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ON ISOMORPHISMS OF GEOMETRICALLY FINITE MÖBIUS GROUPS

by PEKKA TUKIA

1. INTRODUCTION

1A. A Möbius group G of $\overline{\mathbf{R}}^n$ is a group of Möbius transformations of $\overline{\mathbf{R}}^n = \mathbf{R}^n \cup \{\infty\}$; it is a *Kleinian group* if it acts discontinuously somewhere in $\overline{\mathbf{R}}^n$. The action of G extends to the $(n+1)$-dimensional hyperbolic space

$$H^{n+1} = \{(x_1, \ldots, x_{n+1}) \in \mathbf{R}^{n+1} : x_{n+1} > 0\}$$

and G is said to be *geometrically finite* if it is a discrete Möbius group and has a hyperbolic fundamental domain in H^{n+1} with a finite number of faces, cf. 1B.

In this paper we consider two geometrically finite Möbius groups G and G′ and an isomorphism $\varphi : G \to G'$; φ is *type-preserving* if it carries parabolic elements (see 1C) of G bijectively onto parabolic elements of G′. If $A, A' \subset \overline{\mathbf{R}}^n \cup H^{n+1}$, A is G-invariant and A′ is G′-invariant, we say that a map $f : A \to A'$ *induces* φ if $f(g(x)) = \varphi(g)(f(x))$ for every $g \in G$ and $x \in A$; we say also that f is G-*compatible*.

In section 3 we show that any type-preserving isomorphism $\varphi : G \to G'$ is induced by a unique homeomorphism $f_\varphi : L(G) \to L(G')$ of the limit sets (Theorem 3.3). It satisfies a property similar to quasiconformality, called quasisymmetry. If $L(G) = L(G') = \overline{\mathbf{R}}^n$, it has been long known (see 3C) that such a homeomorphism f_φ always exists and that this homeomorphism is quasiconformal if $n > 1$. The existence of such a map f_φ was essential for Mostow's rigidity theorem which is equivalent to the fact that f_φ is a Möbius transformation (if $L(G) = L(G') = \overline{\mathbf{R}}^n$). The proof of the existence of f_φ in our situation is essentially the same as the proof in case that the limit set is $\overline{\mathbf{R}}^n$.

Quasiconformal maps of $\overline{\mathbf{R}}^n$, $n > 1$, are absolutely continuous. It was this fact that made possible Mostow's proof that f_φ is then a Möbius transformation. More generally, if one knows that f_φ is absolutely continuous with respect to some measures of L(G) and L(G′) (which are the Lebesgue measures if $L(G) = L(G') = \overline{\mathbf{R}}^n$), then f_φ is the restriction of a Möbius transformation ([34]).

In addition, [35, Theorem D] implies that if there is $x \in L(G)$ which is not fixed by a parabolic $g \in G$ (x is then a radial point of G (cf. [35, (A1)]) by Theorem 2.4 below) such that f_φ is so regular at x that the differential of f_φ at x can be defined and has a non-zero Jacobian, then f_φ is the restriction of a Möbius transformation.

We then show that if $L(G) \neq \overline{\mathbf{R}}^n$ and if $f: \overline{\mathbf{R}}^n \setminus L(G) \to \overline{\mathbf{R}}^n \setminus L(G)$ is a homeomorphism inducing φ, then φ is type-preserving if $n > 1$ and that then f and f_φ define together a homeomorphism f' inducing φ. (This is true also if $n = 1$, but now one must assume separately that φ is type-preserving.) In addition, f' is quasiconformal if f is (when $n > 1$) and the dilatation is not increased in the extension to the limit set (Theorem 3.8).

This latter fact allows the following complement of Mostow's rigidity theorem for the case $L(G) \neq \overline{\mathbf{R}}^n$, $n > 1$. If f is conformal (that is, f is 1-quasiconformal and for $n \geq 3$ this means that f is a Möbius transformation when restricted to some component of $\overline{\mathbf{R}}^n \setminus L(G)$), then the extension f' is also conformal. But conformal homeomorphisms of $\overline{\mathbf{R}}^n$ are Möbius transformations and hence so is f'. Clearly, this result is consistent with Mostow's theorem for $L(G) = \overline{\mathbf{R}}^n$, as was already observed by Marden for $n = 2$ [14, Theorem 8.1].

In the final section we consider Kleinian groups of $\overline{\mathbf{R}}^2$ and examine when an isomorphism $\varphi: G \to G'$ of two geometrically finite groups is induced by a homeomorphism F of $\overline{H}^3 = H^3 \cup \overline{\mathbf{R}}^2$. We call such an isomorphism φ *geometric*. We first give a new proof (Theorem 4.2) of a theorem originally due to Marden [14] according to which φ is geometric if there is a homeomorphism $f: \overline{\mathbf{R}}^2 \setminus L(G) \to \overline{\mathbf{R}}^2 \setminus L(G')$ inducing φ. Our proof is based on the above mentioned result on the quasiconformality of the map defined by f and f_φ as well as on a theorem according to which a quasiconformal and G-compatible map of $\overline{\mathbf{R}}^2$ can be extended to a quasiconformal and G-compatible homeomorphism of $\overline{H}^3$ (cf. Reimann [26], Thurston [29, chapter 11] and [32, 1E]).

Finally (Theorem 4.7), we characterize geometric isomorphisms of geometrically finite groups of $\overline{\mathbf{R}}^2$ by properties which are generalizations of the Fenchel-Nielsen intersecting-axis condition for Fuchsian groups (cf. 4C).

1B. Geometrically finite groups. — We now give a precise definition of a geometrically finite group. First observe that the action of a Möbius group G of $\overline{\mathbf{R}}^n$ can be automatically extended to $\overline{H}^{n+1} = H^{n+1} \cup \overline{\mathbf{R}}^n$ and therefore we do not distinguish between Möbius groups of $\overline{\mathbf{R}}^n$ and $\overline{H}^{n+1}$. Then G acts as a group of isometries of H^{n+1} in the hyperbolic metric. It is well-known that a Möbius group of $\overline{\mathbf{R}}^n$ is discrete if and only if it acts discontinuously in H^{n+1}.

A *polyhedron* D of H^{n+1} is a subset of H^{n+1} such that $D = \mathrm{cl}(\mathrm{int}\, D)$ (closure and interior in H^{n+1}) and that $\partial D \subset B$ where B is a locally finite union of hyperbolic n-planes of H^{n+1}; if this union is finite, D is *finite-sided*. Note that we do not require that D is connected. A *face* (or an n-face) of D is a set F such that $F \subset H$ for some hyperbolic n-plane $H \subset H^{n+1}$, that $\text{int}_H F$ is a component of $\text{int}_H(H \cap \partial D)$ and that $F = \text{cl}_H(\text{int}_H F)$. Then F is a polyhedron of H and we define inductively an i-face of D as a face of an $(i+1)$-face of D (and an $(n+1)$-face of D as the closure of a component of int D).

If G is a discrete Möbius group of $\overline{\mathbf{R}}^n$, we say that a polyhedron D of H^{n+1} is a *fundamental polyhedron* of G if $g(D)$, $g \in G$, is a locally finite cover of H^{n+1} and if $g(\text{int } D) \cap \text{int } D = \varnothing$ for $g \in G \backslash \{\text{id}\}$. The group G is *geometrically finite*, if it has a finite-sided fundamental polyhedron D such that $g(D) \cap D \neq \varnothing$ for only finitely many $g \in G$.

There is an important case in which G is always geometrically finite. Let

$$M_G = (\overline{H}^{n+1} \backslash L(G))/G,$$

i.e. M_G is the orbit space associated to G. We say that a discrete group is of *compact type* if M_G is compact. For any discrete G, the Dirichlet fundamental polyhedron with center $x \in H^{n+1}$ defined by

$$D = \{y \in H^{n+1}: \ d(y, x) \leq d(g(y), x) \quad \text{for } g \in G\} \tag{1.1}$$

is a fundamental polyhedron for G (Marden [14, 4.1]); D is also sometimes called the Poincaré fundamental polyhedron. If G is of compact type, D is always finite-sided and $g(D) \cap D \neq \varnothing$ for only finitely many $g \in G$, as one easily sees.

Thus groups of compact type are geometrically finite. However, not all geometrically finite groups are of compact type, but then M_G can be compactified by the addition of a finite number of points corresponding to the conjugacy classes of parabolic elements of G (Theorem 2.4). In fact, a geometrically finite group is of compact type if and only if it does not contain parabolic elements (Corollary 2.5).

Many of our proofs could be considerably simplified for groups of compact type. For instance, entire Section 2 can then be omitted. We have written this paper in such a way that the parts needed only in the non-compact case can be easily skipped if wished. Also, if one is interested only of the case $n = 2$, many complications can be avoided. Then most of the theorems of Section 2 can be either omitted or their proofs simplified. This is due to the fact that orientation preserving Möbius transformation of $\overline{\mathbf{R}}^2$ fixing ∞ are just the translations $x \mapsto x + a$ of $\mathbf{R}^2$.

1C. Definitions and notations. — In this paper a Möbius transformation can be either orientation preserving or reversing. As usual, a Möbius transformation g of $\overline{\mathbf{R}}^n$ is called *loxodromic* if it can be conjugated by a Möbius transformation to the form

$$g(x) = \lambda\alpha(x) \tag{1.2}$$

if $x \in \mathbf{R}^n$, where $\lambda > 1$ and $\alpha \in O(n)$ = the orthogonal group of $\mathbf{R}^n$; g is *hyperbolic* if $\alpha = \text{id}$ in (1.2). If g (or its canonical extension to $\overline{\mathbf{R}}^{n+1}$) can be conjugated to the form (1.2) with $\lambda = 1$ (but $\alpha \neq \mathrm{id}$), then g is *elliptic*. The map g is *parabolic*, if it can be conjugated to the form

$$(1.3) \qquad g(x) = \alpha(x) + a$$

for $x \in \mathbf{R}^n$, where $a \in \mathbf{R}^n \setminus \{0\}$, $\alpha \in \mathrm{O}(n)$ and $\alpha(a) = a$. Every Möbius transformation $g \neq \mathrm{id}$ falls into one of the above types.

The number $\lambda > 1$ in (1.2) is the *multiplier* $\mathrm{mul}\, g$ of a loxodromic Möbius transformation g; for non-loxodromic g we set $\mathrm{mul}\, g = 1$.

If G is a Möbius group of $\overline{\mathrm{H}}^{n+1}$ and $\mathrm{A} \subset \overline{\mathrm{H}}^{n+1}$, we set $\mathrm{G}_\mathrm{A} = \{g \in \mathrm{G} : g(\mathrm{A}) = \mathrm{A}\}$ and $\mathrm{G}_{\{v\}} = \mathrm{G}_v$. They are the *stabilizers* of A and v. The set A is G-*invariant* if $\mathrm{G}_\mathrm{A} = \mathrm{G}$. The *limit set* L(G) of a discrete G lies entirely in $\overline{\mathbf{R}}^n$ and we set

$$\Omega(\mathrm{G}) = \overline{\mathbf{R}}^n \setminus \mathrm{L}(\mathrm{G});$$

it is the *ordinary set* of G. A *parabolic fixed point* of G is a point fixed by some parabolic $g \in \mathrm{G}$. A discrete Möbius group is *elementary* if L(G) consists of at most two points.

Let $\mathrm{X} \subset \overline{\mathbf{R}}^n$. Then the *hyperbolic convex hull* Co(X) of Xis the small est closed and (hyperbolically) convex subset of H^{n+1} such that

$$(1.4) \qquad \mathrm{X} \subset \mathrm{cl}\, \mathrm{Co}(\mathrm{X}).$$

This is well-defined if $\mathrm{X} \neq \{x\}$ in which case we set $\mathrm{Co}(\mathrm{X}) = \varnothing$.

The *hyperbolic convex hull* $\mathrm{H}_\mathrm{G} \subset \mathrm{H}^{n+1}$ of a discrete Möbius group G of $\overline{\mathbf{R}}^n$ is defined by

$$(1.5) \qquad \mathrm{H}_\mathrm{G} = \mathrm{Co}(\mathrm{L}(\mathrm{G})).$$

If G is of compact type, then $\mathrm{H}_\mathrm{G}/\mathrm{G}$ is compact. In fact, if L(G) consists of at least two points, then G is of compact type if and only if $\mathrm{H}_\mathrm{G}/\mathrm{G}$ is compact, as a simple argument shows. However, we do not need this fact.

We denote by $e_1, \ldots, e_n$ the standard basis of $\mathbf{R}^n$ and $\mathbf{R}^k$, $k \leq n$, is regarded as a subspace of $\mathbf{R}^n$ with basis $e_1, \ldots, e_k$. The euclidean distance of two points is $|a - b|$. The *closed* euclidean ball with center x and radius $r \geq 0$ is denoted by $\mathrm{B}(x, r)$ or $\mathrm{B}^n(x, r)$ if we wish to emphasize the dimension of $\mathbf{R}^n$. We set

$$\mathrm{B}^n(r) = \mathrm{B}^n(0, r) \quad \text{and} \quad \mathrm{B}^n = \mathrm{B}^n(1).$$

A *ball* of $\overline{\mathbf{R}}^n$ is a set of the form $g(\mathrm{B}^n)$ where g is some Möbius transformation. Similarly, a k-*sphere* of $\overline{\mathbf{R}}^n$ is of the form $g(\partial \mathrm{B}^k)$, $k \leq n$.

In addition to the euclidean metric we will use several other metrics in this paper. The *hyperbolic metric* of H^{n+1} is denoted by d and the *spherical metric* of $\overline{\mathbf{R}}^n$ by q; q is normalized in such a way that the q-diameter of $\overline{\mathbf{R}}^n$ equals 1. In Section 3 we will also consider the *quasihyperbolic* metric k_U of a proper subdomain U of $\mathbf{R}^n$. The diameter of a set A is $d(\mathrm{A})$, $q(\mathrm{A})$, etc., and the distance of a point a from A is $d(a, \mathrm{A})$, $q(a, \mathrm{A})$, etc. As is customary we denote by $d(\mathrm{A})$ and by $d(a, \mathrm{A})$ also the euclidean diameter of a set and the euclidean distance of a from A. If it is not clear from the context whether we mean the euclidean or the hyperbolic metric, we will indicate which one we mean.

The boundary of a set A is bd A or, if sufficiently regular, ∂A. The interior of A is int A, the closure cl A. These operations are mostly taken in $\overline{\mathbf{R}}^n$, $\overline{\mathbf{R}}^{n+1}$ or H^{n+1} and sometimes we use subscripts, bd_A, cl_A, etc., to denote the space where they are taken, if this is not otherwise clear.

We use the following slight extension of the notion of quasiconformality: Let $X \subset \overline{\mathbf{R}}^n$, $n > 1$, be a set such that $X = \text{cl}(\text{int}\, X) \neq \varnothing$ and let $f: X \to \overline{\mathbf{R}}^n$ be an embedding. Then f is *quasiconformal* (or K-*quasiconformal*) if there is $K \geq 1$ such that f is in each component of int X an (orientation reversing or preserving) K-quasiconformal embedding [40]. A map $f: X \to \overline{\mathbf{R}}^n$ is *conformal* if it is 1-quasiconformal.

The identity map of a set is id and we extend affine maps of $\mathbf{R}^n$ to $\overline{\mathbf{R}}^n$ by the rule $\infty \mapsto \infty$.

2. PARABOLIC CUSPS

In this section we study stabilizers of parabolic fixed point of a discrete group G and the action of the group near a parabolic fixed point. Much of it is known (e.g. Theorem 2.1) at least in principle although I have not always found it in published form for general n (e.g. Theorem 2.4). Here we group together these results for easy reference and prove some additional results needed in the sequel.

After this paper was completed, I was informed of B. Apanasov's work [2, 3, 4] which contains results partly overlapping with the beginning of this section, especially with Theorem 2.4. His definition of a geometrically finite group is different but leads to the same class of groups.

2A. Stabilizer of a point. — We now examine the groups that can occur as stabilizers of a point for a discrete group. These are well-known groups and we summarize the results we need in Theorem 2.1.

We say that a group is *loxodromic* or *parabolic* if every element of infinite order is loxodromic or parabolic, respectively, and if there are elements of infinite order.

Theorem **2.1.** — *Let* G *be a discrete Möbius group of* $\overline{\mathbf{R}}^n$ *and suppose that there is a point* $v \in \overline{\mathbf{R}}^n$ *fixed by every* $g \in G$. *Then* G *is either finite, or loxodromic, or parabolic.*

If G *is loxodromic, then there is* $v' \in \overline{\mathbf{R}}^n \setminus \{v\}$ *such that every* $g \in G$ *fixes also* v' *and* G *has an infinite cyclic subgroup of finite index.*

If G *is parabolic and* $v = \infty$, *then we have*:

a) *There is a* G-*invariant k-plane* $V \subset \mathbf{R}^n$, $0 < k \leq n$, *such that* V/G *is compact. If* $V' \subset \mathbf{R}^n$ *is another* G-*invariant k'-plane, then* V *and* V′ *are parallel,* $k' \geq k$ *and* V'/G *is compact if and only if* $k = k'$.

b) *If* a) *is true for* $V = \mathbf{R}^k$, *then the action of* $g \in G$ *in* $\overline{H}^{n+1} \setminus \{\infty\}$ *has the form*

$$(2.1) \qquad g(x, y, t) = (h(x), \alpha(y), t)$$

for $x \in \mathbf{R}^k$, $y \in \mathbf{R}^{n-k}$ *and* $t \geq 0$ *and where* h *is a non-loxodromic Möbius transformation of* $\overline{\mathbf{R}}^k$ *fixing* ∞ *and* $\alpha \in O(n-k)$.

c) *There is a free abelian subgroup* $H \subset G$ *of finite index such that if* a) *is true for* $V = \mathbf{R}^k$, *then* $\mathbf{R}^{n-k}$ *can be decomposed into a sum* $\mathbf{R}^{n-k} = W + W_1 + \ldots + W_p$ *of orthogonal spaces where* $0 \leq \dim W \leq n - k$, *and every* W_i *is 2-dimensional and with the following property. If* $h \in H$, *then*

$$(2.2) \qquad h(x, y, y_1, \ldots, y_p, t) = (x + a, y, \beta_1(y_1), \ldots, \beta_p(y_p), t)$$

for $x \in \mathbf{R}^k$, $y \in W$, $y_i \in W_i$ *and* $t \geq 0$, *where* $a \in \mathbf{R}^k$, $a \neq 0$ *if* $h \neq \mathrm{id}$, *and where* β *is a rotation of* W_i. *Furthermore, for every* i *there is* $h \in H$ *such that this rotation* $\beta_i \neq \mathrm{id}$.

d) *Let* $H' \subset G$ *be a subgroup of finite index and let* $V' \subset \mathbf{R}^n$ *be an* H'*-invariant* q*-plane such that* V'/H' *is compact. Let* $g_1, \ldots, g_s \in G$ *be representatives of the cosets in* G/H'. *Then* $V = \{(g_1(x_1) + \ldots + g_s(x_s))/s : x_i \in V'\}$ *is a* G*-invariant* q*-plane such that* V/G *is compact.*

Proof. — We can assume that $v = \infty$. If every $g \in G$ is of finite order, then $G \setminus \{\mathrm{id}\}$ consists of elliptic elements. Thus $g \mid \mathbf{R}^n$, $g \in G$, is a euclidean isometry and Wolf [42, 3.2.8] implies that G is finite.

