ - k\mu_X^-)/\lambda_X$.

Then, under the above hypotheses, the definition of H_r implies that $H_r^*(Z|_{H_r(W_r)}) = \widetilde{X}|_{W_r}$ and $H_r(v) = v$ for $|v_+| \geqslant 1$ — by Step 1 (i) and the fact that $Z = \widetilde{X} = X_0$ in $\{x \in W_r : |x_+| \geqslant 1\}$. Therefore, if D_r denotes the X-saturated domain of $H_0 \circ H_r = \widetilde{H}_r$, we obtain

$$\begin{cases} \widetilde{H}_r^*(\widetilde{Y}|_{H_r(D_r)}) = \widetilde{X}|_{D_r} \\ x \in D_r \quad \text{and} \quad \widetilde{H}_r(x) = \widetilde{H}(x) \quad \text{for every } x \in \omega^+ \text{ with } |x_+| \text{ large enough.} \end{cases}$$

Thus, by Step 1 (i), we have that $\widetilde{H}_r = \widetilde{H}$ in $W_r \cap \omega^+$, hence in particular

$$U = \{x \in W_r : |x_+| < r\} \subset \Omega \quad \text{and} \quad H|_U = \widetilde{H}_r|_U$$

if r is small enough for U and $\widetilde{H}_r(U)$ to lie in B′ and be contained in $\{f_X^t(\omega) : t \geqslant 0\} \cup W^-$ (which is a neighbourhood of Σ in Q by the Isolating Block Lemma (iii), applied in $\{x \in \overline{W}_R : |x_+| \leqslant R\}$ for some $R \leqslant 1$); indeed, by Step 1 (i) and the definition of Ω, $H|_{U \cap \Omega}$ has to be well-defined and equal to $\widetilde{H}_r|_U$ in the whole domain of definition of $\widetilde{H}_r|_U$, i.e. U. This proves Lemma 1. ∎

We shall now see how to construct C^k-conjugacies using the Extension Lemma. The " Cauchy problem method " introduced here will be omnipresent in the sequel:

Corollary 2. — *Let k, m and $\mathcal{N}_0$ be as in the Extension Lemma. If two elements* X *and* Y *of $\mathcal{N}_0$ have m-th order contact along* $W^+ \cup W^-$, *then, they are C^k-conjugate in a neighbourhood of* Σ. *More precisely, let* V *be the open subset of* $(\partial^+ B \setminus \partial^- B) \times [0, \infty)$ *consisting of those* (x, t) *such that both* $f^t_X(x)$ *and* $f^t_Y(x)$ *are well-defined and lie in* $B \setminus \partial^- B$, *and let* φ *and* ψ *denote the smooth embeddings of* V *into* $B \setminus \partial^- B$ *given by* $\varphi(x, t) = f^t_X(x)$ *and* $\psi(x, t) = f^t_Y(x)$; *then, the hypotheses of the Extension Lemma* (ii) *are satisfied by* $\omega = \varphi(V)$ *and* $h = \psi \circ \varphi^{-1}$.

Proof. — By the Extension Lemma (i) and the Isolating Block Lemma, φ and ψ are smooth embeddings of V into $B \setminus (W^- \cup \partial^- B)$, the images of which are open neighbourhoods of $W^+_1 \setminus \Sigma$. Moreover, as X and Y have m-th order contact along W^+_1 and are tangent to it, φ and ψ have m-th order contact along $\varphi^{-1}(W^+) = \psi^{-1}(W^+)$, hence Corollary 2. ∎

Proof of Sternberg's theorem. — Given a positive integer k and a (real) hyperbolic $R \in \mathfrak{gl}(n, \mathbf{R})$, with stable subspace E^+ and unstable subspace E^-, we shall prove *there exists an integer* $m \geq k$ *with the following property: if a real* R*-vector field* Z *has m-th order contact with a real* R*-normal form* $R + N$ *along* $E^+ \cup E^-$, *then* Z *is C^k-conjugate to* $R + N$.

This and the Preparation Lemma (2.1) clearly imply Sternberg's theorem. Now, there obviously exists a euclidean structure on $\mathbf{R}^n$ for which E^+ and E^- are orthogonal, and such that the hypotheses of the Isolating Block Lemma are satisfied with $\xi = R$, $Q = \mathbf{R}^n = E^+ \oplus E^- \simeq E^+ \times E^-$, $W^+ = E^+$, $W^- = E^-$ and $\Sigma = \{0\}$. Let $\mathcal{N}_0$, m be as in the Extension Lemma (with this choice of ξ), and let Z be a real R-vector field having m-th order contact with a real R-normal form $R + N$ along $E^+ \cup E^-$; if $\widetilde{Z}$ denotes a representative of Z, there exists (*see* the end of (4.4) below) $A \in GL(n, \mathbf{R})$, with $A^* R = R$, such that $X = (A^*(R + N))|_B$ and $Y = (A^* \widetilde{Z})|_B$ satisfy the hypotheses of Corollary 2, hence our result. ∎

Important remark. — In Corollary 2, we obtained a local conjugacy h between X and Y as the solution of the Cauchy Problem " $h = \mathrm{Id}$ on $\partial^+ B \setminus \partial^- B$ ", which was well-posed because every flow-line of X (or Y) which lies outside $W^- \cup (\partial^+ B \cap \partial^- B)$ intersects $\partial^+ B$ transversally, at exactly one point. When ξ is a (real) hyperbolic R, as in Sternberg's theorem, the essential reason for this is that the cylinder $Q' = \{x \in Q = \mathbf{R}^n : |x_+| = 1\}$ is " a quotient of the f_R-invariant open subset $Q \setminus W^-$ by f_R ", meaning that each flow-line of R which is not contained in W^- intersects Q' transversally, at precisely one point. Now, such a quotient (of course diffeomorphic to the orbit space of $f_R|_{\mathbf{R} \times (Q \setminus W^-)}$) can be constructed in many other ways; here is one: for simplicity, assume that R has only real eigenvalues $c_1, \ldots, c_n$; then, there exists a system $(x_1, \ldots, x_n)$ of real linear coordinates on $\mathbf{R}^n$ such that R is the gradient of the non-degenerate quadratic form $F = \Sigma c_j x_j^2/2$ with respect to the euclidean metric $\Sigma\, dx_j^2$, and Q′ can be replaced by $F^{-1}(c)$ for every *negative c*. Under the hypotheses of Corollary 2 with $\xi = R$ —restricting $\mathcal{N}$ if necessary—, if c is close enough to 0 for the sphere $F^{-1}(c) \cap W^+$ to lie in the interior of B, a local conjugacy between X and Y can be obtained by extending to W^- the unique solution of the Cauchy problem " H = Id in V ", where V is a neighbourhood of $F^{-1}(c) \cap W^+$ in $F^{-1}(c)$.

(2.3) General idea of our proofs

Here and in the sequel, we again denote by S a diagonalisable element of $\mathfrak{gl}(n, \mathbf{C})$. If we wish to prove Theorem 2 as the Sternberg theorem, we have to answer the following two questions:

Question 1. — In the complex case, what would a good Preparation Lemma be—in particular, is there any natural analogue of the stable and unstable subspaces?

Question 2. — If Question 1 admits a positive answer, can we establish Theorem 2 by a " Cauchy problem method " as in the proof of Corollary 2?

The answer to Question 1 is very simple: define a *strongly invariant manifold* (s.i.m.) of S to be a subspace of $\mathbf{C}^n$ which is the unstable subspace E_a^- of aS, *viewed as an element of* $\mathfrak{gl}(2n, \mathbf{R})$, for some $a \in \mathbf{C}$ —in other words, E_a^- is the direct sum of those eigenspaces of S associated to eigenvalues c with $\mathcal{R}e(ac) > 0$, which shows, in particular, that the s.i.m.'s of S are *complex* vector subspaces of $\mathbf{C}^n$ and *there is but a finite number of them.* Moreover, *every* S-*normal form is tangent to every s.i.m.* W *of* S *at each of its points* (*proof*: if $W = E_a^-$, then, considering $\mathbf{C}^n$ as $\mathbf{R}^{2n}$, $a(S + N)$ is a real aS-normal form; therefore, it is tangent to E_a^- at each of its points, hence our result, for E_a^- is a complex subspace).

THROUGHOUT THE SEQUEL, $\mathcal{V}$ DENOTES THE UNION OF THE S.I.M.'S OF S.

Here comes the answer to Question 1:

Complex Preparation Lemma. — For every $k \in \mathbf{N}$, *each* S-*vector field is (holomorphically) conjugate to an* S-*vector field* X *which has k-th order contact with an* S-*normal form along* $\mathcal{V}$*—in particular, by the above remark,* X *is tangent to every s.i.m. of* S *" at each of its points ".*

This result was announced in [Ch 85]; its detailed proof is contained in [Ch 86*a*], together with further information on strongly invariant manifolds.

Notice that S *is in the Poincaré domain if and only if it admits* $\mathbf{C}^n$ *itself as a s.i.m.* Thus, the Poincaré-Dulac theorem is a particular case of the Complex Preparation Lemma.

It will take us the next two sections to see that the answer to Question 2 is *yes*. The general idea is the following: if $(x_1, \ldots, x_n)$ denotes a system of complex linear coordinates on $\mathbf{C}^n$ in which the matrix of S is diagonal, $x_j \circ S = c_j x_j$, $1 \leqslant j \leqslant n$, the role of the quadratic form introduced in the final remark of (2.2) will be played by the *complex* function $\mathbf{F} : \mathbf{C}^n \to \mathbf{C}$ given by

$$\mathbf{F}(v) = \Sigma c_j \, | x_j(v) |^2/2. \tag{5}$$

More precisely, we shall obtain a C^k-conjugacy H between a "prepared" S-vector field and the corresponding S-normal form as the solution of the Cauchy problem "$\mathrm{H} = \mathrm{Id}$ in V" in some kind of an isolating block for the complex flow under study; here, V denotes a suitable neighbourhood of the (compact) subset $\mathbf{F}^{-1}(b) \cap \mathscr{V}$ in $\mathbf{F}^{-1}(b)$ for some regular value b of $\mathbf{F}$.

The proof of Theorem 1 rests on the same idea, but requires a finer analysis; the reason why Theorem 2 is obtained by softer methods is that we do not strive for normal forms of the least possible degree—which can be obtained from Theorem 2, using "explicit" calculations as in the proof of Theorem 1 (*see* [Ch 86*c*]).

3. THE COMPLEX FLOW OF A WEAKLY HYPERBOLIC S

(3.1) Introduction and notation

We assume S weakly hyperbolic and denote by $c_1, \ldots, c_n$ its eigenvalues (repeated according to their multiplicities), and by $(x_1, \ldots, x_n)$ a system of coordinates as in (5) above. We shall study the holomorphic flow σ defined by

$$\sigma(t, v) = e^{t\mathrm{S}}\, v, \qquad t \in \mathbf{C}, \qquad v \in \mathbf{C}^n,$$

with the help of the function $\mathbf{F}$ defined by (5). In the sequel, we denote by a a complex number such that $a\mathrm{S} \in \mathfrak{gl}(2n, \mathbf{R})$ is (real) hyperbolic, and by σ_1 the real flow $f_{a\mathrm{S}}$ of aS. We let $\mathbf{F}_1$ be the real quadratic form $v \mapsto \mathscr{R}e(a\mathbf{F}(v))$.

The idea is to "split" our complex (indeed) flow into two comprehensible *real* flows: the action σ_1, and the action of $\mathbf{C}/a\mathbf{R}$ on the orbit space of σ_1 (1) defined by σ, which will be studied in (3.3). Most of the proofs are straightforward and will be only sketched, for they can be found—in a more general setting—in Section 5 of [Ch 86].

(3.2) First properties

We shall start with a collection of essentially trivial results, *which remain valid when* S *is not weakly hyperbolic, provided it has no zero eigenvalue* (so that some aS can be (real) hyperbolic).

(1) Restricted to the complementary subset of the unstable subspace of aS.

Proposition 1. — *The function* $\mathbf{F}_1$ *is a "Lyapunov function" for* σ_1: *its only critical point is* o, *and, for each* $v \in \mathbf{C}^n \backslash \{0\}$,

(i) *the function* $\psi_v : \mathbf{R} \circlearrowleft$ *given by* $\psi_v(t) = \mathbf{F}_1(\sigma_1(t, v))$ *is increasing, and*

(ii) ψ_v *is bounded above (resp. below) if and only if* v *belongs to the stable (resp. unstable) subspace* $W_0^+ = \bigoplus_{\mathcal{R}e(ac_j)<0} Ox_j$ *(resp.* $W_0^- = \bigoplus_{\mathcal{R}e(ac_j)>0} Ox_j$*) of* aS, *where* $Ox_j = \bigcap_{k \neq j} x_k^{-1}(0)$.

Proof. — This is trivial, since $\psi_v(t) = \Sigma \mathcal{R}e(ac_j) \mid e^{ac_j t} x_j(v)|^2/2$ and (therefore) $\psi_v'(t) = \Sigma(\mathcal{R}e(ac_j))^2 \mid e^{ac_j t} x_j(v)|^2$. ■

Corollary 3. — *For each negative real number* c, $Q_c = \mathbf{F}_1^{-1}(c)$ *is a quotient of the* σ*-invariant open subset* $E_c = \mathbf{C}^n \backslash W_0^-$ *by* σ_1 *in the sense of* (2.2), *final remark.* ■

Here comes the "splitting" of σ mentioned at the end of (3.1):

Corollary 4 ([Ch 86], (5.2), *proposition 3*). — *Under the hypotheses of Corollary 3, the mapping* $\varphi_c : Q_c \times \mathbf{R} \to \mathbf{C}^n$ *given by* $\varphi_c(x, t) = \sigma_1(t, x)$ *is a diffeomorphism onto* E_c. *For every* $(x, s) \in Q_c \times \mathbf{R}$ *and every* $(s', t) \in \mathbf{R}^2$,

$$(\varphi_c^* e^{(s'+it)aS})(x, s) = \left(\Phi_c^t(x), s + s' + \int_0^t g_c \circ \Phi_c^u(x)\, du\right),$$

where the flow (Φ_c^t) *on* Q_c *and the function* $g_c : Q_c \to \mathbf{R}$ *are defined as follows: if* ξ_c *denotes the infinitesimal generator of* (Φ_c^t), *then, for each* $x \in Q_c$, $\xi_c(x) = ia\mathrm{S}(x) - g_c(x)\, a\mathrm{S}(x)$. *Thus, the flowlines of* ξ_c *are precisely the intersections of* Q_c *with the orbits of* $\sigma|_{\mathbf{C} \times E_c}$. ■

(3.3) Structure of the flow (Φ_c^t)

Hypotheses and notation. — We assume S weakly hyperbolic, and endow $\mathbf{C}^n$ with the hermitian norm $v \mapsto |v|$ defined by $|v|^2 = \Sigma |x_j(v)|^2$. For each $\mathrm{I} \subset \{1, \ldots, n\}$, we let $E_\mathrm{I} = \bigoplus_{j \in \mathrm{I}} Ox_j$. We still assume $c < 0$, and a is fixed as in (3.1).

