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QUADRATIC VECTOR FIELDS IN THE PLANE HAVE A FINITE NUMBER OF LIMIT CYCLES

by RODRIGO BAMÓN

INTRODUCTION

An isolated periodic orbit of a vector field in $\mathbf{R}^2$ is called a *limit cycle*. Part of Hilbert's 16-th problem is to find an upper bound for the number of limit cycles of polynomial vector fields of a given degree. Still today, very little is known about these upper bounds. Moreover it is not known if an arbitrary polynomial vector field has a finite number of limit cycles.

In 1923, Dulac [D] claimed that all *graphs* (see definitions in Chapter 1) of analytic vector fields in the plane are *finite* (i.e. they are not accumulated by limit cycles). From this result follows the finiteness of limit cycles for polynomial vector fields. Recently, Il'yašenko [I] gave a strikingly simple counterexample to one of Dulac's main assertions, and gave a correct proof for the fact that all *hyperbolic graphs* (see Chapter 1) of analytic vector fields are finite. This represents a major step and is essential for the result in this paper.

Around 1956 Petrovskii and Landis [P-L_1, P-L_2] claimed that quadratic vector fields in the plane have at most 3 limit cycles. In 1959 they withdrew their proof [P-L_3]. Later in 1979, the chinese mathematician Shi Song Ling [Sh_1, Sh_2] produced examples of quadratic vector fields with 4 limit cycles, disproving the estimate of Petrovskii and Landis.

For our work we start from the fact that a polynomial vector field with infinitely many limit cycles must contain a graph (*bounded* or *unbounded*, see Chapter 1) which is accumulated by limit cycles. This follows from the Poincaré-Bendixon Theorem.

Taking into account very special qualitative properties of quadratic systems, all of which are recalled in Chapter 1, we prove the following theorem

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Theorem B. — *All graphs (bounded or unbounded) of quadratic vector fields in* $\mathbf{R}^2$ *are finite.*

From the fact we mentioned above follows immediately

Theorem A. — *Every quadratic vector field in* $\mathbf{R}^2$ *has a finite number of limit cycles.*

In 1983, Chicone and Shafer [Ch-S] proved the finiteness of bounded graphs. Here we give an alternative proof.

In Chapter 1 we give definitions and general information. We also recall properties of quadratic systems. In Chapter 2 we prove Theorem B.

We will denote by χ^2 the space of quadratic vector fields endowed with the topology of the coefficients.

Acknowledgements. — I am indebted to a number of people for helping me in several ways to achieve the results in this paper, among them A. Lins Neto, R. Moussu, J. Palis, J. Sotomayor. I also acknowledge N. Yus for his earlier stimulus in the area of Dynamical Systems.

1. Preliminaries

1.1. *General definitions*

Let X be a differentiable vector field in $\mathbf{R}^2$.

Definition 1. — An orbit $\varphi(t) = (x(t), y(t))$ of X is called a *separatrix of* X if its ω-limit set (or its α-limit set) is a singular point $p = (x_0, y_0)$ of X and

1) $\lim\limits_{t\to\infty} \dfrac{x(t) - x_0}{y(t) - y_0}$ $\left(\text{or } \lim\limits_{t\to-\infty} \dfrac{x(t) - x_0}{y(t) - y_0}\right)$ exists and belongs to $\mathbf{R} \cup \{\pm\infty\}$;
2) there exists $\varepsilon_1 > 0$ and $T > 0$ $(T < 0)$ with $|\varphi(t) - p| < \varepsilon_1$ for all $t \geqslant T$ $(t \leqslant T)$ such that for all $\varepsilon > 0$ there exist an orbit $\psi(t)$ of X and $\widetilde{T} > T$ $(\widetilde{T} < T)$ such that $|\psi(t) - \varphi(t)| < \varepsilon$ for all $t \in [0, T]$ $(t \in [T, 0])$ and $|\psi(\widetilde{T}) - p| > \varepsilon_1$.

Definition 2. — A closed curve Γ in $\mathbf{R}^2$ is called a *graph* of X if it can be parametrized by $\alpha : [0, 1] \to \mathbf{R}^2$ of class C^1 with $\alpha(0) = \alpha(1)$ satisfying:

1) if $\alpha'(t) = 0$ then $X(\alpha(t)) = 0$;
2) if $\alpha'(t) \neq 0$ then $\alpha(t)$ belongs to a separatrix of X and there exists $\lambda > 0$ such that $\alpha'(t) = \lambda X(\alpha(t))$.

We will say that a graph Γ of X has a *return map* if for all cross sections Σ of X intersecting Γ, there exists $p \in \Sigma$ such that $\omega(p) \cap \Sigma \neq \Phi$ or $\alpha(p) \cap \Sigma \neq \Phi$.

1.2. *Poincaré's compactification*

Let $S^2 = \{(x, y, z) \in \mathbf{R}^3 \mid x^2 + y^2 + z^2 = 1\}$ and let H^+ and H^- be the hemispheres $\{(x, y, z) \in S^2 \mid z > 0\}$ and $\{(x, y, z) \in S^2 \mid z < 0\}$, respectively.

To a differentiable vector field X in $\mathbf{R}^2$ we can associate vector fields X^+ and X^- in H^+ and H^- in the following way: first we define in $\{(x,y,z) \mid z = 1\}$ the vector field $\widetilde{X}(x,y,z) = (X(x,y), 0)$ and then we project it by central projection onto H^+ and H^-. When X is a polynomial vector field of degree n, it happens that multiplying X^+ and X^- by z^{n-1} we obtain a vector field in $H^+ \cup H^-$ that can be continuously extended to $S^1 = \{(x,y,z) \in S^2 \mid z = 0\}$ in such a way that the resulting vector field on S^2 is analytic.

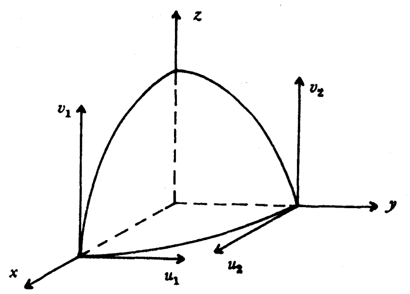
This is the so-called *Poincaré compactification* of X and is denoted by $\mathscr{P}(X)$. This construction allows us to study the flow of a polynomial vector field far away from the origin. The equator S^1 of S^2 represents the " points at infinity " and it happens (by construction) that points at infinity remain at infinity under the flow of $\mathscr{P}(X)$ (i.e. S^1 is invariant for $\mathscr{P}(X)$).

A graph of X containing separatrices (and thus singularities) at infinity will be called an *unbounded graph.*

To study a polynomial vector field at infinity we consider the following coordinates

$$\begin{cases} u_1 = y/x \\ v_1 = 1/x \end{cases} \quad \text{and} \quad \begin{cases} u_2 = x/y \\ v_2 = 1/y. \end{cases}$$

Geometrically they can be represented as follows:



If $X \in \chi^2$ is given by

$$(1) \qquad X : \begin{cases} \dot{x} = P(x,y) = \alpha + mx + ny + ax^2 + bxy + cy^2 \\ \dot{y} = Q(x,y) = \bar{\alpha} + \bar{m}x + \bar{n}y + \bar{a}x^2 + \bar{b}xy + \bar{c}y^2 \end{cases}$$

then in the coordinates u_1, v_1 it is expressed in the form

$$(2) \qquad X_1 : \begin{cases} \dot{u}_1 = P_1(u_1, v_1) \\ \dot{v}_1 = Q_1(u_1, v_1) \end{cases}$$

with
$$P_1(u_1, v_1) = \bar{a} + (\bar{b} - a)\, u_1 + \bar{m}v_1 + (\bar{c} - b)\, u_1^2 + (\bar{n} - m)\, u_1 v_1 + \bar{\alpha}v_1^2 - cu_1^3 - nu_1^2 v_1 - \alpha u_1 v_1^2$$

$$Q_1(u_1, v_1) = - av_1 - bu_1 v_1 - mv_1^2 - cu_1^2 v_1 - nu_1 v_1^2 - \alpha v_1^3$$

and in the coordinates u_2, v_2 it is expressed in the form

$$(3) \qquad X_2 : \begin{cases} \dot{u}_2 = P_2(u_2, v_2) \\ \dot{v}_2 = Q_2(u_2, v_2) \end{cases}$$

with
$$P_2(u_2, v_2) = c + (b - \bar{c})\, u_2 + n v_2 + (a - \bar{b})\, u_2^2 + (m - \bar{n})\, u_2 v_2 + \alpha v_2^2 - \bar{a} u_2^3 - \bar{m} u_2^2 v_2 - \bar{\alpha} u_2 v_2^2$$
$$Q_2(u_2, v_2) = -\bar{c} v_2 - \bar{b} u_2 v_2 - \bar{n} v_2^2 - \bar{a} u_2^2 v_2 - \bar{m} u_2 v_2^2 - \bar{\alpha} v_2^3.$$

Lemma **1.1.** — *Every* $X \in \chi^2$ *has one, two or three pairs of symmetric singularities at infinity or the whole infinity is filled up with singularities.*

Proof. — If $X \in \chi^2$ is given by (1), using the expressions of X_1 and X_2 we have that the singularities at infinity are obtained by solving
$$P_1(u_1, 0) = \bar{a} + (\bar{b} - a)\, u_1 + (\bar{c} - b)\, u_1^2 - c u_1^3 = 0$$
and
$$P_2(u_2, 0) = c + (b - \bar{c})\, u_2 + (a - \bar{b})\, u_2^2 - \bar{a} u_2^3 = 0.$$
From these expressions the lemma follows immediately.

Note that $u_1 \neq 0$ and $u_2 = u_1^{-1}$ represent the same point at infinity. □

Remark. — By a rotation of coordinates we can always carry one of the pairs of symmetric singularities at infinity to the pair of points $p = (0, 1, 0)$ and $-p = (0, -1, 0)$. Hence we can always suppose that the origin in the (u_2, v_2)-plane is a singularity of X_2 or, in other words, we can always suppose $c = 0$.

The following corollary is now clear.

Corollary **1.2.** — *Let* $X \in \chi^2$ *be given by* (1) *with* $c = 0$. *Then:*

I) X *has one pair of symmetric singularities at infinity if and only if* $[(\bar{b} - a)^2 - 4\bar{a}(\bar{c} - b) < 0]$ *or* $[\bar{b} - a = \bar{c} - b = 0$ and $\bar{a} \neq 0]$;

II) X *has two pairs of symmetric singularities at infinity if and only if*
$$[(\bar{b} - a)^2 - 4\bar{a}(\bar{c} - b) = 0 \textit{ and } \bar{c} - b \neq 0]$$
or
$$[\bar{c} - b = 0 \textit{ and } \bar{b} - a \neq 0];$$

III) X *has three pairs of symmetric singularities at infinity if and only if* $(\bar{b} - a)^2 - 4\bar{a}(\bar{c} - b) > 0$ *and* $\bar{c} - b = 0$;

IV) *the whole infinity is filled with singularities of* X *if and only if* $\bar{c} - b = \bar{b} - a = \bar{a} = 0$.