Assume then that there is a loxodromic $g \in G$. We can assume that the fixed points of g are 0 and ∞. We show that every $h \in G$ also fixes 0 and ∞. Assume that there is $h \in G$ not fixing 0 (it fixes ∞ by assumption). Then $g' = hgh^{-1}$ is loxodromic with the same multiplier as g and does not fix 0. Let $g_i = g'^i g g'^{-i}$ and let g_i fix $a_i \in \mathbf{R}^n$. We can assume that $a_i \to 0$ as $i \to \infty$. Since g_i and g have the same multiplier, a contradiction with discreteness follows. Thus every $g \in G$ of infinite order is loxodromic. It follows that there is such a tripartite division of the groups G as claimed.

If G is loxodromic, and if every $g \in G$ fixes 0 and ∞ as we can assume, then every $g \in G$ is of the form (1.2). Let φ be the map $g \mapsto \lambda$, $\lambda > 0$ as in (1.2). Then φ is a homomorphism of G to the multiplicative group of positive real numbers and the image $\varphi(G)$ is infinite cyclic by discreteness. Hence there is an infinite cyclic subgroup H such that $\varphi \mid H$ is injective. Then G/H is finite.

So, to conclude the proof, we must now only prove cases a) — d) for parabolic G. Then every $g \mid \mathbf{R}^{n+1}$ is a euclidean isometry. Thus $g(x, t) = (g(x), t)$ if $g \in G$, $x \in \mathbf{R}^n$ and $t \geq 0$ and to prove our theorem, it suffices to consider the action of G only in $\mathbf{R}^n$. Then Wolf [42, 3.2.8 and 3.2.9] imply that G has a free abelian subgroup G^* of finite index and that there is a G^*-invariant k-plane $V^* \subset \mathbf{R}^n$, $0 \leq k \leq n$, such that V^*/G^* is compact and that the map $g \mapsto g \mid V^*$ is an injective map into the translations of V^*. Since G contains parabolic elements, $k \neq 0$. If $V^* = \mathbf{R}^k$, then every $g \in G^*$ has the form (2.1) by (1.3).

We next prove d). We can assume that $G^* \supset H'$ and that $V' = \mathbf{R}^q$ for some q. Let $V_i' = g_i(V')$ which is invariant for $H_i' = g_i H' g_i^{-1}$. If $h \in H'$, $g_i h g_i^{-1} | V_i'$ is a translation $x \mapsto x + a$. Since a power of $g_i h g_i^{-1}$ is in H', $a \in \mathbf{R}^q = V'$. Since $\mathbf{R}^q$ has a basis of vectors of this form, it follows that V_i' is parallel to $\mathbf{R}^q$. Consequently the "barycenter" V, which is obviously G-invariant, is parallel to $\mathbf{R}^q$, too. It follows by (1.3) that V/H' is compact and hence also V/G. Thus d) is true.

Now, applying d) to G^* and V^*, we find a G-invariant k-plane V, $0 < k \leq n$, such that V/G is compact. If $V = \mathbf{R}^k$, then (1.2) and (1.3) imply that every $g \in G$ has the form (2.1). This expression also implies that if V' is another G-invariant k'-plane, then V and V' are parallel, that $k' \geq k$ and that V'/G is compact if and only if $k = k'$. We have proved a) and b).

Finally, c) is true for $H = G^*$ since G^* is free abelian. To get the decomposition $\mathbf{R}^{n-k} = W + W_1 + \ldots + W_p$, embed $\mathbf{R}^{n-k}$ into $\mathbf{C}^{n-k}$ and consider the complex eigenspaces of the orthogonal maps of $\mathbf{R}^{n-k}$ defined by (2.1) for $h \in H$. Since elements of H commute, we can find a decomposition not depending on $h \in H$. The theorem is proved.

2B. Rank of parabolic elements. — Let G be a discrete Möbius group of $\overline{\mathbf{R}}^n$ and let $v \in L(G)$ be a point fixed by some parabolic $g \in G$. Then G_v is a parabolic group whose elements fix $v \in \overline{\mathbf{R}}^n$. Hence Theorem 2.1 can be applied and we define that the number k in Theorem 2.1 a) is the *rank* of v. If $g \in G_v$ is parabolic, we also say that k is the *rank* of g. Thus the rank of v or g depends also on G. The next lemma shows that elements of G of rank $k > 1$ can be characterized algebraically.

If H is a group containing a free abelian subgroup H_0 of finite index, we say that the *rank* k of H_0 is also the rank of H. Obviously, this does not depend on H_0.

Lemma **2.2.** — *Let* G *be a discrete Möbius group of* $\overline{\mathbf{R}}^n$. *Let* $g \in G$ *be of infinite order and let* $V \subset \overline{\mathbf{R}}^n$ *be the set of fixed points of* g, *consisting of one or two points. Then the stabilizer* G_V *can be characterized as the maximal subgroup of* G *containing* g *which has a free abelian subgroup of finite index.*

Let k *be the rank of* G_V. *Then* $k \geq 1$ *and if* $k > 1$, *then* g *is parabolic of rank* k. *If* $k = 1$, *then* g *is either loxodromic or parabolic of rank* 1.

Proof. — Obviously $g \in G_V$. If g is parabolic, Theorem 2.1 c) implies that G_V has a free abelian subgroup of finite index. If g is loxodromic, choose $v \in V$ and consider G_v. Theorem 2.1 implies that $G_v \subset G_V$. Now, every $h \in G_V$ either fixes the points of V or interchanges them. Thus G_v is of finite index in G_V and Theorem 2.1 then implies that G_V has a free abelian subgroup H_0 of finite index. Moreover, the rank of H_0 is now 1. Consequently G_V can have rank $k > 1$ only if g is parabolic and then k is the rank of g. Obviously always $k \geq 1$.

To show the maximality of G_V, let $H \ni g$ be another subgroup of G containing a free abelian subgroup of finite index. Now, if two Möbius transformations of infinite order commute, then they have the same fixed points. Thus every $h \in H$ of infinite order is in G_V. If $h \in H$, then hgh^{-1} is of infinite order and thus it fixes V. It follows that $h(V) = V$ for all $h \in H$ and consequently $H \subset G_V$. We have proved the maximality of G_V.

2C. Cusps. — If G is a geometrically finite Möbius group, it is possible to associate to parabolic fixed points of G certain open sets, called G-cusp neighbourhoods. Here we define these cusps as such, without reference to a Möbius group.

Let $0 \leq k < n$. Then an open set U of $\overline{\mathbf{R}}^n$ is a *cusp* (or a *k-cusp*) if there is a Möbius transformation α such that

$$\alpha(U) = \mathbf{R}^n \setminus (\mathbf{R}^k \times B^{n-k})$$

where $B^{n-k} = B^{n-k}(0, 1)$. The k-sphere $\alpha^{-1}(\overline{\mathbf{R}}^k)$ $(\overline{\mathbf{R}}^0 = \{0, \infty\})$, which lies in the complement of U, does not depend on α and is called the *center* of the cusp. Similarly, $\alpha^{-1}(\infty)$ does not depend on α and it is the *vertex* of the cusp. An open set $U \subset \overline{H}^{n+1}$ is a *k-cusp of* $\overline{H}^{n+1}$, $0 \leq k < n+1$, if $U = V \cap \overline{H}^{n+1}$ for some k-cusp V of $\overline{\mathbf{R}}^{n+1}$ whose center and vertex lie in $\overline{\mathbf{R}}^n$. The center and vertex of U are the ones of V.

Figure 1 shows a 1-cusp of $\overline{\mathbf{R}}^2$ with vertex $v \neq \infty$. The cusp consists of two components (shaded in the figure) and the circle arc in the unshaded part of the figure is a subset of the center of the cusp.

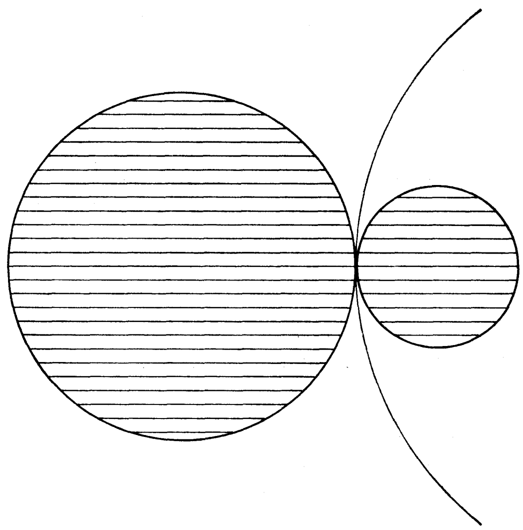

Fig. 1

We will need the fact that in an infinite collection of disjoint cusps the spherical diameters tend to zero. This is a consequence of

Lemma **2.3**. — a) *Let* $n \geq 1$ *and let* U *be a cusp of* $\overline{\mathbf{R}}^n$. *Then there is a ball* B *of* $\overline{\mathbf{R}}^n$ *such that* $B \subset U$ *and that* $q(U) \leq 2q(B)$ *in the spherical metric* q.

b) *Let* U_i *be a sequence of disjoint cusps of* $\overline{\mathbf{R}}^n$. *Then*

$$\sum_i q(U_i)^n < \infty.$$

Proof. — In case a) we can assume, by performing a spherical isometry, that U is of the form

$$U = \mathbf{R}^n \setminus (te_n + \mathbf{R}^k \times B^{n-k}(r))$$

for some $r > 0$ and $t \geq 0$.

Let $B \subset U$ be the open ball (in the spherical metric) such that ∞ and $(t-r)\, e_n$ are diagonal points of B. Suppose first that $t - r \leq 0$. Then

$$q(B) = q(\infty, (t-r)\, e_n).$$

Since $q(\infty, x)$ increases as $|x|$ decreases, $q(\infty, x) < q(\infty, (t-r)\, e_n)$ for all $x \in U$. Hence

$$q(U) \leq 2q(\infty, (t-r)\, e_n) = 2q(B).$$

Thus a) is true if $t - r \leq 0$. If $t - r \geq 0$, then it is also true since now

$$q(B) = q(U) = q(\overline{\mathbf{R}}^n).$$

Claim b) now follows from a) since there is $M \geq 1$ such that if B is a ball of $\overline{\mathbf{R}}^n$, then the spherical volume $V_q(B)$ of B satisfies $V_q(B)/q(B)^n \in [1/M, M]$.

Remark. — The lemma also holds for cusps of $\overline{H}^{n+1}$ if we replace the inequality of a) by $q(U) \leq 4q(B)$ where $B \subset U$ is some ball of $\overline{\mathbf{R}}^{n+1}$. The above reasoning is valid.

2D. Parabolic cusps. — Let G be a discrete Möbius group of $\overline{\mathbf{R}}^n$ and let v be a parabolic fixed point of G of rank k. Then a G-*cusp neighbourhood* of v in $\overline{H}^{n+1}$ is a k-cusp U of $\overline{H}^{n+1}$ such that $\mathrm{cl}\, U \cap L(G) = \{v\}$, U is G_v-invariant and that $g(U) \cap U = \varnothing$ for $g \in G \setminus G_v$. A G-cusp neighbourhood V of v in $\overline{\mathbf{R}}^n$ is defined similarly if $k < n$. Then U and V have vertex v and their center is a G_v-invariant k-sphere.

If G is fixed, we often say simply that U and V are *cusp neighbourhoods* of v. Always, when we speak of cusp neighbourhoods, we mean G-cusp neighbourhoods for some G.

If v is a parabolic fixed point of rank n, then cusp neighbourhoods of v in $\overline{\mathbf{R}}^n$ are not defined since we have not defined n-cusps of $\overline{\mathbf{R}}^n$ (these would be empty sets). However, cusp neighbourhoods in $\overline{H}^{n+1}$ are well-defined. In fact, a parabolic fixed point of rank n has always cusp neighbourhoods in $\overline{H}^{n+1}$, cf. Wielenberg [41, Proposition 4].

We consider cusp neighbourhoods also in the orbit spaces

$$M_G = (\overline{H}^{n+1} \setminus L(G))/G \quad \text{and} \quad \widetilde{M}_G = \Omega(G)/G = (\overline{\mathbf{R}}^n \setminus L(G))/G.$$

Then a cusp neighbourhood of the class of a parabolic fixed point v in M_G is U/G when U is a cusp neighbourhood of v in $\overline{H}^{n+1}$. The definition is similar for cusp neighbourhoods in $\widetilde{M}_G$; in $\widetilde{M}_G$ only classes of parabolic fixed points of rank $< n$ have cusp neighbourhoods.

The next theorem shows that if G is geometrically finite, every parabolic fixed point has cusp neighbourhoods. Furthermore, M_G and $\widetilde{M}_G$ become compact if we add to them the equivalence classes of parabolic fixed points of G; for $\widetilde{M}_G$ we add only the equivalence classes of parabolic fixed points of rank $< n$. These classes are finite in number and we let such a point Gv have as a basis of neighbourhoods sets of the form $(U \cup \{v\})/G$ or $(V \cup \{v\})/G$ when U and V run through the set of cusp neighbourhoods of v in $\overline{H}^{n+1}$ or in $\overline{\mathbf{R}}^n$, respectively. The compact space so obtained is called the *cusp-compactification* of M_G or $\widetilde{M}_G$.

Theorem **2.4.** — *Let* G *be a geometrically finite Möbius group of* $\overline{\mathbf{R}}^n$ *and let* D *be a finite-sided fundamental polyhedron for* G *in* H^{n+1} *such that* $g(D) \cap D \neq \varnothing$ *for only finitely many* $g \in G$. *Then*

a) *setting* $L_D = \mathrm{cl}\, D \cap L(G)$, L_D *is a finite set and every* $v \in L_D$ *is a parabolic fixed point of* G *which has a* G*-cusp neighbourhood in* $\overline{H}^{n+1}$;

b) *if* U_v *is a* G*-cusp neighbourhood in* $\overline{H}^{n+1}$ *of* v *for* $v \in L_D$, *then* $\mathrm{cl}\, D \setminus (L_D \cup (\cup\{U_v : v \in L_D\}))$ *is compact and does not intersect* $L(G)$; *furthermore, given compact* $X \subset \mathrm{cl}\, D \setminus L(G)$, *there is compact* Y *such that* $X \subset Y \subset \mathrm{cl}\, D \setminus L(G)$ *and that every component of* $(\overline{H}^{n+1} \setminus L(G)) \setminus GY$ *is a* G*-cusp neighbourhood (in* $\overline{H}^{n+1}$*) of some* $v \in GL_D$;

c) *every parabolic fixed point of* G *is conjugate to some* $v \in L_D$ *(and thus has a* G*-cusp neighbourhood).*

Proof. — This theorem is for the most part folklore on Möbius groups but we give some explanation.

We prove first that the set L_D of a) is finite. Let $v \in L_D$. Let F_1 be an i_1-face of D of minimal dimension i_1 such that $v \in \mathrm{cl}\, F_1$. Let $g_1(D), \ldots, g_r(D)$ be the elements of $\{g(D) : g \in G\}$ such that $v \in \mathrm{cl}(F_1 \cap g(D))$; their number is finite since $g(D) \cap D \neq \varnothing$ for only finitely many many $g \in G$.

There is $g_i(D)$ such that $g_i(D)$ has an i_2-face F_2 for which $v \in \mathrm{cl}\, F_2$ and $F_1 \cap F_2 = \varnothing$ since otherwise there would be a neighbourhood V of v such that $V \cap g(D) = \varnothing$ for $g \in G \setminus \{g_1, \ldots, g_r\}$. This is impossible since $v \in L(G)$. We can assume that i_2 is minimal for F_2 satisfying these conditions. Let H_j be the hyperbolic i_j-plane such that $F_j \subset H_j$. Then the minimality of i_1 and i_2 implies that there is a neighbourhood U of v such that

$$H_j \cap U = F_j \cap U.$$

This and the minimality of i_1 and i_2 again imply that $H_1 \cap H_2 = \varnothing$. This means that ∂H_1 and ∂H_2 are tangent to each other at $v \in \mathrm{cl}\, H_1 \cap \mathrm{cl}\, H_2$. Then

$$\mathrm{cl}\, H_1 \cap \mathrm{cl}\, H_2 = \mathrm{cl}\, F_1 \cap \mathrm{cl}\, F_2 = \{v\}. \tag{2.3}$$

Now the number of faces of D is finite. So is the number of $g \in G$ such that $g(D) \cap D \neq \varnothing$. In (2.3) F_1 is a face of D and F_2 is a face of some $g(D)$ such that $g(D) \cap D \neq \varnothing$. It follows that L_D is finite.

Let $X = \{g \in G : v \in \mathrm{cl}\, g(D)\}$. If X is finite, then v has a neighbourhood V such that $V \cap g(D) = \varnothing$ for $g \in G \backslash X$ which is impossible since $v \in L(G)$. Since we now know that L_D is finite, this fact implies that the stabilizer G_v of v is infinite. Thus G must contain a loxodromic or parabolic element g fixing v, cf. Theorem 2.1.

However, g cannot be loxodromic. To see this, let L be the hyperbolic line joining the points fixed by g in case g is loxodromic. Then every neighbourhood of every $y \in L$ intersects infinitely many $g^k(D)$, $k \in \mathbf{Z}$. This is impossible and thus v is a parabolic fixed point of G.

We then show that v has cusp neighbourhoods. We can assume that $v = \infty$. Let $v_0 = v, v_1, \ldots, v_r$ be the points of L_D conjugate to v, $v = g_i(v_i)$ for some $g_i \in G$ $(g_0 = \mathrm{id})$ and set

$$\begin{aligned}\mathscr{F} &= \{\mathrm{cl}\, gg_i(D) \backslash \{v\} :\ g \in G_v, i \leq r\} \\ &= \{\mathrm{cl}\, g(D) \backslash \{v\} :\ v \in \mathrm{cl}\, g(D), g \in G\}.\end{aligned}$$

We first prove:

(∗) *The family $\mathscr{F}$ is locally finite in $\overline{H}^{n+1} \backslash \{v\}$ and there is $N > 0$ such that if $D' \in \mathscr{F}$, $D' \cap D'' \neq \varnothing$ for at most N sets $D'' \in \mathscr{F}$.*

To prove (∗), note first that $\mathscr{F}$ is locally finite in H^{n+1} by the definition of a fundamental polyhedron. We prove that it is also locally finite in $\overline{\mathbf{R}}^n \backslash \{v\} = \mathbf{R}^n$. The finite-sidedness of $g_i(D)$ implies that there is some (small) $u > 0$ such that if $y \in \mathbf{R}^n \cap \mathrm{cl}\, gg_i(D)$, $g \in G_v$, $i \leq r$, then there is $y' \in B^n(y, u)$ such that $(y', u) \in \mathrm{int}\, gg_i(D)$ (remember that elements of G_v are euclidean isometries of $\mathbf{R}^n$). Thus, if $\mathscr{F}$ were not locally finite at $y \in \mathbf{R}^n$, we could find a point $y'' \in B^{n+1}(y, 2u) \cap H^{n+1}$ such that $\mathscr{F}$ is not locally finite at y''. Since $\mathscr{F}$ is locally finite at $y \in H^{n+1}$, as we observed above, this is a contradiction and $\mathscr{F}$ is locally finite in $\overline{H}^{n+1} \backslash \{v\}$.

By assumption there is $N > 0$ such that $g(D) \cap D \neq \varnothing$ for at most N elements $g \in G$. If $D', D'' \in \mathscr{F}$ and $D' \cap D'' \neq \varnothing$, then $D' \cap D'' \cap H^{n+1} \neq \varnothing$ by finite-sidedness. Hence this N is valid for (∗) as well. This concludes the proof of (∗).

We set

$$D_v = \cup\, \mathscr{F} \backslash L(G) = \cup \{\mathrm{cl}\, gg_i(D) \backslash L(G) :\ g \in G_v, i \leq r\}$$

and show that D_v contains a cusp neighbourhood of v.