Proposition 2 (see *Fig. 2*). — (i) *Let* $\mathcal{K}$ *be the set of all equivalence classes of the equivalence relation* $c_k \in \mathbf{R}c_j$ *between elements* j, k *of* $\{1, \ldots, n\}$. *Each* $\mathrm{I} \in \mathcal{K}$ *determines a unique* $g_\mathrm{I} \in \mathbf{R}$ *such that*

$$a\mathbf{R}c_j = \mathbf{R}(1 - ig_\mathrm{I}) \; (= \{u \in \mathbf{C} : \mathcal{R}e(u(i - g_\mathrm{I})) = 0\}) \quad \textit{for every } j \in \mathrm{I}.$$

(ii) *For each* $\mathrm{I} \in \mathcal{K}$, *let* I_0^+ *and* I_0^- *be the subsets of* $\{1, \ldots, n\}$ *given by* $\mathrm{I}_0^\pm = \{j : \pm\mathcal{R}e(ac_j(i - g_\mathrm{I})) \leqslant 0\}$. *Then,* $E_{\mathrm{I}_0^+}$ *and* $E_{\mathrm{I}_0^-}$ *are s.i.m.'s of* S. *Conversely, every s.i.m. of* S *which is maximal for the inclusion is both of the form* $E_{\mathrm{I}_0^+}$, $\mathrm{I} \in \mathcal{K}$ *and of the form* $E_{\mathrm{J}_0^-}$, $\mathrm{J} \in \mathcal{K}$. ■

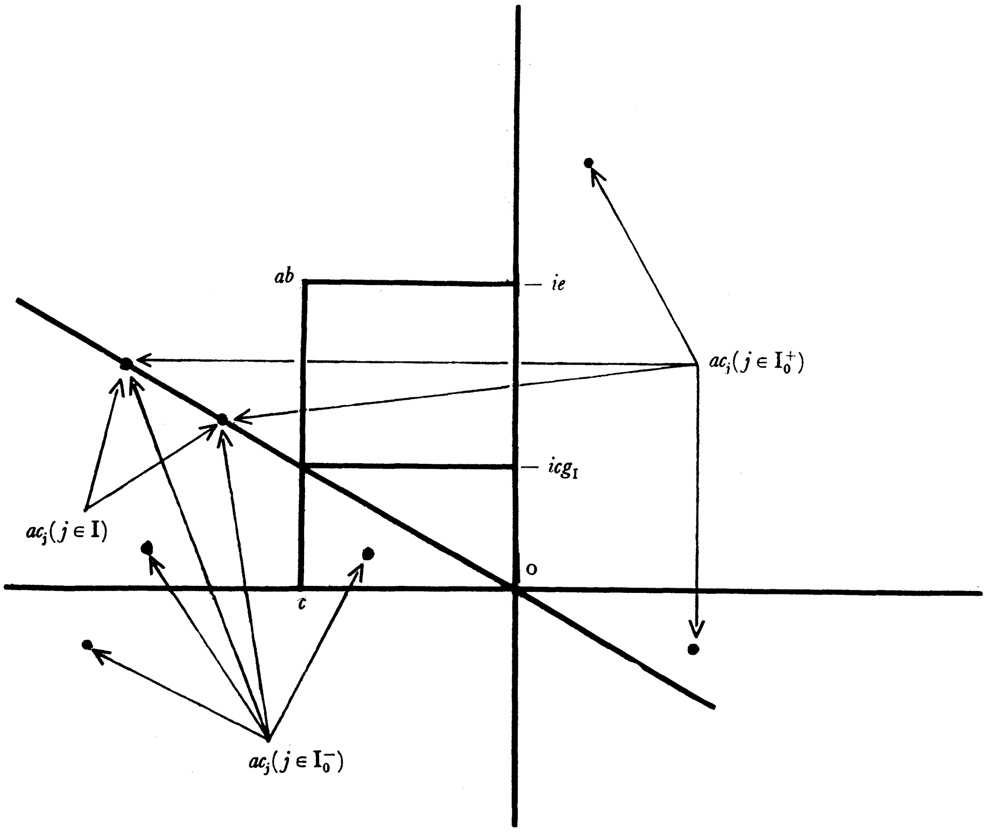

Fig. 2

We can now see how the s.i.m.'s of S appear in the structure of (Φ_c^t).

Notation. — Let $\mathscr{I}_c$ be the set of all $I \in \mathscr{K}$ with $\mathscr{Re}(ac_j) < 0$ for some (and therefore every) $j \in I$; equivalently, $\mathscr{I}_c$ is the set of those $I \in \mathscr{K}$ such that

$$\Sigma_{c,I} = E_I \cap Q_c$$

is not empty—in which case it is a sphere. For each $I \in \mathscr{I}_c$, let $E_{c,I}$ (resp. $Q_{c,I}$) denote the set of those $v \in \mathbf{C}^n$ whose orthogonal projection v_I onto E_I lies in E_c (resp. Q_c), and let $I^\pm = \{j : \pm\mathscr{Re}(ac_j(i - g_I)) < 0\} = I_0^\pm \backslash I$. Clearly, *we can identify* $Q_{c,I}$ *to* $\Sigma_{c,I} \times E_{I^+} \times E_{I^-}$ *via* the canonical isomorphism of $\mathbf{C}^n = E_I \oplus E_{I^+} \oplus E_{I^-}$ onto $E_I \times E_{I^+} \times E_{I^-}$, and the open subset $E_{c,I}$ of $\mathbf{C}^n$ in σ-invariant. Therefore, it is easy to check that both $Q_{c,I}$ and $Q_c \cap E_{c,I}$ are quotients of $E_{c,I}$ by σ_1 in the sense of (2.2), last remark. We let

$$\mathscr{C}_{c,I} : Q_c \cap E_{c,I} \to Q_{c,I}$$

denote the canonical diffeomorphism obtained by following the orbits of σ_1.

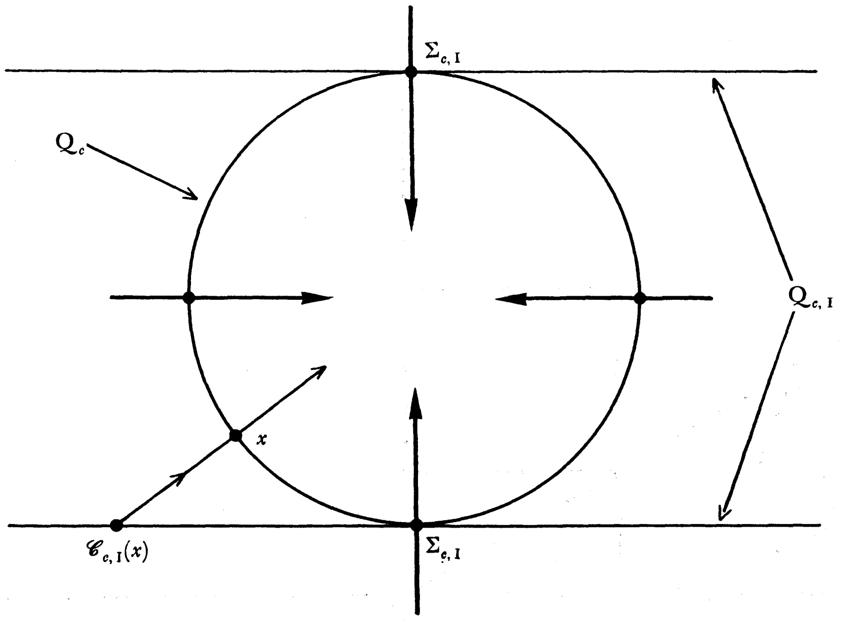

FIG. 3

Proposition 3 ([Ch 86], (5.2), *Proposition 6 and Théorème 1*). — *For each* $\mathrm{I} \in \mathscr{I}_c$, ξ_c *is "normally hyperbolic at* $\Sigma_{c,\mathrm{I}}$, *with global stable manifold* $\mathrm{W}^+_{c,\mathrm{I}} = \mathrm{E}_{c,\mathrm{I}} \cap \mathrm{Q}_c \cap \mathrm{E}_{\mathrm{I}_0^+}$ *and global unstable manifold* $\mathrm{W}^-_{c,\mathrm{I}} = \mathrm{E}_{c,\mathrm{I}} \cap \mathrm{Q}_c \cap \mathrm{E}_{\mathrm{I}_0^-}$". *More precisely,* $\xi_{c,\mathrm{I}} = (\mathscr{C}_{c,\mathrm{I}})_* \xi_c$ *is given by* $\xi_{c,\mathrm{I}} = (i - g_\mathrm{I})\, a\mathrm{S}|_{\mathrm{Q}_{c,\mathrm{I}}}$; *thus, if we write* $\mathrm{Q}_{c,\mathrm{I}}$ *as* $\Sigma_{c,\mathrm{I}} \times \mathrm{E}_{\mathrm{I}^+} \times \mathrm{E}_{\mathrm{I}^-}$, $\xi_{c,\mathrm{I}}$ *splits into a vector field on* $\Sigma_{c,\mathrm{I}}$ *(namely,* $\xi_c|_{\Sigma_{c,\mathrm{I}}}$*), a linear vector field on* $\mathrm{E}_{\mathrm{I}^+}$ *whose stable subspace is* $\mathrm{E}_{\mathrm{I}^+}$ *itself, and a linear vector field on* $\mathrm{E}_{\mathrm{I}^-}$ *whose unstable subspace is the whole of* $\mathrm{E}_{\mathrm{I}^-}$. *In particular, the hypotheses of the Isolating Block Lemma* (2.2) *are satisfied by* $(\mathrm{Q}, \mathrm{M}, \mathrm{E}^+, \mathrm{E}^-, \xi) = (\mathrm{Q}_{c,\mathrm{I}}, \Sigma_{c,\mathrm{I}}, \mathrm{E}_{\mathrm{I}^+}, \mathrm{E}_{\mathrm{I}^-}, \xi_{c,\mathrm{I}})$. ■

Thus, in each "chart" $\mathscr{C}_{c,\mathrm{I}}$, the situation is quite simple; we shall now see how these simple situations are glued together. Here is a first step:

Proposition 4 ([Ch 86], (5.2), *Proposition 6 and Théorème 1*). — *We have that*

$$\bigcup_{\mathrm{I} \in \mathscr{I}_c} \mathrm{E}_{c,\mathrm{I}} = \mathrm{E}_c \quad \textit{and} \quad \bigcup_{\mathrm{I} \in \mathscr{I}_c} (\mathrm{W}^+_{c,\mathrm{I}} \cup \mathrm{W}^-_{c,\mathrm{I}}) = \mathrm{Q}_c \cap \mathscr{V}. \ ■$$

We can now state an analogue of Proposition 1:

Proposition 5 ([Ch 86], (5.2), *Théorème 2*). — *The function* $\mathbf{f}_c : \mathrm{Q}_c \to \mathbf{R}$ *given by* $\mathbf{f}_c(x) = \mathscr{R}e(ia\mathbf{F}(x))$ *is a "Lyapunov function" for the flow* (Φ^t_c): *it is proper on* $\mathrm{Q}_c \cap \mathscr{V}$, *its critical set is* $\Sigma_c = \bigcup_{\mathrm{I} \in \mathscr{I}_c} \Sigma_{c,\mathrm{I}}$ *and, for each* $x \in \mathrm{Q}_c \setminus \Sigma_c$,

(i) *the function* $\psi_{c,x}: \mathbf{R} \circlearrowleft$ *defined by* $\psi_{c,x}(t) = \mathbf{f}_c(\Phi_c^t(x))$ *is increasing and*

(ii) $\psi_{c,x}$ *is bounded from above (resp. below) if and only if* x *belongs to* $W_{c,I}^+$ *(resp.* $W_{c,I}^-$*) for some* $I \in \mathscr{I}_c$. ■

Proposition 6. — *For each* $I \in \mathscr{I}_c$, $\mathbf{f}_c(\Sigma_{c,I}) = \{cg_I\}$ *(therefore, these critical values are distinct*—see *Fig. 2); moreover, we have* $cg_I \leqslant cg_J$ *if and only if* $W_{c,I}^- \cap W_{c,J}^+$ *is not empty, in which case it is contained in the* (Φ_c^t)*-invariant sphere* $W_0^+ \cap Q_c$. *Thus, the latter is the set of those* $x \in Q_c$ *for which the function* $\psi_{c,x}$ *of Proposition 5 is bounded, and*

$$\mathbf{f}_c(W_0^+ \cap Q_c) = [\min_{I \in \mathscr{I}_c} cg_I, \max_{I \in \mathscr{I}_c} cg_I].$$

By Proposition 4, for each $I \in \mathscr{I}_c$, *we have that*

$$W_{c,I}^- = (Q_c \cap E_{I_0^-}) \setminus \bigcup_{g_J < g_I} W_{c,J}^- \quad \textit{and} \quad W_{c,I}^+ = (Q_c \cap E_{I_0^+}) \setminus \bigcup_{g_J > g_I} W_{c,J}^+.$$

Proof. — For each $x \in \Sigma_{c,I}$, the definition of g_I implies that

$$\begin{aligned}\mathbf{f}_c(x) &= \sum_j \mathscr{R}e(a(i - g_I)\, c_j)\, |x_j(x)|^2/2 + \mathscr{R}e(g_I\, a\mathbf{F}(x)) \\ &= \mathscr{R}e(g_I\, a\mathbf{F}(x)) = g_I\, \mathbf{F}_1(x) = cg_I.\end{aligned}$$

For the other assertions, *see* Figure 2 and Propositions 2, 4 and 5. ■

Proposition 7. — *The critical points of* $\mathbf{F}$ *are those* $v \in \mathbf{C}^n$ *with* $\sum_{x_j(v) \neq 0} \mathbf{R} c_j \neq \mathbf{C}$. *Thus, the set of its critical values is* $\bigcup_{1 \leqslant j \leqslant n} \mathbf{R}_+ c_j$. ■

Corollary 5. — *A real number* e *is a regular value of* $\mathbf{f}_c$ *if and only if* $b = (c - ie)/a$ *is a regular value of* $\mathbf{F}$, *and* $Q_b' = \mathbf{F}^{-1}(b)$ *then equals* $\mathbf{f}_c^{-1}(e)$; *if we have* $e < \min\{cg_I : I \in \mathscr{I}_c\}$, *then*

(i) *the union of those s.i.m.'s of* S *which do not intersect* Q_b' *is*

$$\hat{\mathscr{V}}_b = W_0^- \cup \bigcup_{I \in \mathscr{I}_c} E_{I_0^-}, \quad \textit{and}$$

(ii) Q_b' *is a quotient of* $Q_c \setminus \bigcup_{I \in \mathscr{I}_c} W_I^-$ *by* (Φ_c^t) *in the sense of* (2.2), *and*

(iii) *each* $Q_b' \cap E_{I_0^+}$ *with* $I \in \mathscr{I}_c$ *is a compact submanifold of* Q_c, *and*

$$Q_b' \cap \mathscr{V} = \bigcup_{I \in \mathscr{I}_c} (Q_b' \cap W_I^+) = \bigcup_{I \in \mathscr{I}_c} (Q_b' \cap E_{I_0^+}).$$

Proof. — The first assertion is clear. Under the hypothesis of (i), it is easily checked ([Ch 86], (5.1), Corollaire 2) that a s.i.m. E_I of S, $I \subset \{1, \ldots, n\}$, is contained in $\hat{\mathscr{V}}_b$ if and only if b does not lie in $\sum_{j \in I} \mathbf{R}_+ c_j$, hence (i) (*see* Fig. 2). Assertion (ii) is a straightforward consequence of Propositions 5 and 6. For each $I \in \mathscr{I}_c$, $Q_b' \cap E_{I_0^+}$ is a submanifold because b is a regular value of $\mathbf{F}|_{E_{I_0^+}}$ (by Proposition 7, applied to $\sigma|_{\mathbf{C} \times E_{I_0^+}}$ instead of σ); it is compact because $\mathbf{f}_c|_{Q_c \cap \mathscr{V}}$ is proper. The last but one equality is a consequence of (ii) and Proposition 4, and the last equality follows from it and the inclusions $W_I^+ \subset E_{I_0^+} \subset \mathscr{V}$, $I \in \mathscr{I}_c$. ■

Let us now see what Corollary 5 implies for σ itself:

Corollary 6. — For every regular value b of $\mathbf{F}$,

(i) *there exists* $a \in \mathbf{C}$ *such that* aS *is (real) hyperbolic, that* $c = \mathcal{R}e(ab)$ *is negative and that (choosing* this a *and* this c *in the above theory)* $e = \mathcal{R}e(iab)$ *is less than* cg_{I} *for every* $\mathrm{I} \in \mathcal{I}_c$, *and*

(ii) *if* $\hat{\mathcal{V}}_b$ *denotes the (σ-invariant) union of those s.i.m.'s of* S *which do not intersect* $\mathrm{Q}'_b = \mathbf{F}^{-1}(b)$, *then* Q'_b *is a quotient by* σ *of the open σ-invariant subset* $\mathrm{E}_b = \mathbf{C}^n \backslash \hat{\mathcal{V}}_b$, *in the sense of the final remark of* (2.2).