The proof of Theorem B in the next chapter will consider separately each one of the cases arising from this corollary.

1.3. *General properties of quadratic vector field in* $\mathbf{R}^2$

Among the properties of planar quadratic vector fields there is a simple but basic one that we will use throughout the paper. It will be refered to as the *periodic orbit property* and says the following:

If a planar quadratic vector field has a periodic orbit Y, then in the compact region bounded by Y there is a unique singularity. Moreover, the linear part of the vector field at this singularity has conjugate complex eigenvalues.

This property can be used to find simple expressions for the vector field. In fact, if a quadratic vector field X has a periodic orbit, it is clear that there exists an afine change of coordinates such that

$$X: \begin{cases} \dot{x} = mx - y + P_2(x,y) \\ \dot{y} = x + my + Q_2(x,y), \end{cases}$$

where P_2 and Q_2 are homogeneous polynomial vector fields of degree two. In some cases we will use the fact that this form of X is invariant under rotations.

Let us recall another basic property of planar quadratic vector fields.

Contact property. — Let $X \in \chi^2$ be a quadratic vector field in $\mathbf{R}^2$. Then every straight line ℓ in $\mathbf{R}^2$ is either invariant or has at most two *contacts* with X (i.e. points where ℓ is not transversal to X).

This property enables us to find geometric properties for bounded or unbounded graphs of quadratic systems. For example we can prove:

(i) all graphs with return map enclose a convex region;

(ii) a graph with return map and with at least two singularities must contain the straight line segment joining two adjacent singularities. In this case the quadratic vector field has an invariant line.

With these properties we obtain simple expressions for the vector field. In fact, if a quadratic vector field X has an invariant line, then there exists an afine change of coordinates such that

$$X: \begin{cases} \dot{x} = x(m + ax + by) \\ \dot{y} = Q(x,y) \end{cases}$$

and if X has two transversal invariant lines then

$$X: \begin{cases} \dot{x} = x(m + ax + by) \\ \dot{y} = y(\bar{n} + \bar{b}x + \bar{c}y). \end{cases}$$

Whenever necessary we will use these forms of X.

Let us recall two more properties of quadratic systems.

Invariant line property. — If a quadratic system has an invariant line then it has at most one limit cycle.

Two invariant lines property. — The quadratic system

$$\begin{cases} \dot{x} = x(m + ax + by) \\ \dot{y} = y(\bar{n} + \bar{b}x + \bar{c}y) \end{cases}$$

has no limit cycle.

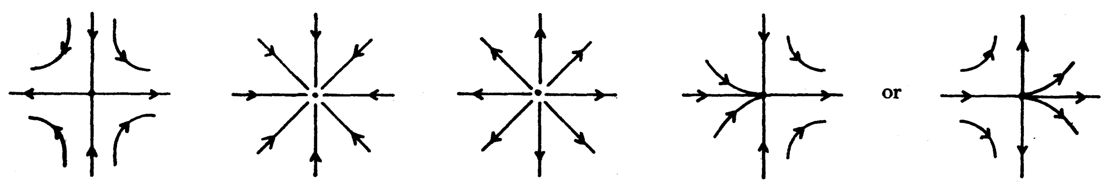
Although the invariant line property may substantially simplify the proofs, we are not going to use it, because we do not know of a good reference for its proof. However, we shall make use of the two invariant lines property, first proved by Bautin (see [C]).

1.4. *Singularities and periodic orbits*

Let p be a singular point of a vector field X in $\mathbf{R}^2$ (i.e. $X(p) = 0$) and let λ_1, λ_2 be the eigenvalues of $DX(p)$. We say that p is *hyperbolic* if $\operatorname{Re}\lambda_i \neq 0$, $i = 1, 2$, that it is *semi-hyperbolic* if $\lambda_1 \lambda_2 = 0$ but $\lambda_1 + \lambda_2 \neq 0$ and that it is a *degenerate singularity* if $\lambda_1 = \lambda_2 = 0$. Moreover, we say that p is a *center type* singularity if λ_1 and λ_2 are complex conjugate with zero real part.

For the moment we are interested in the description of the flow of X in a neighborhood of p (the *topological type* of p).

First we define some types of singularities by their geometrical features:



saddle attractor repellor saddle-node

If the orbits of an attractor (repellor) spiral around the singularity we speak of a *focus*; if not we speak of a *node*.

Let us now relate hyperbolicity with the above topological types.

If p is a hyperbolic singularity of X and λ_1, λ_2 are the eigenvalues of $DX(p)$, then p is a saddle if $\lambda_1 \lambda_2 < 0$, an attractor if $\operatorname{Re}\lambda_i < 0$, $i = 1, 2$, and a repellor if $\operatorname{Re}\lambda_i > 0$, $i = 1, 2$. Moreover, p is a focus if and only if λ_1 and λ_2 are complex conjugate numbers. These are simple basic facts about dynamical systems and can be seen for example in [P-M].

If p is a semi-hyperbolic singularity of X there are two local invariant differentiable curves intersecting transversally at p, such that the behavior of X along these curves determines the topological type of p. The tangent lines to these curves at p are the lines generated by the eigenvectors of $DX(p)$. The invariant curve whose tangent line at p is generated by the eigenvector associated with the vanishing eigenvalue is called the *center manifold* of p. The flow of X on the center manifold of p is given by the first nonzero derivative $f^{(k)}(0)$ of an associate one-dimensional differential equation $\dot{x} = f(x)$ for which $f(0) = f'(0) = 0$. The flow on the other invariant curve is determined by the sign of λ, the nonzero eigenvalue. With this, the topological type of p is a saddle-node if k is even, a saddle if k is odd and $\lambda . f^{(k)}(0) < 0$ and a node if k is odd and $\lambda . f^{(k)}(0) > 0$ (attractor if $\lambda < 0$ and repellor if $\lambda > 0$).

The center manifold theory needed here can be found in [H-P-S] or [Ca].

The topological type in the degenerate case can be obtained by means of the *blowing-up method.* This method consists in " opening " (blowing-up) the singularity into a circle using for example the map

$$\varphi : \mathbf{R} \times S^1 \to \mathbf{R}^2$$
$$(r, \theta) \mapsto (r \cos \theta, r \sin \theta).$$

(We suppose that the vector field X is defined in an open set of $\mathbf{R}^2$ and has the degenerate singularity p at the origin.) An important fact is that there exists a vector field $\overline{X}$ in $\mathbf{R} \times S^1$ (the *blow-up* of X) verifying

$$D\varphi_{(r,\theta)}(\overline{X}(r, \theta)) = X(\varphi(r, \theta)) \quad \text{for } (r, \theta) \in \overline{U}$$

and leaving $\{0\} \times S^1$ invariant. The set $\overline{U}$ is open in $\mathbf{R} \times S^1$ and contains $\{0\} \times S^1$. If we know the flow of $\overline{X}$ in a neighborhood of $\{0\} \times S^1$ (for example if all the singularities of $\overline{X}$ in $\{0\} \times S^1$ are hyperbolic or semi-hyperbolic), then " blowing down " $\overline{X}$ we obtain the topological type of p. If $\overline{X}$ has degenerate singularities along $\{0\} \times S^1$, we blow-up again each one of these singularities and observe if we can determine the corresponding flows. If not, we blow-up again and again. Fortunately this process ends; in fact (X being analytic) we know that after a finite number of blowings-up we only get hyperbolic and semi-hyperbolic singularities. This allows us to describe the topological type of p.

In sections 2.1.1, 2.2 (I.2) and 2.2 (II.2), we will give the topological types of all degenerate singularities which will be needed.

The blowing-up method can be seen in detail in [A], [Du] and [T].

Finally, we recall that the topological type of a center type singularity p of an analytic vector field is either a focus or a *center* (all orbits in a neighborhood of p are periodic).

A periodic orbit γ is called an *attractor (repellor)* if it is the ω-limit set (α-limit set) of all points in a neighborhood of γ.

Let X be a vector field in $\mathbf{R}^2$ and let γ be a periodic orbit of X of period T. The number

$$c = \int_0^T \operatorname{div} X(\gamma(t))\, dt$$

is called the *characteristic exponent* of γ.

It is a well known fact (see [A], [S]) that for $c > 0$ the orbit γ is a repellor and for $c < 0$ it is an attractor.

1.5. *Il'yašenko's Theorem* ([I])

Definition. — A graph Γ of a vector field X in $\mathbf{R}^2$ is called a *hyperbolic graph* of X if all singularities of X contained in Γ are hyperbolic.

Theorem (Il'yašenko). — *Every hyperbolic graph of an analytic vector field in* $\mathbf{R}^2$ *is finite (i.e. not accumulated by limit cycles).*

This theorem is crucial for our result because it allows us to consider only the non-hyperbolic graphs.

1.6. *Dulac's Proposition*

Definition. — A semi-hyperbolic singularity p of a vector field X in $\mathbf{R}^2$ is called *contractive* if $\operatorname{div} X(p) < 0$ and *expansive* if $\operatorname{div} X(p) > 0$. Notice that in this case $\operatorname{div} X(p)$ is the nonzero eigenvalue of $DX(p)$.

Proposition (Dulac [D]). — *Every graph of an analytic vector field in* $\mathbf{R}^2$ *which contains only hyperbolic or contractive (expansive) semi-hyperbolic singularities is finite.*

The proof of this fact is straightforward.

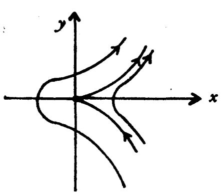
2. Proof of Theorem B

2.1. We will first prove that all bounded graphs of quadratic vector fields are finite. To do this we observe that a quadratic vector field has at most four singularities in the plane. Since we are interested in periodic orbits we may suppose that one of the singularities is a focus or a center (see 1.3 and 1.4). Hence bounded graphs contain one, two or three singularities (this is true for every quadratic vector field; see Berlinskii's Theorem in [C]).

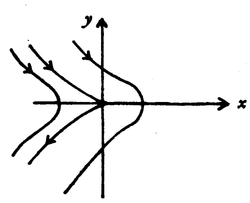
2.1.1. Let us first consider graphs with one singularity. If the singularity is either hyperbolic or semi-hyperbolic the graph is finite. This follows from Il'yašenko's Theorem and Dulac's Proposition, respectively. Suppose now that the singularity is at the origin and that both eigenvalues are zero. If the linear part of the vector field at (0, 0) is identically zero then the vector field is homogeneous and there is no limit cycle. We may then suppose that after a linear change of coordinates the vector field has the form

$$\begin{cases} \dot{x} = y + ax^2 + bxy + cy^2 \\ \dot{y} = \bar{a}x^2 + \bar{b}xy + \bar{c}y^2. \end{cases}$$

Following the blowing-up method we observe that if $\bar{a} \neq 0$ the topological type of the origin is



$\bar{a} > 0$



$\bar{a} < 0$

Since the line $y = 0$ is transversal except in (0, 0) this singularity does not belong to any graph. Thus, necessarily $\bar{a} = 0$.