Since D is finite-sided and $g(\mathrm{D}) \cap \mathrm{D} \neq \varnothing$ for only finitely many $g \in \mathrm{G}$, there is $t_0 > 0$ such that if $(x, t) \in \mathbf{R}^n \times [t_0, \infty)$ and $(x, t) \in g_i(\mathrm{D})$ for some $i \leq r$, then

$$\{x\} \times [t_0, \infty) \subset h(\mathrm{D})$$

for all such $h(\mathrm{D})$, $h \in \mathrm{G}$, for which $(x, t) \in h(\mathrm{D})$. It follows that if $h(\mathrm{D})$, $h \in \mathrm{G}$, touches some $gg_i(\mathrm{D})$, $g \in \mathrm{G}_v$ and $i \leq r$, at a point $(x, t) \in \mathbf{R}^n \times [t_0, \infty)$, then $v \in \mathrm{cl}\, h(\mathrm{D})$. Hence $h(\mathrm{D}) = h' g_j(\mathrm{D})$ for some $h' \in \mathrm{G}_v$ and $j \leq r$. One now sees that

(2.4) $$(x, t) \in \mathrm{D}_v$$

if $x \in \mathbf{R}^n$ and $t \geq t_0$.

Let k be the rank of v. Then we can assume that $\mathbf{R}^k$ is G_v-invariant and that every $g \in \mathrm{G}_v$ has the representation (2.1). Let F_j, $j \leq q$, be the faces of the polyhedra $g_i(\mathrm{D})$, $i \leq r$, such that, if p_j is the dimension of F_j, then $p_j > 0$ and F_j is not contained in a euclidean p_j-plane. Thus there are euclidean p_j-balls $\mathrm{B}_j \subset \mathbf{R}^{n+1}$ such that $\mathrm{F}_i \subset \partial \mathrm{B}_j$. There is $\mathrm{M} > 0$ such that if $(x, t) \in \mathrm{B}_j$ for some j, $x \in \mathbf{R}^n$ and $t \geq 0$, then

(2.5) $$d(x, \mathbf{R}^k) \leq \mathrm{M}.$$

In addition, we can assume that M is so big that (2.5) is true for all

$$x \in \{g_i(v) : i \leq r, v \in \mathrm{L_D}\} \cap \mathbf{R}^n$$

which is a finite set.

We now claim that if $y \in \mathbf{R}^n$ and $d(y, \mathbf{R}^k) > \mathrm{M}$ then

(2.6) $$(y, t) \in \mathrm{D}_v$$

for all $t \in [0, \infty)$. To prove (2.6), let $t' \in [0, \infty)$ be the minimal number such that $\{y\} \times (t', \infty) \subset \mathrm{D}_v$. By (2.4) there is such a t'. If $t' > 0$, then $(y, t') \in g(\partial \mathrm{B}_j)$ for some $g \in \mathrm{G}_v$ and $j \leq q$. We can assume by (2.1) that $g = \mathrm{id}$. Then $d(y, \mathbf{R}^k) \leq \mathrm{M}$ which is a contradiction. Thus $t' = 0$ and $(y, t) \in \mathrm{D}_v$ for $t > 0$.

We show that $y \in \mathrm{D}_v$, too. By (*) $\mathscr{F}$ is locally finite at y. Hence

$$y \in \mathrm{cl}\, gg_i(\mathrm{D}) \setminus \mathrm{L(G)}$$

for some $g \in \mathrm{G}_v$, $i \leq r$. If $y \notin \mathrm{D}_v$, then $y \in gg_i(\mathrm{L_D})$. As above, we can assume that $g = \mathrm{id}$ and we get again a contradiction by (2.5).

If $t > 0$, let

$$\mathrm{U}_t = \overline{\mathrm{H}}^{n+1} \setminus (\mathbf{R}^k \times \mathrm{B}^{n+1-k}(t) \cup \{\infty\}).$$

Then U_t is a G_v-invariant k-cusp. If t is big enough,

$$\mathrm{cl}\, \mathrm{U}_t \subset \mathrm{D}_v \cup \{v\} \subset (\overline{\mathrm{H}}^{n+1} \setminus \mathrm{L(G)}) \cup \{v\}$$

by (2.4) and (2.6). Also, for big t,

$$\mathrm{U}_t \cap g_i(\mathrm{U}_t) = \varnothing$$

for $0 < i \leq r$. This implies that $g(U_t) \cap U_t = \varnothing$ for $g \in G \backslash G_v$. Thus U_t is a cusp neighbourhood of v for big t and we have proved a).

To prove b), it suffices to show that if $v \in L_D$ and if U is a cusp neighbourhood of v, then v has an ordinary neighbourhood V in $\overline{H}^{n+1}$ such that

$$(\mathrm{cl}\, D \backslash (U \cup \{v\})) \cap V = \varnothing.$$

To see this, let $M_v = (\overline{H}^{n+1} \backslash (U \cup \{v\}))/G_v$ and note that M_v is compact. This follows by (2.1) since in (2.1) M_v corresponds to a set of the form

$$\overline{H}^{n+1} \cap (\mathbf{R}^k \times B^{n+1-k})/G$$

which is compact since $\mathbf{R}^k/G$ is.

Consider the projection p: $\overline{H}^{n+1} \backslash \{v\} \to (\overline{H}^{n+1} \backslash \{v\})/G_v \supset M_v$. Let D′ be the image of $\mathrm{cl}\, D \backslash (U \cup \{v\})$ in this projection. Since $g(\mathrm{cl}\, D)$, $g \in G_v$, is locally finite in $\overline{H}^{n+1} \backslash \{v\}$ by (*), $D' \subset M_v$ is closed and hence compact. By (*), each $p^{-1}(x) \cap D$, $x \in D'$, contains at most N points for some $N > 0$. It follows that $\mathrm{cl}\, D \backslash (U \backslash \{v\})$ is compact. Hence there is such a neighbourhood V as claimed and b) is proved.

Finally, to prove c), observe that if v is a parabolic fixed point of G and if $g \in G_v$ is parabolic, then there is $x \in H^{n+1}$ such that the hyperbolic distance of x and $g(x)$ is arbitrarily small. It follows that g can be obtained by lifting a loop J outside any given compact set X of the orbit space $(\overline{H}^{n+1} \backslash L(G))/G$. If X is sufficiently big, then J must be contained in a cusp neighbourhood of M_G corresponding to a point of GL_D, cf. case b). Then $g \in G_{h(v')}$ for some $v' \in L_D$ and $h \in G$, proving c).

Theorem 2.4 has the following immediate

Corollary **2.5**. — *A geometrically finite group* G *of* $\overline{\mathbf{R}}^n$ *is of compact type if and only if* G *does not have parabolic elements.*

2E. The convex hull of L(G). — In this section we study the hyperbolic convex hull H_G of the limit set L(G) (see (1.5)) near a parabolic fixed point of G.

We first prove the following lemma in which, for $X \subset \mathbf{R}^n$,

$$\mathrm{Co}_{\mathrm{euc}}(X)$$

is the euclidean convex hull of X, *i.e.* the smallest convex set of $\mathbf{R}^n$ containing X. The hyperbolic convex hull Co(X) was defined in (1.4).

Lemma **2.6**. — a) *Let* $X \subset \mathbf{R}^n$ *be finite. Then there is* $m > 0$ *such that*

$$\mathrm{Co}_{\mathrm{euc}}(X) \times [m, \infty) \subset \mathrm{Co}(X \cup \{\infty\}).$$

b) *Let* β *be an orthogonal map of* $\mathbf{R}^k$ *and assume that* $\beta(x) \neq x$ *for all* $x \in \mathbf{R}^k \backslash \{0\}$. *Then there are integers* n_i *and numbers* $\lambda_i > 0$ *for* $0 \leq i \leq q$ *with* $\sum_i \lambda_i = 1$ *such that*

$$\sum_i \lambda_i \beta^{n_i}(x) = 0$$

for all $x \in \mathbf{R}^n$.

Proof of a). — We can assume that $\mathbf{R}^n$ is generated affinely by X. Then

$$\mathrm{Co}_{\mathrm{euc}}(\mathrm{X}) = \cup\{\Delta : \Delta \text{ a non-degenerate } n\text{-simplex with vertices in X}\}.$$

Since the number of n-simplexes Δ above is finite, we can assume that $\mathrm{X} = \{x_0, \ldots, x_n\}$ where x_i are the vertices of a non-degenerate simplex. Then $\mathrm{Co}(\mathrm{X} \cup \{\infty\})$ is the non-euclidean simplex with vertices $x_0, \ldots, x_n$ and ∞ and a) follows.

In b), let $\mathbf{R}^n = \mathrm{V}_1 + \ldots + \mathrm{V}_q$ where V_i are one- or two-dimensional subspaces orthogonal to each other and such that $\beta(\mathrm{V}_i) = \mathrm{V}_i$. If V_i is one-dimensional, then $\beta(x) = -x$ for $x \in \mathrm{V}_i$ and if V_i is two-dimensional, $\beta \mid \mathrm{V}_i \neq \mathrm{id}$ and is a rotation through the angle b_i.

Let $x = x_1 + \ldots + x_q$, $x_i \in \mathrm{V}_i$. If V_1 is one-dimensional or if b_1/π is rational, then

$$0 = (\beta(x_1) + \beta^2(x_1) + \ldots + \beta^k(x_1))/k$$

when k is the period of β. If b_1/π is irrational, then there are integers p_i and numbers $\mu_i > 0$ for $i \leq 3$ with $\mu_1 + \mu_2 + \mu_3 = 1$ such that, regardless of x_1,

$$0 = \mu_1 \beta^{p_1}(x_1) + \mu_2 \beta^{p_2}(x_1) + \mu_3 \beta^{p_3}(x_1).$$

Thus always

$$0 = \sum_i \nu_i \beta^{q_i}(x_1)$$

for some q_i and $\nu_i > 0$, $\Sigma \nu_i = 1$, which do not depend on x_1.

If we replace x by $\sum_i \nu_i \beta^{q_i}(x)$, then $x_1 = 0$. Repeating this process, we obtain $x_2 = 0$. After q steps we obtain b).

Theorem **2.7**. — *Let* G *be a discrete, non-elementary Möbius group of* $\overline{\mathbf{R}}^n$ *and let* v *be a parabolic fixed point of rank* k *of* G. *Then there is a Möbius transformation* α *of* $\overline{\mathbf{R}}^n$ *such that* $\alpha(v) = \infty$, *that* $\mathbf{R}^k$ *is* $\alpha \mathrm{G}_v \alpha^{-1}$*-invariant and that*

$$\mathbf{R}^k \times \{0\} \times [1, \infty) \subset \alpha(\mathrm{H}_{\mathrm{G}}) \tag{2.7}$$

(here $0 = (0, \ldots, 0) \in \mathbf{R}^{n-k}$*). If, in addition,* v *has a* G*-cusp neighbourhood, we can assume that*

$$\alpha(\mathrm{H}_{\mathrm{G}}) \subset \mathbf{R}^k \times \mathrm{B}^{n-k} \times (0, \infty). \tag{2.8}$$

Proof. — We assume that $v = \infty$, $\mathbf{R}^k$ is G_v-invariant and $\mathbf{R}^k/\mathrm{G}_v$ is compact. Let $\mathrm{H} \subset \mathrm{G}_v$ be a subgroup of finite index such that $\mathbf{R}^{n-k}$ can be decomposed as $\mathbf{R}^{n-k} = \mathrm{W} + \mathrm{W}_1 + \ldots + \mathrm{W}_p$ in such a way that then (2.2) is true for $h \in \mathrm{H}$. Pick $x \in \mathrm{L}(\mathrm{G}) \cap \mathbf{R}^n$. Applying now Lemma 2.6 b), possibly more than once, we find elements $h_1, \ldots, h_p \in \mathrm{H}$ such that there is

$$y = u + w \in \mathrm{Co}_{\mathrm{euc}}(\{h_1(x), \ldots, h_p(x)\}) \tag{2.9}$$

where $u \in \mathbf{R}^k$ and $w \in W$. Now $y + \mathbf{R}^k$ is H-invariant, but it need not be G_v-invariant. However, if $g_1, \ldots, g_k$ are representativies from the cosets G_v/H and if

$$y' = (g_1(y) + \ldots + g_k(y))/k,$$

then $y' + \mathbf{R}^k$ is G_v-invariant.

It follows that, possibly by changing the origin, we can assume that there are points $x_1, \ldots, x_q \in L(G) \cap \mathbf{R}^n$ such that

$$\mathbf{R}^k \cap \mathrm{Co}_{\mathrm{euc}}(\{x_1, \ldots, x_q\}) \neq \varnothing. \tag{2.10}$$

Now $\mathbf{R}^k/G_v$ is compact. Using this fact and (2.10), we next find points $y_1, \ldots, y_r \in L(G) \cap \mathbf{R}^n$ such that

$$G_v(\mathbf{R}^k \cap \mathrm{Co}_{\mathrm{euc}}(\{y_1, \ldots, y_r\})) = \mathbf{R}^k, \tag{2.11}$$

and, in view of Lemma 2.6 a), (2.7) follows.

Since (2.8) is trivial, the theorem is proved.

Remark. — Unless $\beta(L(G)) \subset \overline{\mathbf{R}}^m$ for some $m < n$ and some Möbius transformation β, we can choose α in such a way that instead of (2.7) we have the stronger inclusion

$$\mathbf{R}^k \times B^{n-k} \times [1, \infty) \subset \alpha(H_G). \tag{2.12}$$

To see this, choose first in the above proof a non-degenerate n-simplex Δ with vertices in $L(G) \cap \mathbf{R}^n$ and pick $x \in \operatorname{int} \Delta$. Replacing the number x in (2.9) with this x and arguing as above, we get the result since then (2.11) can be strengthened to

$$G_v(\mathbf{R}^k \times B^{n-k}(r) \cap \mathrm{Co}_{\mathrm{euc}}(\{y_1, \ldots, y_r\})) = \mathbf{R}^k \times B^{n-k}(r)$$

for some $r > 0$.

2F. A convergence theorem. — We now prove a theorem which we need later when we, in Theorem 3.8, extend a quasiconformal map $\Omega(G) \to \Omega(G')$ of ordinary sets to the limit set in case there are parabolic elements in the groups.

We first prove the following simple lemma.

Lemma **2.8.** — *Let $a_1, \ldots, a_r \in \mathbf{R}/2\pi\mathbf{Z}$ and let V be a neighbourhood of 0 in $\mathbf{R}/2\pi\mathbf{Z}$. Then there is an integer $N \geq 0$ such that for every integer p there is an integer q with $|p - q| \leq N$ for which $qa_i \in V$ if $i \leq r$.*

Proof. — By compactness of $(\mathbf{R}/2\pi\mathbf{Z})^r$ we can find integers $q_1, \ldots, q_s$ such that for every integer p there is q_j such that $(p - q_j)\, a_i \in V$ for all $i \leq r$. Let $N = \max_{j \leq s} |q_j|$. Then the lemma is true with this N since $|p - (p - q_j)| = |q_j| \leq N$.

Lemma **2.9.** — *Let $n > 1$ and $0 < k < n$. Let G_1 and G_2 be discrete groups of Möbius transformations of $\overline{\mathbf{R}}^n$ such that every $g \in G_i$ fixes ∞ and that there are parabolic $g \in G_i$, $i = 1, 2$. Assume further that $\mathbf{R}^k$ is G_i-invariant and that $\mathbf{R}^k/G_i$ is compact. Let f be a homeomorphism* *of* $\mathbf{R}^n$ *such that* f *induces an isomorphism* $\varphi : G_1 \to G_2$ *and that* f *is quasiconformal in a* G_1*-cusp neighbourhood of* ∞. *Then there are* $M \geq 1$, $r > 0$ *and an affine map* α *of* $\mathbf{R}^k$ *such that for every* $(x, y) \in \mathbf{R}^k \times \mathbf{R}^{n-k}$

$$(2.13) \qquad |f(x,y) - \alpha(x)| \leq M$$

if $|y| \leq r$, *and if* $|y| \geq r$, *then*

$$(2.14) \qquad |y|/M \leq d(f(x,y), \mathbf{R}^k) \leq |f(x,y) - \alpha(x)| \leq |y| M.$$

Proof. — We can assume by Theorem 2.1 that f is quasiconformal in a set

$$U = \mathbf{R}^n \setminus (\mathbf{R}^k \times B^{n-k}(r'))$$

for some $r' > 0$. Observe that sets of this form are always G_i-invariant by (2.1 since $\mathbf{R}^k$ is.

By Theorem 2.1 c), there are subgroups $H_i \subset G_i$ of finite index such that $\varphi(H_1) = H_2$ and that H_i restricted to $\mathbf{R}^k$ is a free abelian group of rank k of translations of $\mathbf{R}^k$. Then there is a uniquely determined affine map α of $\mathbf{R}^k$ such that

$$(2.15) \qquad \alpha(x) = f(x)$$

for $x \in H_1(0)$. Thus α induces $\varphi \mid H_1$ (when H_1 and H_2 are regarded as acting in $\mathbf{R}^k$).

We show that α is the required map. Let $B_r = \mathbf{R}^k \times B^{n-k}(r)$ for $r \geq 0$ and let $f_r : B_r \to \mathbf{R}$ be the map $f_r(x,y) = |f(x,y) - \alpha(x)|$. Now B_r is H_1-invariant and B_r/H_1 is compact by (2.1). Thus the H_1-invariant map f_r attains a maximal value M'_r. Then (2.13) is true for $M = \max(1, M'_r)$ if $|y| \leq r$. We see that it suffices to find M and r for which (2.14) is true.

We then prove (2.14). Let

$$M_r = \sup\{d(f(z), \mathbf{R}^k) : z \in \mathbf{R}^n, d(z, \mathbf{R}^k) = r\},$$

and

$$m_r = \inf\{d(f(z), \mathbf{R}^k) : z \in \mathbf{R}^n, d(z, \mathbf{R}^k) = r\}$$

for $r > 0$. By compactness of $\mathbf{R}^k \times \partial B^{n-k}(r)/G_1$, both are finite and tend to ∞ as $r \to \infty$. In particular, for big r, approximately

$$d(z, \mathbf{R}^k) \approx d(z, \partial U) \quad \text{and} \quad d(f(z), \mathbf{R}^k) \approx d(f(z), \partial f(U)).$$

This fact and Väisälä [40, 18.1] applied to $f \mid U$ and $f^{-1} \mid f(U)$ imply that there are positive numbers $r_0 \geq r'$, a_1, b_1, a_2, b_2 such that $0 < a_i < b_i \leq 1/2$ and that if $w, z \in U$ and $d(z, \mathbf{R}^k) \geq r_0$, then

$$(2.16) \qquad |w - z|/d(z, \mathbf{R}^k) \in [a_1, b_1] \quad \text{implies}$$

$$|f(w) - f(z)|/d(f(z), \mathbf{R}^k) \in [a_2, b_2].$$

Pick now $h_1 \in H_1 \setminus \{\text{id}\}$ and let $h_2 = \varphi(h_1)$. These are of the form

$$(2.17) \qquad h_i(x,y) = (x + u_i, \beta_i(y)),$$

$(x,y) \in \mathbf{R}^k \times \mathbf{R}^{n-k}$, where $u_i \in \mathbf{R}^k \setminus \{0\}$ and $\beta_i \in O(n-k)$, cf. (2.2). Let h'_i be the map defined by the right-hand side of (2.17) when $\beta_i = \mathrm{id}$, that is, h'_i is the translation

$$h'_i(z) = z + u_i,$$

$z \in \mathbf{R}^n$. Lemma 2.8 implies that there is an integer $N > 0$ such that for every integer p there is an integer q with $|p - q| \leq N$ for which

(2.18) $$|h_i'^q(z) - h_i^q(z)| \leq \min((b_1 - a_1)/3, a_2/2)\, d(z, \mathbf{R}^k)$$

if $z \in \mathbf{R}^n$, $i = 1, 2$.