Proof. — (i) is clear, and (ii) can therefore be deduced from Corollary 5 (i), using the σ-invariance of $\hat{\mathcal{V}}_b$, Corollaries 3 and 4, and the last assertion of Proposition 6 (*see* [Ch 86], (5.1), Corollaire 3 for a simple direct proof). ■

(3.4) Concluding remarks

With the above notation, the gradient of $\mathbf{f}_c$ with respect to the ambiant metric of course has roughly the same properties as ξ_c. This can be used to study the topology of the cone $\mathbf{F}^{-1}(0)$—and, more generally (using paragraph (5.1)) of [Ch 86] with $r = 2$), of the intersection of two real projective quadrics " in general position ": the advantage of this method would be to avoid the h-cobordism theorem (and its " bad dimensions "), so far necessary in the proof of such results [LM].

Going back to our subject, the advantage of replacing the whole of $\mathbf{C}^n$ by E_b (notations of Corollary 6) is clear, since the orbit space of $\sigma|_{\mathbf{C}\times \mathrm{E}_b}$ is a Hausdorff manifold, diffeomorphic to Q'_b, whereas the orbit space of σ itself is a non-Hausdorff stratified set—the structure of which can easily be investigated *via* the above theory. As there is much current interest in the " quotient structures " induced by foliations or group actions, these non-trivial examples might prove useful. Anyway, it would certainly be interesting to use the above method in the hunting of topological invariants such as that defined by the Camacho-Kuiper-Ladis-Palis theorem (*see* (1.3), Corollary 1).

4. NON-LINEAR COMPLEX FLOWS PROOF OF THEOREM 2

(4.1) Hypotheses and notation

We still assume S weakly hyperbolic, and denote by b an arbitrary regular value of the function $\mathbf{F}$ introduced in (3.1); in order to avoid empty statements, we assume that $\mathrm{Q}'_b = \mathbf{F}^{-1}(b)$ is nonempty—or, equivalently, that b lies in $\sum \mathbf{R}_+ c_j$ (which is always the case when S is in the Siegel domain). The notations will be those of Section 3, with a, c and e as in Corollary 6 (i).

As aS is (real) hyperbolic, the hypotheses of the Isolating Block Lemma (2.2) are satisfied by $(Q, E^+, E^-, \xi) = (\mathbf{C}^n, W_0^+, W_0^-, aS)$, $\mathbf{C}^n$ being endowed with the hermitian metric introduced at the beginning of (3.3). Let B denote the corresponding "isolating block"; replacing b by kb (or the hermitian norm $|.|$ by $|.|/k$) for some positive k, small enough, *we* may—and *shall—assume that the compact* (since $\mathbf{f}_c|_{Q_c \cap \mathscr{V}}$ is proper) *subset* $\mathscr{V} \cap \mathbf{f}_c^{-1}([e, \max_{I \in \mathscr{I}_c} cg_I])$ *lies in the interior of* B; we choose a real number $e' > \max_{I \in \mathscr{I}_c} cg_I$ such that the compact subset $\mathbf{f}_c^{-1}([e, e']) \cap \mathscr{V}$ lies in the interior of B, and let

$$L = \mathbf{f}_c^{-1}([e, e']).$$

(4.2) "Isolating blocks" for the complex flow of S near the origin

Proposition 8 (see *Fig. 4*). — (i) *For each compact tubular neighbourhood* V *of the compact subset* $Q'_b \cap \mathscr{V}$ *in* Q'_b (see *Corollaries 5 and 6*), *the set* $\tilde{V} = L \cap (\bigcup_{t \in \mathbf{R}} \Phi_c^t(V) \cup \bigcup_{I \in \mathscr{I}_c} W_{c,I}^-)$ *is a compact neighbourhood of* $W_0^+ \cap Q_c$ *in* Q_c.

(ii) *Such sets* $\tilde{V}$ *form a basis of the filter of neighbourhoods of* $\mathscr{V} \cap L$ *in* L.

(iii) *For each such* V, $\tilde{\tilde{V}} = B \cap (W_0^- \cup \bigcup_{t \in \mathbf{R}} \sigma_1(\{t\} \times \tilde{V}))$ *is a compact neighbourhood of* 0 *in* $\mathbf{C}^n$.

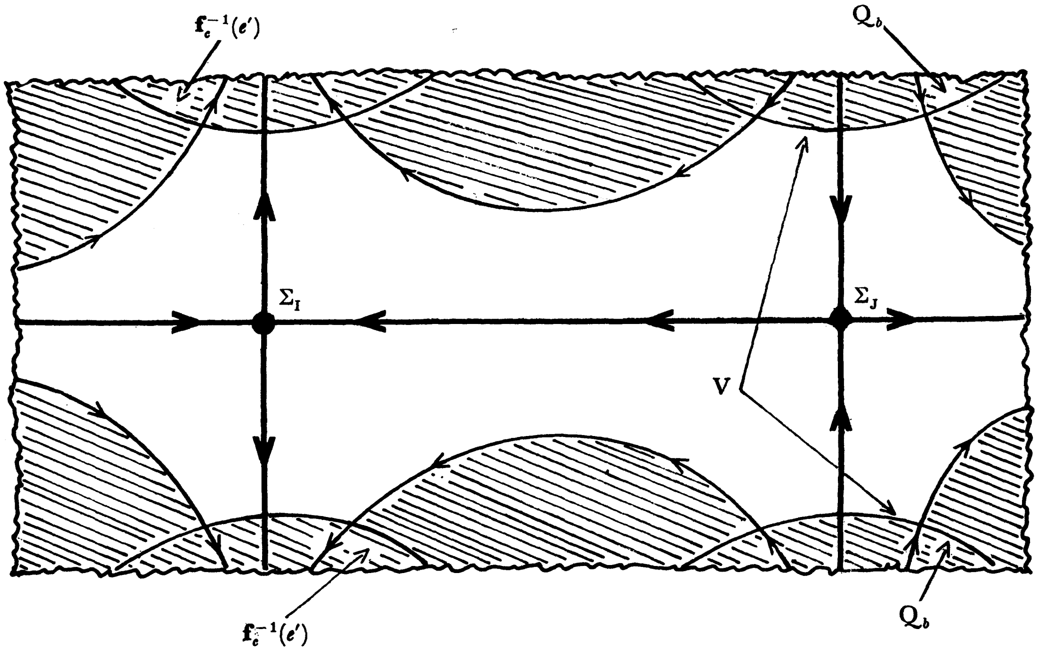

FIG. 4

Proof. — (i) Given $\mathrm{I} \in \mathscr{I}_c$, assume that $\widetilde{\mathrm{V}}$ is a neighbourhood of $\mathrm{L} \cap \mathscr{V} \setminus \bigcup_{g_\mathrm{J} \leqslant g_\mathrm{I}} \mathrm{W}^-_{c,\mathrm{J}}$ —an induction hypothesis which *is* satisfied if cg_I is minimal, by Corollary 5 (ii) and Propositions 3-6. Then, by the last assertion of Proposition 3 and the Isolating Block Lemma (2.2), $\widetilde{\mathrm{V}}$ is a neighbourhood of $\Sigma_{c,\mathrm{I}}$, hence of $\mathrm{L} \cap \mathscr{V} \setminus \bigcup_{g_\mathrm{I} > g_\mathrm{J}} \mathrm{W}^-_{c,\mathrm{J}}$ by Proposition 3. By induction, this proves that $\widetilde{\mathrm{V}}$ is a neighbourhood of $\mathrm{L} \cap \mathscr{V}$, which contains $\mathrm{W}^+_0 \cap \mathrm{Q}_c$ by Proposition 6.

To see that $\widetilde{\mathrm{V}}$ is compact, we shall use

Lemma 2. — Let $(a_m) = (u_m, v_m)$ be a sequence in $\mathbf{C} \times \mathbf{C}^n$, such that v_m converges to a limit v and that $\mathbf{F}(\sigma(a_m))$ is bounded. Then, there is an increasing sequence (m_k) in $\mathbf{N}$ such that $\sigma(a_{m_k})$ converges to a limit w; moreover, if (u_m) is unbounded, then v lies in $\mathscr{V}$, and so does w for a suitable choice of (m_k).

Proof of Lemma 2. — If (u_m) is bounded, we can extract from (a_m) a convergent sequence. If (u_m) is not bounded, then, taking subsequences, we may assume that $|u_m| \to +\infty$, that $u_m/|u_m| \to u \in \mathbf{C}$ and that $\mathbf{F}(\sigma(a_m)) \to \mathrm{L} \in \mathbf{C}$. As S is weakly hyperbolic, there exists $u' \in \mathbf{C}$ such that we have $\mathscr{R}e(u'\, c_j) < 0$ if and only if $\mathscr{R}e(uc_j) < 0$, and $\mathscr{R}e(u'\, c_j) > 0$ if and only if $\mathscr{R}e(uc_j) \geqslant 0$. Now, $\mathscr{R}e(u'\, \mathbf{F}(\sigma(a_m))) = \sum_j \mathscr{R}e(u'\, c_j)\, |x_j(\sigma(a_m))|^2/2$, and

$$(6) \qquad |x_j(\sigma(a_m))|^2 = |x_j(v_m)|^2\, e^{2|u_m|\, \mathscr{R}e((u-(u-u_m/|u_m|))\, c_j)};$$

therefore, we obtain

$$(7) \qquad \lim x_j(\sigma(a_m)) = 0 \quad \text{for } \mathscr{R}e(uc_j) < 0,$$

hence $\mathscr{R}e(u'\, \mathrm{L}) = \lim \sum_{\mathscr{R}e(u'\, c_j) > 0} \mathscr{R}e(u'\, c_j)\, |x_j(\sigma(a_m))|^2$. This proves that $\sigma(a_m)$ is bounded; therefore, by (6), $\mathscr{R}e(uc_j) > 0$ implies $x_j(v) = 0$, hence $v \in \mathscr{V}$; moreover, we can extract from $\sigma(a_m)$ a convergent subsequence, the limit of which belongs to $\mathscr{V}$ by (7), hence our lemma. □

Now, let (y_m) be a sequence in $\widetilde{\mathrm{V}}$. If it has infinitely many terms in the compact subset $\mathscr{V} \cap \mathrm{L}$, then it has a convergent subsequence with limit in $\mathscr{V} \cap \mathrm{L} \subset \widetilde{\mathrm{V}}$. In the remaining case, extracting subsequences, we may assume that every y_m is of the form $\Phi_c^{t_m}(v_m)$, alias $\sigma(u_m, v_m)$, $v_m \in \mathrm{V}$, $t_m \in \mathbf{R}$, $u_m \in \mathbf{C}$, and (since V is compact) that v_m converges to some $v \in \mathrm{V}$. If t_m is bounded, we may assume that it converges to some t, and then $\lim y_m = \Phi_c^t(v)$ lies in L (which is closed), hence in $\widetilde{\mathrm{V}}$. If t_m, hence u_m (*see* Corollary 4), is unbounded, then, as $\mathbf{F}$ is bounded on L, the hypotheses of Lemma 2 are satisfied by (u_m, v_m); therefore, extracting subsequences, we may assume that y_m tends to some $y \in \mathscr{V}$, which has to lie in L, hence in $\widetilde{\mathrm{V}}$, which is therefore compact.

Proof of (ii). — We just have to show that if a sequence $y_m = \Phi_c^{t_m}(v_m)$ in $L\backslash\mathscr{V}$, $v_m \in Q'_b\backslash\mathscr{V}$, $t_m \geqslant 0$, is such that v_m tends to $\mathscr{V} \cap Q'_b$, then some subsequence of (y_m) converges to $\mathscr{V}$; now, extracting subsequences, we may assume that v_m converges to some $v \in \mathscr{V} \cap Q'_b$, hence (ii) by Lemma 2.

Proof of (iii). — Following the orbits of σ_1, we get a diffeomorphism of Q_c onto the cylinder $\{|x_+| = 1\}$ (notations of the Isolating Block Lemma, with the above choice of B), hence (iii), by (i) and the Isolating Block Lemma (iii). ■

(4.3) " Isolating Blocks " for perturbations of the complex flow of S near 0

Hypotheses and notation. — We denote by V a fixed compact tubular neighbourhood of $\mathscr{V} \cap Q'_b$ in Q'_b, small enough for $\widetilde{V}$ to lie in the interior of B. For each $I \in \mathscr{I}_c$, we endow $Q_{c,I} = \Sigma_{c,I} \times E_{I^+} \times E_{I^-}$ with the riemannian metric obtained by multiplying the ambiant hermitian norm by a positive constant α_I such that, if

$$B_{c,I} = \{v \in Q_{c,I} : \alpha_I\,|v_{I^+}| \leqslant 1 \quad \text{and} \quad \alpha_I\,|v_{I^-}| \leqslant 1\},$$

$\mathscr{C}_{c,I}^{-1}(B_{c,I})$ lies in the interior of $\widetilde{V}$ in Q_c (of course, the hypotheses of the Isolating Block Lemma are satisfied by $\xi = \xi_{c,I}$ in $B = B_{c,I}$ for this new metric).

Complex Isolating Block Lemma, First Part. — *There exists a* C^1*-neighbourhood* $\mathscr{N}'$ *of* $aS|_B$ *in the space of smooth vector fields on* B *holomorphic in the interior of* B *such that, denoting by* $\mathscr{N}'_0$ *the set of those elements of* $\mathscr{N}'$ *which are tangent to every s.i.m. of* S, *each* $X \in \mathscr{N}'_0$ *has the following properties:*

(i) *The Hypotheses of the Isolating Block Lemma* (2.2) *are satisfied if* $\xi|_B = X$ *(considered as a real vector field), and the function* F_1 *is increasing along every nonzero real flowline of* X.

(ii) *Let* X_c *denote the vector field on* $Q_c \cap B$ *obtained from* X *as* ξ_c *was obtained from* aS *in Corollary 4. Then, the set*

$$\widetilde{V}_X = L \cap \Big(\bigcup_{t\geqslant 0} f^t_{X_c}(V) \cup \bigcup_{I\in\mathscr{I}_c} W^-_{c,I}\Big)$$

is a compact X_c*-saturated neighbourhood of* $\mathscr{V} \cap L$ *in* L, *contained in the interior of* B.

(iii) *The set* $\widetilde{\widetilde{V}}_X = B \cap (W^-_0 \cup \bigcup_t f^t_X(\widetilde{V}_X))$ *is a compact neighbourhood of* 0 *in* $\mathbf{C}^n$.