In this case ($\bar{a} = 0$) the line $y = 0$ is invariant and the existence of a periodic orbit implies both $\bar{b} \neq 0$ and the existence of another singularity which must be a focus or a center. Changing coordinates by $(x,y) \mapsto (\bar{b}x + \bar{c}y, \lambda y)$ with an appropriate λ, we obtain the following form for X:

$$X : \begin{cases} \dot{x} = ny + ax^2 + bxy - ny^2 \\ \dot{y} = xy \end{cases}$$

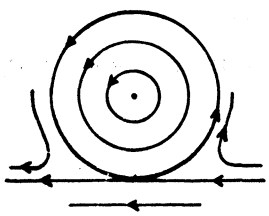
with (0, 0) and (0, 1) as singularities. For (0, 1) to be a focus or a center it is necessary that $b^2 < 4n$. We also need $a \neq 0$.

Suppose $b = 0$. Then the vector field verifies $A_* X = -X$ for $A(x,y) = (-y, x)$. It follows that (0, 1) is a center and that there is no limit cycle.

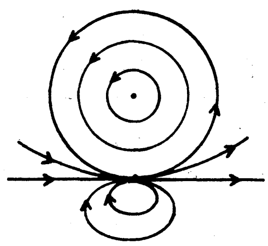
Now denote by X_b the vector field

$$X_b : \begin{cases} \dot{x} = ny + ax^2 + bxy - ny^2 \\ \dot{y} = xy. \end{cases}$$

Then, $X_b = X_0 + b \begin{pmatrix} xy \\ 0 \end{pmatrix}$ and $\det(X_0, X_b) = -bx^2y^2$. It follows that the orbits of X_b are topologically transverse to the ones of X_0 and since X_0 has a center and is symmetric with respect to the y-axis we see that X_b, $b \neq 0$, does not have any periodic orbit. Moreover, calculating the topological type of (0, 0) (by the blowing-up method) we conclude that X_0 has the following bounded graphs according to the values of the coefficient a.



$a < 0$



$0 < a < 1/2$

Also, the vector fields X_0 for $a \geqslant 1/2$ and X_b for $b \neq 0$ do not have any bounded graph.

This settles the case of bounded graphs with one singularity.

2.1.2. Let us now consider bounded graphs with two singularities. Since we are interested in graphs with return map we may suppose (by the contact property) that both singularities are points of an invariant line for the vector field. By a linear change of coordinates we can carry these points to (0, 0) and (0, 1). The vector field then takes the form

$$X : \begin{cases} \dot{x} = x(m + ax + by) \\ \dot{y} = \bar{c}y(y - 1) + \bar{m}x + \bar{a}x^2 + \bar{b}xy \qquad \bar{c} \neq 0. \end{cases}$$

The eigenvalues of the linear part of X are $\{m, -\bar{c}\}$ at (0, 0) and $\{m + b, \bar{c}\}$ at (0, 1). Since $\bar{c} \neq 0$ each singularity is either hyperbolic or semi-hyperbolic. If at least one of them is hyperbolic then the graph is finite (by Il'yašenko's Theorem or by Dulac's Proposition). If both singularities are semi-hyperbolic then $m = b = 0$. But in this case $\dot{x} = ax^2$ and there is no bounded graph.

2.1.3. Finally, if there is a graph with three singularities and with return map then, by the contact property, there are three invariant lines and by the two invariant lines property we know that there is no limit cycle.

This conclude the proof for bounded graphs.

2.2. Now we prove that all unbounded graphs are finite. We proceed by considering separately each one of the relations in Corollary 1.2.

2.2 (I.1) $(\bar{b} - a)^2 - 4\bar{a}(\bar{c} - b) < 0$

Let $X \in \chi^2$ be given by (1) with $c = 0$ and verify the relation above. In this case $p = (0, 1, 0)$ and $-p$ are the unique singularities at infinity. Since X is expressed in coordinates (u_2, v_2) (see 1.2) as

$$X_2 : \begin{cases} \dot{u}_2 = (b - \bar{c})\, u_2 + nv_2 + \dots \\ \dot{v}_2 = \qquad\qquad -\bar{c}v_2 + \dots \end{cases}$$

the point p is hyperbolic for X restricted to infinity (u_2-axis) and it is hyperbolic if and only if $\bar{c} \neq 0$.

Lemma **2.1.** — *If* X *has an unbounded graph* Γ, *then:*

(i) *p and $-p$ are saddles (and so $b \neq 0$) and they belong to Γ. An arc at infinity joining p and $-p$ must be contained in Γ.*

(ii) *The line $\ell : x = -n/b$ is invariant and contained in Γ.*

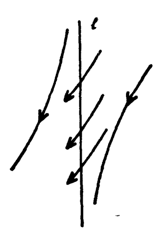
(iii) *There exist coordinates in which we can write*

$$X : \begin{cases} \dot{x} = xy \\ \dot{y} = Q(x,y) \end{cases} \qquad \text{with } \bar{b}^2 - 4\bar{a}(\bar{c} - 1) < 0.$$

Proof. — Part (i) is clear. To prove (ii) we notice that

$$\langle X(-n/b, y), (1, 0)\rangle = \alpha - mn/b + an^2/b^2$$

proving that ℓ is invariant or transversal to the flow. If it is transversal, the separatrices of the saddles p and $-p$ must be on different sides of ℓ (see the figure below). Therefore, there is no unbounded graph



We now prove (iii). By translating coordinates we carry ℓ to the line $x = 0$. The vector field is now given by

$$\begin{cases} \dot{x} = x(m + ax + by) \\ \dot{y} = Q(x,y). \end{cases}$$

Since $b \neq 0$ (p is a saddle), we obtain the desired form for X by changing coordinates: $(x,y) \mapsto (x, m + ax + by)$ □

Let $X \in \chi^2$ be given by

$$X: \begin{cases} \dot{x} = xy \\ \dot{y} = Q(x,y) \end{cases} \qquad \bar{b}^2 - 4\bar{a}(\bar{c} - 1) < 0.$$

Since in this case

$$X_2: \begin{cases} \dot{u}_2 = u_2[1 - \bar{c} - \bar{b}u_2 - \bar{n}v_2 - \bar{a}u_2^2 - \bar{m}u_2 v_2 - \bar{\alpha}v_2^2] \\ \dot{v}_2 = Q_2(u_2, v_2) = - \bar{c}v_2 + \dots \end{cases}$$

we conclude that the v_2-axis is invariant and that the flow along this line is given by

$$\dot{v}_2 = Q_2(0, v_2) = - \bar{c}v_2 - \bar{n}v_2^2 - \bar{\alpha}v_2^3.$$

Thus, the origin $(u_2, v_2) = (0, 0)$ (and hence p) is a saddle if and only if $(1 - \bar{c})\,\bar{c} > 0$ (hyperbolic case) or $\bar{c} = \bar{n} = 0$ and $\bar{\alpha} > 0$ (non-hyperbolic case).

For $X \in \chi^2$ as above we will prove the following scheme:

$$\bar{c}(1 - \bar{c}) > 0 \begin{cases} \bar{n}^2 - 4\bar{\alpha}\bar{c} > 0 : \text{there is no unbounded graph} \\ \bar{n}^2 - 4\bar{\alpha}\bar{c} = 0 \begin{cases} \bar{n} \neq 0 : \text{graph as in Fig. 1} \\ \bar{n} = 0 \begin{cases} \bar{b} \neq 0 : \text{graph as in Fig. 2} \\ \bar{b} = 0 : \text{graph as in Fig. 3} \end{cases} \end{cases} \\ \bar{n}^2 - 4\bar{\alpha}\bar{c} < 0 : \text{graph as in Fig. 4} \end{cases}$$

$$\bar{c} = \bar{n} = 0 \text{ and } \bar{\alpha} > 0 \begin{cases} \bar{b} \neq 0 : \text{graph as in Fig. 5} \\ \bar{b} = 0 : \text{graph as in Fig. 6} \end{cases}$$

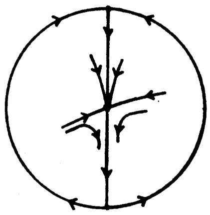


Fig. 1. — The graph does not have a return map and so it is finite

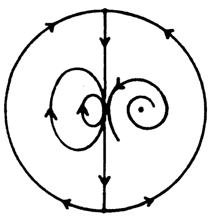


Fig. 2. — Calculating the characteristic exponent it follows that three is no periodic orbit

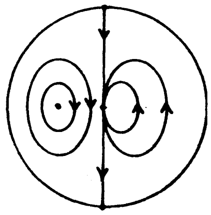


FIG. 3. — There is symmetry, therefore the graph is finite

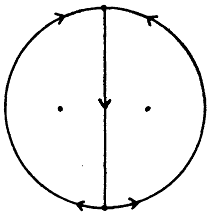


FIG. 4. — By Il'yašenko's theorem, the graph is finite

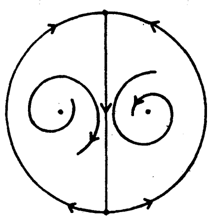


FIG. 5. — Same as in Fig. 2

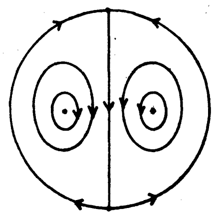


FIG. 6. — Same as in Fig. 3

Let us consider the case $\bar{c}(1 - c) > 0$. We first look at the singularities of X on the invariant line $\ell : x = 0$. These singularities are given by the roots of

$$Q(0,y) = \bar{\alpha} + \bar{n}y + \bar{c}y^2 = 0.$$

If $\bar{n}^2 - 4\bar{a}\bar{c} > 0$, there are two singularities on ℓ which are hyperbolic for X restricted to ℓ. In this case there is no unbounded graph. For $\bar{n}^2 - 4\bar{a}\bar{c} = 0$, there is a unique singularity on ℓ, namely $p_0 = (0, -\bar{n}/2\bar{c})$, which has eigenvalues $-\bar{n}/2\bar{c}$ and 0. Thus, for $\bar{n} \neq 0$, p_0 is a saddle-node and we obtain the graph of Figure 1.