Next we find $r_1 > r_0$ such that for every $z \in \mathbf{R}^n$ with $d(z, \mathbf{R}^k) \geq r_1$ there is an integer q for which

(2.19) $$|h_1'^q(z) - z|/d(z, \mathbf{R}^k) \in [a_1, b_1] \quad \text{and} \quad |h_1^q(z) - z|/d(z, \mathbf{R}^k) \in [a_1, b_1]$$

and for which (2.18) is true. The first inequality is obvious and the second follows from the first using (2.18).

Let $z' = f(z)$. Then $f(h_1^q(z)) = h_2^q(z')$. Now (2.16) and (2.19) imply

(2.20) $$|h_2^q(z') - z'|/d(z', \mathbf{R}^k) \in [a_2, b_2].$$

This and (2.18) imply

(2.21) $$|h_2'^q(z') - z'|/d(z', \mathbf{R}^k) \in [a_2/2, 2b_2].$$

Let $L = |u_2|/|u_1| > 0$; we have

(2.22) $$|h_2'^q(z') - z'| = L\, |h_1'^q(z) - z|$$

regardless of q.

We now apply these inequalities. Relations (2.19), (2.21) and (2.22) imply that, if $d(z, \mathbf{R}^k) \geq r_1$,

(2.23) $$d(z', \mathbf{R}^k)/d(z, \mathbf{R}^k) \in [a_1 L/2b_2, 2b_1 L/a_2]$$

and thus the leftmost inequality of (2.14) is true for $|y| \geq r_1$ if $1/M \leq \min(1, a_1 L/2b_2)$. The middle inequality is trivial and the rightmost inequality is proved as follows.

If $|y| = d(z, \mathbf{R}^k) \geq 2r_1$, define a sequence $z_0 = z, \ldots, z_m$ by setting $z_i = (x, y/2^i)$; the number m is defined by the requirement that $|y|/2^m \in [r_1, 2r_1)$. Then (2.23) is true for $z = z_i$ and $z' = f(z_i) = z'_i$. Now, for big r_1, approximatively

$$d(z_i, \mathbf{R}^k) \approx d(z_i, \partial U) \quad \text{and} \quad d(z'_i, \mathbf{R}^k) \approx d(z'_i, \partial f(U)).$$

Then Väisälä [40, 18.1] implies the existence of $H > 0$, depending only on K and n, such that for $i < m$

(2.24) $$|z'_{i+1} - z'_i| \leq H d(z'_i, \mathbf{R}^k)$$

since $|z_{i+1} - z_i| = d(z_i, \mathbf{R}^k)/2$. We can assume that r_1 is so big that (2.24) is true. Let $M_1 = 2b_2/La_1$. Then $d(z'_i, \mathbf{R}^k) \leq M_1 d(z_i, \mathbf{R}^k)$ by (2.23) and then (2.24) implies

$$\begin{aligned} |z'_m - z'_0| &\leq HM_1(d(z_{m-1}, \mathbf{R}^k) + \ldots + d(z_0, \mathbf{R}^k)) \\ &= HM_1 d(z, \mathbf{R}^k)(1 + 1/2 + \ldots + 1/2^{m-1}) \\ &< 2HM_1 d(z, \mathbf{R}^k). \end{aligned}$$

Now $d(z_m, \mathbf{R}^k) < 2r_1$ and then, as we observed in the proof of (2.13), $|z'_m - \alpha(x)| \leq M_2$ for some $M_2 = M_2(2r_1)$. Thus

$$|z'_0 - \alpha(x)| = |f(z) - \alpha(x)| \leq 2HM_1 d(z, \mathbf{R}^k) + M_2$$

and we finally get that the rightmost inequality of (2.14) is true for $M = 2HM_1 + M_2/r_1$ if $|y| \geq r_1$. This proves the lemma.

As a consequence we obtain the following uniform continuity property for f.

Lemma **2.10.** — *Let the situation be as in Lemma* 2.9. *Then, given* $r' > 0$, *there are* $M' \geq 1$ *and homeomorphisms* $\mu, \mu' : [0, \infty) \to [0, \infty)$ *such that for every* $x, y \in \mathbf{R}^n$ *with* $d(x, \mathbf{R}^k) \leq r'$

$$\mu'(|x-y|) \leq |f(x) - f(y)| \leq \mu(|x-y|). \tag{2.25}$$

If, in addition, $|x-y| \geq r'$, *then*

$$|x-y|/M' \leq |f(x) - f(y)| \leq M'|x-y|. \tag{2.26}$$

Proof. — Inequalities (2.25) follow from the facts that $\max_{|e|=t} |f(x) - f(x+e)|$ and $\min_{|e|=t} |f(x) - f(x+e)|$ are G_1-invariant and that $\mathbf{R}^k \times B^{n-k}(s)/G_1$ is compact for every $t, s \geq 0$. Inequalities (2.26) follow from (2.13) and (2.14) for big $|x-y|$ and can be extended for all $|x-y| \geq r'$ by (2.25), possibly with bigger M'.

Our efforts in Section 2F have had as aim to prove

Theorem **2.11.** — *Let* $n > 1$ *and* $0 < k < n$. *Let* $U, V \subset \overline{\mathbf{R}}^n$ *be k-cusps with center* S_1 *and vertex* u_1 *such that* $V \subset U \neq V$. *Let* S_2 *be a k-sphere of* $\overline{\mathbf{R}}^n$ *and let* $u_2 \in S_2$. *For* $i = 1, 2$ *let* G_i *be a discrete group of Möbius transformations of* $\overline{\mathbf{R}}^n$ *such that* $g(u_i) = u_i$ *and* $g(S_i) = S_i$ *for every* $g \in G_i$, *that there is a parabolic* $g \in G_i$, *and that* $(S_i \setminus \{u_i\})/G_i$ *is compact. Let* $\varphi : G_1 \to G_2$ *be an isomorphism and let f be a homeomorphism of* $\overline{\mathbf{R}}^n$ *inducing* φ *such that* $f | U$ *is quasiconformal.*

Let then g_{ij}, $i = 1, 2$, $j > 0$, *be Möbius transformations of* $\overline{\mathbf{R}}^n$ *and set* $f_j = g_{2j}^{-1} f g_{1j}$, $U_j = g_{1j}^{-1}(U)$ *and* $V_j = g_{1j}^{-1}(V)$. *Assume that there are* $s > 0$ *and points* $a_{1j}, a_{2j} \in \overline{\mathbf{R}}^n \setminus U_j$ *and* $a_{3j} \in \overline{\mathbf{R}}^n$ *such that, setting* $b_{ij} = f_j(a_{ij})$, *we have*

$$q(a_{ij}, a_{kj}) \geq s \quad \text{and} \quad q(b_{ij}, b_{kj}) \geq s \tag{2.27}$$

for all j and distinct i and k in the spherical metric q.

Then by passing to a subsequence we obtain that as $j \to \infty$ *either*

a) $q(V_j) \to 0$ *and* $q(f_j(V_j)) \to 0$, *or*

b) *there is a homeomorphism* h *of* $\overline{\mathbf{R}}^n$ *such that* $f_j \to h$ *uniformly in the spherical metric.*

Proof. — In the following, uniform convergence means uniform convergence in the spherical metric. We also pass several times to subsequences which are denoted in the same manner as the original sequences.

By passing to a subsequence, we obtain that

$$a_{ij} \to a_i \in \overline{\mathbf{R}}^n \quad \text{and} \quad b_{ij} \to b_i \in \overline{\mathbf{R}}^n$$

as $j \to \infty$. By (2.27), the points a_i are distinct and so are b_i. It is convenient to assume that

$$(2.28) \qquad a_{ij} \neq \infty \neq b_{ij}$$

which can always be obtained by slightly changing the points a_{ij} (and by slightly decreasing s). However, it still may be that $a_i = \infty$ or $b_i = \infty$.

We can assume that $u_1 = u_2 = \infty$, $S_1 = S_2 = \overline{\mathbf{R}}^k$ and that $\overline{\mathbf{R}}^n \setminus U = \mathbf{R}^k \times B^{n-k}$ which means that we have the situation of Lemma 2.9. Furthermore, there are Möbius transformations h_{ij} and h_i such that $h_{ij} \to h_i$ uniformly as $j \to \infty$ and that $h_{ij} g_{ij}(\infty) = \infty$. Then, if we replace g_{ij} by $h_{ij} g_{ij}$, we can assume that $g_{ij}(\infty) = \infty$ for all i, j. Observe that now all maps f, f_j and g_{ij} fix ∞. Obviously, (2.28) can be assumed to be still valid.

Thus $g_{ij} \mid \mathbf{R}^n$ is a similarity. Hence there are numbers $c_{ij} > 0$ such that

$$|g_{ij}(x) - g_{ij}(y)| = c_{ij} |x - y|$$

for all $x, y \in \mathbf{R}^n$. By passing to a subsequence we obtain that

$$(2.29) \qquad c_{ij} \to c_i \in [0, \infty]$$

as $j \to \infty$. We shall show that one of the following cases occurs:

(i) $\quad c_1 = c_2 = 0$,

(ii) $\quad 0 < c_1 < \infty$ and $0 < c_2 < \infty$,

(iii) $\quad c_1 = c_2 = \infty$.

The proof is based on the fact that by (2.25) there are homeomorphisms μ' and μ of $[0, \infty)$ such that

$$(2.30) \qquad \mu'(c_{1j} |a_{ij} - a_{kj}|)/c_{2j} \leq |b_{ij} - b_{kj}| \leq \mu(c_{1j} |a_{ij} - a_{kj}|)/c_{2j};$$

we can apply (2.25) since $a_{1j}, a_{2j} \in \overline{\mathbf{R}}^n \setminus U_j$.

It follows by (2.27) that

$$(2.31) \qquad |a_{ij} - a_{kj}| \geq s \quad \text{and} \quad |b_{ij} - b_{kj}| \geq s$$

if $i \neq k$ since $|x-y| \geq q(x,y)$. On the other hand, (2.27) also implies that there is $s' < \infty$ such that for every j we can find indices $i, k, i', k' \in \{1, 2, 3\}$, $i \neq k$, $i' \neq k'$, such that

$$(2.32) \qquad |a_{ij} - a_{kj}| \leq s' \quad \text{and} \quad |b_{i'j} - b_{k'j}| \leq s'.$$

If $c_1 = 0$ and $c_2 > 0$, then (2.30)-(2.32) would imply that for every big j we could find distinct $i, k \in \{1, 2, 3\}$ such that

$$(2.33) \qquad |b_{ij} - b_{kj}| < s$$

which is a contradiction by (2.31). Thus $c_1 = 0$ implies $c_2 = 0$. Similarly, $c_2 = 0$ implies $c_1 = 0$. As easily one sees that $c_1 = \infty$ if and only if $c_2 = \infty$. Thus indeed one of the cases (i)-(iii) always occurs.

We now consider separately cases (i)-(iii) and show that the conclusion of the theorem is valid for each of them.

In case (i), we first fix $k \in \{1, 2, 3\}$ such that

$$(2.34) \qquad a_k \neq \infty \neq b_k.$$

By passing to a subsequence, we can find $t > 0$ such that for all j either

(α) $\qquad d(g_{1j}(a_{kj}), \mathbf{R}^n \setminus \mathrm{U}) \geq t$, or

(β) $\qquad d(g_{1j}(a_{kj}), \mathrm{V}) \geq t$.

If (α) is true, then $\operatorname{int} \mathrm{B}^n(g_{1j}(a_{kj}), t) \cap (\overline{\mathbf{R}}^n \setminus \mathrm{U}) = \varnothing$ and hence

$$\operatorname{int} \mathrm{B}^n(a_{kj}, t/c_{1j}) \cap (\overline{\mathbf{R}}^n \setminus \mathrm{U}_j) = \varnothing.$$

Letting $j \to \infty$, $a_{kj} \to a_k \neq \infty$ and $t/c_{1j} \to \infty$. Now $a_{1j}, a_{2j} \in \overline{\mathbf{R}}^n \setminus \mathrm{U}_j$. Hence $k = 3$ and $a_{1j}, a_{2j} \notin \operatorname{int} \mathrm{B}^n(a_{kj}, t/c_{1j})$. It would follow that $q(a_{1j}, a_{2j}) \to 0$ as $j \to \infty$, a contradiction with (2.27). Hence (α) is impossible.

If (β) is true, then similarly

$$\operatorname{int} \mathrm{B}^n(a_{kj}, t/c_{1j}) \cap \mathrm{V}_j = \varnothing$$

and reasoning as above we get that

$$q(\mathrm{V}_j) \to 0$$

as $j \to \infty$. Now, (2.25) implies the existence of $t' > 0$ such that

$$\operatorname{int} \mathrm{B}^n(f(g_{1j}(a_{kj})), t') \cap f(\mathrm{V}) = \varnothing$$

for all j. Remember that $f_j = g_{2j}^{-1} f g_{1j}$ and hence

$$\operatorname{int} \mathrm{B}^n(b_{kj}, t'/c_{2j}) \cap f_j(\mathrm{V}_j) = \varnothing.$$

If now $j \to \infty$, this again implies that

$$q(f_j(\mathrm{V}_j)) \to 0.$$

Consequently, if (i) is true, the conclusion a) of our theorem is valid.

Next we consider case (ii). First, choose k as in (2.34). There is $m \neq k$ such that also $a_m \neq \infty$. Hence the distances $|a_{kj} - a_{mj}|$ are bounded. Then (ii) and (2.30) imply that also the distances $|b_{kj} - b_{mj}|$ are bounded and thus also $b_m \neq \infty$. Since $\{k, m\} \cap \{1, 2\} \neq \varnothing$, we can assume, possibly by changing notation that

$$(2.35) \qquad a_1 \neq \infty \neq b_1.$$

Let $D_1 \subset \mathbf{R}^n \setminus U$ be a compact set such that $G_1 D_1 = \mathbf{R}^n \setminus U$. Let $D_2 = f(D_1)$ which is also compact. For every j there is $g_j \in G_1$ such that $g_j(g_{1j}(a_{1j})) \in D_1$. Let $g'_{1j} = g_j g_{1j}$ and $g'_{2j} = \varphi(g_j) g_{2j}$. Then $f_j = g'^{-1}_{2j} f g'_{1j}$, $U_j = g'^{-1}_{1j}(U)$ and $V_j = g'^{-1}_{1j}(V)$. This means that we can replace g_{ij} by g'_{ij}.

If this replacement is made, $g_{1j}(a_{1j})$ varies in the compact set D_1. Similarly, $g_{2j}(b_{2j}) = f(g_{1j}(a_{1j}))$ varies in the compact set D_2. Since the numbers c_{ij} are bounded away from 0 and ∞, we can obtain by passing to a subsequence that there are similarities g_1 and g_2 of $\mathbf{R}^n$ such that $g_{ij} \to g_i$ uniformly in the spherical metric as $j \to \infty$. It follows that

$$f_j = g_{2j}^{-1} f g_{1j} \to g_2^{-1} f g_1$$

uniformly in the spherical metric. Hence, in case (ii), conclusion b) of our theorem is valid.

Finally we consider case (iii). This is the most complicated case. We first show that by passing to a subsequence we can obtain that

$$(2.36) \qquad c_{1j}/c_{2j} \to c \in (0, \infty)$$

as $j \to \infty$. In any case there is a subsequence such that (2.36) is true for some $c \in [0, \infty]$. We show that in fact $c \in (0, \infty)$.

We use (2.26). In view of (2.31) and (iii) this implies the existence of $M' \geq 1$ such that

$$(2.37) \qquad (c_{1j}/c_{2j})\, |a_{ij} - a_{kj}| / M' \leq |b_{ij} - b_{kj}| \leq M'(c_{1j}/c_{2j})\, |a_{ij} - a_{kj}|$$

for all i, j, k. By (2.31) and (2.32), it is impossible that $c = 0$ or $c = \infty$. Hence (2.36) is indeed true.

Exactly as in case (ii), using (2.37) and (2.36) instead of (2.30), we see now that we can assume that (2.35) is true also in case (iii).

Let $r_{ij} = d(0, g_{ij}^{-1}(\mathbf{R}^k))$. Then $r_{1j} \leq |a_{1j}| + d(a_{1j}, g_{1j}^{-1}(\mathbf{R}^k)) = |a_{1j}| + 1/c_{1j}$ since $g_{1j}(a_{1j}) \in \mathbf{R}^k \times B^{n-k}$. Similarly $r_{2j} \leq b_{1j} + t/c_{2j}$ for some $t \geq 0$. Since $a_1 \neq \infty \neq b_1$, it follows that the numbers r_{ij} are bounded. Then, arguing as in the third paragraph of this proof, we can assume that

$$g_{ij}(\mathbf{R}^k) = \mathbf{R}^k.$$

Choose now an affine map α of $\mathbf{R}^k$, $r > 0$ and $M \geq 1$ such that (2.13) and (2.14) are true. Define

$$\alpha_j = g_{2j}^{-1}\, \alpha(g_{1j} \mid \mathbf{R}^k)$$

which is an affine map of $\mathbf{R}^k$. By (2.36) there is $m \geq 1$ such that

(2.38) $$|\alpha_j(x) - \alpha_j(y)|/|x - y| \in [1/m, m].$$

whenever $x, y \in \mathbf{R}^k$ are distinct. In the following it is convenient to regard an affine map β of $\mathbf{R}^k$ as extended to $\mathbf{R}^n$ by $\beta(x, y) = \beta(x)$ if $(x, y) \in \mathbf{R}^k \times \mathbf{R}^{n-k}$. Then (2.13) and (2.14) imply that

(2.39) $$|f_j(a_{1j}) - \alpha_j(a_{1j})| \leq \mathrm{M}/c_{2j}.$$

Now $a_{1j} \to a_1 \neq \infty$ and $b_{1j} \to b_1 \neq \infty$ as $j \to \infty$. Then (2.38) and (2.39) imply that there is an affine map β of $\mathbf{R}^k$ such that for a subsequence

(2.40) $$\alpha_j \to \beta$$

uniformly on compact subsets of $\mathbf{R}^k$ as $j \to \infty$.

It follows by (2.40), (2.13) and (2.14) that β has the following property. Let $\mathrm{A} \subset \mathbf{R}^k$ be compact and $\varepsilon > 0$. Then there is an integer $j_0 > 0$ such that if $j \geq j_0$ and $(x, y) \in \mathrm{A} \times \mathbf{R}^{n-k}$, then

(2.41) $$|f_j(x, y) - \beta(y)| \leq \varepsilon$$

if $|y| \leq r/c_{1j}$ and if $|y| \geq r/c_{1j}$, then

(2.42) $$c\,|y|/2\mathrm{M} - \varepsilon \leq d(f_j(x, y), \mathbf{R}^k) - \varepsilon \leq |f_j(x, y) - \beta(x)| \leq 2c\mathrm{M}\,|y| + \varepsilon.$$

Now we are in a position to apply the compactness properties of quasiconformal mappings. We apply Väisälä [40, 19.2 and 20.5] to the maps $f_j \mid \mathrm{U}_j$; these theorems assume that f_j is defined on a fixed set U_j but since, in an obvious topology of subsets of $\overline{\mathbf{R}}^n$, U_j tends to $\mathbf{R}^n \backslash \mathbf{R}^k$ as $j \to \infty$, we can easily modify them to fit the present case. Also, if $k = n - 1$, we must consider separately the two components of U_j. Then these theorems imply that there is a map $h : \mathbf{R}^n \backslash \mathbf{R}^k \to \overline{\mathbf{R}}^n$ such that $f_j(x) \to h(x)$ for every $x \in \mathbf{R}^n \backslash \mathbf{R}^k$ and that the convergence is uniform on every compact set of $\mathbf{R}^n \backslash \mathbf{R}^k$. Furthermore, [40, 21.1] implies that h restricted to a component A of $\mathbf{R}^n \backslash \mathbf{R}^k$ is either an embedding or a constant. However, (2.42) implies that $h \mid \mathrm{A}$ cannot be a constant. In addition, (2.42) also implies that $h(\mathbf{R}^n \backslash \mathbf{R}^k) \subset \mathbf{R}^n$. Obviously, h is an embedding also if $k = n - 1$.