(iv) *All the properties of* ξ_c *stated in Propositions 3, 5, 6 and Corollary 5 remain true if we replace* Q_c *by* $\widetilde{V}_X$, ξ_c *by* $X_c|_{\widetilde{V}_X}$ *and each* $W^{\pm}_{c,I}$ *by its intersection with* L, *with the following modifications:*

— *In Propositions 5-6, replace " bounded from above (resp. below) " by " bounded from above by a real number* $\alpha < e'$ *(resp. bounded from below by a real number* $\alpha > e$*) ".*

— *In Proposition 3, replace the assertions after " More precisely... " by the following: " Moreover, if* $\mathscr{C}_{X,I} : (\widetilde{V}_X, \Sigma_{c,I}) \to (Q_{c,I}, \Sigma_{c,I})$ *is the local diffeomorphism which consists* *in following the real orbits of* X *and if* X_I *denotes the image of* X_c *by* $\mathscr{C}_{X,I}$, *then, the image of* $\mathscr{C}_{X,I}$ *contains* $B_{c,I}$, *the hypotheses of the Isolating Block Lemma* (2.2) *are satisfied in* $B_{c,I}$ *by* $(Q, M, E^+, E^-, \xi) = (Q_{c,I}, \Sigma_{c,I}, E_{I^+}, E_{I^-}, X_I)$, *and* $X_I|_{B_{c,I}}$ *tends to* $\xi_{c,I}|_{B_{c,I}}$ *in the* C^1*-topology if* X *tends to* $aS|_B$ *in the* C^1*-topology.* "

Proof. — Let X be a smooth vector field on B, holomorphic in the interior of B and tangent to every s.i.m. of S. If X is C^1-close enough to $aS|_B$, then, by the Extension Lemma (i), assertion (i) is true; therefore, (iii) follows from (ii) as in the proof of Proposition 8 (iii). Checking (ii) and (iv) takes several steps, in which the locution " if X is C^1-close enough to $aS|_B$ " will be implicit:

Step 1. — The second (i.e. local) part of our perturbed Proposition 3 is true. Indeed, as X is arbitrarily C^1-close to $aS|_B$, the image of $\mathscr{C}_{X,I}$ does contain $B_{c,I}$; now, for each $v \in Q_{c,I}$, $X_I(v)$ is the image of $iX(v)$ by the linear projection $\mathbf{C}^n \to T_v Q_{c,I}$ with kernel $\mathbf{R}X(v)$, which proves that $X_I|_{B_{c,I}}$ is tangent to $\Sigma_{c,I} \times E_{I^+}$ and $\Sigma_{c,I} \times E_{I^-}$ and tends to $\xi_{c,I}|_{B_{c,I}}$ in the C^1-topology when X tends to $aS|_B$ in the C^1-topology, hence Step 1 by the Extension Lemma (i).

Step 2. — The Lie derivative $L_{X_c}\mathbf{f}_c$ *is positive outside* Σ_c: as X_c is arbitrarily C^1-close to $\xi_c|_{Q_c \cap B}$, this is true outside any fixed neighbourhood U of Σ_c. To construct a neighbourhood U of Σ_c in which it is also true, notice that, for each $I \in \mathscr{I}_c$, X_c is tangent to $W^+_{c,I}$ and to $W^-_{c,I}$, hence to $\Sigma_{c,I}$, and that $\mathbf{f}_c$ can be written $\mathbf{f}_c(v) = cg_I + \sum_{j \notin I} \mathscr{R}e((i - g_I)\, ac_j)\, |x_j(v)|^2/2$; therefore, by Taylor's Formula, there does exist a compact tubular neighbourhood U_I of $\Sigma_{c,I}$ in Q_c, independent of X, such that $L_{X_c}\mathbf{f}_c|_{U_I}$ is of the form $v \mapsto q_X(v)\,((x_j(v))_{j \notin I})$, where $q_X(v)$ denotes a quadratic form, depending continuously on v and X (in the C^1-topology), and such that every $q_X(v)$ is positive definite if $X = aS|_B$ —hence if X is C^1-close enough to $aS|_B$.

Step 3. — For each $I \in \mathscr{I}_c$, $W^+_{c,I} \cap L$ *(resp.* $W^-_{c,I} \cap L$*) is the global stable (resp. unstable) manifold of* $X_c|_{B \cap L}$ *at* $\Sigma_{c,I}$: indeed, given $v \in L \cap W^+_{c,I}$, *every* $f^t_{X_c}(v)$ *lies in* $W^+_{c,I}$ (since it belongs to $E_{I_0^+}$ by Proposition 2 (ii) and our hypothesis on X, this is deduced from the last equality of Proposition 6, using induction on the value of cg_I). Moreover, $f^t_{X_c}(v)$ *exists and lies in* $L \cap W^+_{c,I}$ *for every* $t \geqslant 0$, since it can escape from the compact subset $L \cap \mathscr{V} \cap B$ neither through ∂B—which $L \cap \mathscr{V}$ does not intersect—, nor through $\mathbf{f}_c^{-1}(\{e, e'\})$—because of Step 2 and the inequality

$$(8) \qquad e < \min \mathbf{f}_c(W^-_{c,J}) = cg_J = \max \mathbf{f}_c(W^+_{c,J}) < e', \quad J \in \mathscr{I}_c,$$

obvious from Propositions 3-6. Now, for each positive $\varepsilon < \min_{J \neq K} |cg_J - cg_K|/2$, the last equalities of Proposition 6 and (8) imply that $\Sigma^+_{\varepsilon,J} = \mathbf{f}_c^{-1}(cg_J - \varepsilon) \cap W^+_{c,J}$ (resp. $\Sigma^-_{\varepsilon,J} = \mathbf{f}_c^{-1}(cg_J + \varepsilon) \cap W^-_{c,J}$) is the compact submanifold $\mathbf{f}_c^{-1}(cg_J - \varepsilon) \cap E_{I_0^+}$ (resp. $\mathbf{f}_c^{-1}(cg_J + \varepsilon) \cap E_{I_0^-}$) for every $J \in \mathscr{I}_c$. Moreover, as

$$W^+_{c,J} \cap \mathbf{f}_c^{-1}(cg_J) = \Sigma_{c,J} = W^-_{c,J} \cap \mathbf{f}_c^{-1}(cg_J),$$

again by (8), we can choose ε so small that $\Sigma^+_{\varepsilon,J}$ and $\Sigma^-_{\varepsilon,J}$ lie in the interior of $\mathscr{C}^{-1}_{X,J}(B_{c,J})$ for every J. Then, by Steps 1-2 and the Isolating Block Lemma, the interior of each $\mathscr{C}^{-1}_{X,J}(B_{c,J})$ in Q_c contains a compact neighbourhood $C_{X,J}$ of $\Sigma_{c,J}$, of the form (*see* Fig. 5 below)

$$C_{X,J} = \mathbf{f}_c^{-1}([cg_J - \varepsilon, cg_J + \varepsilon]) \cap (W^-_{c,J} \cup \bigcup_{t\geqslant 0} f^t_{X_c}(K_J))$$

for some compact tubular neighbourhood K_J of $\Sigma^+_{\varepsilon,J}$ in $\mathbf{f}_c^{-1}(cg_J - \varepsilon)$. By Step 2, $L_{X_c}\mathbf{f}_c$ is bounded from below by a positive constant in $\mathscr{V} \cap L \backslash \bigcup_J C_{X,J}$, where $f^t_{X_c}(v)$ can therefore spend only a finite amount of non-negative time. Moreover, for every $J \in \mathscr{I}_c$, each flowline of X_c which does not lie in $W^+_{c,J}$ and enters $C_{X,J}$ clearly has to leave it through $\mathbf{f}_c^{-1}(cg_J + \varepsilon)$—hence forever, by Step 2—after a finite time. Therefore, $f^t_{X_c}(v)$ has to be in $C_{X,I}$ for every large enough t, hence our result by Step 1 (the case of $W^-_{c,I}$ is of course entirely analogous).

Step 4. — End of the proof: let ∂V denote the boundary of our tube V in Q'_b. As X_c is arbitrarily C^1-close to $\xi_c|_{Q_c \cap B}$, $\mathbf{f}_c$ takes every value between e and e' on each flowline of X_c through ∂V, and $L \cap \bigcup_{t\geqslant 0} f^t_{X_c}(\partial V)$ is arbitrarily C^1-close to $L \cap \bigcup_{t\geqslant 0} \Phi^t_c(\partial V)$. Let $\widetilde{V}_X$ denote the "box" in Q_c, bounded laterally by $L \cap \bigcup_{t\geqslant 0} f^t_{X_c}(\partial V)$, at one of its ends by V, and at its other end by the perturbed version of $\mathbf{f}_c^{-1}(e') \cap \widetilde{V}$ (i.e. the submanifold of $\mathbf{f}_c^{-1}(e')$ with boundary $\mathbf{f}_c^{-1}(e') \cap \bigcup_{t\geqslant 0} f^t_{X_c}(\partial V)$ which contains $\mathscr{V} \cap \mathbf{f}_c^{-1}(e')$). We may assume that the compact subset $\widetilde{V}_X$ contains every $\mathscr{C}^{-1}_{X,I}(B_{c,I})$, and our problem is to show that $\widetilde{V}_X$ does admit the definition given in (ii), which is easy: given compact subsets $C_{X,I}$, $I \in \mathscr{I}_c$, as in Step 3, Steps 1-3 imply that, for each $v \in V$ and each $I \in \mathscr{I}_c$, either v belongs to $W^+_{c,I}$, or there exists a real number $t_I(v) \geqslant 0$ such that $f^t_{X_c}(v)$ lies outside $C_{X,I}$ for $t > t_I(v)$. Now, $f^t_{X_c}(v)$ can leave $\widetilde{V}_X$ for $t > 0$ only through $\mathbf{f}_c^{-1}(e')$, and *has* to leave $\widetilde{V}_X$ if v does not belong to $\bigcup_I W^+_{c,I}$, for $L_{X_c}\mathbf{f}_c$ is bounded from below by a positive constant in $\widetilde{V}_X \backslash \bigcup_I C_{X,I}$, where $f^t_{X_c}(v)$ lies for $t > \max_I t_I(v)$. This proves what we wished. ■

We can now give a complex analogue of the Isolating Block Lemma (i)-(ii); the estimates in (vii) will not be needed before Section 5:

Complex Isolating Block Lemma, Second Part. — With the notation of the Complex Isolating Block Lemma, First Part, if we choose $\mathscr{N}'$ small enough, then there exists a positive constant C *such that every* $X \in \mathscr{N}'_0$ *has the following properties*:

(v) $X|_{\widetilde{\widetilde{V}}_X}$ *defines a global complex flow* R_X *on* $\widetilde{\widetilde{V}}_X$, *which means the following*: *if* U_1 *and* U_2 *are connected 2-dimensional submanifolds with corners of* **C**, *containing* 0, *if* $\gamma_j : U_j \to \widetilde{\widetilde{V}}_X$, $j = 1, 2$, *denote two smooth mappings which are holomorphic and satisfy* $\gamma'_j(u) = X(\gamma_j(u))$

inside U_j *and if, moreover,* $\gamma_1(0) = \gamma_2(0) = v \in \widetilde{\widetilde{V}}_X$, *then,* $\gamma_1(u) = \gamma_2(u) = R_X(u, v) = R_X^u(v)$ *(which is thus defined) for every* $u \in U_1 \cap U_2$.

(vi) *For each* $v \in \widetilde{\widetilde{V}}_X$, *the set of those* $u \in \mathbf{C}$ *for which* $R_X^u(v) \in \widetilde{\widetilde{V}}_X$ *is defined (in the sense of* (i)*) is a connected 2-dimensional submanifold with corners of* $\mathbf{C}$, *which is compact if and only if* v *does not lie in* $\mathscr{V}$. *More precisely, let* φ_X *be the diffeomorphism of* $\Omega_X = \{(x, t) \in \widetilde{V}_X \times \mathbf{R} : f_X^t(x) \in \widetilde{\widetilde{V}}_X\}$ *onto* $\widetilde{\widetilde{V}}_X \backslash W_0^-$ *given by* $\varphi_X(x, t) = f_X^t(x)$, *and let* $g_X : \widetilde{V}_X \to \mathbf{R}$ *be the smooth function defined by* $iX(x) = X_c(x) + g_X(x)\, X(x)$, $x \in \widetilde{V}_X$. *Then, for every* $(x, s) \in \Omega_X$ *and every* $(s', t) \in \mathbf{R}^2$,

$$R_X^{s'+it}(\varphi_X(x, s)) = \varphi_X\Big(f_{X_c}^t(x), s + s' + \int_0^t g_X \circ f_{X_c}^u(x)\, du\Big), \tag{9}$$

meaning in particular that both sides have the same definition domain.

(vii) *One defines a smooth map* $r_X : \widetilde{\widetilde{V}}_X \backslash \widehat{\mathscr{V}}_b \to \mathbf{C}$, *analytic* [1] *off* ∂B, *by*

$$R_X^{-r_X(v)}(v) \in V.$$

For each $v \in \widetilde{\widetilde{V}}_X \backslash \widehat{\mathscr{V}}_b$, *let* $\gamma_{X,v}$ *be the path from* 0 *to* $r_X(v)$ *in* $\mathbf{C}$ *such that (for some parametrisation of* $\gamma_{X,v}$*) the path* $s \mapsto R_X^{-\gamma_{X,v}(s)}(v)$ *consists in following first the (real) orbit* $\{f_X^t(v)\}$ *from* v *to its intersection* x *with* $\widetilde{V}_X$, *and then the orbit* $\{f_{X_c}^t(x)\}$ *from* x *to its intersection* $R_X^{-r_X(v)}(v)$ *with* V; *then, denoting by* d *the hermitian distance in* $\mathbf{C}^n$, *we have*

$$|\gamma_{X,v}(s)| \leqslant C \operatorname{Log} \frac{2}{d(v, \widehat{\mathscr{V}}_b)} \quad \textit{for every } s \textit{ and every } v \in \widetilde{\widetilde{V}}_X \backslash \widehat{\mathscr{V}}_b. \tag{10}$$

Proof. — Assertions (v)-(vi) follow from the definition of $\widetilde{\widetilde{V}}_X$. Under the hypotheses and with the notation of (vii), let $s_X : \widetilde{\widetilde{V}}_X \backslash W_0^- \to \mathbf{R}$ and $t_X : \widetilde{V}_X \backslash \widehat{\mathscr{V}}_b \to \mathbf{R}$ be defined (*see* (i)-(vi)) by $f_X^{-s_X(v)}(v) \in Q_c$ and $f_{X_c}^{-t_X(x)}(x) \in Q'_b$. As X (resp. X_c) is transversal to Q_c (resp. Q'_b) at every point of $\widetilde{V}_X$ (resp. V), these two functions are smooth, and analytic off ∂B. Since (9) yields

$$(11) \quad \begin{cases} r_X(v) = s_X(v) + \displaystyle\int_0^{-t_X(x)} g_X \circ f_{X_c}^u(x)\, du + it_X(x), & \text{where } x = f_X^{-s_X(v)}(v) \\ |\gamma_{X,v}(s)| \leqslant |s_X(v)| + |t_X(x)|\,(1 + \max\limits_{y \in \widetilde{V}_X} |g_X(y)|) & \text{for every } s, \end{cases}$$

r_X is smooth, and analytic off ∂B. We shall now prove (10):

Step 1. — *Restricting* $\mathscr{N}'$ *if necessary, there exists a positive constant* k *such that, for* $X \in \mathscr{N}'_0$, *we have* $d(f_X^{-t}(v), \widehat{\mathscr{V}}_b) \geqslant e^{-k|t|}\, d(v, \widehat{\mathscr{V}}_b)$, $t \in \mathbf{R}$, $v \in B$, *and* $d(f_{X_c}^{-t}(x), \widehat{\mathscr{V}}_b) \geqslant e^{-kt}\, d(x, \widehat{\mathscr{V}}_b)$, $x \in \widetilde{V}_X$, $0 \leqslant t \leqslant t_X(x)$: let $h : Q_c \to [0, 1]$ be smooth, compactly supported, equal to 1 in a neighbourhood of $\widetilde{V}$, let $Y = hX_c$, and let $k_X = \max\{|dY(x)| : x \in Q_c\}$. If X is

[1] Analytic means *real* analytic.

C^1-close enough to aS, then, for $x \in \widetilde{V}_X$ and $0 \leqslant t \leqslant t_X(x)$, we have $f_{X_c}^{-t}(x) = f_Y^{-t}(x)$ and, for every smooth path $\gamma : [0, 1] \to Q_c$ from $f_{X_c}^{-t}(x)$ to $\widehat{\mathscr{V}}_b \cap Q_c$,

$$\frac{d}{du}\,|(f_Y^u \circ \gamma)'(s)|^2 = 2((f_Y^u \circ \gamma)'(s) \mid dY(f_Y^u \circ \gamma(s)).(f_Y^u \circ \gamma)'(s)) \leqslant 2k_X\,|(f_Y^u \circ \gamma)'(s)|^2, \quad 0 \leqslant u \leqslant t,$$

hence, by integration, $\int_0^1 |(f_Y^t \circ \gamma)'(s)|\,ds \leqslant e^{k_X t}\int_0^1 |\gamma'(s)|\,ds$. Let d_c denote the geodesic distance on Q_c; as $f_Y^t \circ \gamma$ is a path from x to $\widehat{\mathscr{V}}_b \cap Q_c$, we get

$$d_c(x, \widehat{\mathscr{V}}_b \cap Q_c) \leqslant e^{k_X t}\, d_c(f_{X_c}^{-t}(x), \widehat{\mathscr{V}}_b \cap Q_c);$$

now, k_X depends continuously on X in the C^1-topology, and, on the compact subset $h^{-1}(1)$, the distances d and d_c are equivalent. This proves that—restricting $\mathscr{N}'$ if necessary—our second inequality holds for some positive constant k; modifying the latter, the same argument yields our first inequality.