For $\bar{n} = 0$, we necessarily have $\bar{\alpha} = 0$. Notice that if there is a periodic orbit $\gamma = (\gamma_1, \gamma_2)$, it must be contained in $\{x > 0\}$ or in $\{x < 0\}$. Also, from the expression for X, we have $\gamma_2 = \dot{\gamma}_1/\gamma_1$. Calculating the characteristic exponent of a periodic orbit of period T we obtain:

$$\begin{aligned}\int_0^T \operatorname{div} X(\gamma(t))\,dt &= \int_0^T \bar{b}\gamma_1(t)\,dt + \int_0^T (2\bar{c}+1)\,\gamma_2(t)\,dt\\ &= \bar{b}\int_0^T \gamma_1(t)\,dt + (2\bar{c}+1)\int_0^T \frac{\dot{\gamma}_1(t)}{\gamma_1(t)}\,dt\\ &= \bar{b}\int_0^T \gamma_1(t)\,dt.\end{aligned}$$

So, if $\bar{b} \neq 0$, all possible periodic orbits in the same half-plane as well as the singularities must be of the same type: all repellors or all attractors. Since this is impossible, there is no periodic orbit at all. Finally, for $\bar{b} = 0$, X has the form

$$X = \begin{cases} \dot{x} = xy \\ \dot{y} = \bar{m}x + \bar{a}x^2 + \bar{c}y^2 \end{cases}$$

and we easily show that $A_* X = -X$ for $A(x,y) = (x, -y)$. The flow of X is then given in Figure 3.

When $\bar{n}^2 - 4\bar{a}\bar{c} < 0$ there are no singularities on ℓ and we get the graph of Figure 4 with hyperbolic singularities. By Il'yašenko's Theorem we know that these graphs are finite.

Let us now consider the case $\bar{c} = \bar{n} = 0$ and $\bar{\alpha} > 0$. In this case there are no singularities on the invariant line $\ell : x = 0$, and there are two singularities lying on different sides of ℓ. Both of them are center-type singularities. If there is a periodic orbit $\gamma = (\gamma_1, \gamma_2)$ of period T we calculate its characteristic exponent and obtain the number $\bar{b}\int_0^T \gamma_1(t)\,dt$. As before, we see that there is no periodic orbit if $\bar{b} \neq 0$. For $\bar{b} = 0$ the vector field verifies $A_* X = -X$ for $A : (x,y) \to (x, -y)$ and so we obtain the graph of Figure 6. This proves Theorem B in the case (I.1).

2.2 (I.2) $\bar{b} - a = \bar{c} - b = 0$ *and* $\bar{a} \neq 0$.

Let $X \in \chi^2$ be given by (1) with $c = 0$ and verify the relations above. In this case $p = (0, 1, 0)$ and $-p$ are the unique singularities at infinity. Let X_2 be the expression of X in coordinates (u_2, v_2) (see 1.2). Since

$$X_2 : \begin{cases} \dot{u}_2 = nv_2 + \dots \\ \dot{v}_2 = -bv_2 + \dots, \end{cases}$$

the point p is not hyperbolic for X restricted to infinity (u_2-axis) and it is semi-hyperbolic if and only if $b \neq 0$.

Lemma **2.2.** — *Let* $b \neq 0$. *If* X *has an unbounded graph* Γ, *then*:

(i) p *and* $-p$ *are saddles and they belong to* Γ. *An arc at infinity joining* p *and* $-p$ *is contained in* Γ.

(ii) *The line* $\ell : x = -n/b$ *is invariant and contained in* Γ.

(iii) *There are coordinates in which*

$$X: \begin{cases} \dot{x} = xy \\ \dot{y} = \bar{\alpha} + \bar{m}x + \bar{n}y + \bar{a}x^2 + y^2. \end{cases}$$

Proof. — Since p is semi-hyperbolic and is a node along the center manifold (the infinite line), it is either a node or a saddle. If it is a node, no unbounded graph is possible.

The rest of the proof goes as for Lemma 2.1. □

Lemma **2.3.** — *Let* $b = 0$. *If* X *has a periodic orbit then there are coordinates in which*

$$X: \begin{cases} \dot{x} = mx - y + ax^2 \\ \dot{y} = x + my + \bar{a}x^2 + ayx \end{cases} \qquad \bar{a} \neq 0.$$

Proof. — We first note that the conditions $\bar{b} - a = 0$, $b = \bar{c} = 0$ and $\bar{a} \neq 0$ are invariant under affine change of coordinates that keep $p = (0, 1, 0)$ fixed. The lemma then follows by the periodic orbit property. □

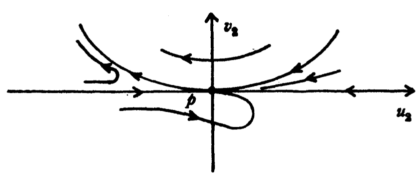
Notice that we are only interested in quadratic vector fields which have periodic orbits. Therefore, we will frequently use the coordinates given by Lemma 2.3.

If $a \neq 0$, using $(x, y) \mapsto (\varepsilon_1 x, \varepsilon_2 y)$, $\varepsilon_i = \pm 1$, $i = 1, 2$ if necessary, we can suppose $\bar{a} > 0$ and $a > 0$.

Lemma **2.4.** — *Let* $X \in \chi^2$ *be given by*

$$\begin{cases} \dot{x} = mx - y + ax^2 & \bar{a} > 0 \\ \dot{y} = x + my + \bar{a}x^2 + axy & a > 0. \end{cases}$$

Then the singularity $p = (0, 1, 0)$ *at infinity has the following topological type*:



Proof. — By blowing-up the singularity at the origin for the vector field

$$X_2: \begin{cases} \dot{u}_2 = -v_2 - \bar{a}u_2^3 - u_2^2 v_2 \\ \dot{v}_2 = -au_2 v_2 - mv_2^2 - \bar{a}u_2^2 v_2 - u_2 v_2^2, \end{cases}$$

we recognize the above topological type. □

We will now prove the following scheme:

$$
b \neq 0 \begin{cases} \bar{a}b > 0 : \text{ there is no unbounded graph} \\ \bar{a}b < 0 \text{ (coordinates as in Lemma 2.2)} \begin{cases} \bar{n} \neq 0 : \text{graph as in Fig. 7} \\ \bar{n} = 0 : \text{graph as in Fig. 8} \end{cases} \end{cases}
$$

$$
b = 0 \text{ (coordinates as in Lemma 2.3)} \begin{cases} a = 0 \begin{cases} m = 0 : \text{X is Hamiltonian} \\ m \neq 0 : \text{since } \operatorname{div} X(x,y) \equiv 2m, \text{ there is no periodic orbit} \end{cases} \\ a > 0 \begin{cases} 2am - \bar{a} = 0 : \text{graph as in Fig. 9} \\ 2am - \bar{a} \neq 0 : \text{there is no unbounded graph.} \end{cases} \end{cases}
$$

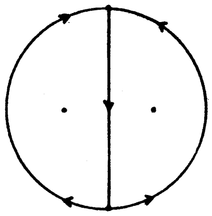


FIG. 7. — The characteristic exponent is nonzero, therefore there is no periodic orbit

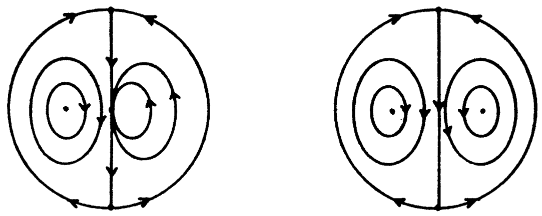


FIG. 8. — There is symmetry, hence graphs are finite

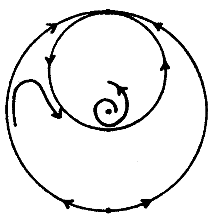


FIG. 9. — There is a Liapunov function inside the graph. Thus, there is no periodic orbit

Let us consider the different cases:

When $b \neq 0$ and $\bar{a}b > 0$, p is a node and therefore there is no unbounded graph. For $b \neq 0$ and $\bar{a}b < 0$, p is a saddle. Let us consider coordinates as in Lemma 2.2. If there exists a periodic orbit of period T, we calculate its characteristic exponent obtaining $\bar{n}T$. Thus, as before, there is no periodic orbit if $\bar{n} \neq 0$. When $\bar{n} = 0$, the vector field is symmetric with respect to $(x,y) \mapsto (x, -y)$, and there are no limit cycles.

We now suppose $b = 0$. Take coordinates as in Lemma 2.3 with $\bar{a} > 0$ and $a > 0$. Let $y = y(x)$ be the parabola

$$y = y(x) = (a/2)\, x^2 - mx - (1 + m^2)/2a$$

and let $\alpha_N(x)$ be its normal vector $\alpha_N(x) = (ax - m, -1)$.

Easy calculations show that

$$\langle X(x, y(x)), \alpha_N(x) \rangle = (2am - \bar{a})\, x^2.$$

Fix $\bar{a} > 0$, $a > 0$ and consider m as a parameter. Let m_0 be given by $2am_0 - \bar{a} = 0$ and let $y = y_0(x)$ be the parabola

$$y = y_0(x) = (a/2)\, x^2 - m_0\, x - (1 + m_0^2)/2a.$$

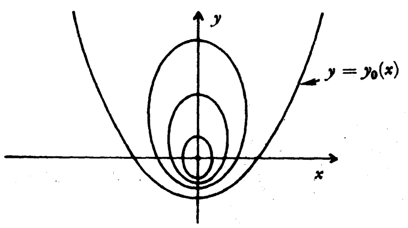
For $m = m_0$ the parabola $y = y_0(x)$ is invariant under the flow of X. Let

$$b(x) = -m_0\, x - (1 + m_0^2)/a.$$

By straightforward calculation we obtain:

(i) $b(x) < y_0(x)$ for all $x \in \mathbf{R}$

(ii) The function $f(x,y) = (y - y_0(x))/(y - b(x))^2$ has the origin as a maximum and in the region $\Omega = \{(x,y) / y > y_0(x)\}$ this is the only critical point. That is, f has the following level curves in Ω:



(iii) $Xf(x,y) = \left(\dfrac{\partial f}{\partial x} P + \dfrac{\partial f}{\partial x} Q\right)(x,y) = -2\bar{a}\, \dfrac{y - y_0(x)}{(y - b(x))^3}\, x^2 < 0$ for each $(x,y) \in \Omega$ and $x \neq 0$.

In this way, if $2am - \bar{a} = 0$, the origin is a repellor and there are no periodic orbits. For $m \neq m_0$ (i.e. $2am - \bar{a} \neq 0$), the parabola $y = y_0(x)$ is transversal to X. Since $X = X_m = X_{m_0} + (m - m_0)\, R$ where X_m is the vector field in Lemma 2.3 and $R(x,y) = (x,y)$ is the radial vector field, we observe that the separatrices at p move to different sides of $y = y_0(x)$ when m changes. Thus there is no unbounded graph when $m \neq m_0$. The proof of Theorem B in case (I) is complete.