Next we extend h to the whole $\overline{\mathbf{R}}^n$ by setting $h(\infty) = \infty$ and $h \mid \mathbf{R}^k = \beta$. Then (2.41) and (2.42) imply that h is a homeomorphism of $\overline{\mathbf{R}}^n$. To show that $f_j \to h$ uniformly in $\overline{\mathbf{R}}^n$ it suffices to show that if x_j, $x \in \mathbf{R}^n$ and $x_j \to x$, then $f_j(x_j) \to h(x)$. If $x \in \mathbf{R}^n \backslash \mathbf{R}^k$ this follows from the above and if $x \in \mathbf{R}^k$, this is a consequence of (2.41) and (2.42).

Hence in case (iii) the conclusion b) is valid and the theorem is proved.

Remarks 1. — The assumption that $f \mid \mathrm{U}$ is quasiconformal was needed only in case (iii) which can occur if and only if $\limsup_{j \to \infty} (\sup\{q(x, \mathrm{U}_j) : x \in \overline{\mathbf{R}}^n\}) = 0$.

2. — If $n \neq 4$, one could simplify the above proof using Sullivan's theorem that homeomorphisms of quasiconformal manifolds can be approximated by quasiconformal ones ([28] and [38, 4.4]). In the present proof (after the normalizations of the first paragraph) these theorems (we can assume that $\mathbf{R}^n/G_i$ is a manifold by passing to a subgroup of finite index) would imply that there is a quasiconformal homeomorphism $g : \mathbf{R}^n \to \mathbf{R}^n$ inducing φ such that $|g(x) - f(x)| \leq 1$ for all $x \in \mathbf{R}^n$ and that

$$g \mid \mathbf{R}^n \backslash (\mathbf{R}^k \times B^{n-k}(2)) = f \mid \mathbf{R}^n \backslash (\mathbf{R}^k \times B^{n-k}(2)).$$

Now the compactness properties of quasiconformal mappings would simplify case (iii) of the above proof and in addition Lemmas 2.8 and 2.9 could be omitted; from Lemma 2.10 we would need only (2.25) which is independent of Lemma 2.9.

3. THE MAP OF THE LIMIT SETS

In this section we consider a type-preserving isomorphism $\varphi : G \to G'$ of two geometrically finite groups of $\overline{\mathbf{R}}^n$ and show that there is always a homeomorphism f_φ of the limit sets inducing φ (Theorem 3.3). The proof of the existence of f_φ is the same as in the case of finite hyperbolic volume. Thus we consider first hyperbolic pseudo-isometries and then extend them to the boundary. Our second main theorem in this section is Theorem 3.8 where we show that a homeomorphism $f : \Omega(G) \to \Omega(G')$ of the ordinary sets which induces φ can always be extended to the limit set to a homeomorphism of $\overline{\mathbf{R}}^n$ inducing φ; moreover, the extension is quasiconformal if f is.

3A. Quasisymmetric maps. — Let $X \subset \mathbf{R}^n$. An embedding $f : X \to \mathbf{R}^m$ is said to be *quasisymmetric* if there is a homeomorphism $\eta : [0, \infty) \to [0, \infty)$ such that

(3.1) $$|f(y) - f(x)| \leq \eta(\rho)\, |f(z) - f(x)|$$

whenever $x, y, z \in X$ and $|y - x| \leq \rho\, |z - x|$. Then

(3.2) $$|f(y) - f(x)| \geq \eta'(\rho)\, |f(z) - f(x)|$$

if $|y - x| \geq \rho\, |z - x|$ and $\eta'(\rho) = \eta(\rho^{-1})^{-1}$ ($\eta'(0) = 0$). We also say that f is *η-quasisymmetric* if it satisfies (3.1) with this particular η. The embedding f is said to be *locally quasisymmetric* if every $x \in X$ has a neighbourhood U such that $f \mid U$ is quasisymmetric. By means of auxiliary Möbius transformations we can extend the definition of local quasisymmetry to embeddings $X \to \overline{\mathbf{R}}^m$, $X \subset \overline{\mathbf{R}}^n$.

Quasisymmetric embeddings were discussed in [37] and they are a natural generalization of quasiconformal maps. For instance, a homeomorphism of $\mathbf{R}^n$ is quasiconformal if and only if it is quasisymmetric. If $n = 1$, our definition of quasisymmetric maps of $\mathbf{R}^1$ can be shown to be equivalent with the usual definition of quasisymmetric maps [13, II.7.1] except that one customarily considers only increasing maps.

3B. Pseudo-isometric maps. — Let $X, Y \subset H^{n+1}$ and let $F : X \to Y$ be a map. The map F is a *pseudo-isometry* if there are constants $c_1 \geq 1$ and $c_2 \geq 0$ such that in the hyperbolic metric

$$c_1^{-1} d(x,y) - c_2 \leq d(F(x), F(y)) \leq c_1 d(x,y) \tag{3.3}$$

for all $x, y \in X$; in this case we also say that F is a (c_1, c_2)*-pseudo-isometry*. Thus pseudo-isometries are always Lipschitz maps in the hyperbolic metric.

Pseudo-isometries were introduced by Mostow [23]. Their importance comes from the following extension theorem which is essentially due to Efremovitch and Tihomirova [6].

Theorem **3.1.** — *Let* $X, Y \subset H^{n+1}$ *and assume that* X *is convex in the hyperbolic geometry. Let* $X' = \mathrm{cl}\, X \cap \overline{\mathbf{R}}^n$ *and* $Y' = \mathrm{cl}\, Y \cap \overline{\mathbf{R}}^n$. *Let* $F : X \to Y$ *be a* (c_1, c_2)*-pseudo-isometry. Then* F *can be extended to a continuous map* $X \cup X' \to Y \cup Y'$, *also denoted by* F, *such that* $F \mid X'$ *is an embedding. If* $d(z, F(X))$ *is bounded for* $z \in Y$, *then* $F(X') = Y'$. *If* $\infty \notin X' \cup F(X')$ *or if* $F(\infty) = \infty$, *then* $F \mid X' \cap \mathbf{R}^n$ *is quasisymmetric. In the case* $F(\infty) = \infty \in X'$, F *is moreover* η*-quasisymmetric for some* η *depending only on* c_1 *and* c_2.

Proof. — This can be proved exactly as in [29, 5.9.2-5.9.5]. We observe only that obviously $F(X') = Y'$ if $d(z, F(X))$ is bounded for $z \in Y$ and add some remarks concerning quasisymmetry of $F' \mid X' \cap \mathbf{R}^n$. If $\infty \notin X' \cup F(X')$, then it suffices to show that $F \mid X'$ is locally quasisymmetric by [37, 2.23]. Thus we can assume, by composing with auxiliary Möbius transformations, that $\infty \in X'$ and that $F(\infty) = \infty$. Then the argument in [29, 5.9.4 and 5.9.6] can easily be adapted to show that $F \mid X' \cap \mathbf{R}^n$ is η-quasisymmetric for some η depending only on c_1 and c_2.

Remark. — Actually, it would suffice in Theorem 3.1 that F is a (c_1, c_2)-pseudo-isometry $C \to Y$ where $C \subset H^{n+1}$ is a set such that $d(z, X) \leq M$ for some $M > 0$ and all $z \in X$. Of course, now η would depend also on M, in addition to c_1 and c_2. This can be proved like the above weaker version.

3C. The map of the limit sets. — If G and G′ are geometrically finite, non-elementary groups of $\overline{\mathbf{R}}^n$ and if $\varphi : G \to G'$ is a type-preserving isomorphism, it is fairly easy to construct a pseudo-isometry $F : H_G \to H_{G'}$ of the convex hulls of the limit sets (cf. (1.5)) which induces φ. Then we get by Theorem 3.1 the homeomorphism $f_\varphi : L(G) \to L(G')$ inducing φ whose construction is the central point of this section.

The idea of the map of the limit sets can be traced back to Nielsen [24] (if $n = 1$ and H^2/G compact). Mostow [22] realized the importance of f_φ for the rigidity of hyperbolic space forms and Margulis [17] showed that f_φ exists whenever H^{n+1}/G and H^{n+1}/G' are compact; the general finite-volume case $(n > 1)$ follows by Prasad [25]. (If $n = 1$, one must assume in addition that φ is type-preserving in the general finite-volume case.) If $n = 2$, Floyd [7] implies the existence of f_φ in the compact case.

We need the following lemma on type-preserving isomorphisms. It explains why in Mostow's rigidity theorem one need not assume that the isomorphism is type-preserving (in this case all parabolic elements have rank n). Next lemma is an immediate consequence of Lemma 2.2.

Lemma 3.2. — *Let* φ *be an isomorphisms of two discrete Möbius groups. Then* φ *is type-preserving if and only if* φ *and* φ^{-1} *carry parabolic elements of rank one onto parabolic elements and in this case* φ *preserves the rank of a parabolic element.*

Theorem 3.3. — *Let* G *and* G′ *be geometrically finite groups of* $\overline{\mathbf{R}}^n$ *and let* $\varphi : \mathrm{G} \to \mathrm{G}'$ *be a type-preserving isomorphism. Then there is a homeomorphism* $f_\varphi : \mathrm{L}(\mathrm{G}) \to \mathrm{L}(\mathrm{G}')$ *of the limit sets inducing* φ *which is unique if* G *is non-elementary. Moreover,* $f_\varphi \mid \mathrm{L}(\mathrm{G}) \cap \mathbf{R}^n$ *is quasisymmetric if either* $\infty \notin \mathrm{L}(\mathrm{G}) \cup \mathrm{L}(\mathrm{G}')$ *or* $f_\varphi(\infty) = \infty$.

Let then $\mathrm{A} \subset \Omega(\mathrm{G})$ *be a* G*-invariant set such that* A/G *is finite and let* $f : \mathrm{A} \to \Omega(\mathrm{G}')$ *be an embedding inducing* φ. *Then* f *and* f_φ *define a homeomorphism* $h : \mathrm{L}(\mathrm{G}) \cup \mathrm{A} \to \mathrm{L}(\mathrm{G}') \cup f(\mathrm{A})$ *such that* $h \mid (\mathrm{L}(\mathrm{G}) \cup \mathrm{A}) \cap \mathbf{R}^n$ *is quasisymmetric if either* $\infty \notin \mathrm{L}(\mathrm{G}) \cup \mathrm{L}(\mathrm{G}') \cup \mathrm{A} \cup f(\mathrm{A})$ *or* $h(\infty) = \infty$; h *is unique if* $\mathrm{L}(\mathrm{G}) \cup \mathrm{A}$ *is infinite.*

Proof. — Setting $\mathrm{A} = \varnothing$, we get the first part of the theorem from the second. If $\mathrm{L}(\mathrm{G}) \cup \mathrm{A}$ is infinite, then h maps necessarily the attractive fixed point of a loxodromic $g \in \mathrm{G}$ to the attractive fixed point of $\varphi(g)$ (which is also loxodromic). Since these points are dense in L(G) (except if L(G) = a point), the map h is unique; obviously h is also unique if L(G) = a point.

We assume now that $\mathrm{L}(\mathrm{G}) \cup \mathrm{A}$ is infinite since otherwise either G is finite or $\mathrm{A} = \varnothing$ and L(G) contains at most two points, and these cases are easily dealt with.

Denote by

$$\mathrm{H} = \mathrm{Co}(\mathrm{L}(\mathrm{G}) \cup \mathrm{A}),$$
$$\mathrm{H}' = \mathrm{Co}(\mathrm{L}(\mathrm{G}') \cup f(\mathrm{A}))$$

the convex hulls (cf. (1.4)). Then, in view of Theorem 3.1, our theorem follows from

Lemma 3.4. — *There is a pseudo-isometry* $\mathrm{F} : \mathrm{H} \to \mathrm{H}'$ *inducing* φ *such that* $d(z, \mathrm{F}(\mathrm{H}))$ *is bounded for* $z \in \mathrm{H}'$.

Proof. — This can be proved like [29, 5.9.1]. Since our situation differs from that of Thurston and since we also allow parabolic and elliptic elements, we describe the proof in some detail. For groups of compact type it could be simplified as in Margulis [17] (if $\mathrm{A} = \varnothing$) since then $\mathrm{H}_\mathrm{G}/\mathrm{G}$ is compact.

Let P(G) and P(G′) be the set of parabolic fixed points of G and G′, respectively. If $v \in \mathrm{A}$, let $v' = f(v)$. If $v \in \mathrm{P}(\mathrm{G})$, let $v' \in \mathrm{P}(\mathrm{G}')$ be the point such that if $g \in \mathrm{G}_v$ is parabolic, then $\varphi(g) \in \mathrm{G}'_{v'}$. By Lemma 2.2, v' does not depend on the choice of g.

Then the map $v \mapsto v'$ is a bijection $P(G) \cup A \to P(G') \cup f(A)$. We fix for every $v \in P(G) \cup A$ closed $(n+1)$-balls B_v and B'_v such that

(i) $\operatorname{int} B_v \subset H^{n+1}$, $\operatorname{int} B'_v \subset H^{n+1}$, ∂B_v is tangent to $\overline{\mathbf{R}}^n$ at v, and $\partial B'_v$ is tangent to $\overline{\mathbf{R}}^n$ at v',

(ii) $B_{g(v)} = g(B_v)$ and $B'_{g(v)} = \varphi(g)(B'_v)$ for $g \in G$,

(iii) the families $\{B_v : v \in P(G) \cup A\}$ and $\{B'_v : v \in P(G) \cup A\}$ are disjoint,

(iv) $B_v/G_v = B_v/G$ and $B'_v/G'_{v'} = B'_v/G'$,

(v) let k_v be the rank of v if $v \in P(G)$, and if $v \in A$, let $k_v = 0$; then $(H \cap \partial B_v) \cup \{v\}$ contains a k-sphere S_v and $(H' \cap \partial B'_v) \cup \{v'\}$ contains a k-sphere S'_v such that $v \in S_v$, $v' \in S'_v$, S_v is G_v-invariant, and that S'_v is $G'_{v'}$-invariant.

The existence of such balls and spheres follows by the existence of cusp neighbourhoods, Theorem 2.4, and Theorems 2.1 and 2.7. Observe that, if $v \in P(G)$, then v and v' have the same rank by Lemma 3.2.

We set

$$B = \cup\{B_v : v \in P(G) \cup A\}, \quad B' = \cup\{B'_v : v \in P(G) \cup A\},$$

$$H_0 = H \setminus \operatorname{int} B, \quad H'_0 = H' \setminus \operatorname{int} B'.$$

Then H_0 is G-invariant, H'_0 is G'-invariant and H_0/G and H'_0/G' are compact. This follows easily from Theorem 2.4 b).

Next we find a G-invariant triangulation K of a neighbourhood of the pair $(H_0, \partial H_0)$ in $(H^{n+1} \setminus \operatorname{int} B, \partial B)$ such that K/G is a finite triangulation and that every simplex of K touches at most one B_v. Thus every $T \in K$ has an affine structure. We will now define a map $F_1 : K \to H'$ which induces φ as follows. In order that F_1 be Lipschitz (in hyperbolic metric), K must regular enough, for instance a C^∞-triangulation. If $P(G) = \varnothing$, we could assume that simplexes of K are hyperbolic simplexes.

Let K' be the barycentric subdivision of K. We first define $F_1 : K \to H'$ for vertices of K'. These can be defined otherwise arbitrarily, one only takes care that F_1 induces φ and that $F_1(x) \in S'_v$ if $x \in \partial B_v$. Observe that the hyperbolic element of length induces on $\partial B_v \setminus \{v\}$ and $\partial B'_v \setminus \{v'\}$ metrics which are similar to the euclidean metric of $\mathbf{R}^n$; then the groups G_v and $G'_{v'}$ act as groups of similarities on these sets, respectively. Thus, if $T \subset \partial B_v$ is a simplex of K', we can let $F_1 \mid T$ be the unique affine extension. Then $F_1(T) \subset S'_v \subset H'$. Suppose then that we have defined F_1 in all $(i-1)$-simplexes of K. Let $T' \not\subset \partial B$ be an i-simplex of K' such that $T' \subset T$ where T is an i-simplex of K. Then there is a vertex a of T' such that $a \in \operatorname{int} T$ and that the face T'' of T' opposite to a lies in the $(i-1)$-skeleton of K. If $x \in T''$, let s_x be the segment of T' (in the affine structure of T') with endpoints a and x. If we set now $F_1(s_x) =$ the hyperbolic segment with endpoints $F_1(a)$ and $F_1(x)$ and that $F_1 \mid s_x$ is a linear stretch (in the respective structures), we get an extension of F_1 to T'. It is obvious that in this manner we get a map $F_1 : K \to H'$ which is locally Lipchitz (if K is regular enough) and for which

$$(3.4) \qquad F_1(\partial B \cap K) \subset \bigcup\{S'_v : v \in P(G) \cup A\}.$$

We then extend F_1 to $K_1 = K \cup \{x \in H^{n+1} : x$ is on a hyperbolic ray with endpoints $v \in P(G) \cup A$ and $u \in \partial B_v \cap K\}$. By Theorem 2.7 we can choose K so big that $K_1 \supset H$, which we now assume. Choose $v_1, \ldots, v_k \in P(G) \cup A$ such that every $v \in P(G) \cup A$ is conjugate under G to exactly one v_i. Let α_i and β_i be Möbius transformations of $\overline{H}^{n+1}$ such that $\alpha_i(\partial B_{v_i}) = \mathbf{R}^n \times \{1\} \cup \{\infty\} = \beta_i(\partial B'_{v_i})$. We extend now F_1 to $K_1 \cap B_{v_i}$ by requiring that, if $\alpha_i^{-1}(x, 1) \in K$, $\beta_i F_1 \alpha_i^{-1} \mid \{x\} \times [1, \infty)$ is of the form $(x, t) \mapsto (y, t)$, $t \geq 1$. Then $F_1 \mid K_1 \cap B_{v_i}$ is G_{v_i}-compatible and the extension to other sets $B_v \cap K_1$ is by G-compatibility. Then obviously we get a map $F_1 : K_1 \to H'$ which induces φ.

We claim that there is $L > 0$ such that

$$(3.5) \qquad d(F_1(x), F_1(y)) \leq L\, d(x, y)$$

for all $x, y \in H$. It is obvious by the definition of F_1 that, given $z \in H$, there is $L = L_z$ and a neighbourhood $U = U_z$ of z such that (3.5) is true for all $x, y \in U \cap H$. We can assume that $L = L_z$ is bounded if z varies in a set X such that X/G is compact, for instance if $z \in H_0$. In view of the definition of F_1 in B, we can now find an $L > 0$ such that (3.5) is true locally in H. Then the convexity of H implies that (3.5) is true globally in H. That is, F_1 is Lipschitz.