Step 2. — Reduction of the problem: clearly, for each $X \in \mathscr{N}'_0$, s_X is bounded below, and its minimum depends continuously on X; moreover, s_X is bounded above by the function r_+ associated to $\xi = X$ by the Isolating Block Lemma, which proves that $v \mapsto \left| s_X(v)/\mathrm{Log}\,\frac{2}{|v_+|}\right|$ has a finite supremum M_X; by the proof of the Isolating Block Lemma, M_X depends continuously on X, and we can therefore choose $\mathscr{N}'$ so that $M = \max\{M_X : X \in \mathscr{N}'_0\}$ is finite. As $|v_+| = d(v, W_0^-) \geqslant d(v, \widehat{\mathscr{V}}_b)$, we get $|s_X(v)| \leqslant M\,\mathrm{Log}\,\frac{2}{d(v, \widehat{\mathscr{V}}_b)}$, hence, by Step 1, $d(f_X^{-s_X(v)}(v), \widehat{\mathscr{V}}_b) \geqslant 2^{-kM}\, d(v, \widehat{\mathscr{V}}_b)^{1+kM}$ for each $v \in \widetilde{\widetilde{V}}_X \backslash W_0^-$. Therefore, by (11), we just have to prove

Step 3. — The non-negative function $x \mapsto -\,t_X(x)/\mathrm{Log}\,d(x, \widehat{\mathscr{V}}_b)$ is bounded on $\widetilde{V}_X \backslash \widehat{\mathscr{V}}_b$, and its supremum depends continuously on $X \in \mathscr{N}'_0$ in the C^1-topology: let the compact subsets $C_{X,I}$, $I \in \mathscr{I}_c$, be as in Step 3 of the proof of the Complex Isolating Block Lemma, First Part. Let the elements $I_1, \ldots, I_m$ of $\mathscr{I}_c$ be so ordered as to satisfy $cg_{I_j} > cg_{I_k}$ for $j < k$; for $0 \leqslant j \leqslant m$, let $p_j^+, p_{j+1}^- : \widetilde{V}_X \backslash \widehat{\mathscr{V}}_b \to \widetilde{V}_X$ and $t_j^+, t_{j+1}^- : \widetilde{V}_X \backslash \widehat{\mathscr{V}}_b \to \mathbf{R}_+$ be defined inductively as follows (*see* Fig. 5):

$$\begin{cases} t_0^+(x) = 0, \quad t_{m+1}^-(x) = t_X(x) \\ \text{for } 0 < j \leqslant m, \quad [t_j^-(x), t_j^+(x)] = \begin{cases} \{t : f_{X_c}^{-t}(x) \in C_{X,I_j}\} & \text{if non-empty} \\ \{t_{j-1}^+(x)\} & \text{in the remaining case} \end{cases} \\ \text{for } 0 \leqslant j \leqslant m, \quad p_j^+(x) = f_{X_c}^{-t_j^+(x)}(x) \quad \text{and} \quad p_{i+1}^- = f_{X_c}^{-t_{i+1}^-(x)}(x). \end{cases}$$

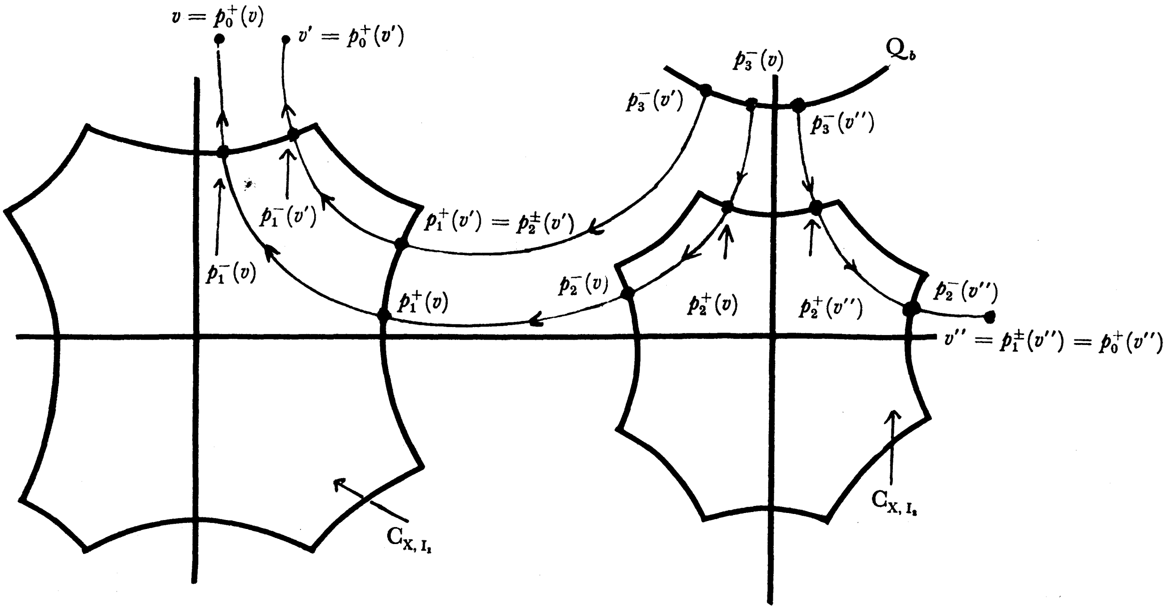

FIG. 5

As $L_{X_c} \mathbf{f}_c$ has a positive lower bound on $\widetilde{V}_X \setminus \bigcup_I C_{X,I}$ and $\mathbf{f}_c|_{\widetilde{V}_X}$ is bounded,

(12) the length of $[0, t_X(x)] \setminus \bigcup_j [t_j^-(x), t_j^+(x)]$ is uniformly bounded with respect to x and (choosing $\mathcal{N}'$ small enough) X.

Therefore, by Step 1, $x \mapsto -t_1^-(x)/\mathrm{Log}\, d(x, \hat{\mathscr{V}}_b)$ is bounded on $\widetilde{V}_X \setminus \hat{\mathscr{V}}_b$, uniformly with respect to X. Let us make the induction hypothesis that so is $x \to -t_j^-(x)/\mathrm{Log}\, d(x, \hat{\mathscr{V}}_b)$ for some $j \leqslant m$; then by Step 1, there is a positive constant α such that we have $d(p_j^-(x), \hat{\mathscr{V}}_b) \geqslant d(x, \hat{\mathscr{V}}_b)^\alpha$, $x \in \widetilde{V}_X \setminus \hat{\mathscr{V}}_b$, $X \in \mathcal{N}_0'$. Thus, as $d(p_j^-(x), W_{c,I_j}^- \cap C_{X,I_j})$ is not less than $d(p_j^-(x), \hat{\mathscr{V}}_b)$ and, on C_{X,I_j}, the distance induced by that of Q_{c,I_j} *via* $\mathscr{C}_{X,I_j}$ is equivalent to d, it follows from the Extension Lemma (i)—applied to X_{I_j} in B_{c,I_j}—that $-t_j^+(x)/\mathrm{Log}\, d(x, \hat{\mathscr{V}}_b)$ is bounded in $\widetilde{V}_X \setminus \hat{\mathscr{V}}_b$, uniformly with respect to X; therefore, by (12) and Step 1, so is $-t_{j+1}^-(x)/\mathrm{Log}\, d(x, \hat{\mathscr{V}}_b)$, hence our lemma. ■

(4.4) Conjugacy results; proof of Theorem 2

Notation. — We assume $\mathcal{N}'$ small enough for every $X \in \mathcal{N}_0'$ to satisfy properties (i)-(vii) of the Complex Isolating Block Lemma. For each $X \in \mathcal{N}_0'$, we let

$$\dot{V}_X = \{x \in \widetilde{V}_X : \mathbf{f}_c(x) < e'\}$$

and $$\dot{\mathbf{V}}_X = (\{v \in \widetilde{\widetilde{V}}_X \setminus W_0^- : \varphi_X^{-1}(v) \in \dot{V}_X \times \mathbf{R}\} \cup W_0^-) \setminus \partial B.$$

We shall construct conjugacies as in Corollary 2; here is the easy part:

Proposition 9. — (i) *For each* $X \in \mathcal{N}'_0$, *the set* ω_X *of those* $(x, u) \in V \times \mathbf{C}$ *such that* $R^u_X(x)$ *is well-defined and lies in* $\dot{V}_X$ *is an open subset of* $V \times (\mathbf{R} + i\mathbf{R}_+)$, *and the mapping* $\Phi_X : \omega_X \to \dot{V}_X \backslash \hat{\mathscr{V}}_b$ *given by* $\Phi_X(x, u) = R^u_X(x)$ *is an analytic diffeomorphism, the inverse of which is* $v \mapsto (R_X^{-r_X(v)}(v), r_X(v))$.

(ii) *Given* X *and* Y *in* $\mathcal{N}'_0$, *the mapping* $h := \Phi_Y \circ \Phi_X^{-1} : \Phi_X(\omega_X \cap \omega_Y) \to \Phi_Y(\omega_X \cap \omega_Y)$ *is an analytic diffeomorphism, sending the restriction of* zX *onto that of* zY *for every* $z \in \mathbf{C}$, *and, by* (i),

$$h(v) = R_Y^{r_X(v)} \circ R_X^{-r_X(v)}(v) \quad \text{and} \quad h^{-1}(v) = R_X^{r_Y(v)} \circ R_Y^{-r_Y(v)}(v). \quad \blacksquare \tag{13}$$

We can now prove a complex analogue of Corollary 2:

Complex Extension Lemma. — *For every positive integer* k, *there exist an integer* $p \geqslant k$ *and a* C^1-*neighbourhood* $\mathcal{N}''$ *of* $aS|_B$ *in* $\mathcal{N}'$ *such that, if two elements* X, Y *of* $\mathcal{N}'' \cap \mathcal{N}'_0$ *have* p-*th order contact along* $\mathscr{V} \cap B$, *then, with the notation of Proposition 9,*

(i) *the sets* $U = \Phi_X(\omega_X \cap \omega_Y) \cup (\dot{V}_X \cap \hat{\mathscr{V}}_b)$ *and* $U' = \Phi_Y(\omega_Y \cap \omega_X) \cup (\dot{V}_Y \cap \hat{\mathscr{V}}_b)$ *are open neighbourhoods of* $\mathscr{V} \cap \dot{V}_X = \mathscr{V} \cap \dot{V}_Y$ *in* $\dot{V}_X$ *and* $\dot{V}_Y$ *respectively, and*

(ii) *the mapping* $H : U \to U'$, *equal to* h *off* $\hat{\mathscr{V}}_b$ *and to* Id *on* $\hat{\mathscr{V}}_b$, *is a* C^k-*diffeomorphism, such that* $H_*(zX|_U) = zY|_{U'}$ *for every* $z \in \mathbf{C}$.

Proof. — Let $\delta > 0$ be small enough for $\sigma([0, \delta] \times V)$ to lie in the interior of B, let $\mathbf{T} = \mathbf{R}/\delta\mathbf{Z}$, and let $q : Q_c \times \mathbf{R} \to Q_c \times \mathbf{T}$ denote the canonical projection. For every $X \in \mathcal{N}'_0$, close enough to $aS|_B$, we have $\tilde{V}_X \times [0, \delta] \subset \varphi_X^{-1}(B \backslash \partial B)$; thus, as $[X, iX] = 0$, there is a unique analytic vector field $\mathbf{X}_c$ on $\tilde{V}_X \times \mathbf{T}$ such that $q^* \mathbf{X}_c = \varphi_X^*(iX|_{\tilde{\tilde{V}}_X \backslash W_0^-})$, the flow of which is given by

$$f^t_{\mathbf{X}_c} \circ q(x, s) = q\left(f^t_{X_c}(x), s + \int_0^t g_X \circ f^u_{X_c}(x)\, du\right), \tag{14}$$

meaning in particular that both sides of this identity are defined in the same domain (recall that the definition domain of g_X is $\tilde{V}_X$).

In the rest of the proof, we assume $X, Y \in \mathcal{N}'_0$ close enough to $aS|_B$. The general idea is as follows: using the " charts " φ_X and φ_Y, we transform our initial complex conjugacy problem into another one, concerning *real* flows (Steps 1 and 2). We can then use the Extension Lemma (2.2) to solve this reduced problem (Step 3); this is why we have introduced $\mathbf{T}$ and q, as a more direct approach would make use of a less simple tool.

Step 1. — *Let* $U_b = U \backslash \hat{\mathscr{V}}_b$, $U'_b = U' \backslash \hat{\mathscr{V}}_b$, *let* d_0 *be the set of those* $x \in \dot{V}_X$ *such that, with the notation of* (11), $f_{Y_c}^{t_X(x)} \circ f_{X_c}^{-t_X(x)}(x)$ *is defined and lies in* $\dot{V}_Y$, *and let* $h_0 : d_0 \times \mathbf{T} \to \dot{V}_Y \times \mathbf{T}$ *and* $q^* h_0 : d_0 \times \mathbf{R} \to \dot{V}_Y \times \mathbf{R}$ *be given by*

$$h_0(x, \theta) = f_{\mathbf{Y}_c}^{t_X(x)} \circ f_{\mathbf{X}_c}^{-t_X(x)}(x, \theta) \quad \text{and} \quad q^* h_0(x, s) = f_{q^*\mathbf{Y}_c}^{t_X(x)} \circ f_{q^*\mathbf{X}_c}^{-t_X(x)}(x, s).$$

Then, $U_b = \varphi_X(\Omega_X \cap (q^* h_0)^{-1}(\Omega_Y))$ *and*

$$h = \varphi_Y \circ (q^* h_0) \circ \varphi_X^{-1}|_{U_b}. \tag{15}$$

This is obvious from (9), (11) and (13), since

$$f^t_{q^* X_c}(x, s) = \Big(f^t_{X_c}(x), s + \int_0^t g_X \circ f^u_{X_c}(x)\, du\Big). \quad \square$$

Step 2. — Let $m \geqslant k$ *and* $\mathcal{N}_0$ *be as in the Extension Lemma with* $\xi = a$S; *assume that*

(i) X *and* Y *belong to* $\mathcal{N}_0$ *and have m-th order contact along* $\mathscr{V}$,

(ii) $D_0 = d_0 \cup (\widehat{\mathscr{V}}_b \cap \dot{V}_X)$ *is an open neighbourhood of* $\mathscr{V} \cap \dot{V}_X$ *in* $\dot{V}_X$, *and*

(iii) h_0 *extends to a* C^m*-embedding* $H_0 : D_0 \times \mathbf{T} \to \dot{V}_Y \times \mathbf{T}$, *having m-th order contact with the identity along* $(\mathscr{V} \cap \dot{V}_X) \times \mathbf{T}$.

Then, the conclusion of the Complex Extension Lemma is satisfied.