2.2 (II.1) *$(\bar{b} - a)^2 - 4\bar{a}(\bar{c} - b) = 0$ and $\bar{c} - b \neq 0$.*

In this case there are two pairs of singularities at infinity. The one different from $\{p = (0, 1, 0), -p\}$ is a pair of saddle-nodes for X restricted to infinity. This is clear from the equation of X restricted to infinity:

$$\dot{u}_1 = \bar{a} + (\bar{b} - a)\, u_1 + (\bar{c} - b)\, u_1^2.$$

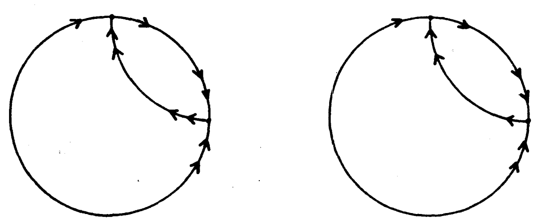
By rotating coordinates we can carry this pair of singularities to $\{p, -p\}$, leading us to the next case:

2.2 (II.2) $b - \bar{c} = 0$ *and* $\bar{b} - a \neq 0$.

Besides $\{p, -p\}$ there is another pair of singularities at infinity. When we restrict X to infinity, p and $-p$ are non-hyperbolic while the other pair is hyperbolic.

As before, let X_2 be the expression of X in coordinates (u_2, v_2). The linear part of X_2 at the origin is $\begin{pmatrix} 0 & n \\ 0 & -b \end{pmatrix}$. So p is semi-hyperbolic if and only if $b \neq 0$. If $b \neq 0$ and X has an unbounded graph Γ, then Γ must contain two adjacent singularities at infinity and the corresponding arc between them.

Also, for $b \neq 0$, if X has an unbounded graph without singularities in the plane, it must be of one of the two following types:



By Dulac's Proposition these graphs are finite.

Now, if $b \neq 0$ and X has an unbounded graph which contains singularities in the plane and which has a return map, then by the contact property it is proved that the separatrices of the graph are contained in invariant lines. By changing coordinates we put these invariant lines in the axes and so the vector field takes the form

$$\begin{cases} \dot{x} = x(m + ax + by) \\ \dot{y} = y(\bar{n} + \bar{b}x + by). \end{cases}$$

By the two invariant lines property this graph is finite.

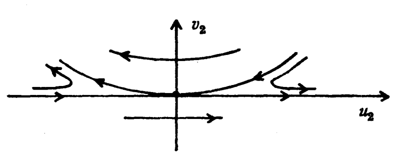
Suppose now $b = 0$.

Lemma **2.5.** — *If* X *has a periodic orbit then there exist coordinates in which*

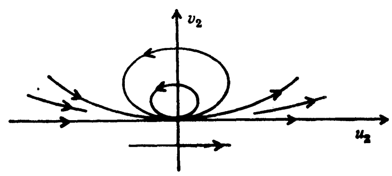
$$X : \begin{cases} \dot{x} = mx - y + ax^2 & a - \bar{b} > 0 \\ \dot{y} = x + my + \bar{a}x^2 + \bar{b}xy & \bar{a} \geq 0. \end{cases}$$

Proof. — The same as in Lemma 2.3 and the remark following it. □

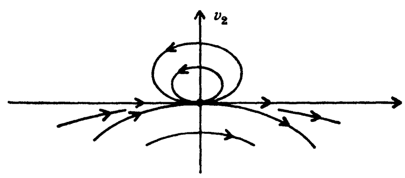
Lemma **2.6.** — *Let* X *be given as in the lemma above. Then the topological type of the singularity* $p = (0, 1, 0)$ *at infinity is one of the following*:

(i) 

if $0 < \bar{b} < a$;

(ii) 

if $\bar{b} \leqslant 2a$; $\bar{b} < 0$;

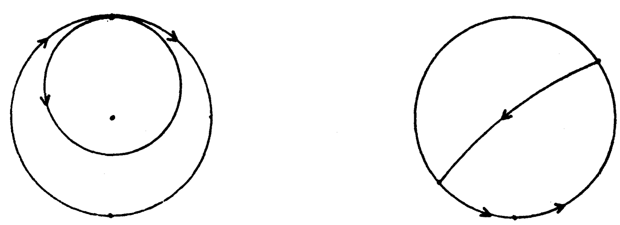
(iii) 

if $2a < \bar{b} < a < 0$;

(iv) if $\bar{b} = 0$ then the topological types are as in (i), (ii) or as in the following figures

Proof. — As for Lemma 2.4. □

From the possible topological types for p, we obtain that unbounded graphs may have 1 or 3 singularities at infinity. For example the following graphs can exist:



Unbounded graphs with two singularities at infinity cannot exist because graphs with a return map must enclose a convex region.

Suppose that $X \in \chi^2$ as in Lemma 2.5 has a graph with a return map and with three singularities at infinity. Then, one of them is p (or $-p$) and the others are the adjacent ones which are themselves symmetric. This pair of symmetric singularities are in the direction $y/x = \bar{a}/(a - \bar{b})$. Since they are contained in a graph they must be saddles and, by the contact property, the separatrix in the plane must be an invariant straight line ℓ of the form

$$y = y(x) = \frac{\bar{a}}{a - \bar{b}}\, x + \mathrm{N}.$$

From the equation

$$\langle \mathrm{X}(x, y(x)), (\bar{a}, \bar{b} - a) \rangle = \bar{a}(a - \bar{b})\, x^2 + \left(\bar{b}\mathrm{N}(\bar{b} - a) - \frac{\bar{a}^2}{a - \bar{b}} + \bar{b} - a\right) x$$
$$- \bar{a}\mathrm{N} + m\mathrm{N}(\bar{b} - a) \equiv 0,$$

where $(\bar{a}, \bar{b} - a)$ is a vector normal to ℓ, it follows that $\bar{a} = m = 0$ and $\mathrm{N} = -1/\bar{b}$.

Thus, X can be expressed in the form

$$X : \begin{cases} \dot{x} = -y + ax^2 \\ \dot{y} = x + \bar{b}xy. \end{cases}$$

Since this vector field has the symmetry $A_* X = -X$ for $A(x,y) = (-x,y)$, we see that the origin is a center, and so there is no limit cycle.

We will now prove the assertions in the following scheme for the case $b = 0$.

$$0 < \bar{b} < a \begin{cases} 3am - \bar{b}m - \bar{a} = 0 \begin{cases} \bar{a} = 0 : \text{graph as in Fig. 10} \\ \bar{a} > 0 : \text{graph as in Fig. 11} \end{cases} \\ 3am - \bar{b}m - \bar{a} \neq 0 : \text{there is no unbounded graph} \end{cases}$$

$$\bar{b} \leqslant 2a,\ \bar{b} < 0 : \text{there is no unbounded graph with return map}$$

$$2a < \bar{b} < a < 0 \begin{cases} 3am - \bar{b}m - \bar{a} = 0 \begin{cases} \bar{a} = 0 : \text{graph as in Fig. 12} \\ \bar{a} > 0 : \text{graph as in Fig. 13} \end{cases} \\ 3am - \bar{b}m - \bar{a} \neq 0 : \text{there is no unbounded graph} \end{cases}$$

$$\bar{b} = 0 \begin{cases} 3am - \bar{a} = 0 \begin{cases} \bar{a} = 0 : \text{graph as in Fig. 14} \\ \bar{a} > 0 : \text{there is no unbounded graph} \end{cases} \\ 3am - \bar{a} \neq 0 : \text{there is no unbounded graph.} \end{cases}$$

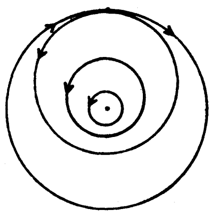


FIG. 10. — There is a first integral inside the graph, therefore it is finite

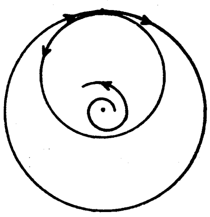


FIG. 11. — There is a Liapunov function inside the graph. Thus, there are no limit cycles

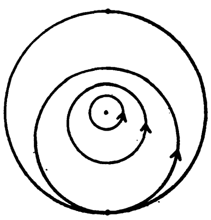


FIG. 12. — Same as in Fig. 10

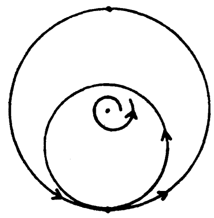


Fig. 13. — Same as in Fig. 11

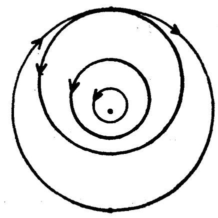


Fig. 14. — The graph is symmetric, therefore it is finite

To prove the above assertions, consider the parabola

$$y = y(x) = \frac{2a - \bar{b}}{2} x^2 - mx - \frac{1 + m^2}{2a}$$

and its normal vector $\alpha_N(x) = ((2a - \bar{b})\, x - m, -1)$.

Easy calculations give

$$\langle X(x, y(x)), \alpha_N(x) \rangle = (3am - \bar{b}m - \bar{a})\, x^2.$$

Let m_0 be given by the relation $3am_0 - \bar{b}m_0 - \bar{a} = 0$ and let $y = y_0(x)$ be the parabola

$$y = y_0(x) = \frac{2a - \bar{b}}{2} x^2 - m_0\, x - \frac{1 + m_0^2}{2a}.$$

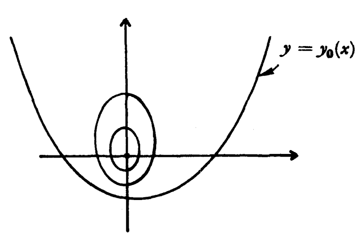
Fix a, $\bar{b}$ and $\bar{a}$ satisfying $0 < \bar{b} < a$ and $\bar{a} \geqslant 0$.

Notice that $m_0 = 0$ for $\bar{a} = 0$. Let us consider m as a parameter.

If $m = m_0$ the parabola $y = y_0(x)$ is invariant and forms an unbounded graph. Take $r = \bar{b}/a$ and let $b(x) = -m_0\, x - (1 + m_0^2)/\bar{b}$. Then:

(i) $b(x) < y_0(x)$ for all $x \in \mathbf{R}$.

(ii) The function $f(x,y) = (y - y_0(x))^r/(y - b(x))^2$ has the origin as a maximum and in the region $\Omega = \{(x,y)/y > y_0(x)\}$ this is the only critical point. The level curves of f in Ω are



(iii) $Xf(x, y) = \left(\frac{\partial f}{\partial x} P + \frac{\partial f}{\partial y} Q\right)(x, y) = -4(2a - \bar{b})\, m_0 \frac{(y - y_0(x))^r}{(y - b(x))^3} x^2 < 0$
for all $(x, y) \in \Omega$ with $x \neq 0$.

Therefore, for $\bar{a} = 0$ the origin is a center (Fig. 10) and if $\bar{a} > 0$ (so that $m_0 \neq 0$) there are no periodic orbits inside the graph (Fig. 11).

For $m \neq m_0$ (i.e. $3am - \bar{b}m - \bar{a} \neq 0$), we have as before that the relation:

$$X = X_m = X_{m_0} + (m - m_0) R$$

is satisfied, and so there is no unbounded graph. This ends the proof in case $0 < \bar{b} < a$.