We repeat the above process and get a Lipschitz map $F_2 : K_2 \to H$ inducing φ^{-1}, where $K_2 \supset H'$. Then $F_2 F_1$ induces $\mathrm{id} : G \to G$. Since K/G is compact, there is $M > 0$ such that

$$(3.6) \qquad d(F_2 F_1(x), x) \leq M$$

for all $x \in K$. If $x \in H \setminus K$, $x \in B_v$ for some $v \in P(G) \cup A$. Let $y \in \partial B_v$ be the point such that x is on the hyperbolic ray r with endpoints v and y. Then $y \in K$ by the choice of K. In addition, $F_2 F_1(x)$ lies on the ray with endpoints v and $F_2 F_1(y)$ and $F_2 F_1 \mid r$ is an isometry. It follows that (3.6) is true for all $x \in H \setminus K$ and hence for all $x \in H$. Now (3.6) and the fact that both $F_1 \mid H$ and $F_2 \mid H'$ are Lipschitz maps imply that if we set $F = F_1 \mid H$, then F is a pseudo-isometry $H \to H'$.

Finally, to conclude the proof of the lemma, we observe that one shows like (3.6) that $d(F_1 F_2(z), z) \leq M'$ for some $M' > 0$ and for all $z \in H'$. It follows that $d(z, F(H)) \leq M'$ for all $z \in H'$.

Remarks 1. — Since the isomorphism φ of Theorem 3.3 is induced by a pseudo-isometry $H \to H'$, one easily sees that there is $k \geq 1$ such that the multiplier of $g \in G$ (cf. 1C) satisfies

$$(3.7) \qquad (\mathrm{mul}\, g)^{1/k} \leq \mathrm{mul}\, \varphi(g) \leq (\mathrm{mul}\, g)^k$$

since $\log \mathrm{mul}\, g = \inf\{d(x, g(x)) : x \in H^{n+1}\}$.

2. — If $\infty \notin L(G) \cup L(G')$ or if $f_\varphi(\infty) = \infty$, then [37] implies that $f_\varphi \mid L(G) \cap \mathbf{R}^n$ is η-quasisymmetric for some η of the form $\eta(t) = C \max(t^\alpha, t^{1/\alpha})$, $C > 0$ and $\alpha \geq 1$, and that f_φ satisfies a two-sided Hölder condition of the form

$$C' \, |x - y|^{\alpha'} \leq |f_\varphi(x) - f_\varphi(y)| \leq C'' \, |x - y|^{\alpha''}$$

for positive C', C'', α' and α'' on bounded sets of $L(G) \cap \mathbf{R}^n$. This follows from [37, 3.10] and [37, 3.14] since $L(G) \cap \mathbf{R}^n$ is homogeneously dense [37, 3.8] if G is non-elementary by [36, Corollary C1].

3D. The quasihyperbolic metric. — In the next section we will extend G-compatible maps to the limit set of G. For this, the notion of a quasihyperbolic metric is a handy tool. Let U be a proper subdomain of $\mathbf{R}^n$, $n \geq 1$. Then the *quasihyperbolic metric* k_U of U is defined by the element of length

$$|dx|/d(x, \partial U);$$

if U is a component of $\mathbf{R}^n \setminus \mathbf{R}^{n-1}$, then k_U is the usual hyperbolic metric of U. This metric was first studied by Gehring and Palka in [8]. We need the following two properties of the quasihyperbolic metric.

Lemma **3.5.** — *Let* U *be a proper subdomain of* $\mathbf{R}^n$ *and let* $M \geq 0$. *If* $z \in \mathbf{R}^n \setminus U$, $x, y \in U$ *and* $k_U(x,y) \leq M$, *then*

$$e^{-M} \leq |x - z|/|y - z| \leq e^{M}.$$

Proof. — We can assume that $|x - z| \geq |y - z| > 0$. Let $\varepsilon > 0$. Then there is a rectifiable path γ joining y and x such that $k_U(x,y) + \varepsilon \geq \int_\gamma |du|/d(u, \partial U)$. Thus

$$M + \varepsilon \geq \int_\gamma |du|/d(u, \partial U) \geq \int_\gamma |du|/|u - z| \geq \log(|x - z|/|y - z|). \tag{3.8}$$

This implies the lemma since $\varepsilon > 0$ was arbitrary.

Lemma **3.6.** — *Let* U *and* V *be proper subdomains of* $\mathbf{R}^n$, $n \geq 1$, *and let* $f : U \to V$ *be a Möbius transformation such that* $f(U) = V$. *Then, if* $x, y \in U$,

$$k_U(x,y)/2 \leq k_V(f(x), f(y)) \leq 2 k_U(x,y). \tag{3.9}$$

Proof. — See Gehring-Palka [8, Corollary 2.5].

3E. Extension to the limit set. — Now we can extend a G-compatible homeomorphism $\Omega(G) \to \Omega(G')$ of the ordinary sets to the limit sets. In the compact case our extension theorem is contained almost entirely in

Lemma **3.7.** — *Let* G *and* G′ *be geometrically finite Möbius groups of* $\overline{\mathbf{R}}^n$ *and let* $\varphi : G \to G'$ *be a type-preserving isomorphism. Further, let* $B \subset \Omega(G)$ *be a* G*-invariant set such that* B/G *is compact and let* $f : B \to \Omega(G')$ *be a continuous map inducing* φ. *Then* f *and the map* f_φ *of*

Theorem 3.3 *define together a continuous map* $f' : \mathrm{L(G)} \cup \mathrm{B} \to \overline{\mathbf{R}}^n$ *which is an embedding if* f *is.*

If, in addition, $\infty \in \mathrm{L(G)}$ *and* $f_\varphi(\infty) = \infty$, *then there is a homeomorphism* $\eta_0 : [0, \infty) \to [0, \infty)$ *such that, setting* $\eta_0'(\rho) = \eta_0(\rho^{-1})^{-1} (\eta_0(0) = 0)$,

$$\eta_0'(\rho) \, |f'(x) - f'(z)| \leq |f'(y) - f'(z)| \leq \eta_0(\rho) \, |f'(x) - f'(z)| \tag{3.10}$$

whenever $z \in \mathrm{L(G)} \cap \mathbf{R}^n$, $x, y \in (\mathrm{L(G)} \cup \mathrm{B}) \cap \mathbf{R}^n$ *and* $|y - z| = \rho \, |x - z|$.

Proof. — We can assume that $\varnothing \neq \mathrm{L(G)} \ni \infty$ and that $f_\varphi(\infty) = \infty$, f_φ as in Theorem 3.3. Since G and G′ are geometrically finite, there is a set $\mathrm{A} \subset \mathrm{B}$ such that A/G is finite and that, for every $u \in \mathrm{B}$, there is $v \in \mathrm{A}$ such that u and v are in a component of $\Omega(\mathrm{G})$ and that $f'(u)$ and $f'(v)$ are in a component of $\Omega(\mathrm{G}')$. Furthermore, possibly by adding points to B, we can assume that $f' \mid \mathrm{A}$ is an embedding. It now follows by Theorem 3.3 that there is a homeomorphism $\eta : [0, \infty) \to [0, \infty)$ such that $f' \mid (\mathrm{L(G)} \cup \mathrm{A}) \cap \mathbf{R}^n$ is an η-quasisymmetric embedding.

We first assume that $\mathrm{L(G)} \neq \{\infty\}$; that is, L(G) contains at least two points. If U is now a component of $\Omega(\mathrm{G})$, it is a proper subdomain of $\mathbf{R}^n$ and thus the quasihyperbolic metric is well-defined. If x, y are in such a component U, we denote by $k(x, y)$ their distance in the quasihyperbolic metric of U. Similarly, if $x, y \in \mathrm{U}'$ where U′ is a component of $\Omega(\mathrm{G}')$, we denote by $k'(x, y)$ their distance in the quasihyperbolic metric of U′. If $g \in \mathrm{G}$, $g(\mathrm{U}) = \mathrm{U}$ and if $\varphi(g)(\mathrm{U}') = \mathrm{U}'$, then, by Lemma 3.6, $g \mid \mathrm{U}$ and $\varphi(g) \mid \mathrm{U}'$ are 2-bilipschitz in the metrics k and k', respectively. Thus, remembering that B/G is compact, we can find for every $u \in \mathrm{B}$ an element $v \in \mathrm{A}$ such that u and v are in a component of $\Omega(\mathrm{G})$, $f'(u)$ and $f'(v)$ are in a component of $\Omega(\mathrm{G}')$, and that $k(u, v) \leq \mathrm{M}_0$ and $k'(f'(u), f'(v)) \leq \mathrm{M}_0$ for some fixed M_0.

We first show that (3.10) is true. It suffices to prove only the right-hand inequality which then implies the left-hand inequality. Let x, y, z be as in (3.10). We can assume that $y \neq z \neq x$. Define $y' \in \mathrm{L(G)} \cup \mathrm{A}$ as follows. If $y \in \mathrm{L(G)}$, we set $y' = y$. Otherwise, we let $y' \in \mathrm{A}$ be a point such that y and y' are in a component U of $\Omega(\mathrm{G})$, $f'(y)$ and $f'(y')$ are in a component of $\Omega(\mathrm{G}')$ and that $k(y, y') \leq \mathrm{M}_0$ and $k'(f(y), f'(y')) \leq \mathrm{M}_0$; we have seen above that there is always such y'. Define x' similarly. Then Lemma 3.5 implies that there is $\mathrm{M}_1 = \mathrm{M}_1(\mathrm{M}_0) \geq 1$ such that all the ratios $|y - z| / |y' - z|$, $|x - z| / |x' - z|$, $|f'(y) - f'(z)| / |f'(y') - f'(z)|$ and $|f'(x) - f'(z)| / |f'(x') - f'(z)|$ lie in the interval $[1/\mathrm{M}_1, \mathrm{M}_1]$. Thus

$$\frac{|f'(y) - f'(z)|}{|f'(x) - f'(z)|} \leq \mathrm{M}_1^2 \frac{|f'(y') - f'(z)|}{|f'(x') - f'(z)|} \leq \mathrm{M}_1^2 \, \eta(\mathrm{M}_1^2 \, \rho)$$

since $f' \mid (\mathrm{L(G)} \cup \mathrm{A}) \cap \mathbf{R}^n$ is η-quasisymmetric. Consequently, (3.10) is true with $\eta_0(\rho) = \mathrm{M}_1^2 \, \eta(\mathrm{M}_1^2 \, \rho)$.

We then show that f' is continuous. It suffices to show that f' is continuous at an arbitrary point $x \in \mathrm{L(G)}$. We have assumed that L(G) contains more than one point. Then we can assume that $x \neq \infty$; if $x = \infty$, we only change the normalization of f_φ.

Now, by Theorem 3.3, $f' \mid L(G) \cup A$ is continuous, and then the continuity of f' follows from (3.10) since, as we have seen above, for every $y \in B$ there is $y' \in A$ such that $|y-x|/|y'-x|$ and $|f'(y)-f'(x)|/|f'(y')-f'(x)|$ lie in the interval $[1/M_1, M_1]$.

If $L(G) = $ a point $= \{\infty\}$, then every $g \in G \cup G'$ is an isometry of $\mathbf{R}^n$ (Theorem 2.1). Since B/G and $f(B)/G'$ are compact, the continuity of $f' \mid A \cup \{\infty\}$ easily implies the continuity of f'.

Obviously, it now follows that f' is an embedding if f is.

Remark. — One could show in addition that, if $\infty \in L(G)$ and if $f_\varphi(\infty) = \infty$, then $f' \mid (L(G) \cup B) \cap \mathbf{R}^n$ is quasisymmetric whenever f is a locally quasisymmetric embedding. However, the proof would be fairly complicated and since (3.10) suffices for Theorem 3.8 (only for $n = 1$ it would simplify the proof), we omit it.

Theorem **3.8.** — *Let* G *and* G′ *be geometrically finite Möbius groups of* $\overline{\mathbf{R}}^n$, $n \geq 1$, *and let* $\varphi : G \to G'$ *be an isomorphism. If* $n = 1$, *assume in addition that* φ *is type-preserving. Let* $f : \Omega(G) \to \Omega(G')$ *be a homeomorphism of the ordinary sets inducing* φ. *Then* φ *is always type-preserving and, if* $f_\varphi : L(G) \to L(G')$ *is the map of Theorem* 3.3, *then* f *and* f_φ *define together a homeomorphism* f' *of* $\overline{\mathbf{R}}^n$ *inducing* φ.

We have in addition: If $n \geq 2$ *and* $\Omega(G) = \varnothing$, *then* f' *is a Möbius transformation. If* $n \geq 2$, $\Omega(G) \neq \varnothing$ *and if* f *is* K*-quasiconformal for some* $K \geq 1$, *then* f' *is* K*-quasiconformal, too. If* $n = 1$, $f'(\infty) = \infty$ *and if* f *is locally quasisymmetric (this includes the case that* $\Omega(G) = \varnothing$*), then* $f' \mid \mathbf{R}$ *is quasisymmetric.*

Proof. — We first remark that if $\Omega(G)/G$ and $\Omega(G')/G'$ are compact, the theorem is almost entirely contained in Lemma 3.7. Indeed, by Theorem 2.4 c), every parabolic $g \in G$ or $g' \in G'$ must have rank n; then Lemma 3.2 implies that φ is type-preserving also if $n \geq 2$. Now it follows by Lemma 3.7 that f' is an embedding and hence a homeomorphism of $\overline{\mathbf{R}}^n$ inducing φ.

If $n \geq 2$, then f' is quasiconformal if the linear dilatation

$$(3.11) \qquad H(z) = \limsup_{|x-z|=|y-z|\to 0} \frac{|f'(x)-f'(z)|}{|f'(y)-f'(z)|}$$

of f' is bounded for $z \in \overline{\mathbf{R}}^n \setminus \{\infty, f'^{-1}(\infty)\}$. If f is quasiconformal, then $H(z)$ is bounded for $z \in \Omega(G) \setminus \{\infty, f'^{-1}(\infty)\}$; and it is bounded for $z \in L(G) \setminus \{\infty, f'^{-1}(\infty)\}$ by (3.10). Hence f' is quasiconformal if f is quasiconformal or if $\Omega(G) = \varnothing$. If $\Omega(G) = \varnothing$, then f' is in fact a Möbius transformation by Mostow's rigidity theorem [22, 23]. This is due to the fact that quasiconformal maps are absolutely continuous. In fact, f_φ is the restriction of a Möbius transformation whenever it is absolutely continuous with respect to some measures of $L(G)$ and $L(G')$ [34]. If $\Omega(G) \neq \varnothing$ and if f is K-quasiconformal, then f' is, too, K-quasiconformal since $L(G)$ has now zero measure [1, 5, 33] by [40, 34.1, 32.4 and 34.6].

The quasisymmetric case $(n = 1)$ requires some additional considerations which we postpone. Observe that if $n = 1$, then $\Omega(G)/G$ and $\Omega(G')/G'$ are always compact.

We now show that φ is type-preserving also in the remaining cases. Then $n \geq 2$ and Lemma 3.2 implies that it suffices to show that $g \in G$ is parabolic of rank $< n$ if and only if $\varphi(g) \in G'$ is parabolic of rank $< n$. Define $\widetilde{M}_G = \Omega(G)/G$ and let $p : \Omega(G) \to \widetilde{M}_G$ be the canonical projection. Define $\widetilde{M}_{G'}$ and $p' : \Omega(G') \to \widetilde{M}_{G'}$ similarly and let $\tilde{f} : \widetilde{M}_G \to \widetilde{M}_{G'}$ be the map defined by f. By Theorem 2.4, there is a compact set $C' \subset \widetilde{M}_{G'}$ such that if $g' \in G'$ is an element of infinite order obtained by lifting a loop in a component of $M_{G'} \backslash C'$, then g' is parabolic of rank $< n$.

Let $g \in G$ be parabolic of rank $< n$. By Theorem 2.4, g can be obtained by lifting a loop in $M_G \backslash f^{-1}(C')$. Hence $\varphi(g)$ can be obtained by lifting a loop in $M_{G'} \backslash C'$ and ist hus parabolic of rank $< n$. Similarly one sees that if $g' \in G'$ is parabolic of rank $< n$, then $\varphi^{-1}(g)$ is parabolic of rank $< n$.

Thus φ is type-preserving and the map f_φ of Theorem 3.3 always exists. We show that the map f' defined by f and f_φ is continuous also if G contains parabolic elements of rank $< n$. Let $B \subset \Omega(G)$ be a set such that B/G is compact and that every component of $\Omega(G) \backslash B$ is a cusp neighbourhood of G and that every component of $\Omega(G') \backslash f(B)$ is contained in a cusp neighbourhood of G'; such a set B exists by Theorem 2.4. Then Lemma 2.3 b) implies that, given $d > 0$, there is only a finite number of components of $\Omega(G) \backslash B$ and of $\Omega(G') \backslash f(B)$ whose diameter exceeds d. Since $f' \mid L(G) \cup B$ is continuous by Lemma 3.7 this fact implies that f' is continuous at all points $z \in L(G)$ which are not parabolic fixed points of G. If z is a parabolic fixed point of G, there is a well-defined component V of $\Omega(G) \backslash B$ such that $\{z\} = \operatorname{cl} V \cap L(G)$. It is not difficult to see that $f \mid \operatorname{cl} V$ is continuous. We can conclude that f' is continuous at L(G) and thus everywhere, $f' \mid \Omega(G)$ being continuous by assumption. Obviously, f' is a bijection and consequently a homeomorphism of $\overline{\mathbf{R}}^n$ inducing φ.

As above, if $n \geq 2$ and $\Omega(G) = \varnothing$, then f_φ is a Möbius transformation. We then assume that $n \geq 2$, $\Omega(G) \neq \varnothing$ and that f is K-quasiconformal. We show that f' is K-quasiconformal, too. We can assume that $f'(\infty) = \infty$. We have already observed in the first paragraph of this proof that f' is K-quasiconformal as soon as it is quasiconformal. We also observed that the quasiconformality of f' follows from the existence of a constant $M \in [1, \infty)$ such that

$$\textbf{(3.12)} \qquad 1/M \leq \frac{|f'(x) - f'(z)|}{|f'(y) - f'(z)|} \leq M$$

for all $z \in L(G) \cap \mathbf{R}^n$, $x, y \in \mathbf{R}^n$, $|x - z| = |y - z| > 0$.

We now prove (3.12). We first fix a G-invariant set $B \subset \Omega(G)$ such that B/G is compact and that every component of $\Omega(G) \backslash B$ is a cusp neighbourhood of some parabolic $g \in G$. By (3.10), (3.12) is true for some $M = M(B) \geq 1$ if, in addition, $x, y \in B \cup L(G)$. If $x \in \Omega(G) \backslash B$, there is $x' \in B$ such that $|x - z| = |x' - z|$. One now easily sees that it suffices to consider in (3.12) only triples x, y, z such that in addition $x \in \Omega(G)\setminus B$ and $y \in B$; then we get (3.12) for all triples by increasing M. If this is not true we can find sequences $z_j \in L(G)\setminus \mathbf{R}^n$, $x_j \in (\Omega(G)\setminus B) \cap \mathbf{R}^n$ and $y \in B \cap \mathbf{R}^n$ with $|x_j - z_j| = |y_j - z_j|$ such that

$$\textbf{(3.13)} \qquad H_j = |f'(x_j) - f'(z_j)|/|f'(y_j) - f'(z_j)| \to H \in \{0, \infty\}$$

as $j \to \infty$.

Since the number of components of $(\Omega(G)\setminus B)/G$ is finite, we can assume that there is a component U of $\Omega(G)\setminus B$ such that $x_j \in g_j^{-1}(U)$ for some $g_j \in G$. We can assume, by passing to a subsequence, that there is a smaller cusp $V \subset U \neq V$ with the same center S_1 and vertex u_1 as U for which $x_j \in g_j^{-1}(V)$, too. Otherwise there would be a G-invariant set $B' \subset \Omega(G)$ such that B'/G is compact and that $B' \supset B \cup \{x_j : j > 0\}$ and then (3.10) would imply a contradiction with (3.13).