Indeed, $q^* h_0$ clearly extends to a C^m-embedding $q^* H_0 : D_0 \times \mathbf{R} \to \dot{V}_Y \times \mathbf{R}$, having m-th order contact with Id along $(\mathscr{V} \cap \dot{V}_X) \times \mathbf{R}$; as X and Y have m-th order contact along every s.i.m. of S and are tangent to it, it follows that

$$U_c := U \backslash W_0^- = \varphi_X(\Omega_X \cap (q^* H_0)^{-1}(\Omega_Y)) \tag{16}$$

is an open neighbourhood of $\dot{V}_X \cap \mathscr{V} \backslash W_0^-$ in $\dot{V}_X$, and that $H|_{U_c} = \varphi_Y \circ (q^* H_0) \circ \varphi_X^{-1}|_{U_c}$ is a C^m-diffeomorphism onto $U' \backslash W_0^-$, having m-th order contact with Id along $\dot{V}_X \cap \mathscr{V} \backslash W_0^-$, hence in particular along $W_0^+ \backslash \{0\}$. As U_c is X-saturated by (16), we can apply the Extension Lemma (2.2) with $(\omega, h) = (\mathring{U}_c, H|_{\mathring{U}_c})$; since the set Ω provided by the Extension Lemma clearly satisfies $\Omega \backslash W_0^- = \mathring{U}_c$, this prove Step 2. $\square$

For each $I \in \mathscr{I}_c$, let $q_I : Q_{c,I} \times \mathbf{R} \to Q_{c,I} \times \mathbf{T} = \mathbf{Q}_{c,I}$ be the canonical projection, and let $\varphi_{c,I} : Q_{c,I} \times \mathbf{R} \to E_{c,I}$ be defined by $\varphi_{c,I}(x, s) = \sigma_1(s, x)$. As $[a\mathrm{S}, ia\mathrm{S}] = 0$, there exists a unique analytic vector field $\boldsymbol{\xi}_{c,I}$ on $Q_{c,I} \times \mathbf{T}$ such that $q_I^* \boldsymbol{\xi}_{c,I} = \varphi_{c,I}^*(ia\mathrm{S})$, given by $\boldsymbol{\xi}_{c,I}(x, \theta) = (\xi_{c,I}(x), g_I)$; therefore, setting $\mathbf{B}_{c,I} = B_{c,I} \times \mathbf{T}$ and $\boldsymbol{\Sigma}_{c,I} = \Sigma_{c,I} \times \mathbf{T}$, the hypotheses of the Isolating Block Lemma (2.2) are satisfied by $(Q, B, \xi) = (\mathbf{Q}_{c,I}, \mathbf{B}_{c,I}, \boldsymbol{\xi}_{c,I})$. Here is our last step:

Step 3. — Let the elements $I_1, \ldots, I_\ell$ *of* $\mathscr{I}_c$ *be so numbered as to satisfy* $cg_{I_j} < cg_{I_{j'}}$ *for* $j < j'$, *and let the integers* $m = m_{\ell+1} \leqslant m_\ell \leqslant \ldots \leqslant m_1$ *be defined as follows: m is the same as in Step 2 and, for* $1 \leqslant j \leqslant \ell$, m_j *is the integer m associated to* $(B, \xi, \Sigma, k) = (\mathbf{B}_{c,I_j}, \boldsymbol{\xi}_{c,I_j}, \boldsymbol{\Sigma}_{c,I_j}, m_{j+1})$ *by the Extension Lemma* (2.2). *Then, the hypotheses of Step 2 are satisfied if* X *and* Y *have* m_1*-th order contact along* $\mathscr{V}$.

The proof is by induction. As above, set $\mathbf{Q}_e = Q_e \times \mathbf{T}$, $\mathbf{d}_0 = d_0 \times \mathbf{T}$, etc. Let $\dot{L} = L \backslash \mathbf{f}_e^{-1}(e')$, let $H_0 : \mathbf{D}_0 \to \dot{V}_Y$ equal h_0 off $\hat{\mathscr{V}}_b$ and Id on $\hat{\mathscr{V}}_b$, and let $\mathscr{W}_j = \dot{L} \cap \mathscr{V} \backslash \bigcup_{j' \geqslant j} W^-_{e, I_{j'}}$, $1 \leqslant j \leqslant \ell + 1$. We shall establish that, for $1 \leqslant j \leqslant \ell + 1$,

$(17)_j$ *$d_0 \cup \mathscr{W}_j$ is an open neighbourhood of $\mathscr{W}_j$ in $\dot{L}$, and $H_0|_{d_0 \cup \mathscr{W}_j}$ is of class C^{m_j} and has m_j-th order contact with Id along $\mathscr{W}_j$.*

This will imply our result for $j = \ell + 1$, since $m_{\ell+1} = m$ and $\mathscr{W}_{\ell+1} = \mathscr{V} \cap \dot{L}$.

Notice that (14) and the Complex Isolating Block Lemma (iv) yield $(17)_1$; indeed, $\mathbf{V}$ is a quotient of $\tilde{V}_X \backslash \hat{\mathscr{V}}_b$ (resp. $\tilde{V}_Y \backslash \hat{\mathscr{V}}_b$) by f_{X_e} (resp. f_{Y_e}), and h_0 is that conjugacy between $\mathbf{X}_e$ and $\mathbf{Y}_e$ which is the " maximal " solution of the Cauchy problem " $h_0 = \mathrm{Id}$ on $\mathbf{V}$ " with domain in $\dot{\mathbf{V}}_X$ and range in $\dot{\mathbf{V}}_Y$; therefore, since each $\mathbf{W}_I^+ \cap \mathbf{L}$ is the stable manifold at $\boldsymbol{\Sigma}_{e, I}$ of both $\mathbf{X}_e$ and $\mathbf{Y}_e$, which have m_1-th order contact along it, $(17)_1$ is true.

Let us make the induction hypothesis that so is $(17)_j$ for some $j \leqslant \ell$. Using the Extension Lemma (2.2), we shall prove that $d_0 \cup \mathscr{W}_{j+1}$ is a neighbourhood of $\mathscr{W}_j \cup \Sigma_{e, I_j}$ in $\dot{L}$ and that, near $\boldsymbol{\Sigma}_{e, I_j}$, H_0 has m_{j+1}-th order contact with Id along $\mathbf{W}^-_{e, I_j}$. Since $\mathbf{W}^-_{e, I_j} \cap \mathbf{L}$ is the global unstable manifold at $\boldsymbol{\Sigma}_{e, I_j}$ of both $\mathbf{X}_e$ and $\mathbf{Y}_e$, which have m_{j+1}-th order contact along it, this will imply $(17)_{j+1}$, as H_0 is a conjugacy between $\mathbf{X}_e$ and $\mathbf{Y}_e$ on and off $\mathscr{V}$.

Let $\varphi_{X,j} : (Q_{e, I_j} \times \mathbf{R}, \Sigma_{e, I_j} \times \mathbf{R}) \to E_{e, I_j}$ be the local diffeomorphism $(x, s) \mapsto f_X^s(x)$, let $\mathscr{C}_{X,j} : (\mathbf{Q}_e, \boldsymbol{\Sigma}_{e, I_j}) \to (\mathbf{Q}_{e, I_j}, \boldsymbol{\Sigma}_{e, I_j})$ be the local " chart " such that $q_{I_j} \circ \varphi_{X,j}^{-1} \circ \varphi_X = \mathscr{C}_{X,j} \circ q$, and let $\mathscr{C}_{Y,j}$ be associated to Y in the same fashion. By the perturbed version of Proposition 3 in the Complex Isolating Block Lemma (iv), $\mathbf{B}_{e, I_j}$ lies in the interior of the images of $\mathscr{C}_{X,j}$ and $\mathscr{C}_{Y,j}$; moreover, $\mathbf{X}_{e,j} = (\mathscr{C}_{X,j})_* \mathbf{X}_e|_{\mathbf{B}_{e, I_j}}$ and $\mathbf{Y}_{e,j} = (\mathscr{C}_{Y,j})_* \mathbf{Y}_e|_{\mathbf{B}_{e, I_j}}$ are C^1-small perturbations of $\xi_{e, I_j}|_{\mathbf{B}_{e, I_j}}$, having m_j-th order contact along the image of $\mathbf{W}^+_{e, I_j} \cup \mathbf{W}^-_{e, I_j}$ by $\mathscr{C}_{X,j}$ and $\mathscr{C}_{Y,j}$, i.e. along $\boldsymbol{\Sigma}_{e, I_j} \times (E^+_{I_j} \cup E^-_{I_j})$. Therefore, we can apply part (ii) of the Extension Lemma (2.2) with

$$(B, \xi, X, Y, k, m, \omega, h) = (\mathbf{B}_{e, I_j}, \xi_{e, I_j}, \mathbf{X}_{e,j}, \mathbf{Y}_{e,j}, m_{j+1}, m_j, \boldsymbol{\omega}_j, \mathbf{h}_j),$$

where $\mathbf{h}_j = \mathscr{C}_{Y,j} \circ (H_0|_{d_0 \cup \mathscr{W}_j}) \circ \mathscr{C}_{X,j}^{-1}|_{\omega_j}$ and $\boldsymbol{\omega}_j$ is defined as follows: given C_{X, I_j} and C_{Y, I_j} as in Step 3 of the proof of the Complex Isolating Block Lemma, First Part, $\boldsymbol{\omega}_j$ is the ($\mathbf{X}_{e,j}$-saturated) set of those $\mathbf{x}$ in the interior of $\mathbf{C}_{X, I_j}$ such that $\mathbf{h}_j(\mathbf{x})$ lies in the interior of $\mathbf{C}_{Y, I_j}$. As the set Ω provided by the Extension Lemma in this situation clearly satisfies $\mathscr{C}_{X,j}^{-1}(\Omega) \subset D_0$, we conclude that $\mathbf{D}_0$ and H_0 have the required properties near $\boldsymbol{\Sigma}_{e, I_j}$. ∎

Corollary 7. — Theorem 2 is true.

Proof. — Let k be a positive integer, and let $p \geqslant k$ be as in the Complex Extension Lemma. By the Complex Preparation Lemma (2.3), we just have to prove the following: if an S-vector field has p-th order contact with an S-normal form $S + N$ along $\mathscr{V}$, then Z is C^k-conjugate to $S + N$. This will be a consequence of the Complex Extension Lemma if we can prove that there exists $A \in GL(n, \mathbf{C})$, with $A^* S = S$, such that—given a representative $\widetilde{Z}$ of Z— $aA^* \widetilde{Z}|_B$ and $aA^*(S + N)|_B$ are well-defined and lie in $\mathscr{N}'' \cap \mathscr{N}'_0$.

Now, this is quite easy: for each λ in the spectrum Λ of S, let $d(\lambda)$ be the dimension of the corresponding eigenspace $E(\lambda)$ of S, and let $(x^j_\lambda)_{1 \leqslant j \leqslant d(\lambda)}$ be a system of complex linear coordinates on $E(\lambda)$ in which the nilpotent endomorphism $N^1|_{E(\lambda)}$ has an upper triangular matrix. If $d = \max d(\lambda)$, define A_ε, $\varepsilon > 0$, by

$$A_\varepsilon E(\lambda) = E(\lambda) \quad \text{and} \quad x^j_\lambda \circ A|_{E(\lambda)} = \varepsilon^{j+d} x^j_\lambda, \quad 1 \leqslant j \leqslant d(\lambda), \quad \lambda \in \Lambda.$$

For every small enough ε, $A = A_\varepsilon$ fulfils our requirements. Indeed, A_ε preserves S and (therefore) each one of its s.i.m.s', and both $A^*_\varepsilon \widetilde{Z}$ and $A^*_\varepsilon(S + N)$ tend to S on B in the C^1-topology when ε tends to 0. ■

5. PROOF OF THEOREM 1

(5.1) Algebraic background: normal forms and their formal flows

Hypotheses and notations. — Given a (not necessarily weakly hyperbolic) S, we let σ and $c_j, x_j, 1 \leqslant j \leqslant n$, be as in (3.1), and $\sigma^u(v) = \sigma(u, v)$. For each $p \in \mathbf{N}^n$, we denote by $p_1, \ldots, p_n$ its coordinates, and let $|p| = \Sigma p_j$ and $x^p = x_1^{p_1} \ldots x_n^{p_n}$. For each $X \in \mathbf{d}$, we denote by $\hat{R}_X$ the Taylor expansion of its complex flow at (0, 0), viewed as a convergent power series in the variables $u, x_1, \ldots, x_n$, where u is the (complex) parameter of the flow; we let dom $\hat{R}_X$ be the strict convergence domain of $\hat{R}_X$—if $\hat{R}_X = \Sigma a_{m,p} u^m x^p$, recall that dom $\hat{R}_X$ is the (open) set of those $(u', v') \in \mathbf{C} \times \mathbf{C}^n$ such that, for some $(u, v) \in \mathbf{C} \times \mathbf{C}^n$ with $|u| > |u'|$ and $|x_j(v)| > |x_j(v')|$ for every j, the set $\{a_{m,p} u^m x^p(v)\}$ is bounded; for each $(u, v) \in \text{dom}\, \hat{R}_X$, *we shall write* $\hat{R}_X(u, v) = \hat{R}_X(u, x_1(v), \ldots, x_n(v)) = \Sigma a_{m,p} u^m x^p(v)$—which is well-defined, by Abel's lemma.

Proposition 10. — For every S-*normal form* $S + N$,

(i) *the polynomial vector field* $N - N^1$ *is a linear combination of the monomials* $x^p \dfrac{\partial}{\partial x_j}$, $p \in P_j = \{p \in \mathbf{N}^n : |p| > 1 \text{ and } \Sigma p_k c_k = c_j\}$, $1 \leqslant j \leqslant n$,

(ii) *if we view* $\hat{R}_N$ *and* $\hat{R}_{N^1}$ *as elements of* $\mathbf{C}\{u\}[[x_1, \ldots, x_n]]^n$, *then*

$$x_j \circ (\hat{R}_N - \hat{R}_{N^1}) = \sum_{p \in P_j} a_{N,p}(u)\, x^p \quad \textit{for } 1 \leqslant j \leqslant n,$$

where each $a_{N,p} : \mathbf{C} \circlearrowleft$ *is* polynomial; *moreover, seeting* $\hat{R}^u_N(v) = \hat{R}_N(u, v)$, *we have*

$$\hat{R}^u_{S+N} = \sigma^u \circ \hat{R}^u_N = \hat{R}^u_N \circ \sigma^u \quad \textit{for each } u, \textit{ and}$$

(iii) *if* $N^1 = 0$, *then the degree of each* $a_{N,p}$ *is less than* $|p|$.

Proof. — Since $[S, N - N^1] = 0$ and $S = \Sigma c_j x_j \frac{\partial}{\partial x_j}$, (i) is clear. To prove the rest, we shall use a linear representation: denote by $\mathscr{E}$ the complex algebra of all germs at $0 \in \mathbf{C}^n$ of holomorphic complex functions, by $\mathfrak{m}$ its maximal ideal $\{f \in \mathscr{E} : f(0) = 0\}$ and by $\mathscr{E}_k$ the finite dimensional algebra $\mathscr{E}/\mathfrak{m}^{k+1}$ for each non-negative integer k. Each $X \in \mathbf{d}$ defines the derivation $X^* : f \mapsto L_X f$ of $\mathscr{E}$, hence a derivation X_k^* of $\mathscr{E}_k$. Denoting by J^k the canonical projection of $\mathscr{E}$ onto $\mathscr{E}_k$, we clearly have that $J^k(x_j \circ \hat{R}_N^u) = e^{uN_k^*}(J^k x_j)$ for every (j, k, u) [1]. Now, as N^1 is nilpotent, so is N_k^* for each k; therefore, each $u \mapsto J^k x_j \circ \hat{R}_N^u$ is polynomial, and its degree is less than k if $N^1 = 0$ (because this means that $N^* \mathfrak{m}^q \subset \mathfrak{m}^{q+1}$ for every positive integer q). From this and the fact that each $\hat{R}_N^u$ preserves S, assertions (ii)-(iii) follow at once. ∎

The following obvious result ([Ch 86], (4.3.2), Lemme 4) will be useful:

Proposition 11. — Let $\mathbf{N}^n$ *be equipped with the ordering* $\leqslant$ *defined by "* $p \leqslant q$ *if and only if* $p_j \leqslant q_j$ *for every* j*"; then, for each* $A \subset \mathbf{N}^n$*, the minimal set* min A *of* A *is finite, and every* $p \in A$ *satisfies* $p \geqslant q$ *for some* $q \in \min A$*. Therefore, if* $P_0 = \{p \in \mathbf{N}^n \backslash \{0\} : \Sigma p_k c_k = 0\}$ *and if* $P_1, \ldots, P_n$ *are as in Proposition 10, then, for* $0 \leqslant j \leqslant n$*, each element of* P_j *can be written—in a non-unique fashion in general—as the sum of one element of* $\min P_j$ *and finitely many elements of* $\min P_0$. ∎

These two results yield the easy part of the Poincaré-Dulac theorem:

Corollary 8. — If P_0 *is empty—which of course is the case when* S *is in the Poincaré domain—then*

(i) *each* P_j *is finite, and, for every* $X \in \mathbf{d}$ *with* $[S, X] = 0$, $X - X^1$ *is a linear combination of the monomials* $x^p \frac{\partial}{\partial x_j}$, $p \in P_j$, $1 \leqslant j \leqslant n$, *and*

(ii) *every* S-*normal form* $S + N$ *generates a global holomorphic action of* $\mathbf{C}$ *on* $\mathbf{C}^n$*; more precisely, by Proposition 10* (ii), *the complex flow of* N *is an algebraic* $\mathbf{C}$*-action on* $\mathbf{C}^n$. ∎

We shall now see what happens if P_0 is nonempty. For simplicity, we consider only *special* S-*normal forms*, in the following sense: call an S-vector field *special* if $X^1 = S$ (of course, such an X has nothing special in the " generic " case when S has only simple eigenvalues!).