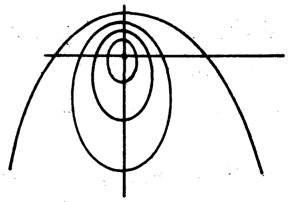
Now fix a, $\bar{b}$ *and* $\bar{a}$ *satisfying* $2a < \bar{b} < a < 0$ *and* $\bar{a} \geqslant 0$.

Recall that if $\bar{a} = 0$ then $m_0 = 0$. Let us again consider m as a parameter.

First let $m = m_0$. In this case the parabola $y = y_0(x)$ is invariant and forms an unbounded graph. Take r and $b(x)$ as before. Then:

(i) $y_0(x) < b(x)$ for all $x \in \mathbf{R}$.

(ii) The function $f(x,y) = (y_0(x) - y)^r/(b(x) - y)^2$ has the origin as a maximum and in the region $\Omega = \{(x,y)/y < y_0(x)\}$ this is the only critical point. The level curves of f in Ω are



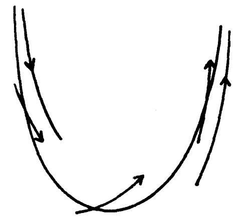
(iii) $Xf(x,y) = 4(2a - \bar{b})\, m_0 \frac{(y_0(x) - y)^r}{(b(x) - y)^2} x^2 < 0$
for all $(x,y) \in \Omega$ with $x \neq 0$.

Therefore, if $\bar{a} = 0$ there is a first integral and the origin is a center (Fig. 12) and if $\bar{a} > 0$ (so that $m_0 \neq 0$) there is a Liapunov function and there is no periodic orbit inside the graph (Fig. 13).

To conclude the case above, we now let $m \neq m_0$ (i.e. $3am - \bar{b}m - \bar{a} \neq 0$). The same arguments as in the previous case prove that there is no unbounded graph.

If $\bar{b} \leqslant 2a$ *and* $\bar{b} < 0$, then from the topological type of p we conclude that no unbounded graph is possible.

Fix a, $\bar{b}$ *and* $\bar{a}$ *satisfying* $\bar{b} = 0 < a$ *and* $\bar{a} > 0$. Consider m as a parameter. If $m = m_0$ the parabola $y = y_0(x)$ is invariant and it is easily shown that there is a saddle on the parabola. Hence there is no unbounded graph. When $m \neq m_0$ the parabola $y = y_0(x)$ is transversal to X. Suppose that the topological type of p is the one in (i) of Lemma 2.6. (This is the only possibility when $\bar{b} = 0$ for the existence of an unbounded graph.) Since the separatrices at infinity bound hyperbolic sectors, the transversal parabola $y = y_0(x)$ must leave the separatrices at different sides. The following picture illustrates the situation:



Thus, there is no unbounded graph.

Finally *fix* a, $\bar{b}$ *and* $\bar{a}$ *satisfying* $\bar{b} = 0 < a$ *and* $\bar{a} = 0$, so that $m_0 = 0$. When $m = m_0 = 0$ the vector field verifies $A_* X = -X$ for $A(x, y) = (-x, y)$, the origin is a center and there are no limit cycles. If $m \neq m_0$, as before there is no unbounded graph.

Thus, all assertions concerning the case $b = 0$ are proved and case (II) is settled.

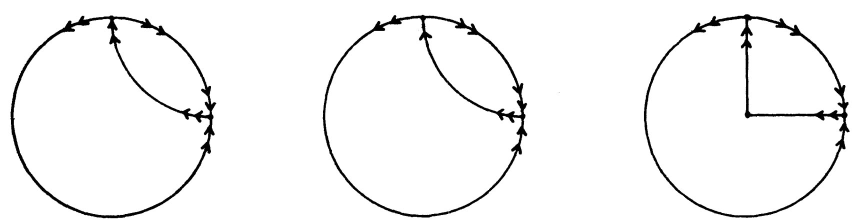
2.2 (III) $(\bar{b} - a)^2 - 4\bar{a}(\bar{c} - b) > 0$ *and* $\bar{c} - b \neq 0$.

We now come to the most difficult part of the proof of our main result.

For $X \in \chi^2$ given by (1) (see Chapter 1) with $c = 0$ and satisfying the relation above, there are three pairs of symmetric singularities at infinity.

Lemma **2.7.** — *If* $X \in \chi^2$ *has three pairs of symmetric singularities at infinity and if two of them are hyperbolic, then* X *has a finite number of limit cycles.*

Proof. — When a quadratic vector field X has three pairs of symmetric singularities at infinity, all of them are hyperbolic for the restriction of X to infinity (see Lemma 1.1). So the only possible unbounded graphs with a return map are of the following types:



(double arrows indicate hyperbolicity)

The first one is finite by Il'yašenko's Theorem. The middle one is not accumulated by periodic orbits (Dulac's Proposition). The last one, with singularities in the plane, must have separatrices contained in invariant lines, and as explained before, in this case there are no limit cycles. □

Lemma **2.8.** — *If* $X \in \chi^2$ *has three pairs of symmetric singularities at infinity and two of them are not hyperbolic, then:*

(i) *the two pairs of non-hyperbolic singularities are semi-hyperbolic;*

(ii) *the third pair of symmetric singularities consists of hyperbolic nodes;*

(iii) *there exist coordinates in which the hyperbolic pair of singularities is*

$$\{r = 1/2\sqrt{2}(-1, 1, 0), -r\}$$

and the semi-hyperbolic pairs of singularities are

$$\{p = (0, 1, 0); -p\} \quad \text{and} \quad \{q = (1, 0, 0), -q\}.$$

In these coordinates the vector field has an expression as in (1) *with* $a = 0$, $c = 0$, $\bar{a} = 0$, $\bar{c} = 0$, $b + \bar{b} = 0$ *and* $b \neq 0$.

Proof. — We first observe that given any order in the pairs of singularities, there are coordinates in $\mathbf{R}^2$ such that the first pair is $\{p, -p\}$, the second pair is $\{q, -q\}$ and the third one is $\{r, -r\}$. In fact, with a rotation of coordinates we carry the first pair to $\{p, -p\}$; with a linear change of coordinates of the form $A(x,y) = (x, \lambda x + y)$ (which fixes p) the second pair is taken to $\{q, -q\}$; and finally with a change of coordinates $(x,y) \mapsto (x, \lambda y)$, $\lambda \neq 0$, the third one is taken to $\{r, -r\}$.

If X is expressed as in (1), then in the coordinates above the following relations are true $c = 0$, $\bar{a} = 0$, $\bar{c} - b + a - \bar{b} = 0$ and $b - \bar{c} \neq 0$ (this follows from (2) and (3) in 1.2, by imposing the conditions $P_2(0, 0) = P_1(0, 0) = P_1(-1, 0) = 0$). Moreover we have the following table

singularities	eigenvalues
$p = (0, 1, 0)$	$-\bar{c}, b - \bar{c} \neq 0$
$q = (1, 0, 0)$	$-a, \bar{b} - a = \bar{c} - b \neq 0$
$r = 1/2\sqrt{2}(-1, 1, 0)$	$\bar{b} - \bar{c}, \bar{c} - b \neq 0.$

If we suppose that p and q are not hyperbolic then $a = \bar{c} = 0$ and $\bar{b} = -b \neq 0$. The lemma now follows directly. □

By the two lemmas above we can restrict ourselves to quadratic vector fields X with expression

$$X : \begin{cases} \dot{x} = \alpha + mx + ny + bxy \\ \dot{y} = \bar{\alpha} + \bar{m}x + \bar{n}y - bxy \end{cases} \quad b \neq 0.$$

Moreover, by translating the coordinates we can suppose $n = \bar{m} = 0$. Also, if necessary, the change of coordinates $(x,y) \mapsto (y, x)$ makes $b > 0$.

Lemma 2.9. — *Let* $X \in \chi^2$ *be given by*

$$X: \begin{cases} \dot{x} = \alpha + mx + bxy \\ \dot{y} = \bar{\alpha} + \bar{n}y - bxy \end{cases}$$

with $b > 0$. *Then*:

(i) $p = (0, 1, 0)$ *and* $q = (0, -1, 0)$ *are semi-hyperbolic singularities of* X *at infinity (hyperbolic for the restriction of* X *to infinity) and* $r = 1/2\sqrt{2}(-1, 1, 0)$ *is a hyperbolic node*;

(ii) *if* X *has an unbounded graph then it must contain* p *and* q *(or* $-p$ *and* $-q$*) and the corresponding arc between them*; *moreover* p *and* q *(*$-p$ *and* $-q$*) must be saddles or saddles-nodes*;

(iii) *if* X *has an unbounded graph that contains singularities in the plane and has a return map, then the separatrices are contained in invariant lines. In this case the vector field does not have limit cycles.*

Proof. — Parts (i) and (ii) are clear from Lemmas 2.7 and 2.8. If X satisfies the hypothesis in (iii) then by the contact property it follows that there exist two invariant lines that must contain the separatrices. We know that in this case there is no limit cycle. □

In what follows we will consider $X \in \chi^2$ to be given by

$$X: \begin{cases} \dot{x} = \alpha + mx + bxy \\ \dot{y} = \bar{\alpha} + \bar{n}y - bxy \end{cases}$$

with $b > 0$, and we will study, in terms of the coefficients, when X can have unbounded graphs without singularities in the plane. We will prove the following assertions:

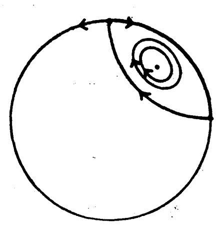
(i) if $\bar{\alpha}\alpha = 0$ there is no graph without singularities in the plane;

(ii) if either $\bar{\alpha} < 0$, $\alpha > 0$ or $m\bar{n} > 0$ there is no unbounded graph.

Notice that when $m > 0$ and $\bar{n} < 0$, we can change coordinates $(x, y) \mapsto (-y, -x)$ so that we may suppose $m < 0$ and $\bar{n} > 0$.

Let $\bar{\alpha} > 0$, $m \leqslant 0$, $\alpha < 0$ and $\bar{n} \geqslant 0$. Then:

(iii) if $\alpha + \bar{\alpha} = m + \bar{n} = 0$ the only possible graph is as follows



(iv) if $(\alpha + \bar{\alpha})(m + \bar{n}) = 0$ but $\alpha + \bar{\alpha} + m + \bar{n} \neq 0$, there is no unbounded graph;

(v) if $\alpha + \bar{\alpha} \neq 0$ and $m + \bar{n} \neq 0$, then any graph without singularities in the plane is finite.

To prove (i) suppose $\bar{\alpha} = 0$. The line $y = 0$ is invariant and contains the center manifold of q. So, there is no graph, as required. The same happens if $\alpha = 0$.