We next choose similarities α_j and β_j of $\mathbf{R}^n$ such that $\alpha_j(0) = z_j$, $\alpha_j(e_1) = y_j$, $\beta_j^{-1} f' \alpha_j(0) = 0$ and $\beta_j^{-1} f' \alpha_j(e_1) = e_1$. Let

$$f_j = \beta_j^{-1} f' \alpha_j = g_{2j}^{-1} f' g_{1j}$$

where $g_{1j} = g_j \alpha_j$, $g_{2j} = \varphi(g_j) \beta_j$ and where we extend α_j and β_j to $\overline{\mathbf{R}}^n$ by setting $\alpha_j(\infty) = \infty = \beta_j(\infty)$. Then every f_j is a homeomorphism of $\overline{\mathbf{R}}^n$ fixing 0, e_1 and ∞ such that

$$\textbf{(3.14)} \qquad H_j = |f_j(x_j')|$$

where $x_j' = \alpha_j^{-1}(x_j)$ and $|x_j'| = 1$.

We can now apply Theorem 2.11 with substitutions

$$f \mapsto f', \quad u_2 \mapsto f'(u_1), \quad G_1 \mapsto G_{u_1}, \quad G_2 \mapsto G'_{u_2}, \quad \varphi \mapsto \varphi \mid G_1,$$
$$a_{1j} \mapsto 0, \quad a_{2j} \mapsto e_1, \quad a_{3j} \mapsto \infty.$$

Let k be the rank of the parabolic fixed points u_1 and u_2 of G and G′, respectively. Then k does not depend on i by Lemma 3.2. For S_2 we can take any G_2'-invariant k-sphere through u_2. Now $b_{1j} = f_j(a_{1j}) = 0$, $b_{2j} = f_j(a_{2j}) = e_1$ and $b_{3j} = f_j(a_{3j}) = \infty$. Hence the condition (2.27) is true and since also

$$a_{1j}, a_{2j} \in \alpha_j^{-1}(L(G) \cup B) \subset \overline{\mathbf{R}}^n \setminus U_j \quad (U_j = \alpha_j^{-1} g_j^{-1}(U) = g_{1j}^{-1}(U)),$$

the conditions of Theorem 2.11 are satisfied. Consequently, we can assume that one of the cases a) or b) of Theorem 2.11 occurs.

Assume that we have the case a) of Theorem 2.11. Let v_j be the vertex of the cusp V_j. Then $v_j \to x_j'$ and thus, since $|x_j'| = 1$, $|v_j| \in [1/2, 2]$ for big j. Since $\alpha_j(0) \in L(G)$, $\alpha_j(v_j) \in L(G)$ and $\alpha_j(e_1) \in B$, this and (3.10) now imply that there is $m \geq 1$ such that $|f_j(v_j)| \in [1/m, m]$ if j is big. Now, $x_j' \in V_j$ and since $q(f_j(V_j)) \to 0$, a contradiction with (3.13) and (3.14) follows.

In case b) of Theorem 2.11, $f_j \to h$ uniformly in the spherical metric for some homeomorphism h of $\overline{\mathbf{R}}^n$. Then $h(0) = 0$ and $h(\infty) = \infty$. Thus there is $m \geq 1$

such that $1/m \leq |h(z)| \leq m$ if $|z| = 1$, implying a contradiction with (3.13) and (3.14).

These contradictions prove that (3.12) is true for some M and we have proved that f' is quasiconformal.

To conclude the proof, we must still consider the case that $n = 1$, $f'(\infty) = \infty$ and f' is locally quasisymmetric. Then $f' \mid \mathbf{R}$ is quasisymmetric if we can show that there is a quasiconformal extension F of f' to $\overline{H}^2$. If $\Omega(G) = \overline{\mathbf{R}}$, this can be seen as in [39, 3.15.4] since a locally quasisymmetric embedding of a compact space is quasisymmetric [37, 2.23]. If $\Omega(G) = \overline{\mathbf{R}}\backslash\{x\}$, then x is a parabolic fixed point of G and if $x = \infty$, as we can assume, then one easily sees (cf. [13, Remark in II.7.1]) that $f \mid \mathbf{R}$ is quasisymmetric and hence such an extension exists.

Thus we can assume that L(G) consists of more than one point. We can also assume that every $g \in G$ is orientation preserving. We utilize the well-known result that there is a quasiconformal map $F_0 : \overline{H}^2 \to \overline{H}^2$ inducing φ. Then $F_0 \mid L(G) = f' \mid L(G)$. We show that we can modify F_0 in such a way that $F_0 \mid \overline{\mathbf{R}} = f'$. Let I be a component of $\Omega(G)$. Let J be the hyperbolic line with the same endpoints as I and let D be the domain of H^2 bounded by $\mathrm{cl}(I \cup J)$. Then there is a hyperbolic $g \in G$ such that if $h \in G$, then $h(D) = D$ if $h = g^k$ for some $k \in \mathbf{Z}$ and otherwise $h(D) \cap D = \varnothing$. Let $\langle g \rangle$ be the group generated by g. Now $D/\langle g \rangle$ and $F_0(D)/\langle \varphi(g) \rangle$ are conformally equivalent to annular domains of the form $1 \leq |z| \leq r$, and then Kelingos [12, Theorem 1] implies that there is a quasiconformal $F_1 : \overline{H}^2 \to \overline{H}^2$ inducing φ such that

$$F_1 \mid \overline{H}^2\backslash D = F_0 \mid \overline{H}^2\backslash D$$

and

$$F_1 \mid I = f' \mid I$$

if we can show that f' can be extended to a quasiconformal map $F' : U \to \overline{H}^2$ inducing $\varphi \mid \langle g \rangle$ for some $\langle g \rangle$-invariant neighbourhood U of I in $\overline{H}^2$.

We show that there is such F′. We can assume that $I = f'(I) = (0, \infty)$ and $f'(\infty) = \infty$. Then g and $\varphi(g)$ are of the form $z \mapsto \lambda z$. Let $F' : \overline{H}^2 \to \overline{H}^2$ be the Beurling-Ahlfors extension of f' ([13, II.6.5]). Then F′ is always a homeomorphism of $\overline{H}^2$ which is quasiconformal if f' is quasisymmetric. The analytic expression of F′ implies that F′ induces $\varphi \mid \langle g \rangle$. One sees also from this expression that if $f' \mid [a, b]$ is quasisymmetric, then F′ is quasiconformal in the euclidean triangle of $\mathbf{R}^2$ with vertices $(a, 0)$, $(b, 0)$ and $((a + b)/2, (b - a)/2)$. This, and the fact that F′ induces $\varphi \mid \langle g \rangle$, imply that F′ is quasiconformal in such a neighbourhood of I as claimed.

Since the number of components of $\Omega(G)/G$ is finite, we see that we can modify F_0 to a quasiconformal map F inducing φ such that $F \mid \overline{\mathbf{R}} = f'$.

We have now completely proved the theorem.

Remarks 1. — If we assume in Theorem 3.8 that $f : \Omega(G) \to \Omega(G')$ is a continuous map inducing φ and wish to show that φ is type-preserving and that the map f' defined by f_φ and f is a *continuous* map of $\overline{\mathbf{R}}^n$, then the above proof applies if we assume the following. If $n = 1$, we assume as in Theorem 3.8 that φ is type-preserving. If $n > 1$, we assume that the map $\Omega(G)/G \to \Omega(G')/G'$ defined by f can be extended to a continuous map of the cusp-compactifications (see 2D) in such a way that the equivalence classes of parabolic fixed points of G of rank $< n$ are mapped surjectively onto the equivalence classes of parabolic fixed points of G′ of rank $< n$. Thus if $n > 1$ and $\Omega(G)/G$ and $\Omega(G')/G'$ are compact, no additional assumption is necessary.

2. — It is interesting to note that if the map $f: \Omega(G) \to \Omega(G')$ of Theorem 3.8 satisfies a Lipschitz type condition, then φ is either a conjugation by a Möbius transformation or comes very near of being it.

Suppose that $\Omega(G) \neq \varnothing$ and that for some $L > 1$

$$|x-y|/L \leq |f(x)-f(y)| \leq L\,|x-y| \tag{3.15}$$

which is valid for all $x \in \Omega(G)\setminus\{\infty, f^{-1}(\infty)\}$ and for all y in some neighbourhood U_x of x. Then f is quasiconformal (if $n > 1$) and so is the extension f' of f to $\overline{\mathbf{R}}^n$ by Theorem 3.8. Thus, in particular, f' is ACL and it follows, since L(G) has zero measure in the geometrically finite Kleinian case, that (3.15) is true for f' and for all $x, y \in \mathbf{R}^n$ (it follows that $f'(\infty) = \infty$). This is valid also if $n = 1$.

Thus we get a bilipschitz map of $\overline{\mathbf{R}}^n$ and ([34]) it follows that φ preserves multipliers (see 1C) which has the consequence ([34]) that φ is a conjugation by a Möbius transformation, at least if $L(G) \subset h(\overline{\mathbf{R}}^k)$ for no $k < n$ and for no Möbius transformation h.

4. GEOMETRIC ISOMORPHISMS OF KLEINIAN GROUPS OF $\overline{\mathbf{R}}^2$

4A. Introduction and some definitions. — We now apply the results of Section 3 to the isomorphism problem of Kleinian groups of $\overline{\mathbf{R}}^2$. Let $\varphi: G \to G'$ be an isomorphism between two geometrically finite, non-elementary Kleinian groups of $\overline{\mathbf{R}}^2$. We seek conditions guaranteeing the existence of a homeomorphism $F: \overline{H}^3 \to \overline{H}^3$ of hyperbolic spaces inducing φ; in this case φ is said to be *geometric*.

Marden [14] has shown that φ is always geometric if there is a homeomorphism $f: \Omega(G) \to \Omega(G')$ of the ordinary sets inducing φ; he assumes that G and G′ are torsionless and do not contain orientation reversing elements. In the first part of this section we give a new proof of Marden's theorem which is valid for all geometrically finite G and G′. It is based on Theorem 3.8 which allows the extension of f to the limit set of G to a homeomorphism of $\overline{\mathbf{R}}^2$ which is quasiconformal if f is. We then refer to a theorem which asserts that if f is quasiconformal, then there is a quasiconformal extension of f to $\overline{H}^3$ inducing φ (Theorem 4.1). Since f can always be modified to a quasiconformal map, we get our theorem.

In the latter part of this section we seek conditions which would imply the existence of a homeomorphism $\Omega(G) \to \Omega(G')$ inducing φ and hence also the geometricity of φ (Theorem 4.7). Our conditions are reminiscent of the Fenchel-Nielsen intersecting axis condition for Fuchsian groups, stated in terms of axes of loxodromic elements ([15, 30, 31]), but we now consider a loxodromic $g \in G$ and a quasi-Fuchsian $H \subset G$ (cf. 4C and 4D).

In this part we rely much on earlier work, most of it being due to Maskit; especially this is so if L(G) is non-connected; if it is connected, then our presentation is little more self-contained. The existence of the map f_φ of the limit sets is helpful but perhaps not crucial. Possibly in this part our main contribution to the isomorphism problem of Kleinian groups is the statement of these conditions for the geometricity of φ which assume a fairly simple form if there are no parabolic nor elliptic elements.

It should be remarked that the proof of our theorem is especially simple if L(G) is connected and if the groups do not contain parabolic elements, cf. Remark 1 following Theorem 4.7.

If every component of $\Omega(G)$ is simply connected, Marden and Maskit [16] have given conditions implying the geometricity of φ. Johannson's results [11] imply that there is a large class of groups for which every isomorphism is geometric (cf. also Jaco [10, X.15]). Maskit [20] has considered the case of function groups (i.e. $\Omega(G)$ has an invariant component). It is essential to us if $\Omega(G)$ has non-simply-connected components.

We say that a Kleinian group G of $\overline{\mathbf{R}}^2$ is *quasi-Fuchsian* if G is finitely generated and if L(G) is a topological circle. If g is loxodromic, we let $P(g)$ be the attractive fixed point of g and $N(g)$ the repelling fixed point; if g is parabolic, we let $P(g) = N(g)$ be the fixed point of g. An isomorphism $\varphi: G \to G'$ of two Kleinian groups of $\overline{\mathbf{R}}^2$ is *strongly type-preserving* if it is type-preserving (cf. 1A) and if, whenever $g \in G$ is elliptic and orientation preserving, also $\varphi(g)$ is and they correspond to rotations through angles of the same absolute value. Usually the words "type-preserving" in the literature refer to this stronger condition.

I wish to thank Albert Marden for pointing out an error in an earlier version of this section. He also informed me about Johannson's results mentioned above.

4B. Extension of a map of ordinary sets. — In our case the extension to $\overline{H}^3$ is obtained by

Theorem **4.1.** — *Let* G *and* G′ *be Möbius groups of* $\overline{\mathbf{R}}^2$ *and let* f *be a quasiconformal homeomorphism of* $\overline{\mathbf{R}}^2$ *inducing an isomorphism* $\varphi: G \to G'$. *Then there is a quasiconformal homeomorphism* F *of* $\overline{H}^3$ *which extends* f *and induces* φ.

A proof of this theorem has been given by Reimann [26]. Another proof is to be included in chapter of 11 of Thurston's book [29]; according to it, $F \mid H^3$ is not only quasiconformal but even bilipschitz in the hyperbolic metric of H^3. A third approach to this theorem, which should work for all discrete G and G′, has been indicated in [32, 1E].

Theorems 3.8 and 4.1 have the following corollary which gives an improved version of Marden's isomorphism theorem for geometrically finite groups.

Theorem **4.2.** — *Let* G *and* G' *be geometrically finite Kleinian groups of* $\overline{\mathbf{R}}^2$ *and let* $f:\Omega(G)\to\Omega(G')$ *be a homeomorphism inducing an isomorphism* $\varphi: G \to G'$. *Then* f *can be extended to a homeomorphism* F *of* $\overline{H}^3$ *inducing* φ. *In addition,* F *is quasiconformal if* f *is quasiconformal and* F *is a Möbius transformation if* f *is conformal or if* $\Omega(G) = \varnothing$. *Here quasiconformal means that* f *is an orientation preserving or reversing quasiconformal embedding independently in each component of* $\Omega(G)$ *and in the conformal case it is allowed that* f *is anticonformal in some components of* $\Omega(G)$.

Proof. — Theorem 3.8 implies that f can be extended to a homeomorphism f_0 of $\overline{\mathbf{R}}^2$ inducing φ which is quasiconformal if f is. Thus, if f is quasiconformal, there is by Theorem 4.1 a quasiconformal extension F of f to $\overline{H}^3$ inducing φ. If f is conformal or if $\Omega(G) = \varnothing$, then f_0 is a Möbius transformation by Theorem 3.8 and then the unique Möbius transformation F extending f_0 to $\overline{H}^3$ induces also φ.

Thus we must only consider the case in which f is a homeomorphism, not necessarily quasiconformal. The quotient $\Omega(G)/G$ is obtained from a compact Riemann surface S by the removal of a finite number of points. S may be non-connected and it may have non-orientable or bordered components if G contains orientation reversing elements. In addition to the removed points, there is a finite number of marked points of S which correspond to fixed points in $\Omega(G)$ of elliptic elements of G. In the same manner, $\Omega(G')/G'$ is obtained from a compact surface S' by the removal of a finite number of points. We can regard S and S' as PL 2-manifolds, the PL structures being compatible with the conformal structures in the sense that a PL homeomorphism $S \to S'$ is always quasiconformal.

The homeomorphism f induces a homeomorphism $\bar{f}: S \to S'$. It is well-known that $\bar{f}$ is isotopic to a PL map $\bar{f}_1 : S \to S'$ such that $\bar{f}_1$ and the isotopy respect the removed and marked points. Since $\bar{f}$ and $\bar{f}_1$ are isotopic by such an isotopy we can lift $\bar{f}_1$ to a homeomorphism $f_1 : \Omega(G) \to \Omega(G')$ which induces φ. Since $\bar{f}_1$ is quasiconformal, f_1 is also. Thus there is, as we have shown, an extension F_1 of f_1 to a homeomorphism of $\overline{H}^3$ inducing φ. The map F_1 induces a homeomorphism

$$\overline{F}_1 : M_G = (\overline{H}^3\setminus L(G))/G \to M_{G'} = (\overline{H}^3\setminus L(G'))/G'$$

such that $\overline{F}_1 \mid S = \bar{f}_1$. Since $\bar{f}$ and $\bar{f}_1$ are isotopic, we can, using a collar of S in M_G, deform $\overline{F}_1$ to a homeomorphism $\overline{F}_2 : M_G \to M_{G'}$ such that $\overline{F}_2 = \bar{f}$ in S.

Since the isotopy respects the marked and removed points, $\overline{F}_2$ lifts to a homeomorphism $F_2 : \overline{H}^3\setminus L(G) \to \overline{H}^3\setminus L(G)$ inducing φ for which $F_2 \mid \Omega(G) = f$. By a reflection on $\overline{\mathbf{R}}^2$ we can extend F_2 to a homeomorphism $F_3 : \overline{\mathbf{R}}^3\setminus L(G) \to \overline{\mathbf{R}}^3\setminus L(G')$ inducing φ. Since G and G' are also geometrically finite when considered as groups of $\overline{\mathbf{R}}^3$, Theorem 3.8 implies that F_2 can be extended to a homeomorphism F of $\overline{H}^3$ inducing φ. Then $F \mid \Omega(G) = f$ and F is the desired extension.

Remarks 1. — Our theorem is stronger than Marden's in the respect that in our case the groups may contain elliptic and orientation reversing elements. On the other hand, Marden assumes that only G is geometrically finite. We can also drop the assumption on geometrical finiteness of G′ using Marden's results. These imply that also G′ is geometrically finite since now, by a theorem of Selberg [27], we can pass to subgroups of finite index in such a way that everything is orientation preserving and torsionless. Then Marden's theorem ([14, Theorem 8.1] and [16, p. 10]) implies that a subgroup of G′ of finite index is geometrically finite. Hence also G′ is and thus the assumptions of Theorem 4.2 are valid. Except for this improvement, Theorem 4.2 is independent of Marden's theorem and the 3-dimensional topology used by him.

2. — An analogue of Theorem 4.1 is valid for Möbius groups of $\overline{\mathbf{R}}^1$; in it one assumes that f fixes ∞ and that $f \mid \mathbf{R}$ is quasisymmetric (cf. [32]). Also Theorem 4.2 can be modified for $n = 1$; now one must assume that φ is type-preserving (which follows from the assumptions if $n = 2$) and the analogue of the condition that f is quasiconformal is that f is locally quasisymmetric. However, the extension need not be a Möbius transformation even if f is locally a Möbius transformation or if $\Omega(G) = \varnothing$. This theorem is naturally nothing new in the theory of Fuchsian groups but the interesting point is that this proof is in principle non-topological, that is, we do not have to examine the explicit surface topology of H^2/G or of H^2/G'; recall that this was true of the proof of [32]. It is true that our proof of Theorem 3.8 refers to results which make use of the topology but this could be avoided, cf. the remark following Lemma 3.7.