Corollary 9. — There exists a positive integer k *such that, if* $z : \mathbf{C}^n \to \mathbf{C}^{\min P_0}$ *and* $w : \mathbf{C} \times \mathbf{C}^n \to \mathbf{C}^{\{1, \ldots, k\} \times \min P_0}$ *are given by*

$$\begin{cases} z(v) = (z_p(v))_{p \in \min P_0} & \textit{and} \quad z_p = x^p, \quad p \in \min P_0 \\ w(u, v) = (w_{m,p}(u, v))_{1 \leqslant m \leqslant k,\, p \in \min P_0}, & \textit{where } w_{m,p} = u^m z_p, \end{cases}$$

[1] Viewing $\hat{R}_N^u$ as a germ at $0 \in \mathbf{C}^n$ of a holomorphic diffeomorphism.

the following hold: for each special S*-normal form* $S + N$, *there are polynomial functions* $M_{N,p} : \mathbf{C} \circlearrowleft$ *and convergent power series* $R_{N,p}$ *at* $0 \in \mathbf{C}^{\min P_0} \times \mathbf{C}^{\{1,\dots,k\} \times \min P_0}$, $p \in \bigcup_{j>0} \min P_j$, *such that*

$$x_j(\hat{R}_N(u, v) - v) = \sum_{p \in \min P_j} (uM_{N,p}(u) + R_{N,p}(z(v), w(u, v)))\, x^p(v), \quad 1 \leqslant j \leqslant n \tag{18}$$

and

$$R_{N,p} = 0 \quad \text{on } \mathbf{C}^{\min P_0} \times \{0\}. \tag{19}$$

Identity (18) *means in particular that we have* $(u, v) \in \operatorname{dom} \hat{R}_N$ *if and only if* $(z(v), w(u, v))$ *lies in the strict convergence domain* $\operatorname{dom} R_{N,p}$ *for every* p.

Proof. — By Proposition 10 (ii)-(iii), $x_j \circ \hat{R}_N = x_j + \sum_{p \in P_j, 0<m<|p|} a_{m,p} u^m x^p$, $1 \leqslant j \leqslant n$. Now, by Proposition 11, there does exist a positive integer k such that each $u^m x^p$, $p \in \bigcup_{j>0} P_j \backslash \min P_j$ and $0 < m < |p|$, can be written $z^{K(m,p)} w^{L(m,p)} x^{Q(p)}$, with $Q(p) \in \bigcup_{j>0} \min P_j$ and $|L(m, p)| > 0$. We shall stick to the following

Rule. — *Choose exactly one such mapping* $(m, p) \xrightarrow{J} (K(m, p), L(m, p), Q(p))$.

Clearly, J is injective. Therefore, if we set $b_{r,\ell,q} = a_{m,p}$ if $(r, \ell, q) = J(m, p)$ and $b_{r,\ell,q} = 0$ in the remaining case, our corollary follows at the formal level, with $uM_{N,p}(u) = \sum_{0<m<|p|} a_{m,p} u^m$ and $R_{N,p}(z, w) = \sum_{r,\ell} b_{r,\ell,p} z^r w^\ell$.

Now, given (u, v) and (u', v') in $\mathbf{C} \times \mathbf{C}^n$,

a) the set $\{a_{m,p} u^m x^p(v)\}$ is bounded if and only if so is $\{b_{r,\ell,q} z^r(v) w^\ell(u, v) x^q(v)\}$, and

b) the inequalities $|u'| < |u|$ and $|x_j(v')| < |x_j(v)|$ for every j imply $|z_p(v')| < |z_p(v)|$ and $|w_{m,p}(u', v')| < |w_{m,p}(u, v)|$ for every (m, p); this and *a)* show that (u', v') dom $\hat{R}_N$ implies $(z(v'), w(u', v')) \in \operatorname{dom} R_{N,p}$ for every p;

c) conversely, if we have $|z_p(v')| < |z_p(v)|$ and $|w_{m,p}(u', v')| < |w_{m,p}(u, v)|$ for every (m, p), then, assuming that $\{b_{r,\ell,q} z^r(v) w^\ell(u, v)\}$ is bounded for every q, we get the following: as $(u, v) \mapsto (z(v), w(u, v))$ is continuous, there exists $(u'', v'') \in \mathbf{C} \times \mathbf{C}^n$ with $|u''| > |u'|$ and $|x_j(v'')| > |x_j(v')|$ for every j, satisfying $|z_p(v'')| < |z_p(v)|$ and $|w_{m,p}(u'', v'')| < |w_{m,p}(u, v)|$ for every (m, p). In other words, by *a)*, $(z(v'), w(u', v')) \in \operatorname{dom} R_{N,p}$ for every p implies $(u', v') \in \operatorname{dom} \hat{R}_N$. ∎

Corollary 10. — *If* $S + N$ *and* $S + N'$ *are two special* S*-normal forms, then, there are polynomial functions* $Q_{N,N',p} : \mathbf{C} \circlearrowleft$ *and convergent power series* $T_{N,N',p}$ *at* $0 \in \mathbf{C}^{\min P_0} \times \mathbf{C}^{\{1,\dots,k\} \times \min P_0}$, $p \in \bigcup_{j>0} \min P_j$, *such that, for* $1 \leqslant j \leqslant n$,

$$\begin{aligned} x_j(\hat{R}^u_{S+N'} \circ \hat{R}^{-u}_{S+N}(v)) &= x_j(\hat{R}^u_{N'} \circ \hat{R}^{-u}_N(v)) \\ &= x_j(v) + \sum_{p \in \min P_j} (uQ_{N,N',p}(u) + T_{N,N',p}(z(v), w(u, v)))\, x^p(v) \end{aligned} \tag{20}$$

and

$$(21) \qquad T_{N,N',p} = 0 \quad \textit{on } \mathbf{C}^{\min P_\bullet} \times \{0\}.$$

Identity (20) *means in particular that, if we have* $(-u, v) \in \operatorname{dom} \hat{R}_N$ *and* $(u, \hat{R}_N^{-u}(v)) \in \operatorname{dom} \hat{R}_{N'}$, *then* $(z(v), w(u, v))$ *belongs to* $\operatorname{dom} T_{N,N',p}$ *for each* p.

Proof. — Since N and N′ lie in the centraliser of S, $\hat{R}^u_{S+N'} \circ \hat{R}^{-u}_{S+N} = \hat{R}^u_{N'} \circ \hat{R}^{-u}_N$. Now, Proposition 10 (ii)-(iii) and straightforward calculations yield

$$x_j \circ \hat{R}^u_{N'} \circ \hat{R}^{-u}_N = x_j + \sum_{p \in P_j} b_{N,N',p}(u)\, x^p,$$

where each $b_{N,N',p} : \mathbf{C} \circlearrowleft$ is polynomial, of degree less than $|p|$, and vanishes at 0. We conclude as in the proof of Corollary 9. ■

(5.2) A generalisation of Theorem 1

If S is weakly hyperbolic, call an S-normal form $S + N$ *reduced* if N is a linear combination of monomials $x^p \dfrac{\partial}{\partial x_j}$, $p \in P_j$, $1 \leqslant j \leqslant n$, such that every c_k with $p_k \neq 0$ lies in $\mathbf{R}c_j$. In particular, if S is hyperbolic, the only reduced S-normal form is S itself—and, of course, every S-normal form is special. Therefore, Theorem 1 is a particular case of

Theorem 3. — If S *is weakly hyperbolic, then every special* S*-vector field is* C^0*-conjugate to a reduced special* S*-normal form.*

Before proving this result, let us explain why it provides " good " normal forms:

Proposition 12. — If S *is weakly hyperbolic, then every reduced* S*-normal form* $S + N$ *generates a holomorphic* **C***-action* R_{S+N} *of the form* $R^u_{S+N} = R^u_N \circ \sigma^u = \sigma^u \circ R^u_N$, *where* R_N *is the* algebraic **C***-action* $(u, v) \mapsto R^u_N(v)$, $u \in \mathbf{C}$, $v \in \mathbf{C}^n$, *generated by* N. *More precisely, denoting by* v_J *the canonical projection of* $v \in \mathbf{C}^n = \bigoplus_{I \in \mathscr{K}} E_I$ *onto* E_J, $J \in \mathscr{K}$, *the hypothesis that* $S + N$ *is reduced means that* $N(v) = \sum_{I \in \mathscr{K}} N(v_I)$ *for each* $v \in \mathbf{C}^n$; *for every* I, $N_I = N|_{E_I}$ *and* $S_I = S|_{E_I}$ *are vector fields on* E_I, *and* S_I *is in the Poincaré domain. Therefore, by Corollary 8, the flow* R_{N_I} *of each* N_I *is an algebraic action, and the actions* R_N *and* R_{S+N} " *split* " *as follows: for each* $v \in \mathbf{C}^n$,

$$R_N(u, v) = \sum_{I \in \mathscr{K}} R_{N_I}(u, v_I),$$

hence

$$R^u_{S+N}(v) = \sum_{I \in \mathscr{K}} R^u_{S_I} \circ R^u_{N_I}(v_I) = \sum_{I \in \mathscr{K}} R^u_{N_I} \circ R^u_{S_I}(v_I).$$

Proof. — For each $I \in \mathscr{K}$, S and N are tangent to each of the two s.i.m. $E_{I_0^+}$ and $E_{I_0^-}$ of S, hence to their intersection E_I. As every S_I is in the Poincaré domain by weak hyperbolicity, one just has to check our characterisation of reduced S-normal forms, which is easy. ■

Remark. — *When* S *in only weakly hyperbolic, a reduced special* S*-normal form* S + N *may be* C^0*-unequivalent to* S: for example, if $n = 2$, $c_1 = 1$, $c_2 = 2$ and $N = x_1^2 \frac{\partial}{\partial x_2}$, then every leaf of the foliation defined by S (*see* the beginning of (1.3)) is a punctured complex line with puncture at $0 \in \mathbf{C}^2$, whereas the leaves of the foliation defined by S + N are injectively immersed complex lines, with the sole exception of $Oy\backslash\{0\}$.

(5.3) Proof of Theorem 3

Still assuming S weakly hyperbolic, we shall establish

Lemma 3. — *For each regular value b of* $\mathbf{F}$, *every special* S*-normal form* $S + N_0$ *is* C^0*-conjugate to the (special)* S*-normal form* $S + N_1$ *obtained from* $S + N_0$ *by cancelling the coefficient of every monomial* $x^p \frac{\partial}{\partial x_j}$, $p \in P_j$, *such that* $b \in \sum_{p_k \neq 0} \mathbf{R}_+ c_k$.

This implies Theorem 3: by Theorem 2, every special S-vector field is C^1-conjugate to a special S-normal form S + N (to see that it is indeed special, just notice that the conjugacies constructed in the proofs of the Complex Preparation Lemma [Ch 86*a*] and of Corollary 7 are tangent to the identity at 0). Now, N is a linear combination of monomials $x^p \frac{\partial}{\partial x_j}$, $p \in P_j$, $1 \leqslant j \leqslant n$, and, for each such monomial, Proposition 7 yields the following: either $\sum_{p_k \neq 0} \mathbf{R}_+ c_k$ is a half-line, or it contains a regular value of $\mathbf{F}$. Therefore, applying Lemma 3 finitely many times, we obtain that S + N is C^0-conjugate to S + N′, where N′ is obtained from N by cancelling the coefficients of those $x^p \frac{\partial}{\partial x_j}$ such that $\sum_{p_k \neq 0} \mathbf{R}_+ c_k$ is not a half-line, hence Theorem 3 by weak hyperbolicity. □

Proof of Lemma 3. — We may assume $b \in \Sigma \mathbf{R} c_j$, as $N_0 = N_1$ if this is not the case. Our other hypotheses and notations will be those of (4.1).

Step 1. — *For each* $p \in \mathbf{N}^n$, *we have* $b \in \sum_{p_k \neq 0} \mathbf{R}_+ c_k$ *if and only if* x^p *vanishes identically on* $\hat{\mathscr{V}}_b$. Indeed, a s.i.m. E_I of S, $I \subset \{1, \ldots, n\}$, is contained in $\hat{\mathscr{V}}_b$ if and only if $b \notin \sum_{j \in I} R_+ c_j$. Therefore, we have $b \in \sum_{p_k \neq 0} R_+ c_k$ if and only if, for every such I, there exists $k \notin I$ with $p_k \neq 0$, hence our result.

Hypotheses and notation. — For each $\varepsilon > 0$, let $\tilde{\varepsilon} = \varepsilon \, \mathrm{Id} \in \mathrm{GL}(n, \mathbf{C})$. In the sequel, we let $N = \tilde{\varepsilon}^* N_0$, $N' = \tilde{\varepsilon}^* N_1$, $X = a\tilde{\varepsilon}^*(S + N_0)|_B = a(S + N)|_B$, $Y = a\tilde{\varepsilon}^*(S + N_1)|_B = a(S + N')|_B$, and the locution "for every small enough ε" is implicit. When ε tends to 0, so do N and N′ in the C^1-topology. Therefore, *we* may (and *shall*) *assume that* X *and* Y *belong to the set* $\mathcal{N}'_0$ defined in the Complex Isolating Block Lemma. To simplify notations, *we assume* $a = 1$, which of course is no restriction.

We shall prove that *the mapping h associated to* X, Y *by Proposition 9* (ii) *fulfils conditions* (i)-(ii) *of the Complex Extension Lemma with* $k = 0$.

This extension problem will be solved using Corollaries 9-10, which will allow us to obtain our conjugacy as the sum of a convergent power series in variables which are continuous functions. More precisely, we shall see that the mapping obtained by replacing R_X and R_Y by $\hat{R}_X : \mathrm{dom}\, \hat{R}_X \to \mathbf{C}^n$ and $\hat{R}_Y : \mathrm{dom}\, \hat{R}_Y \to \mathbf{C}^n$ in Proposition 9 (13) can be extended continuously by Id on $\hat{\mathcal{V}}_b \cap \dot{V}_X$ and coincides with h near $\mathcal{V} \cap \dot{V}_X$.

Step 2. — *For each* $(m, p) \in \mathbf{N} \times \mathbf{N}^n$, *if* x^p *vanishes identically on* $\hat{\mathcal{V}}_b$, *then*

(i) *the mapping* $\tilde{\tilde{V}}_X \ni v \mapsto \begin{cases} \gamma_{X,v}(s)^m \, x^p(v) & \text{if } v \notin \hat{\mathcal{V}}_b \\ 0 & \text{if } v \in \hat{\mathcal{V}}_b \end{cases}$

is uniformly bounded with respect to s *and (small enough)* ε, *and*

(ii) *the mapping* $v \mapsto \begin{cases} r_X(v)^m \, x^p(v) & \text{if } v \notin \hat{\mathcal{V}}_b \\ 0 & \text{if } v \in \hat{\mathcal{V}}_b \end{cases}$

is continuous on $\tilde{\tilde{V}}_X$.