To prove the other assertions let us consider X expressed in the coordinates at infinity:

$$X_1 : \begin{cases} \dot{u}_1 = -\, bu_1 - bu_1^2 + (\bar{n} - m)\, u_1 v_1 + \bar{\alpha} v_1^2 - \alpha u_1 v_1^2 \\ v_1 = -\, bu_1 v_1 - m v_1^2 - \alpha v_1^3, \end{cases}$$

$$X_2 : \begin{cases} \dot{u}_2 = bu_2 + bu_2^2 + (m - \bar{n})\, u_2 v_2 + \alpha v_2^2 - \bar{\alpha} u_2 v_2^2 \\ \dot{v}_2 = bu_2 v_2 - \bar{n} v_2^2 - \bar{\alpha} v_2^3. \end{cases}$$

In both systems, the origin is a semi-hyperbolic singularity.

The center manifold for X_1 has the form

$$u_1 = h_1(v_1) = (\bar{\alpha}/b)\, v_1^2 + \theta(v_1^3)$$

and the flow along it is given by

$$\dot{v}_1 = -\, m v_1^2 - (\alpha + \bar{\alpha})\, v_1^3 + \theta(v_1^4).$$

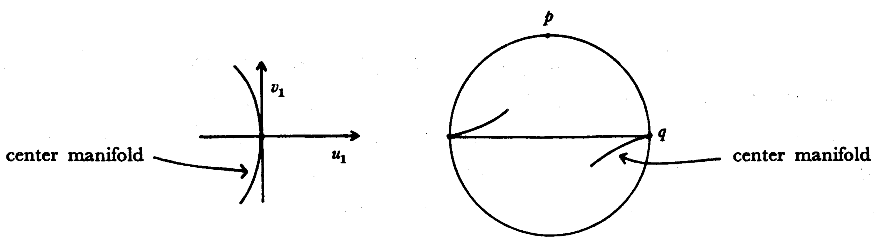
Similarly, the center manifold for X_2 has the form

$$u_2 = h_2(v_2) = -\, (\alpha/b)\, v_2^2 + \theta(v_2^3)$$

and the flow is given by

$$\dot{v}_2 = -\, \bar{n} v_2^2 - (\alpha + \bar{\alpha})\, v_2^3 + \theta(v_2^4).$$

We can now prove (ii). Suppose $\bar{a} < 0$. Since $b > 0$, the center manifold of X_1 is locally contained in the half plane $u_1 \leqslant 0$. That is:



From the contact property it follows that all graphs with return map must enclose a convex region. On the other hand, by Lemma 2.9 any such graph must contain the adjacent singularities p and q (or $-p$ and $-q$). But this is impossible because of the location of the center manifold (see figure above). The same happens when $\alpha > 0$.

If $m\bar{n} > 0$ then p or q is a node, and thus there is no unbounded graph.

Now suppose $\bar{\alpha} > 0$, $m \leqslant 0$, $\alpha < 0$ *and* $\bar{n} \geqslant 0$. Notice that from the expressions for the center manifolds of p and q we have the following situations:

(double arrows indicate hyperbolicity)

Besides proving (iii) to (v) we will see that these are the only cases where we can have unbounded graphs without singularities in the plane. In fact, consider the hyperbola $y = y(x) = -\alpha/(bx)$ and its normal vector $\alpha_N(x) = (-\alpha, bx^2)$. Easy calculations show that

$$\langle X(x, y(x)), \alpha_N(x) \rangle = x(b(\alpha + \bar{\alpha})\, x - \alpha(m + \bar{n})).$$

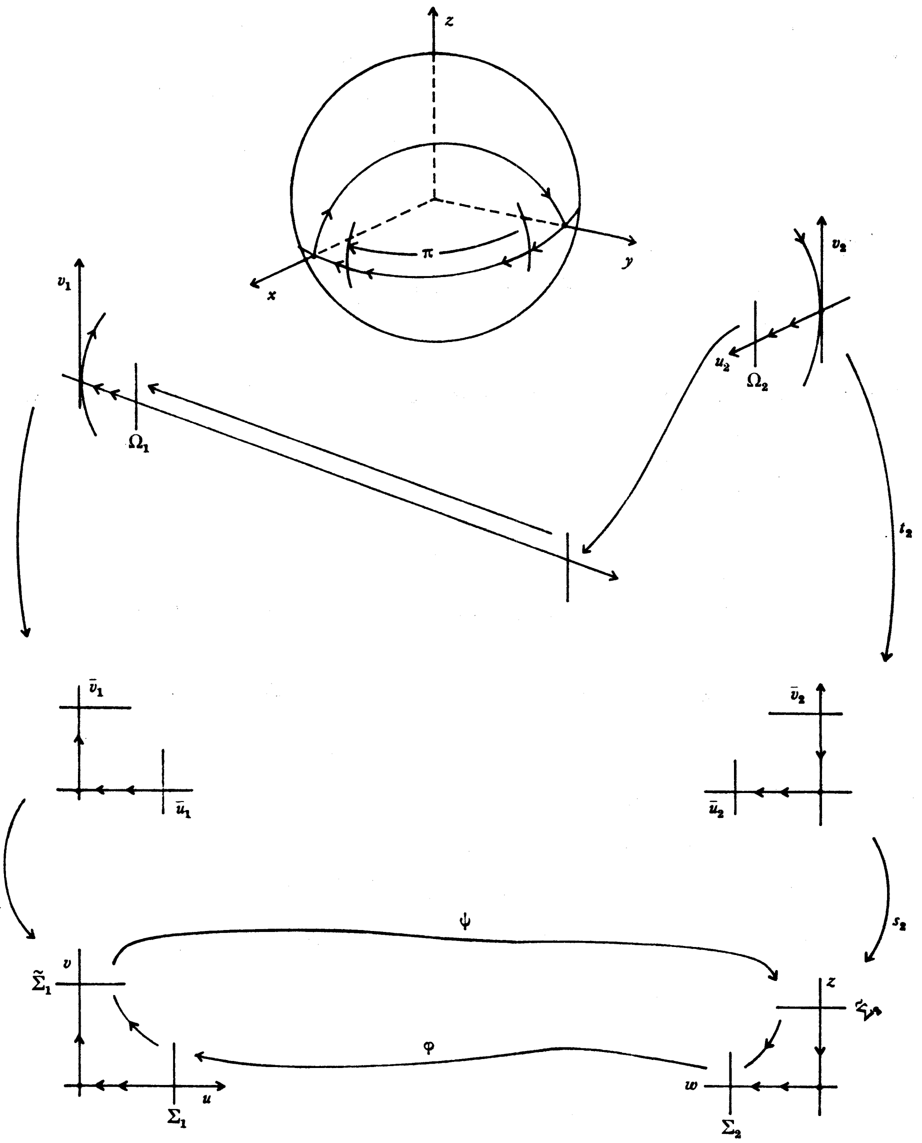
To prove (iii), observe that, since $\alpha + \bar{\alpha} = m + \bar{n} = 0$, the hyperbola is invariant, the vector field verifies $A_* X = -X$ for $A(x, y) = (y, x)$ and we obtain the graph indicated above.

To prove (iv), i.e. when $(\alpha + \bar{\alpha})(m + \bar{n}) = 0$ but $\alpha + \bar{\alpha} + m + \bar{n} \neq 0$, notice that the hyperbola $y = y(x)$ is transversal to X and no unbounded graph can exist (the separatrices of p and q must be on different sides of $y = y(x)$).

Let us now prove (v). We suppose $\alpha + \bar{\alpha} \neq 0$ and $m + \bar{n} \neq 0$. We recall that $\bar{\alpha} > 0$, $m \leqslant 0$, $\alpha < 0$ and $\bar{n} \geqslant 0$. The relations $m = 0$ and $\alpha + \bar{\alpha} > 0$ or $\bar{n} = 0$ and $\alpha + \bar{\alpha} < 0$ imply respectively that q or p are nodes (see the expressions for the center manifolds), and thus in these cases there are no unbounded graphs.

Now we arrive at the hardest part of the proof of Theorem B. There are three cases to consider: $m\bar{n} \neq 0$; $m = 0$, $\bar{n} > 0$, $\alpha + \bar{\alpha} < 0$; and $m < 0$, $\bar{n} = 0$, $\alpha + \bar{\alpha} > 0$. As shown in the figures above in the three cases there can exist an unbounded graph without singularities in the plane. We will prove now that if such a graph exists, then it is finite. For that purpose we will analyse return maps (Poincaré maps) associated to these graphs and show that these maps have isolated fixed points.

To help clarify our arguments let us consider the following figure and diagrams:



Let us explain the notation. For $0 < \varepsilon < \delta$, let

$$\Sigma_1 = \{(\delta, v)/|v| < \varepsilon\}, \qquad \Sigma_2 = \{(\delta, z)/|z| < \varepsilon\},$$

$$\tilde{\Sigma}_1 = \{(u, \delta)/|u| < \varepsilon\}, \qquad \tilde{\Sigma}_2 = \{(w, \delta)/|w| < \varepsilon\},$$

$$\Omega_1 = \{(\delta, v_1)/|v_1| < \varepsilon\}, \qquad \Omega_2 = \{(\delta, v_2)/|v_2| < \varepsilon\},$$

be transversal sections as indicated.

Let ρ_1 denote the change of coordinates from the (u_2, v_2)-plane to the (u_1, v_2)-plane. Since $v_1 = 1/x$ and $v_2 = 1/y$ we have $v_2/v_2 = x/y = u_1$ and so $\rho_1(u_2, v_2) = (1/u_2, v_2/u_2)$. Moreover, $\rho_1(\Omega_2) = \{(1/\delta, v_1)/|v_1| < \varepsilon/\delta\}$.

Let $\rho_2 : \rho_1(\Omega_2) \to \Omega_1$; $\varphi : \Sigma_2 \to \Sigma_1$ and $\psi : \widetilde{\Sigma}_1 \to \widetilde{\Sigma}_2$ be the Poincaré maps naturally defined by X.