4C. Intersection preserving isomorphisms. — The Fenchel-Nielsen intersecting-axis theorem for Fuchsian groups can be stated as follows. If g is a hyperbolic transformation of the hyperbolic plane H^2, let $Ax(g)$ be the hyperbolic line joining the points fixed by g; if h is another such map, we say that g and h *intersect* if $Ax(g) \neq Ax(h)$ and if $Ax(g) \cap Ax(h) \neq \varnothing$. If now G and G′ are two Fuchsian groups acting in H^2 and $\varphi : G \to G'$ is an isomorphism, then $\varphi(g) = fgf^{-1}$ for some homeomorphism f of H^2 if and only if φ preserves intersection. That is, $g, h \in G$ are hyperbolic elements with intersecting axes if and only if $\varphi(g), \varphi(h) \in G'$ are also.

We now state a natural generalization of this condition for Kleinian groups of $\overline{\mathbf{R}}^2$. Let H be a quasi-Fuchsian group and let g be a loxodromic Möbius transformation of $\overline{\mathbf{R}}^2$. Then we say that g and H *intersect* if and only if the fixed points of g are in different components of $\overline{\mathbf{R}}^2 \setminus L(H)$. If $\varphi : G \to G'$ is a type-preserving isomorphism of two geometrically finite Möbius groups of $\overline{\mathbf{R}}^2$, we say that φ *preserves intersection* if, whenever $g \in G$ is loxodromic and $H \subset G$ is a quasi-Fuchsian subgroup, g and H intersect if and only if $\varphi(g)$ and $\varphi(H)$ intersect. Observe that then $\varphi(g)$ is loxodromic and since $L(\varphi(H)) = f_\varphi(L(G))$ = a topological circle, $\varphi(H)$ is also quasi-Fuchsian. Obviously, if φ preserves intersection, so does φ^{-1}.

We have the following characterization for intersection preserving isomorphisms. Here and in the following $f_\varphi : L(G) \to L(G')$ is the map of Theorem 3.3.

Lemma **4.3.** — *Let* $\varphi : G \to G'$ *be a type-preserving isomorphism of geometrically finite Möbius groups of* $\overline{\mathbf{R}}^2$. *Then* φ *is intersection preserving if and only if it has the following property. Let* $S \subset L(G)$ *be a circle such that* $S = L(H)$ *for some quasi-Fuchsian* $H \subset G$ *and let* $x, y \in L(G) \setminus S$. *Then* x *and* y *are in different components of* $\overline{\mathbf{R}}^2 \setminus S$ *if and only if* $f_\varphi(x)$ *and* $f_\varphi(y)$ *are in different components of* $\overline{\mathbf{R}}^2 \setminus f_\varphi(S)$.

Proof. — This lemma is a consequence of the following fact. Let U and V be open sets of $\overline{\mathbf{R}}^2$ intersecting with $L(G)$. Then there is a loxodromic $g \in G$ with one fixed point in U and the other in V. The proof is simple. The argument for the Fuchsian case given in [30, Proposition 1.4 and 3.1] applies now as well. The lemma follows.

The importance of intersection preserving isomorphisms is due to the fact that, if $\varphi : G \to G'$ is such an isomorphism, then, in the non-quasi-Fuchsian case, it defines a bijection of the components of $\Omega(G)$ onto the components of $\Omega(G')$ by

Lemma **4.4.** — *Let* $\varphi : G \to G'$ *be an intersection preserving isomorphism of two non-quasi-Fuchsian, geometrically finite Möbius groups of* $\overline{\mathbf{R}}^2$. *Then, if* D *is a component of* $\Omega(G)$, *there is a unique component* D' *of* $\Omega(G')$ *such that* $f_\varphi(\partial D) = \partial D'$. *The stabilizers of these components satisfy*

$$\varphi(G_D) = G'_{D'}. \tag{4.1}$$

Furthermore, if D' *is a component of* $\Omega(G')$, *then there is a unique component* D *of* $\Omega(G)$ *such that* $f_\varphi^{-1}(\partial D') = \partial D$; *for these components* (4.1) *is true.*

Proof. — Obviously, we need to prove only the first paragraph. We can also assume that every $g \in G \cup G'$ is orientation preserving.

Let D_i, $i \in I$, be the components of $\Omega(G) \setminus D$ and set $H = G_D$, $H_i = G_{D_i}$, $H' = \varphi(H)$ and $H'_i = \varphi(H_i)$. By [18, Theorem 3] H_i is quasi-Fuchsian and hence $L(H_i) = \partial D_i$ is a circle. Also, since $D/H = D/G$ is a Riemann surface of finite type, $L(H) = \partial D$. Lemma 4.3 implies that there is a component D'_i of $\overline{\mathbf{R}}^2 \setminus f_\varphi(\partial D_i)$ such that

$$\begin{aligned} D'_i \cap L(G') &= f_\varphi(D_i \cap L(G)), \quad \text{and} \\ D'_i \cap L(H') &= f_\varphi(D_i \cap L(H)) = \varnothing. \end{aligned} \tag{4.2}$$

Set now

$$\begin{aligned} A &= L(H) \cup (\bigcup_{i \in I} D_i) = \overline{\mathbf{R}}^2 \setminus D, \quad \text{and} \\ A' &= L(H') \cup (\bigcup_{i \in I} D'_i) \end{aligned}$$

and define a map $f : A \to A'$ by $f \mid L(H) = f_\varphi \mid L(H)$ and by $f \mid \mathrm{cl}\, D_i = f_i$ where $f_i : \mathrm{cl}\, D_i \to \mathrm{cl}\, D'_i$ is a homeomorphism extending $f_\varphi \mid \partial D_i$. Then (4.2) implies that f is a bijection and since only finitely many of the spherical diameters of D_i and D'_i exceed a given positive number by Ahlfors' finiteness theorem and by [19, Theorem 6] (or by [36, Corollary E]), f is continuous and hence a homeomorphism.

Let $D' = \overline{\mathbf{R}}^2 \setminus A' = \overline{\mathbf{R}}^2 \setminus f(A)$. We show that D' is a component of $\Omega(G')$. Obviously D' is open and since $A \neq \mathbf{R}^2$, $D' \neq \varnothing$. Since A does not separate $\overline{\mathbf{R}}^2$, neither does A' by [9, p. 101]. Thus D' is connected. Since $L(G) \subset A$, $D' \cap L(G) = \varnothing$ by (4.2) and since

$$\textbf{(4.3)} \qquad \partial D' = \partial A' = f_\varphi(\partial D) = f_\varphi(L(H)) = L(H') \subset L(G'),$$

D' is a component of $\Omega(G')$.

To prove the remaining parts of the lemma we need to know that for no components $\widetilde{D}$ of $\Omega(G) \setminus D$ or $\widetilde{D}'$ of $\Omega(G') \setminus D'$ it is true that

$$\textbf{(4.4)} \qquad \partial D = \partial \widetilde{D} \quad \text{or} \quad \partial D' = \widetilde{D}'.$$

We need to consider only the first case in (4.4). Suppose that there is such a component $\widetilde{D}$ of $\Omega(G) \setminus D$ for which (4.4) is true. Then $\partial \widetilde{D} \subset \operatorname{cl} D_i$ for some $i \in I$. Since $D_i \cap D = \varnothing$, actually now $\partial \widetilde{D} \subset \partial D_i \subset \partial D$. Hence, by (4.4), $\partial \widetilde{D} = \partial D_i = \partial D$ and it would follow that $L(G) \subset \partial D_i$. That is, G would be quasi-Fuchsian contrary to our assumption and (4.4) follows.

We get by (4.4) immediately that D' is unique and that

$$G_D = G_{\partial D} \quad \text{and} \quad G'_{D'} = G'_{\partial D'}$$

which imply (4.1). The lemma is proved.

Remarks 1. — Actually, in Lemmas 4.3 and 4.4 G and G' can be any finitely generated discrete Möbius groups of $\mathbf{R}^2$ (in Lemma 4.3, they need not be even finitely generated), provided that one knows that there is a homeomorphism $f_\varphi : L(G) \to L(G')$ inducing intersection preserving φ; if f_φ is known to exist, then the intersection preserving property can be defined as above.

2. — It follows by Lemma 4.4 that in Lemma 4.3 the set S can be any closed subset of $L(G)$: Furthermore, if it is known that there is a homeomorphism $f_\varphi : L(G) \to L(G')$ inducing an intersection preserving φ, then by the preceding remark, G and G' can be any finitely generated discrete Möbius groups of $\overline{\mathbf{R}}^2$.

3. — Actually, it is sufficient for Lemma 4.4 that φ satisfies the intersection preserving property for all pairs (g, H) where $g \in G$ is loxodromic and H is a quasi-Fuchsian subgroup such that $H \subset G_D$ for some component D of $\Omega(G)$ or $\varphi(H) \subset G'_{D'}$ for some component D' of $\Omega(G')$.

4. — Marden and Maskit [16, Theorem 1] have given another set of conditions on φ implying (4.1). Observe that their assumption that every component of $\Omega(G)$ is simply connected is not needed in the proof of (4.1) ([16, pp. 12-13]).

4D. Orientation consistent isomorphisms. — If we would consider only groups G such that $L(G)$ is connected, the intersection preserving property defined above would suffice for our main theorem 4.7. However, if L(G) is non-connected, we must add conditions concerning orientation. We would need something like oriented intersection. Since they may be quasi-Fuchsian $H \subset G$ for which $L(G) \cap D = \varnothing$ for some component D of $\overline{\mathbf{R}}^2 \setminus L(H)$, we give it in the following form. We must take care also of subgroups of G corresponding to parabolic fixed points of rank two.

Let $\varphi : G \to G'$ be a type-preserving isomorphism of geometrically finite groups of $\overline{\mathbf{R}}^2$. Then we say that φ is *orientation preserving* if it is true that

a) if $H \subset G$ is a quasi-Fuchsian subgroup and $g \in G$ is loxodromic, then there is an orientation preserving homeomorphism f of $\overline{\mathbf{R}}^2$ extending $f_\varphi \mid L(H)$ such that $f(P(g)) = P(\varphi(g))$ ($P(g)$ is the attracting fixed point), and

b) if $H = G_v$ for some parabolic fixed point of G of rank two, then there is an orientation preserving homeomorphism of $\overline{\mathbf{R}}^2$ inducing $\varphi \mid H$.

The isomorphism φ is *orientation reversing* if a) and b) are true with the words " orientation preserving " replaced by the words " orientation reversing " and φ is *orientation consistent* if it is either orientation preserving or reversing. This terminology is adapted from Maskit [20].

The following lemma is obvious.

Lemma **4.5.** — *Let* $\varphi : G \to G'$ *be an isomorphism of geometrically finite groups of* $\overline{\mathbf{R}}^2$. *Then* φ *is intersection preserving if it is orientation consistent.*

4E. Isomorphisms of function groups. — Our main theorem 4.7 is based on the following theorem due to Maskit in case L(G) is non-connected. A *function group* is a Möbius group G which has a G-invariant component D of $\Omega(G)$.

Theorem **4.6.** — *Let* G *and* G′ *be geometrically finite, non-elementary function groups of* $\overline{\mathbf{R}}^2$ *with invariant domains* D *and* D′, *respectively, and let* $\varphi : G \to G'$ *be an isomorphism. If* L(G) *is connected, assume that* φ *is type-preserving. If* L(G) *is not connected, assume that* φ *is strongly type-preserving and orientation consistent, that every element of* G *and* G′ *is orientation preserving and that every finite subgroup of* G *is cyclic. Then there is a quasiconformal homeomorphism* $f : D \to D'$ *inducing* φ.

Proof. — One sees as in the proof of Theorem 4.2 that it suffices to find a homeomorphism $f : D \to D'$ inducing φ since then f can be modified to a quasiconformal homeomorphism inducing φ. We now find such an f.

Assume that L(G) is non-connected. Then this is Maskit's theorem in [20, 1.8]. We must only verify that Maskit's conditions are satisfied. Otherwise this is clear but we must show that φ is orientation consistent in Maskit's sense. Let φ be, say, orientation preserving (in our sense). Let H be a factor subgroup of G ([20, 1.5]). Let D_H be the component of $\Omega(H)$ such that $D_H \supset D$ and let D'_H be the component of $\Omega(\varphi(H))$

such that $D'_H \supset D'$. We must show that there is an orientation preserving homeomorphism $h : D_H \to D'_H$ inducing $\varphi \mid H$. We consider separately the following cases.

a) L(H) consists of more than one point.

b) L(H) consists of a point.

c) $L(H) = \varnothing$.

In case a) H is non-elementary and then [18, Theorem 4] implies that H is quasi-Fuchsian or degenerate. By [21, 1.11 and Theorem 9] H is in fact quasi-Fuchsian. Then, as is well-known [15, 31], there is an orientation preserving homeomorphism h of $\overline{\mathbf{R}}^2$ extending $f_\varphi \mid L(H)$ and inducing $\varphi \mid H$. Then condition a) of 4D implies that $h(D_H) = D'_H$ since $L(G) \neq L(H)$.

In case b), $H = G_v$ for some parabolic fixed point v of G, cf. Theorem 2.1. If H has rank two, then b) of 4D implies that there is an orientation preserving homeomorphism $h : \overline{\mathbf{R}}^2 \backslash \{v\} \to \overline{\mathbf{R}}^2 \backslash \{f_\varphi(v)\}$ inducing $\varphi \mid H$. If H has rank one, then, as is well-known (cf. [21, 2.4]), H and $\varphi(H)$ are either cyclic groups generated by a parabolic element or are conjugate (in the group of Möbius transformations) to the group whose elements are of the form $z \to \pm z + k$, $k \in \mathbf{Z}$. Regardless of the case at hand, there is now always both an orientation preserving and reversing homeomorphism $\overline{\mathbf{R}}^2 \backslash \{v\} \to \overline{\mathbf{R}}^2 \backslash \{f_\varphi(v)\}$ inducing $\varphi \mid H$.

In case c) H is finite and hence cyclic by our assumptions. Thus H is generated by an elliptic element and since φ is strongly type-preserving, there is again both an orientation preserving and reversing homeomorphism of $\overline{\mathbf{R}}^2$ inducing $\varphi \mid H$.

The case that φ is orientation reversing is completely similar and thus the theorem is true if L(G) is non-connected.

We then consider the case that L(G) is connected. If G and G′ are quasi-Fuchsian (this is true always if there are no parabolic elements), then our theorem is the classical Fenchel-Nielsen theorem [15, 31]. If L(G) is not a circle, then the existence of such an f is also more or less known although I have not found it in the literature. Using Maskit's theorem we can reason as follows. Let $H \subset G$, $H' = \varphi(H) \subset G'$ be torsionless subgroups of finite index such that every $g \in H \cup H'$ is orientation preserving; by [27, Lemma 8] there are such groups. Thus D is H-invariant and $L(H) = L(G)$. Then H has a connected structure [20, 4.1] (the stabilizers of structure loops are now parabolic groups) and by [20, 6.4 and 1.8] there is a homeomorphism $h : D \to D'$ inducing $\varphi \mid H$.

Since G and G′ are non-elementary, D and D′ are conformally equivalent to H^2. Consequently we can transform the situation by conformal mappings to the following one: G and G′ are finitely generated Möbius groups of H^2 such that there is a homeomorphism h of H^2 inducing $\varphi \mid H$ where H is a subgroup of finite index. It follows that φ satisfies the Fenchel-Nielsen intersecting-axis condition and thus there is a homeomorphism of H^2 inducing φ ([15, 31]). This homeomorphism can then be transferred back to a homeomorphism $D \to D'$.

4F. The isomorphism theorem. — We can now prove easily our second main theorem in this section.

Theorem **4.7.** — *Let* $\varphi : G \to G'$ *be a type-preserving isomorphism of two non-elementary, geometrically finite groups of* $\overline{\mathbf{R}}^2$. *In addition,*

a) *if* L(G) *is a circle, assume that g is orientation reversing if and only if* $\varphi(g)$ *is,*
b) *if* L(G) *is connected but not a circle, assume that* φ *preserves intersection,*
c) *if* L(G) *is non-connected, assume that elements of* G *and* G′ *are orientation preserving, that every finite subgroup of* G *is cyclic and that* φ *is strongly type-preserving and orientation consistent.*

Then φ *is induced by a quasiconformal homeomorphism* F *of* $\overline{\mathrm{H}}^3$.

Proof. — By Theorem 4.2 it suffices to find a quasiconformal homeomorphism $f : \Omega(G) \to \Omega(G')$ inducing φ. We now show that in every case there is such a homeomorphism.

In case a) we show first that $g \in G$ preserves the components of $\Omega(G)$ if and only if $\varphi(g)$ preserves the components of $\Omega(G')$. This is true since we can characterize the elements of G preserving the components of $\Omega(G)$ as the elements $g \in G$ for which g and $g \mid L(G)$ (considered as a homeomorphism of L(G)) are both either orientation preserving or both orientation reversing. Similar characterization is valid for elements of G′. Now a) and the fact that $\varphi(g) \mid L(G') = f_\varphi g f_\varphi^{-1} \mid L(G')$ imply our claim. Using Lemma 4.6 we now find the quasiconformal homeomorphism $f : \Omega(G) \to \Omega(G')$ inducing φ.

We then prove b) and c) together. In these cases φ is intersection preserving (see Lemma 4.5) and hence by Lemma 4.4, given a component D of $\Omega(G)$, there is a unique component D′ of $\Omega(G')$ such that $f_\varphi(\partial D) = \partial D'$ and that $\varphi(G_D) = G'_{D'}$, and vice versa. By Marden [14, Corollary 6.5], G_D and $G'_{D'}$ are geometrically finite. Hence we can apply Theorem 4.6 and find a quasiconformal homeomorphism $f_D : D \to D'$ inducing $\varphi \mid G_D$. We choose for every component D of $\Omega(G)$ such a map f_D and obviously we can choose them in such a way that they define together a homeomorphism $f : \Omega(G) \to \Omega(G')$ inducing φ. Since the number of components of $\Omega(G)/G$ is finite, f is quasiconformal. The theorem follows.

Remarks 1. — What we essentially need in case b) of Theorem 4.7 is that Lemma 4.4 is true. And (see Remark 3 following Lemma 4.4) this lemma is true even if the assumption that φ preserves intersection is somewhat weakened.

This weakening is especially striking if G does not contain parabolic elements. Then G_D is quasi-Fuchsian for every component D of $\Omega(G)$ (see [18, Theorem 4] and [21, Theorem 9 and 1.11]). Hence, by the above-mentioned remark, it would suffice in this case to assume that φ preserves intersection for all loxodromic $g \in G$ and all quasi-Fuchsian $H \subset G$ of the form G_D or $\varphi^{-1}(G'_{D'})$ for some components D of $\Omega(G)$ or D′ of $\Omega(G')$. This condition comes very near the condition that φ preserves boundary transformations of Marden-Maskit [16, Theorem 2]; a boundary transformation of G is an element g such that $g \in G_D$ for some component D of $\Omega(G)$.

It should be noted that in this special case the proof of Theorem 4.7 b) is especially simple. Now the simple Lemma 4.3 implies almost immediately that components of $\Omega(G)$ and $\Omega(G')$ are in one-to-one correspondence (Lemma 4.4) and the existence of the homeomorphism $f: \Omega(G) \to \Omega(G')$ inducing φ follows from the Fenchel-Nielsen intersecting-axis theorem for Fuchsian groups, cf. 4C.

2. — In cases b) and c), if φ is orientation preserving or reversing, then F can also be chosen to be orientation preserving or reversing, respectively. In case a), and sometimes in case c), F can be chosen to be either orientation reversing or preserving. Cf. [20, 6.1 and 6.5].

3. — If G does not contain parabolic nor torsion elements, one can characterize the intersection preserving property using products in cohomology of the quotient manifold $M_G = (\overline{H}^{n+1} \setminus L(G))/G$ with local coefficients modulo the boundary. Thus, if L(G) is in addition connected, we can give a cohomological characterization for the geometricity of an isomorphism. We hope to return to this.

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