Indeed, for every $v \in \tilde{\tilde{V}}_X \backslash \hat{\mathcal{V}}_b$, our hypothesis on p yields $|x^p(v)| \leqslant d(v, \hat{\mathcal{V}}_b)$, hence (i) and—since r_X is continuous—(ii), by the Complex Isolating Block Lemma (vii).

Step 3. — (i) *The mapping* $\tilde{\tilde{V}}_X \ni v \mapsto \begin{cases} (z(v), w(r_X(v), v)) & \text{if } v \notin \hat{\mathcal{V}}_b \\ (0, 0) & \text{if } v \in \hat{\mathcal{V}}_b \end{cases}$

is continuous.

(ii) *We have* $(-r_X(v), v) \in \mathrm{dom}\, \hat{R}_N$ *and* $\hat{R}_{S+N}^{-r_X(v)}(v) = R_X^{-r_X(v)}(v)$ *for each* $v \in \tilde{\tilde{V}}_X \backslash \hat{\mathcal{V}}_b$.

Indeed, by Step 1, each x^p with $p \in P_0$ vanishes identically on $\hat{\mathcal{V}}_b$ (as $\Sigma p_j c_j = 0$, the origin lies in the convex hull of $\{c_j : p_j \neq 0\}$, hence in its interior by weak hyperbolicity), and thus Step 2 (ii) yields (i). Moreover, by Step 2 (i), the mapping $v \mapsto (z(v), w(-\gamma_{X,v}(s), v))$ is uniformly bounded on $\tilde{\tilde{V}}_X \backslash \hat{\mathcal{V}}_b$ with respect to s and (small enough) ε. Now, we have $\mathrm{dom}\, R_{N,p} \supset \varepsilon^{-1} \mathrm{dom}\, R_{N_0,p}$ for each $p \in \bigcup_{j>0} \min P_j$;

therefore, if ε is small enough, Corollary 9 implies $(-\gamma_{X,v}(s), v) \in \operatorname{dom} \hat{R}_N$ for every $v \in \tilde{\tilde{V}}_X \setminus \hat{\mathscr{V}}_b$ and every s. This proves (ii), for both $R_X^{-r_X(v)}(v)$ and $\hat{R}_X^{-r_X(v)}(v)$ are obtained by " integrating $S + N$ along the path $-\gamma_{X,v}$ ". □

Step 4. — (i) *For each* $v \in \tilde{\tilde{V}}_X \setminus \hat{\mathscr{V}}_b$, *we have* $(r_X(v), R_X^{-r_X(v)}(v)) \in \operatorname{dom} \hat{R}_{N'}$.

(ii) *The mapping* $\hat{H} : \dot{V}_X \to \mathbf{C}^n$, *equal to* Id *on* $\hat{\mathscr{V}}_b$ *and to* $v \mapsto \hat{R}^{r_X(v)}_{S+N'} \circ R_X^{-r_X(v)}(v)$ *off* $\hat{\mathscr{V}}_b$, *is continuous.*

Indeed, for each $q \in \min P_0$ and each $v \in \tilde{\tilde{V}}_X \setminus \hat{\mathscr{V}}_b$,

$$z_q(R_X^{-r_X(v)}(v)) = z_q(\hat{R}_{S+N}^{-r_X(v)}(v)) = z_q(\hat{R}_N^{-r_X(v)}(v))$$

by Step (ii) and Proposition 10 (ii); now, Corollary 9 (18) implies that $z_q(\hat{R}_N^{-u}(v))$ is a polynomial in the variables $R_{N,p}(z(v), w(u, v))$ and $uM_{N,p}(u)$, all of whose coefficients are of the form $z_{p'}(v)$, $p' \in P_0$; moreover, the coefficients of each $M_{N,p}$ tend to 0 with ε, and so does each $\sup_{x \in \tilde{\tilde{V}}_X} |R_{N,p}(z(v), w(-r_X(v), v))|$. Therefore, we obtain (i) as Step 3 (ii).

By Step 3 (ii) and Proposition 10 (ii),

$$\hat{H}(v) = \hat{R}^{r_X(v)}_{S+N'} \circ \hat{R}^{-r_X(v)}_{S+N}(v) = \hat{R}^{r_X(v)}_{N'} \circ \hat{R}_N^{-r_X(v)}(v)$$

off $\hat{\mathscr{V}}_b$, hence, for $1 \leqslant j \leqslant n$,

$$x_j \circ \hat{H}(v) = x_j(v) + \sum_{p \in \min P_j} (r_X(v)\, Q_{N,N',p}(r_X(v)) + T_{N,N',p}(z(v), w(r_X(v), v)))\, x^p(v) \tag{22}$$

by Corollary 10 (20). Therefore, by Step 2 (ii) and Corollary 10 (21), we just have to prove that $Q_{N,N',p} = 0$ if x^p does not vanish identically on $\hat{\mathscr{V}}_b$, which is clear: for each s.i.m. $W \subset \hat{\mathscr{V}}_b$ of S, we have indeed

$$x_j(\hat{R}^u_{N'} \circ \hat{R}_N^{-u}(v) - v) = \sum_{p \in \min P_j,\ x^p|_W \neq 0} uQ_{N,N',p}(u)\, x^p(v) = 0$$

for $1 \leqslant j \leqslant n$ and $v \in W$, since $N = N'$ on W. □

Step 5. — *The connected component* U_0 *of* 0 *in* $\hat{H}^{-1}(\dot{V}_Y)$ *contains* $\hat{\mathscr{V}}_b \cap \dot{V}_X$ *and is contained in* U *(notation of the Complex Extension Lemma). For every* $v \in U_0 \setminus \hat{\mathscr{V}}_b$, $r_Y(\hat{H}(v)) = r_X(v)$, *and* $\hat{H}(v) = h(v)$ *(notation of Proposition 9).*

Indeed, $\hat{\mathscr{V}}_b \cap V_X = \hat{\mathscr{V}}_b \cap V_Y = \hat{\mathscr{V}}_b \cap \dot{L}$, hence (as $X = Y$ on $\hat{\mathscr{V}}_b$) $\hat{\mathscr{V}}_b \cap \dot{V}_X = \hat{\mathscr{V}}_b \cap \dot{V}_Y$; since this is a connected set containing 0, on which $\hat{H} = \mathrm{Id}$, it lies in U_0.

By the same argument as in the proof of Step 4 (i), (22) yields

$$(-r_X(v), \hat{H}(v)) \in \operatorname{dom} \hat{R}_{N'} \quad \text{and} \quad \hat{R}_Y^{-r_X(v)}(\hat{H}(v)) = R_X^{-r_X(v)}(v), \quad v \in U_0 \setminus \hat{\mathscr{V}}_b.$$

Moreover, replacing X by Y in Step 3, we get

$$(-r_Y(\hat{H}(v)), \hat{H}(v)) \in \operatorname{dom} \hat{R}_{N'} \quad \text{and} \quad \hat{R}_Y^{-r_Y(\hat{H}(v))}(\hat{H}(v)) = R_Y^{-r_Y(\hat{H}(v))}(\hat{H}(v)), \quad v \in U_0 \backslash \hat{\mathscr{V}}_b.$$

Thus, if we can show that $r_X = r_Y \circ \hat{H}$ in $U_0\backslash\hat{\mathscr{V}}_b$, we shall have that $R_Y^{-r_X(v)}(\hat{H}(v)) = R_X^{-r_X(v)}(v)$, hence $\hat{H}(v) = R_Y^{r_X(v)} \circ R_X^{-r_X(v)}(v)$, for each $v \in U_0\backslash\hat{\mathscr{V}}_b$. By Proposition 9, this will prove our result. Now, both r_X and $r_Y \circ \hat{H}$ are analytic in the connected set $U_0\backslash\hat{\mathscr{V}}_b$, and they clearly coincide—because so do h and $\hat{H}$—in a neighbourhood of V (the inclusion $V \subset U_0$ comes from the fact V and $\dot{V}_X \cap \mathscr{V} = \dot{V}_Y \cap \mathscr{V}$ are connected); therefore, $r_X = r_Y \circ \hat{H}$ in $U_0\backslash\hat{\mathscr{V}}_b$. □

With the notation of the Complex Extension Lemma, we have proved that U is an open neighbourhood of $\hat{\mathscr{V}}_b\backslash\dot{V}_X$ in $\dot{V}_X$ and that the mapping $H : U \to \mathbf{C}^n$, equal to Id on $\hat{\mathscr{V}}_b$ and to h off $\hat{\mathscr{V}}_b$, is continuous, hence a homeomorphism onto its image. Now, we have that $H^*(zY) = zX$ for every $z \in \mathbf{C}$, on and off $\hat{\mathscr{V}}_b$: this is true off $\hat{\mathscr{V}}_b$ by Proposition 9, and on $\hat{\mathscr{V}}_b$ because $X|_W = Y|_W$ is tangent to W for every s.i.m. $W \subset \hat{\mathscr{V}}_b$ of S, and $H|_W = Id|_W$. Therefore, the germ of H at 0 is a C^0-conjugacy between X and Y, hence Lemma 3. ■

6. CONCLUDING REMARKS

(6.1) C^k-linearisations

The following result is stated as Theorem B in [DR]:

Theorem 4. — For each positive integer k, there is an open and dense subset V_k *of* $\mathfrak{gl}(n, \mathbf{C})$, *the complementary subset of which has codimension one, such that every* S*-vector field with* $S \in V_k$ *is* C^k*-conjugate to* S.

Proof. — If $S_0 \in \mathfrak{gl}(n, \mathbf{C})$ is weakly hyperbolic and has only simple eigenvalues, then ([Ch 86a]), for every $k \in \mathbf{N}$, there exist an integer $\ell \geqslant k$ and a neighbourhood U_k of S_0 in $\mathfrak{gl}(n, \mathbf{C})$ such that, for every $S \in U_k$, the S-normal forms in the Complex Preparation Lemma (2.3) can be chosen of degree ℓ; now, the set U'_k of those $S \in U_k$ such that the only S-normal form of degree ℓ is S itself has a closed, one-codimensional complementary subset in U_k—*see* (5.1).

Given a positive integer k, let $a \in \mathbf{C}$ and $p \geqslant k$ be as in the Complex Extension Lemma (4.4) with $S = S_0$. As the eigenvalues, eigenspaces and maximal s.i.m.'s of $T \in \mathfrak{gl}(n, \mathbf{C})$ depend analytically on T near S_0, there is an open neighbourhood U'' of S_0 in $\mathfrak{gl}(n, \mathbf{C})$ such that, for each $T \in U'' \cap U'_p$ and each T-vector field Z, having p-th order contact with T along its s.i.m.'s we have the following: given a representative $\tilde{Z}$

of Z, there exists an $A \in GL(n, \mathbf{C})$ such that the hypotheses of the Complex Extension Lemma are fulfiled by $Y = aA^* \widetilde{Z}|_B$ and $X = aA^* T|_B$; thus, every T-vector field with $T \in U'' \cap U'_p$ is C^k-conjugate to T, hence Theorem 4. ■

Although the set V_k we obtain in this fashion is larger than that of [DR], it is still rather " small ", and its definition is not really simple; the purpose of [Ch 86*c*] is to construct a much more reasonable V_1.

(6.2) Families

Given a diagonalisable $S \in \mathfrak{gl}(n, \mathbf{C})$, an (S, p)*-unfolding* is a germ X at $(0, 0) \in \mathbf{C}^n \times \mathbf{C}^p$ of a holomorphic vector field of the form $(x, u) \mapsto (X_u(x), 0)$, where each X_u is a local holomorphic vector field on $\mathbf{C}^n$, such that $X_0(0) = 0$ and that S is the semi-simple part of $dX_0(0)$. If 0 is not an eigenvalue of S, an (S, p)*-normal form* is an (S, p)-unfolding having a representative $(x, u) \mapsto (Sx + N_u(x), 0)$ such that each N_u is polynomial and commutes with S, and $S + N_0$ is an S-normal form. An (S, p)-unfolding and a (T, p)-unfolding are C^k*-conjugate*, $k \in \mathbf{N}$, when there exists a C^k-conjugacy $h : (\mathbf{C}^n \times \mathbf{C}^p, (0, 0))$ ⤾ between them (in the sense of (1.3)) admitting a representative of the form $(x, u) \mapsto (h_u(x,) u)$. The above methods yield the following result [Ch 86*b*]:

Theorem 5. — (i) *If* S *is weakly hyperbolic, then, for each pair* (k, p) *of positive integers, every* (S, p)*-unfolding is* C^k*-conjugate to an* (S, p)*-normal form.*

(ii) *Let* $(z_1, \dots, z_n)$ *be the canonical coordinate system on* $\mathbf{C}^n$. If S *is hyperbolic, then, for each* $p \in \mathbf{N}$, *every* (S, p)*-unfolding is* C^0*-conjugate to a normal form represented by* $(x, u) \mapsto ((c_j(u)\, z_j(x))_{1 \leqslant j \leqslant n}, 0)$, *where* $c_1, \dots, c_n : (\mathbf{C}^p, 0) \to \mathbf{C}$ *are local holomorphic functions.*

(iii) *For each pair* (k, p) *of positive integers and each* S *in the set* V_k *of Theorem 4, every* (S, p)*-unfolding is* C^k*-conjugate to a normal form of the same type as in* (ii).

Of course, (i), (ii) and (iii) are generalisations of Theorems 2, 1 and 4 respectively. In [Ch 86*c*], we prove (iii) for $k = 1$ and a reasonable V_1; as the germs of $c_1, \dots, c_n$ are C^1-conjugacy invariants, this provides universal unfoldings ([A]) for C^1-conjugacy. Noticing that, as in the proof of Theorem 4, the degree of the (S, p)-normal forms in (i) with respect to the $\mathbf{C}^n$-variable has a bound which depends only on S, we obtain versal unfoldings for C^k-conjugacy, under the sole weak hyperbolicity hypothesis. The problem of finding *universal* unfoldings in this general case seems very difficult—*see* [Ch 86*c*].

If we replace C^k-conjugacy by (holomorphic) conjugacy, (i) and (iii) are true *in the Poincaré domain*, where the set which corresponds to V_k is simply the set of those S which have only simple eigenvalues and for which $P_1, \dots, P_n$ are empty—*see* [A], § 36, C. In the Siegel domain, there is no hope for such results—which are false even at the formal level; this is why Theorem 5 is interesting.

(6.3) Historical and technical comments

The proof of Theorem 2 originated in an attempt to understand and generalise the excellent—and curiously underrated—work of Dumortier and Roussarie ([DR]). My main contributions are the Complex Preparation Lemma—which would have simplified their proof of Theorem 4—the weakening of Hyperbolicity, and the statement and proof of a general normal form theorem instead of a linearisation result. The " Lyapunov function " **F**, introduced in [C], is not really necessary here (it is not used in [DR]), but—besides being crucial in the proof of Theorem 1—it makes everything twork just as well for general smooth germs of $\mathbf{Z}^k \times \mathbf{R}^m$-actions, yielding the generalisation of [DR] I was aiming for (*see* Chapter 3 of [Ch 86] and [Ch 86*d*]).

The Complex Preparation Lemma (2.3) and the Extension Lemma (2.2) came from a geometric reading of Nelson's nice (almost) proof of Sternberg's theorem ([N]). A difference with [N] is that I localise everything in " isolating blocks ", which makes the situation geometrically clearer—but technically worse, due to problems of definition domains; this formulation allowed the direct study of holomorphic flows in Section 4—otherwise, they should have been extended to global smooth $\mathbf{R}^2$-actions (as in Section 6 of [Ch 86]).

The author is entirely responsible for the rather simple-minded proof of Theorem 1. In [Ch 86*c*], similarly, C^1-conjugacies are constructed as the sums of convergent power series in variables which are functions of class C^1.

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