Let t_1, s_1, t_2, s_2 be local changes of coordinates given as follows. First take

$$\text{(i)} \qquad t_1 : \begin{cases} \bar{u}_1 = u_1 - h_1(v_1) = u_1 - (\bar{\alpha}/b)\, v_1^2 - \theta(v_1^3) \\ \bar{v}_1 = v_1. \end{cases}$$

In these new coordinates, X_1 has the form

$$\begin{cases} \dot{\bar{u}}_1 = -b\bar{u}_1 - b\bar{u}_1^2 + \bar{u}_1 \bar{v}_1 g_1(\bar{u}_1, \bar{v}_1) \\ \dot{\bar{v}}_1 = -b\bar{u}_1 \bar{v}_1 - m\bar{v}_1^2 - (\alpha + \bar{\alpha})\, \bar{v}_1^3 + \bar{v}_1^4 f_1(\bar{v}_1). \end{cases}$$

Now, to avoid the term $-b\bar{u}_1 \bar{v}_1$ in the component of the $\bar{v}_1$-axis we take

$$s_1 : \begin{cases} u = \bar{u}_1 \\ v = \bar{v}_1/(1 + \bar{u}_1), \end{cases}$$

obtaining

$$X_1 : \begin{cases} \dot{u} = -bu - bu^2 + uv\widetilde{g}_1(u, v) \\ \dot{v} = -mv^2 - (\alpha + \alpha)\, v^3 + v^4 \widetilde{f}_1(u, v) + uv^2 \widetilde{h}_1(u, v). \end{cases}$$

Now take

$$\text{(ii)} \qquad t_2 : \begin{cases} \bar{u}_2 = u_2 - h_2(v_2) = u_2 + (\alpha/b)\, v_2^2 - \theta(v_2^3) \\ \bar{v}_2 = v_2. \end{cases}$$

In these coordinates X_2 is expressed as

$$\begin{cases} \dot{\bar{u}}_2 = b\bar{u}_2 + b\bar{u}_2^2 + \bar{u}_2 \bar{v}_2 g_2(\bar{u}_2, \bar{v}_2) \\ \dot{\bar{v}}_2 = b\bar{u}_2 \bar{v}_2 - \bar{n}\bar{v}_2^2 - (\alpha + \bar{\alpha})\, \bar{v}_2^3 + \bar{v}_2^4 f_2(\bar{v}_2^4). \end{cases}$$

Again, to avoid the term $b\bar{u}_2 \bar{v}_2$ in the component of the $\bar{v}_2$-axis we take

$$s_2 : \begin{cases} w = \bar{u}_2 \\ z = \bar{v}_2/(1 + \bar{u}_2) \end{cases}$$

obtaining

$$\widetilde{X}_2 \begin{cases} \dot{w} = bw + bw^2 + wz\widetilde{g}_2(w, z) \\ \dot{z} = -\bar{n}z^2 - (\alpha + \bar{\alpha})\, z^3 + z^4 \widetilde{f}_2(w, z) + wz^2 \widetilde{h}_2(w, z). \end{cases}$$

Finally, let $u = u(v)$ be the Poincaré map from $\Sigma_1^+ = \{(\delta, v) \in \Sigma_1 / v > 0\}$ to $\widetilde{\Sigma}_1$ defined by $\widetilde{X}_1$ and let $z = z(w)$ be the Poincaré map from $\widetilde{\Sigma}_2^+ = \{(w, \delta) \in \widetilde{\Sigma}_2 / w > 0\}$ to Σ_2 defined by $\widetilde{X}_2$.

To prove the finiteness of the graphs we will compare the Poincaré maps above to other ones defined by auxiliary vector fields.

To do this consider the following vector fields

$$X_1' : \begin{cases} \dot{u} = -bu \\ \dot{v} = -m' v^2 \end{cases} \qquad \text{and} \qquad X_2' : \begin{cases} \dot{w} = bw \\ \dot{z} = -\bar{n}' z^2 \end{cases}$$

with $m' < 0$ and $\bar{n}' > 0$. Let $\tilde{u}: \Sigma_1^+ \to \tilde{\Sigma}_1$ and $\tilde{z}: \tilde{\Sigma}_2^+ \to \Sigma_2$ be the Poincaré maps associated to X_1' and X_2' respectively. We will use the following expressions

$$\det(\tilde{X}_1, X_1') = uv^2[b(m' - m) + F_1(u, v)],$$
$$\det(\tilde{X}_2, X_2') = wz^2[b(\bar{n} - \bar{n}') + F_2(w, z)],$$

where $F_1(0, 0) = 0$ and $F_2(0, 0) = 0$, to compare the flows of $\tilde{X}_i$ and X_i', $i = 1$ or 2. Notice for example that if $|m| > |m'|$ then $\det(\tilde{X}_1, X_1^{1'}) > 0$ for $u > 0$, and so $\tilde{u}(v) < u(v)$ for small enough $v > 0$.

Let us first calculate $u = \tilde{u}(v)$, $z = \tilde{z}(w)$ and show that $\varphi'(0) = 1$. We have

$$(4) \quad \begin{cases} u = \tilde{u}(v) = \delta e^{-b/(m'\delta)} e^{b/(m'v)} = k_1 e^{b/(m'v)} \\ z = \tilde{z}(w) = \dfrac{\delta}{1 + (\bar{n}'\delta/b)\ln\delta - (\bar{n}'\delta/b)\ln w} = \dfrac{\delta}{k_2 - (\bar{n}'\delta/b)\ln w} \end{cases}$$

Now, to calculate $\varphi'(0)$ we use the following lemma.

Lemma **2.10.** — *We have* $\rho_2'(0) = \delta$.

Proof. — We calculate $\rho_2'(0)$ by the following formula (see [A])

$$\rho_2'(0) = \frac{|X_1(1/\delta, 0)|}{|X_1(\delta, 0)|} \exp \int_0^T \operatorname{div} X_1(\gamma(t))\, dt,$$

where $\gamma(t)$ is the orbit of X_1 that goes from $(1/\delta, 0)$ to $(\delta, 0)$ in time T. The vector field X_1, when restricted to $v_1 = 0$, has the equation $\dot{u}_1 = -bu_1 - bu_1^2$. Integrating we obtain

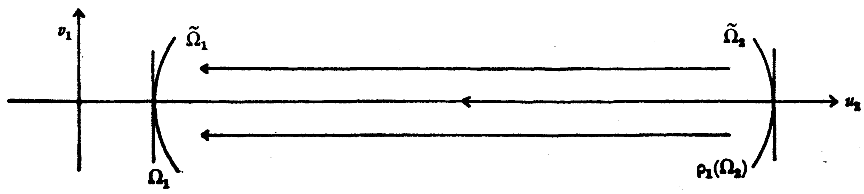
$$\gamma(t) = \left(\frac{e^{-bt}}{1 + \delta - e^{-bt}}, 0\right).$$

From $\gamma(T) = \delta$ we obtain $T = -1/b \ln \delta$. Since $\operatorname{div} X_1(u_1, 0) = -b - 3bu_1$ we calculate

$$\int_0^T \operatorname{div} X_1(\gamma(t))\, dt = -bT - 3\int_0^T \frac{be^{-bt}}{1 + \delta - e^{-bt}}\, dt$$
$$= \ln\delta - 3\ln(1 + \delta - e^{-bt})\Big|_0^T$$
$$= \ln \delta^4.$$

Finally, since $|X_1(\delta, 0)| = b\delta(1 + \delta)$ and $|X_1(1/\delta, 0)| = b(1 + \delta)/\delta^2$ we obtain $\rho_2'(0) = \delta$, proving the lemma. □

Remark. — If $\tilde{\Omega}_1$ and $\tilde{\Omega}_2$ are cross sections for X_1 with $\tilde{\Omega}_1$ tangent to Ω_1 at $(\delta, 0)$ and $\tilde{\Omega}_2$ tangent to $\rho_1(\Omega_2)$ at $(1/\delta, 0)$ and they are parametrized by the projection to the v_1-axis,



then the corresponding Poincaré map $\tilde{\rho}_2 : \tilde{\Omega}_2 \to \tilde{\Omega}_1$ satisfies $\tilde{\rho}_2'(0) = \rho_2'(0) = \delta$.

From the above remark and from the fact that $\rho_1(\delta, v_2) = (1/\delta, v_2/\delta)$ we finally obtain $\varphi'(0) = 1$.

We can now give the expression of our modified Poincaré map.

Lemma **2.13.** — *Let* $u = \tilde{u}(v)$ *and* $z = \tilde{z}(w)$ *be as in* (4). *Let* $v = \tilde{\varphi}(z) = \lambda z$ *and* $w = \tilde{\psi}(u) = \mu u$. *Then*

$$(\tilde{u} \circ \tilde{\varphi} \circ \tilde{z} \circ \tilde{\psi})(u) = k u^{\bar{n}'/|m'|\lambda}$$

where k *is a positive real number.*

Proof. — The formula is obtained by composing the maps. □

Let us now compare the return map of X with the modified return map and prove that if there exists a graph then it is finite (we will prove slightly more: they are not accumulated by periodic orbits). Suppose $0 < (\bar{n}/|m|) < 1$ or else $\bar{n} = 0$. Chose m', $\bar{n}'$, λ and μ such that: $m' < 0$, $|m'| < m$, $\bar{n}' > \bar{n}$, $\lambda < 1$, $(\bar{n}'/|m'|\lambda) < 1$ and $0 < \mu u < \psi(u)$ for all $u > 0$ small enough. Then for $\tilde{u}$, $\tilde{z}$, $\tilde{\psi}$ and $\tilde{\varphi}$ defined in (4) and in Lemma 2.13 we have

$$\begin{aligned} &\tilde{u}(v) < u(v) && \text{since, for } u > 0,\ \det(\tilde{X}_1, X_1') > 0 \text{ near } (0, 0),\\ &\tilde{z}(w) < z(w) && \text{since, for } w > 0,\ \det(\tilde{X}_2, X_2') < 0 \text{ near } (0, 0),\\ &\tilde{\psi}(u) < \psi(u) && \text{for } u > 0 \text{ small enough},\\ &\tilde{\varphi}(z) < \varphi(z) && \text{for } z > 0 \text{ small enough}. \end{aligned}$$

Thus

$$(\tilde{u} \circ \tilde{\rho} \circ \tilde{z} \circ \tilde{\psi})(u) = k u^{\bar{n}'/|m'|\lambda} < (u \circ \varphi \circ z \circ \psi)(u),$$

and since $(\bar{n}'/|m'|\lambda) < 1$ we conclude that $u < (u \circ \varphi \circ z \circ \psi)(u)$ for all u small enough. With this it is proved that if there is an unbounded graph then it is a repellor (i.e. it is the α-limit set of some orbit).

In the same way if $(\bar{n}/|m|) > 1$ or $m = 0$ we prove that if there is an unbounded graph it is an attractor (i.e. it is the ω-limit set of some orbit). This ends the proof of Case (III).

2.2 (IV) $\bar{c} - b = \bar{b} - a = \bar{a} = 0$.

Let $X \in \chi^2$ be given by (1) (see Section 1) with $c = \bar{c} - b = \bar{b} - a = \bar{a} = 0$. In this case, X is transversal to infinity with the exception of two symmetric points.

Since the relations above are invariant under any affine change of coordinates, the periodic orbit property allows us to restrict ourselves to vector fields of the form

$$\begin{cases} \dot{x} = mx - y + ax^2 + bxy \\ \dot{y} = x + my + axy + by^2. \end{cases}$$

For these vector fields it is not hard to prove that with a rotation of coordinates we can make $b = 0$. We then obtain

$$\begin{cases} \dot{x} = mx - y + ax^2 \\ \dot{y} = x + my + axy. \end{cases}$$

With this last expression we see that the origin is the only singularity and therefore, there is a finite number of limit cycles.

The proof of Theorem B is now complete.

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